Part 1

a)

$$S(u)=\int_{-\infty}^{\infty}\cos(2\pi ax)\,rect(bx)\,e^{-j2\pi ux}dx$$
$$=\frac{1}{2}\int_{-\infty}^{\infty}\left(e^{j2\pi(a-u)x}+e^{-j2\pi(a+u)x}\right)rect(bx)dx$$
$$=\frac{1}{2b}\left(sinc\left(\frac{\pi u-\pi a}{b}\right)+sinc\left(\frac{\pi u+\pi a}{b}\right)\right)$$
$$\approx\pi\left(\delta(u-a)+\delta(u+a)\right)$$

$F(u)=\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\sigma_f^2}}e^{-\frac{x^2}{2\sigma_f^2}}e^{-j2\pi(xu)}dx$, by differentiating f(x) and applying differentiation properties.

$$=e^{-2\pi^2u^2\sigma_f^2}$$

(Consulted Google) **- Procedures -**

$$\frac{df(x)}{dx}=-\frac{x}{\sigma^2}f(x)$$

$j\omega F(\omega)=\frac{1}{j\sigma^2}\frac{dF(\omega)}{d\omega}$ **, and then do integral on both side to obtain the result.**

$$-\omega\sigma^2=\frac{\frac{dF(\omega)}{d\omega}}{F(\omega)}$$

$$N(u)=\sigma_n\cdot rect(Au)$$

b)

The filtered signal.

$$SF(u)=S(u)\cdot F(u)$$
$$=\frac{1}{2b}\left(sinc\left(\frac{\pi u-\pi a}{b}\right)+sinc\left(\frac{\pi u+\pi a}{b}\right)\right)\cdot e^{-2\pi^2u^2\sigma_f^2}$$

$$E\left(SF(u)\right)=\int_{-\infty}^{\infty}SF(u)^2\,du$$
$$=\frac{1}{4b^2}\int_{-\infty}^{\infty}\left(\left(sinc\left(\frac{\pi u-\pi a}{b}\right)+sinc\left(\frac{\pi u+\pi a}{b}\right)\right)\cdot e^{-2\pi^2u^2\sigma_f^2}\right)^2du$$

When b is small enough, the sinc function can be approximately treated as a delta function. Therefore, the above integral.

$$E\left(SF\left(u\right)\right)$$
$$=\frac{1}{4b^2}\int_{-\infty}^{\infty}\left(\left(\delta\left(\frac{\pi u-\pi a}{b}\right)+\delta\left(\frac{\pi u+\pi a}{b}\right)\right)\cdot e^{-2\pi^2u^2\sigma_f^2}\right)^2 du$$
$$=\frac{1}{2b^2}e^{-4\pi^2a^2\sigma_f^2}$$

Edit -
The assumption above is not quite correct, but it won’t affect the result too much (only constant item). By saying b is small, we can have. See https://calculus.subwiki.org/wiki/Sinc-squared_function.

$$E\left(SF\left(u\right)\right)=\int_{-\infty}^{\infty}SF\left(u\right)^2 du$$
$$=\frac{1}{4b^2}\cdot e^{-4\pi^2a^2\sigma_f^2}\int_{-\infty}^{\infty}\left(\left(sinc\left(\frac{\pi u-\pi a}{b}\right)+sinc\left(\frac{\pi u+\pi a}{b}\right)\right)\right)^2 du$$
$$=\frac{1}{4b^2}\cdot e^{-4\pi^2a^2\sigma_f^2}\cdot 4\pi=\frac{\pi}{b^2}\cdot e^{-4\pi^2a^2\sigma_f^2}$$

The filtered noise.

$$NF\left(u\right)=N\left(u\right)\cdot F\left(u\right)$$
$$=\sigma_n\cdot rect\left(Au\right)\cdot e^{-2\pi^2u^2\sigma_f^2}$$

And the integral of NF is

$$E\left(NF\left(u\right)\right)=\int_{-\infty}^{\infty}NF\left(u\right)^2 du$$
$$=\int_{-\infty}^{\infty}\left(\sigma_n\cdot rect\left(Au\right)\cdot e^{-2\pi^2u^2\sigma_f^2}\right)^2 du$$
$$=\sigma_n^2\int_{-\frac{1}{A}}^{\frac{1}{A}}\left(e^{-2\pi^2u^2\sigma_f^2}\right)^2 du$$
$$=\sigma_n^2\int_{-\frac{1}{A}}^{\frac{1}{A}}e^{-4\pi^2u^2\sigma_f^2} du$$
$$=\sqrt{\frac{1}{4\pi}}\frac{\sigma_n^2}{\sigma_f}$$

Hence the energy of the noise is a constant. That makes sense, since our noise is band limited and finite. The integral is a Gaussian integral. To make this case valid, A should be small.

c)

The SNR.

$$SNR = \frac{P\left(SF\left(u\right)\right)}{P\left(NF\left(u\right)\right)}$$

$$= \frac{\lim_{T\to\infty} \frac{1}{T}\int_0^T SF\left(u\right)^2 du}{\lim_{T\to\infty} \frac{1}{T}\int_0^T NF\left(u\right)^2 du}$$

$$= \frac{\frac{\pi}{b^2} e^{-4\pi^2 a^2 \sigma_f^2}}{\sqrt{\frac{1}{4\pi} \frac{\sigma_n^2}{\sigma_f}}}$$

$$= c \cdot \frac{\sigma_f e^{-4\pi^2 a^2 \sigma_f^2}}{\sigma_n^2}$$

Seemingly not correct…Pretty confused about the T in the definition of P, since both filtered signal and noise are finite in frequency domain. Also, we can just calculate SNR by dividing energy.

d)

$SNR\left(\sigma_f\right) = c_1 \cdot \sigma_f e^{-c_2 \sigma_f^2}$, where $c_1 = \frac{\sqrt{4\pi^3}}{\sigma_n^2 b^2}$ (might miss some constants when FT and not be totally correct), and $c_2 = 4\pi^2 a^2$

$$\frac{d}{d\sigma_f} SNR\left(\sigma_f\right) = c_1 \cdot \sigma_f e^{-c_2 \sigma_f^2}$$

$$= c_1 e^{-c_2 \sigma_f^2}\left(1 - 2c_2 \sigma_f^2\right)$$

Hence, when $\left(1 - 2c_2 \sigma_f^2\right) = 0, \sigma_f^2 = \frac{1}{2c_2} = \frac{1}{8\pi^2 a^2}, \sigma_f = \frac{1}{2\sqrt{2}\pi a}$. The SNR reaches its maximum.

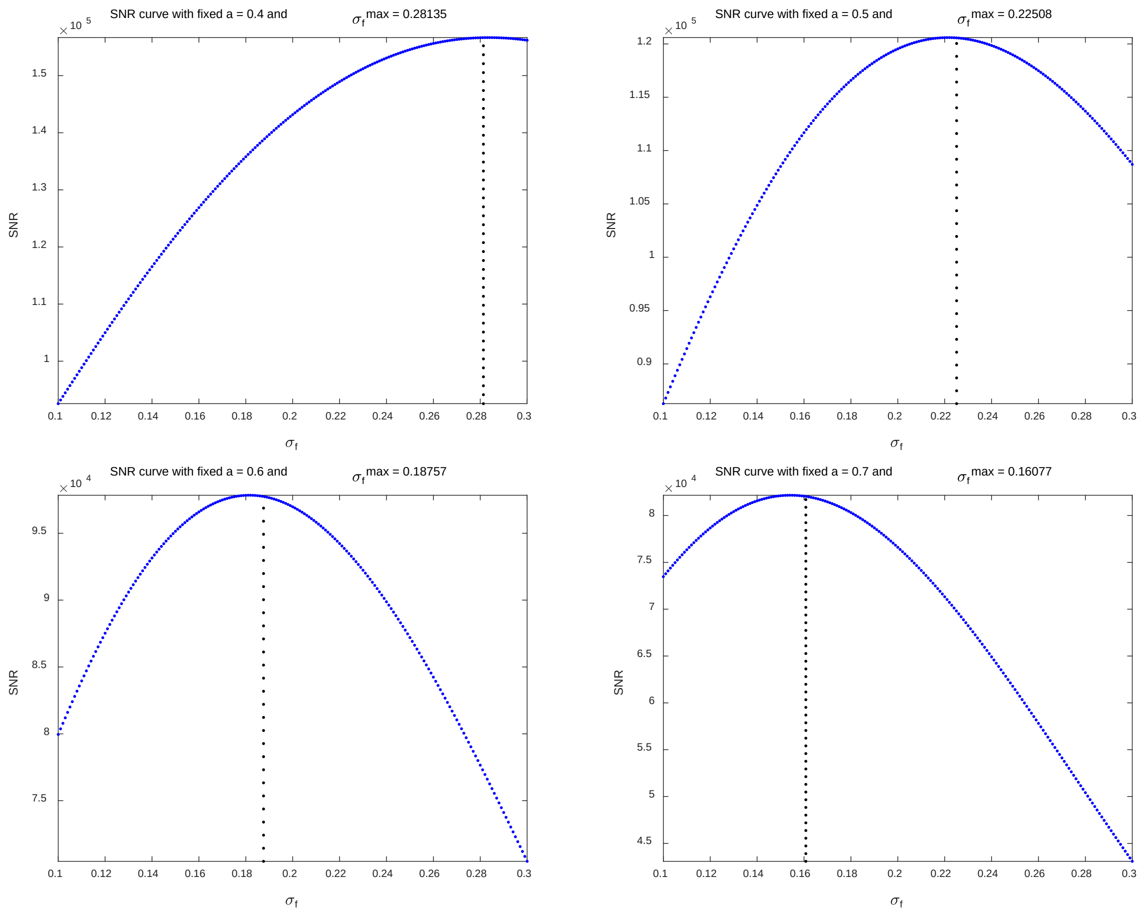
Part 2

a) When a = 0, the signal is a constant in time domain, hence in frequency domain it is a delta function at zero (or a very slim sinc function). If σ_f increases, SNR should increase monotonically. This result makes sense, since a “stronger Gaussian filter” in time domain will lead to a smoother image. Smoother means there are less fluctuations and the image is more constant.

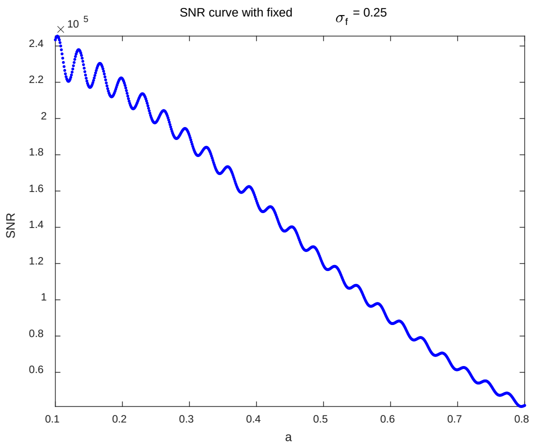
b) When σ_f is fixed and a increases, the signal has a higher frequency. In frequency domain for the signal, the two delta function pair will separate more from each other, and the filtered signal will be smaller, which leads to a worse SNR. This also make sense, since our filter has a low-pass profile and does not deal well with high frequency signal.

c) When a is fixed and σ_f decreases, the filter in frequency domain becomes more tolerant for higher frequency. SNR will first increase until σ_f reaches the stationary point, then decreases. This is tricky, as I cannot imagine by intuition why it happens. It has something to do between the portion of signal and noise after applying the filter. I’ll compare with numerical results in MATLAB.

d) Some result figures with MATLAB.



For figures above, we could interpret that when a is fixed, SNR increases first and then decrease, while there will always be a stationary point σ_f that maximizes SNR. The point is close to my deduction, but I'm still not sure if my approximation is correct as there is still discrepancy between numerical result and deduction.



For fixed σ_f, the result is pretty tricky. I suspect that the fluctuation comes from the periodical signal. We might discuss that later.

Overall, those results make sense.