CTf22: Solutions

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1 Categories

2 Functors and Constructions on Categories

3 Natural Transformations and Equivalences

4 Limits and Colimits

Exercise 21

Show that a full and faithful functor reflects the property of being a terminal (initial object).

Solution

Proof. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a fully faithful functor, and $X \in \operatorname{ob}\mathcal{C}$ such that FX is the terminal object of $\mathcal{D}$. Then for any $Y \in \mathcal{C}$ we have $\{*\} \cong \operatorname{Hom}_{\mathcal{D}}(FY, FX) \cong \operatorname{Hom}_{\mathcal{C}}(Y, X)$ so X is terminal in $\mathcal{C}$. We Similarly show that F reflects the property of being initial. $\square$

Exercise 22

Show that every equalizer is monic.

Solution

Proof. Let $e\colon E \to X$ be the equalizer of $f_1, f_2\colon X \rightrightarrows Y$, and take $g_1, g_2\colon T \to E$ such that $eg_1 = eg_2$. Since both g_1 and g_2 make

$$\begin{tikzcd} E \arrow[r, "e"] & X \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & Y \\ T \arrow[u, dashed] \arrow[ur, "eg_1=eg_2"'] & & \end{tikzcd}$$

commute, by uniqueness it follows that $g_1 = g_2$, so e is monic. $\square$

Exercise 23

Let $E \xrightarrow{e} X \underset{f_2}{\overset{f_1}{\rightrightarrows}} Y$ be an equalizer diagram. Show that e is iso if and only if $f_1 = f_2$.

Solution

Proof. ($\Rightarrow$) Immediate, since

$$f_1 = f_1 \operatorname{id}_X = f_1 e e^{-1} = f_2 e e^{-1} = f_2 \operatorname{id}_X = f_2.$$

($\Leftarrow$) If $f_1 = f_2$ then the there exists a unique $k\colon X \to E$ such that $ek = \operatorname{id}_X$. Then $eke = e \operatorname{id}_E$ and since e is monic by the previous exercise it follows that $ke = 1_E$, so e is an isomorphism.

$\square$

Exercise 24

Show that in Set, every monomorphism fits into an equalizer diagram.

Solution

Proof. Let $f : X \to Y$ be a monomorphism in Set. Define $g_1, g_2\colon Y \rightrightarrows \{0,1\}$ by $g_1(y) = 1$ and

$$g_2(y) = \begin{cases} 1, & y \in \operatorname{im} f \\ 0, & \text{otherwise} \end{cases}.$$

It is clear that $g_2 f = g_1 f$. Let $h\colon T \to Y$ be any map such that $g_1 h = g_2 h$. It follows that $\operatorname{im} h \subseteq \operatorname{im} f$ and so we can define $k\colon T \to X$ by $k(t) = f^{-1}h(t)$. Then $fk = h$ and k is unique, so f is the equalizer of $g_1, g_2\colon Y \rightrightarrows \{0,1\}$. $\square$

Exercise 25

Let

$$\begin{tikzcd} A \arrow[r, "b"] \arrow[d, "a"] & B \arrow[d, "f"] \\ X \arrow[r, "g"] & Y \end{tikzcd}$$

be a pullback diagram with f monic. Show that a is also monic. Also, if f is iso, so is a.

Solution

Proof. Let $t_1, t_2 \colon T \rightrightarrows A$ such that $at_1 = at_2$. Then $fbt_1 = gat_1 = gat_2 = fbt_2$ and since f is monic, $bt_1 = bt_2$. Hence, there exists a unique $k \colon T \to A$ such that

$$\begin{tikzcd}
T \arrow[dr, dashed, "k"] \arrow[drr, bend left, "bt_1=bt_2"] \arrow[ddr, bend right, "at_1=at_2"'] & & \\
& A \arrow[r, "b"] \arrow[d, "a"] & B \arrow[d, "f"] \\
& X \arrow[r, "g"] & Y
\end{tikzcd}$$

commute. Since either $k = t_1$ or $k = t_2$ would work, it must be the case that $t_1 = t_2$. □

Exercise 26

Given two commuting squares

$$\begin{tikzcd}
A \arrow[r, "b"] \arrow[d, "a"] & B \arrow[r, "c"] \arrow[d, "f"] & C \arrow[d, "d"] \\
X \arrow[r, "g"] & Y \arrow[r, "h"] & Z
\end{tikzcd}.$$

Show that

i. if both squares are pullback squares, then so is the composite square;

ii. if the right square and the composite square are pullbacks, then so is the left square.

Solution

i. *Proof.* □

ii. *Proof.* □

Exercise 27

Solution

Proof. □

Exercise 28

Solution

Proof. □

Exercise 29

Solution

Proof. ∎

Exercise 30

Solution

Proof. ∎

5 Complete Categories

Exercise 42

Take one of your favourite categories ($\mathsf{Top}, \mathsf{Pos}, \mathsf{Rng}, \mathsf{Mon}, \mathsf{Grp}, \mathsf{Grph}, \mathsf{Cat}$) and show that it is both complete and cocomplete.

Solution Consider Grp. By Proposition 5.1, it is enough to prove that Grp has all small products and equalizers. The equalizer of $f, g\colon G \rightrightarrows H$ is the pair (G', i) where $G' = \{a \in G \mid f(a) = g(a)\}$ and i is the inclusion map. Note the G' is a group since f and g are group homomorphisms. For a set of groups $\{G_i\}_{i\in I}$, the product $\prod_{i\in I} G_i$ has a group structure by performing the group operations pointwise.

Exercise 43

Show that if $\mathcal{C}$ is complete, then $F\colon \mathcal{C} \to \mathcal{D}$ preserves all limits if F preserves products and equalizers. This no longer holds if $\mathcal{C}$ is not complete: F may preserve all products and equalizers which exists in $\mathcal{C}$, yet not preserve all limits which exists in $\mathcal{C}$.

Solution I assume the question means *small* limits, since I don't think this holds for all limits.

Proof. Suppose $\mathcal{C}$ is complete and $F\colon \mathcal{C} \to \mathcal{D}$ preserves products and equalizers. Let $G\colon \mathcal{I} \to \mathcal{C}$ be a small diagram. Then $\lim_{\mathcal{I}} G$ can be expressed as

$$\lim_{\mathcal{I}} G \longrightarrow \textstyle\prod_{i\in \mathrm{ob}\,\mathcal{I}} Gi \underset{d}{\overset{c}{\rightrightarrows}} \textstyle\prod_{f\in \mathrm{mor}\,\mathcal{I}} G(\mathrm{cod}\, f),$$

and since F preserves equalizers and products, $F(\lim_{\mathcal{I}} G) = \lim_{\mathcal{I}}(FG)$. ∎

Exercise 44

Suppose a category $\mathcal{C}$ has limits of shape $\mathcal{I}$. Show that the operation which assigns each diagram $\mathcal{I} \to \mathcal{C}$ to its limit in $\mathcal{C}$ is part of a functor $F\colon [\mathcal{I}, \mathcal{C}] \to \mathcal{C}$.

Solution

Proof. Let $\eta\colon G \Rightarrow H$ be a morphism in $[\mathcal{I}, \mathcal{C}]$, $\mu\colon \Delta_{FG} \Rightarrow G$, $\varepsilon\colon \Delta_{FH} \Rightarrow H$ and $f\colon C \to C'$ a morphism in $\mathcal{I}$. This is summarized in the following diagram

$$\begin{tikzcd}
& FG \arrow[dl, "\mu_C"'] \arrow[dr, "\mu_{C'}"] & & & FH \arrow[dl, "\varepsilon_C"'] \arrow[dr, "\varepsilon_{C'}"] & \\
GC \arrow[rr, "Gf"] \arrow[rrr, bend right, "\eta_C"'] & & GC' \arrow[rrr, bend right, "\eta_{C'}"'] & HC \arrow[rr, "Hf"] & & HC'
\end{tikzcd}.$$

Note also that the diagram commutes. Thus, $(FG, \eta\mu)$ is a cone for the diagram H, so there is a unique morphism $g\colon FG \to FH$ such that $\varepsilon\Delta_g = \eta\mu$. We define $F\eta := g$. It is straightforward to verify that the uniqueness of g turns F into a functor. $\square$

Exercise 45

Let $\mathcal{C}, \mathcal{D}$ and $\mathcal{E}$ be categories. Show that the following categories are isomorphic:

$$[\mathcal{E}, [\mathcal{C}, \mathcal{D}]] \cong [\mathcal{E} \times \mathcal{C}, \mathcal{D}] \cong [\mathcal{C}, [\mathcal{E}, \mathcal{D}]].$$

Use this and the previous exercise to give a more elegant proof of Theorem 4.5.

Solution

Proof. Consider the functors $[\mathcal{E}, [\mathcal{C}, \mathcal{D}]] \xrightarrow{F_1} [\mathcal{E} \times \mathcal{C}, \mathcal{D}] \xrightarrow{F_2} [\mathcal{C}, [\mathcal{E}, \mathcal{D}]]$ given by

- On objects: for $G\colon \mathcal{E} \to [\mathcal{C}, \mathcal{D}]$, $H\colon \mathcal{E} \times \mathcal{C} \to \mathcal{D}$, $f\colon E \to E'$ in $\mathcal{E}$ and $g\colon C \to C'$ in $\mathcal{C}$ we have

$$\begin{aligned}
&F_1(G)(E, C) = G(E)(C)\\
&F_1(G)(f, g) = (G(E')(g))(Gf)_C\\
&F_2(H)(C)(E) = H(E, C)\\
&F_2(H)(C)(f) = H(f, \mathrm{id}_C)\\
&(F_2(H)(g))_E = H(\mathrm{id}_E, f).
\end{aligned}$$

- On morphisms: for $G_1, G_2\colon \mathcal{E} \rightrightarrows [\mathcal{C}, \mathcal{D}], H_1, H_2\colon \mathcal{E} \times \mathcal{C} \rightrightarrows \mathcal{D}, \eta\colon G_1 \Rightarrow G_2$ and $\varepsilon\colon H_1 \Rightarrow H_2$ we have $(F_1\eta)_{(E,C)} = (\eta_E)_C$ and $((F_2\varepsilon)_C)_E = \varepsilon_{(E,C)}$.

Since these functors are clearly invertible, they are isomorphisms of categories. $\square$

Exercise 46

Show that a full and faithful functor reflects the property of being a terminal (or initial) object. Deduce that equivalences preserve the terminal (or initial) object.

Solution

Proof. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a fully faithful functor and $X \in \operatorname{ob}\mathcal{C}$ such that FX is terminal and take any $Y \in \operatorname{ob}\mathcal{C}$. Then $\operatorname{Hom}_{\mathcal{C}}(Y, X) \cong \operatorname{Hom}_{\mathcal{D}}(FY, FX) \cong \{*\}$, so X is terminal in $\mathcal{C}$. Similarly, we show that fully faithful functors reflect the property of being initial.

Hence, if F is an equivalence, $X \in \operatorname{ob} C$ is terminal, $Z \in \operatorname{ob}\mathcal{D}$ is any object and $Y \in \operatorname{ob}\mathcal{C}$ is chosen such that $FY \cong Z$ we have that $\{*\} = \operatorname{Hom}_{\mathcal{C}}(Y, X) \cong \operatorname{Hom}_{\mathcal{D}}(FY, FX) \cong \operatorname{Hom}_{\mathcal{D}}(Z, FX)$ so FX is terminal in $\mathcal{D}$. $\square$

6 Cartesian Closed Categories

Exercise 47

Show that in a ccc, there are natural isomorphisms

i. $1^X \cong 1$,

ii. $(Y \times Z)^X \cong Y^X \times Z^X$,

iii. $(Y^Z)^X = Y^{Z\times X}$.

Solution Let $\mathcal{C}$ be a ccc category and X, Y, Z and A objects in $\mathcal{C}$.

i. *Proof.* Since 1 is terminal we have unique morphisms $1 \times X \to 1$, $A \times X \to 1$ and $f\colon A \to 1$, so

$$\begin{tikzcd} 1 \times X \arrow[r] & 1 \\ & A \times X \arrow[ul, "f\times 1_X"'] \arrow[u] \end{tikzcd}$$

trivially commutes. Hence, $1^X \cong 1$. Moreover, this isomorphism is natural as a morphism $1^{(-)} \Rightarrow 1$ since all the maps in the square

$$\begin{tikzcd} 1^X \arrow[r] \arrow[d] & 1^Y \arrow[d] \\ 1 \arrow[r] & 1 \end{tikzcd}$$

are the identity maps. $\square$

ii. *Proof.* Let $\eta_{(Y,Z)}\colon Y^X \times Z^X \to (Y \times Z)^X$ be the uniqe map which makes

$$\begin{tikzcd} (Y \times Z)^X \times X \arrow[r, "\mathrm{ev}_{X,Y\times Z}"] & Y \times Z \\ & Y^X \times Z^X \times X \arrow[ul, "\eta_{(Y,Z)}\times 1_X"] \arrow[u, "h"'] \end{tikzcd}$$

commute, where $h = (\mathrm{ev}_{X,Y}(\pi_{Y^X} \times 1_X), \mathrm{ev}_{X,Z}(\pi_{Z^X} \times 1_X))$. To construct an inverse, let $f_Y\colon (Y \times Z)^X \to Y^X$ be the unique map such that

$$\begin{tikzcd} Y^X \times X \arrow[r, "\mathrm{ev}_{X,Y}"] & Y \\ & (Y \times Z)^X \times X \arrow[ul, "f_Y\times 1_X"] \arrow[u, "\pi_Y\,\mathrm{ev}_{X,Y\times Z}"'] \end{tikzcd}$$

commutes. Analogously, we define $f_Z\colon (Y\times Z)^X \to Z^X$. We claim that $\mu_{(Y,Z)} = (f_Y, f_Z)\colon (Y\times Z)^X \to Y^X \times Z^X$ is the inverse of $\eta_{(Y,Z)}$. Since

$$\begin{tikzcd}
& (Y\times Z)^X\times X \arrow[dl, "f_Y\times 1_X"'] \arrow[d, dashed, "\mu_{(Y,Z)}\times 1_X"] \arrow[dr, "f_Z\times 1_X"] & \\
Y^X\times X \arrow[d, "\mathrm{ev}_{X,Y}"'] & Y^X\times Z^X\times X \arrow[l, "\pi_{Y^X}\times 1_X"] \arrow[r, "\pi_{Z^X}\times 1_X"'] \arrow[d, "h"] & Z^X\times X \arrow[d, "\mathrm{ev}_{X,Z}"] \\
Y & Y\times Z \arrow[l, "\pi_Y"] \arrow[r, "\pi_Z"'] & Z
\end{tikzcd}$$

commutes, it follows that

$$\begin{aligned}
h(\mu_{(Y,Z)}\times 1_X) &= (\mathrm{ev}_{X,Y}(f_Y\times 1_X), \mathrm{ev}_{X,Z}(f_Z\times 1_X)) \\
&= (\pi_Y\,\mathrm{ev}_{X,Y\times Z}, \pi_Z\,\mathrm{ev}_{X,Y\times Z}) \\
&= \mathrm{ev}_{X,Y\times Z}
\end{aligned}$$

and so

$$\begin{tikzcd}
(Y\times Z)^X\times X \arrow[rr, "\mathrm{ev}_{X,Y\times Z}"] & & Y\times Z \\
& Y^X\times Z^X\times X \arrow[ul, "\eta_{(Y,Z)}\times 1_X"] \arrow[ur, "h"'] & \\
& & (Y\times Z)^X\times X \arrow[ul, "\mu_{(Y,Z)}\times 1_X"] \arrow[uu, "\mathrm{ev}_{X,Y\times Z}"']
\end{tikzcd}$$

commutes (the top left triangle commutes by definition of $\eta_{(Y,Z)}$ and we've just show that the bottom right triangle commute, so the big triangle commutes as well). By uniqueness, $\eta_{(Y,Z)}\mu_{(Y,Z)} = \mathrm{id}_{(Y\times Z)^X}$. By definition of f_Y and $\eta_{(Y,Z)}$,

$$\begin{tikzcd}
Y^X\times X \arrow[rr, "\mathrm{ev}_{X,Y}"] & & Y \\
& (Y\times Z)^X\times X \arrow[ul, "f_Y\times 1_X"'] \arrow[ur, "\pi_Y\,\mathrm{ev}_{X,Y\times Z}"] \arrow[r, "\mathrm{ev}_{X,Y\times Z}"] & Y\times Z \arrow[u, "\pi_Y"'] \\
& & Y^X\times Z^X\times X \arrow[ul, "\eta_{(Y,Z)}\times 1_X"] \arrow[u, "h"']
\end{tikzcd}$$

commutes and so both $\pi_{Y^X}\times 1_X$ and $(f_Y\eta_{Y,Z})\times 1_X$ make

$$\begin{tikzcd}
Y^X\times X \arrow[r, "\mathrm{ev}_{X,Y}"] & Y \\
& Y^X\times Z^X\times X \arrow[ul] \arrow[u, "\pi_Y h"']
\end{tikzcd}$$

commute, so $f_Y \eta_{(Y,Z)} = \pi_{Y^X}$. Analogously, $f_Z \eta_{(Y,Z)} = \pi_{Z^X}$. Thus,

$$\begin{tikzcd}
& Y^X \times Z^X \arrow[d, "\eta_{(Y,Z)}"] \arrow[ddl, bend right, "\pi_{Y^X}"'] \arrow[ddr, bend left, "\pi_{Z^X}"] & \\
& (Y \times Z)^X \arrow[dl, "f_Y"'] \arrow[d, "\mu_{(Y,Z)}"] \arrow[dr, "f_Z"] & \\
Y^X & Y^X \times Z^X \arrow[l, "\pi_{Y^X}"] \arrow[r, "\pi_{Z^X}"'] & Z^X
\end{tikzcd}$$

commutes, and by uniqueness it follows that $\mu_{(Y,Z)}\eta_{(Y,Z)} = 1_{Y^X \times Z^X}$.

It is left to show that this gives a natural isomorphism $\eta\colon (-)^X \times (-)^X \Rightarrow (- \times -)^X$ of functors $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$. Since a morphism of bifunctors is natural if and only if it is natural in each component and the definitions are symmetric in their components, it suffices to check that

$$\begin{tikzcd}
Y^X \times Z^X \arrow[r, "f^X \times 1_{Z^X}"] \arrow[d, "\eta_{(Y,Z)}"] & (Y')^X \times Z^X \arrow[d, "\eta_{(Y',Z)}"] \\
(Y \times Z)^X \arrow[r, "(f \times 1_Z)^X"] & (Y' \times Z)^X
\end{tikzcd}$$

commutes. Since $\mathrm{ev}_{X,-}\colon (-)^X \times X \Rightarrow 1_{\mathcal{C}}$ is a natural transformation, we have that

$$\begin{aligned}
\mathrm{ev}_{X,Y' \times Z} \circ (f \times 1_Z)^X \times 1_X \circ \eta_{Y,Z} \times 1_X &= (f \times 1_Z) \circ \mathrm{ev}_{X,Y \times Z} \circ \eta_{(Y,Z) \times 1_X} \\
&= (f \times 1_Z)h
\end{aligned}$$

and

$$\begin{tikzcd}
& Y^X \times Z^X \times X \arrow[d, "f^X \times 1_{Z^X} \times 1_X"] \arrow[ddl, bend right, "f \circ \mathrm{ev}_{X,Y}(\pi_{Y^X} \times 1_X)"'] \arrow[ddr, bend left, "\mathrm{ev}_{X,Z}(\pi_{Z^X} \times 1_X)"] & \\
& (Y')^X \times Z^X \times X \arrow[dl, "\mathrm{ev}_{X,Y'}(\pi_{(Y')^K} \times 1_X)"'] \arrow[d, "h'"] \arrow[dr, "\mathrm{ev}_{X,Z}(\pi_{Z^X} \times 1_X)"] & \\
Y' & Y' \times Z \arrow[l] \arrow[r] & Z
\end{tikzcd}$$

commutes, so

$$\begin{aligned}
\mathrm{ev}_{X,Y' \times Z} \circ \eta_{(Y',Z)} \times 1_X \circ f^X \times 1_{Z^X} \times 1_X &= h' \circ f^X \times 1_{Z^X} \times 1_X \\
&= (f \times 1_Z)h.
\end{aligned}$$

It follows that both $(f \times 1_Z)^X \eta_{(Y,Z)}$ and $\eta_{(Y',Z)}(f^X \times 1_{Z^X})$ make

$$
\begin{tikzcd}
(Y' \times Z)^X \times X \arrow[r, "\mathrm{ev}_{X,Y' \times Z}"] & Y' \times Z \\
& Y^X \times Z^X \times X \arrow[ul, "- \times 1_X"] \arrow[u, "(f \times 1_Z)h"']
\end{tikzcd}
$$

commute, and so they are equal. □

iii. *Proof.* Define $f = \mathrm{ev}_{Z,Y} \circ \mathrm{ev}_{X,Y^Z} \times 1_Z \colon (Y^Z)^X \times X \times Z \to Y$. Then for any $h \colon A \times X \times Z \to Y$ we get a map $H' \colon A \times X \to Y^Z$ such that $\mathrm{ev}_{Z,Y} \circ H' \times 1_Z = h$ which gives a unique map $H \colon A \to (Y^Z)^X$ such that $\mathrm{ev}_{X,Y^Z} \circ H \times 1_X = H'$. Since

$$
\begin{aligned}
f \circ H \times 1_{X \times Z} &= \mathrm{ev}_{Z,Y} \circ \mathrm{ev}_{X,Y^Z} \times 1_Z \circ H \times 1_{X \times Z} \\
&= \mathrm{ev}_{Z,Y} \circ (\mathrm{ev}_{X,Y^Z} \circ H \times 1_X) \times 1_Z \\
&= \mathrm{ev}_{Z,Y} \circ H' \times 1_Z \\
&= h,
\end{aligned}
$$

we conclude that $(Y^Z)^X \cong Y^{Z \times X}$. □

Exercise 48

If a ccc has coproducts, we have

i. $X \times (Y + Z) \cong (X \times Y) + (X \times Z)$

ii. $Y^{Z+X} = Y^Z \times Y^X$.

Solution

i. *Proof.* Suppose $\mathcal{C}$ is a ccc which has coproducts, and let X, Y and Z be objects in $\mathcal{C}$. Consider any cocone

$$
\begin{tikzcd}
X \times Y \arrow[dr, "\iota_1"'] & & X \times Z \arrow[dl, "\iota_2"] \\
& D &
\end{tikzcd}
.
$$

Exponentiating, we get

$$
\begin{tikzcd}
X \times Y \arrow[r, "\overline{\iota_1} \times 1_X"] \arrow[dr, "\iota_1"'] & D^X \times X \arrow[d, "\mathrm{ev}_{X,D}"] & X \times Z \arrow[l, "\overline{\iota_2} \times 1_X"'] \arrow[dl, "\iota_2"] \\
& D &
\end{tikzcd}
$$

where $\overline{\iota_1}$ and $\overline{\iota_2}$ are the transposes of ι_1 and ι_2 respectively. Hence, we have the cocone

$$\begin{tikzcd}
Y \arrow[dr, "\overline{\iota_1}"] & & Z \arrow[dl, "\overline{\iota_2}"'] \\
& D^X &
\end{tikzcd}$$

and since $Y+Z$ is limiting, we get a unique map $h\colon Y+Z \to D^X$ so that

$$\begin{tikzcd}
Y \arrow[r, "\iota_Y"] \arrow[dr, "\overline{\iota_1}"'] & Y+Z \arrow[d, "h"] & Z \arrow[l, "\iota_Z"'] \arrow[dl, "\overline{\iota_2}"] \\
& D^X &
\end{tikzcd}$$

commutes. Then

$$\begin{tikzcd}
Y \times X \arrow[r, "\iota_Y \times 1_X"] \arrow[dr, "\overline{\iota_1} \times 1_X"'] & (Y+Z) \times X \arrow[d, "h \times 1_X"] & Z \times X \arrow[l, "\iota_Z \times 1_X"'] \arrow[dl, "\overline{\iota_2} \times 1_X"] \\
& D^X \times X \arrow[d, "\mathrm{ev}_{X,D}"] & \\
& D &
\end{tikzcd}$$

commutes and it follows that $(Y+Z) \times X$ is the coproduct of $Y \times X$ and $Z \times X$. Thus $Y \times X + Z \times X \cong (Y+Z) \times X$ $\square$

ii. *Proof.* Note that $Y^Z \times Y^X(Z+X) \cong Y^X \times Y^Z \times Z + Y^Z \times Y^X \times X$ and let $f\colon Y^Z \times Y^X(Z+X) \to Y$ be given by the composition

$$Y^X \times Y^Z \times Z + Y^Z \times Y^X \times X \xrightarrow{1_{Y^X} \times \mathrm{ev}_{Z,Y} + 1_{Y^Z} \times \mathrm{ev}_{X,Y}}$$
$$Y^X \times Y + Y^Z \times Y \cong (Y^X + Y^Z) \times Y \xrightarrow{\pi_Y} Y.$$

Let $h\colon A \times (Z+X) \to Y$ be any morphism. Since $A \times (Z+X) \cong A \times Z + A \times X$ we have unique maps $H_Z\colon A \to Y^Z$ and $H_X\colon A \to Y^X$ such that

$$\begin{tikzcd}
Y^Z \times Z \arrow[r, "\mathrm{ev}_{Z,Y}"] & Y \\
& A \times Z \arrow[ul, "H_Z \times 1_Z"] \arrow[u, "\iota_{A \times Z} h"']
\end{tikzcd}
\qquad
\begin{tikzcd}
Y^X \times X \arrow[r, "\mathrm{ev}_{X,Y}"] & Y \\
& A \times X \arrow[ul, "H_X \times 1_X"] \arrow[u, "\iota_{A \times X} h"']
\end{tikzcd}$$

commute, where $A \times Z \xrightarrow{\iota_{A \times Z}} A \times Z + A \times X \xleftarrow{\iota_{A \times X}} A \times X$ are the inclusion maps. Let $H = (H_Z, H_X)\colon A \to Y^Z \times Y^X$. Since

$$\pi_Y \circ (1_{Y^X} \times \mathrm{ev}_{Z,Y}) \circ (H \times 1_Z) = \pi_Y \circ H_X \times (\mathrm{ev}_{Z,Y} \circ H_Z \times 1_Z) = \iota_{A \times Z} h$$

and similarly $\pi_Y \circ (1_{Y^Z} \times \mathrm{ev}_{X,Y}) \circ (H \times 1_X) = \iota_{A\times X} h$, it follows that

$$\begin{aligned} h &= \iota_{A\times Z} h + \iota_{A\times X} h \\ &= \pi_Y \circ (1_{Y^X} \times \mathrm{ev}_{Z,Y}) \circ (H \times 1_Z) + \pi_Y \circ (1_{Y^Z} \times \mathrm{ev}_{X,Y}) \circ (H \times 1_X) \\ &= f \circ (H \times 1_Z + H \times 1_X) \\ &= f \circ (H \times 1_{Z+X}). \end{aligned}$$

Hence,

$$\begin{tikzcd} Y^X \times Y^Z \times (Z+X) \arrow[r, "f"] & Y \\ & A \times (Z+X) \arrow[ul, "H\times 1_{Z+X}"] \arrow[u, "h"'] \end{tikzcd}$$

commutes and so $Y^{Z+X} \cong Y^Z \times Y^X$. □

Exercise 49

In a ccc, prove that the transpose of a composite $Z \xrightarrow{g} W \xrightarrow{f} Y^X$ is

$$Z \times X \xrightarrow{g \times 1_X} W \times X \xrightarrow{\bar{f}} Y,$$

if $\bar{f}$ is the transpose of f.

Solution

Proof. We need to find a morphism $\overline{fg}\colon Z \times X \to Y$ such that $\mathrm{ev}_{X,Y} \circ (f \circ g \times 1_X) = \overline{fg}$. Well,

$$\mathrm{ev}_{X,Y} \circ (f \circ g \times 1_X) = \mathrm{ev}_{X,Y} \circ (f \times 1_X) \circ (g \times 1_X) = \bar{f} \circ (g \times 1_X),$$

so we're done. □

Exercise 50

Suppose $\mathcal{C}, \mathcal{D}, \mathcal{E}$ are categories, such that

1. For each pair of objects $(C, D) \in \mathrm{ob}(\mathcal{C} \times \mathcal{D})$, an object $F_0(C, D)$ in $\mathcal{E}$;
2. For each object $C \in \mathrm{ob}\,\mathcal{C}$, a functor $F_C\colon \mathcal{D} \to \mathcal{E}$ satisfying $F_C(D) = F_0(C, D)$ for each $D \in \mathrm{ob}\,\mathcal{D}$;
3. For each object $D \in \mathrm{ob}\,\mathcal{D}$, a functor $F_D\colon \mathcal{C} \to \mathcal{E}$ satisfying $F_D(C) = F_0(C, D)$ for each object $C \in \mathcal{D}$;

such that for each pair of morphism $f\colon C \to C'$ in $\mathcal{C}$ and $g\colon D \to D'$ in $\mathcal{D}$ we have a commuting square

$$\begin{tikzcd} F_0(C, D) \arrow[r, "F_D(f)"] \arrow[d, "F_C(g)"] & F_0(C', D) \arrow[d, "F_{C'}(g)"] \\ F_0(C, D') \arrow[r, "F_{D'}(f)"] & F_0(C', D') \end{tikzcd}$$

in $\mathcal{E}$.

Show that there is a unique functor $F\colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ whose operation on objects in F_0, while $F(1_C, g) = F_C(g)$ and $F(f, 1_D) = F_D(f)$.

Solution

Proof. For $f\colon C \to C'$ in $\mathcal{C}$ and $g\colon D \to D'$ in $\mathcal{D}$ we define $F(f,g)\colon F_0(C,D) \to F_0(C',D')$ to be any composition around the commuting square

$$\begin{tikzcd} F_0(C,D) \arrow[r, "F_D(f)"] \arrow[d, "F_C(g)"] & F_0(C',D) \arrow[d, "F_{C'}(g)"] \\ F_0(C,D') \arrow[r, "F_{D'}(f)"] & F_0(C',D') \end{tikzcd}.$$

This is easily checked to give a functor $F\colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}$. $\square$

Exercise 51

An object Y in a category with finite product is called *exponentiating* if the exponential X^Y exists for each $Y \in \operatorname{ob}\mathcal{C}$. Show that if X is exponentiating, the assignment $Y \mapsto X^Y$ is the object part of a functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{C}$.

Solution

Proof. For $f\colon Y \to Z$, we define $X^f\colon X^Z \to X^Y$ to be the unique map which makes

$$\begin{tikzcd} X^Y \times Y \arrow[rr, "\mathrm{ev}_{Y,X}"] & & X \\ & X^Z \times Z \arrow[ur, "\mathrm{ev}_{Z,X}"'] & \\ X^Z \times Y \arrow[uu, "X^f \times 1_Y"] \arrow[ur, "1_{X^Z} \times f"'] & & \end{tikzcd}$$

commute. From uniqueness, we get that $X^{1_Y} = 1_{X^Y}$ and for $X^{gf} = X^f X^g$ for $Y \xrightarrow{f} Z \xrightarrow{g} W$. Hence, $X^{(-)}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ is a functor. $\square$

Exercise 52

Show that for every cartesian closed category $\mathcal{C}$ there is a functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$, assigning Y^X to (X,Y).

Solution

Proof. Checking the conditions of Exercise 50:

1. for each object (X, Y) in $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ we have the object Y^X in $\mathcal{C}$;
2. for each $X \in \mathrm{ob}\, \mathcal{C}^{\mathrm{op}}$ we have the functor $(-)^X \colon \mathcal{C} \to \mathcal{C}$;
3. for each $Y \in \mathrm{ob}\, \mathcal{C}^{\mathrm{op}}$ we have the functor $Y^{(-)} \colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$.

It is left to show that for any $f \colon X_2 \to X_1$ and $g \colon Y_1 \to Y_2$ in $\mathcal{C}$,

$$\begin{tikzcd}
Y_1^{X_1} \arrow[r, "Y_1^f"] \arrow[d, "g^{X_1}"] & Y_1^{X_2} \arrow[d, "g^{X_2}"] \\
Y_2^{X_1} \arrow[r, "Y_2^f"] & Y_2^{X_2}
\end{tikzcd}$$

commutes[1]. Since

$$\begin{tikzcd}
Y_1^{X_1} \times X_2 \arrow[r, "Y_1^f \times 1_{X_2}"] \arrow[d, "1_{Y_1^{X_1}} \times f"] & Y_1^{X_2} \times X_2 \arrow[r, "g^{X_2} \times 1_{X_2}"] \arrow[d, "\mathrm{ev}_{X_2,Y_1}"] & Y_2^{X_2} \times X_2 \arrow[d, "\mathrm{ev}_{X_2,Y_2}"] \\
Y_1^{X_1} \times X_1 \arrow[r, "\mathrm{ev}_{X_1,Y_1}"'] & Y_1 \arrow[r, "g"'] & Y_2
\end{tikzcd}$$

commutes[2], we have that

$$\mathrm{ev}_{X_2,Y_2}(g^{X_2} Y_1^f \times 1_{X_2}) = g\, \mathrm{ev}_{X_2,Y_1}(Y_1^f \times 1_{X_2}) = g\, \mathrm{ev}_{X_1,Y_1}(1_{Y_1^{X_1}} \times f),$$

and since

$$\begin{tikzcd}
Y_2^{X_2} \times X_2 \arrow[r, "\mathrm{ev}_{X_2,Y_2}"] & Y_2 & \\
Y_2^{X_1} \times X_2 \arrow[u, "Y_2^f \times 1_{X_2}"] \arrow[r, "1_{Y_2^{X_1}} \times f"] & Y_2^{X_1} \times X_1 \arrow[u, "\mathrm{ev}_{X_1,Y_2}"'] & Y_1 \arrow[ul, "g"'] \\
Y_1^{X_1} \times X_2 \arrow[u, "g^{X_1} \times 1_{X_2}"] \arrow[r, "1_{Y_1^{X_1}} \times f"'] & Y_1^{X_1} \times X_1 \arrow[u, "g^{X_1} \times 1_{X_1}"] \arrow[ur, "\mathrm{ev}_{X_1,Y_1}"'] &
\end{tikzcd}$$

commutes[3], then

$$\mathrm{ev}_{X_2,Y_2}(Y_2^f g^{X_1} \times 1_{X_2}) = \mathrm{ev}_{X_1,Y_2}(1_{Y_2^{X_1}} \times f)(g^{X_1} \times 1_{X_1}) = g\, \mathrm{ev}_{X_1,Y_1}(1_{Y_1^{X_1}} \times f).$$

Hence, both $Y_2^f g^{X_1}$ and $g^{X_2} Y_1^f$ make

---

[1]In Set, $Y^X = \mathrm{Hom}(X, Y)$ so the functor acts functions by sending $h \colon X_1 \to Y_1$ to $ghf \colon X_2 \to Y_2$. Thus, the commutativity of the diagram is equivalent to function composition being associative.

[2]left square commutes by definition of Y_1^f, right square commutes by definition of g^{X_2}

[3]top square commutes by definition of Y_2^f, bottom square commutes because $- \times - \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a bifunctor and the right triangle commutes by definition of g^{X_1}

$$
\begin{tikzcd}
Y_2^{X_2} \times X_2 \arrow[r, "\mathrm{ev}_{X_2,Y_2}"] & Y_2 \\
Y_1^{X_1} \times X_2 \arrow[u, "- \times 1_{X_2}"] \arrow[r, "1_{Y_1^{X_1}} \times f"'] & Y_1^{X_1} \times X_1 \arrow[u, "g\,\mathrm{ev}_{X_1,Y_1}"']
\end{tikzcd}
$$

commute, and so by the universal property of $Y_2^{X_2}$ they are equal.

□

Exercise 53

Let A be the unique function making

$$
\begin{tikzcd}
1 \arrow[r, "0"] \arrow[dr, "1_{\mathbb{N}}"'] & \mathbb{N} \arrow[r, "S"] \arrow[d, "A"] & \mathbb{N} \arrow[d, "A"] \\
& \mathbb{N}^{\mathbb{N}} \arrow[r, "S^{\mathbb{N}}"] & \mathbb{N}^{\mathbb{N}}
\end{tikzcd}
$$

commute. Show that the addition function is represented by the transpose of A.

Solution

Proof. Let $a\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be the unique map such that

$$
\begin{tikzcd}
\mathbb{N}^{\mathbb{N}} \times \mathbb{N} \arrow[r, "\mathrm{ev}_{\mathbb{N},\mathbb{N}}"] & \mathbb{N} \\
\mathbb{N} \times \mathbb{N} \arrow[u, "A \times 1_{\mathbb{N}}"] \arrow[ur, "a"']
\end{tikzcd}
$$

commutes. Then for $m, n \in \mathbb{N}$ we have that $a(0,n) = \mathrm{ev}_{\mathbb{N},\mathbb{N}}(A \times 1_{\mathbb{N}})(0,n) = A(0)(n) = n$ and $a(Sn,m) = A(Sn)(m) =$

□

Exercise 54

Solution

Proof. □

7 Presheaves

Exercise 55

Suppose object A and B are such that for every object X in $\mathcal{C}$ there is a bijection $f_X : \mathcal{C}(A,X) \to \mathcal{C}(B,X)$, naturally in a sense you define yourself. Conclude that A and B are isomorphic.

Solution

Proof. The assumption is equivalent to $f\colon \mathcal{C}(A,-) \Rightarrow \mathcal{C}(B,-)$ being a natural isomorphism of functors $\mathcal{C} \to \mathsf{Set}$, i.e. an isomorphism in $\mathsf{Set}^{\mathcal{C}}$.

Let $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}^{\mathsf{Set}}$ be the contravariant hom functor, i.e. $FX = \mathcal{C}(X,-)$. Then the co-Yoneda Lemma states that

$$\mathcal{C}^{\mathsf{Set}}(FX,FY) = \mathcal{C}^{\mathsf{Set}}(\mathcal{C}(X,-),\mathcal{C}(Y,-)) \cong \mathcal{C}(Y,X) = \mathcal{C}^{\mathrm{op}}(X,Y)$$

so F is fully faithful. Since fully faithful functors reflect isomorphism, it follows that if $FA \cong FB$ via f then there exists an isomorphism $g\colon A \to B$ in $\mathcal{C}^{\mathrm{op}}$ such that $Fg = f$. □

Exercise 8.1

Let $\mathcal{C}$ be a category and F a presheaf on $\mathcal{C}$. Show that F is representable if and only if there are C in $\mathcal{C}$ and $x \in F(C)$ such that for any D in $\mathcal{C}$ and $y \in F(D)$ there exists a unique map $\alpha\colon D \to C$ such that $y = x \cdot_F \alpha := F(\alpha)(x)$.

Solution

Proof. ($\Rightarrow$) If F is representable then there exists C in $\mathcal{C}$ and a natural isomorphism $\eta\colon \mathcal{C}(-,C) \Rightarrow F$. Let $x = \eta_C(1_C) \in FC$. For $D \in \mathrm{ob}\,\mathcal{C}$ and $y \in FD$, we have that $\mathcal{C}(D,C) \cong FD$ via η_D so let $\alpha = \eta_D^{-1}(y)$. By commutativity of

$$\begin{tikzcd}
\mathcal{C}(C,C) \arrow[r, "-\circ\alpha"] \arrow[d, "\eta_C"] & \mathcal{C}(D,C) \arrow[d, "\eta_D"] \\
FC \arrow[r, "F\alpha"] & FD
\end{tikzcd}$$

it follows that $y = \eta_D(\alpha) = \eta_D(1_C\alpha) = F(\alpha)(\eta_C(1_C)) = F(\alpha)(x)$.

($\Leftarrow$) Define $\eta\colon \mathcal{C}(-,C) \Rightarrow F$ by $\eta_C(1_C) = x$. Requiring η to be a natural transformation means that

$$\begin{tikzcd}
\mathcal{C}(C,C) \arrow[r, "-\circ\beta"] \arrow[d, "\eta_C"] & \mathcal{C}(D,C) \arrow[d, "\eta_D"] \\
FC \arrow[r, "F\beta"] & FD
\end{tikzcd}$$

commutes for all $D \in \mathrm{ob}\,\mathcal{C}$ and $\beta\colon D \to C$. In other words, $\eta_D(\beta) = F(\beta)(x)$ for all $D \in \mathrm{ob}\,\mathcal{C}$ and $\beta\colon D \to C$. We claim that η is a natural isomorphism. Indeed, for any $\gamma\colon D \to D'$ and $\beta\colon D' \to C$ we have that $F(\gamma)\eta_{D'}(\beta) = F(\gamma)F(\beta)(x) = F(\beta\gamma)(x) = \eta_D(\beta\gamma)$ so that

$$\begin{tikzcd}
\mathcal{C}(D',C) \arrow[r, "-\circ\gamma"] \arrow[d, "\eta_{D'}"] & \mathcal{C}(D,C) \arrow[d, "\eta_D"] \\
FC \arrow[r, "F\gamma"] & FD
\end{tikzcd}$$

commutes. The assumption that for each $y \in FD$ there is a unique $\alpha \in \mathcal{C}(D, C)$ such that $y = F(\alpha)(x) = \eta_D(\alpha)$ implies that η_D is a bijection for all $D \in \operatorname{ob}\mathcal{C}$. $\square$

EXERCISE 56

Show that the following are equivalent for each small category $\mathcal{C}$:

(a) $\mathcal{C}$ has a terminal object 1.

(b) The terminal object in $\mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}$ is representable.

Solution

Proof. $(a) \implies (b)$ Since limits in a functor category are computed pointwise, the terminal object of $\mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}$ is the functor that sends every object to the one element set and every map to the identity. This functor is certainly isomorphic to $\mathcal{C}(-, 1)$.

$(b) \implies (a)$ Let $\mathcal{C}(-, X)$ be the terminal object of $\mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}$. Then for every Y in $\mathcal{C}$, $\mathcal{C}(Y, X) \cong \{*\}$, so there is a unique map $Y \to X$. Thus, X is terminal in $\mathcal{C}$. $\square$

EXERCISE 57

Show that the following are equivalent for each small category $\mathcal{C}$:

(a) $\mathcal{C}$ has binary products.

(b) For each pair of object A and B in $\mathcal{C}$ the presheaf $yA \times yB$ is representable in $\mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}$.

Can you generalize the statement in this and Exercise 56 to general limits?

Solution

Proof. $(a) \implies (b)$ Let A, B be objects of $\mathcal{C}$. Note that for all $C \in \operatorname{ob}\mathcal{C}$ we have an isomorphism $\mathcal{C}(C, A) \times \mathcal{C}(C, B) \cong \mathcal{C}(C, A \times B)$ in Set given by sending (f_1, f_2) to the unique map which makes

$$\begin{tikzcd} & C \arrow[dl, "f_1"'] \arrow[d, dashed] \arrow[dr, "f_2"] & \\ A & A \times B \arrow[l, "\pi_A"] \arrow[r, "\pi_B"'] & B \end{tikzcd}$$

commute. Its inverse is given by sending $f\colon C \to A \times B$ to $(\pi_A f, \pi_B f)$. If $g\colon C' \to C$, $f_1\colon C \to A$ and $f_2\colon C \to B$ are morphisms in $\mathcal{C}$, since there is a unique map which makes

$$\begin{tikzcd} & C' \arrow[dl, "f_1 g"'] \arrow[d, dashed] \arrow[dr, "f_2 g"] & \\ A & A \times B \arrow[l, "\pi_A"] \arrow[r, "\pi_B"'] & B \end{tikzcd}$$

commute, so

$$\begin{tikzcd} \mathcal{C}(C, A) \times \mathcal{C}(C, A) \arrow[r] \arrow[d, "- \circ g"] & \mathcal{C}(C, A \times B) \arrow[d, "- \circ g"] \\ \mathcal{C}(C', A) \times \mathcal{C}(C', B) \arrow[r] & \mathcal{C}(C', A \times B) \end{tikzcd}$$

commutes. Hence, the isomorphism is natural in C.

$(b) \implies (a)$ For $A, B \in \operatorname{ob}\mathcal{C}$ take $C \in \operatorname{ob}\mathcal{C}$ such that there exists a natural isomorphism $\eta\colon \mathcal{C}(-,A)\times\mathcal{C}(-,B) \Rightarrow \mathcal{C}(-,C)$. We claim that $\mathcal{C}$ is the product. Let $(\pi_A,\pi_B) = \eta_C^{-1}(\mathrm{id}_C)$. Then for any $f_1\colon X \to A$ and $f_2\colon X \to B$ in $\mathcal{C}$, we have that

$$\begin{aligned}\eta_X(\pi_A\eta_X(f_1,f_2),\pi_B\eta_X(f_1,f_2)) &\overset{*}{=} \eta_C(\pi_A,\pi_B)\circ\eta_X(f_1,f_2)\\ &= \mathrm{id}_C\circ\eta_X(f_1,f_2) = \eta_X(f_1,f_2),\end{aligned}$$

where $(*)$ holds since η is a natural transformation. Since η_X is an isomorphism,

$$\begin{tikzcd} & X \arrow[dl, "f_1"'] \arrow[d, dashed, "\eta_X(f_1,f_2)"] \arrow[dr, "f_2"] & \\ A & C \arrow[l, "\pi_A"] \arrow[r, "\pi_B"'] & B \end{tikzcd}$$

commutes. Since $\eta_X(f_1,f_2)$ is the unique map which makes this diagram commute, C is the product of A and B. $\square$

The generalization of this and Exercise 56 would be

Proposition 7.1. *For $F: \mathcal{I}\to\mathcal{C}$, $\lim F$ exists in $\mathcal{C}$ if and only $\lim yF$ is representable, where $y\colon \mathcal{C}\to \mathsf{Set}^{\mathcal{C}^{op}}$ is the Yoneda embedding.*

Firs, we prove an important lemma.

Lemma 7.2. *Let $\mathcal{C}$ be a small category. The hom functor $\mathcal{C}(-,-)\colon \mathcal{C}^{op}\times\mathcal{C}\to$ Set preserves limits.*

Proof. Let $F\colon \mathcal{I}\to\mathcal{C}$ be a diagram in $\mathcal{C}$ such that $\lim F$ exists. By definition of limiting cones, we have a natural bijection $\eta\colon \mathcal{C}^{\mathcal{I}}(\Delta_{(-)},F) \Rightarrow \mathcal{C}(-,\lim F)$ sending a natural transformation $\theta\colon \Delta_Y \Rightarrow F$ to the unique map $Y \to \lim F$ such that

$$\begin{tikzcd} & Y \arrow[dl, "\theta_i"'] \arrow[d, "\eta_Y(\theta)"] \arrow[dr, "\theta_j"] & \\ F_i & \lim_i F_i \arrow[l] \arrow[r] & F_j \end{tikzcd}$$

commutes. Naturality follows from the fact that

$$\begin{tikzcd} & Y' \arrow[d, "g"] \arrow[ddl, bend right, "(\theta\Delta_g)_i"'] \arrow[ddr, bend left, "(\theta\Delta_g)_j"] & \\ & Y \arrow[dl, "\theta_i"'] \arrow[d, "\eta_Y(\theta)"] \arrow[dr, "\theta_j"] & \\ F_i & \lim_i F_i \arrow[l] \arrow[r] & F_j \end{tikzcd}$$

commutes and the uniqueness of $\eta_{Y'}(\theta\Delta_g)$. Hence, it remains to show that we have a natural bijection $\lim_i \mathcal{C}(-,F_i)\cong\mathcal{C}^{\mathcal{I}}(\Delta_{(-)},F)$. We define $\mu\colon \mathcal{C}^{\mathcal{I}}(\Delta_{(-)},F)\Rightarrow$ $\lim_i\mathcal{C}(-,F_i)$ by sending a natural transformation $\theta\colon \Delta_Y\to F$ to $g(*)$ where g is the unique map such that

$$\begin{tikzcd}
& * \arrow[d, "g"] \arrow[dl, bend right, "\mathrm{const}_{\theta_j}"'] \arrow[dr, bend left, "\mathrm{const}_{\theta_j}"] & \\
& \lim_i \mathcal{C}(Y, F_i) \arrow[dl] \arrow[dr] & \\
\mathcal{C}(Y, F_i) \arrow[rr] & & \mathcal{C}(Y, F_j)
\end{tikzcd}$$

commutes. This defines a bijection since for every element element x in $\lim_i \mathcal{C}(Y, F_i)$ we can assign a natural transformation $\theta\colon \Delta_Y \Rightarrow F$ by defining θ_i to be the image of x under the map $\lim_i \mathcal{C}(Y, F_i) \to \mathcal{C}(Y, F_i)$. Naturality of θ follows from the commutativity of

$$\begin{tikzcd}
& \lim_i \mathcal{C}(Y, F_i) \arrow[dl] \arrow[dr] & \\
\mathcal{C}(Y, F_i) \arrow[rr] & & \mathcal{C}(Y, F_j)
\end{tikzcd}$$

. To prove naturality in Y, let $f\colon Y' \to Y$ in $\mathcal{C}$ and $f^*\colon \lim_i \mathcal{C}(Y, F_i) \to \lim_i \mathcal{C}(Y', F_i)$ be the map induced by f. Then

$$\begin{tikzcd}
& \lim_i \mathcal{C}(Y, F_i) \arrow[dl, "g_i"'] \arrow[dr, "g_j"] \arrow[drr, "f^*"] & & & \\
\mathcal{C}(Y, F_i) \arrow[rr] \arrow[drr, "-\circ f"'] & & \mathcal{C}(Y, F_j) \arrow[drr, "-\circ f"] & \lim_i \mathcal{C}(Y', F_i) \arrow[dl, "g'_i"'] \arrow[dr, "g'_j"] & \\
& & \mathcal{C}(Y', F_i) \arrow[rr] & & \mathcal{C}(Y', F_j)
\end{tikzcd}$$

commutes, and for $\theta\colon \Delta_Y \Rightarrow F$ we have that

$$g'_i \mu_{Y'}(\theta \Delta_f) = (\theta \Delta f)_i = \theta_i f = g_i \mu_Y(\theta) f = g'_i f^* \mu_Y(\theta)$$

We conclude that $\mathcal{C}(-, \lim_i F_i) \cong \lim_i \mathcal{C}(-, F_i)$.

Since the hom functor is contravariant is the first compoenent, we need to show that $\mathcal{C}(\operatorname{colim}_i F_i, -) \cong \lim_i \mathcal{C}(F_i, -)$, but this is just the dual of the above paragraph. $\square$

Proof of Proposition 7.1. ($\Rightarrow$) We claim that $\lim_i yF_i \cong \mathsf{Set}(-, \lim_i F_i)$. Indeed, we have that $(\lim_i yF_i) \cong \lim_i \mathsf{Set}(-, F_i) \cong \mathsf{Set}(-, \lim_i F_i)$, where the last isomorphism follows from Lemma 7.2.

($\Leftarrow$) Since the Yoneda embedding is fully faithful, it reflects limits, i.e. if $\lim_i yF_i \cong \mathsf{Set}(-, X)$ for some $X \in \operatorname{ob} \mathcal{C}$ then $\lim_i F_i \cong X$. $\square$

Exercise 58

Again, let $\mathcal{C}$ be a small category with binary products, and let A and B be objects in $\mathcal{C}$.

i. Show that the assignment

$$X \mapsto \mathcal{C}(X \times A, B)$$

is part of a functor $\mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$, with the action on morphisms $f\colon X' \to X$ in $\mathcal{C}$ given by precomposition with $f \times \mathrm{id}_A$.

ii. What does it say about $\mathcal{C}$ if the functor in part (i) is representable?

Solution

i. Follows since $\mathrm{id}_X \times \mathrm{id}_A = \mathrm{id}_{X\times A}$ and $fg \times \mathrm{id}_A = (f \times \mathrm{id}_A)(g \times \mathrm{id}_A)$.

ii. Claim: if we have a natural isomorphism$\eta\colon \mathcal{C}(- \times A, B) \Rightarrow \mathcal{C}(-, C)$ for some $C \in \mathrm{ob}\,\mathcal{C}$, then $C \cong B^A$. Indeed, for $f\colon X \times A \to B$ we have $\eta_X(f)\colon X \to C$ and since

$$\begin{tikzcd}
\mathcal{C}(C \times A, B) \arrow[r, "\eta_C"] \arrow[d, "- \circ \eta_X(f) \times \mathrm{id}_A"'] & \mathcal{C}(C, C) \arrow[d, "- \circ \eta_X(f)"] \\
\mathcal{C}(X \times A, B) \arrow[r, "\eta_X"] & \mathcal{C}(X, C)
\end{tikzcd}$$

commutes, it follows that

$$\eta_X(\eta_C^{-1}(\mathrm{id}_C) \circ \eta_X(f) \times \mathrm{id}_A) = \eta_C(\eta_C^{-1}(\mathrm{id}_C)) \circ \eta_X(f) = \eta_X(f),$$

so $\eta_C^{-1}(\mathrm{id}_C) \circ \eta_X(f) \times \mathrm{id}_A = f$ by injectivity of η_X. In particular,

$$\begin{tikzcd}
C \times A \arrow[r, "\eta_C^{-1}(\mathrm{id}_C)"] & B \\
X \times A \arrow[u, "\eta_X(f) \times \mathrm{id}_A"] \arrow[ur, "f"']
\end{tikzcd}$$

commutes.

Exercise 8.3

A forest is a poset $(X, \leq)$ such that for any element $x \in X$ the set $\downarrow x = \{y \in X \mid y \leq x\}$ is finite and linearly ordered by $\leq$. If this set has $n+1$ elements, we say that x has depth n. Write For for the category of forests and monotone, depth-preserving maps.

Show that For is isomorphic to the category of presheaves on $\mathbb{N}$ (considered as a poset in the usual way).

Solution

Proof. We need to define functors $F\colon \mathsf{For} \to \mathsf{Set}^{\mathbb{N}^{\mathrm{op}}}$ and $G\colon \mathsf{Set}^{\mathbb{N}^{\mathrm{op}}} \to \mathsf{For}$ such that $FG = \mathrm{id}_{\mathsf{Set}^{\mathbb{N}^{\mathrm{op}}}}$ and $GF = \mathrm{id}_{\mathsf{For}}$. □

Exercise 59

Prove that $y\colon \mathcal{C} \to \mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}$ preserves all limits which exist in $\mathcal{C}$. Prove also that if $\mathcal{C}$ is cartesian closed, y preserves exponents.

Solution

Proof. The first claim is proved in Proposition 7.1. For the second claim, we want to prove that $y(Y^X) \cong (yY)^{yX}$. Well, using the Yoneda lemma and the fact the y preserves products we have

$$\begin{aligned} y(Y^X)(C) = \mathcal{C}(C, Y^X) \cong \mathcal{C}(C \times X, Y) = yY(C \times X) &\cong \mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}(y(C \times X), yY) \\ \cong \mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}(yC \times yX, yY) \cong \mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}\left(yC, (yY)^{yX}\right) &\cong (yY)^{yX}(C). \end{aligned}$$

□

8 Presheaves as a Topos

Exercise 9.2

Let $\mathcal{C}$ be a category with pullbacks.

i. Show that $\mathrm{Sub}(X)$ is a meet semi-lattice for each object X in $\mathcal{C}$.

ii. Show that if $f\colon Y \to X$ is a morphism in $\mathcal{C}$, then $f^*\colon \mathrm{Sub}(X) \to \mathrm{Sub}(Y)$ is a morphism of meet semi-lattices.

iii. Show that we have a presheaf

$$\mathrm{Sub}\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}.$$

Solution

i. *Proof.* Consider two monomorphisms $A \xrightarrow{n} X \xleftarrow{m} b$. Then we can define $n \wedge m$ to be any composition along the pullback square $\begin{tikzcd} C \arrow[r] \arrow[d] & B \arrow[d, "m"] \\ A \arrow[r, "n"] & X \end{tikzcd}$, and note that pullbacks of monos are monos (Exercise 25). It is straighforward to verify that this defines a meet operation. If $\mathcal{C}$ has an initial object, then $\mathrm{Sub}(X)$ has a bottom element. □

ii. *Proof.* For $n\colon A \to X$ in $\mathrm{Sub}(X)$, define $f^*(n)$ to be the pullback of n along f, i.e. $\begin{tikzcd} C \arrow[r] \arrow[d, "f^*(n)"'] & A \arrow[d, "n"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$. □

iii. *Proof.* To show that $\mathrm{Sub}(fg) = \mathrm{Sub}(g)\,\mathrm{Sub}(f)$ note that we have the commutative diagram

$$\begin{tikzcd}
C \arrow[r] \arrow[d, "(fg)^*(n)"'] & B \arrow[r] \arrow[d, "f^*(n)"] & A \arrow[d, "n"] \\
Z \arrow[r, "g"'] & Y \arrow[r, "f"'] & X
\end{tikzcd}$$

where the right and composite squares are pullbacks, and by Exercise 26 it follows that the left square is a pullback, i.e. $(fg)^*(n) = g^*(f^*(n))$. To show that $\mathrm{Sub}(\mathrm{id}_X) = \mathrm{id}_{\mathrm{Sub}(X)}$ observe that $\begin{tikzcd} A \arrow[r, "\mathrm{id}_A"] \arrow[d, "n"] & A \arrow[d, "n"] \\ X \arrow[r, "\mathrm{id}_X"'] & X \end{tikzcd}$ is a pullback square. $\square$

Exercise 9.3

Show that in Set we have for each set X an isomorphism of posets:

$$\mathrm{Sub}(X) \cong (\mathcal{P}(X(, \subseteq))).$$

Solution

Proof. $\square$