# CTF22: Solutions

Yoav Eshel, 12973351 Universiteit van Amsterdam yoav.eshel@student.uva.nl

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# 1 Categories

- 2 Functors and Constructions on Categories
- 3 Natural Transformations and Equivalences

# 4 Limits and Colimits

# Exercise 21

Show that a full and faithful functor reflects the property of being a terminal (initial object).

## Solution

*Proof.* Let  $F: \mathcal{C} \to \mathcal{D}$  be a fully faithful functor, and  $X \in \text{ob } \mathcal{C}$  such that FX is the terminal object of  $\mathcal{D}$ . Then for any  $Y \in \mathcal{C}$  we have  $\{*\} \cong \text{Hom}_{\mathcal{D}}(FY, FX) \cong \text{Hom}_{\mathcal{C}}(Y, X)$  so X is terminal in  $\mathcal{C}$ . We Similarly show that F reflects the property of being initial.

## Exercise 22

Show that every equalizer is monic.

# Solution

*Proof.* Let  $e: E \to X$  be the equalizer of  $f_1, f_2: X \rightrightarrows Y$ , and take  $g_1, g_2: T \to Y$ 

$$E$$
 such that  $eg_1 = eg_2$ . Since both  $g_1$  and  $g_2$  make  $F \xrightarrow{e} X \xrightarrow{f_1} Y$ 

$$\downarrow f_2 \\ eg_1 = eg_2$$

$$T$$

commute, by uniqueness it follows that  $g_1 = g_2$ , so e is monic.

Exercise 23

Let  $E \xrightarrow{e} X \xrightarrow{f_1} Y$  be an equalizer diagram. Show that e is iso if and only if  $f_1 = f_2$ .

#### Solution

*Proof.*  $(\Rightarrow)$  Immediate, since

$$f_1 = f_1 \operatorname{id}_X = f_1 e e^{-1} = f_2 e e^{-1} = f_2 \operatorname{id}_X = f_2.$$

( $\Leftarrow$ ) If  $f_1 = f_2$  then the there exists a unique  $k \colon X \to E$  such that  $ek = \operatorname{id}_X$ . Then  $eke = e\operatorname{id}_E$  and since e is monic by the previous exercise it follows that  $ke = 1_E$ , so e is an isomorphism.

Exercise 24

Show that in Set, every monomorphism fits into an equalizer diagram.

#### Solution

*Proof.* Let  $f: X \to Y$  be a monomorphism in Set. Define  $g_1, g_2: Y \rightrightarrows \{0, 1\}$  by  $g_1(y) = 1$  and

$$g_2(y) = \begin{cases} 1, & y \in \text{im } f \\ 0, & \text{otherwise} \end{cases}$$
.

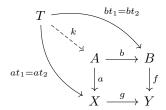
It is clear that  $g_2f = g_1f$ . Let  $h: T \to Y$  be any map such that  $g_1h = g_2h$ . It follows that im  $h \subseteq \text{im } f$  and so we can define  $k: T \to X$  by  $k(t) = f^{-1}h(t)$ . Then fk = h and k is unique, so f is the equalizer of  $g_1, g_2: Y \rightrightarrows \{0, 1\}$ .  $\square$ 

Exercise 25

 $A \xrightarrow{b} B$  Let  $\downarrow a \qquad \downarrow f$  be a pullback diagram with f monic. Show that a is also  $X \xrightarrow{g} Y$ 

monic. Also, if f is iso, so is a.

*Proof.* Let  $t_1, t_2 \colon T \rightrightarrows A$  such that  $at_1 = at_2$ . Then  $fbt_1 = gat_1 = gat_2 = fbt_2$  and since f is monic,  $bt_1 = bt_2$ . Hence, there exists a unique  $k \colon T \to A$  such that



commute. Since either  $k=t_1$  or  $k=t_2$  would work, it must be the case that  $t_1=t_2$ .

## Exercise 26

Given two commuting squares

$$\begin{array}{ccc} A \stackrel{b}{\longrightarrow} B \stackrel{c}{\longrightarrow} C \\ \downarrow^a & \downarrow^f & \downarrow^d \cdot \\ X \stackrel{g}{\longrightarrow} Y \stackrel{h}{\longrightarrow} Z \end{array}$$

Show that

- i. if both squares are pullback squares, then so is the composite square;
- **ii.** if the right square and the composite square are pullbacks, then so is the left square.

# Solution

Exercise 27

## Solution

Proof. 
$$\Box$$

Exercise 28

## Solution

Proof.

Exercise 29 Solution

Proof.

Exercise 30

Solution

Proof.

# 5 Complete Categories

#### Exercise 42

Take one of your favourite categories (Top, Pos, Rng, Mon, Grp, Grph, Cat) and show that it is both complete and cocomplete.

**Solution** Consider Grp. By Proposition 5.1, it is enough to prove that Grp has all small products and equalizers. The equalizer of  $f,g:G \rightrightarrows H$  is the pair (G',i) where  $G'=\{a\in G\mid f(a)=g(a)\}$  and i is the inclusion map. Note the G' is a group since f and g are group homomorphisms. For a set of groups  $\{G_i\}_{i\in I}$ , the product  $\prod_{i\in I}G_i$  has a group structure by performing the group operations pointwise.

## Exercise 43

Show that if  $\mathcal{C}$  is complete, then  $F \colon \mathcal{C} \to \mathcal{D}$  preserves all limits if F preserves products and equalizers. This no longer holds if  $\mathcal{C}$  is not complete: F may preserve all products and equalizers which exists in  $\mathcal{C}$ , yet not preserve all limits which exists in  $\mathcal{C}$ .

**Solution** I assume the question means small limits, since I don't think this holds for all limits.

*Proof.* Suppose  $\mathcal{C}$  is complete and  $F \colon \mathcal{C} \to \mathcal{D}$  preserves products and equalizers. Let  $G \colon \mathcal{I} \to \mathcal{C}$  be a small diagram. Then  $\lim_{\mathcal{L}} G$  can be expressed as

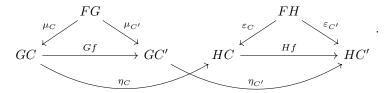
$$\lim_{\mathcal{I}} G \longrightarrow \prod_{i \in \text{ob } \mathcal{I}} Gi \xrightarrow{c} \prod_{f \in \text{mor } \mathcal{I}} G(\text{cod } f),$$

and since F preserves equalizers and products,  $F(\lim_{\mathcal{I}} G) = \lim_{\mathcal{I}} (FG)$ .

#### Exercise 44

Suppose a category  $\mathcal{C}$  has limits of shape  $\mathcal{I}$ . Show that the operation which assigns each diagram  $\mathcal{I} \to \mathcal{C}$  to its limit in  $\mathcal{C}$  is part of a functor  $F: [\mathcal{I}, \mathcal{C}] \to \mathcal{C}$ .

*Proof.* Let  $\eta: G \Rightarrow H$  be a morphism in  $[\mathcal{I}, \mathcal{C}]$ ,  $\mu: \Delta_{FG} \Rightarrow G$ ,  $\varepsilon: \Delta_{FH} \Rightarrow H$  and  $f: C \to C'$  a morphism in  $\mathcal{I}$ . This is summarized in the following diagram



Note also that the diagram commutes. Thus,  $(FG, \eta\mu)$  is a cone for the diagram H, so there is a unique morphism  $g \colon FG \to FH$  such that  $\varepsilon \Delta_g = \eta \mu$ . We define  $F\eta := g$ . It is straightforward to verify that the uniqueness of g turns F into a functor.

#### Exercise 45

Let  $\mathcal{C}, \mathcal{D}$  and  $\mathcal{E}$  be categories. Show that the following categories are isomorphic:

$$[\mathcal{E}, [\mathcal{C}, \mathcal{D}]] \cong [\mathcal{E} \times \mathcal{C}, \mathcal{D}] \cong [\mathcal{C}, [\mathcal{E}, \mathcal{D}]].$$

Use this and the previous exercise to give a more elegant proof of Theorem 4.5.

## Solution

*Proof.* Consider the functors  $[\mathcal{E}, [\mathcal{C}, \mathcal{D}]] \xrightarrow{F_1} [\mathcal{E} \times \mathcal{C}, \mathcal{D}] \xrightarrow{F_2} [\mathcal{C}, [\mathcal{E}, \mathcal{D}]]$  given by

• On objects: for  $G \colon \mathcal{E} \to [\mathcal{C}, \mathcal{D}], H \colon \mathcal{E} \times \mathcal{C} \to \mathcal{D}, f \colon E \to E'$  in  $\mathcal{E}$  and  $g \colon C \to C'$  in  $\mathcal{C}$  we have

$$F_1(G)(E,C) = G(E)(C)$$

$$F_1(G)(f,g) = (G(E')(g))(Gf)_C$$

$$F_2(H)(C)(E) = H(E,C)$$

$$F_2(H)(C)(f) = H(f, id_C)$$

$$(F_2(H)(g))_E = H(id_E, f).$$

• On morphisms: for  $G_1, G_2 : \mathcal{E} \Rightarrow [\mathcal{C}, \mathcal{D}], H_1, H_2 : \mathcal{E} \times \mathcal{C} \Rightarrow \mathcal{D}, \eta : G_1 \Rightarrow G_2$ and  $\varepsilon : H_1 \Rightarrow H_2$  we have  $(F_1\eta)_{(E,C)} = (\eta_E)_C$  and  $((F_2\varepsilon)_C)_E = \varepsilon_{(E,C)}$ .

Since these functors are clearly invertible, they are isomorphisms of categories.

# Exercise 46

Show that a full and faithful functor reflects the property of being a terminal (or initial) object. Deduce that equivalences preserve the terminal (or initial) object.

*Proof.* Let  $F: \mathcal{C} \to \mathcal{D}$  be a fully faithful functor and  $X \in \text{ob } \mathcal{C}$  such that FX is terminal and take any  $Y \in \text{ob } \mathcal{C}$ . Then  $\text{Hom}_{\mathcal{C}}(Y,X) \cong \text{Hom}_{\mathcal{D}}(FY,FX) \cong \{*\}$ , so X is terminal in  $\mathcal{C}$ . Similarly, we show that fully faithful functors reflect the property of being initial.

Hence, if F is an equivalence,  $X \in \text{ob } C$  is terminal,  $Z \in \text{ob } \mathcal{D}$  is any object and  $Y \in \text{ob } \mathcal{C}$  is chosen such that  $FY \cong Z$  we have that  $\{*\} = \text{Hom}_{\mathcal{C}}(Y, X) \cong \text{Hom}_{\mathcal{D}}(FY, FX) \cong \text{Hom}_{\mathcal{D}}(Z, FX)$  so FX is terminal in  $\mathcal{D}$ .

## 6 Cartesian Closed Categories

#### Exercise 47

Show that in a ccc, there are natural isomorphisms

- **i.**  $1^X \cong 1$ .
- ii.  $(Y \times Z)^X \cong Y^X \times Z^X$ ,
- iii.  $(Y^Z)^X = Y^{Z \times X}$ .

**Solution** Let  $\mathcal{C}$  be a ccc category and X, Y, Z and A objects in  $\mathcal{C}$ .

i. Proof. Since 1 is terminal we have unique morphisms  $1 \times X \to 1$ ,  $A \times X \to 1$ 

 $1^X \cong 1$ . Moreover, this isomorphism is natural as a morphism  $1^{(-)} \Rightarrow 1$ 

since all the maps in the square 
$$\begin{array}{c} 1^X \longrightarrow 1^Y \\ \downarrow \\ 1 \longrightarrow 1 \end{array} \text{ are the identity maps. } \square$$

ii. Proof. Let  $\eta_{(Y,Z)} \colon Y^X \times Z^X \to (Y \times Z)^X$  be the unique map which makes

$$(Y\times Z)^X\times X \xrightarrow{\operatorname{ev}_{X,Y\times Z}} Y\times Z$$
 
$$\uparrow_{\eta_{(Y,Z)\times 1_X}} \uparrow_h$$
 
$$Y^X\times Z^X\times X$$

commute, where  $h=(\operatorname{ev}_{X,Y}(\pi_{Y^X}\times 1_X),\operatorname{ev}_{X,Z}(\pi_{Z^X}\times 1_X))$ . To construct an inverse, let  $f_Y\colon (Y\times Z)^X\to Y^X$  be the unique map such that

$$Y^X \times X \xrightarrow{\operatorname{ev}_{X,Y}} Y \\ \uparrow_{Y \times 1_X} \qquad \uparrow^{\pi_Y \operatorname{ev}_{X,Y \times Z}} \\ (Y \times Z)^X \times X$$

commutes. Analogously, we define  $f_Z \colon (Y \times Z)^X \to Z^X$ . We claim that  $\mu_{(Y,Z)} = (f_Y, f_Z) \colon (Y \times Z)^X \to Y^X \times Z^X$  is the inverse of  $\eta_{(Y,Z)}$ . Since

$$(Y \times Z)^{X} \times X$$

$$f_{Y} \times 1_{X} \qquad \mu_{(Y,Z)} \times 1_{X} \qquad f_{Z} \times 1_{X}$$

$$Y^{X} \times X \underset{\pi_{Y} \times \times 1_{X}}{\longleftrightarrow} Y^{X} \times Z^{X} \times X \underset{\pi_{Z} \times \times 1_{X}}{\longleftrightarrow} Z^{X} \times X$$

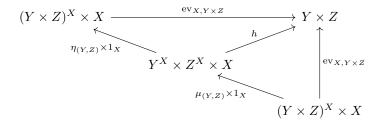
$$ev_{X,Y} \qquad \qquad \downarrow h \qquad \qquad \downarrow ev_{X,Z}$$

$$Y \leftarrow \xrightarrow{\pi_{Y}} Y \times Z \xrightarrow{\pi_{Z}} Z$$

commutes, it follows that

$$h(\mu_{(Y,Z)} \times 1_X) = (ev_{X,Y}(f_Y \times 1_X), ev_{X,Z}(f_Z \times 1_X))$$
$$= (\pi_Y ev_{X,Y \times Z}, \pi_Z ev_{X,Y \times Z})$$
$$= ev_{X,Y \times Z}$$

and so



commutes (the top left triangle commutes by definition of  $\eta_{(Y,Z)}$  and we've just show that the bottom right triangle commute, so the big triangle commutes as well). By uniqueness,  $\eta_{(Y,Z)}\mu_{(Y,Z)}=\mathrm{id}_{(Y\times Z)^X}$ . By definition of  $f_Y$  and  $\eta_{(Y,Z)}$ ,

$$Y^{X} \times X \xrightarrow{\text{ev}_{X,Y}} Y$$

$$(Y \times Z)^{X} \times X \xrightarrow{\text{ev}_{X,Y} \times Z} \uparrow \pi_{Y}$$

$$(Y \times Z)^{X} \times X \xrightarrow{\text{ev}_{X,Y} \times Z} Y \times Z$$

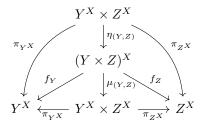
$$\uparrow_{\eta_{(Y,Z)} \times 1_{X}} \uparrow_{h}$$

$$Y^{X} \times Z^{X} \times X$$

commutes and so both  $\pi_{YX} \times 1_X$  and  $(f_Y \eta_{Y,Z}) \times 1_X$  make

$$Y^X \times X \xrightarrow{\operatorname{ev}_{X,Y}} Y \\ \uparrow^{\pi_Y h} \\ Y^X \times Z^X \times X$$

commute, so  $f_Y \eta_{(Y,Z)} = \pi_{Y^X}$ . Analogously,  $f_Z \eta_{(Y,Z)} = \pi_{Z^X}$ . Thus,



commutes, and by uniqueness it follows that  $\mu_{(Y,Z)}\eta_{(Y,Z)} = 1_{Y^X \times Z^X}$ .

It is left to show that this gives a natural isomorphism  $\eta : (-)^X \times (-)^X \Rightarrow (-\times -)^X$  of functors  $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ . Since a morphism of bifunctors is natural if and only if it is natural in each component and the definitions are symmetric in their components, it suffices to check that

$$Y^{X} \times Z^{X} \xrightarrow{f^{X} \times 1_{Z^{X}}} (Y')^{X} \times Z^{X}$$

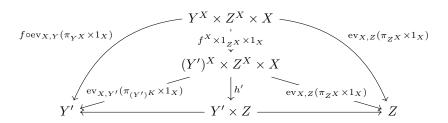
$$\downarrow^{\eta_{(Y,Z)}} \qquad \downarrow^{\eta_{(Y',Z)}}$$

$$(Y \times Z)^{X} \xrightarrow{(f \times 1_{Z})^{X}} (Y' \times Z)^{X}$$

commutes. Since  $\operatorname{ev}_{X,-}: (-)^X \times X \Rightarrow 1_{\mathcal{C}}$  is a natural transformation, we have that

$$\operatorname{ev}_{X,Y'\times Z} \circ (f \times 1_Z)^X \times 1_X \circ \eta_{Y,Z} \times 1_X = (f \times 1_Z) \circ \operatorname{ev}_{X,Y\times Z} \circ \eta_{(Y,Z)\times 1_X}$$
$$= (f \times 1_Z)h$$

and



commutes, so

$$\operatorname{ev}_{X,Y'\times Z} \circ \eta_{(Y',Z)} \times 1_X \circ f^X \times 1_{Z^X} \times 1_X = h' \circ f^X \times 1_{Z^X} \times 1_X$$
$$= (f \times 1_Z)h.$$

It follows that both  $(f \times 1_Z)^X \eta_{(Y,Z)}$  and  $\eta_{(Y',Z)}(f^X \times 1_{Z^X})$  make

$$(Y' \times Z)^X \times X \xrightarrow{\operatorname{ev}_{X,Y' \times Z}} Y' \times Z$$

$$\uparrow (f \times 1_Z)h$$

$$Y^X \times Z^X \times X$$

commute, and so they are equal.

**iii.** Proof. Define  $f = \operatorname{ev}_{Z,Y} \circ \operatorname{ev}_{X,Y^Z} \times 1_Z \colon (Y^Z)^X \times X \times Z \to Y$ . Then for any  $h \colon A \times X \times Z \to Y$  we get a map  $H' \colon A \times X \to Y^Z$  such that  $\operatorname{ev}_{Z,Y} \circ H' \times 1_Z = h$  which gives a unique map  $H \colon A \to (Y^Z)^X$  such that  $\operatorname{ev}_{X,Y^Z} \circ H \times 1_X = H'$ . Since

$$\begin{split} f \circ H \times \mathbf{1}_{X \times Z} &= \operatorname{ev}_{Z,Y} \circ \operatorname{ev}_{X,Y^Z} \times \mathbf{1}_Z \circ H \times \mathbf{1}_{X \times Z} \\ &= \operatorname{ev}_{Z,Y} \circ (\operatorname{ev}_{X,Y^Z} \circ H \times \mathbf{1}_X) \times \mathbf{1}_Z \\ &= \operatorname{ev}_{Z,Y} \circ H' \times \mathbf{1}_Z \\ &= h, \end{split}$$

we conclude that  $(Y^Z)^X \cong Y^{Z \times X}$ .

Exercise 48

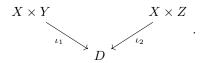
If a ccc has coproducts, we have

i. 
$$X \times (Y + Z) \cong (X \times Y) + (X \times Z)$$

ii. 
$$Y^{Z+X} = Y^Z \times Y^X$$
.

# Solution

i. Proof. Suppose C is a ccc which has coproducts, and let X,Y and Z be objects in C. Consider any cocone

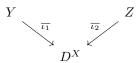


Exponentiating, we get

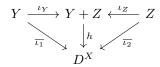
$$X \times Y \xrightarrow{\overline{\iota_1} \times 1_X} D^X \times X \xleftarrow{\overline{\iota_2} \times 1_X} X \times Z$$

$$\downarrow^{\text{ev}_{X,D}} \downarrow^{\text{ev}_{X,D}}$$

where  $\overline{\iota_1}$  and  $\overline{\iota_2}$  are the transposes of  $\iota_1$  and  $\iota_2$  respectively. Hence, we have the cocone



and since Y+Z is limiting, we get a unique map  $h\colon Y+Z\to D^X$  so that



commutes. Then

$$Y \times X \xrightarrow{\iota_{Y} \times 1_{X}} (Y + Z) \times X \xleftarrow{\iota_{Z} \times 1_{X}} Z \times X$$

$$\downarrow h \times 1_{X} \qquad \downarrow h \times 1_{X} \qquad \downarrow ev_{X,D}$$

$$\downarrow D$$

$$\downarrow D$$

commutes and it follows that  $(Y+Z)\times X$  is the coproduct of  $Y\times X$  and  $Z\times X$ . Thus  $Y\times X+Z\times X\cong (Y+Z)\times X$ 

ii. Proof. Note that  $Y^Z \times Y^X(Z+X) \cong Y^X \times Y^Z \times Z + Y^Z \times Y^X \times X$  and let  $f: Y^Z \times Y^X(Z+X) \to Y$  be given by the composition

$$Y^X \times Y^Z \times Z + Y^Z \times Y^X \times X \xrightarrow{1_{Y^X} \times \operatorname{ev}_{Z,Y} + 1_{Y^Z} \times \operatorname{ev}_{X,Y}} Y$$
$$Y^X \times Y + Y^Z \times Y \cong (Y^X + Y^Z) \times Y \xrightarrow{\pi_Y} Y.$$

Let  $h: A \times (Z+X) \to Y$  be any morphism. Since  $A \times (Z+X) \cong A \times Z + A \times X$  we have unique maps  $H_Z: A \to Y^Z$  and  $H_X: A \to Y^X$  such that

$$Y^Z \times Z \xrightarrow{\operatorname{ev}_{Z,Y}} Y \qquad Y^X \times X \xrightarrow{\operatorname{ev}_{X,Y}} Y \\ \uparrow_{L_X \times 1_Z} \uparrow_{\iota_{A \times Z} h} \qquad \uparrow_{L_X \times 1_X} \uparrow_{L_X \times h} \\ A \times Z \qquad A \times X$$

commute, where  $A \times Z \xrightarrow{\iota_{A \times Z}} A \times Z + A \times X \xleftarrow{\iota_{A \times X}} A \times X$  are the inclusion maps. Let  $H = (H_Z, H_X) \colon A \to Y^Z \times Y^X$ . Since

$$\pi_Y \circ (1_{Y^X} \times ev_{Z,Y}) \circ (H \times 1_Z) = \pi_Y \circ H_X \times (ev_{Z,Y} \circ H_Z \times 1_Z) = \iota_{A \times Z} h$$

and similarly 
$$\pi_Y \circ (1_{YZ} \times \text{ev}_{X,Y}) \circ (H \times 1_X) = \iota_{A \times X} h$$
, it follows that

$$\begin{split} h &= \iota_{A \times Z} h + \iota_{A \times X} h \\ &= \pi_Y \circ (1_{Y^X} \times \operatorname{ev}_{Z,Y}) \circ (H \times 1_Z) + \pi_Y \circ (1_{Y^Z} \times \operatorname{ev}_{X,Y}) \circ (H \times 1_X) \\ &= f \circ (H \times 1_Z + H \times 1_X) \\ &= f \circ (H \times 1_{Z+X}). \end{split}$$

$$Y^X \times Y^Z \times (Z+X) \xrightarrow{\quad f \quad } Y$$
 Hence, 
$$\uparrow_h \quad \text{commutes and so } Y^{Z+X} \cong X \times (Z+X)$$
 
$$Y^Z \times Y^X. \qquad \square$$

## Exercise 49

In a ccc, prove that the transpose of a composite  $Z \xrightarrow{g} W \xrightarrow{f} Y^X$  is

$$Z \times X \xrightarrow{g \times 1_X} W \times X \xrightarrow{\bar{f}} Y$$
,

if  $\bar{f}$  is the transpose of f.

#### Solution

*Proof.* We need to find a morphism  $\overline{fg} \colon Z \times X \to Y$  such that  $\operatorname{ev}_{X,Y} \circ (f \circ g \times 1_X) = \overline{fg}$ . Well,

$$\operatorname{ev}_{X,Y}\circ(f\circ g\times 1_X)=\operatorname{ev}_{X,Y}\circ(f\times 1_X)\circ(g\times 1_X)=\bar{f}\circ(g\times 1_X),$$
 so we're done.   

#### Exercise 50

Suppose  $\mathcal{C}, \mathcal{D}, \mathcal{E}$  are categories, such that

- 1. For each pair of objects  $(C, D) \in ob(\mathcal{C} \times \mathcal{D})$ , an object  $F_0(C, D)$  in  $\mathcal{E}$ ;
- 2. For each object  $C \in \text{ob } \mathcal{C}$ , a functor  $F_C \colon \mathcal{D} \to \mathcal{E}$  satisfying  $F_C(D) = F_0(C, D)$  for each  $D \in \text{ob } \mathcal{D}$ ;
- 3. For each object  $D \in \text{ob } \mathcal{D}$ , a functor  $F_D \colon \mathcal{C} \to \mathcal{E}$  satisfying  $F_D(C) = F_0(C, D)$  for each object  $C \in \mathcal{D}$ ;

such that for each pair of morphism  $f: C \to C'$  in  $\mathcal{C}$  and  $g: D \to D'$  in  $\mathcal{D}$  we have a commuting square

$$F_0(C,D) \xrightarrow{F_D(f)} F_0(C',D)$$

$$\downarrow^{F_C(g)} \qquad \downarrow^{F_{C'}(g)}$$

$$F_0(C,D') \xrightarrow{F_{D'}(f)} F_0(C',D')$$

in  $\mathcal{E}$ .

Show that there is a unique functor  $F: \mathcal{C} \times \mathcal{D} \to \mathcal{E}$  whose operation on objects in  $F_0$ , while  $F(1_C, g) = F_C(g)$  and  $F(f, 1_D) = F_D(f)$ .

## Solution

*Proof.* For  $f: C \to C'$  in  $\mathcal{C}$  and  $g: D \to D'$  in  $\mathcal{D}$  we define  $F(f,g): F_0(C,D) \to F_0(C',D')$  to be any composition around the commuting square

$$F_0(C,D) \xrightarrow{F_D(f)} F_0(C',D)$$

$$\downarrow^{F_C(g)} \qquad \downarrow^{F_{C'}(g)} \cdot$$

$$F_0(C,D') \xrightarrow{F_{D'}(f)} F_0(C',D')$$

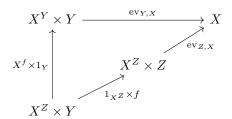
This is easily checked to give a functor  $F: \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ .

#### Exercise 51

An object Y in a category with finite product is called *exponentiating* if the exponential  $X^Y$  exists for each  $Y \in \text{ob } \mathcal{C}$ . Show that if X is exponentiating, the assignment  $Y \mapsto X^Y$  is the object part of a functor  $\mathcal{C}^{\text{op}} \to \mathcal{C}$ .

## Solution

*Proof.* For  $f: Y \to Z$ , we define  $X^f: X^Z \to X^Y$  to be the unique map which makes



commute. From uniqueness, we get that  $X^{1_Y}=1_{X^Y}$  and for  $X^{gf}=X^fX^g$  for  $Y\xrightarrow{f}Z\xrightarrow{g}W$ . Hence,  $X^{(-)}\colon\mathcal{C}^{\mathrm{op}}\to\mathcal{C}$  is a functor.

## Exercise 52

Show that for every cartesian closed category  $\mathcal{C}$  there is a functor  $\mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$ , assigning  $Y^X$  to (X,Y).

*Proof.* Checking the conditions of Exercise 50:

- 1. for each object (X,Y) in  $\mathcal{C}^{op} \times \mathcal{C}$  we have the object  $Y^X$  in  $\mathcal{C}$ ;
- 2. for each  $X \in \text{ob } \mathcal{C}^{\text{op}}$  we have the functor  $(-)^X : \mathcal{C} \to \mathcal{C}$ ;
- 3. for each  $Y \in \text{ob } \mathcal{C}^{\text{op}}$  we have the functor  $Y^{(-)} : \mathcal{C}^{\text{op}} \to \mathcal{C}$ .

It is left to show that for any  $f: X_2 \to X_1$  and  $g: Y_1 \to Y_2$  in  $\mathcal{C}$ ,

$$Y_1^{X_1} \xrightarrow{Y_1^f} Y_1^{X_2}$$

$$\downarrow g^{X_1} \qquad \downarrow g^{X_2}$$

$$Y_2^{X_1} \xrightarrow{Y_2^f} Y_2^{X_2}$$

commutes<sup>1</sup>. Since

$$Y_1^{X_1} \times X_2 \xrightarrow{Y_1^f \times 1_{X_2}} Y_1^{X_2} \times X_2 \xrightarrow{g^{X_2} \times 1_{X_2}} Y_2^{X_2} \times X_2$$

$$\downarrow^{1_{Y_1^{X_1}} \times f} \qquad \qquad \downarrow^{\text{ev}_{X_2, Y_1}} \qquad \downarrow^{\text{ev}_{X_2, Y_2}}$$

$$Y_1^{X_1} \times X_1 \xrightarrow{\text{ev}_{X_1, Y_1}} Y_1 \xrightarrow{g} Y_2$$

commutes<sup>2</sup>, we have that

$$\operatorname{ev}_{X_2,Y_2}(g^{X_2}Y_1^f\times 1_{X_2}) = g\operatorname{ev}_{X_2,Y_1}(Y_1^f\times 1_{X_2}) = g\operatorname{ev}_{X_1,Y_1}(1_{Y_1^{X_1}}\times f),$$

and since

$$\begin{array}{c} Y_2^{X_2} \times X_2 & \xrightarrow{\operatorname{ev}_{X_2,Y_2}} & Y_2 \\ Y_2^f \times 1_{X_2} & \xrightarrow{\operatorname{ev}_{X_1,Y_2}} & g \\ Y_2^{X_1} \times X_2 & \xrightarrow{1_{Y_2^{X_1}} \times f} & Y_2^{X_1} \times X_1 & Y_1 \\ g^{X_1} \times 1_{X_2} & & g^{X_1} \times 1_{X_1} & \operatorname{ev}_{X_1,Y_1} \\ Y_1^{X_1} \times X_2 & \xrightarrow{1_{Y_1^{X_1}} \times f} & Y_1^{X_1} \times X_1 \end{array}$$

commutes<sup>3</sup>, then

$$\operatorname{ev}_{X_2,Y_2}(Y_2^f g^{X_1} \times 1_{X_2}) = \operatorname{ev}_{X_1,Y_2}(1_{Y_2^{X_1}} \times f)(g^{X_1} \times 1_{X_1}) = g \operatorname{ev}_{X_1,Y_1}(1_{Y_1^{X_1}} \times f).$$

Hence, both  $Y_2^f g^{X_1}$  and  $g^{X_2} Y_1^f$  make

<sup>&</sup>lt;sup>1</sup>In Set,  $Y^X = \text{Hom}(X,Y)$  so the functor acts functions by sending  $h: X_1 \to Y_1$  to  $ghf: X_2 \to Y_2$ . Thus, the commutativity of the diagram is equivalent to function composition being associative.

<sup>&</sup>lt;sup>2</sup>left square commutes by definition of  $Y_1^f$ , right square commutes by definition of  $g^{X_2}$ <sup>3</sup>top square commutes by definition of  $Y_2^f$ , bottom square commutes because  $-\times-:\mathcal{C}\times\mathcal{C}\to\mathcal{C}$  is a bifunctor and the right triangle commutes by definition of  $g^{X_1}$ 

commute, and so by the universal property of  $Y_2^{X_2}$  they are equal.

Exercise 53

Let A be the unique function making

commute. Show that the addition function is represented by the transpose of A.

## Solution

*Proof.* Let  $a: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$  be the unique map such that

$$\begin{array}{c}
\mathbb{N}^{\mathbb{N}} \times \mathbb{N} \xrightarrow{\operatorname{ev}_{\mathbb{N},\mathbb{N}}} \mathbb{N} \\
A \times 1_{\mathbb{N}} \uparrow & a \\
\mathbb{N} \times \mathbb{N}
\end{array}$$

commutes. Then for  $m,n\in\mathbb{N}$  we have that  $a(0,n)=\operatorname{ev}_{\mathbb{N},\mathbb{N}}(A\times 1_{\mathbb{N}})(0,n)=A(0)(n)=n$  and a(Sn,m)=A(Sn)(m)=

Exercise 54

Solution

Proof.  $\Box$ 

# 7 Presheaves

Exercise 55

Suppose object A and B are such that for every object X in C there is a bijection  $f_X : C(A, X) \to C(B, X)$ , naturally in a sense you define yourself. Conclude that A and B are isomorphic.

*Proof.* The assumption is equivalent to  $f \colon \mathcal{C}(A,-) \Rightarrow \mathcal{C}(B,-)$  being a natural isomorphism of functors  $\mathcal{C} \to \mathsf{Set}$ , i.e. an isomorphism in  $\mathsf{Set}^{\mathcal{C}}$ .

Let  $F: \mathcal{C}^{\mathrm{op}} \to \mathcal{C}^{\mathsf{Set}}$  be the contravariant hom functor, i.e.  $FX = \mathcal{C}(X, -)$ . Then the co-Yoneda Lemma states that

$$\mathcal{C}^{\mathsf{Set}}(FX, FY) = \mathcal{C}^{\mathsf{Set}}(\mathcal{C}(X, -), \mathcal{C}(Y, -)) \cong \mathcal{C}(Y, X) = \mathcal{C}^{\mathsf{op}}(X, Y)$$

so F is fully faithful. Since fully faithful functors reflect isomorphism, it follows that if  $FA \cong FB$  via f then there exists an isomorphism  $g \colon A \to B$  in  $\mathcal{C}^{\mathrm{op}}$  such that Fg = f.

## Exercise 8.1

Let  $\mathcal{C}$  be a category and F a presheaf on  $\mathcal{C}$ . Show that F is representable if and only if there are C in  $\mathcal{C}$  and  $x \in F(C)$  such that for any D in  $\mathcal{C}$  and  $y \in F(D)$  there exists a unique map  $\alpha \colon D \to C$  such that  $y = x \cdot_F \alpha \coloneqq F(\alpha)(x)$ .

## Solution

*Proof.* ( $\Rightarrow$ ) If F is representable then there exists C in C and a natural isomorphism  $\eta: C(-,C) \Rightarrow F$ . Let  $x = \eta_C(1_C) \in FC$ . For  $D \in \text{ob } C$  and  $y \in FD$ , we have that  $C(D,C) \cong FD$  via  $\eta_D$  so let  $\alpha = \eta_D^{-1}(y)$ . By commutativity of

$$\begin{array}{ccc} \mathcal{C}(C,C) & \stackrel{-\circ\alpha}{\longrightarrow} \mathcal{C}(D,C) \\ & & \downarrow^{\eta_C} & & \downarrow^{\eta_D} \\ FC & \stackrel{F\alpha}{\longrightarrow} FD \end{array}$$

it follows that  $y = \eta_D(\alpha) = \eta_D(1_C \alpha) = F(\alpha)(\eta_C(1_C)) = F(\alpha)(x)$ .

 $(\Leftarrow)$  Define  $\eta: \mathcal{C}(-,C) \Rightarrow F$  by  $\eta_C(1_C) = x$ . Requiring  $\eta$  to be a natural transformation means that

$$\begin{array}{ccc}
\mathcal{C}(C,C) & \xrightarrow{-\circ\beta} \mathcal{C}(D,C) \\
\downarrow^{\eta_C} & & \downarrow^{\eta_D} \\
FC & \xrightarrow{F\beta} FD
\end{array}$$

commutes for all  $D \in \text{ob } \mathcal{C}$  and  $\beta \colon D \to C$ . In other words,  $\eta_D(\beta) = F(\beta)(x)$  for all  $D \in \text{ob } \mathcal{C}$  and  $\beta \colon D \to C$ . We claim that  $\eta$  is a natural isomorphism. Indeed, for any  $\gamma \colon D \to D'$  and  $\beta \colon D' \to C$  we have that  $F(\gamma)\eta_{D'}(\beta) = F(\gamma)F(\beta)(x) = F(\beta\gamma)(x) = \eta_D(\beta\gamma)$  so that

$$\begin{array}{ccc}
\mathcal{C}(D',C) & \xrightarrow{-\circ \gamma} \mathcal{C}(D,C) \\
\downarrow^{\eta_{D'}} & & \downarrow^{\eta_D} \\
FC & \xrightarrow{F\gamma} & FD
\end{array}$$

commutes. The assumption that for each  $y \in FD$  there is a unique  $\alpha \in \mathcal{C}(D, C)$  such that  $y = F(\alpha)(x) = \eta_D(\alpha)$  implies that  $\eta_D$  is a bijection for all  $D \in \text{ob } \mathcal{C}$ .

#### Exercise 56

Show that the following are equivalent for each small category C:

- (a)  $\mathcal{C}$  has a terminal object 1.
- (b) The terminal object in  $\mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}$  is representable.

#### Solution

*Proof.*  $(a) \Longrightarrow (b)$  Since limits in a functor category are computed pointwise, the terminal object of  $\mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}$  is the functor that sends every object to the one element set and every map to the identity. This functor is certainly isomorphic to  $\mathcal{C}(-,1)$ .

(b)  $\Longrightarrow$  (a) Let  $\mathcal{C}(-,X)$  be the terminal object of  $\mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}$ . Then for every Y in  $\mathcal{C}$ ,  $\mathcal{C}(Y,X)\cong \{*\}$ , so there is a unique map  $Y\to X$ . Thus, X is terminal in  $\mathcal{C}$ .

#### Exercise 57

Show that the following are equivalent for each small category C:

- (a)  $\mathcal{C}$  has binary products.
- (b) For each pair of object A and B in  $\mathcal C$  the presheaf  $yA\times yB$  is representable in  $\mathsf{Set}^{\mathcal C^{\mathrm{op}}}.$

Can you generalize the statement in this and Exercise 56 to general limits?

#### Solution

*Proof.* (a)  $\Longrightarrow$  (b) Let A, B be objects of C. Note that for all  $C \in \text{ob } C$  we have an isomorphism  $C(C, A) \times C(C, B) \cong C(C, A \times B)$  in Set given by sending  $(f_1, f_2)$ 

to the unique map which makes  $A \xleftarrow{f_1} C$  commute. Its in-

verse is given by sending  $f: C \to A \times B$  to  $(\pi_A f, \pi_B f)$ . If  $g: C' \to C, f_1: C \to A$  and  $f_2: C \to B$  are morphisms in C, since there is a unique map which makes

$$C' \qquad C(C,A) \times C(C,A) \longrightarrow C(C,A \times B)$$

$$A \xleftarrow{f_1 g} \downarrow \qquad f_2 g \qquad \text{commute, so} \qquad \downarrow \neg \circ g \qquad \downarrow \neg \circ g$$

$$C(C',A) \times C(C',B) \longrightarrow C(C',A \times B)$$

commutes. Hence, the isomorphism is natural in C.

 $(b) \implies (a)$  For  $A, B \in \text{ob } \mathcal{C}$  take  $C \in \text{ob } \mathcal{C}$  such that there exists a natural isomorphism  $\eta \colon \mathcal{C}(-,A) \times \mathcal{C}(-,B) \Rightarrow \mathcal{C}(-,C)$ . We claim that  $\mathcal{C}$  is the product. Let  $(\pi_A, \pi_B) = \eta_C^{-1}(\mathrm{id}_C)$ . Then for any  $f_1 \colon X \to A$  and  $f_2 \colon X \to B$  in  $\mathcal{C}$ , we have that

$$\eta_X(\pi_A \eta_X(f_1, f_2), \pi_B \eta_X(f_1, f_2)) \stackrel{*}{=} \eta_C(\pi_A, \pi_B) \circ \eta_X(f_1, f_2)$$
$$= \mathrm{id}_C \circ \eta_X(f_1, f_2) = \eta_X(f_1, f_2),$$

where (\*) holds since  $\eta$  is a natural transformation. Since  $\eta_X$  is an isomorphism,

$$A \xleftarrow{f_1} \xrightarrow{\eta_X(f_1, f_2)} \xrightarrow{f_2} B$$

commutes. Since  $\eta_X(f_1, f_2)$  is the unique map which makes this diagram commute, C is the product of A and B. 

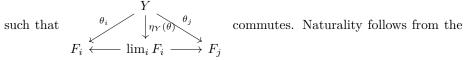
The generalization of this and Exercise 56 would be

**Proposition 7.1.** For  $F: \mathcal{I} \to \mathcal{C}$ ,  $\lim F$  exists in  $\mathcal{C}$  if and only  $\lim yF$  is representable, where  $y: \mathcal{C} \to \mathsf{Set}^{\mathcal{C}^{op}}$  is the Yoneda embedding.

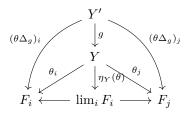
Firs, we prove an important lemma.

**Lemma 7.2.** Let C be a small category. The hom functor  $C(-,-): C^{op} \times C \rightarrow C$ Set preserves limits.

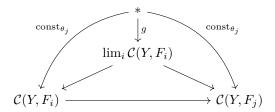
*Proof.* Let  $F: \mathcal{I} \to \mathcal{C}$  be a diagram in  $\mathcal{C}$  such that  $\lim F$  exists. By definition of limiting cones, we have a natural bijection  $\eta: \mathcal{C}^{\mathcal{I}}(\Delta_{(-)}, F) \Rightarrow \mathcal{C}(-, \lim F)$ sending a natural transformation  $\theta \colon \Delta_Y \Rightarrow F$  to the unique map  $Y \to \lim F$ 



fact that



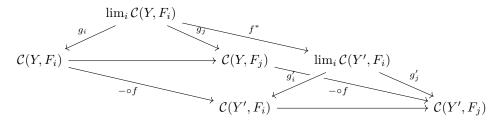
commutes and the uniqueness of  $\eta_{Y'}(\theta \Delta_{\underline{g}})$ . Hence, it remains to show that we have a natural bijection  $\lim_{i} \mathcal{C}(-, F_i) \cong \mathcal{C}^{\mathcal{I}}(\Delta_{(-)}, F)$ . We define  $\mu \colon \mathcal{C}^{\mathcal{I}}(\Delta_{(-)}, F) \Rightarrow \lim_{i} \mathcal{C}(-, F_i)$  by sending a natural transformation  $\theta \colon \Delta_Y \to F$  to g(\*) where gis the unique map such that



commutes. This defines a bijection since for every element element x in  $\lim_i \mathcal{C}(Y, F_i)$  we can assign a natural transformation  $\theta \colon \Delta_Y \Rightarrow F$  by defining  $\theta_i$  to be the image of x under the map  $\lim_i \mathcal{C}(Y, F_i) \to \mathcal{C}(Y, F_i)$ . Naturality of  $\theta$  follows from

the commutativity of  $\mathcal{C}(Y,F_i) \xrightarrow{\text{Inm}_i \, \mathcal{C}(Y,F_i)}. \quad \text{To prove}$ 

naturality in Y, let  $f: Y' \to Y$  in  $\mathcal{C}$  and  $f^*: \lim_i \mathcal{C}(Y, F_i) \to \lim_i \mathcal{C}(Y', F_i)$  be the map induced by f. Then



commutes, and for  $\theta: \Delta_Y \Rightarrow F$  we have that

$$g'_i \mu_{Y'}(\theta \Delta_f) = (\theta \Delta f)_i = \theta_i f = g_i \mu_Y(\theta) f = g'_i f^* \mu_Y(\theta)$$

We conclude that  $\mathcal{C}(-, \lim_i F_i) \cong \lim_i \mathcal{C}(-, F_i)$ .

Since the hom functor is contravariant is the first component, we need to show that  $C(\operatorname{colim}_i F_i, -) \cong \lim_i C(F_i, -)$ , but this is just the dual of the above paragraph.

Proof of Proposition 7.1.  $(\Rightarrow)$  We claim that  $\lim_i yF_i \cong \mathsf{Set}(-,\lim_i F_i)$ . Indeed, we have that  $(\lim_i yF_i) \cong \lim_i \mathsf{Set}(-,F_i) \cong \mathsf{Set}(-,\lim_i F_i)$ , where the last isomorphism follows from Lemma 7.2.

 $(\Leftarrow)$  Since the Yoneda embedding is fully faithful, it reflects limits, i.e. if  $\lim_i y F_i \cong \mathsf{Set}(-,X)$  for some  $X \in \mathsf{ob}\,\mathcal{C}$  then  $\lim_i F_i \cong X$ .

## Exercise 58

Again, let  $\mathcal C$  be a small category with binary products, and let A and B be objects in  $\mathcal C.$ 

i. Show that the assignment

$$X \mapsto \mathcal{C}(X \times A, B)$$

is part of a functor  $\mathcal{C}^{\text{op}} \to \mathsf{Set}$ , with the action on morphisms  $f \colon X' \to X$  in  $\mathcal{C}$  given by precomposition with  $f \times \mathrm{id}_A$ .

ii. What does it say about C if the functor in part (i) is representable?

## Solution

- **i.** Follows since  $\mathrm{id}_X \times \mathrm{id}_A = \mathrm{id}_{X \times A}$  and  $fg \times \mathrm{id}_A = (f \times \mathrm{id}_A)(g \times \mathrm{id}_A)$ .
- **ii.** Claim: if we have a natural isomorphism  $\eta \colon \mathcal{C}(-\times A, B) \Rightarrow \mathcal{C}(-, C)$  for some  $C \in \text{ob}\,\mathcal{C}$ , then  $C \cong B^A$ . Indeed, for  $f \colon X \times A \to B$  we have  $\eta_X(f) \colon X \to C$  and since

$$\begin{array}{ccc}
\mathcal{C}(C \times A, B) & \xrightarrow{\eta_C} \mathcal{C}(C, C) \\
& \xrightarrow{-\circ \eta_X(f) \times \mathrm{id}_A} \downarrow & & \downarrow -\circ \eta_X(f) \\
\mathcal{C}(X \times A, B) & \xrightarrow{\eta_X} \mathcal{C}(X, C)
\end{array}$$

commutes, it follows that

$$\eta_X(\eta_C^{-1}(\mathrm{id}_C)\circ\eta_X(f)\times\mathrm{id}_A)=\eta_C(\eta_C^{-1}(\mathrm{id}_C))\circ\eta_X(f)=\eta_X(f),$$

so  $\eta_C^{-1}(\mathrm{id}_C)\circ\eta_X(f)\times\mathrm{id}_A=f$  by injectivity of  $\eta_X.$  In particular,

$$C \times A \xrightarrow{\eta_C^{-1}(\mathrm{id}_C)} B$$

$$\eta_X(f) \times \mathrm{id}_A \uparrow \qquad f$$

$$X \times A$$

commutes.

#### Exercise 8.3

A forest is a poset  $(X, \leq)$  such that for any element  $x \in X$  the set  $\downarrow x = \{y \in X \mid y \leq x\}$  is finite and linearly ordered by  $\leq$ . If this set has n+1 elements, we say that x has depth n. Write For for the category of forests and monotone, depth-preserving maps.

Show that For is isomorphic to the category of presheaves on  $\mathbb{N}$  (considered as a poset in the usual way).

# Solution

*Proof.* We need to define functors  $F \colon \mathsf{For} \to \mathsf{Set}^{\mathbb{N}^{\mathsf{op}}}$  and  $G \colon \mathsf{Set}^{\mathbb{N}^{\mathsf{op}}} \to \mathsf{For}$  such that  $FG = \mathrm{id}_{\mathsf{Set}^{\mathbb{N}^{\mathsf{op}}}}$  and  $GF = \mathrm{id}_{\mathsf{For}}$ .

# Exercise 59

Prove that  $y \colon \mathcal{C} \to \mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}$  preserves all limits which exist in  $\mathcal{C}$ . Prove also that if  $\mathcal{C}$  is cartesian closed, y preserves exponents.

## Solution

*Proof.* The first claim is proved in Proposition 7.1. For the second claim, we want to prove that  $y(Y^X) \cong (yY)^{yX}$ . Well, using the Yoneda lemma and the fact the y preserves products we have

$$\begin{split} y(Y^X)(C) &= \mathcal{C}(C,Y^X) \cong \mathcal{C}(C \times X,Y) = yY(C \times X) \cong \mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}(y(C \times X),yY) \\ &\cong \mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}(yC \times yX,yY) \cong \mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}\left(yC,(yY)^{yX}\right) \cong (yY)^{yX}(C). \end{split}$$

8 Presheaves as a Topos