Part 1

a) $SNR(I_2)$ approximately equals to $\frac{1}{4}SNR(I_1)$.

b) $\frac{E(I_2)}{E(I_1)} = \frac{E(S_2+N_2)}{E(S_1+N_1)} = \frac{E\left(\frac{1}{2}S_1+N_2\right)}{E(S_1+N_1)}$, which depends on how large is the noise comparing with the signal.

If the noise is large enough, it will dominate and the fraction will be close to 1 (e.g. when noise is much more than signal, say sigma = 5 while max(S) = 1) and $\frac{E(I_2)}{E(I_1)} \approx \frac{E(N_2)}{E(N_1)} = 1$;

When the noise is small, then the fraction will be close to 0.25 as the signal dominates and $\frac{E(I_2)}{E(I_1)} \approx \frac{SNR(I_2)}{SNR(I_1)} = \frac{1}{4}$;

The other cases result in a fraction between 0.25 and 1.

Additional notes -

Since the instructor asked us to come up with a function describing how sigma would affect this fraction, I'll paste my deduction here. Given that the signal S is independent of the noise N (maybe also mutually independent so the covariance is 0?). Let epsilon be the E(S_1).

$$\frac{E(I_2)}{E(I_1)} = \frac{E(S_2+N_2)}{E(S_1+N_1)} = \frac{E\left(\frac{1}{2}S_1+N_2\right)}{E(S_1+N_1)}$$

$$= \frac{\sum_u \sum_v \left(\frac{1}{2}S_1[u,v] + N_2[u,v]\right)^2}{\sum_u \sum_v \left(S_1[u,v] + N_1[u,v]\right)^2}$$

$$= \frac{\sum_u \sum_v \left(\frac{1}{2}S_1[u,v]\right)^2 + \sum_u \sum_v \left(N_2[u,v]\right)^2 + \sum_u \sum_v \left(S_1[u,v]N_2[u,v]\right)}{\sum_u \sum_v \left(S_1[u,v]\right)^2 + \sum_u \sum_v \left(N_1[u,v]\right)^2 + \sum_u \sum_v \left(2S_1[u,v]N_1[u,v]\right)}$$

$$= \frac{\frac{1}{4}\varepsilon + n\sigma^2 + \sum_u \sum_v \left(S_1[u,v]N_2[u,v]\right)}{\varepsilon + n\sigma^2 + \sum_u \sum_v \left(2S_1[u,v]N_1[u,v]\right)}$$

$$\approx \frac{\frac{1}{4}\varepsilon + n\sigma^2}{\varepsilon + n\sigma^2} = 1 - \frac{\frac{3}{4}\varepsilon}{\varepsilon + n\sigma^2} = f(\sigma)$$

Since S[u,v] and N[u,v] are independent from each other, and S[u,v] are given constants, we could roughly assume that $\sum_u \sum_v \left(S[u,v]N[u,v]\right) = \bar{c}\sum_u \sum_v N[u,v]$, where c bar is the average intensity of the image signal and N is the noise. Given that the noise has a 0 mean and sigma squared variance, this cross term has an expectation of 0. Again, this estimation does not conform very well when sigma or the variance of the Gaussian noise is larger, as it would cause some fluctuation in statistics.

c) When there is no noise, then $I[u,v] = S[u,v]$. Since $S_2[u,v] = \frac{1}{2}S_1[u,v]$, then $I_2[u,v] = \frac{1}{2}I_1[u,v]$. In this case, just simply multiply each pixel in I_2 by 2 can make both images have same energy. That is, $f\left(I_2[u,v]\right) = 2\cdot\left(I_2[u,v]\right)$.

d) In this case, although the energy of signal in I_2 is doubled, the noise is also amplified, which results in a not improved SNR. The $RMSE\left(I_1, S_1\right)$ is σ , while

$RMSE\left(\hat{I}_2, S_1\right) = \left(\frac{1}{n}\sum_u \sum_v \left(\hat{I}_2[u,v] - S_1[u,v]\right)^2\right)^{0.5} = \left(\frac{1}{n}\sum_u \sum_v \left(2N_2[u,v]\right)^2\right)^{0.5} = 2\sigma$, which is twice as much as $RMSE\left(I_1, S_1\right)$.

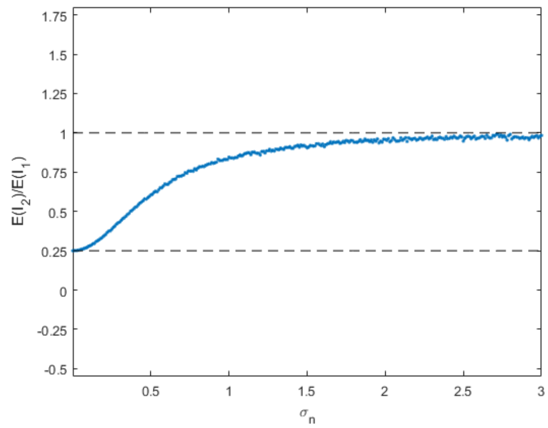
Part 2

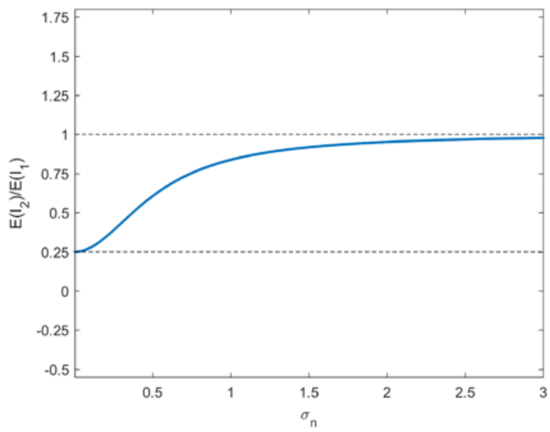
See results and values in .mlx live script file.

As we can see from the results in the scripts, all the values verified our deductions above.

In the results, we can see that

- I2 and I2 hat cannot be distinguished from their looks.
- I2 is clearly noisier than I1.
- Without noise, I2 hat's energy should be equal to that of I1 without noise, which in our result E(S1).
- By changing the sigma of noise, we can observe results similar with analysis above. The corresponding curve as below. On the left goes the actually numerical result and on the right is the approximated result with deduction in Part 1(b).





- After multiplying the pixels in I2 and get I2 hat, the RMSE aligns with our expectation.