Ph 22.2 – Three-Body Problem

-v20140211-

From one, to two, to three

In the previous assignment, you have used your freshly baked Runge–Kutta code to integrate the differential equations of the *one-body problem*: that is, of the motion of a body immersed in a central gravitational potential. You might already know from your study of general physics that a system of two bodies moving under their mutual gravitational attraction (i.e., a *two-body problem*) can be reduced mathematically to the one-body problem, and then solved exactly. This is not true in general of systems with more than two components. Although there are analytical approximations that can be used to explore these more numerous systems, direct numerical computation is often the tool of choice. In this assignment, you will examine a real-world three-body problem using your Runge–Kutta integrator.

The Trojan asteroids

The mathematician Joseph Lagrange studied the so-called *restricted three-body* systems, where one of the three masses is much smaller than the other two. In this case, to a very good approximation, the two large masses (M_1 and M_2, with $M_1 < M_2$) follow a standard two-body Keplerian orbit, while the small mass (m) moves in the gravitational potential generated by M_1 and M_2. Lagrange examined quasicircular orbits: he determined that, in the *corotating frame* where M_1 and M_2 are at rest, there are five points of equilibrium for m, now known as the *Lagrange points* L1–L5. In the inertial frame, the Lagrange points correspond to *stationary orbits*, which have the property that the distance between m and M_1 and M_2 is constant at all times.

In the corotating frame, the Lagrange points L1–L3 sit along the line that joins M_1 and M_2; these points are *unstable*, meaning that a small mass initially placed there will gradually veer off (this did not keep NASA from placing the SOHO solar observatory at L1 in the Earth–Sun system, and from planning the WMAP microwave-radiation anisotropy probe for L2, because unstable orbits can still be corrected with thrusters). On the contrary, L4 and L5 are stable; they sit roughly along the orbit of M_1, respectively sixty degrees ahead and behind this mass. Apart from mathematical arguments, we can find evidence of their stability by looking up at the sky: the L4 and L5 points of the Jupiter–Sun system are home to the two families of *Trojan asteroids* (so called because they were given names associated with the *Iliad*). Trojan asteroids have been found also at the Lagrange points of Mars and Neptune.

When this assignment was originally written, no trojan asteroids of earth had been found. However, a trojan asteroid of earth has now been discovered:

http://news.nationalgeographic.com/news/2011/07/1107128-trojan-asteroid-earth-planet-orbit-nasa-space-science/

Caltech and students from around the globe met at Caltech in the summer of 2011 to compete in the Caltech Space Challenge to design a mission to a near earth asteroid. They considered this new trojan asteroid, 2010 TK7, but as the article explains, it is more difficult to reach than some other known near earth asteroids. Once you are done with this assignment, which considers trojan asteroids around jupiter, you may wish to model the 2010 TK7 orbit.

- Newton's universal gravitational constant: $G = 6.6742 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$
- Sun's mass: $M_2 = 1.989 \times 10^{30}\,\mathrm{kg}$
- Jupiter's mass: $M_1 = 9.548 \times 10^{-4} \times M_2 = 1.899 \times 10^{27}\,\mathrm{kg}$
- Semi major axis of Jupiter's orbit: $R = 778.3 \times 10^9\,\mathrm{m}$
- Period of Jupiter's orbit: $T_J = 4332.589\,\mathrm{days} = 3.743 \times 10^8\,\mathrm{s}$

Box 1: Physical constants for this week's assignment.

The Assignment

1. *In short:* Study the motion of an asteroid that is orbiting around Jupiter, subject also to the gravitational attraction of the Sun. Verify that L4 and L5 [given by Eq. (6) below] are stable positions, and that nearby positions are not. Assume that Jupiter is in a circular orbit around the Sun, and that the mass of the asteroid is negligible. Use the physical constants and astrophysical values given in Box 1.

   In detail: Work in the corotating frame centered on the center of mass of Jupiter (M_1) and the Sun (M_2). In this frame, without loss of generality, Jupiter and the Sun will sit at the fixed positions

   $$\mathbf{r_1} = \left(\frac{M_2 R}{M_1 + M_2}, 0, 0\right), \quad \mathbf{r_2} = \left(-\frac{M_1 R}{M_1 + M_2}, 0, 0\right). \tag{1}$$

   where R is the Jupiter–Sun separation. Remember that we are working in the rotating frame whose angular velocity is the same as that of the Jupiter–Sun system,

   $$\Omega = \sqrt{\frac{G(M_1 + M_2)}{R^3}}; \tag{2}$$

   then the equation of motion for the position $\mathbf{r} \equiv (x, y, 0)$ of the asteroid is

   $$m\mathbf{a} = -\frac{GmM_1}{|\mathbf{r} - \mathbf{r}_1|^3}(\mathbf{r} - \mathbf{r}_1) - \frac{GmM_2}{|\mathbf{r} - \mathbf{r}_2|^3}(\mathbf{r} - \mathbf{r}_2) - 2m\,\boldsymbol{\Omega} \times \mathbf{v} - m\,\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}), \tag{3}$$

   where $\boldsymbol{\Omega} = \Omega(0, 0, 1)$, and where $\mathbf{v} \equiv (v_x, v_y, 0)$ is the velocity of the asteroid. The terms to the right of the gravitational forces are the Coriolis force,

   $$-\,2m\,\boldsymbol{\Omega} \times \mathbf{v} \equiv 2m\Omega\,(v_y, -v_x, 0), \tag{4}$$

   and the centrifugal force,

   $$-\,m\,\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) \equiv m\Omega^2\,(x, y, 0). \tag{5}$$

   To test the stability of the L4 and L5 points, put the asteroid at rest at the initial position

   $$r_{\mathrm{init}} = R\left(\frac{M_2 - M_1}{M_1 + M_2}\cos\alpha, \sin\alpha, 0\right), \tag{6}$$

   where L4 and L5 correspond to $\alpha = \pm\pi/3$; then evolve the orbit of the asteroid for several periods $T = 2\pi/\Omega$ of the Jupiter–Sun system. When $\alpha \approx \pm\pi/3$, you should see small oscillations around the Lagrange points; when α is different, you should see unstable orbits where the asteroid wanders all over the plane.

Use the general-purpose Runge–Kutta routine developed in the last assignment; the dynamical variables are only four (x, y, v_x, and v_y), and you need to write a derivative function `func` to implement the first-order system consisting of Eq. (3) and of the definitions $v_x = dx/dt$, $v_y = dy/dt$. Be sure to use consistent units throughout, and set the timestep to a small fraction of an orbital period.

2. Now abandon the restricted three-body problem for the generic case of comparable masses. Work in the inertial frame, where the equations of motion are

$$M_1\mathbf{a}_1 = -\frac{GM_1M_2}{|\mathbf{r}_1-\mathbf{r}_2|^3}(\mathbf{r}_1-\mathbf{r}_2) - \frac{GM_1M_3}{|\mathbf{r}_1-\mathbf{r}_3|^3}(\mathbf{r}_1-\mathbf{r}_3), \tag{7}$$

$$M_2\mathbf{a}_2 = -\frac{GM_2M_1}{|\mathbf{r}_2-\mathbf{r}_1|^3}(\mathbf{r}_2-\mathbf{r}_1) - \frac{GM_2M_3}{|\mathbf{r}_2-\mathbf{r}_3|^3}(\mathbf{r}_2-\mathbf{r}_3), \tag{8}$$

$$M_3\mathbf{a}_3 = -\frac{GM_3M_1}{|\mathbf{r}_3-\mathbf{r}_1|^3}(\mathbf{r}_3-\mathbf{r}_1) - \frac{GM_3M_2}{|\mathbf{r}_3-\mathbf{r}_2|^3}(\mathbf{r}_3-\mathbf{r}_2). \tag{9}$$

Thus, you must evolve a total of twelve dynamical variables (three times two planar coordinates, plus three times two velocites). Start by verifying the simplest *Lagrange solution* (no relation with the Lagrange points), where three equal masses $M_1 = M_2 = M_3 = M$ move along the vertices of a rotating, equilateral triangle. The side d of the triangle and the velocity v of the masses are related by $v = \sqrt{GM/d}$. Confirm this analytically. Then plot the resulting orbits.

3. Go on to the fascinating *choreographic orbit* describing the motion of three equal-mass bodies along an "8" figure. For motion contained in the x–y plane, and units where $G = M_1 = M_2 = M_3 = 1$, approximate initial conditions are

$$(x_3, y_3) = (0, 0), \qquad (v_3^x, v_3^y) = (-0.93240737, -0.86473146), \tag{10}$$

$$(x_1, y_1) = (0.97000436, -0.24308753), \qquad (v_1^x, v_1^y) = (-v_3^x/2, -v_3^y/2), \tag{11}$$

$$(x_2, y_2) = (-x_1, -y_1), \qquad (v_2^x, v_2^y) = (-v_3^x/2, -v_3^y/2) \tag{12}$$

(and $z_1 = z_2 = z_3 = 0$ at all times). Plot the resulting orbits; if you can, prepare an animation. This is perhaps easiest if you prepare your graphs in *Mathematica*.

Alternatively, look up the paper:

http://arxiv.org/abs/1303.0181

and generate a *yin-yang* orbit using the initial conditions given in Table I of that paper. The paper also describes *butterfly, bumblebee, moth, goggles, dragonfly,* and *yarn* orbits!