Single-server queue with Markov dependent inter-arrival and service times

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Motivation

- In packet communication networks, the dependency between successive service times/arrival times and the dependency between inter-arrival times and service times are well observed.

- Experiments and research have shown that such dependencies may have dominant effects on the waiting time and the queue length.

- In this project, we were trying to find out:
  - How auto-correlations of successive inter-arrival times affect the waiting time and the queue length.
  - How auto-correlations of successive service times affect the waiting time and the queue length
  - How cross-correlations between inter-arrival times and service times affect the waiting time and the queue length

Approach

- Theory
  - Derived and proved remarks and formulas of importance, including
    - The steady-state auto-correlation
    - The steady-state cross-correlation
    - Queuing Model
  - Hands-on numerical examples that indicate the impact of auto-correlation and cross-correlation on the waiting times

- Programming
  - Random number generators for different types of distribution of inter-arrival times and services times
  - Simulations over different auto-correlations between inter-arrival times/service times and cross-correlations between inter-arrival times and service times.
  - Plots that indicate the impact of auto-correlation and cross-correlation on the waiting times

Queueing Model

- Single Server Queueing model
  - Regulated by a Definite Finite Discrete Markov Chain
  - Exponentially Distributed Service Time
  - Markovian Arrival Process
    - Poisson Process with Exponentially distributed Inter-Arrival Times

- Queueing model defined by a Trivariate process
  - Inter-Arrival Process, Service Time Process, State Space

$$P\left(A_{n+1} \leq x, S_n \leq \mathrm{y}, Z_{n+1} \leq \mathrm{j} \mid Z_n = \mathrm{i}, \left(A_{r+1,}S_r, Z_r\right), 0 \leq \mathrm{r} \leq \mathrm{n}-1\right)$$

$$= P(A_1 \leq x, S_0 \leq \mathrm{y}, Z_1 \leq \mathrm{j} \mid Z_0 = \mathrm{i})$$

  - Memory – Less Property Due to Independence given past state information
  - Equivalent representation

$$G_i(y)p_{i,j}(1 - e^{-\lambda_j x})$$

  - Service time distributions, state transition probability, cdf of arrival process

Correlation

- Assumes Cross-correlation and Auto-correlation exists
  - Correlation within sample given some lag and correlation across two samples
- Auto-correlation

$$\rho(S_0, S_{0+n}) = \frac{\sum_{i=1}^{N}\sum_{j=1}^{N} \pi_i (p_{i,j}^{(n)} - \pi_j)\, \gamma_j\, \gamma_i}{\sum_{i=1}^{N} \pi_i\, s_i^2 - (\sum_{i=1}^{N} \pi_i\, \gamma_i)^2}, \quad n \geq 1$$

- Cross-correlation

$$\rho(A_n, S_n) = \frac{\sum_{i=1}^{N} \pi_i (\lambda^{-1}{}_i - \lambda^{-1}) (\gamma_i - \gamma)}{\{\sum_{i=1}^{N} \pi_i (\lambda^{-1}{}_i - \lambda^{-1})^2 - \sum_{i=1}^{N} \pi_i (\gamma_i - \gamma)^2\}^{\frac{1}{2}}}$$

$p_{i,j}^{(n)} = P(Z_n = j \mid Z_0 = i)$, $\lambda^{-1} = \sum_{i=1}^{N} \pi_i \lambda^{-1}{}_i$, and $\gamma = \sum_{i=1}^{N} \pi_i \gamma_i$.

Model Flexibility

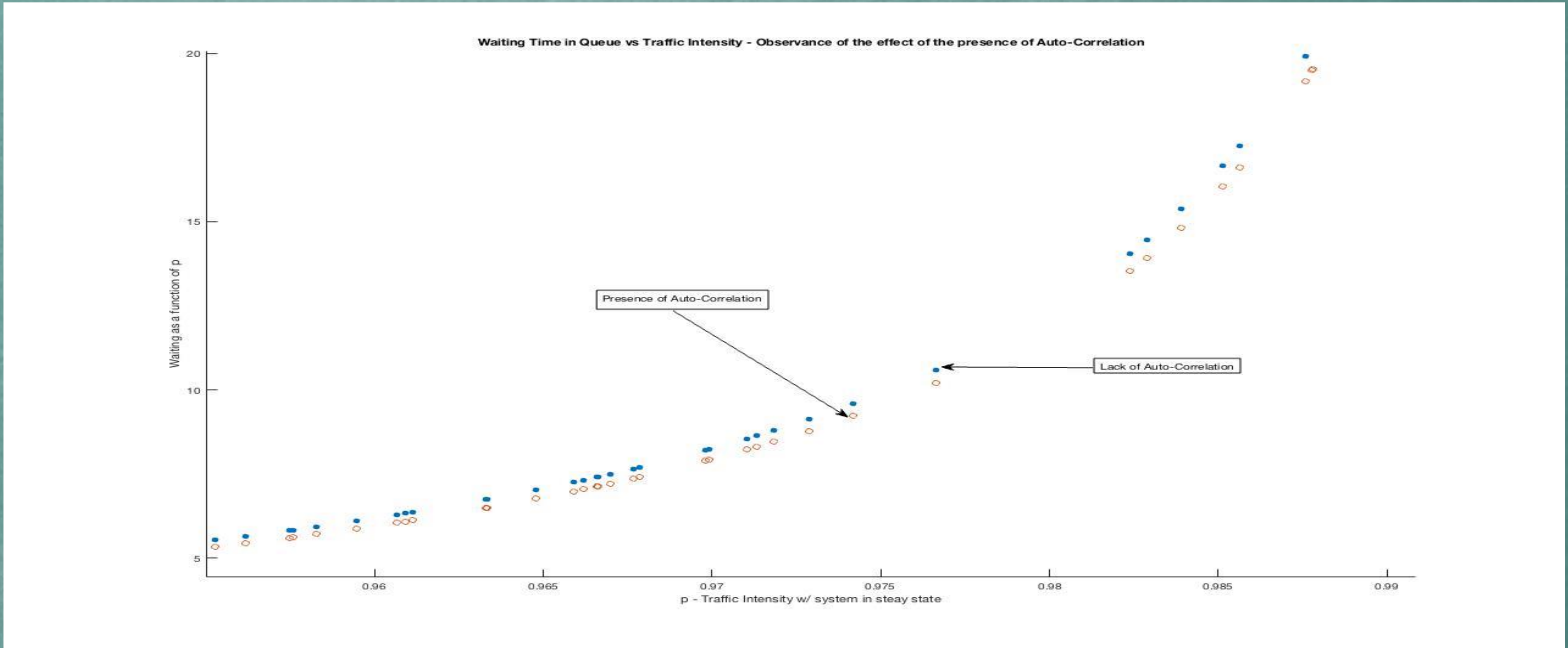
- Can Model State - Transition Without Arrivals
  - How?
- Can Model periodic arrivals
  - How?

Experimental Procedure

- Experimental Design
  - 4 state Markov chain
  - Service time process defined by some exponential distribution multiplied by U
    - U parameter useful for observing effects of mean service time on expected waiting time
  - Arrival process defined by Markovian Poisson arrival process
- Three Trials of Analysis
  - Positive Cross-correlation
  - Negative Cross-correlation
  - Zero Cross-correlation
    - Independence between service time and arrival time process
- Effects On waiting Time?
- Effect Of Cross-correlation on Magnitude of effect Auto-Correlation poses on Waiting time

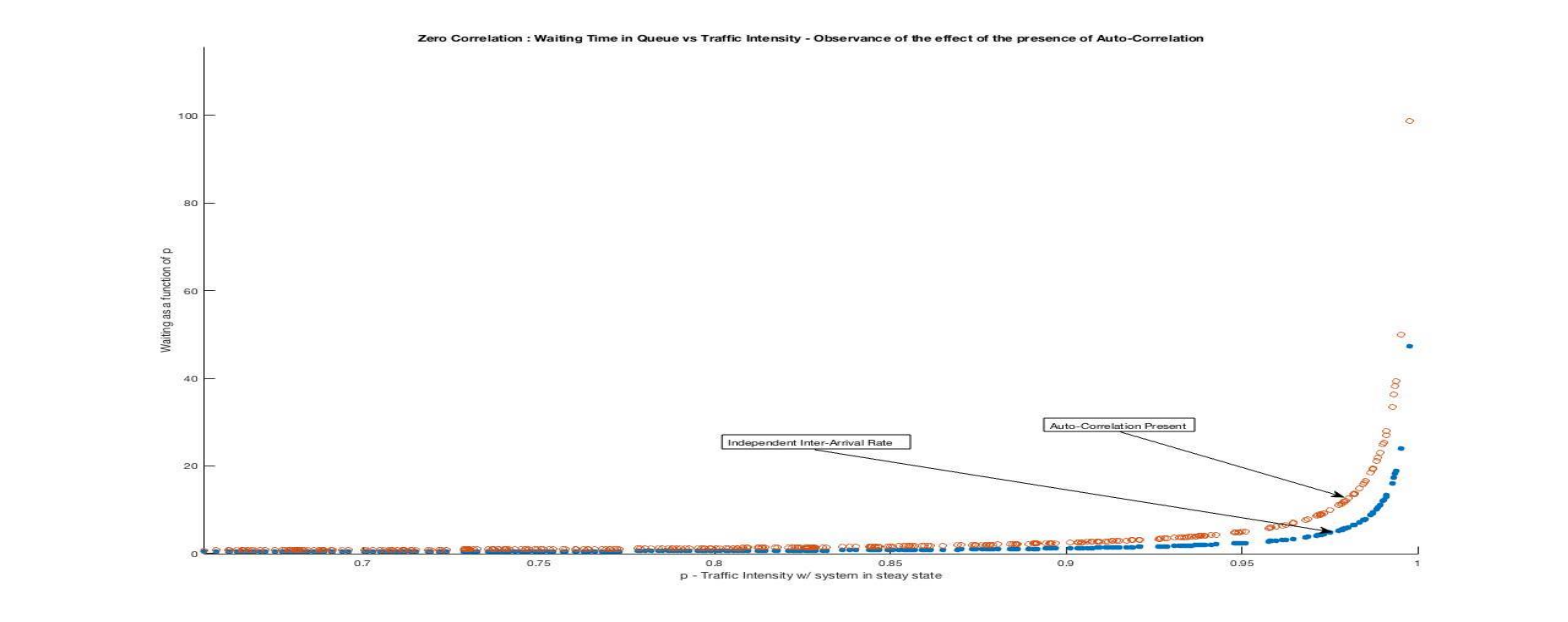
Experimental Procedure

- Positive Cross-Correlation



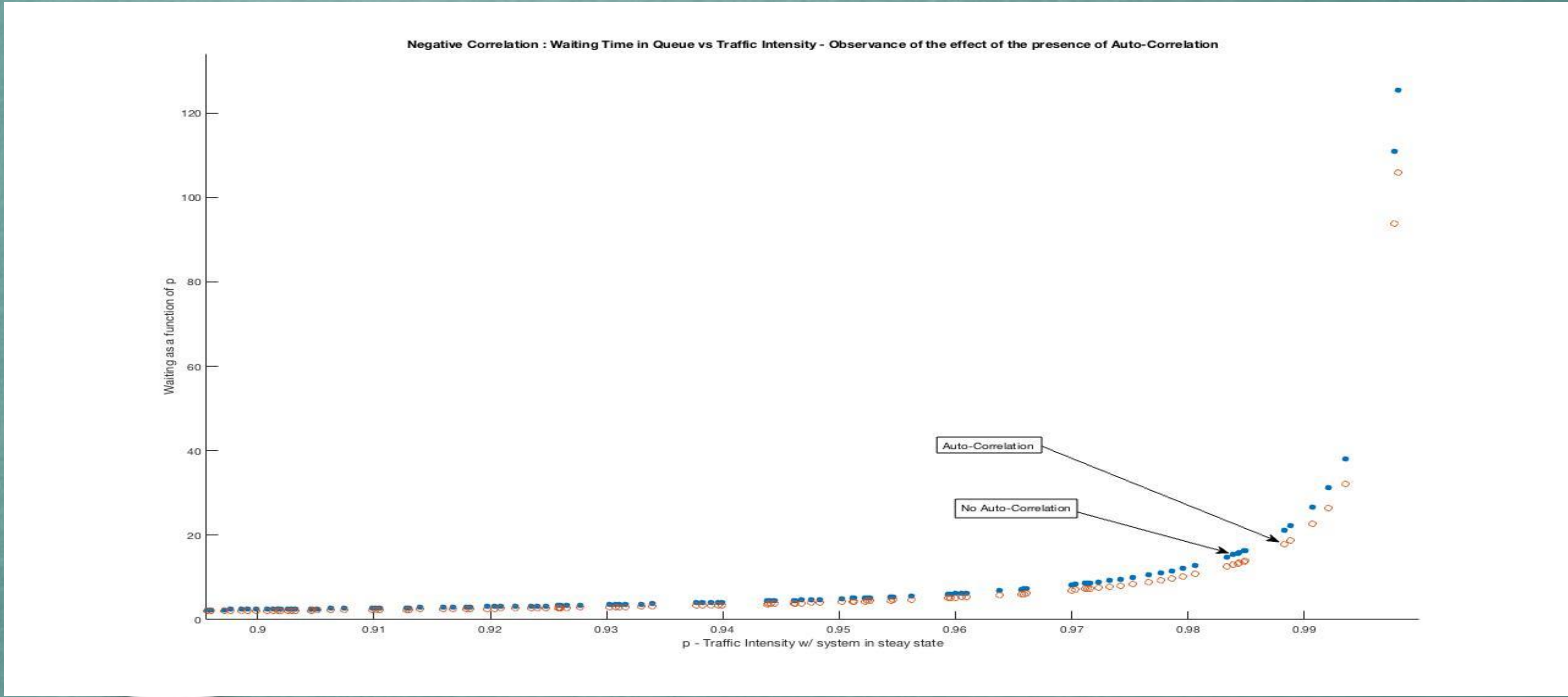
Experimental Procedure

- Zero Cross-Correlation



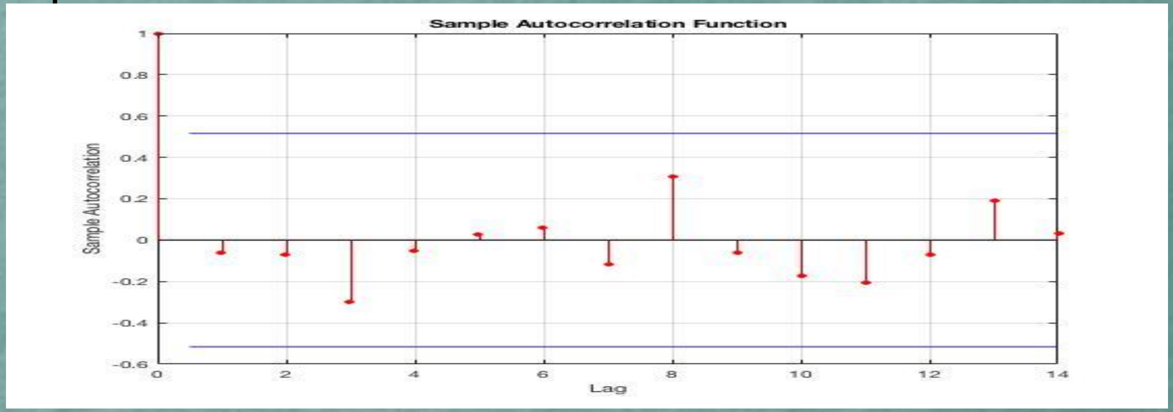
Experimental Procedure

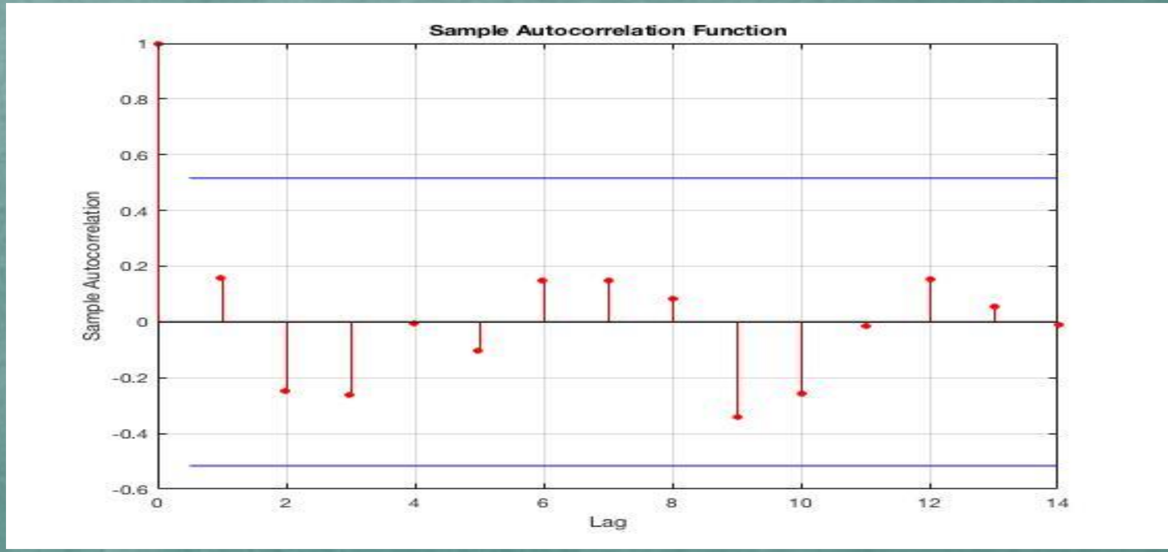
- Negative Cross-Correlation



Auto-Correlative Behavior With Respect To Time

- Auto-Correlation as Time lag Increases for exponentially distributed service time process and Markovian Arrival Process





Expected Waiting Time Process & Traffic Intensity

- Traffic Intensity
  - Mean service time / Mean arrival rate

$$p_{int.} = \frac{\sum_{i=1}^{N} \pi_i \gamma_i}{\sum_{i=1}^{N} \pi_i \lambda^{-1}{}_i}$$

- Waiting Time process Computed by Use of Laplace Stieltjes transform in the complex plane
  - Complex variable whose real valued component is greater than 1
  - Utilize the lebesque-Stieltjes integral to acquire the service time probability density function

$$g(s) = \int_0^\infty e^{-st} \lambda e^{-\lambda t} dt$$

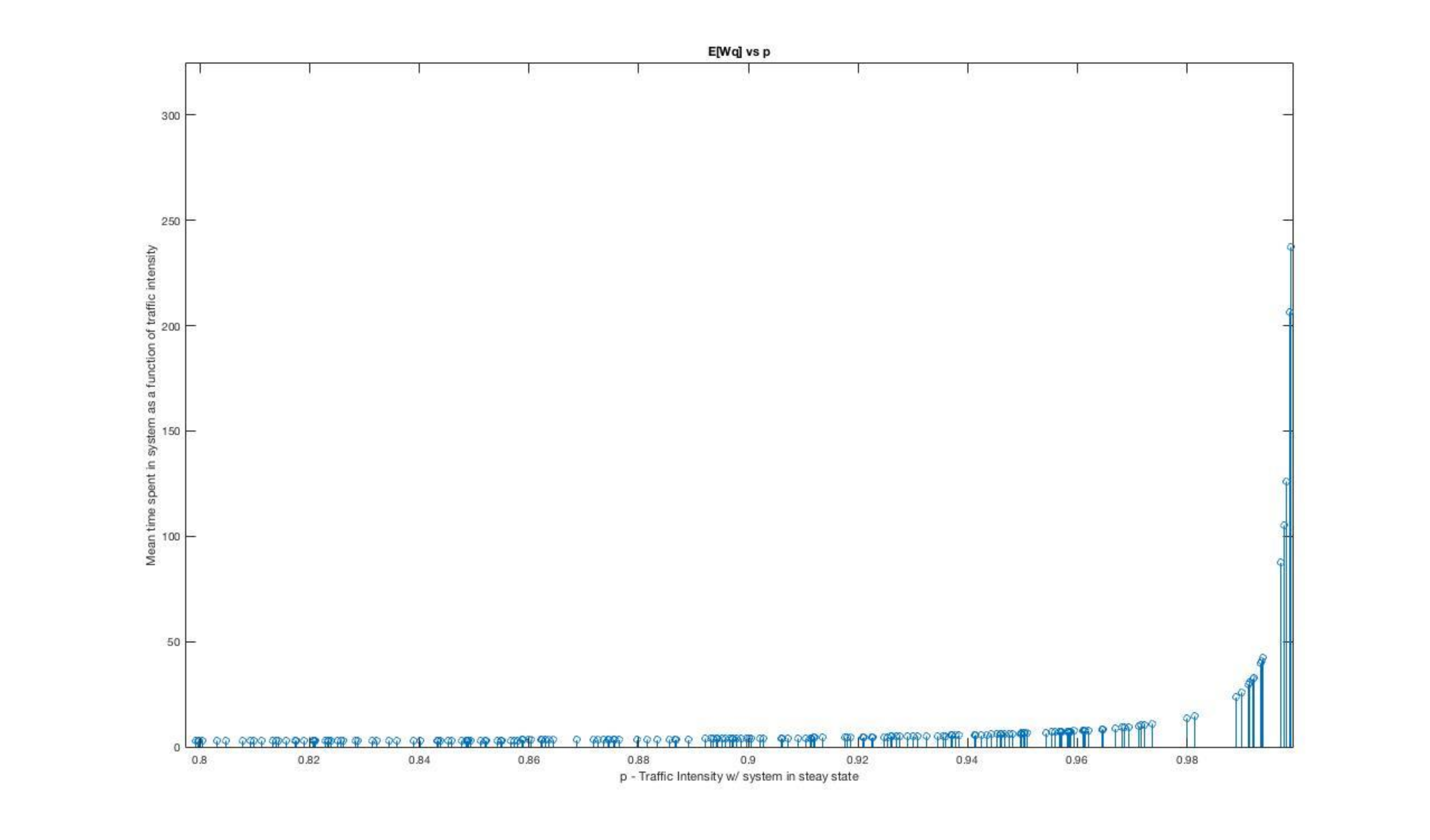
  - Waiting time process obtained as defined by Adan et al. [1]

$$W_N = \frac{(1 - p_{int.}) * s * g(s)}{s - \lambda(1 - g(s))}$$

- Complex plane representation
- Mean Waiting time process

$$E[W_q] = \frac{\lambda E[\gamma^2]}{2(1 - p_{int.})}$$

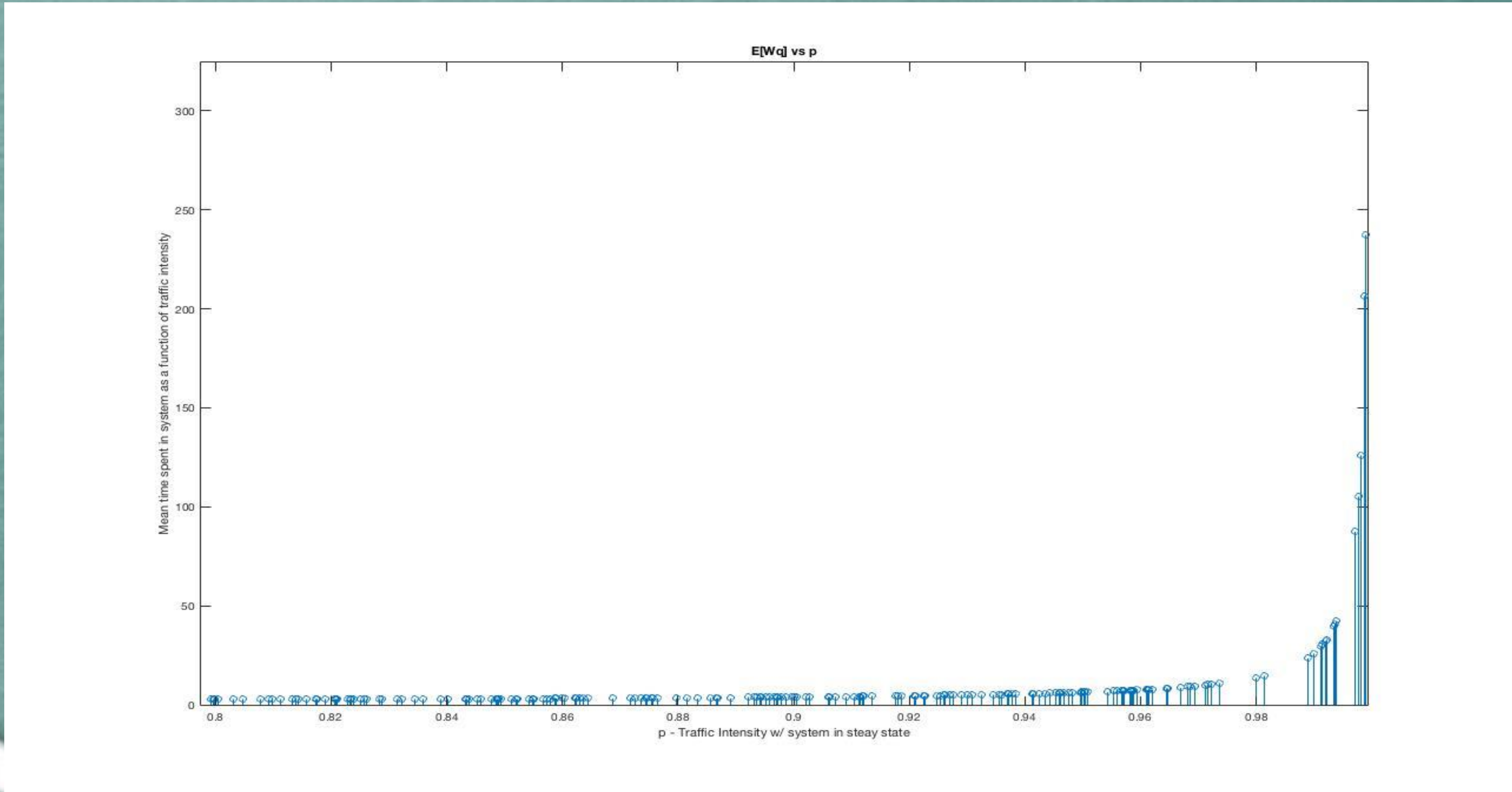
Mean Waiting Time as a function of Traffic Intensity



Further Work

$$E[W_q] = \frac{\lambda E[\gamma^2]}{2(1 - p_{int.})}$$

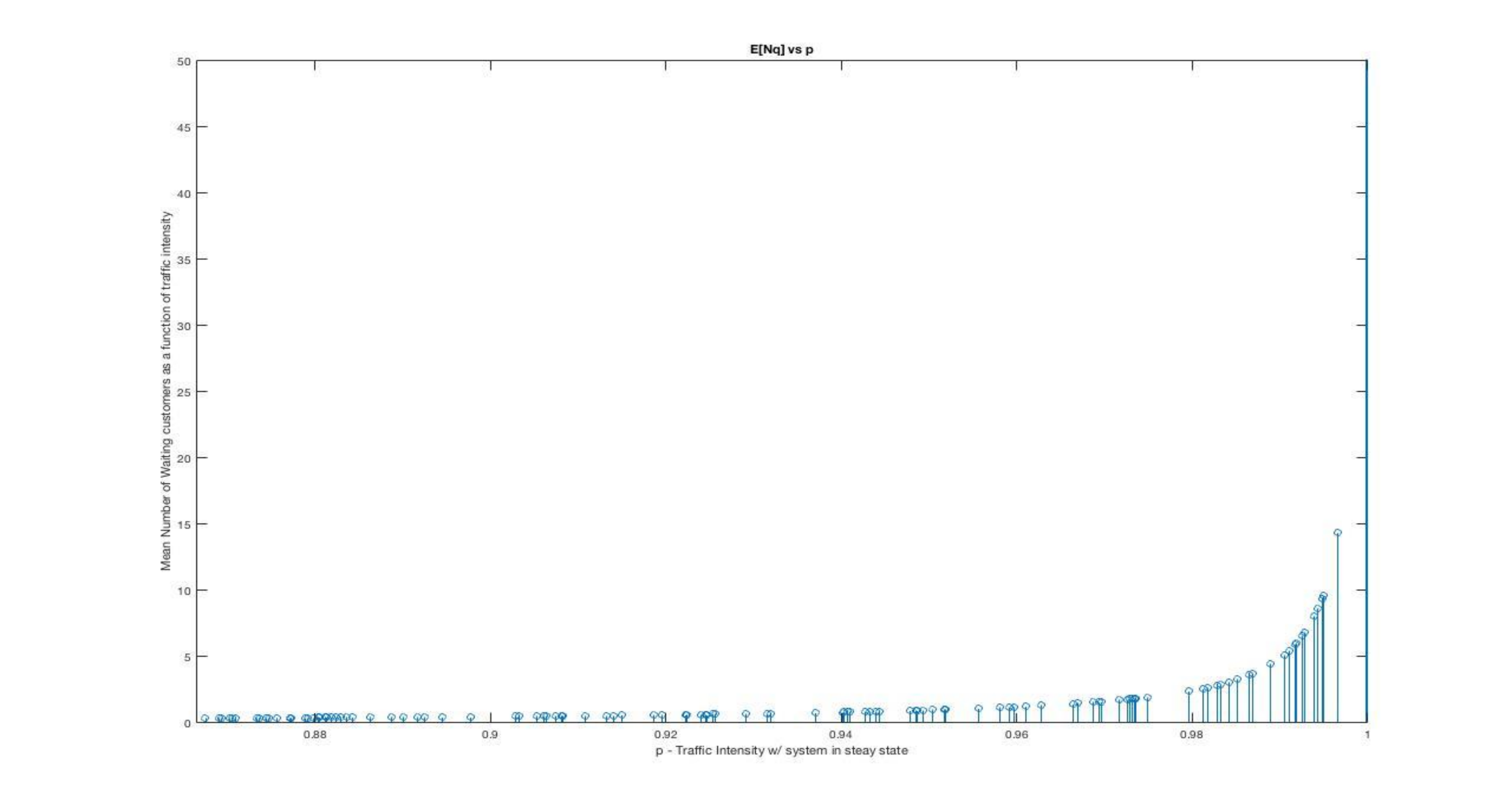
- Mean Time Spent In Queue Given Presence of Cross Correlation



Further Work

$$E[N_q] = \lambda E[W_q] = \frac{\lambda^2 E[\gamma^2]}{2(1 - p_{int.})}$$

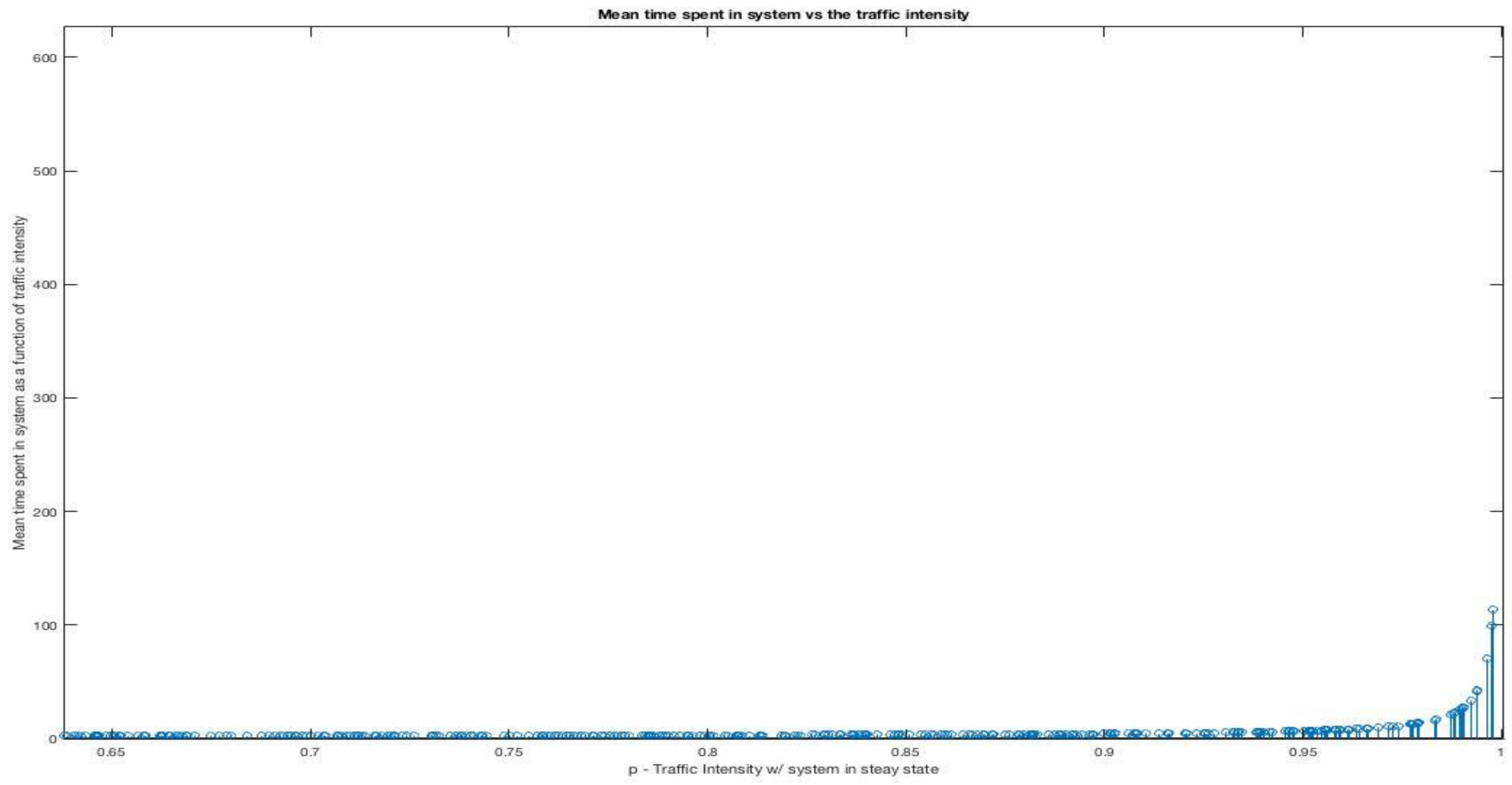
- Mean Number of Cust. In Queue Given Presence of Cross Correlation



Further Work

$$E[W] = E[W_q] + E[\lambda] = \frac{\lambda E[\gamma^2]}{2(1 - p_{int.})} + E[\lambda]$$

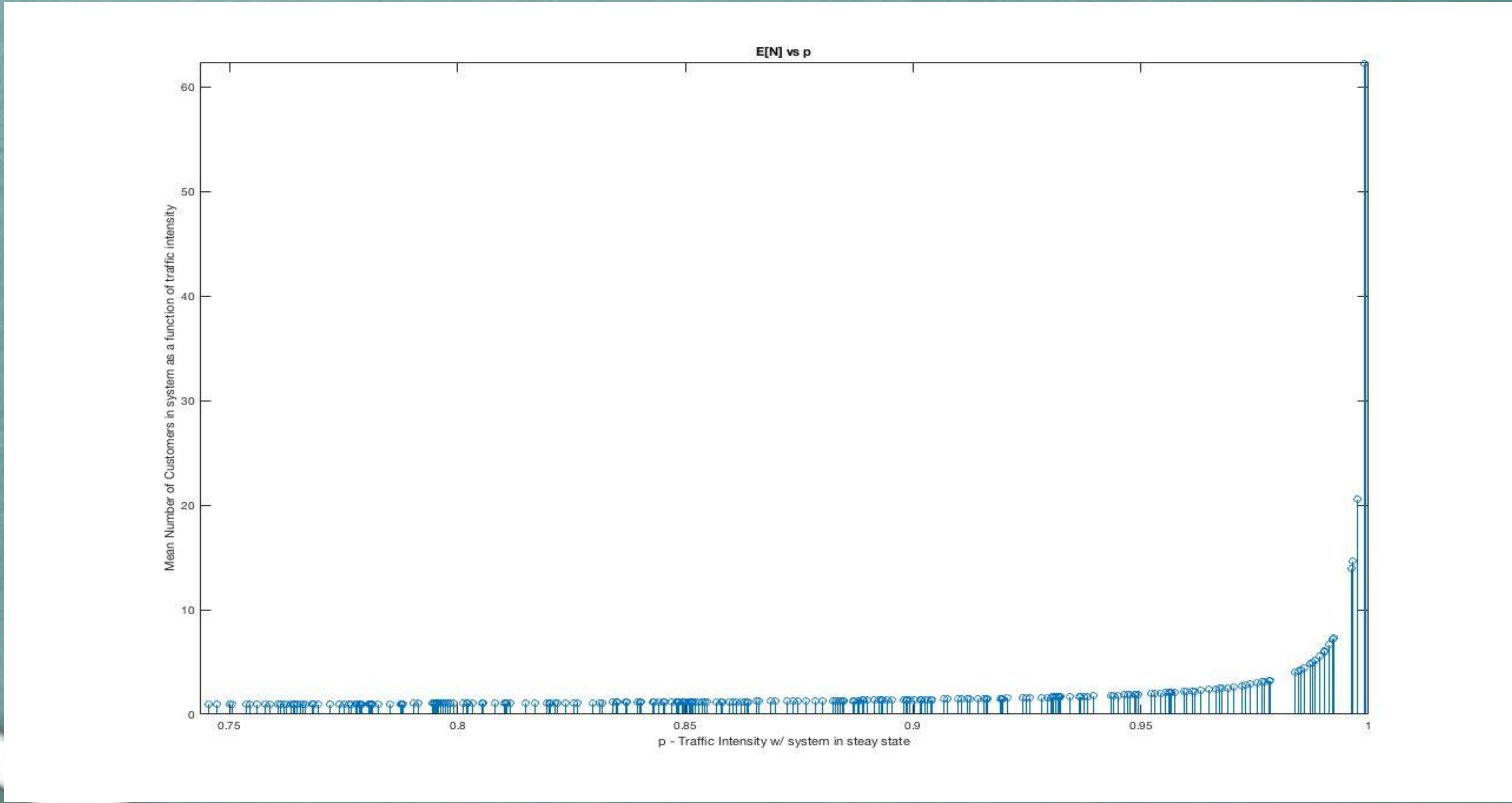
- Mean Time Spent in In System Given Presence of Cross Correlation



Further Work

$$E[N] = \lambda E[W] = \frac{\lambda^2 E[\gamma^2]}{2(1 - p_{int.})} + p_{int.}$$

- Mean Number of Cust. In System Given Presence of Cross Correlation



Citations

[1] I. Cidon, R. Guerin, A. Khamisy and M. Sidi, On queues with inter-arrival times proportional to service times, Technical Report EE PUB. No. 811, Technion (1991).

[2] I. Cidon, R. Guerin, A. Khamisy and M. Sidi, Analysis of a correlated queue in communication systems, Technical Report EE PUB. No. 812, Technion (1991).

[3] B. Conolly and N. Hadidi, A correlated queue, Appl. Probab. 6 (1969) 122–136.

[4] Adan, I.J.B.F., and V.G. Kulkarni. "Single-Server Queue with Markov-Dependent Inter-Arrival and Service Times." SpringerLink, Kluwer Academic Publishers.

[5] Moltchanov, Dmitri. "M/G/1 And M/G/1/K Systems."

Code

https://github.com/yoanyomba123/Markovian-Arrival-Processes-