Topos Theory - Solutions

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February 27, 2023

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Todo list

Exercise 1

If $f, g\colon X \rightrightarrows Y$ are arrows and $\mathrm{graph}(f) = \mathrm{graph}(g)$, then $f = g$.

Solution.

Proof. Since $\langle 1_Y, f\rangle$ and $\langle 1_Y, g\rangle$ are monos and represent the same subobject of $Y \times X$, we have that $\langle 1_Y, f\rangle = \langle 1_Y, g\rangle$. Then

$$f = \pi_X \circ \langle 1_Y, f\rangle = \pi_X \circ \langle 1_Y, g\rangle = g.$$

$\square$

Exercise 2

Show, using Exercise 1, that the singleton map is always monic.

Solution.

1. *Proof.* Let $f, g\colon Y \to X$ such that $\{\cdot\} \circ f = \{\cdot\} \circ g\colon Y \to \Omega^X$. Let $H\colon X \times Y \to \Omega$ be the transpose of $\{\cdot\} \circ f = \{\cdot\} \circ g$. We argue that H classifies both $\langle f, 1_Y\rangle$ and $\langle g, 1_Y\rangle$. Since transposition is a bijection between maps $Y \to \Omega^X$ and $X \times Y \to \Omega$, we have that

$$\begin{tikzcd}
X \times \Omega^X \arrow[r, "\mathrm{ev}_X"] & \Omega \\
X \times X \arrow[u, "1_X \times \{\cdot\}"] \arrow[ur, "\Delta"] & \\
X \times Y \arrow[u, "1_X \times f"] \arrow[uur, "H"'] &
\end{tikzcd}$$

commutes, so in particular $H = \Delta \circ 1_X \times f$. Note that the left and right squares in

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d, "\langle f, 1_Y\rangle"'] & X \arrow[r, "!_X"] \arrow[d, "\delta_X"'] & 1 \arrow[d, "t"] \\
X \times Y \arrow[r, "1_X \times f"'] & X \times X \arrow[r, "\Delta"'] & \Omega
\end{tikzcd}$$

are pullback squares[1], so the outer square is a pullback as well. Hence, H classify $\langle f, 1_Y\rangle$. Arguing analougsly, we conclude that H classifies g as well. Hence $\langle f, 1_Y\rangle = \langle g, 1_Y\rangle$ and by Exercise 1 it follows that $f = g$, so $\{\cdot\}$ is monic. $\square$

[1]The right square is a pullback by definition of Δ. The left square certainly commutes, and it is a pullback since for any diagram $X \times Y \xleftarrow{h_2} Z \xrightarrow{h_1} X$ such that $\langle h_1, h_1\rangle = (1_X \times f) \circ h_@$, we can take $\pi_Y h_2\colon Z \to Y$.

2. *Proof.* Let $f: Y \to X$. Then $\{\cdot\} \circ f: Y \to \Omega^X$ and we can take its transpose $\overline{\{\cdot\} \circ f}: X \times Y \to \Omega$. I claim this map classifies $\langle f, 1_Y \rangle: Y \to X \times Y$. □

EXERCISE 3

Let $f: Y \to X$ be a map.

i. Show that the maps

$$X \times Y \xrightarrow{\{\cdot\} \times 1_Y} \Omega^X \times Y \xrightarrow{1_{\Omega^X} \times f} \Omega^X \times X \xrightarrow{\mathrm{ev}_X} \Omega$$

and

$$X \times Y \xrightarrow{1_X \times f} X \times X \xrightarrow{\Delta} \Omega$$

are equal.

ii. Let $Pf: \Omega^X \to \Omega^Y$ be the exponential transpose of the map

$$\Omega^X \times Y \xrightarrow{1_{\Omega^X} \times f} \Omega^X \times X \xrightarrow{\mathrm{ev}_X} \Omega.$$

Show that the exponential transpose of the map

$$X \xrightarrow{\{\cdot\}} \Omega^X \xrightarrow{Pf} \Omega^Y$$

is the map

$$Y \times X \xrightarrow{f \times 1_X} X \times X \xrightarrow{\Delta} \Omega$$

Solution.

i. *Proof.*

$$\begin{aligned} \mathrm{ev}_X \circ (1_{\Omega_X} \times f) \circ (\{\cdot\} \times 1_Y) &= \mathrm{ev}_X \circ (\{\cdot\} \times f) \\ &= \mathrm{ev}_X \circ (\{\cdot\} \times 1_X) \circ (1_X \times f) \\ &= \Delta \circ (1_X \times f). \end{aligned}$$

□

ii. *Proof.* Since

$$\begin{aligned} \mathrm{ev}_Y \circ (1_Y \times (Pf \circ \{\cdot\})) &= \mathrm{ev}_Y \circ (1_Y \times Pf) \circ (1_Y \times \{\cdot\}) \\ &= \mathrm{ev}_X \circ (1_{\Omega_X} \times f) \circ (1_Y \times \{\cdot\}), \end{aligned}$$

the result follows from part (i). □

Exercise 4

Show that the map $\mathrm{ev}_{X,Y}$, thus defined, is indeed a natural transformation.

Solution.

Proof. Explicitly, we want to show that $\mathrm{ev}_X\colon(-)^X \times X \Rightarrow \Delta_{(-)}$ is a natural transformation between functors $\mathsf{Set}^{\mathcal{C}^{\mathrm{op}}} \to \mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}$. Indeed, for $C \in \mathrm{ob}\,\mathcal{C}$, $\phi\colon y_C \times X \to Y$ in $Y^X C$, $x \in XC$ and $f\colon Y \to Y'$ we have that

$$f\,\mathrm{ev}_{X,Y}(\phi, x) = f\phi_C(1_C, x) = \mathrm{ev}_{X,Y}(f\phi, x) = \mathrm{ev}_{X,Y} \circ (f \times 1_X)(\phi, x).$$

□

Exercise 5

Prove that $y\colon\mathcal{C} \to \mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}$ preserves all limits which exist in $\mathcal{C}$. Prove also, that if $\mathcal{C}$ is cartesian closed, y preserve exponents.

Solution.

Proof. Let $F\colon I \to \mathcal{C}$ be a diagram with limit $(C, \eta\colon \Delta_C \Rightarrow F)$ in $\mathcal{C}$. Functoriality of y implies that $(y_C, y\eta)$ is a cone for yF. Let (X, μ) be any other cone for yF. Since every presheaf is a colimit of representables, there exists a diagram $G\colon J \to \mathcal{C}$ and a natrual transformation $\nu\colon yG \Rightarrow \Delta_X$ such that (X, ν) is a colimiting cocone. Then for each $x \in J$, $(y_{Gx}, (\mu_i\nu_x)_{i\in I})$ is a cone for yF, and since y is full and faithful, it is the image of a cone in $\mathcal{C}$. Thus, for each $x \in J$ we have unique map $Gx \to C$ and these maps assemble into the components of a natural transformation $G \Rightarrow \Delta_C$. Since (X, ν) is colimiting for yG, we get a unique map $X \to yC$, which proves that $(y_C, y\eta)$ is a limiting cone for yF.

Suppose $\mathcal{C}$ is cartesian closed. We want to show that $y_{C^D} \cong (y_C)^{y_D}$ or equivalently, $\mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}(X \times y_D, y_C) \cong \mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}(X, y_{C^D})$ for every presheaf X. If X is representable, then this certainly hold since

$$\begin{aligned}
\mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}(y_B \times y_D, y_C) &\cong \mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}(y_{B\times D}, y_C) && (y \text{ preserves limits})\\
&\cong \mathcal{C}(B \times D, C) && (y \text{ is full and faithful})\\
&\cong \mathcal{C}(B, C^D) &&\\
&\cong \mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}(y_B, y_{C^D}). && (y \text{ is full and faithful})
\end{aligned}$$

For an arbitrary presheaf X, we use again that it is a colimit of some diagram $yG\colon J \to \mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}$. Then arrows $X \to y_{C^D}$ correspond to cocones on yG with vertex y_{C^D}, which, by the isomorphism above, correspond to cocones on the diagram

$$J \xrightarrow{G} \mathcal{C} \xrightarrow{y} \mathsf{Set}^{\mathcal{C}^{\mathrm{op}}} \xrightarrow{(-)\times y_D} \mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}$$

with vertex y_C. Using that $(-)\times y_D$ preserves colimits, these cocones correspond to arrow $X \times y_D \to y_C$, as desired. □

Exercise 6

Let $(P, \leq)$ be a preorder. For $p \in P$, let $\downarrow p = \{q \in P \mid q \leq p\}$. Show that sieves on p can be identified with downwards closed subsets of $\downarrow p$. If we denote the unique arrow $q \to p$ by qp and U is a downwards closed subset of $\downarrow p$, what is $(qp)^*U$?

Solution.

Proof. Let R be a sieve on p. Since arrows are unique, we can identify an arrow $qp \in R$ with its domain q, and since for any $qp \in R$ and $r \leq q$ we have $rp = (rq)(qp) \in R$, we can identify R with a downwards closed subset of $\downarrow p$.

If U is downwards closed subset of $\downarrow p$, then pulling it back along qp is precisely cutting it at q, i.e. $(qp) * U = U \cap \downarrow q$. $\square$

Exercise 7

Show that $\mathcal{P}(X)(C) = \mathrm{Sub}(y_C \times X)$ and that, for $f : C' \to C$, $\mathcal{P}(X)(f)(U) = (y_f \times 1_X)^*(U)$. Prove also, that the element relation, as a subpresheaf $\in_X$ of $\mathcal{P}(X) \times X$, is given by

$$(\in_X)(C) = \{(U, x) \in \mathrm{Sub}(y_C \times X) \times XC \mid (1_C, x) \in UC\}$$

Solution.

Proof. Note that

$$\mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}(Y, \Omega^X) \cong \mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}(Y \times X, \Omega) \cong \mathrm{Sub}(Y \times X)$$

so indeed $\mathcal{P}(X) = \Omega^X$. The isomorphism $\mathcal{P}(X)(C) \cong \mathrm{Sub}(y_C \times X)$ follows from Yoneda

$$\mathrm{Sub}(y_C \times X) \cong \mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}(y_C \times X, \Omega) \cong \mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}(y_C, \Omega^X) \cong \Omega^X(C).$$

Let U be a subpresheaf of $y_C \times X$ and $f : C' \to C$ an arrow in $\mathcal{C}$. Let $\phi : y_c \times X \to \Omega$ be the map calssifying U. Then

$$\Omega^X f(\phi) = \phi \circ (y_f \times 1_X)$$

so, $\mathcal{P}(X)(f)$ should send U to the subobject of $y_{C'} \times X$ classified by $\phi \circ (y_f \times 1_X)$, or equivalently, to the pullback of U along $(y_f \times 1_X)$.

$$\begin{tikzcd}
(y_f \times 1_X)^* U \arrow[r] \arrow[d, hook] & U \arrow[r] \arrow[d, hook] & 1 \arrow[d, "t"] \\
y'_C \times X \arrow[r, "y_f \times 1_X"'] & y_C \times X \arrow[r, "\phi"'] & \Omega
\end{tikzcd}$$

Observe that $\in_X$ is just the subobject of $\Omega^X \times X$ classified by $\mathrm{ev}_X : \Omega^X \times X \to \Omega$. So, if U is a subobject of $y_C \times X$, $\phi : y_C \times X \to \Omega$ classifies U and $x \in XC$ we have that $(U, x) \in \in_X C$ if and only if $\phi_C(1_C, x)$ is the maximal sieve on C if and only if $(1_C, x) \in UC$. $\square$

Exercise 8

Let $\mathcal{E}$ be a topos with subobject classifier $1 \xrightarrow{t} \Omega$. Recall that an object C of a category $\mathcal{C}$ is called injective if any diagram $N \xleftarrow{m} M \xrightarrow{f} C$ with m mono, admits an extension by an arrow $g : N \to C$ satisfying $gm = f$.

i. Prove that Ω is injective.

ii. Prove that every object of the form Ω^X is injective.

iii. Conclude that $\mathcal{E}$ has enough injectives.

Solution.

i. *Proof.* Let $i : A \to M$ be the subobject of M classified by f. Then $mi : A \to N$ is monic, so there's a $\phi : N \to \Omega$ which classifies it. I claim this ϕ does the trick. Consider the diagram

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[d, "i"] & A \arrow[r] \arrow[d, "im"] & 1 \arrow[d, "t"] \\ M \arrow[r, "f"] & N \arrow[r, "\phi"] & \Omega \end{tikzcd}$$

and note that the right square is a pullback by definition of ϕ and the left square is a pullback since m is monic. So the composite square is a pullback and in particular, ϕm classifies i so $\phi m = f$. $\square$

ii. *Proof.* Consider the diagram $N \xleftarrow{m} M \xrightarrow{f} \Omega^X$ with m monic. Then we have a diagram $N \times X \xleftarrow{m \times 1_X} M \times X \xrightarrow{\bar{f}} \Omega$ where $\bar{f}$ is the transpose of f, and by the previous part there's a map $\bar{h} : N \times X \to \Omega$ such that $\bar{f} = \bar{h} \circ (m \times 1_X)$. Let $h : N \to \Omega^X$ be the transpose of $\bar{h}$. Then the diagram

$$\begin{tikzcd} \Omega^X \times X \arrow[drr] & & \\ N \times X \arrow[u, "h \times 1_X"] \arrow[rr, "\bar{h}"] & & \Omega \\ M \times X \arrow[u, "m \times 1_X"] \arrow[urr, "\bar{f}"'] & & \end{tikzcd}$$

commutes, since the top and bottom squares commute, and it follows that $f = hm$. $\square$

iii. That depends on the definition of enough.

Exercise 9

Let $\mathcal{C}$ be a regular category, and P an object in $\mathcal{C}$. Prove that the following are equivalent:

(i) For every regular epi $f: A \to B$, any arrow $P \to B$ factors through f.

(ii) Every regular epi with codomain P has a section.

Solution.

Proof. $(i) \Rightarrow (ii)$ Since the identity $P \to P$ factors through every regular epi $A \to P$ by assumption, the result follows.

$(ii) \Rightarrow (i)$ Let $f: A \to B$ be a regular epi, and $g: P \to B$ any morphism. We can take the pullback

$$\begin{tikzcd} X \arrow[r, "b"] \arrow[d, "a"] & A \arrow[d, "f"] \\ P \arrow[r, "g"] & B \end{tikzcd}$$

and note that a is a regular epi since f is (by definition of a regular category). So a admits a section s, and we have that $fbs = gas = g$. □

Exercise 10

Show that if $\mathcal{C}$ has equalizers, $\mathcal{C}$ is Cauchy complete

Solution.

Proof. Let $e: C \to C$ be idempotent and $i: D \to C$ be the equalizer of $1_C, e: C \rightrightarrows C$. Then there exists and $r: C \to D$ such that

$$\begin{tikzcd} D \arrow[r, "i"] & C \arrow[r, shift left, "e"] \arrow[r, shift right, "1_C"'] & C \\ C \arrow[u, "r"] \arrow[ur, "e"'] & & \end{tikzcd}$$

commutes.

Since $iri = ei = i$ it follows that

$$\begin{tikzcd} D \arrow[r, "i"] & C \arrow[r, shift left, "e"] \arrow[r, shift right, "1_C"'] & C \\ D \arrow[u, "ri"] \arrow[ur, "i"'] & & \end{tikzcd}$$

commutes, so $ri = 1_D$. □

Exercise 11

For a nonempty set A, let F_A be the following presheaf on the real numbers $\mathbb{R}$:

$$F_A(U) = \begin{cases} A, & 0 \in U \\ \{*\}, & \text{otherwise} \end{cases}.$$

Show that F_A is a sheaf, and give a concrete presentation of the étale space corresponding to F_A.

Solution. Let $(U_i, x_i)_{i\in I}$ be a compatible family and $V = \bigcup U_i$. Suppose $0 \notin V$, so that $F(V) = \{*\}$ and we only have one choice for the amalgamation of the x_i's. On the other hand, if $0 \in V$, then we can choose any of the x_i such that $0 \in U_i$ (that is becuause if $0 \in U_i \cap U_j$ then $x_i = x_j$) To do

Exercise 12

Show that for a presheaf F and the associated local homeomorphism $\pi\colon \coprod_{x\in X} G_x \to X$ that we have constructed, the following holds: every morphism of presheaves $F \to H$, where H is a sheaf, factors uniquely through the sheaf corresponding to $\pi\colon \coprod_{x\in X} G_x \to X$. Conclude that $\pi\colon \coprod_{x\in X} G_x \to X$ is the associated sheaf of F. Conclude that the inclusion of categories $\mathsf{Sh}(X) \to \mathsf{Set}^{\mathcal{O}_X{}^{\mathrm{op}}}$ has a left adjoint.

Solution.

Proof. We have the sheaf $\mathcal{F}$ given by

$$\mathcal{F}(U) = \{s\colon U \to \coprod_{x\in X} G_x \mid s \text{ continuous and } \pi s = 1_U\}.$$

Let H be a sheaf and $\eta\colon F \to H$ a map of presheaves. □ To do

Exercise 13

Show that the category $\mathsf{Sh}(X)$ is closed under finite limits in $\mathsf{Set}^{\mathcal{O}_X{}^{\mathrm{op}}}$, and that the left adjoint of Exercise 12 preserve finite limits.

Solution.

Proof. □ To do

Exercise 14

F is separated if and only if each compatible family in F, indexed by a covering sieve, has at most one amalgamation.

Solution.

Proof. □ To do

Exercise 15

Suppose G is a subpresheaf of F. If G is a sheaf, then G is closed in $\mathrm{Sub}(F)$. Conversely, every closed subpresheaf of a sheaf is a sheaf.

Solution.

Proof. □ To do

Exercise 16

Prove that F is a sheaf if and only if for every presheaf X and every dense subpresheaf A of X, any arrow $A \to F$ has a unique extension to an arrow $X \to F$.

Solution.

Proof. □ To do

Exercise 17

Show that every split fork is a coequalizer diagram, and moreover a coequalizer which is preserved by any functor (this is called an *absolute* coequalizer).

Solution. To do

Exercise 18

Suppose D_1 is the diagram $a \overset{f}{\underset{g}{\rightrightarrows}} b \xrightarrow{h} c$ in a category $\mathcal{C}$, and D_2 is the $a' \overset{f'}{\underset{g'}{\rightrightarrows}} b' \xrightarrow{h'} c'$ diagram in $\mathcal{C}$. Assume that D_2 is a retract of D_1 in the category of diagrams in $\mathcal{C}$ of type $\bullet \rightrightarrows \bullet \longrightarrow \bullet$. Prove that if D_1 is a split fork, then so it D_2.

Solution. To do

1 Elementary Toposes