On the Decomposition of Persistence Modules

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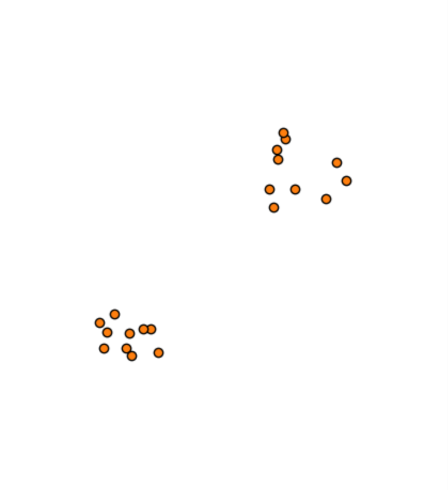
Contents

Persistent Homology

One-parameter persistent homology

- Start with a point cloud data X.



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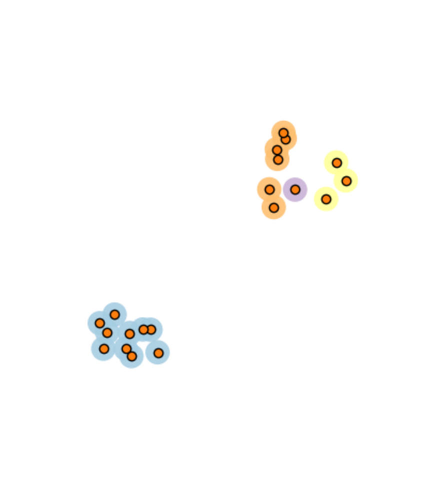

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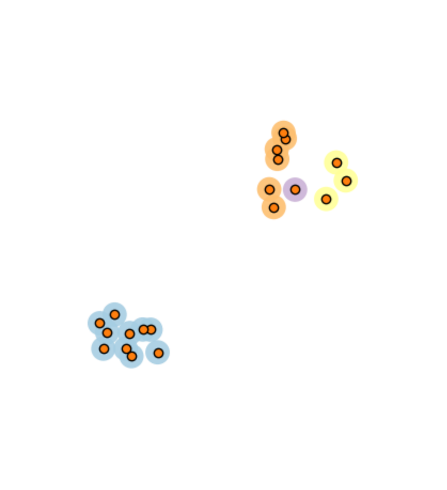

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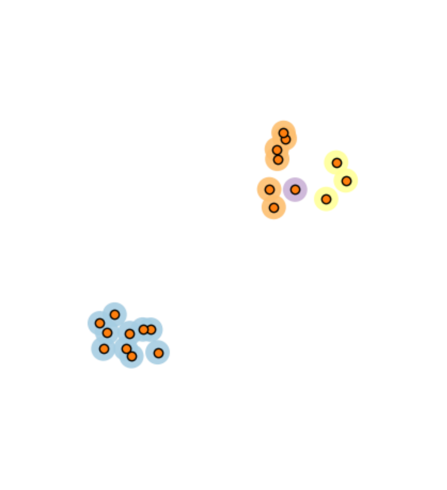
  Idea: if $H_i(X_\varepsilon) \neq 0$, then X_ε has an i-dimensional hole.



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Definition

The collection $\{H_i(X_\varepsilon), \iota_{\delta,\varepsilon}\}$ is called the *i-dimensional persistent homology* of X.

Persistence modules

Definition

A persistence module (over $\mathbb{R}$) is a functor $M : \mathbb{R} \to \mathsf{Vec}_{\mathbf{k}}$, where we view the partially ordered set (poset) $\mathbb{R}$ as a category in the natural way ($a \leq b \iff a \to b$)

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Example

$$\mathbb{R} : \cdots \to -1 \to 0 \to 1 \to 0 \to \cdots$$

$$M : \cdots \to \mathbf{k} \xrightarrow{[1,0]^T} \mathbf{k}^2 \xrightarrow{[1,1]} \mathbf{k} \to \cdots$$

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We say that M is pointwise finite dimensional (pfd) if $\dim M(t) < \infty$ for all $t \in \mathbb{R}$

Structure theorem

Definition

Let $[a,b) \subset \mathbb{R}$. An interval module $\mathbb{I}_{[a,b)}$ is the persistence module which has $\mathbb{I}_{[a,b]}(t) = \mathbf{k}$ for $t \in [a,b)$ (and 0 otherwise) and $\mathbb{I}_{[a,b)}(t \to s) = \mathrm{id}_{\mathbf{k}}$ for $a \leq t \leq s < b$ (and 0 otherwise).

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Theorem

Let M be a pfd persistence module. Then there exists a unique colletion of intervals $\mathcal{B} = \{[a_i, b_i) \subset \mathbb{R} \mid i = 1, \ldots, n\}$ such that

$$M \cong \bigoplus_{A \in \mathcal{B}} \mathbb{I}_A.$$

[Botnan and Crawley-Boevey, 2018]

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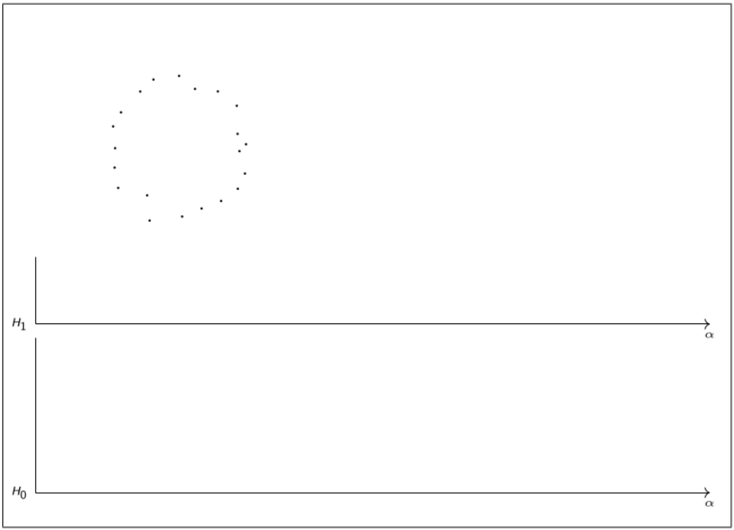
We call $\mathcal{B}$ the barcode of M.

An example



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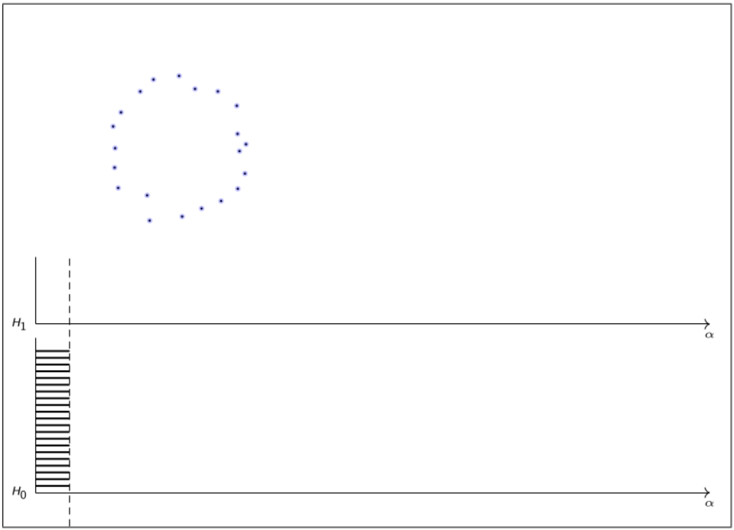

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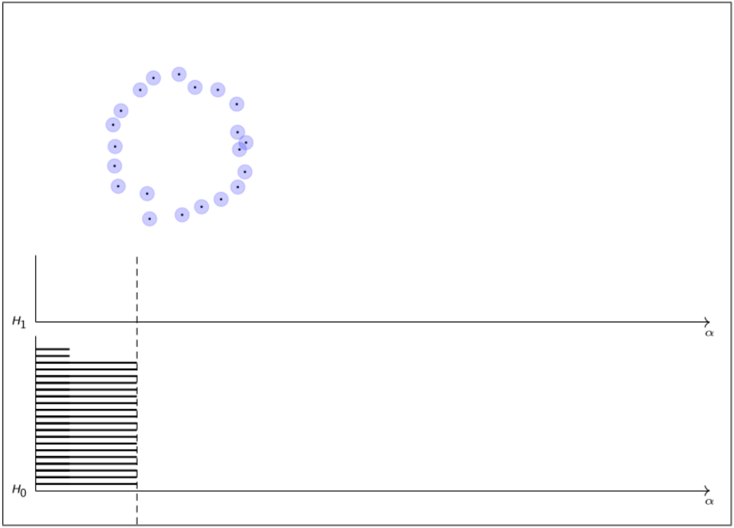

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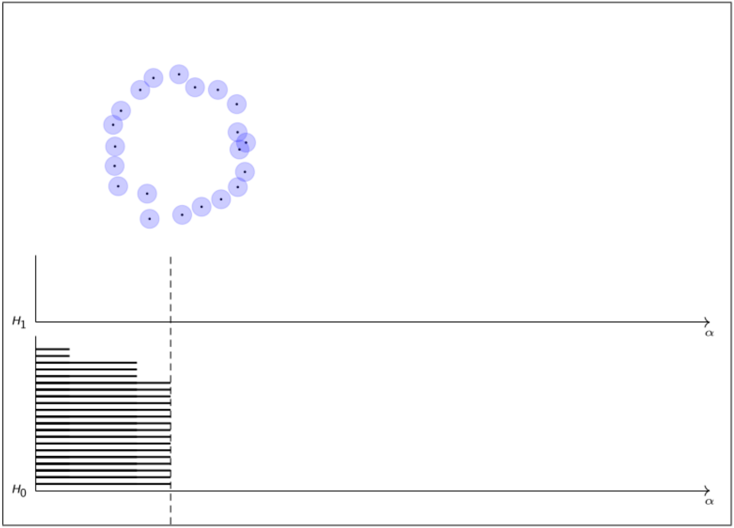

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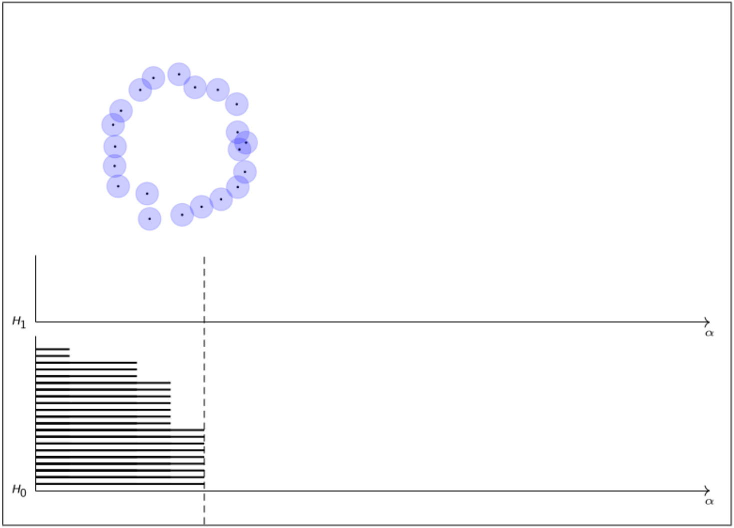

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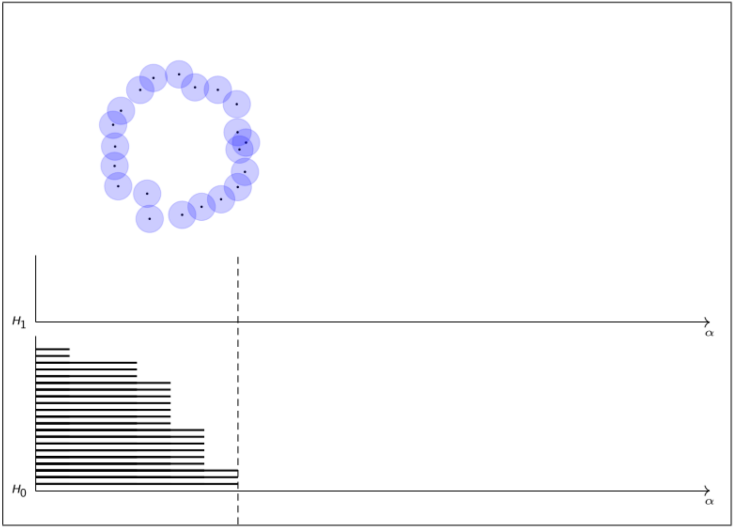

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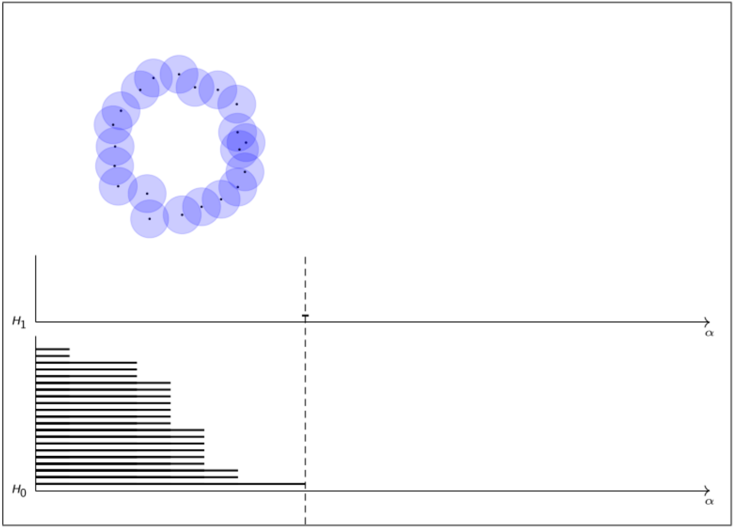

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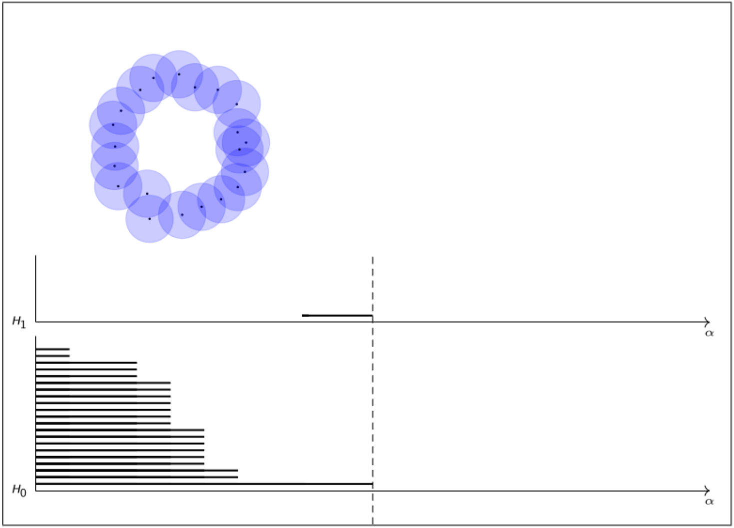

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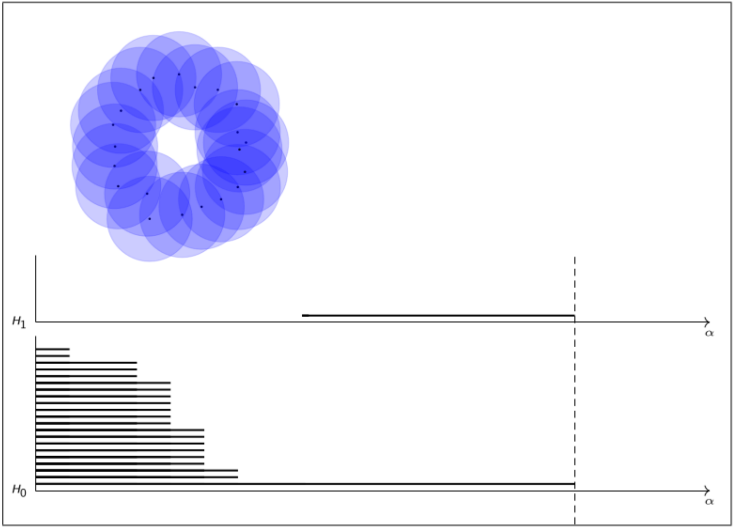

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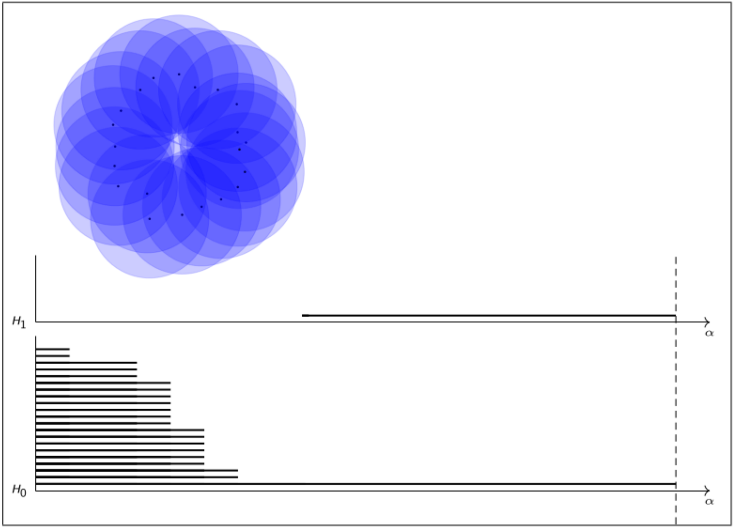

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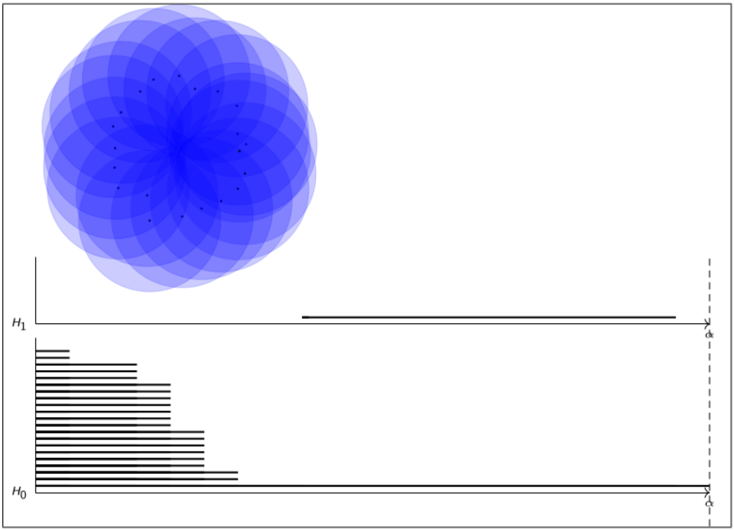

Limitations of one parameter persistent homology

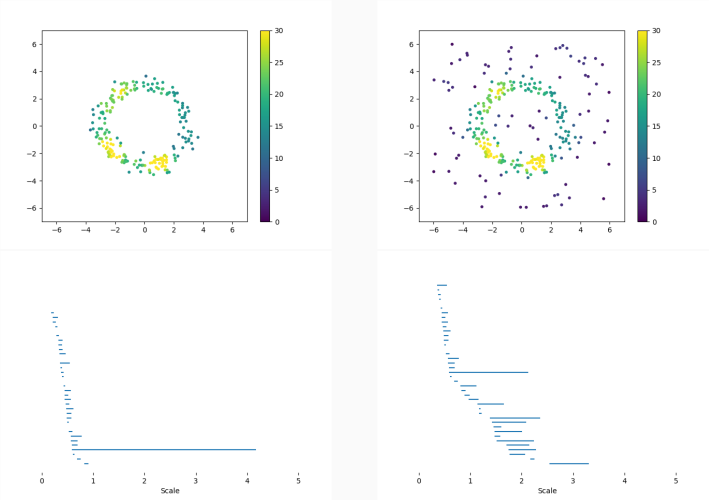

Figure 1: [Botnan and Lesnick, 2022]

Multiparameter persistence

Recall: a persistence module is a functor $M\colon \mathbb{R} \to \mathsf{Vec}_{\mathbf{k}}$.

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Multiparameter persistence

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What if we replace $\mathbb{R}$ by $\mathbb{R}^2$? Or $\mathbb{N}^2$? Or $\mathbb{R}^n$? Do we still have a "barcode"?

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
(0,2) \arrow[u] \arrow[r] & (1,2) \arrow[u] \arrow[r] & (2,2) \arrow[u] \arrow[r] & \cdots \\
(0,1) \arrow[u] \arrow[r] & (1,1) \arrow[u] \arrow[r] & (2,1) \arrow[u] \arrow[r] & \cdots \\
(0,0) \arrow[u] \arrow[r] & (1,0) \arrow[u] \arrow[r] & (2,0) \arrow[u] \arrow[r] & \cdots
\end{tikzcd}$$

(a) The poset $\mathbb{N}^2$

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
\mathbf{k} \arrow[u] \arrow[r, "1"] & \mathbf{k} \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & \cdots \\
\mathbf{k} \arrow[u, "1"] \arrow[r, "{[1,0]^T}"] & \mathbf{k}^2 \arrow[u, "{[1,0]}"'] \arrow[r, "{[1,1]}"] & \mathbf{k} \arrow[u] \arrow[r] & \cdots \\
0 \arrow[u] \arrow[r] & \mathbf{k} \arrow[u, "{[1,0]^T}"'] \arrow[r, "1"] & \mathbf{k} \arrow[u, "1"] \arrow[r] & \cdots
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(b) A persistence module over $\mathbb{N}^2$

Main theorem

Unfortunately, there is no similar complete discrete invariant for multidimensional persistence [Carlsson and Zomorodian, 2007].

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Theorem

Let P be a small category and $M\colon P \to \mathrm{Vec}_{\mathbf{k}}$ a pfd persistence module. Then M has an essentially unique decomposition into indecomposable summands.

Existence of Decomposition

Overview

The Tensor Product

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$$\begin{tikzcd}
& \text{The Tensor Product} \arrow[dl] \\
\text{Tensor-Hom Adjunction} &
\end{tikzcd}$$

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Overview

- The Tensor Product
  - Tensor-Hom Adjunction
  - Pure-Exact Sequences
    - Pure-Injective Module
    - Direct Summand is a Pure Submodule
    - Union of Chain of Pure Submodules is Pure

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 & \text{Pure-Injective Module} \arrow[dl] & \text{Direct Summand is a Pure Submodule} & \text{Union of Chain of Pure Submodules is Pure} \\
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\end{tikzcd}
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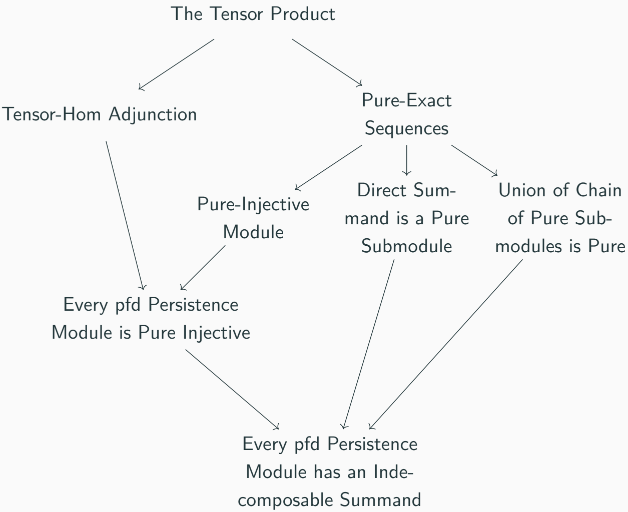
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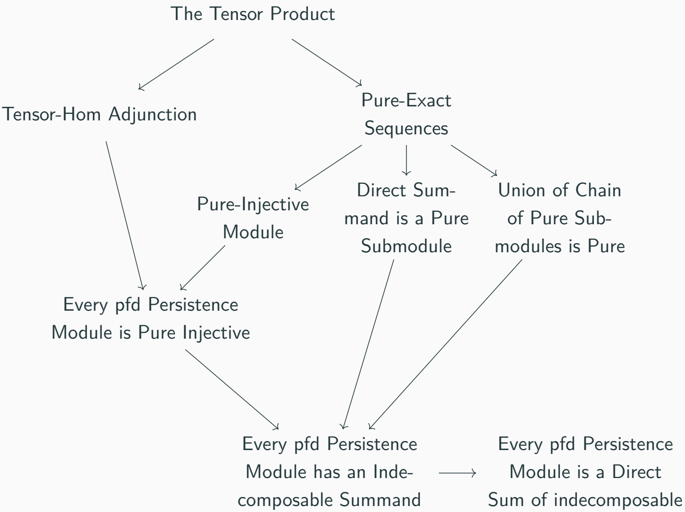
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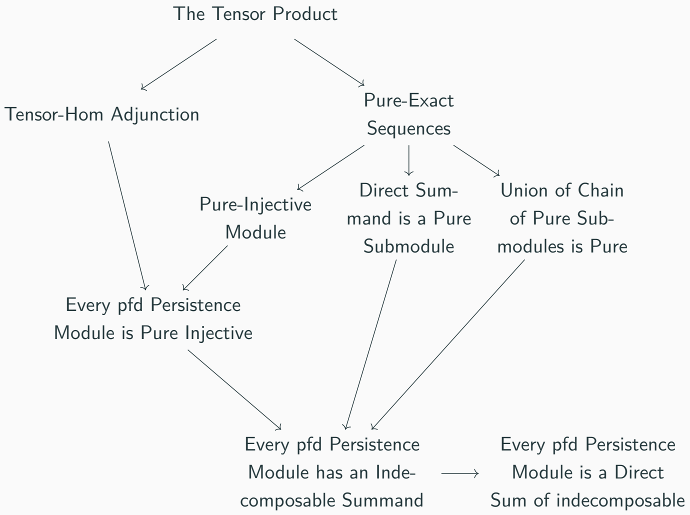

Based on idea by William Crawley-Boevey

The Tensor Product and Duality

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Pointwise duality gives a contravariant, exact functor
$D : \mathrm{Vec}_{\mathbf{k}}^{P} \to \mathrm{Vec}_{\mathbf{k}}^{P^{op}}$

$$M : \qquad \mathbf{k} \longrightarrow \mathbf{k}^2 \longrightarrow \mathbf{k}^3$$

$$DM : \qquad \mathbf{k}^* \longleftarrow (\mathbf{k}^2)^* \longleftarrow (\mathbf{k}^3)^*$$

Tensor-Hom Adjunction

And we have a tensor-hom adjunction

$$\mathrm{Hom}_{\mathbf{k}}(N \otimes_P M, \mathbf{k}) \cong \mathrm{Hom}_P(M, DN)$$

Pure submodules and pure-exact sequences

A short exact sequence $0 \to A \to B \to C \to 0$ is pure-exact if

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is exact for all M.

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A submodule $M' \subseteq M$ is pure in M if the sequence

$$0 \to M' \to M \to M/M' \to 0$$

is pure-exact.

Pure-injectivity

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1. $\mathrm{Hom}_P(-, M)$ preserves pure-exact sequences
2. Every pure exact sequence $0 \to M \to M' \to M'' \to 0$ splits.

Pure-injectivity

Lemma

Every pfd persistence module M is pure injective

Every pfd persistence module has an indecomposable summand

Let $x \in M$. The set $S = \{N \subseteq M \mid N \text{ is pure in } M \text{ and } x \notin N\}$ has a maximal element N.

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M/N is non-zero since $x \in M/N$ and indecomposable by the maximality of N.

Uniqueness of Decomposition

Uniqueness

Theorem

If $M = \bigoplus_{i \in I} M_i = \bigoplus_{j \in J} N_j$ *are two direct sum decompositions of* M *into indecomposables, then there exists an isomorphism* $\varphi \colon I \to J$ *such that* $M_i \cong N_{\varphi(i)}$ *for all* $i \in I$.

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- An adaptation of the Krull-Remak-Schmidt-Azumaya Theorem

Conclusion

Thank you for your attention

Botnan, M. B. and Crawley-Boevey, W. (2018).
Decomposition of persistence modules.

Botnan, M. B. and Lesnick, M. (2022).
An introduction to multiparameter persistence.

Carlsson, G. and Zomorodian, A. (2007).
The theory of multidimensional persistence.
Discrete and Computational Geometry, 42:71–93.