

# CTF22: SOLUTIONS

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## 1 CATEGORIES

## 2 FUNCTORS AND CONSTRUCTIONS ON CATEGORIES

## 3 NATURAL TRANSFORMATIONS AND EQUIVALENCES

## 4 LIMITS AND COLIMITS

### EXERCISE 21

Show that a full and faithful functor reflects the property of being a terminal (initial object).

#### Solution

*Proof.* Let  $F: \mathcal{C} \rightarrow \mathcal{D}$  be a fully faithful functor, and  $X \in \text{ob } \mathcal{C}$  such that  $FX$  is the terminal object of  $\mathcal{D}$ . Then for any  $Y \in \mathcal{C}$  we have  $\{*\} \cong \text{Hom}_{\mathcal{D}}(FY, FX) \cong \text{Hom}_{\mathcal{C}}(Y, X)$  so  $X$  is terminal in  $\mathcal{C}$ . We Similarly show that  $F$  reflects the property of being initial.  $\square$

### EXERCISE 22

Show that every equalizer is monic.

#### Solution

*Proof.* Let  $e: E \rightarrow X$  be the equalizer of  $f_1, f_2: X \rightrightarrows Y$ , and take  $g_1, g_2: T \rightarrow$

$$E \text{ such that } eg_1 = eg_2. \text{ Since both } g_1 \text{ and } g_2 \text{ make } \begin{array}{ccc} E & \xrightarrow{e} & X \xrightleftharpoons[f_2]{f_1} Y \\ \uparrow \scriptstyle \text{---} & \nearrow \scriptstyle eg_1=eg_2 & \\ T & & \end{array}$$

commute, by uniqueness it follows that  $g_1 = g_2$ , so  $e$  is monic.  $\square$

#### EXERCISE 23

Let  $E \xrightarrow{e} X \xrightleftharpoons[f_2]{f_1} Y$  be an equalizer diagram. Show that  $e$  is iso if and only if  $f_1 = f_2$ .

#### Solution

*Proof.*  $(\Rightarrow)$  Immediate, since

$$f_1 = f_1 \text{id}_X = f_1 e e^{-1} = f_2 e e^{-1} = f_2 \text{id}_X = f_2.$$

$(\Leftarrow)$  If  $f_1 = f_2$  then there exists a unique  $k: X \rightarrow E$  such that  $ek = \text{id}_X$ . Then  $eke = e \text{id}_E$  and since  $e$  is monic by the previous exercise it follows that  $ke = 1_E$ , so  $e$  is an isomorphism.  $\square$

#### EXERCISE 24

Show that in **Set**, every monomorphism fits into an equalizer diagram.

#### Solution

*Proof.* Let  $f: X \rightarrow Y$  be a monomorphism in **Set**. Define  $g_1, g_2: Y \rightrightarrows \{0, 1\}$  by  $g_1(y) = 1$  and

$$g_2(y) = \begin{cases} 1, & y \in \text{im } f \\ 0, & \text{otherwise} \end{cases}.$$

It is clear that  $g_2 f = g_1 f$ . Let  $h: T \rightarrow Y$  be any map such that  $g_1 h = g_2 h$ . It follows that  $\text{im } h \subseteq \text{im } f$  and so we can define  $k: T \rightarrow X$  by  $k(t) = f^{-1}h(t)$ . Then  $f k = h$  and  $k$  is unique, so  $f$  is the equalizer of  $g_1, g_2: Y \rightrightarrows \{0, 1\}$ .  $\square$

#### EXERCISE 25

Let  $\begin{array}{ccc} A & \xrightarrow{b} & B \\ \downarrow a & & \downarrow f \\ X & \xrightarrow{g} & Y \end{array}$  be a pullback diagram with  $f$  monic. Show that  $a$  is also monic. Also, if  $f$  is iso, so is  $a$ .

**Solution**

*Proof.* Let  $t_1, t_2: T \rightrightarrows A$  such that  $at_1 = at_2$ . Then  $fbt_1 = gat_1 = gat_2 = fbt_2$  and since  $f$  is monic,  $bt_1 = bt_2$ . Hence, there exists a unique  $k: T \rightarrow A$  such that

$$\begin{array}{ccccc}
 T & & \xrightarrow{bt_1=bt_2} & & B \\
 & \searrow k & & & \downarrow f \\
 & & A & \xrightarrow{b} & B \\
 & & \downarrow a & & \downarrow f \\
 & & X & \xrightarrow{g} & Y
 \end{array}$$

$at_1=at_2$  (curved arrow from  $T$  to  $X$ )

commute. Since either  $k = t_1$  or  $k = t_2$  would work, it must be the case that  $t_1 = t_2$ .  $\square$

**EXERCISE 26**

Given two commuting squares

$$\begin{array}{ccccc}
 A & \xrightarrow{b} & B & \xrightarrow{c} & C \\
 \downarrow a & & \downarrow f & & \downarrow d \\
 X & \xrightarrow{g} & Y & \xrightarrow{h} & Z
 \end{array}$$

Show that

- i. if both squares are pullback squares, then so is the composite square;
- ii. if the right square and the composite square are pullbacks, then so is the left square.

**Solution**

i. *Proof.*  $\square$

ii. *Proof.*  $\square$

**EXERCISE 27**

**Solution**

*Proof.*  $\square$

**EXERCISE 28**

**Solution**

*Proof.*  $\square$

EXERCISE 29

**Solution**

*Proof.*

□

EXERCISE 30

**Solution**

*Proof.*

□

## 5 COMPLETE CATEGORIES

EXERCISE 42

Take one of your favourite categories (**Top**, **Pos**, **Rng**, **Mon**, **Grp**, **Grph**, **Cat**) and show that it is both complete and cocomplete.

**Solution** Consider **Grp**. By Proposition 5.1, it is enough to prove that **Grp** has all small products and equalizers. The equalizer of  $f, g: G \rightrightarrows H$  is the pair  $(G', i)$  where  $G' = \{a \in G \mid f(a) = g(a)\}$  and  $i$  is the inclusion map. Note the  $G'$  is a group since  $f$  and  $g$  are group homomorphisms. For a set of groups  $\{G_i\}_{i \in I}$ , the product  $\prod_{i \in I} G_i$  has a group structure by performing the group operations pointwise.

EXERCISE 43

Show that if  $\mathcal{C}$  is complete, then  $F: \mathcal{C} \rightarrow \mathcal{D}$  preserves all limits if  $F$  preserves products and equalizers. This no longer holds if  $\mathcal{C}$  is not complete:  $F$  may preserve all products and equalizers which exists in  $\mathcal{C}$ , yet not preserve all limits which exists in  $\mathcal{C}$ .

**Solution** I assume the question means *small* limits, since I don't think this holds for all limits.

*Proof.* Suppose  $\mathcal{C}$  is complete and  $F: \mathcal{C} \rightarrow \mathcal{D}$  preserves products and equalizers. Let  $G: \mathcal{I} \rightarrow \mathcal{C}$  be a small diagram. Then  $\lim_{\mathcal{I}} G$  can be expressed as

$$\lim_{\mathcal{I}} G \longrightarrow \prod_{i \in \text{ob } \mathcal{I}} G_i \xrightarrow[d]{c} \prod_{f \in \text{mor } \mathcal{I}} G(\text{cod } f),$$

and since  $F$  preserves equalizers and products,  $F(\lim_{\mathcal{I}} G) = \lim_{\mathcal{I}} (FG)$ . □

EXERCISE 44

Suppose a category  $\mathcal{C}$  has limits of shape  $\mathcal{I}$ . Show that the operation which assigns each diagram  $\mathcal{I} \rightarrow \mathcal{C}$  to its limit in  $\mathcal{C}$  is part of a functor  $F: [\mathcal{I}, \mathcal{C}] \rightarrow \mathcal{C}$ .

**Solution**

*Proof.* Let  $\eta: G \Rightarrow H$  be a morphism in  $[\mathcal{I}, \mathcal{C}]$ ,  $\mu: \Delta_{FG} \Rightarrow G$ ,  $\varepsilon: \Delta_{FH} \Rightarrow H$  and  $f: C \rightarrow C'$  a morphism in  $\mathcal{I}$ . This is summarized in the following diagram

$$\begin{array}{ccccc}
 & FG & & FH & \\
 \mu_C \swarrow & & \searrow \mu_{C'} & \varepsilon_C \swarrow & \searrow \varepsilon_{C'} \\
 GC & \xrightarrow{Gf} & GC' & HC & \xrightarrow{Hf} & HC' \\
 & \searrow \eta_C & & \swarrow \eta_{C'} & \\
 & & & & 
 \end{array}$$

Note also that the diagram commutes. Thus,  $(FG, \eta\mu)$  is a cone for the diagram  $H$ , so there is a unique morphism  $g: FG \rightarrow FH$  such that  $\varepsilon\Delta_g = \eta\mu$ . We define  $F\eta := g$ . It is straightforward to verify that the uniqueness of  $g$  turns  $F$  into a functor.  $\square$

EXERCISE 45

Let  $\mathcal{C}, \mathcal{D}$  and  $\mathcal{E}$  be categories. Show that the following categories are isomorphic:

$$[\mathcal{E}, [\mathcal{C}, \mathcal{D}]] \cong [\mathcal{E} \times \mathcal{C}, \mathcal{D}] \cong [\mathcal{C}, [\mathcal{E}, \mathcal{D}]].$$

Use this and the previous exercise to give a more elegant proof of Theorem 4.5.

**Solution**

*Proof.* Consider the functors  $[\mathcal{E}, [\mathcal{C}, \mathcal{D}]] \xrightarrow{F_1} [\mathcal{E} \times \mathcal{C}, \mathcal{D}] \xrightarrow{F_2} [\mathcal{C}, [\mathcal{E}, \mathcal{D}]]$  given by

- On objects: for  $G: \mathcal{E} \rightarrow [\mathcal{C}, \mathcal{D}]$ ,  $H: \mathcal{E} \times \mathcal{C} \rightarrow \mathcal{D}$ ,  $f: E \rightarrow E'$  in  $\mathcal{E}$  and  $g: C \rightarrow C'$  in  $\mathcal{C}$  we have

$$\begin{aligned}
 F_1(G)(E, C) &= G(E)(C) \\
 F_1(G)(f, g) &= (G(E')(g))(Gf)_C \\
 F_2(H)(C)(E) &= H(E, C) \\
 F_2(H)(C)(f) &= H(f, \text{id}_C) \\
 (F_2(H)(g))_E &= H(\text{id}_E, f).
 \end{aligned}$$

- On morphisms: for  $G_1, G_2: \mathcal{E} \Rightarrow [\mathcal{C}, \mathcal{D}]$ ,  $H_1, H_2: \mathcal{E} \times \mathcal{C} \Rightarrow \mathcal{D}$ ,  $\eta: G_1 \Rightarrow G_2$  and  $\varepsilon: H_1 \Rightarrow H_2$  we have  $(F_1\eta)_{(E, C)} = (\eta_E)_C$  and  $((F_2\varepsilon)_C)_E = \varepsilon_{(E, C)}$ .

Since these functors are clearly invertible, they are isomorphisms of categories.  $\square$

EXERCISE 46

Show that a full and faithful functor reflects the property of being a terminal (or initial) object. Deduce that equivalences preserve the terminal (or initial) object.

### Solution

*Proof.* Let  $F: \mathcal{C} \rightarrow \mathcal{D}$  be a fully faithful functor and  $X \in \text{ob } \mathcal{C}$  such that  $FX$  is terminal and take any  $Y \in \text{ob } \mathcal{C}$ . Then  $\text{Hom}_{\mathcal{C}}(Y, X) \cong \text{Hom}_{\mathcal{D}}(FY, FX) \cong \{*\}$ , so  $X$  is terminal in  $\mathcal{C}$ . Similarly, we show that fully faithful functors reflect the property of being initial.

Hence, if  $F$  is an equivalence,  $X \in \text{ob } \mathcal{C}$  is terminal,  $Z \in \text{ob } \mathcal{D}$  is any object and  $Y \in \text{ob } \mathcal{C}$  is chosen such that  $FY \cong Z$  we have that  $\{*\} = \text{Hom}_{\mathcal{C}}(Y, X) \cong \text{Hom}_{\mathcal{D}}(FY, FX) \cong \text{Hom}_{\mathcal{D}}(Z, FX)$  so  $FX$  is terminal in  $\mathcal{D}$ .  $\square$

## 6 CARTESIAN CLOSED CATEGORIES

### EXERCISE 47

Show that in a ccc, there are natural isomorphisms

- i.  $1^X \cong 1$ ,
- ii.  $(Y \times Z)^X \cong Y^X \times Z^X$ ,
- iii.  $(Y^Z)^X = Y^{Z \times X}$ .

**Solution** Let  $\mathcal{C}$  be a ccc category and  $X, Y, Z$  and  $A$  objects in  $\mathcal{C}$ .

- i. *Proof.* Since  $1$  is terminal we have unique morphisms  $1 \times X \rightarrow 1$ ,  $A \times X \rightarrow 1$  and  $f: A \rightarrow 1$ , so

$$\begin{array}{ccc} 1 \times X & \xrightarrow{\quad} & 1 \\ & \nwarrow f \times 1_X & \uparrow \\ & & A \times X \end{array} \quad \text{trivially commutes. Hence,}$$

$1^X \cong 1$ . Moreover, this isomorphism is natural as a morphism  $1^{(-)} \Rightarrow 1$

$$\begin{array}{ccc} 1^X & \xrightarrow{\quad} & 1^Y \\ \downarrow & & \downarrow \\ 1 & \xrightarrow{\quad} & 1 \end{array} \quad \text{since all the maps in the square are the identity maps. } \square$$

- ii. *Proof.* Let  $\eta_{(Y,Z)}: Y^X \times Z^X \rightarrow (Y \times Z)^X$  be the unique map which makes

$$\begin{array}{ccc} (Y \times Z)^X \times X & \xrightarrow{\text{ev}_{X,Y \times Z}} & Y \times Z \\ & \nwarrow \eta_{(Y,Z)} \times 1_X & \uparrow h \\ & & Y^X \times Z^X \times X \end{array}$$

commute, where  $h = (\text{ev}_{X,Y}(\pi_{Y^X} \times 1_X), \text{ev}_{X,Z}(\pi_{Z^X} \times 1_X))$ . To construct an inverse, let  $f_Y: (Y \times Z)^X \rightarrow Y^X$  be the unique map such that

$$\begin{array}{ccc} Y^X \times X & \xrightarrow{\text{ev}_{X,Y}} & Y \\ & \nwarrow f_Y \times 1_X & \uparrow \pi_Y \text{ ev}_{X,Y \times Z} \\ & & (Y \times Z)^X \times X \end{array}$$

commutes. Analogously, we define  $f_Z: (Y \times Z)^X \rightarrow Z^X$ . We claim that  $\mu_{(Y,Z)} = (f_Y, f_Z): (Y \times Z)^X \rightarrow Y^X \times Z^X$  is the inverse of  $\eta_{(Y,Z)}$ . Since

$$\begin{array}{ccccc}
 & (Y \times Z)^X \times X & & & \\
 & \swarrow f_Y \times 1_X & \downarrow \mu_{(Y,Z)} \times 1_X & \searrow f_Z \times 1_X & \\
 Y^X \times X & \xleftarrow{\pi_{Y^X} \times 1_X} & Y^X \times Z^X \times X & \xrightarrow{\pi_{Z^X} \times 1_X} & Z^X \times X \\
 \downarrow \text{ev}_{X,Y} & & \downarrow h & & \downarrow \text{ev}_{X,Z} \\
 Y & \xleftarrow{\pi_Y} & Y \times Z & \xrightarrow{\pi_Z} & Z
 \end{array}$$

commutes, it follows that

$$\begin{aligned}
 h(\mu_{(Y,Z)} \times 1_X) &= (\text{ev}_{X,Y}(f_Y \times 1_X), \text{ev}_{X,Z}(f_Z \times 1_X)) \\
 &= (\pi_Y \text{ev}_{X,Y \times Z}, \pi_Z \text{ev}_{X,Y \times Z}) \\
 &= \text{ev}_{X,Y \times Z}
 \end{aligned}$$

and so

$$\begin{array}{ccc}
 (Y \times Z)^X \times X & \xrightarrow{\text{ev}_{X,Y \times Z}} & Y \times Z \\
 \nwarrow \eta_{(Y,Z)} \times 1_X & & \nearrow h \\
 & Y^X \times Z^X \times X & \\
 \nwarrow \mu_{(Y,Z)} \times 1_X & & \nearrow \text{ev}_{X,Y \times Z} \\
 & (Y \times Z)^X \times X &
 \end{array}$$

commutes (the top left triangle commutes by definition of  $\eta_{(Y,Z)}$  and we've just show that the bottom right triangle commute, so the big triangle commutes as well). By uniqueness,  $\eta_{(Y,Z)}\mu_{(Y,Z)} = \text{id}_{(Y \times Z)^X}$ . By definition of  $f_Y$  and  $\eta_{(Y,Z)}$ ,

$$\begin{array}{ccccc}
 Y^X \times X & \xrightarrow{\text{ev}_{X,Y}} & Y & & \\
 \nwarrow f_Y \times 1_X & & \nearrow \pi_Y \text{ev}_{X,Y \times Z} & \nearrow \pi_Y & \\
 & (Y \times Z)^X \times X & \xrightarrow{\text{ev}_{X,Y \times Z}} & Y \times Z & \\
 & \nwarrow \eta_{(Y,Z)} \times 1_X & & \nearrow h & \\
 & & Y^X \times Z^X \times X & &
 \end{array}$$

commutes and so both  $\pi_{Y^X} \times 1_X$  and  $(f_Y \eta_{Y,Z}) \times 1_X$  make

$$\begin{array}{ccc}
 Y^X \times X & \xrightarrow{\text{ev}_{X,Y}} & Y \\
 \nwarrow & & \nearrow \pi_Y h \\
 & Y^X \times Z^X \times X &
 \end{array}$$



commute, so  $f_Y \eta_{(Y,Z)} = \pi_{Y^X}$ . Analogously,  $f_Z \eta_{(Y,Z)} = \pi_{Z^X}$ . Thus,

$$\begin{array}{ccccc}
 & & Y^X \times Z^X & & \\
 & \swarrow \pi_{Y^X} & \downarrow \eta_{(Y,Z)} & \searrow \pi_{Z^X} & \\
 & & (Y \times Z)^X & & \\
 & \swarrow f_Y & \downarrow \mu_{(Y,Z)} & \searrow f_Z & \\
 Y^X & \xleftarrow{\pi_{Y^X}} & Y^X \times Z^X & \xrightarrow{\pi_{Z^X}} & Z^X
 \end{array}$$

commutes, and by uniqueness it follows that  $\mu_{(Y,Z)} \eta_{(Y,Z)} = 1_{Y^X \times Z^X}$ .

It is left to show that this gives a natural isomorphism  $\eta: (-)^X \times (-)^X \Rightarrow (- \times -)^X$  of functors  $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ . Since a morphism of bifunctors is natural if and only if it is natural in each component and the definitions are symmetric in their components, it suffices to check that

$$\begin{array}{ccc}
 Y^X \times Z^X & \xrightarrow{f^X \times 1_{Z^X}} & (Y')^X \times Z^X \\
 \downarrow \eta_{(Y,Z)} & & \downarrow \eta_{(Y',Z)} \\
 (Y \times Z)^X & \xrightarrow{(f \times 1_Z)^X} & (Y' \times Z)^X
 \end{array}$$

commutes. Since  $\text{ev}_{X,-}: (-)^X \times X \Rightarrow 1_{\mathcal{C}}$  is a natural transformation, we have that

$$\begin{aligned}
 \text{ev}_{X,Y' \times Z} \circ (f \times 1_Z)^X \times 1_X \circ \eta_{Y,Z} \times 1_X &= (f \times 1_Z) \circ \text{ev}_{X,Y \times Z} \circ \eta_{(Y,Z)} \times 1_X \\
 &= (f \times 1_Z) h
 \end{aligned}$$

and

$$\begin{array}{ccccc}
 & & Y^X \times Z^X \times X & & \\
 & \swarrow f \circ \text{ev}_{X,Y} (\pi_{Y^X} \times 1_X) & \downarrow f^X \times 1_{Z^X} \times 1_X & \searrow \text{ev}_{X,Z} (\pi_{Z^X} \times 1_X) & \\
 & & (Y')^X \times Z^X \times X & & \\
 & \swarrow \text{ev}_{X,Y'} (\pi_{(Y')^X} \times 1_X) & \downarrow h' & \searrow \text{ev}_{X,Z} (\pi_{Z^X} \times 1_X) & \\
 Y' & \xleftarrow{\quad} & Y' \times Z & \xrightarrow{\quad} & Z
 \end{array}$$

commutes, so

$$\begin{aligned}
 \text{ev}_{X,Y' \times Z} \circ \eta_{(Y',Z)} \times 1_X \circ f^X \times 1_{Z^X} \times 1_X &= h' \circ f^X \times 1_{Z^X} \times 1_X \\
 &= (f \times 1_Z) h.
 \end{aligned}$$

It follows that both  $(f \times 1_Z)^X \eta_{(Y,Z)}$  and  $\eta_{(Y',Z)} (f^X \times 1_{Z^X})$  make

$$\begin{array}{ccc}
(Y' \times Z)^X \times X & \xrightarrow{\text{ev}_{X,Y'} \times Z} & Y' \times Z \\
& \nwarrow - \times 1_X & \uparrow (f \times 1_Z)h \\
& & Y^X \times Z^X \times X
\end{array}$$

commute, and so they are equal.  $\square$

- iii. *Proof.* Define  $f = \text{ev}_{Z,Y} \circ \text{ev}_{X,Y^Z} \times 1_Z: (Y^Z)^X \times X \times Z \rightarrow Y$ . Then for any  $h: A \times X \times Z \rightarrow Y$  we get a map  $H': A \times X \rightarrow Y^Z$  such that  $\text{ev}_{Z,Y} \circ H' \times 1_Z = h$  which gives a unique map  $H: A \rightarrow (Y^Z)^X$  such that  $\text{ev}_{X,Y^Z} \circ H \times 1_X = H'$ . Since

$$\begin{aligned}
f \circ H \times 1_{X \times Z} &= \text{ev}_{Z,Y} \circ \text{ev}_{X,Y^Z} \times 1_Z \circ H \times 1_{X \times Z} \\
&= \text{ev}_{Z,Y} \circ (\text{ev}_{X,Y^Z} \circ H \times 1_X) \times 1_Z \\
&= \text{ev}_{Z,Y} \circ H' \times 1_Z \\
&= h,
\end{aligned}$$

we conclude that  $(Y^Z)^X \cong Y^{Z \times X}$ .  $\square$

#### EXERCISE 48

If a ccc has coproducts, we have

- i.  $X \times (Y + Z) \cong (X \times Y) + (X \times Z)$
- ii.  $Y^{Z+X} = Y^Z \times Y^X$ .

#### Solution

- i. *Proof.* Suppose  $\mathcal{C}$  is a ccc which has coproducts, and let  $X, Y$  and  $Z$  be objects in  $\mathcal{C}$ . Consider any cocone

$$\begin{array}{ccc}
X \times Y & & X \times Z \\
& \searrow \iota_1 & \swarrow \iota_2 \\
& D &
\end{array}$$

Exponentiating, we get

$$\begin{array}{ccc}
X \times Y & \xrightarrow{\overline{\iota_1} \times 1_X} & D^X \times X \xleftarrow{\overline{\iota_2} \times 1_X} X \times Z \\
& \searrow \iota_1 & \downarrow \text{ev}_{X,D} \swarrow \iota_2 \\
& & D
\end{array}$$

where  $\overline{\iota_1}$  and  $\overline{\iota_2}$  are the transposes of  $\iota_1$  and  $\iota_2$  respectively. Hence, we have the cocone

$$\begin{array}{ccc}
Y & & Z \\
& \searrow \overline{\iota_1} & \swarrow \overline{\iota_2} \\
& D^X &
\end{array}$$

and since  $Y + Z$  is limiting, we get a unique map  $h: Y + Z \rightarrow D^X$  so that

$$\begin{array}{ccccc}
Y & \xrightarrow{\iota_Y} & Y + Z & \xleftarrow{\iota_Z} & Z \\
& \searrow \overline{\iota_1} & \downarrow h & \swarrow \overline{\iota_2} & \\
& & D^X & &
\end{array}$$

commutes. Then

$$\begin{array}{ccccc}
Y \times X & \xrightarrow{\iota_Y \times 1_X} & (Y + Z) \times X & \xleftarrow{\iota_Z \times 1_X} & Z \times X \\
& \searrow \overline{\iota_1} \times 1_X & \downarrow h \times 1_X & \swarrow \overline{\iota_2} \times 1_X & \\
& & D^X \times X & & \\
& & \downarrow \text{ev}_{X,D} & & \\
& & D & &
\end{array}$$

commutes and it follows that  $(Y + Z) \times X$  is the coproduct of  $Y \times X$  and  $Z \times X$ . Thus  $Y \times X + Z \times X \cong (Y + Z) \times X$   $\square$

- ii. *Proof.* Note that  $Y^Z \times Y^X(Z + X) \cong Y^X \times Y^Z \times Z + Y^Z \times Y^X \times X$  and let  $f: Y^Z \times Y^X(Z + X) \rightarrow Y$  be given by the composition

$$\begin{aligned}
Y^X \times Y^Z \times Z + Y^Z \times Y^X \times X &\xrightarrow{1_{Y^X} \times \text{ev}_{Z,Y} + 1_{Y^Z} \times \text{ev}_{X,Y}} \\
&Y^X \times Y + Y^Z \times Y \cong (Y^X + Y^Z) \times Y \xrightarrow{\pi_Y} Y.
\end{aligned}$$

Let  $h: A \times (Z + X) \rightarrow Y$  be any morphism. Since  $A \times (Z + X) \cong A \times Z + A \times X$  we have unique maps  $H_Z: A \rightarrow Y^Z$  and  $H_X: A \rightarrow Y^X$  such that

$$\begin{array}{ccc}
Y^Z \times Z & \xrightarrow{\text{ev}_{Z,Y}} & Y \\
\swarrow H_Z \times 1_Z & \uparrow \iota_{A \times Z} h & \\
& A \times Z &
\end{array}
\quad
\begin{array}{ccc}
Y^X \times X & \xrightarrow{\text{ev}_{X,Y}} & Y \\
\swarrow H_X \times 1_X & \uparrow \iota_{A \times X} h & \\
& A \times X &
\end{array}$$

commute, where  $A \times Z \xrightarrow{\iota_{A \times Z}} A \times Z + A \times X \xleftarrow{\iota_{A \times X}} A \times X$  are the inclusion maps. Let  $H = (H_Z, H_X): A \rightarrow Y^Z \times Y^X$ . Since

$$\pi_Y \circ (1_{Y^X} \times \text{ev}_{Z,Y}) \circ (H \times 1_Z) = \pi_Y \circ H_X \times (\text{ev}_{Z,Y} \circ H_Z \times 1_Z) = \iota_{A \times Z} h$$

and similarly  $\pi_Y \circ (1_{Y^Z} \times \text{ev}_{X,Y}) \circ (H \times 1_X) = \iota_{A \times X} h$ , it follows that

$$\begin{aligned} h &= \iota_{A \times Z} h + \iota_{A \times X} h \\ &= \pi_Y \circ (1_{Y^X} \times \text{ev}_{Z,Y}) \circ (H \times 1_Z) + \pi_Y \circ (1_{Y^Z} \times \text{ev}_{X,Y}) \circ (H \times 1_X) \\ &= f \circ (H \times 1_Z + H \times 1_X) \\ &= f \circ (H \times 1_{Z+X}). \end{aligned}$$

Hence,

$$\begin{array}{ccc} Y^X \times Y^Z \times (Z+X) & \xrightarrow{f} & Y \\ & \nwarrow H \times 1_{Z+X} & \uparrow h \\ & & A \times (Z+X) \end{array} \quad \text{commutes and so } Y^{Z+X} \cong Y^Z \times Y^X.$$

□

#### EXERCISE 49

In a ccc, prove that the transpose of a composite  $Z \xrightarrow{g} W \xrightarrow{f} Y^X$  is

$$Z \times X \xrightarrow{g \times 1_X} W \times X \xrightarrow{\bar{f}} Y,$$

if  $\bar{f}$  is the transpose of  $f$ .

#### Solution

*Proof.* We need to find a morphism  $\bar{f}g: Z \times X \rightarrow Y$  such that  $\text{ev}_{X,Y} \circ (f \circ g \times 1_X) = \bar{f}g$ . Well,

$$\text{ev}_{X,Y} \circ (f \circ g \times 1_X) = \text{ev}_{X,Y} \circ (f \times 1_X) \circ (g \times 1_X) = \bar{f} \circ (g \times 1_X),$$

so we're done. □

#### EXERCISE 50

Suppose  $\mathcal{C}, \mathcal{D}, \mathcal{E}$  are categories, such that

1. For each pair of objects  $(C, D) \in \text{ob}(\mathcal{C} \times \mathcal{D})$ , an object  $F_0(C, D)$  in  $\mathcal{E}$ ;
2. For each object  $C \in \text{ob}\mathcal{C}$ , a functor  $F_C: \mathcal{D} \rightarrow \mathcal{E}$  satisfying  $F_C(D) = F_0(C, D)$  for each  $D \in \text{ob}\mathcal{D}$ ;
3. For each object  $D \in \text{ob}\mathcal{D}$ , a functor  $F_D: \mathcal{C} \rightarrow \mathcal{E}$  satisfying  $F_D(C) = F_0(C, D)$  for each object  $C \in \mathcal{C}$ ;

such that for each pair of morphism  $f: C \rightarrow C'$  in  $\mathcal{C}$  and  $g: D \rightarrow D'$  in  $\mathcal{D}$  we have a commuting square

$$\begin{array}{ccc} F_0(C, D) & \xrightarrow{F_D(f)} & F_0(C', D) \\ \downarrow F_C(g) & & \downarrow F_{C'}(g) \\ F_0(C, D') & \xrightarrow{F_{D'}(f)} & F_0(C', D') \end{array}$$

in  $\mathcal{E}$ .

Show that there is a unique functor  $F: \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E}$  whose operation on objects in  $F_0$ , while  $F(1_C, g) = F_C(g)$  and  $F(f, 1_D) = F_D(f)$ .

**Solution**

*Proof.* For  $f: C \rightarrow C'$  in  $\mathcal{C}$  and  $g: D \rightarrow D'$  in  $\mathcal{D}$  we define  $F(f, g): F_0(C, D) \rightarrow F_0(C', D')$  to be any composition around the commuting square

$$\begin{array}{ccc} F_0(C, D) & \xrightarrow{F_D(f)} & F_0(C', D) \\ \downarrow F_C(g) & & \downarrow F_{C'}(g) \\ F_0(C, D') & \xrightarrow{F_{D'}(f)} & F_0(C', D') \end{array}$$

This is easily checked to give a functor  $F: \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E}$ . □

EXERCISE 51

An object  $Y$  in a category with finite product is called *exponentiating* if the exponential  $X^Y$  exists for each  $Y \in \text{ob } \mathcal{C}$ . Show that if  $X$  is exponentiating, the assignment  $Y \mapsto X^Y$  is the object part of a functor  $\mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$ .

**Solution**

*Proof.* For  $f: Y \rightarrow Z$ , we define  $X^f: X^Z \rightarrow X^Y$  to be the unique map which makes

$$\begin{array}{ccc} X^Y \times Y & \xrightarrow{\text{ev}_{Y, X}} & X \\ \uparrow X^f \times 1_Y & & \nearrow \text{ev}_{Z, X} \\ & X^Z \times Z & \\ \uparrow & \nearrow 1_{X^Z} \times f & \\ X^Z \times Y & & \end{array}$$

commute. From uniqueness, we get that  $X^{1_Y} = 1_{X^Y}$  and for  $X^{gf} = X^f X^g$  for  $Y \xrightarrow{f} Z \xrightarrow{g} W$ . Hence,  $X^{(-)}: \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$  is a functor. □

EXERCISE 52

Show that for every cartesian closed category  $\mathcal{C}$  there is a functor  $\mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{C}$ , assigning  $Y^X$  to  $(X, Y)$ .

### Solution

*Proof.* Checking the conditions of Exercise 50:

1. for each object  $(X, Y)$  in  $\mathcal{C}^{\text{op}} \times \mathcal{C}$  we have the object  $Y^X$  in  $\mathcal{C}$ ;
2. for each  $X \in \text{ob } \mathcal{C}^{\text{op}}$  we have the functor  $(-)^X: \mathcal{C} \rightarrow \mathcal{C}$ ;
3. for each  $Y \in \text{ob } \mathcal{C}^{\text{op}}$  we have the functor  $Y^{(-)}: \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$ .

It is left to show that for any  $f: X_2 \rightarrow X_1$  and  $g: Y_1 \rightarrow Y_2$  in  $\mathcal{C}$ ,

$$\begin{array}{ccc} Y_1^{X_1} & \xrightarrow{Y_1^f} & Y_1^{X_2} \\ \downarrow g^{X_1} & & \downarrow g^{X_2} \\ Y_2^{X_1} & \xrightarrow{Y_2^f} & Y_2^{X_2} \end{array}$$

commutes<sup>1</sup>. Since

$$\begin{array}{ccccc} Y_1^{X_1} \times X_2 & \xrightarrow{Y_1^f \times 1_{X_2}} & Y_1^{X_2} \times X_2 & \xrightarrow{g^{X_2} \times 1_{X_2}} & Y_2^{X_2} \times X_2 \\ \downarrow 1_{Y_1^{X_1}} \times f & & \downarrow \text{ev}_{X_2, Y_1} & & \downarrow \text{ev}_{X_2, Y_2} \\ Y_1^{X_1} \times X_1 & \xrightarrow{\text{ev}_{X_1, Y_1}} & Y_1 & \xrightarrow{g} & Y_2 \end{array}$$

commutes<sup>2</sup>, we have that

$$\text{ev}_{X_2, Y_2}(g^{X_2} Y_1^f \times 1_{X_2}) = g \text{ev}_{X_2, Y_1}(Y_1^f \times 1_{X_2}) = g \text{ev}_{X_1, Y_1}(1_{Y_1^{X_1}} \times f),$$

and since

$$\begin{array}{ccccc} Y_2^{X_2} \times X_2 & \xrightarrow{\text{ev}_{X_2, Y_2}} & Y_2 & & \\ \uparrow Y_2^f \times 1_{X_2} & & \uparrow \text{ev}_{X_1, Y_2} & \swarrow g & \\ Y_2^{X_1} \times X_2 & \xrightarrow{1_{Y_2^{X_1}} \times f} & Y_2^{X_1} \times X_1 & & Y_1 \\ \uparrow g^{X_1} \times 1_{X_2} & & \uparrow g^{X_1} \times 1_{X_1} & \nearrow \text{ev}_{X_1, Y_1} & \\ Y_1^{X_1} \times X_2 & \xrightarrow{1_{Y_1^{X_1}} \times f} & Y_1^{X_1} \times X_1 & & \end{array}$$

commutes<sup>3</sup>, then

$$\text{ev}_{X_2, Y_2}(Y_2^f g^{X_1} \times 1_{X_2}) = \text{ev}_{X_1, Y_2}(1_{Y_2^{X_1}} \times f)(g^{X_1} \times 1_{X_1}) = g \text{ev}_{X_1, Y_1}(1_{Y_1^{X_1}} \times f).$$

Hence, both  $Y_2^f g^{X_1}$  and  $g^{X_2} Y_1^f$  make

<sup>1</sup>In **Set**,  $Y^X = \text{Hom}(X, Y)$  so the functor acts functions by sending  $h: X_1 \rightarrow Y_1$  to  $ghf: X_2 \rightarrow Y_2$ . Thus, the commutativity of the diagram is equivalent to function composition being associative.

<sup>2</sup>left square commutes by definition of  $Y_1^f$ , right square commutes by definition of  $g^{X_2}$

<sup>3</sup>top square commutes by definition of  $Y_2^f$ , bottom square commutes because  $- \times -: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  is a bifunctor and the right triangle commutes by definition of  $g^{X_1}$

$$\begin{array}{ccc}
Y_2^{X_2} \times X_2 & \xrightarrow{\text{ev}_{X_2, Y_2}} & Y_2 \\
\uparrow - \times 1_{X_2} & & \uparrow g^{\text{ev}_{X_1, Y_1}} \\
Y_1^{X_1} \times X_2 & \xrightarrow{1_{Y_1^{X_1}} \times f} & Y_1^{X_1} \times X_1
\end{array}$$

commute, and so by the universal property of  $Y_2^{X_2}$  they are equal.  $\square$

### EXERCISE 53

Let  $A$  be the unique function making

$$\begin{array}{ccccc}
1 & \xrightarrow{0} & \mathbb{N} & \xrightarrow{S} & \mathbb{N} \\
& \searrow 1_{\mathbb{N}} & \downarrow A & & \downarrow A \\
& & \mathbb{N}^{\mathbb{N}} & \xrightarrow{S^{\mathbb{N}}} & \mathbb{N}^{\mathbb{N}}
\end{array}$$

commute. Show that the addition function is represented by the transpose of  $A$ .

### Solution

*Proof.* Let  $a: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  be the unique map such that

$$\begin{array}{ccc}
\mathbb{N}^{\mathbb{N}} \times \mathbb{N} & \xrightarrow{\text{ev}_{\mathbb{N}, \mathbb{N}}} & \mathbb{N} \\
\uparrow A \times 1_{\mathbb{N}} & \nearrow a & \\
\mathbb{N} \times \mathbb{N} & & 
\end{array}$$

commutes. Then for  $m, n \in \mathbb{N}$  we have that  $a(0, n) = \text{ev}_{\mathbb{N}, \mathbb{N}}(A \times 1_{\mathbb{N}})(0, n) = A(0)(n) = n$  and  $a(Sn, m) = A(Sn)(m) =$

$\square$

### EXERCISE 54

### Solution

*Proof.*  $\square$

## 7 PRESHEAVES

### EXERCISE 55

Suppose object  $A$  and  $B$  are such that for every object  $X$  in  $\mathcal{C}$  there is a bijection  $f_X: \mathcal{C}(A, X) \rightarrow \mathcal{C}(B, X)$ , naturally in a sense you define yourself. Conclude that  $A$  and  $B$  are isomorphic.

### Solution

*Proof.* The assumption is equivalent to  $f: \mathcal{C}(A, -) \Rightarrow \mathcal{C}(B, -)$  being a natural isomorphism of functors  $\mathcal{C} \rightarrow \mathbf{Set}$ , i.e. an isomorphism in  $\mathbf{Set}^{\mathcal{C}}$ .

Let  $F: \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}^{\mathbf{Set}}$  be the contravariant hom functor, i.e.  $FX = \mathcal{C}(X, -)$ . Then the co-Yoneda Lemma states that

$$\mathcal{C}^{\mathbf{Set}}(FX, FY) = \mathcal{C}^{\mathbf{Set}}(\mathcal{C}(X, -), \mathcal{C}(Y, -)) \cong \mathcal{C}(Y, X) = \mathcal{C}^{\text{op}}(X, Y)$$

so  $F$  is fully faithful. Since fully faithful functors reflect isomorphism, it follows that if  $FA \cong FB$  via  $f$  then there exists an isomorphism  $g: A \rightarrow B$  in  $\mathcal{C}^{\text{op}}$  such that  $Fg = f$ .  $\square$

### EXERCISE 8.1

Let  $\mathcal{C}$  be a category and  $F$  a presheaf on  $\mathcal{C}$ . Show that  $F$  is representable if and only if there are  $C$  in  $\mathcal{C}$  and  $x \in F(C)$  such that for any  $D$  in  $\mathcal{C}$  and  $y \in F(D)$  there exists a unique map  $\alpha: D \rightarrow C$  such that  $y = x \cdot_F \alpha := F(\alpha)(x)$ .

### Solution

*Proof.* ( $\Rightarrow$ ) If  $F$  is representable then there exists  $C$  in  $\mathcal{C}$  and a natural isomorphism  $\eta: \mathcal{C}(-, C) \Rightarrow F$ . Let  $x = \eta_C(1_C) \in FC$ . For  $D \in \text{ob } \mathcal{C}$  and  $y \in FD$ , we have that  $\mathcal{C}(D, C) \cong FD$  via  $\eta_D$  so let  $\alpha = \eta_D^{-1}(y)$ . By commutativity of

$$\begin{array}{ccc} \mathcal{C}(C, C) & \xrightarrow{-\circ \alpha} & \mathcal{C}(D, C) \\ \downarrow \eta_C & & \downarrow \eta_D \\ FC & \xrightarrow{F\alpha} & FD \end{array}$$

it follows that  $y = \eta_D(\alpha) = \eta_D(1_C \alpha) = F(\alpha)(\eta_C(1_C)) = F(\alpha)(x)$ .

( $\Leftarrow$ ) Define  $\eta: \mathcal{C}(-, C) \Rightarrow F$  by  $\eta_C(1_C) = x$ . Requiring  $\eta$  to be a natural transformation means that

$$\begin{array}{ccc} \mathcal{C}(C, C) & \xrightarrow{-\circ \beta} & \mathcal{C}(D, C) \\ \downarrow \eta_C & & \downarrow \eta_D \\ FC & \xrightarrow{F\beta} & FD \end{array}$$

commutes for all  $D \in \text{ob } \mathcal{C}$  and  $\beta: D \rightarrow C$ . In other words,  $\eta_D(\beta) = F(\beta)(x)$  for all  $D \in \text{ob } \mathcal{C}$  and  $\beta: D \rightarrow C$ . We claim that  $\eta$  is a natural isomorphism. Indeed, for any  $\gamma: D \rightarrow D'$  and  $\beta: D' \rightarrow C$  we have that  $F(\gamma)\eta_{D'}(\beta) = F(\gamma)F(\beta)(x) = F(\beta\gamma)(x) = \eta_D(\beta\gamma)$  so that

$$\begin{array}{ccc} \mathcal{C}(D', C) & \xrightarrow{-\circ \gamma} & \mathcal{C}(D, C) \\ \downarrow \eta_{D'} & & \downarrow \eta_D \\ FC & \xrightarrow{F\gamma} & FD \end{array}$$



commutes. The assumption that for each  $y \in FD$  there is a unique  $\alpha \in \mathcal{C}(D, C)$  such that  $y = F(\alpha)(x) = \eta_D(\alpha)$  implies that  $\eta_D$  is a bijection for all  $D \in \text{ob } \mathcal{C}$ .  $\square$

#### EXERCISE 56

Show that the following are equivalent for each small category  $\mathcal{C}$ :

- (a)  $\mathcal{C}$  has a terminal object 1.
- (b) The terminal object in  $\mathbf{Set}^{\mathcal{C}^{\text{op}}}$  is representable.

#### Solution

*Proof.* (a)  $\implies$  (b) Since limits in a functor category are computed pointwise, the terminal object of  $\mathbf{Set}^{\mathcal{C}^{\text{op}}}$  is the functor that sends every object to the one element set and every map to the identity. This functor is certainly isomorphic to  $\mathcal{C}(-, 1)$ .

(b)  $\implies$  (a) Let  $\mathcal{C}(-, X)$  be the terminal object of  $\mathbf{Set}^{\mathcal{C}^{\text{op}}}$ . Then for every  $Y$  in  $\mathcal{C}$ ,  $\mathcal{C}(Y, X) \cong \{*\}$ , so there is a unique map  $Y \rightarrow X$ . Thus,  $X$  is terminal in  $\mathcal{C}$ .  $\square$

#### EXERCISE 57

Show that the following are equivalent for each small category  $\mathcal{C}$ :

- (a)  $\mathcal{C}$  has binary products.
- (b) For each pair of object  $A$  and  $B$  in  $\mathcal{C}$  the presheaf  $yA \times yB$  is representable in  $\mathbf{Set}^{\mathcal{C}^{\text{op}}}$ .

Can you generalize the statement in this and Exercise 56 to general limits?

#### Solution

*Proof.* (a)  $\implies$  (b) Let  $A, B$  be objects of  $\mathcal{C}$ . Note that for all  $C \in \text{ob } \mathcal{C}$  we have an isomorphism  $\mathcal{C}(C, A) \times \mathcal{C}(C, B) \cong \mathcal{C}(C, A \times B)$  in  $\mathbf{Set}$  given by sending  $(f_1, f_2)$

to the unique map which makes

$$\begin{array}{ccccc} & & C & & \\ & f_1 \swarrow & \downarrow & \searrow f_2 & \\ A & \xleftarrow{\pi_A} & A \times B & \xrightarrow{\pi_B} & B \end{array}$$

commute. Its in-

verse is given by sending  $f: C \rightarrow A \times B$  to  $(\pi_A f, \pi_B f)$ . If  $g: C' \rightarrow C$ ,  $f_1: C \rightarrow A$  and  $f_2: C \rightarrow B$  are morphisms in  $\mathcal{C}$ , since there is a unique map which makes

$$\begin{array}{ccc} \mathcal{C}(C, A) \times \mathcal{C}(C, A) & \longrightarrow & \mathcal{C}(C, A \times B) \\ \downarrow - \circ g & & \downarrow - \circ g \\ \mathcal{C}(C', A) \times \mathcal{C}(C', B) & \longrightarrow & \mathcal{C}(C', A \times B) \end{array}$$

commutes. Hence, the isomorphism is natural in  $C$ .

(b)  $\implies$  (a) For  $A, B \in \text{ob } \mathcal{C}$  take  $C \in \text{ob } \mathcal{C}$  such that there exists a natural isomorphism  $\eta: \mathcal{C}(-, A) \times \mathcal{C}(-, B) \Rightarrow \mathcal{C}(-, C)$ . We claim that  $\mathcal{C}$  is the product. Let  $(\pi_A, \pi_B) = \eta_C^{-1}(\text{id}_C)$ . Then for any  $f_1: X \rightarrow A$  and  $f_2: X \rightarrow B$  in  $\mathcal{C}$ , we have that

$$\begin{aligned} \eta_X(\pi_A \eta_X(f_1, f_2), \pi_B \eta_X(f_1, f_2)) &\stackrel{*}{=} \eta_C(\pi_A, \pi_B) \circ \eta_X(f_1, f_2) \\ &= \text{id}_C \circ \eta_X(f_1, f_2) = \eta_X(f_1, f_2), \end{aligned}$$

where  $(*)$  holds since  $\eta$  is a natural transformation. Since  $\eta_X$  is an isomorphism,

$$\begin{array}{ccccc} & & X & & \\ & f_1 \swarrow & \downarrow \eta_X(f_1, f_2) & \searrow f_2 & \\ A & \xleftarrow{\pi_A} & C & \xrightarrow{\pi_B} & B \end{array}$$

commutes. Since  $\eta_X(f_1, f_2)$  is the unique map which makes this diagram commute,  $C$  is the product of  $A$  and  $B$ .  $\square$

The generalization of this and Exercise 56 would be

**Proposition 7.1.** *For  $F: \mathcal{I} \rightarrow \mathcal{C}$ ,  $\lim F$  exists in  $\mathcal{C}$  if and only if  $\lim yF$  is representable, where  $y: \mathcal{C} \rightarrow \text{Set}^{\mathcal{C}^{op}}$  is the Yoneda embedding.*

Firs, we prove an important lemma.

**Lemma 7.2.** *Let  $\mathcal{C}$  be a small category. The hom functor  $\mathcal{C}(-, -): \mathcal{C}^{op} \times \mathcal{C} \rightarrow \text{Set}$  preserves limits.*

*Proof.* Let  $F: \mathcal{I} \rightarrow \mathcal{C}$  be a diagram in  $\mathcal{C}$  such that  $\lim F$  exists. By definition of limiting cones, we have a natural bijection  $\eta: \mathcal{C}^{\mathcal{I}}(\Delta(-), F) \Rightarrow \mathcal{C}(-, \lim F)$  sending a natural transformation  $\theta: \Delta_Y \Rightarrow F$  to the unique map  $Y \rightarrow \lim F$

such that

$$\begin{array}{ccccc} & & Y & & \\ & \theta_i \swarrow & \downarrow \eta_Y(\theta) & \searrow \theta_j & \\ F_i & \xleftarrow{\quad} & \lim_i F_i & \xrightarrow{\quad} & F_j \end{array}$$

fact that

commutes. Naturality follows from the

$$\begin{array}{ccccc} & & Y' & & \\ & (\theta \Delta_g)_i \swarrow & \downarrow g & \searrow (\theta \Delta_g)_j & \\ & & Y & & \\ & \theta_i \swarrow & \downarrow \eta_Y(\theta) & \searrow \theta_j & \\ F_i & \xleftarrow{\quad} & \lim_i F_i & \xrightarrow{\quad} & F_j \end{array}$$

commutes and the uniqueness of  $\eta_{Y'}(\theta \Delta_g)$ . Hence, it remains to show that we have a natural bijection  $\lim_i \mathcal{C}(-, F_i) \cong \mathcal{C}^{\mathcal{I}}(\Delta(-), F)$ . We define  $\mu: \mathcal{C}^{\mathcal{I}}(\Delta(-), F) \Rightarrow \lim_i \mathcal{C}(-, F_i)$  by sending a natural transformation  $\theta: \Delta_Y \rightarrow F$  to  $g(*)$  where  $g$  is the unique map such that

$$\begin{array}{ccc}
& * & \\
\text{const}_{\theta_j} \swarrow & \downarrow g & \searrow \text{const}_{\theta_j} \\
& \lim_i \mathcal{C}(Y, F_i) & \\
\swarrow & & \searrow \\
\mathcal{C}(Y, F_i) & \xrightarrow{\quad} & \mathcal{C}(Y, F_j)
\end{array}$$

commutes. This defines a bijection since for every element  $x$  in  $\lim_i \mathcal{C}(Y, F_i)$  we can assign a natural transformation  $\theta: \Delta_Y \Rightarrow F$  by defining  $\theta_i$  to be the image of  $x$  under the map  $\lim_i \mathcal{C}(Y, F_i) \rightarrow \mathcal{C}(Y, F_i)$ . Naturality of  $\theta$  follows from

$$\begin{array}{ccc}
& \lim_i \mathcal{C}(Y, F_i) & \\
\swarrow & & \searrow \\
\mathcal{C}(Y, F_i) & \xrightarrow{\quad} & \mathcal{C}(Y, F_j)
\end{array}$$

the commutativity of . To prove

naturality in  $Y$ , let  $f: Y' \rightarrow Y$  in  $\mathcal{C}$  and  $f^*: \lim_i \mathcal{C}(Y, F_i) \rightarrow \lim_i \mathcal{C}(Y', F_i)$  be the map induced by  $f$ . Then

$$\begin{array}{ccccc}
& \lim_i \mathcal{C}(Y, F_i) & & & \\
g_i \swarrow & & g_j \searrow & f^* \searrow & \\
\mathcal{C}(Y, F_i) & \xrightarrow{\quad} & \mathcal{C}(Y, F_j) & \xrightarrow{\quad} & \lim_i \mathcal{C}(Y', F_i) \\
& \searrow - \circ f & & \swarrow g'_i & \searrow g'_j \\
& & \mathcal{C}(Y', F_i) & \xrightarrow{\quad} & \mathcal{C}(Y', F_j)
\end{array}$$

commutes, and for  $\theta: \Delta_Y \Rightarrow F$  we have that

$$g'_i \mu_{Y'}(\theta \Delta_f) = (\theta \Delta_f)_i = \theta_i f = g_i \mu_Y(\theta) f = g'_i f^* \mu_Y(\theta)$$

We conclude that  $\mathcal{C}(-, \lim_i F_i) \cong \lim_i \mathcal{C}(-, F_i)$ .

Since the hom functor is contravariant is the first component, we need to show that  $\mathcal{C}(\text{colim}_i F_i, -) \cong \lim_i \mathcal{C}(F_i, -)$ , but this is just the dual of the above paragraph.  $\square$

*Proof of Proposition 7.1.* ( $\Rightarrow$ ) We claim that  $\lim_i yF_i \cong \mathbf{Set}(-, \lim_i F_i)$ . Indeed, we have that  $(\lim_i yF_i) \cong \lim_i \mathbf{Set}(-, F_i) \cong \mathbf{Set}(-, \lim_i F_i)$ , where the last isomorphism follows from Lemma 7.2.

( $\Leftarrow$ ) Since the Yoneda embedding is fully faithful, it reflects limits, i.e. if  $\lim_i yF_i \cong \mathbf{Set}(-, X)$  for some  $X \in \text{ob } \mathcal{C}$  then  $\lim_i F_i \cong X$ .  $\square$

#### EXERCISE 58

Again, let  $\mathcal{C}$  be a small category with binary products, and let  $A$  and  $B$  be objects in  $\mathcal{C}$ .

- i. Show that the assignment

$$X \mapsto \mathcal{C}(X \times A, B)$$

is part of a functor  $\mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$ , with the action on morphisms  $f: X' \rightarrow X$  in  $\mathcal{C}$  given by precomposition with  $f \times \text{id}_A$ .

- ii. What does it say about  $\mathcal{C}$  if the functor in part (i) is representable?

### Solution

- i. Follows since  $\text{id}_X \times \text{id}_A = \text{id}_{X \times A}$  and  $fg \times \text{id}_A = (f \times \text{id}_A)(g \times \text{id}_A)$ .
- ii. Claim: if we have a natural isomorphism  $\eta: \mathcal{C}(- \times A, B) \Rightarrow \mathcal{C}(-, C)$  for some  $C \in \text{ob } \mathcal{C}$ , then  $C \cong B^A$ . Indeed, for  $f: X \times A \rightarrow B$  we have  $\eta_X(f): X \rightarrow C$  and since

$$\begin{array}{ccc} \mathcal{C}(C \times A, B) & \xrightarrow{\eta_C} & \mathcal{C}(C, C) \\ - \circ \eta_X(f) \times \text{id}_A \downarrow & & \downarrow - \circ \eta_X(f) \\ \mathcal{C}(X \times A, B) & \xrightarrow{\eta_X} & \mathcal{C}(X, C) \end{array}$$

commutes, it follows that

$$\eta_X(\eta_C^{-1}(\text{id}_C) \circ \eta_X(f) \times \text{id}_A) = \eta_C(\eta_C^{-1}(\text{id}_C)) \circ \eta_X(f) = \eta_X(f),$$

so  $\eta_C^{-1}(\text{id}_C) \circ \eta_X(f) \times \text{id}_A = f$  by injectivity of  $\eta_X$ . In particular,

$$\begin{array}{ccc} C \times A & \xrightarrow{\eta_C^{-1}(\text{id}_C)} & B \\ \eta_X(f) \times \text{id}_A \uparrow & \nearrow f & \\ X \times A & & \end{array}$$

commutes.

### EXERCISE 8.3

A forest is a poset  $(X, \leq)$  such that for any element  $x \in X$  the set  $\downarrow x = \{y \in X \mid y \leq x\}$  is finite and linearly ordered by  $\leq$ . If this set has  $n + 1$  elements, we say that  $x$  has depth  $n$ . Write  $\mathbf{For}$  for the category of forests and monotone, depth-preserving maps.

Show that  $\mathbf{For}$  is isomorphic to the category of presheaves on  $\mathbb{N}$  (considered as a poset in the usual way).

### Solution

*Proof.* We need to define functors  $F: \mathbf{For} \rightarrow \mathbf{Set}^{\mathbb{N}^{\text{op}}}$  and  $G: \mathbf{Set}^{\mathbb{N}^{\text{op}}} \rightarrow \mathbf{For}$  such that  $FG = \text{id}_{\mathbf{Set}^{\mathbb{N}^{\text{op}}}}$  and  $GF = \text{id}_{\mathbf{For}}$ .  $\square$

### EXERCISE 59

Prove that  $y: \mathcal{C} \rightarrow \mathbf{Set}^{\mathcal{C}^{\text{op}}}$  preserves all limits which exist in  $\mathcal{C}$ . Prove also that if  $\mathcal{C}$  is cartesian closed,  $y$  preserves exponents.

#### Solution

*Proof.* The first claim is proved in Proposition 7.1. For the second claim, we want to prove that  $y(Y^X) \cong (yY)^{yX}$ . Well, using the Yoneda lemma and the fact the  $y$  preserves products we have

$$\begin{aligned} y(Y^X)(C) &= \mathcal{C}(C, Y^X) \cong \mathcal{C}(C \times X, Y) = yY(C \times X) \cong \mathbf{Set}^{\mathcal{C}^{\text{op}}}(y(C \times X), yY) \\ &\cong \mathbf{Set}^{\mathcal{C}^{\text{op}}}(yC \times yX, yY) \cong \mathbf{Set}^{\mathcal{C}^{\text{op}}}(yC, (yY)^{yX}) \cong (yY)^{yX}(C). \end{aligned}$$

□

## 8 PRESHEAVES AS A TOPOS