

On the Decomposition of Persistence Modules

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Persistent Homology

Existence of Decomposition

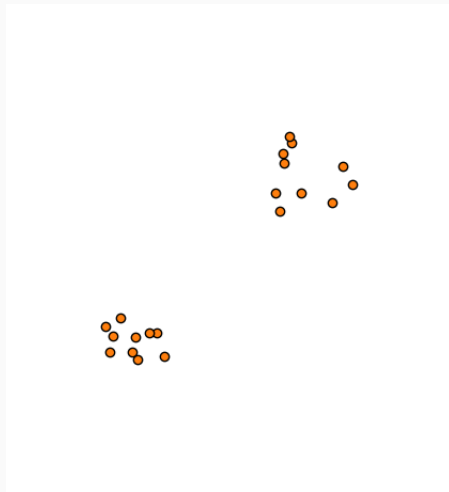
Uniqueness of Decomposition

Conclusion

Persistent Homology

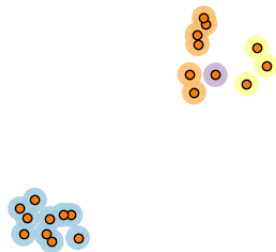
One-parameter persistent homology

- Start with a point cloud data X .



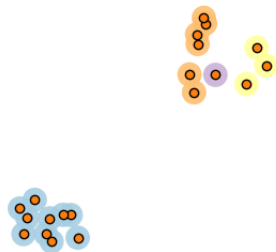
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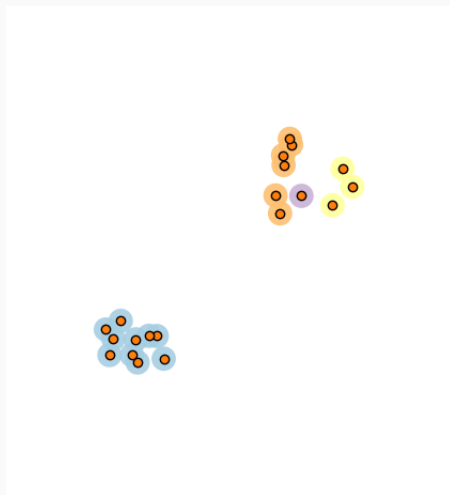
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Idea: if $H_i(X_\varepsilon) \neq 0$, then X_ε has an i -dimensional hole.



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Definition

The collection $\{H_i(X_\varepsilon), \iota_{\delta,\varepsilon}\}$ is called the *i-dimensional persistent homology* of X .

Definition

A **persistence module** (over \mathbb{R}) is a functor $M : \mathbb{R} \rightarrow \text{Vec}_k$, where we view the partially ordered set (poset) \mathbb{R} as a category in the natural way ($a \leq b \iff a \rightarrow b$)

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Example

$$\mathbb{R} : \cdots \rightarrow -1 \rightarrow 0 \rightarrow 1 \rightarrow 0 \rightarrow \cdots$$

$$M : \cdots \rightarrow \mathbf{k} \xrightarrow{[1,0]^T} \mathbf{k}^2 \xrightarrow{[1,1]} \mathbf{k} \rightarrow \cdots$$

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We say that M is **pointwise finite dimensional (pfd)** if $\dim M(t) < \infty$ for all $t \in \mathbb{R}$

Structure theorem

Definition

Let $[a, b) \subset \mathbb{R}$. An **interval module** $\mathbb{I}_{[a,b)}$ is the persistence module which has $\mathbb{I}_{[a,b)}(t) = \mathbf{k}$ for $t \in [a, b)$ (and 0 otherwise) and $\mathbb{I}_{[a,b)}(t \rightarrow s) = \text{id}_{\mathbf{k}}$ for $a \leq t \leq s < b$ (and 0 otherwise).

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Theorem

Let M be a pfd persistence module. Then there exists a unique collection of intervals $\mathcal{B} = \{[a_i, b_i) \subset \mathbb{R} \mid i = 1, \dots, n\}$ such that

$$M \cong \bigoplus_{A \in \mathcal{B}} \mathbb{I}_A.$$

[Botnan and Crawley-Boevey, 2018]

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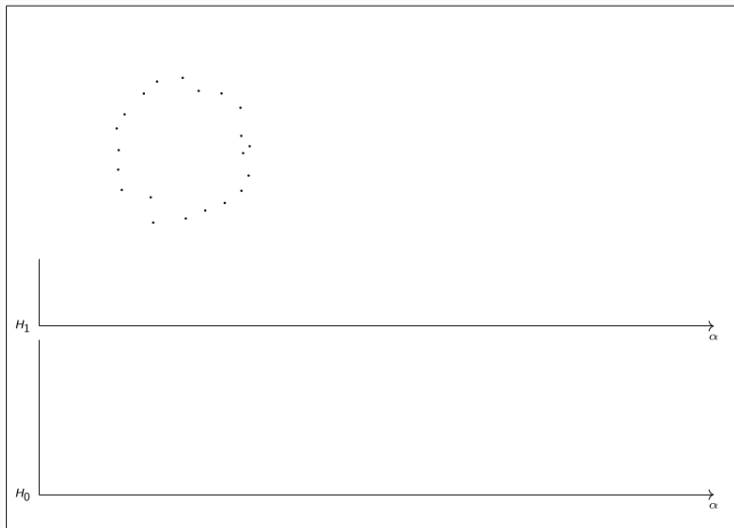
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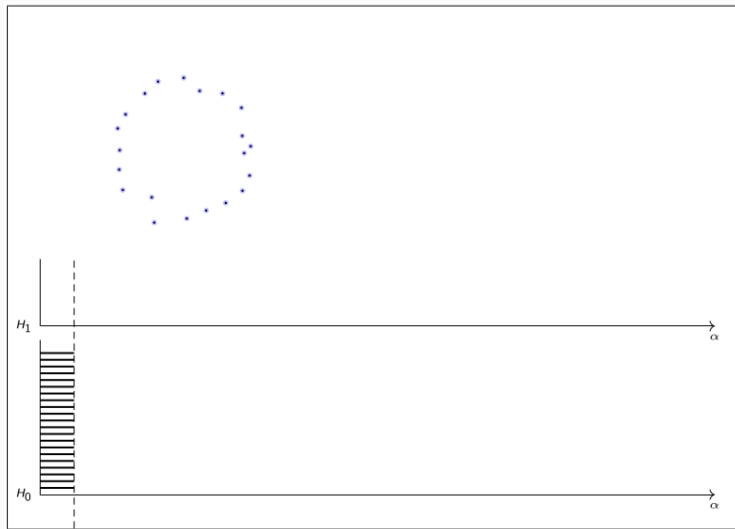
[Botnan and Crawley-Boevey, 2018]

We call \mathcal{B} the **barcode** of M .

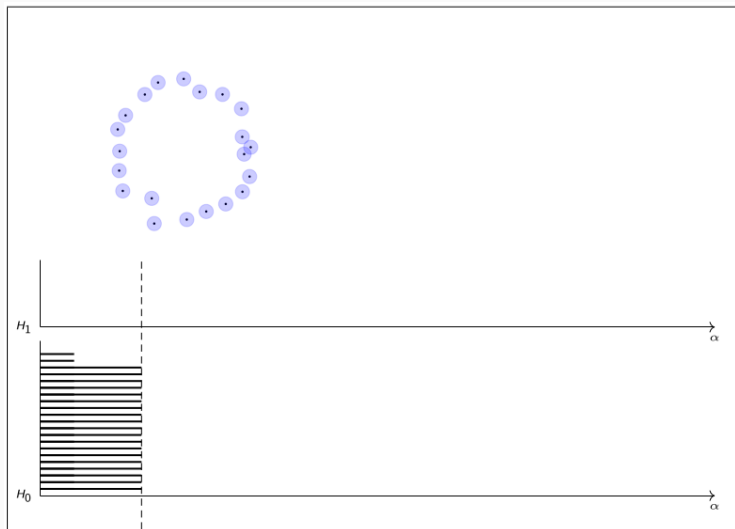
An example



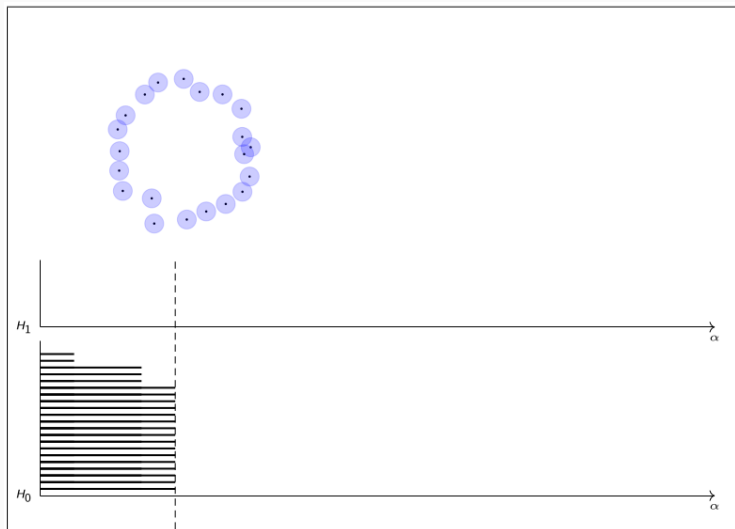
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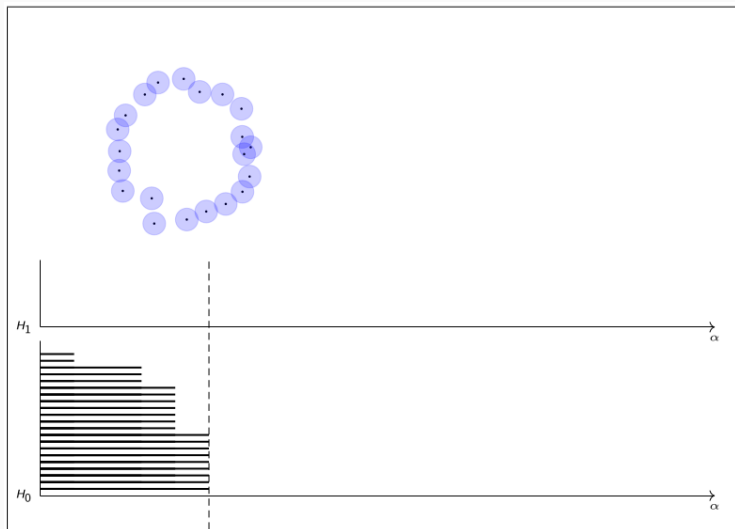
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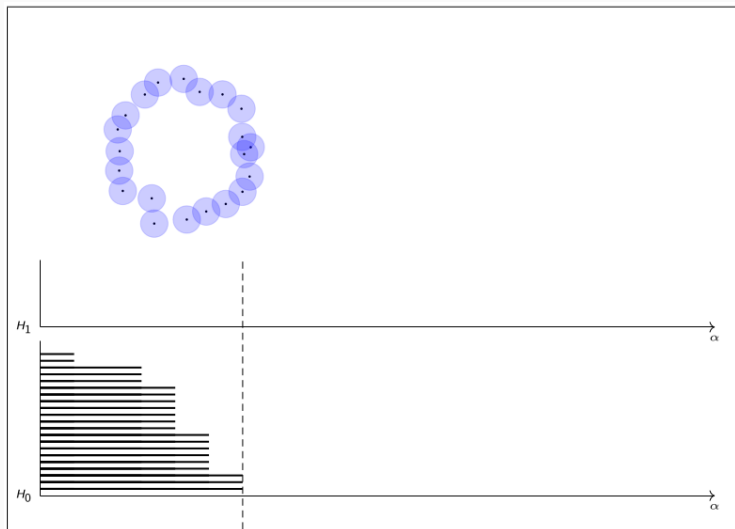
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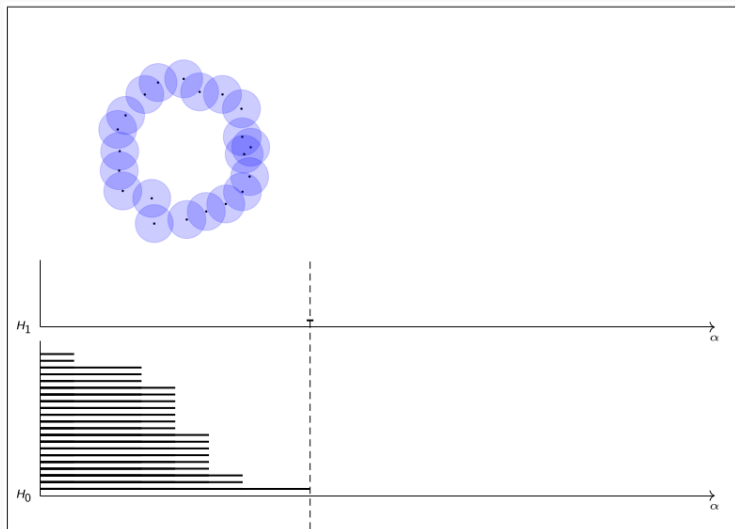
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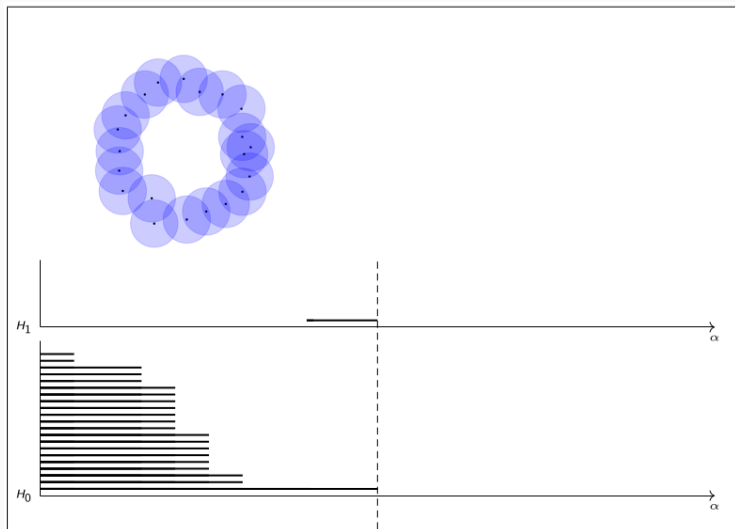
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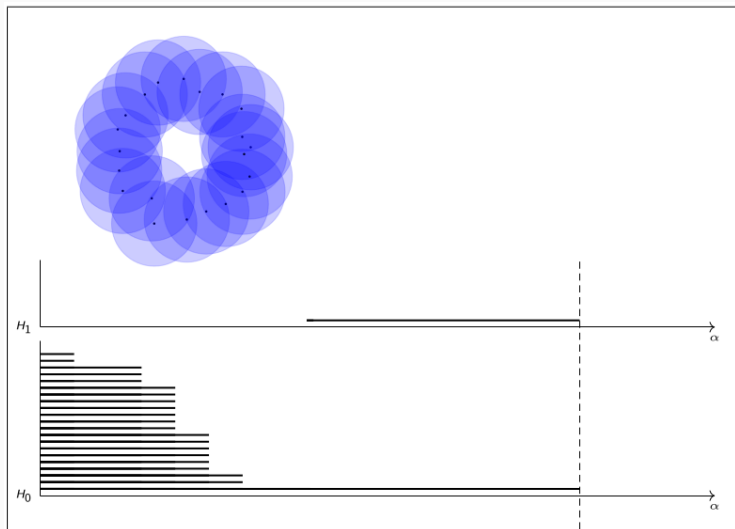
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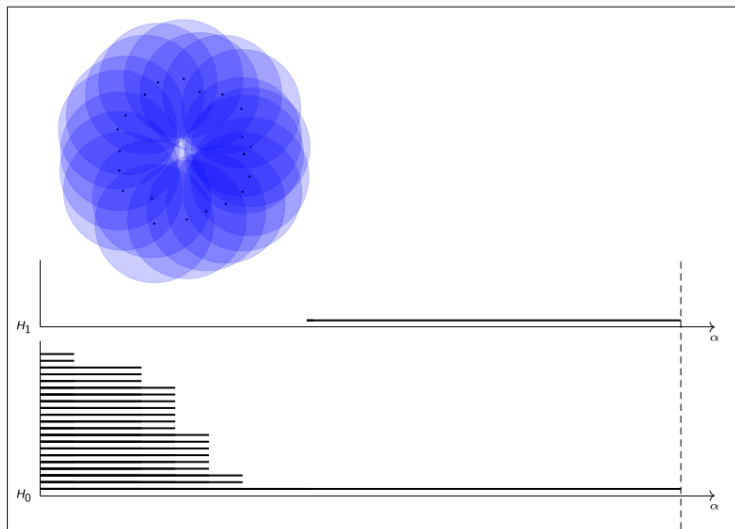
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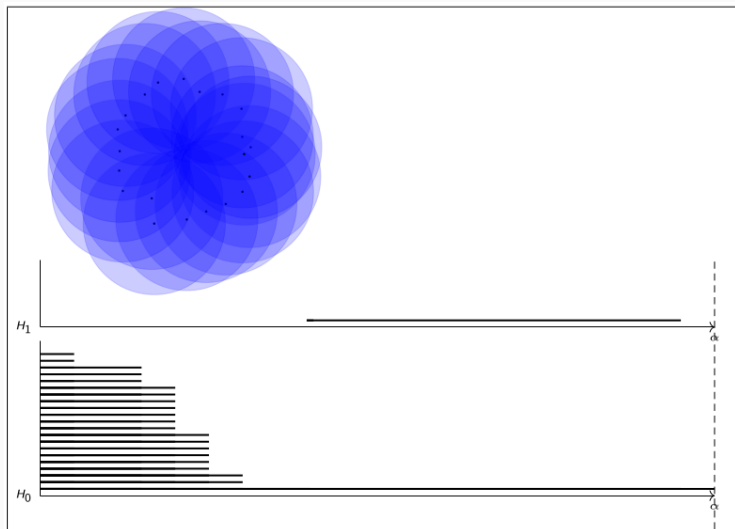
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Limitations of one parameter persistent homology

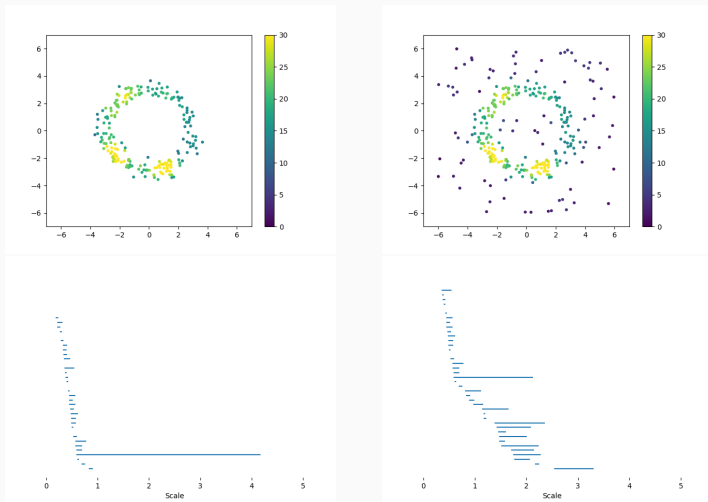


Figure 1: [Botnan and Lesnick, 2022]

Multiparameter persistence

Recall: a persistence module is a functor $M: \mathbb{R} \rightarrow \text{Vec}_{\mathbf{k}}$.

Multiparameter persistence

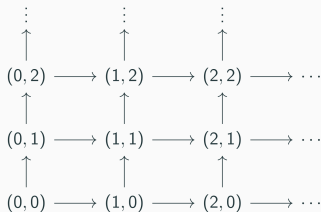
Recall: a persistence module is a functor $M: \mathbb{R} \rightarrow \text{Vec}_k$.

What if we replace \mathbb{R} by \mathbb{R}^2 ? Or \mathbb{N}^2 ? Or \mathbb{R}^n ? Do we still have a "barcode"?

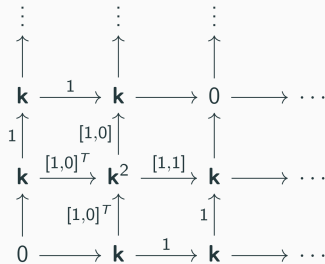
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(a) The poset \mathbb{N}^2



(b) A persistence module over \mathbb{N}^2

Unfortunately, there is no similar complete discrete invariant for multidimensional persistence [Carlsson and Zomorodian, 2007].

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Theorem

Let P be a small category and $M: P \rightarrow \text{Vec}_{\mathbf{k}}$ a pfd persistence module. Then M has an essentially unique decomposition into indecomposable summands.

Existence of Decomposition

Overview

The Tensor Product

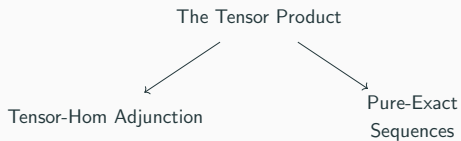
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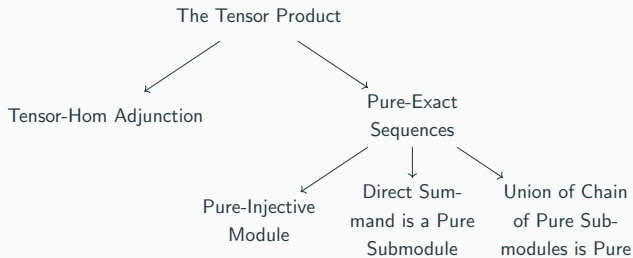


Tensor-Hom Adjunction

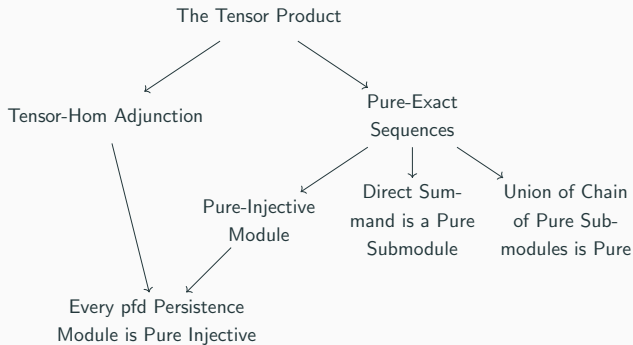
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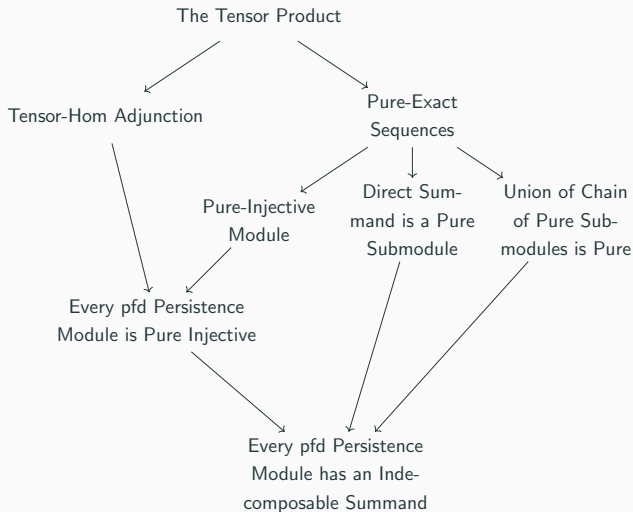
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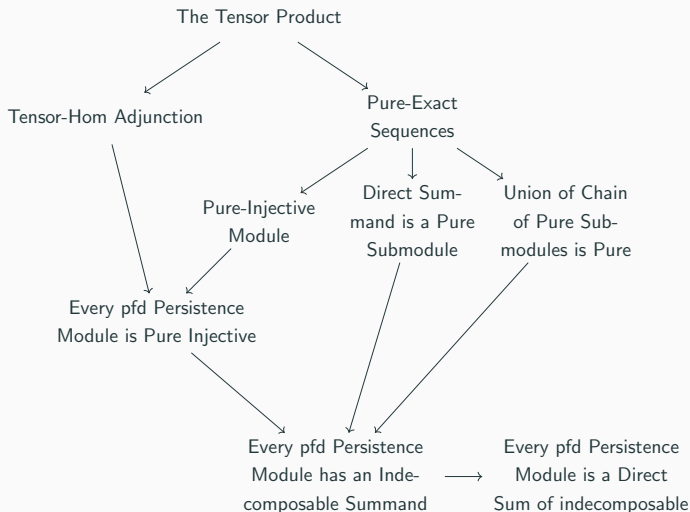
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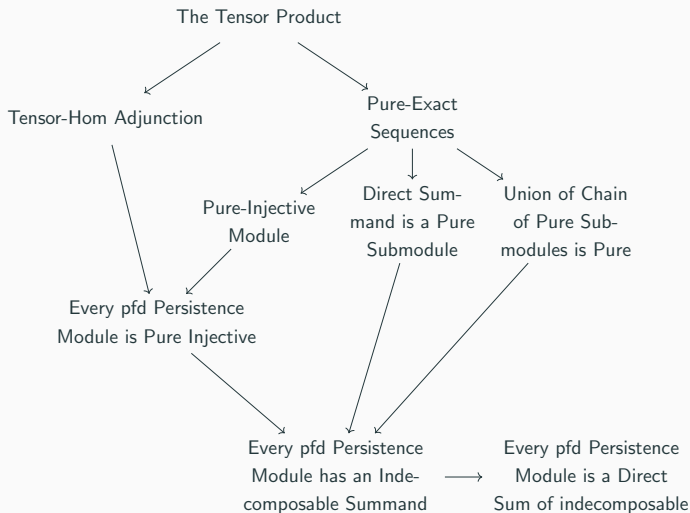
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Based on idea by William Crawley-Boevey

The Tensor Product and Duality

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The tensor product $N \otimes_P M$ is defined by the universal property

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Pointwise duality gives a contravariant, exact functor

$$D: \text{Vec}_{\mathbf{k}}^P \rightarrow \text{Vec}_{\mathbf{k}}^{P^{\text{op}}}$$

$$M: \quad \mathbf{k} \longrightarrow \mathbf{k}^2 \longrightarrow \mathbf{k}^3$$

$$DM: \quad \mathbf{k}^* \longleftarrow (\mathbf{k}^2)^* \longleftarrow (\mathbf{k}^3)^*$$

Tensor-Hom Adjunction

And we have a tensor-hom adjunction

$$\mathrm{Hom}_{\mathbf{k}}(N \otimes_P M, \mathbf{k}) \cong \mathrm{Hom}_P(M, DN)$$

Pure submodules and pure-exact sequences

A short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is **pure-exact** if

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A submodule $M' \subseteq M$ is **pure** in M if the sequence

$$0 \rightarrow M' \rightarrow M \rightarrow M/M' \rightarrow 0$$

is pure-exact.

Pure-injectivity

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2. Every pure exact sequence $0 \rightarrow M \rightarrow M' \rightarrow M'' \rightarrow 0$ splits.

Lemma

Every pfd persistence module M is pure injective

Every pfd persistence module has an indecomposable summand

Let $x \in M$. The set $S = \{N \subseteq M \mid N \text{ is pure in } M \text{ and } x \notin N\}$ has a maximal element N .

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M/N is non-zero since $x \in M/N$ and indecomposable by the maximality of N .

Uniqueness of Decomposition

Theorem

If $M = \bigoplus_{i \in I} M_i = \bigoplus_{j \in J} N_j$ are two direct sum decompositions of M into indecomposables, then there exists an isomorphism $\varphi: I \rightarrow J$ such that $M_i \cong N_{\varphi(i)}$ for all $i \in I$.

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- An adaptation of the Krull-Remak-Schmidt-Azumaya Theorem

Conclusion

Thank you for your attention

-  Botnan, M. B. and Crawley-Boevey, W. (2018).
Decomposition of persistence modules.
-  Botnan, M. B. and Lesnick, M. (2022).
An introduction to multiparameter persistence.
-  Carlsson, G. and Zomorodian, A. (2007).
The theory of multidimensional persistence.
Discrete and Computational Geometry, 42:71–93.