

CTF22: SOLUTIONS

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1 CATEGORIES

2 FUNCTORS AND CONSTRUCTIONS ON CATEGORIES

3 NATURAL TRANSFORMATIONS AND EQUIVALENCES

4 LIMITS AND COLIMITS

EXERCISE 21

Show that a full and faithful functor reflects the property of being a terminal (initial object).

Solution

Proof. Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a fully faithful functor, and $X \in \text{ob } \mathcal{C}$ such that FX is the terminal object of \mathcal{D} . Then for any $Y \in \mathcal{C}$ we have $\{*\} \cong \text{Hom}_{\mathcal{D}}(FY, FX) \cong \text{Hom}_{\mathcal{C}}(Y, X)$ so X is terminal in \mathcal{C} . We Similarly show that F reflects the property of being initial. \square

EXERCISE 22

Show that every equalizer is monic.

Solution

Proof. Let $e: E \rightarrow X$ be the equalizer of $f_1, f_2: X \rightrightarrows Y$, and take $g_1, g_2: T \rightarrow E$

such that $eg_1 = eg_2$. Since both g_1 and g_2 make

$$\begin{array}{ccc} E & \xrightarrow{e} & X \xrightleftharpoons[f_2]{f_1} Y \\ \uparrow \scriptstyle{eg_1=eg_2} & \nearrow & \\ T & & \end{array}$$

commute,

by uniqueness it follows that $g_1 = g_2$, so e is monic. \square

EXERCISE 23

Let $E \xrightarrow{e} X \xrightleftharpoons[f_2]{f_1} Y$ be an equalizer diagram. Show that e is iso if and only if $f_1 = f_2$.

Solution

Proof. (\Rightarrow) Immediate, since

$$f_1 = f_1 \text{id}_X = f_1 e e^{-1} = f_2 e e^{-1} = f_2 \text{id}_X = f_2.$$

(\Leftarrow) If $f_1 = f_2$ then there exists a unique $k: X \rightarrow E$ such that $ek = \text{id}_X$. Then $ek e = e \text{id}_E$ and since e is monic by the previous exercise it follows that $ke = 1_E$, so e is an isomorphism. \square

EXERCISE 24

Show that in **Set**, every monomorphism fits into an equalizer diagram.

Solution

Proof. Let $f: X \rightarrow Y$ be a monomorphism in **Set**. Define $g_1, g_2: Y \rightrightarrows \{0, 1\}$ by $g_1(y) = 1$ and

$$g_2(y) = \begin{cases} 1, & y \in \text{im } f \\ 0, & \text{otherwise} \end{cases}.$$

It is clear that $g_2 f = g_1 f$. Let $h: T \rightarrow Y$ be any map such that $g_1 h = g_2 h$. It follows that $\text{im } h \subseteq \text{im } f$ and so we can define $k: T \rightarrow X$ by $k(t) = f^{-1}h(t)$. Then $f k = h$ and k is unique, so f is the equalizer of $g_1, g_2: Y \rightrightarrows \{0, 1\}$. \square

EXERCISE 25

Let $\begin{array}{ccc} A & \xrightarrow{b} & B \\ \downarrow a & & \downarrow f \\ X & \xrightarrow{g} & Y \end{array}$ be a pullback diagram with f monic. Show that a is also monic. Also, if f is iso, so is a .

Solution

Proof. Let $t_1, t_2: T \rightrightarrows A$ such that $at_1 = at_2$. Then $fbt_1 = gat_1 = gat_2 = fbt_2$ and since f is monic, $bt_1 = bt_2$. Hence, there exists a unique $k: T \rightarrow A$ such that

$$\begin{array}{ccccc}
 & & T & & \\
 & & \searrow^{k} & \nearrow^{bt_1=bt_2} & \\
 & A & \xrightarrow{b} & B & \\
 \swarrow^{at_1=at_2} & \downarrow a & & \downarrow f & \\
 & X & \xrightarrow{g} & Y &
 \end{array}$$

commute. Since either $k = t_1$ or $k = t_2$ would work, it must be the case that $t_1 = t_2$. \square

EXERCISE 26

Given two commuting squares

$$\begin{array}{ccccc}
 A & \xrightarrow{b} & B & \xrightarrow{c} & C \\
 \downarrow a & & \downarrow f & & \downarrow d \\
 X & \xrightarrow{g} & Y & \xrightarrow{h} & Z
 \end{array}$$

Show that

- i. if both squares are pullback squares, then so is the composite square;
- ii. if the right square and the composite square are pullbacks, then so is the left square.

Solution

- i. *Proof.* \square
- ii. *Proof.* \square

EXERCISE 27

Solution

Proof. \square

EXERCISE 28

Solution

Proof. \square

EXERCISE 29

Solution

Proof.

□

EXERCISE 30

Solution

Proof.

□

5 COMPLETE CATEGORIES

EXERCISE 42

Take one of your favourite categories (**Top**, **Pos**, **Rng**, **Mon**, **Grp**, **Grph**, **Cat**) and show that it is both complete and cocomplete.

Solution Consider **Grp**. By Proposition 5.1, it is enough to prove that **Grp** has all small products and equalizers. The equalizer of $f, g: G \rightrightarrows H$ is the pair (G', i) where $G' = \{a \in G \mid f(a) = g(a)\}$ and i is the inclusion map. Note the G' is a group since f and g are group homomorphisms. For a set of groups $\{G_i\}_{i \in I}$, the product $\prod_{i \in I} G_i$ has a group structure by performing the group operations pointwise.

EXERCISE 43

Show that if \mathcal{C} is complete, then $F: \mathcal{C} \rightarrow \mathcal{D}$ preserves all limits if F preserves products and equalizers. This no longer holds if \mathcal{C} is not complete: F may preserve all products and equalizers which exists in \mathcal{C} , yet not preserve all limits which exists in \mathcal{C} .

Solution I assume the question means *small* limits, since I don't think this holds for all limits.

Proof. Suppose \mathcal{C} is complete and $F: \mathcal{C} \rightarrow \mathcal{D}$ preserves products and equalizers. Let $G: \mathcal{I} \rightarrow \mathcal{C}$ be a small diagram. Then $\lim_{\mathcal{I}} G$ can be expressed as

$$\lim_{\mathcal{I}} G \longrightarrow \prod_{i \in \text{ob } \mathcal{I}} G_i \xrightarrow[c]{\quad} \prod_{f \in \text{mor } \mathcal{I}} G(\text{cod } f),$$

and since F preserves equalizers and products, $F(\lim_{\mathcal{I}} G) = \lim_{\mathcal{I}} (FG)$. □

EXERCISE 44

Suppose a category \mathcal{C} has limits of shape \mathcal{I} . Show that the operation which assigns each diagram $\mathcal{I} \rightarrow \mathcal{C}$ to its limit in \mathcal{C} is part of a functor $F: [\mathcal{I}, \mathcal{C}] \rightarrow \mathcal{C}$.

Solution

Proof. Let $\eta: G \Rightarrow H$ be a morphism in $[\mathcal{I}, \mathcal{C}]$, $\mu: \Delta_{FG} \Rightarrow G$, $\varepsilon: \Delta_{FH} \Rightarrow H$ and $f: C \rightarrow C'$ a morphism in \mathcal{I} . This is summarized in the following diagram

$$\begin{array}{ccccc}
 & FG & & FH & \\
 \mu_C \swarrow & & \searrow \mu_{C'} & \varepsilon_C \swarrow & \searrow \varepsilon_{C'} \\
 GC & \xrightarrow{Gf} & GC' & HC & \xrightarrow{Hf} & HC' \\
 & \searrow \eta_C & & \swarrow \eta_{C'} & \\
 & & & &
 \end{array}$$

Note also that the diagram commutes. Thus, $(FG, \eta\mu)$ is a cone for the diagram H , so there is a unique morphism $g: FG \rightarrow FH$ such that $\varepsilon\Delta_g = \eta\mu$. We define $F\eta := g$. It is straightforward to verify that the uniqueness of g turns F into a functor. \square

EXERCISE 45

Let \mathcal{C}, \mathcal{D} and \mathcal{E} be categories. Show that the following categories are isomorphic:

$$[\mathcal{E}, [\mathcal{C}, \mathcal{D}]] \cong [\mathcal{E} \times \mathcal{C}, \mathcal{D}] \cong [\mathcal{C}, [\mathcal{E}, \mathcal{D}]].$$

Use this and the previous exercise to give a more elegant proof of Theorem 4.5.

Solution

Proof. Consider the functors $[\mathcal{E}, [\mathcal{C}, \mathcal{D}]] \xrightarrow{F_1} [\mathcal{E} \times \mathcal{C}, \mathcal{D}] \xrightarrow{F_2} [\mathcal{C}, [\mathcal{E}, \mathcal{D}]]$ given by

- On objects: for $G: \mathcal{E} \rightarrow [\mathcal{C}, \mathcal{D}]$, $H: \mathcal{E} \times \mathcal{C} \rightarrow \mathcal{D}$, $f: E \rightarrow E'$ in \mathcal{E} and $g: C \rightarrow C'$ in \mathcal{C} we have

$$\begin{aligned}
 F_1(G)(E, C) &= G(E)(C) \\
 F_1(G)(f, g) &= (G(E')(g))(Gf)_C \\
 F_2(H)(C)(E) &= H(E, C) \\
 F_2(H)(C)(f) &= H(f, \text{id}_C) \\
 (F_2(H)(g))_E &= H(\text{id}_E, f).
 \end{aligned}$$

- On morphisms: for $G_1, G_2: \mathcal{E} \Rightarrow [\mathcal{C}, \mathcal{D}]$, $H_1, H_2: \mathcal{E} \times \mathcal{C} \Rightarrow \mathcal{D}$, $\eta: G_1 \Rightarrow G_2$ and $\varepsilon: H_1 \Rightarrow H_2$ we have $(F_1\eta)_{(E, C)} = (\eta_E)_C$ and $((F_2\varepsilon)_C)_E = \varepsilon_{(E, C)}$.

Since these functors are clearly invertible, they are isomorphisms of categories. \square

EXERCISE 46

Show that a full and faithful functor reflects the property of being a terminal (or initial) object. Deduce that equivalences preserve the terminal (or initial) object.

Solution

Proof. Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a fully faithful functor and $X \in \text{ob } \mathcal{C}$ such that FX is terminal and take any $Y \in \text{ob } \mathcal{C}$. Then $\text{Hom}_{\mathcal{C}}(Y, X) \cong \text{Hom}_{\mathcal{D}}(FY, FX) \cong \{*\}$, so X is terminal in \mathcal{C} . Similarly, we show that fully faithful functors reflect the property of being initial.

Hence, if F is an equivalence, $X \in \text{ob } \mathcal{C}$ is terminal, $Z \in \text{ob } \mathcal{D}$ is any object and $Y \in \text{ob } \mathcal{C}$ is chosen such that $FY \cong Z$ we have that $\{*\} = \text{Hom}_{\mathcal{C}}(Y, X) \cong \text{Hom}_{\mathcal{D}}(FY, FX) \cong \text{Hom}_{\mathcal{D}}(Z, FX)$ so FX is terminal in \mathcal{D} . \square

6 CARTESIAN CLOSED CATEGORIES

EXERCISE 47

Show that in a ccc, there are natural isomorphisms

- i. $1^X \cong 1$,
- ii. $(Y \times Z)^X \cong Y^X \times Z^X$,
- iii. $(Y^Z)^X = Y^{Z \times X}$.

Solution Let \mathcal{C} be a ccc category and X, Y, Z and A objects in \mathcal{C} .

- i. *Proof.* Since 1 is terminal we have unique morphisms $1 \times X \rightarrow 1$, $A \times X \rightarrow 1$

$$\text{and } f: A \rightarrow 1, \text{ so } \begin{array}{ccc} 1 \times X & \longrightarrow & 1 \\ & \nwarrow f \times 1_X & \uparrow \\ & & A \times X \end{array} \text{ trivially commutes. Hence, } 1^X \cong 1.$$

Moreover, this isomorphism is natural as a morphism $1^{(-)} \Rightarrow 1$ since all

$$\text{the maps in the square } \begin{array}{ccc} 1^X & \longrightarrow & 1^Y \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & 1 \end{array} \text{ are the identity maps. } \square$$

- ii. *Proof.* Let $\eta_{(Y,Z)}: Y^X \times Z^X \rightarrow (Y \times Z)^X$ be the unique map which makes

$$\begin{array}{ccc} (Y \times Z)^X \times X & \xrightarrow{\text{ev}_{X, Y \times Z}} & Y \times Z \\ & \nwarrow \eta_{(Y,Z)} \times 1_X & \uparrow h \\ & & Y^X \times Z^X \times X \end{array}$$

commute, where $h = (\text{ev}_{X,Y}(\pi_{Y^X} \times 1_X), \text{ev}_{X,Z}(\pi_{Z^X} \times 1_X))$. To construct an inverse, let $f_Y: (Y \times Z)^X \rightarrow Y^X$ be the unique map such that

$$\begin{array}{ccc} Y^X \times X & \xrightarrow{\text{ev}_{X,Y}} & Y \\ & \nwarrow f_Y \times 1_X & \uparrow \pi_Y \text{ ev}_{X, Y \times Z} \\ & & (Y \times Z)^X \times X \end{array}$$

commutes. Analogously, we define $f_Z: (Y \times Z)^X \rightarrow Z^X$. We claim that $\mu_{(Y,Z)} = (f_Y, f_Z): (Y \times Z)^X \rightarrow Y^X \times Z^X$ is the inverse of $\eta_{(Y,Z)}$. Since

$$\begin{array}{ccccc}
 & (Y \times Z)^X \times X & & & \\
 & \swarrow f_Y \times 1_X & \downarrow \mu_{(Y,Z)} \times 1_X & \searrow f_Z \times 1_X & \\
 Y^X \times X & \xleftarrow{\pi_{Y^X} \times 1_X} & Y^X \times Z^X \times X & \xrightarrow{\pi_{Z^X} \times 1_X} & Z^X \times X \\
 \downarrow \text{ev}_{X,Y} & & \downarrow h & & \downarrow \text{ev}_{X,Z} \\
 Y & \xleftarrow{\pi_Y} & Y \times Z & \xrightarrow{\pi_Z} & Z
 \end{array}$$

commutes, it follows that

$$\begin{aligned}
 h(\mu_{(Y,Z)} \times 1_X) &= (\text{ev}_{X,Y}(f_Y \times 1_X), \text{ev}_{X,Z}(f_Z \times 1_X)) \\
 &= (\pi_Y \text{ev}_{X,Y \times Z}, \pi_Z \text{ev}_{X,Y \times Z}) \\
 &= \text{ev}_{X,Y \times Z}
 \end{aligned}$$

and so

$$\begin{array}{ccc}
 (Y \times Z)^X \times X & \xrightarrow{\text{ev}_{X,Y \times Z}} & Y \times Z \\
 \swarrow \eta_{(Y,Z)} \times 1_X & & \nearrow h \\
 & Y^X \times Z^X \times X & \\
 \nwarrow \mu_{(Y,Z)} \times 1_X & & \nearrow \text{ev}_{X,Y \times Z} \\
 & (Y \times Z)^X \times X &
 \end{array}$$

commutes (the top left triangle commutes by definition of $\eta_{(Y,Z)}$ and we've just show that the bottom right triangle commute, so the big triangle commutes as well). By uniqueness, $\eta_{(Y,Z)}\mu_{(Y,Z)} = \text{id}_{(Y \times Z)^X}$. By definition of f_Y and $\eta_{(Y,Z)}$,

$$\begin{array}{ccccc}
 Y^X \times X & \xrightarrow{\text{ev}_{X,Y}} & Y & & \\
 \swarrow f_Y \times 1_X & & \nearrow \pi_Y \text{ev}_{X,Y \times Z} & & \\
 & (Y \times Z)^X \times X & \xrightarrow{\text{ev}_{X,Y \times Z}} & Y \times Z & \\
 \nwarrow \eta_{(Y,Z)} \times 1_X & & \nearrow h & & \\
 & Y^X \times Z^X \times X & & &
 \end{array}$$

commutes and so both $\pi_{Y^X} \times 1_X$ and $(f_Y \eta_{Y,Z}) \times 1_X$ make

$$\begin{array}{ccc}
 Y^X \times X & \xrightarrow{\text{ev}_{X,Y}} & Y \\
 \swarrow & & \nearrow \pi_Y h \\
 & Y^X \times Z^X \times X &
 \end{array}$$

commute, so $f_Y \eta_{(Y,Z)} = \pi_{Y^X}$. Analogously, $f_Z \eta_{(Y,Z)} = \pi_{Z^X}$. Thus,

$$\begin{array}{ccccc}
 & & Y^X \times Z^X & & \\
 & \swarrow \pi_{Y^X} & \downarrow \eta_{(Y,Z)} & \searrow \pi_{Z^X} & \\
 & & (Y \times Z)^X & & \\
 & \swarrow f_Y & \downarrow \mu_{(Y,Z)} & \searrow f_Z & \\
 Y^X & \xleftarrow{\pi_{Y^X}} & Y^X \times Z^X & \xrightarrow{\pi_{Z^X}} & Z^X
 \end{array}$$

commutes, and by uniqueness it follows that $\mu_{(Y,Z)} \eta_{(Y,Z)} = 1_{Y^X \times Z^X}$.

It is left to show that this gives a natural isomorphism $\eta: (-)^X \times (-)^X \Rightarrow (- \times -)^X$ of functors $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$. Since a morphism of bifunctors is natural if and only if it is natural in each component and the definitions are symmetric in their components, it suffices to check that

$$\begin{array}{ccc}
 Y^X \times Z^X & \xrightarrow{f^X \times 1_{Z^X}} & (Y')^X \times Z^X \\
 \downarrow \eta_{(Y,Z)} & & \downarrow \eta_{(Y',Z)} \\
 (Y \times Z)^X & \xrightarrow{(f \times 1_Z)^X} & (Y' \times Z)^X
 \end{array}$$

commutes. Since $\text{ev}_{X,-}: (-)^X \times X \Rightarrow 1_{\mathcal{C}}$ is a natural transformation, we have that

$$\begin{aligned}
 \text{ev}_{X,Y' \times Z} \circ (f \times 1_Z)^X \times 1_X \circ \eta_{Y,Z} \times 1_X &= (f \times 1_Z) \circ \text{ev}_{X,Y \times Z} \circ \eta_{(Y,Z)} \times 1_X \\
 &= (f \times 1_Z) h
 \end{aligned}$$

and

$$\begin{array}{ccccc}
 & & Y^X \times Z^X \times X & & \\
 & \swarrow f \circ \text{ev}_{X,Y} (\pi_{Y^X} \times 1_X) & \downarrow f^X \times 1_{Z^X} \times 1_X & \searrow \text{ev}_{X,Z} (\pi_{Z^X} \times 1_X) & \\
 & & (Y')^X \times Z^X \times X & & \\
 & \swarrow \text{ev}_{X,Y'} (\pi_{(Y')^X} \times 1_X) & \downarrow h' & \searrow \text{ev}_{X,Z} (\pi_{Z^X} \times 1_X) & \\
 Y' & \xleftarrow{\quad} & Y' \times Z & \xrightarrow{\quad} & Z
 \end{array}$$

commutes, so

$$\begin{aligned}
 \text{ev}_{X,Y' \times Z} \circ \eta_{(Y',Z)} \times 1_X \circ f^X \times 1_{Z^X} \times 1_X &= h' \circ f^X \times 1_{Z^X} \times 1_X \\
 &= (f \times 1_Z) h.
 \end{aligned}$$

It follows that both $(f \times 1_Z)^X \eta_{(Y,Z)}$ and $\eta_{(Y',Z)} (f^X \times 1_{Z^X})$ make

$$\begin{array}{ccc}
(Y' \times Z)^X \times X & \xrightarrow{\text{ev}_{X,Y' \times Z}} & Y' \times Z \\
& \nwarrow -\times 1_X & \uparrow (f \times 1_Z)h \\
& & Y^X \times Z^X \times X
\end{array}$$

commute, and so they are equal. \square

- iii. *Proof.* Define $f = \text{ev}_{Z,Y} \circ \text{ev}_{X,Y^Z} \times 1_Z: (Y^Z)^X \times X \times Z \rightarrow Y$. Then for any $h: A \times X \times Z \rightarrow Y$ we get a map $H': A \times X \rightarrow Y^Z$ such that $\text{ev}_{Z,Y} \circ H' \times 1_Z = h$ which gives a unique map $H: A \rightarrow (Y^Z)^X$ such that $\text{ev}_{X,Y^Z} \circ H \times 1_X = H'$. Since

$$\begin{aligned}
f \circ H \times 1_{X \times Z} &= \text{ev}_{Z,Y} \circ \text{ev}_{X,Y^Z} \times 1_Z \circ H \times 1_{X \times Z} \\
&= \text{ev}_{Z,Y} \circ (\text{ev}_{X,Y^Z} \circ H \times 1_X) \times 1_Z \\
&= \text{ev}_{Z,Y} \circ H' \times 1_Z \\
&= h,
\end{aligned}$$

we conclude that $(Y^Z)^X \cong Y^{Z \times X}$. \square

EXERCISE 48

If a ccc has coproducts, we have

- i. $X \times (Y + Z) \cong (X \times Y) + (X \times Z)$
- ii. $Y^{Z+X} = Y^Z \times Y^X$.

Solution

- i. *Proof.* Suppose \mathcal{C} is a ccc which has coproducts, and let X, Y and Z be objects in \mathcal{C} . Consider any cocone

$$\begin{array}{ccc}
X \times Y & & X \times Z \\
& \searrow \iota_1 & \swarrow \iota_2 \\
& D &
\end{array}$$

Exponentiating, we get

$$\begin{array}{ccccc}
X \times Y & \xrightarrow{\overline{\iota_1} \times 1_X} & D^X \times X & \xleftarrow{\overline{\iota_2} \times 1_X} & X \times Z \\
& \searrow \iota_1 & \downarrow \text{ev}_{X,D} & \swarrow \iota_2 & \\
& & D & &
\end{array}$$

where $\overline{\iota_1}$ and $\overline{\iota_2}$ are the transposes of ι_1 and ι_2 respectively. Hence, we have the cocone

$$\begin{array}{ccc}
Y & & Z \\
& \searrow \overline{\iota_1} & \swarrow \overline{\iota_2} \\
& D^X &
\end{array}$$

and since $Y + Z$ is limiting, we get a unique map $h: Y + Z \rightarrow D^X$ so that

$$\begin{array}{ccccc}
Y & \xrightarrow{\iota_Y} & Y + Z & \xleftarrow{\iota_Z} & Z \\
& \searrow \overline{\iota_1} & \downarrow h & \swarrow \overline{\iota_2} & \\
& & D^X & &
\end{array}$$

commutes. Then

$$\begin{array}{ccccc}
Y \times X & \xrightarrow{\iota_Y \times 1_X} & (Y + Z) \times X & \xleftarrow{\iota_Z \times 1_X} & Z \times X \\
& \searrow \overline{\iota_1} \times 1_X & \downarrow h \times 1_X & \swarrow \overline{\iota_2} \times 1_X & \\
& & D^X \times X & & \\
& & \downarrow \text{ev}_{X,D} & & \\
& & D & &
\end{array}$$

commutes and it follows that $(Y + Z) \times X$ is the coproduct of $Y \times X$ and $Z \times X$. Thus $Y \times X + Z \times X \cong (Y + Z) \times X$ \square

- ii. *Proof.* Note that $Y^Z \times Y^X (Z + X) \cong Y^X \times Y^Z \times Z + Y^Z \times Y^X \times X$ and let $f: Y^Z \times Y^X (Z + X) \rightarrow Y$ be given by the composition

$$\begin{aligned}
Y^X \times Y^Z \times Z + Y^Z \times Y^X \times X &\xrightarrow{1_{Y^X} \times \text{ev}_{Z,Y} + 1_{Y^Z} \times \text{ev}_{X,Y}} \\
&Y^X \times Y + Y^Z \times Y \cong (Y^X + Y^Z) \times Y \xrightarrow{\pi_Y} Y.
\end{aligned}$$

Let $h: A \times (Z + X) \rightarrow Y$ be any morphism. Since $A \times (Z + X) \cong A \times Z + A \times X$ we have unique maps $H_Z: A \rightarrow Y^Z$ and $H_X: A \rightarrow Y^X$ such that

$$\begin{array}{ccc}
Y^Z \times Z & \xrightarrow{\text{ev}_{Z,Y}} & Y \\
\swarrow H_Z \times 1_Z & & \uparrow \iota_{A \times Z} h \\
& A \times Z &
\end{array}
\quad
\begin{array}{ccc}
Y^X \times X & \xrightarrow{\text{ev}_{X,Y}} & Y \\
\swarrow H_X \times 1_X & & \uparrow \iota_{A \times X} h \\
& A \times X &
\end{array}$$

commute, where $A \times Z \xrightarrow{\iota_{A \times Z}} A \times Z + A \times X \xleftarrow{\iota_{A \times X}} A \times X$ are the inclusion maps. Let $H = (H_Z, H_X): A \rightarrow Y^Z \times Y^X$. Since

$$\pi_Y \circ (1_{Y^X} \times \text{ev}_{Z,Y}) \circ (H \times 1_Z) = \pi_Y \circ H_X \times (\text{ev}_{Z,Y} \circ H_Z \times 1_Z) = \iota_{A \times Z} h$$

and similarly $\pi_Y \circ (1_{Y^Z} \times \text{ev}_{X,Y}) \circ (H \times 1_X) = \iota_{A \times X} h$, it follows that

$$\begin{aligned} h &= \iota_{A \times Z} h + \iota_{A \times X} h \\ &= \pi_Y \circ (1_{Y^X} \times \text{ev}_{Z,Y}) \circ (H \times 1_Z) + \pi_Y \circ (1_{Y^Z} \times \text{ev}_{X,Y}) \circ (H \times 1_X) \\ &= f \circ (H \times 1_Z + H \times 1_X) \\ &= f \circ (H \times 1_{Z+X}). \end{aligned}$$

Hence,

$$\begin{array}{ccc} Y^X \times Y^Z \times (Z+X) & \xrightarrow{f} & Y \\ & \nwarrow H \times 1_{Z+X} & \uparrow h \\ & & A \times (Z+X) \end{array} \quad \text{commutes and so } Y^{Z+X} \cong Y^Z \times Y^X.$$

□

EXERCISE 49

In a ccc, prove that the transpose of a composite $Z \xrightarrow{g} W \xrightarrow{f} Y^X$ is

$$Z \times X \xrightarrow{g \times 1_X} W \times X \xrightarrow{\bar{f}} Y,$$

if \bar{f} is the transpose of f .

Solution

Proof. We need to find a morphism $\bar{f}g: Z \times X \rightarrow Y$ such that $\text{ev}_{X,Y} \circ (f \circ g \times 1_X) = \bar{f}g$. Well,

$$\text{ev}_{X,Y} \circ (f \circ g \times 1_X) = \text{ev}_{X,Y} \circ (f \times 1_X) \circ (g \times 1_X) = \bar{f} \circ (g \times 1_X),$$

so we're done. □

EXERCISE 50

Suppose $\mathcal{C}, \mathcal{D}, \mathcal{E}$ are categories, such that

1. For each pair of objects $(C, D) \in \text{ob}(\mathcal{C} \times \mathcal{D})$, an object $F_0(C, D)$ in \mathcal{E} ;
2. For each object $C \in \text{ob} \mathcal{C}$, a functor $F_C: \mathcal{D} \rightarrow \mathcal{E}$ satisfying $F_C(D) = F_0(C, D)$ for each $D \in \text{ob} \mathcal{D}$;
3. For each object $D \in \text{ob} \mathcal{D}$, a functor $F_D: \mathcal{C} \rightarrow \mathcal{E}$ satisfying $F_D(C) = F_0(C, D)$ for each object $C \in \mathcal{C}$;

such that for each pair of morphism $f: C \rightarrow C'$ in \mathcal{C} and $g: D \rightarrow D'$ in \mathcal{D} we have a commuting square

$$\begin{array}{ccc} F_0(C, D) & \xrightarrow{F_D(f)} & F_0(C', D) \\ \downarrow F_C(g) & & \downarrow F_{C'}(g) \\ F_0(C, D') & \xrightarrow{F_{D'}(f)} & F_0(C', D') \end{array}$$

in \mathcal{E} .

Show that there is a unique functor $F: \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E}$ whose operation on objects in F_0 , while $F(1_C, g) = F_C(g)$ and $F(f, 1_D) = F_D(f)$.

Solution

Proof. For $f: C \rightarrow C'$ in \mathcal{C} and $g: D \rightarrow D'$ in \mathcal{D} we define $F(f, g): F_0(C, D) \rightarrow F_0(C', D')$ to be any composition around the commuting square

$$\begin{array}{ccc} F_0(C, D) & \xrightarrow{F_D(f)} & F_0(C', D) \\ \downarrow F_C(g) & & \downarrow F_{C'}(g) \\ F_0(C, D') & \xrightarrow{F_{D'}(f)} & F_0(C', D') \end{array}$$

This is easily checked to give a functor $F: \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E}$. □

EXERCISE 51

An object Y in a category with finite product is called *exponentiating* if the exponential X^Y exists for each $Y \in \text{ob } \mathcal{C}$. Show that if X is exponentiating, the assignment $Y \mapsto X^Y$ is the object part of a functor $\mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$.

Solution

Proof. For $f: Y \rightarrow Z$, we define $X^f: X^Z \rightarrow X^Y$ to be the unique map which makes

$$\begin{array}{ccc} X^Y \times Y & \xrightarrow{\text{ev}_{Y,X}} & X \\ \uparrow X^f \times 1_Y & \nearrow \text{ev}_{Z,X} & \\ X^Z \times Y & \xrightarrow{1_{X^Z} \times f} & X^Z \times Z \end{array}$$

commute. From uniqueness, we get that $X^{1_Y} = 1_{X^Y}$ and for $X^{gf} = X^f X^g$ for $Y \xrightarrow{f} Z \xrightarrow{g} W$. Hence, $X^{(-)}: \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$ is a functor. □

EXERCISE 52

Show that for every cartesian closed category \mathcal{C} there is a functor $\mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{C}$, assigning Y^X to (X, Y) .

Solution

Proof. Checking the conditions of Exercise 50:

1. for each object (X, Y) in $\mathcal{C}^{\text{op}} \times \mathcal{C}$ we have the object Y^X in \mathcal{C} ;
2. for each $X \in \text{ob } \mathcal{C}^{\text{op}}$ we have the functor $(-)^X: \mathcal{C} \rightarrow \mathcal{C}$;
3. for each $Y \in \text{ob } \mathcal{C}^{\text{op}}$ we have the functor $Y^{(-)}: \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$.

It is left to show that for any $f: X_2 \rightarrow X_1$ and $g: Y_1 \rightarrow Y_2$ in \mathcal{C} ,

$$\begin{array}{ccc} Y_1^{X_1} & \xrightarrow{Y_1^f} & Y_1^{X_2} \\ \downarrow g^{X_1} & & \downarrow g^{X_2} \\ Y_2^{X_1} & \xrightarrow{Y_2^f} & Y_2^{X_2} \end{array}$$

commutes¹. Since

$$\begin{array}{ccccc} Y_1^{X_1} \times X_2 & \xrightarrow{Y_1^f \times 1_{X_2}} & Y_1^{X_2} \times X_2 & \xrightarrow{g^{X_2} \times 1_{X_2}} & Y_2^{X_2} \times X_2 \\ \downarrow 1_{Y_1^{X_1}} \times f & & \downarrow \text{ev}_{X_2, Y_1} & & \downarrow \text{ev}_{X_2, Y_2} \\ Y_1^{X_1} \times X_1 & \xrightarrow{\text{ev}_{X_1, Y_1}} & Y_1 & \xrightarrow{g} & Y_2 \end{array}$$

commutes², we have that

$$\text{ev}_{X_2, Y_2}(g^{X_2} Y_1^f \times 1_{X_2}) = g \text{ev}_{X_2, Y_1}(Y_1^f \times 1_{X_2}) = g \text{ev}_{X_1, Y_1}(1_{Y_1^{X_1}} \times f),$$

and since

$$\begin{array}{ccccc} Y_2^{X_2} \times X_2 & \xrightarrow{\text{ev}_{X_2, Y_2}} & Y_2 & & \\ \uparrow Y_2^f \times 1_{X_2} & & \uparrow \text{ev}_{X_1, Y_2} & \swarrow g & \\ Y_2^{X_1} \times X_2 & \xrightarrow{1_{Y_2^{X_1}} \times f} & Y_2^{X_1} \times X_1 & & Y_1 \\ \uparrow g^{X_1} \times 1_{X_2} & & \uparrow g^{X_1} \times 1_{X_1} & \nearrow \text{ev}_{X_1, Y_1} & \\ Y_1^{X_1} \times X_2 & \xrightarrow{1_{Y_1^{X_1}} \times f} & Y_1^{X_1} \times X_1 & & \end{array}$$

commutes³, then

$$\text{ev}_{X_2, Y_2}(Y_2^f g^{X_1} \times 1_{X_2}) = \text{ev}_{X_1, Y_2}(1_{Y_2^{X_1}} \times f)(g^{X_1} \times 1_{X_1}) = g \text{ev}_{X_1, Y_1}(1_{Y_1^{X_1}} \times f).$$

Hence, both $Y_2^f g^{X_1}$ and $g^{X_2} Y_1^f$ make

¹In **Set**, $Y^X = \text{Hom}(X, Y)$ so the functor acts functions by sending $h: X_1 \rightarrow Y_1$ to $ghf: X_2 \rightarrow Y_2$. Thus, the commutativity of the diagram is equivalent to function composition being associative.

²left square commutes by definition of Y_1^f , right square commutes by definition of g^{X_2}

³top square commutes by definition of Y_2^f , bottom square commutes because $- \times -: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ is a bifunctor and the right triangle commutes by definition of g^{X_1}

$$\begin{array}{ccc}
Y_2^{X_2} \times X_2 & \xrightarrow{\text{ev}_{X_2, Y_2}} & Y_2 \\
\uparrow -\times 1_{X_2} & & \uparrow g \text{ev}_{X_1, Y_1} \\
Y_1^{X_1} \times X_2 & \xrightarrow{1_{Y_1^{X_1}} \times f} & Y_1^{X_1} \times X_1
\end{array}$$

commute, and so by the universal property of $Y_2^{X_2}$ they are equal. \square

EXERCISE 53

Let A be the unique function making

$$\begin{array}{ccccc}
1 & \xrightarrow{0} & \mathbb{N} & \xrightarrow{S} & \mathbb{N} \\
& \searrow 1_{\mathbb{N}} & \downarrow A & & \downarrow A \\
& & \mathbb{N}^{\mathbb{N}} & \xrightarrow{S^{\mathbb{N}}} & \mathbb{N}^{\mathbb{N}}
\end{array}$$

commute. Show that the addition function is represented by the transpose of A .

Solution

Proof. Let $a: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ be the unique map such that

$$\begin{array}{ccc}
\mathbb{N}^{\mathbb{N}} \times \mathbb{N} & \xrightarrow{\text{ev}_{\mathbb{N}, \mathbb{N}}} & \mathbb{N} \\
A \times 1_{\mathbb{N}} \uparrow & \nearrow a & \\
\mathbb{N} \times \mathbb{N} & &
\end{array}$$

commutes. Then for $m, n \in \mathbb{N}$ we have that $a(0, n) = \text{ev}_{\mathbb{N}, \mathbb{N}}(A \times 1_{\mathbb{N}})(0, n) = A(0)(n) = n$ and $a(Sn, m) = A(Sn)(m) =$

\square

EXERCISE 54

Solution

Proof.

\square

7 PRESHEAVES

EXERCISE 55

Suppose object A and B are such that for every object X in \mathcal{C} there is a bijection $f_X: \mathcal{C}(A, X) \rightarrow \mathcal{C}(B, X)$, naturally in a sense you define yourself. Conclude that A and B are isomorphic.

Solution

Proof. The assumption is equivalent to $f: \mathcal{C}(A, -) \Rightarrow \mathcal{C}(B, -)$ being a natural isomorphism of functors $\mathcal{C} \rightarrow \mathbf{Set}$, i.e. an isomorphism in $\mathbf{Set}^{\mathcal{C}}$.

Let $F: \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}^{\text{Set}}$ be the contravariant hom functor, i.e. $FX = \mathcal{C}(X, -)$. Then the co-Yoneda Lemma states that

$$\mathcal{C}^{\text{Set}}(FX, FY) = \mathcal{C}^{\text{Set}}(\mathcal{C}(X, -), \mathcal{C}(Y, -)) \cong \mathcal{C}(Y, X) = \mathcal{C}^{\text{op}}(X, Y)$$

so F is fully faithful. Since fully faithful functors reflect isomorphism, it follows that if $FA \cong FB$ via f then there exists an isomorphism $g: A \rightarrow B$ in \mathcal{C}^{op} such that $Fg = f$. \square

EXERCISE 8.1

Let \mathcal{C} be a category and F a presheaf on \mathcal{C} . Show that F is representable if and only if there are C in \mathcal{C} and $x \in F(C)$ such that for any D in \mathcal{C} and $y \in F(D)$ there exists a unique map $\alpha: D \rightarrow C$ such that $y = x \cdot_F \alpha := F(\alpha)(x)$.

Solution

Proof. (\Rightarrow) If F is representable then there exists C in \mathcal{C} and a natural isomorphism $\eta: \mathcal{C}(-, C) \Rightarrow F$. Let $x = \eta_C(1_C) \in FC$. For $D \in \text{ob } \mathcal{C}$ and $y \in FD$, we have that $\mathcal{C}(D, C) \cong FD$ via η_D so let $\alpha = \eta_D^{-1}(y)$. By commutativity of

$$\begin{array}{ccc} \mathcal{C}(C, C) & \xrightarrow{- \circ \alpha} & \mathcal{C}(D, C) \\ \downarrow \eta_C & & \downarrow \eta_D \\ FC & \xrightarrow{F\alpha} & FD \end{array}$$

it follows that $y = \eta_D(\alpha) = \eta_D(1_C \alpha) = F(\alpha)(\eta_C(1_C)) = F(\alpha)(x)$.

(\Leftarrow) Define $\eta: \mathcal{C}(-, C) \Rightarrow F$ by $\eta_C(1_C) = x$. Requiring η to be a natural transformation means that

$$\begin{array}{ccc} \mathcal{C}(C, C) & \xrightarrow{- \circ \beta} & \mathcal{C}(D, C) \\ \downarrow \eta_C & & \downarrow \eta_D \\ FC & \xrightarrow{F\beta} & FD \end{array}$$

commutes for all $D \in \text{ob } \mathcal{C}$ and $\beta: D \rightarrow C$. In other words, $\eta_D(\beta) = F(\beta)(x)$ for all $D \in \text{ob } \mathcal{C}$ and $\beta: D \rightarrow C$. We claim that η is a natural isomorphism. Indeed, for any $\gamma: D \rightarrow D'$ and $\beta: D' \rightarrow C$ we have that $F(\gamma)\eta_{D'}(\beta) = F(\gamma)F(\beta)(x) = F(\beta\gamma)(x) = \eta_D(\beta\gamma)$ so that

$$\begin{array}{ccc} \mathcal{C}(D', C) & \xrightarrow{- \circ \gamma} & \mathcal{C}(D, C) \\ \downarrow \eta_{D'} & & \downarrow \eta_D \\ FC & \xrightarrow{F\gamma} & FD \end{array}$$

commutes. The assumption that for each $y \in FD$ there is a unique $\alpha \in \mathcal{C}(D, C)$ such that $y = F(\alpha)(x) = \eta_D(\alpha)$ implies that η_D is a bijection for all $D \in \text{ob } \mathcal{C}$. \square

EXERCISE 56

Show that the following are equivalent for each small category \mathcal{C} :

- (i) \mathcal{C} has a terminal object 1.
- (ii) The terminal object in $\mathbf{Set}^{\mathcal{C}^{\text{op}}}$ is representable.

Solution

Proof. (a) \implies (b) Since limits in a functor category are computed pointwise, the terminal object of $\mathbf{Set}^{\mathcal{C}^{\text{op}}}$ is the functor that sends every object to the one element set and every map to the identity. This functor is certainly isomorphic to $\mathcal{C}(-, 1)$.

(b) \implies (a) Let $\mathcal{C}(-, X)$ be the terminal object of $\mathbf{Set}^{\mathcal{C}^{\text{op}}}$. Then for every Y in \mathcal{C} , $\mathcal{C}(Y, X) \cong \{*\}$, so there is a unique map $Y \rightarrow X$. Thus, X is terminal in \mathcal{C} . \square

EXERCISE 57

Show that the following are equivalent for each small category \mathcal{C} :

- (i) \mathcal{C} has binary products.
- (ii) For each pair of objects A and B in \mathcal{C} the presheaf $yA \times yB$ is representable in $\mathbf{Set}^{\mathcal{C}^{\text{op}}}$.

Can you generalize the statement in this and Exercise 56 to general limits?

Solution

Proof. (a) \implies (b) Let A, B be objects of \mathcal{C} . Note that for all $C \in \text{ob } \mathcal{C}$ we have an isomorphism $\mathcal{C}(C, A) \times \mathcal{C}(C, B) \cong \mathcal{C}(C, A \times B)$ in \mathbf{Set} given by sending (f_1, f_2)

to the unique map which makes

$$\begin{array}{ccccc} & & C & & \\ & f_1 \swarrow & \downarrow & \searrow f_2 & \\ A & \xleftarrow{\pi_A} & A \times B & \xrightarrow{\pi_B} & B \end{array}$$

commute. Its in-

verse is given by sending $f: C \rightarrow A \times B$ to $(\pi_A f, \pi_B f)$. If $g: C' \rightarrow C$, $f_1: C \rightarrow A$ and $f_2: C \rightarrow B$ are morphisms in \mathcal{C} , since there is a unique map which makes

$$\begin{array}{ccccc} & & C' & & \\ & f_1 g \swarrow & \downarrow & \searrow f_2 g & \\ A & \xleftarrow{\pi_A} & A \times B & \xrightarrow{\pi_B} & B \end{array}$$

commute, so

$$\begin{array}{ccc} \mathcal{C}(C, A) \times \mathcal{C}(C, A) & \longrightarrow & \mathcal{C}(C, A \times B) \\ \downarrow - \circ g & & \downarrow - \circ g \\ \mathcal{C}(C', A) \times \mathcal{C}(C', B) & \longrightarrow & \mathcal{C}(C', A \times B) \end{array}$$

commutes. Hence, the isomorphism is natural in C .

(b) \implies (a) For $A, B \in \text{ob } \mathcal{C}$ take $C \in \text{ob } \mathcal{C}$ such that there exists a natural isomorphism $\eta: \mathcal{C}(-, A) \times \mathcal{C}(-, B) \Rightarrow \mathcal{C}(-, C)$. We claim that C is the product. Let $(\pi_A, \pi_B) = \eta_C^{-1}(\text{id}_C)$. Then for any $f_1: X \rightarrow A$ and $f_2: X \rightarrow B$ in \mathcal{C} , we have that

$$\begin{aligned} \eta_X(\pi_A \eta_X(f_1, f_2), \pi_B \eta_X(f_1, f_2)) &\stackrel{*}{=} \eta_C(\pi_A, \pi_B) \circ \eta_X(f_1, f_2) \\ &= \text{id}_C \circ \eta_X(f_1, f_2) = \eta_X(f_1, f_2), \end{aligned}$$

where $(*)$ holds since η is a natural transformation. Since η_X is an isomorphism,

$$\begin{array}{ccccc} & & X & & \\ & f_1 \swarrow & \downarrow \eta_X(f_1, f_2) & \searrow f_2 & \\ A & \xleftarrow{\pi_A} & C & \xrightarrow{\pi_B} & B \end{array}$$

commutes. Since $\eta_X(f_1, f_2)$ is the unique map which makes this diagram commute, C is the product of A and B . \square

The generalization of this and Exercise 56 would be

Proposition 7.1. *For $F: \mathcal{I} \rightarrow \mathcal{C}$, $\lim F$ exists in \mathcal{C} if and only if $\lim yF$ is representable, where $y: \mathcal{C} \rightarrow \text{Set}^{\mathcal{C}^{op}}$ is the Yoneda embedding.*

Firs, we prove an important lemma.

Lemma 7.2. *Let \mathcal{C} be a small category. The hom functor $\mathcal{C}(-, -): \mathcal{C}^{op} \times \mathcal{C} \rightarrow \text{Set}$ preserves limits.*

Proof. Let $F: \mathcal{I} \rightarrow \mathcal{C}$ be a diagram in \mathcal{C} such that $\lim F$ exists. By definition of limiting cones, we have a natural bijection $\eta: \mathcal{C}^{\mathcal{I}}(\Delta_{(-)}, F) \Rightarrow \mathcal{C}(-, \lim F)$ sending a natural transformation $\theta: \Delta_Y \Rightarrow F$ to the unique map $Y \rightarrow \lim F$ such that

$$\begin{array}{ccccc} & & Y & & \\ & \theta_i \swarrow & \downarrow \eta_Y(\theta) & \searrow \theta_j & \\ F_i & \xleftarrow{\quad} & \lim_i F_i & \xrightarrow{\quad} & F_j \end{array} \quad \text{commutes. Naturality follows from the fact that}$$

$$\begin{array}{ccccc} & & Y' & & \\ & (\theta \Delta_g)_i \swarrow & \downarrow g & \searrow (\theta \Delta_g)_j & \\ & & Y & & \\ & \theta_i \swarrow & \downarrow \eta_Y(\theta) & \searrow \theta_j & \\ F_i & \xleftarrow{\quad} & \lim_i F_i & \xrightarrow{\quad} & F_j \end{array}$$

commutes and the uniqueness of $\eta_{Y'}(\theta \Delta_g)$. Hence, it remains to show that we have a natural bijection $\lim_i \mathcal{C}(-, F_i) \cong \mathcal{C}^{\mathcal{I}}(\Delta_{(-)}, F)$. We define $\mu: \mathcal{C}^{\mathcal{I}}(\Delta_{(-)}, F) \Rightarrow \lim_i \mathcal{C}(-, F_i)$ by sending a natural transformation $\theta: \Delta_Y \rightarrow F$ to $g(*)$ where g is the unique map such that

$$\begin{array}{ccc}
& * & \\
\text{const}_{\theta_j} \swarrow & \downarrow g & \searrow \text{const}_{\theta_j} \\
& \lim_i \mathcal{C}(Y, F_i) & \\
\swarrow & & \searrow \\
\mathcal{C}(Y, F_i) & \xrightarrow{\quad} & \mathcal{C}(Y, F_j)
\end{array}$$

commutes. This defines a bijection since for every element x in $\lim_i \mathcal{C}(Y, F_i)$ we can assign a natural transformation $\theta: \Delta_Y \Rightarrow F$ by defining θ_i to be the image of x under the map $\lim_i \mathcal{C}(Y, F_i) \rightarrow \mathcal{C}(Y, F_i)$. Naturality of θ follows from

$$\begin{array}{ccc}
& \lim_i \mathcal{C}(Y, F_i) & \\
\swarrow & & \searrow \\
\mathcal{C}(Y, F_i) & \xrightarrow{\quad} & \mathcal{C}(Y, F_j)
\end{array}$$

the commutativity of . To prove

naturality in Y , let $f: Y' \rightarrow Y$ in \mathcal{C} and $f^*: \lim_i \mathcal{C}(Y, F_i) \rightarrow \lim_i \mathcal{C}(Y', F_i)$ be the map induced by f . Then

$$\begin{array}{ccccc}
& \lim_i \mathcal{C}(Y, F_i) & & & \\
g_i \swarrow & & g_j \searrow & f^* \searrow & \\
\mathcal{C}(Y, F_i) & \xrightarrow{\quad} & \mathcal{C}(Y, F_j) & \xrightarrow{\quad} & \lim_i \mathcal{C}(Y', F_i) \\
& \searrow -\circ f & & \swarrow g'_i & \searrow g'_j \\
& & \mathcal{C}(Y', F_i) & \xrightarrow{\quad} & \mathcal{C}(Y', F_j)
\end{array}$$

commutes, and for $\theta: \Delta_Y \Rightarrow F$ we have that

$$g'_i \mu_{Y'}(\theta \Delta_f) = (\theta \Delta_f)_i = \theta_i f = g_i \mu_Y(\theta) f = g'_i f^* \mu_Y(\theta)$$

We conclude that $\mathcal{C}(-, \lim_i F_i) \cong \lim_i \mathcal{C}(-, F_i)$.

Since the hom functor is contravariant in the first component, we need to show that $\mathcal{C}(\text{colim}_i F_i, -) \cong \lim_i \mathcal{C}(F_i, -)$, but this is just the dual of the above paragraph. \square

Proof of Proposition 7.1. (\Rightarrow) We claim that $\lim_i yF_i \cong \text{Set}(-, \lim_i F_i)$. Indeed, we have that $(\lim_i yF_i) \cong \lim_i \text{Set}(-, F_i) \cong \text{Set}(-, \lim_i F_i)$, where the last isomorphism follows from Lemma 7.2.

(\Leftarrow) Since the Yoneda embedding is fully faithful, it reflects limits, i.e. if $\lim_i yF_i \cong \text{Set}(-, X)$ for some $X \in \text{ob } \mathcal{C}$ then $\lim_i F_i \cong X$. \square

EXERCISE 58

Again, let \mathcal{C} be a small category with binary products, and let A and B be objects in \mathcal{C} .

- i. Show that the assignment

$$X \mapsto \mathcal{C}(X \times A, B)$$

is part of a functor $\mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$, with the action on morphisms $f: X' \rightarrow X$ in \mathcal{C} given by precomposition with $f \times \text{id}_A$.

- ii. What does it say about \mathcal{C} if the functor in part (i) is representable?

Solution

- i. Follows since $\text{id}_X \times \text{id}_A = \text{id}_{X \times A}$ and $f g \times \text{id}_A = (f \times \text{id}_A)(g \times \text{id}_A)$.
- ii. Claim: if we have a natural isomorphism $\eta: \mathcal{C}(- \times A, B) \Rightarrow \mathcal{C}(-, C)$ for some $C \in \text{ob } \mathcal{C}$, then $C \cong B^A$. Indeed, for $f: X \times A \rightarrow B$ we have $\eta_X(f): X \rightarrow C$ and since

$$\begin{array}{ccc} \mathcal{C}(C \times A, B) & \xrightarrow{\eta_C} & \mathcal{C}(C, C) \\ -\circ \eta_X(f) \times \text{id}_A \downarrow & & \downarrow -\circ \eta_X(f) \\ \mathcal{C}(X \times A, B) & \xrightarrow{\eta_X} & \mathcal{C}(X, C) \end{array}$$

commutes, it follows that

$$\eta_X(\eta_C^{-1}(\text{id}_C) \circ \eta_X(f) \times \text{id}_A) = \eta_C(\eta_C^{-1}(\text{id}_C)) \circ \eta_X(f) = \eta_X(f),$$

so $\eta_C^{-1}(\text{id}_C) \circ \eta_X(f) \times \text{id}_A = f$ by injectivity of η_X . In particular,

$$\begin{array}{ccc} C \times A & \xrightarrow{\eta_C^{-1}(\text{id}_C)} & B \\ \eta_X(f) \times \text{id}_A \uparrow & \nearrow f & \\ X \times A & & \end{array}$$

commutes.

EXERCISE 8.3

A forest is a poset (X, \leq) such that for any element $x \in X$ the set $\downarrow x = \{y \in X \mid y \leq x\}$ is finite and linearly ordered by \leq . If this set has $n + 1$ elements, we say that x has depth n . Write \mathbf{For} for the category of forests and monotone, depth-preserving maps.

Show that \mathbf{For} is isomorphic to the category of presheaves on \mathbb{N} (considered as a poset in the usual way).

Solution

Proof. We need to define functors $F: \mathbf{For} \rightarrow \mathbf{Set}^{\mathbb{N}^{\text{op}}}$ and $G: \mathbf{Set}^{\mathbb{N}^{\text{op}}} \rightarrow \mathbf{For}$ such that $FG = \text{id}_{\mathbf{Set}^{\mathbb{N}^{\text{op}}}}$ and $GF = \text{id}_{\mathbf{For}}$. \square

EXERCISE 59

Prove that $y: \mathcal{C} \rightarrow \mathbf{Set}^{\mathcal{C}^{\text{op}}}$ preserves all limits which exist in \mathcal{C} . Prove also that if \mathcal{C} is cartesian closed, y preserves exponents.

Solution

Proof. The first claim is proved in Proposition 7.1. For the second claim, we want to prove that $y(Y^X) \cong (yY)^{yX}$. Well, using the Yoneda lemma and the fact the y preserves products we have

$$\begin{aligned} y(Y^X)(C) &= \mathcal{C}(C, Y^X) \cong \mathcal{C}(C \times X, Y) = yY(C \times X) \cong \mathbf{Set}^{\mathcal{C}^{\text{op}}}(y(C \times X), yY) \\ &\cong \mathbf{Set}^{\mathcal{C}^{\text{op}}}(yC \times yX, yY) \cong \mathbf{Set}^{\mathcal{C}^{\text{op}}}(yC, (yY)^{yX}) \cong (yY)^{yX}(C). \end{aligned}$$

□

8 PRESHEAVES AS A TOPOS

EXERCISE 9.2

Let \mathcal{C} be a category with pullbacks.

- i. Show that $\text{Sub}(X)$ is a meet semi-lattice for each object X in \mathcal{C} .
- ii. Show that if $f: Y \rightarrow X$ is a morphism in \mathcal{C} , then $f^*: \text{Sub}(X) \rightarrow \text{Sub}(Y)$ is a morphism of meet semi-lattices.
- iii. Show that we have a presheaf

$$\text{Sub}: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}.$$

Solution

- i. *Proof.* Consider two monomorphisms $A \xrightarrow{n} X \xleftarrow{m} B$. Then we can define

$$n \wedge m \text{ to be any composition along the pullback square } \begin{array}{ccc} C & \longrightarrow & B \\ \downarrow & & \downarrow_m \\ A & \xrightarrow{n} & X \end{array}, \text{ and}$$

note that pullbacks of monos are monos (Exercise 25). It is straightforward to verify that this defines a meet operation. If \mathcal{C} has an initial object, then $\text{Sub}(X)$ has a bottom element. □

- ii. *Proof.* For $n: A \rightarrow X$ in $\text{Sub}(X)$, define $f^*(n)$ to be the pullback of n

$$\text{along } f, \text{ i.e. } \begin{array}{ccc} C & \longrightarrow & A \\ f^*(n) \downarrow & & \downarrow_n \\ Y & \xrightarrow{f} & X \end{array}. \quad \square$$

iii. *Proof.* To show that $\text{Sub}(fg) = \text{Sub}(g)\text{Sub}(f)$ note that we have the commutative diagram

$$\begin{array}{ccccc} C & \longrightarrow & B & \longrightarrow & A \\ (fg)^*(n) \downarrow & & \downarrow f^*(n) & & \downarrow n \\ Z & \xrightarrow{g} & Y & \xrightarrow{f} & X \end{array}$$

where the right and composite squares are pullbacks, and by Exercise 26 it follows that the left square is a pullback, i.e. $(fg)^*(n) = g^*(f^*(n))$. To

show that $\text{Sub}(\text{id}_X) = \text{id}_{\text{Sub}(X)}$ observe that $\begin{array}{ccc} A & \xrightarrow{\text{id}_A} & A \\ \downarrow n & & \downarrow n \\ X & \xrightarrow{\text{id}_X} & X \end{array}$ is a pullback square.

□

EXERCISE 9.3

Show that in **Set** we have for each set X an isomorphism of posets:

$$\text{Sub}(X) \cong (\mathcal{P}(X), \subseteq).$$

Solution

Proof. The isomorphism is given by sending each monomorphism $n: A \rightarrow X$ in $\text{Sub}(X)$ to $\text{im } n$ and each subset $A \subseteq X$ to the inclusion map $i_A: A \rightarrow X$. These maps are morphisms of posets since for $m \leq n$ we have $m(A) = nk(A)$ so

$\text{im } m \subseteq \text{im } n$ and if $A \subseteq B$ then we have the commuting diagram $\begin{array}{ccc} A & \xrightarrow{i} & B \\ \downarrow i_A & \searrow i_B & \\ X & & \end{array}$

of inclusion maps, so $i_A \leq i_B$.

□

EXERCISE 60

Let (P, \leq) be a preorder. For $p \in P$ we let $\downarrow p = \{q \in P \mid q \leq p\}$. Show that sieves on p can be identified with downward closed subsets of $\downarrow p$. If we denote the unique arrow $q \rightarrow p$ by qp and U is a downward closed subset of $\downarrow p$, what is $(qp)^*U$?

Solution

Proof.

□

EXERCISE 61

Let \mathcal{C} be a small category. Show that the power object in the category of presheaves on \mathcal{C} can be identified as follows $\mathcal{P}(X)(C) = \text{Sub}(X \times y_C)$ and that for $f: C' \rightarrow C$, $\mathcal{P}(X)(f)(U) = (\text{id}_X \times y_f)^*(U)$. Prove also that the element relation, as a subpresheaf ϵ_X of $\mathcal{P}(X) \times X$ is given by

$$(\epsilon_X)(C) = \{(U, x) \in \text{Sub}(y_C \times X) \times X(C) \mid (\text{id}_C, x) \in U(C)\}.$$

Solution

Proof.

□

EXERCISE 62

Solution

Proof.

□

EXERCISE 63

Solution

Proof.

□