Online learning using Bregman Divergences

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Material follows Chapter 11 of “Prediction Learning and Games” and “Adaptive game playing using multiplicative weights” by Freund and Schapire.

Outline

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- Here we have stochasticity - but only in **algorithm**, not in **outcome**

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- Theorem (main theorem)

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  - Proof: by combining upper and lower bounds on $\sum_{i=1}^{N} w_i^{T+1}$

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Recall Single step bound for **Hedge**(η)

The total weight has to decrease if the loss is large

$$\sum_{i=1}^{N} w_i^{t+1} \leq \left(\sum_{i=1}^{N} w_i^t \right) \left(1 - (1 - e^{-\eta}) \mathbf{p}^t \cdot \ell_t\right)$$

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- When comparing to single best expert $X = \log N$
- **Intuition:** X should be small if best distribution $\mathbf{q}^*$ is close to initial distribution $\mathbf{p}^0$

Relative Entropy Bound

- Relative Entropy or KL-Divergence:

$$\mathrm{RE}\,(\mathbf{q} \parallel \mathbf{p}) \doteq \sum_{i=1}^{N} \mathbf{q}(i) \ln \frac{\mathbf{q}(i)}{\mathbf{p}(i)}$$

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- For any distribution **q** and any learning rate $\eta > 0$ The cumulative loss of **Hedge**(η) is bounded by:

$$\sum_{t=1}^{T} \mathbf{p}_t \cdot \ell_t \leq a_\eta \sum_{t=1}^{T} \mathbf{q} \cdot \ell_t + c_\eta \,\mathrm{RE}\,(\mathbf{q} \;\|\; \mathbf{p}_0)$$

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- Where $a_\eta = \frac{\eta}{1-e^{-\eta}}$, $c_\eta = \frac{1}{1-e^{-\eta}}$

Per Iteration bound

- For any distribution $\mathbf{q}$ and any iteration t **Hedge**(η):

$$c_\eta \left(\mathrm{RE}\left(\mathbf{q} \parallel \mathbf{p}_t\right) - \mathrm{RE}\left(\mathbf{q} \parallel \mathbf{p}_{t+1}\right)\right) \geq \mathbf{p}_t \cdot \ell_t - a_\eta \, \mathbf{q} \cdot \ell_t$$

Proof (from RE to ratio)

$$\begin{aligned}
&\mathrm{RE}\left(\mathbf{q} \parallel \mathbf{p}_t\right) - \mathrm{RE}\left(\mathbf{q} \parallel \mathbf{p}_{t+1}\right) \\
&= \sum_{i=1}^{N} \mathbf{q}(i) \ln \frac{\mathbf{q}(i)}{\mathbf{p}_t(i)} - \sum_{i=1}^{N} \mathbf{q}(i) \ln \frac{\mathbf{q}(i)}{\mathbf{p}_{t+1}(i)} \\
&= \sum_{i=1}^{N} \mathbf{q}(i) \ln \frac{\mathbf{p}_{t+1}(i)}{\mathbf{p}_t(i)} \\
&= \sum_{i=1}^{N} \mathbf{q}(i) \ln \frac{\exp\left(-\eta \ell_t(i)\right)}{Z_t}
\end{aligned}$$

Because

$$\mathbf{p}_{t+1}(i) = \mathbf{p}_t(i) \frac{e^{-\eta \ell_t(i)}}{Z_t}; \quad Z_t = \sum_{i=1}^{N} \mathbf{p}_t(i) \exp(-\eta \ell_t(i))$$

Proof (from ratio to bound)

$$\begin{aligned}
&\sum_{i=1}^{N} \mathbf{q}(i) \ln \frac{\exp\left(-\eta \ell_t(i)\right)}{Z_t} \\
&= \quad -\eta \mathbf{q} \cdot \ell_t - \ln Z_t \\
&\geq \quad -\eta \mathbf{q} \cdot \ell_t - \ln \left[\sum_{i=1}^{N} \mathbf{p}_t(i) \left(1 - (1 - e^{-\eta}) \ell_t(i)\right) \right] \\
&\qquad \text{because } e^{-\eta x} \leq (1 - (1 - e^{-\eta})x) \text{ for } x \in [0, 1] \\
&\geq \quad (1 - e^{-\eta}) \mathbf{p}_t \cdot \ell_t - \eta \mathbf{q} \cdot \ell_q \\
&\qquad \text{because } \ln(1 - x) \leq -x \text{ for } x < 1
\end{aligned}$$

Divergence as Potential

- For any distribution **q** and any iteration t **Hedge**(η):

$$\mathrm{RE}\left(\mathbf{q} \parallel \mathbf{p}_t\right) - \mathrm{RE}\left(\mathbf{q} \parallel \mathbf{p}_{t+1}\right) \geq \frac{1}{c_\eta}\mathbf{p}_t \cdot \ell_t - \frac{a_\eta}{c_\eta}\,\mathbf{q} \cdot \ell_t$$

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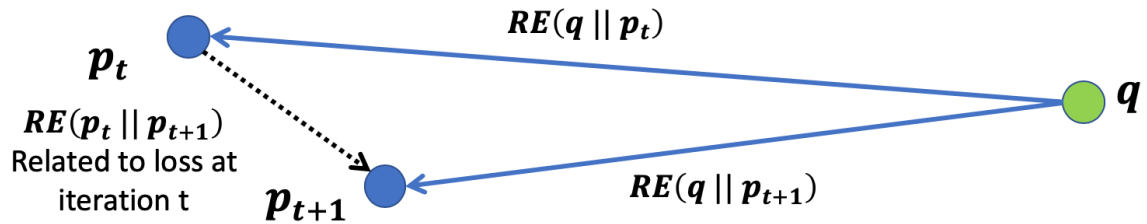

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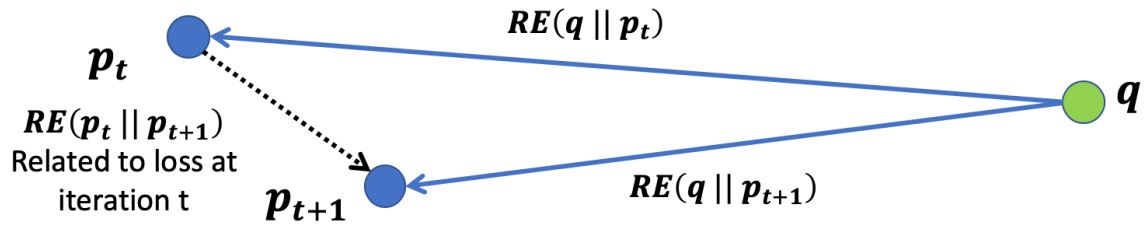

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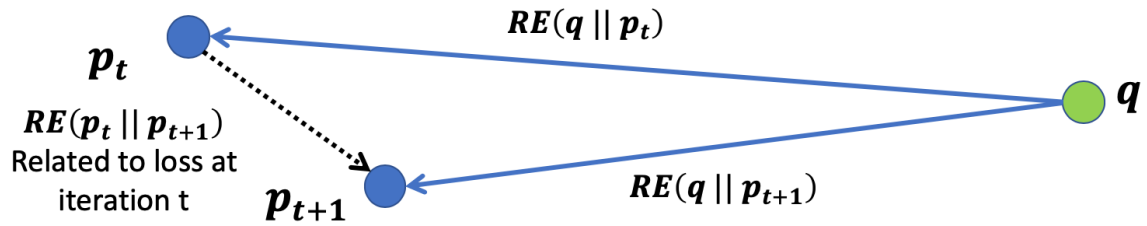

- If the loss of $\mathbf{p}_t$ is larger than that of **q** forces $\mathbf{p}_{t+1}$ do get closer to **q**.
- The distance from $\mathbf{p}_0$ to $\mathbf{q}^*$ bounds the total regret.

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- Small η = small correction = strong regularization.

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- If update is small, we can think of it in terms of gradients:

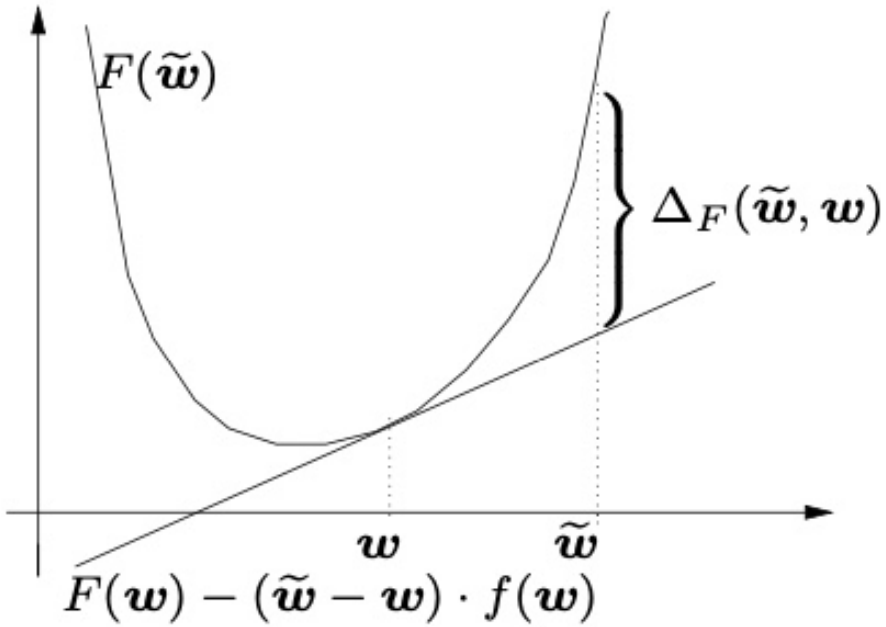
$$\bigtriangledown_\omega(L(\omega, x_t) + \eta D(\omega, \omega_t))$$

Bregman Divergences [Br,CL,Cs]

For any differentiable convex function F

$$\Delta_F(\widetilde{\boldsymbol{w}}, \boldsymbol{w}) = F(\widetilde{\boldsymbol{w}}) - F(\boldsymbol{w}) - (\widetilde{\boldsymbol{w}} - \boldsymbol{w}) \cdot \underbrace{\nabla_{\boldsymbol{w}} F(\boldsymbol{w})}_{f(\boldsymbol{w})}$$

$$= F(\widetilde{\boldsymbol{w}}) - \begin{array}{c}\text{supporting hyperplane} \\ \text{through } (\boldsymbol{w}, F(\boldsymbol{w}))\end{array}$$



Bregman Divergences: Simple Properties

1. $\Delta_F(\widetilde{\boldsymbol{w}}, \boldsymbol{w})$ is convex in $\widetilde{\boldsymbol{w}}$
2. $\Delta_F(\widetilde{\boldsymbol{w}}, \boldsymbol{w}) \geq 0$
   If F convex equality holds iff $\widetilde{\boldsymbol{w}} = \boldsymbol{w}$
3. Usually not symmetric: $\Delta_F(\widetilde{\boldsymbol{w}}, \boldsymbol{w}) \neq \Delta_F(\boldsymbol{w}, \widetilde{\boldsymbol{w}})$
4. Linearity (for $a \geq 0$):
   $\Delta_{F+a\,H}(\widetilde{\boldsymbol{w}}, \boldsymbol{w}) = \Delta_F(\widetilde{\boldsymbol{w}}, \boldsymbol{w}) + a\,\Delta_H(\widetilde{\boldsymbol{w}}, \boldsymbol{w})$
5. Unaffected by linear terms ($a \in \mathbf{R}$, $\boldsymbol{b} \in \mathbf{R}^n$):
   $\Delta_{H+a\widetilde{\boldsymbol{w}}+\boldsymbol{b}}(\widetilde{\boldsymbol{w}}, \boldsymbol{w}) = \Delta_H(\widetilde{\boldsymbol{w}}, \boldsymbol{w})$

Bregman Divergences: more properties

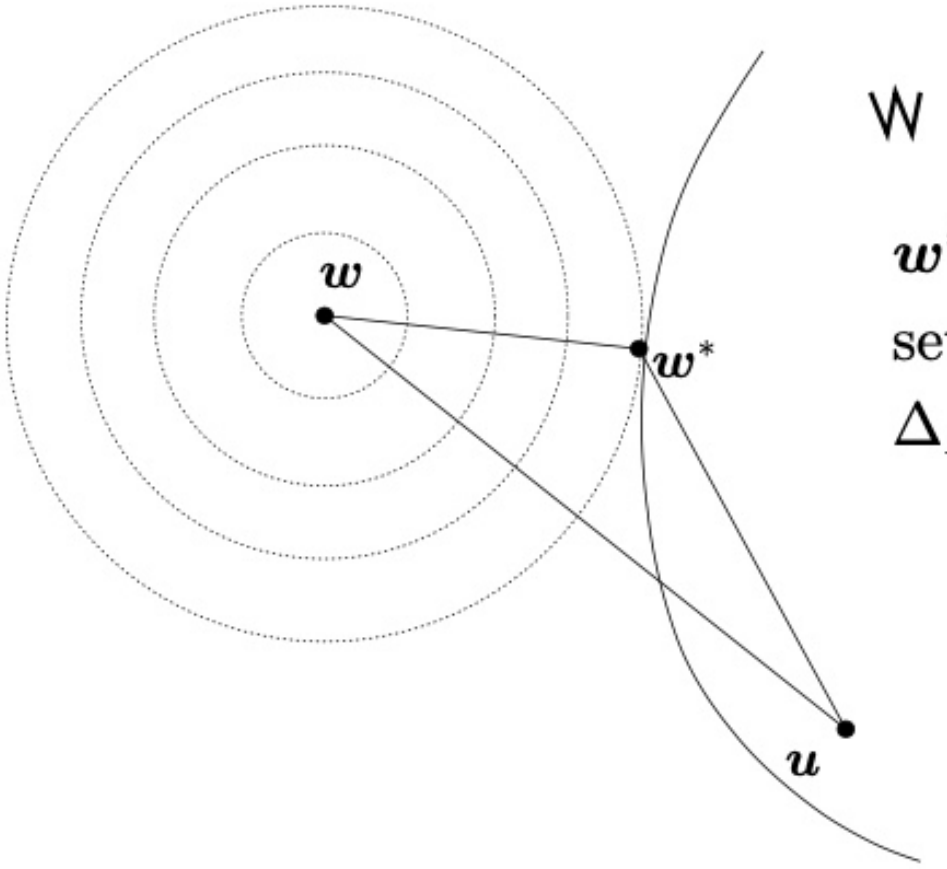
6. $\nabla_{\widetilde{\boldsymbol{w}}} \Delta_F(\widetilde{\boldsymbol{w}}, \boldsymbol{w})$

$$\begin{aligned} &= \nabla F(\widetilde{\boldsymbol{w}}) - \nabla_{\widetilde{\boldsymbol{w}}}(\widetilde{\boldsymbol{w}} \nabla_{\boldsymbol{w}} F(\boldsymbol{w})) \\ &= f(\widetilde{\boldsymbol{w}}) - f(\boldsymbol{w}) \end{aligned}$$

7. $\Delta_F(\boldsymbol{w}_1, \boldsymbol{w}_2) + \Delta_F(\boldsymbol{w}_2, \boldsymbol{w}_3)$

$$\begin{aligned} &= F(\boldsymbol{w}_1) - F(\boldsymbol{w}_2) - (\boldsymbol{w}_1 - \boldsymbol{w}_2) f(\boldsymbol{w}_2) \\ &\quad F(\boldsymbol{w}_2) - F(\boldsymbol{w}_3) - (\boldsymbol{w}_2 - \boldsymbol{w}_3) f(\boldsymbol{w}_3) \\ &= \Delta_F(\boldsymbol{w}_1, \boldsymbol{w}_3) + (\boldsymbol{w}_1 - \boldsymbol{w}_2) \cdot (f(\boldsymbol{w}_3) - f(\boldsymbol{w}_2)) \end{aligned}$$

A Pythagorean Theorem [Br,Cs,A,HW]



$\mathcal{W}$

$\boldsymbol{w}^*$ is projection of $\boldsymbol{w}$ onto convex set $\mathcal{W}$ w.r.t. Bregman divergence Δ_F:

$$\boldsymbol{w}^* = \operatorname*{argmin}_{\boldsymbol{u} \in \mathcal{W}} \Delta_F(\boldsymbol{u}, \boldsymbol{w})$$

Theorem:

$$\Delta_F(\boldsymbol{u}, w) \geq \Delta_F(\boldsymbol{u}, \boldsymbol{w}^*) + \Delta_F(\boldsymbol{w}^*, \boldsymbol{w})$$

Examples

Squared Euclidean Distance

$$\begin{aligned} F(\boldsymbol{w}) &= ||\boldsymbol{w}||_2^2/2 \\ f(\boldsymbol{w}) &= \boldsymbol{w} \\ \Delta_F(\widetilde{\boldsymbol{w}}, \boldsymbol{w}) &= ||\widetilde{\boldsymbol{w}}||_2^2/2 - ||\boldsymbol{w}||_2^2/2 - (\widetilde{\boldsymbol{w}} - \boldsymbol{w}) \cdot \boldsymbol{w} \\ &= ||\widetilde{\boldsymbol{w}} - \boldsymbol{w}||_2^2/2 \end{aligned}$$

(Unnormalized) Relative Entropy

$$\begin{aligned} F(\boldsymbol{w}) &= \sum_i (w_i \ln w_i - w_i) \\ f(\boldsymbol{w}) &= \ln \boldsymbol{w} \\ \Delta_F(\widetilde{\boldsymbol{w}}, \boldsymbol{w}) &= \sum_i \left(\widetilde{w_i} \ln \frac{\widetilde{w_i}}{w_i} + w_i - \widetilde{w_i} \right) \end{aligned}$$

Examples-2 [GLS,GL]

p-norm Algs (q is dual to p: $\frac{1}{p} + \frac{1}{q} = 1$)

$$F(\boldsymbol{w}) = \frac{1}{2}||\boldsymbol{w}||_q^2$$

$$f(\boldsymbol{w}) = \nabla \frac{1}{2}||\boldsymbol{w}||_q^2$$

$$\Delta_F(\widetilde{\boldsymbol{w}}, \boldsymbol{w}) = \frac{1}{2}||\widetilde{\boldsymbol{w}}||_q^2 + \frac{1}{2}||\boldsymbol{w}||_q^2 - \widetilde{\boldsymbol{w}} \cdot f(\boldsymbol{w})$$

When $p = q = 2$ this reduces to squared Euclidean distance (Widrow-Hoff).

General Motivation of Updates [KW]

Trade-off between two term:

$$\boldsymbol{w}_{t+1} = \underset{\boldsymbol{w}}{\operatorname{argmin}}\Big(\underbrace{\Delta_F(\boldsymbol{w}, \boldsymbol{w}_t)}_{weight\ \ domain} + \eta_t \underbrace{L_t(\boldsymbol{w})}_{label\ \ domain} \Big)$$

$\Delta_F(\boldsymbol{w}, \boldsymbol{w}_t)$ is "regularization term" and serves as measure of progress in the analysis.

When loss L is convex (in $\boldsymbol{w}$)

$$\nabla_{\boldsymbol{w}}(\Delta_F(\boldsymbol{w}, \boldsymbol{w}_t) + \eta_t L_t(\boldsymbol{w})) = 0$$

iff

$$f(\boldsymbol{w}) - f(\boldsymbol{w}_t) + \eta_t \underbrace{\nabla L_t(\boldsymbol{w})}_{\approx \nabla L_t(\boldsymbol{w}_t)} = 0$$

$$\Rightarrow \quad \boldsymbol{w}_{t+1} = f^{-1}\left(f(\boldsymbol{w}_t) - \eta_t \nabla L_t(\boldsymbol{w}_t)\right)$$

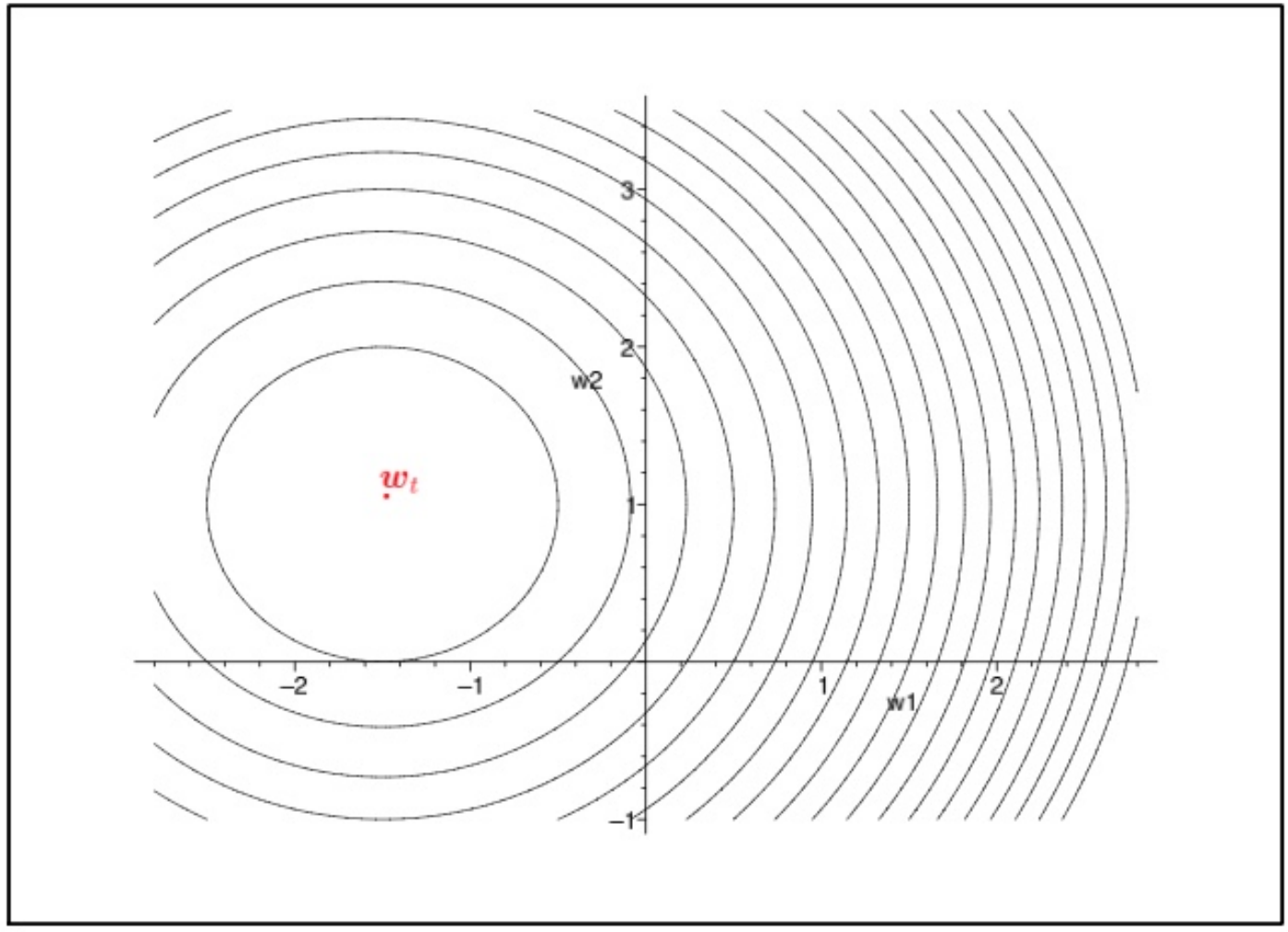
Divergence: Euclidean Distance Squared

$$\Delta_F(\boldsymbol{w}, \boldsymbol{w}_t) = \|\boldsymbol{w} - \boldsymbol{w}_t\|_2^2/2$$

$\boldsymbol{w}_t = (-3/2, 1)$

$\boldsymbol{x}_t = (1, -0.5)$

$y_t = 1$



Second step: Relate $\Delta_F(\boldsymbol{w}_t, \boldsymbol{w}_{t+1})$ to loss $L_t(\boldsymbol{w}_t)$

Loss & divergence are dependent

Get $\Delta_F(\boldsymbol{w}_t, \boldsymbol{w}_{t+1}) \leq$ const. $L_t(\boldsymbol{w}_t)$

Then solve for $\sum_t L_t(\boldsymbol{w}_t)$

Yield bounds of the form

$$\sum_t L_t(\boldsymbol{w}_t) \leq a \sum_t L_t(\boldsymbol{u}) + b\, \Delta_F(\boldsymbol{u}, \boldsymbol{w}_1)$$

a, b constants, $a > 1$.

Regret bounds ($a = 1$):

time changing η, subtler analysis [AG]

How to prove relative loss bounds?

Loss: $L_t(\boldsymbol{w}) = L((\boldsymbol{x}_t, y_t), \boldsymbol{w})$ convex in $\boldsymbol{w}$

Divergence: $\Delta_F(\boldsymbol{u}, \boldsymbol{w}) = F(\boldsymbol{u}) - F(\boldsymbol{w}) - (\boldsymbol{u} - \boldsymbol{w}) \cdot f(\boldsymbol{w})$

Update: $f(\boldsymbol{w}_{t+1}) - f(\boldsymbol{w}_t) = -\eta \nabla_{\boldsymbol{w}} L_t(\boldsymbol{w}_t)$

$$L_t(\boldsymbol{u}) \overbrace{\geq}^{\text{convexity}} L_t(\boldsymbol{w}_t) + (\boldsymbol{u} - \boldsymbol{w}_t) \cdot \underbrace{\nabla_{\boldsymbol{w}} L_t(\boldsymbol{w}_t)}_{\text{update}}$$

$$= L_t(\boldsymbol{w}_t) - \frac{1}{\eta} \underbrace{(\boldsymbol{u} - \boldsymbol{w}_t) \cdot (f(\boldsymbol{w}_{t+1}) - f(\boldsymbol{w}_t))}_{\text{prop. 7 of } \Delta_F}$$

$$= L_t(\boldsymbol{w}_t) + \frac{1}{\eta} (\Delta_F(\boldsymbol{u}, \boldsymbol{w}_{t+1}) - \Delta_F(\boldsymbol{u}, \boldsymbol{w}_t) - \Delta_F(\boldsymbol{w}_t, \boldsymbol{w}_{t+1}))$$

First step: Teleskoping

Summing over t [WJ,KW]

$$\sum_t L_t(\boldsymbol{w}_t) \leq \sum_t L_t(\boldsymbol{u}) + \frac{1}{\eta}\sum_t \Big(\Delta_F(\boldsymbol{u}, \boldsymbol{w}_t) - \Delta_F(\boldsymbol{u}, \boldsymbol{w}_{t+1}) + \Delta_F(\boldsymbol{w}_t, \boldsymbol{w}_{t+1}) \Big)$$

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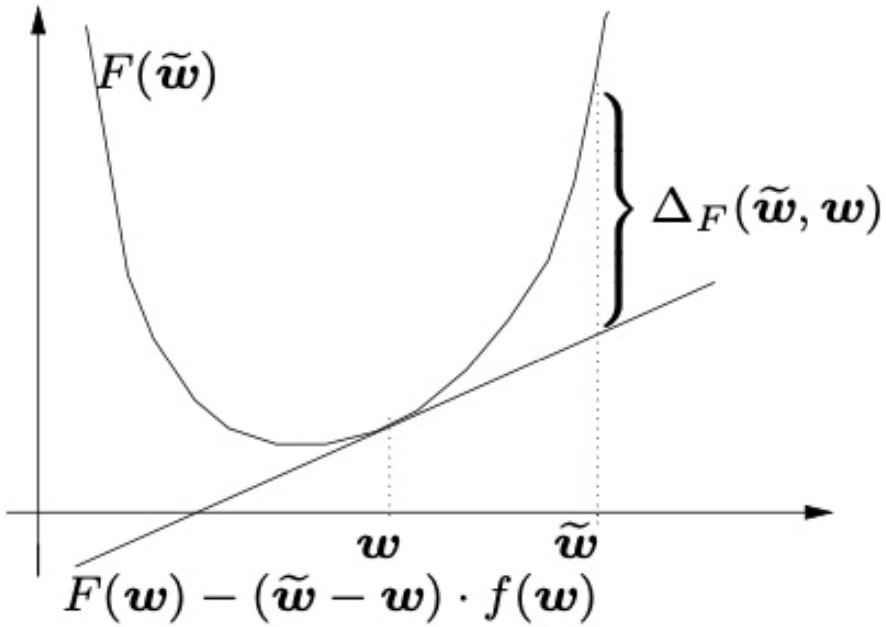

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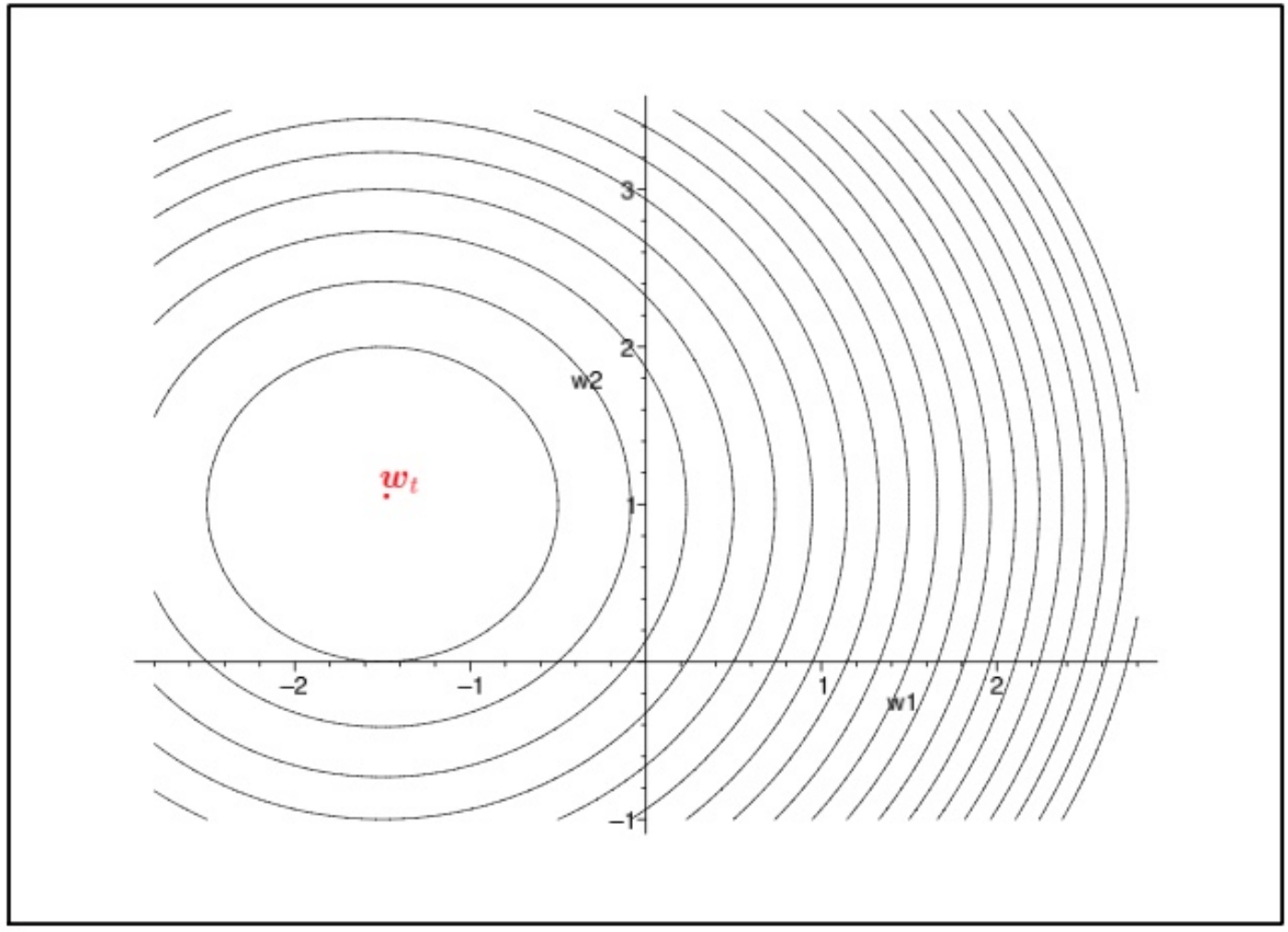

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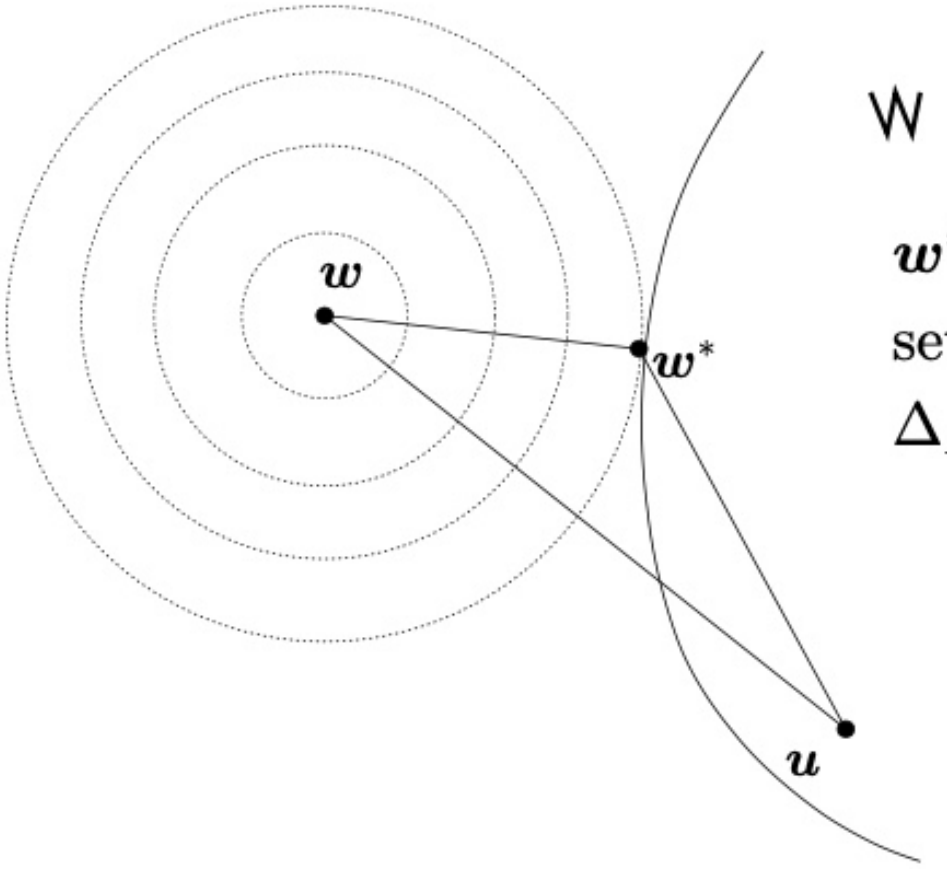

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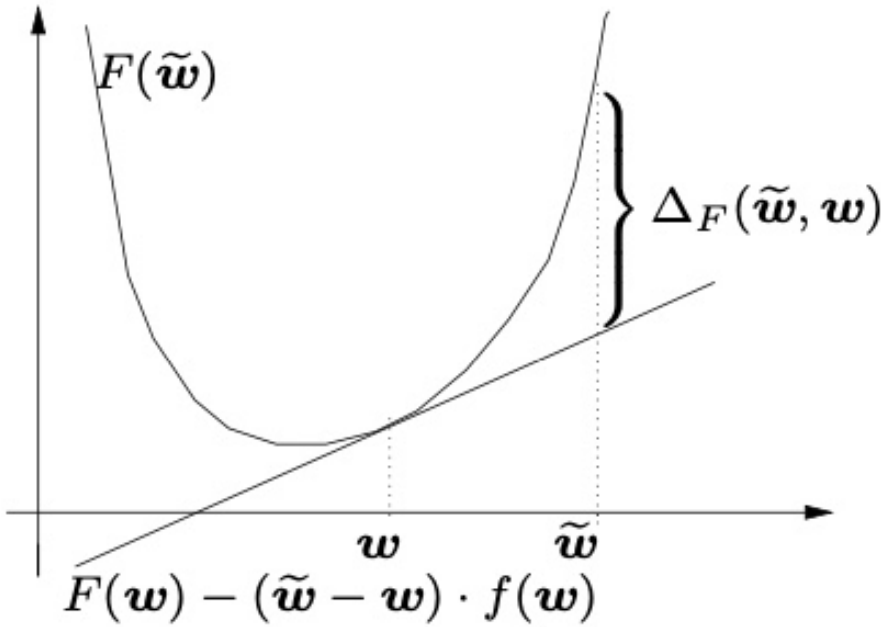

Divergence as Regularization

- Use a divergence term $D(\omega_t, \omega_{t+1})$ to measure the size of the update.
- Find ω_{t+1} to balance loss reduction and divergence:

$$\text{argmin}_\omega = L(\omega, x_t) + \eta D(\omega, \omega_t)$$

- If update is small, we can think of it in terms of gradients:

$$\bigtriangledown_\omega (L(\omega, x_t) + \eta D(\omega, \omega_t))$$

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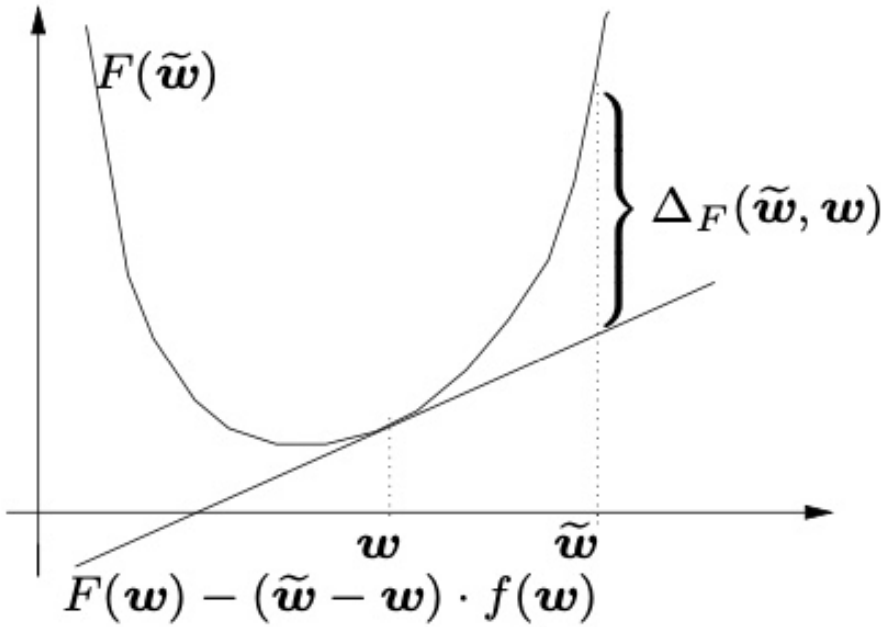

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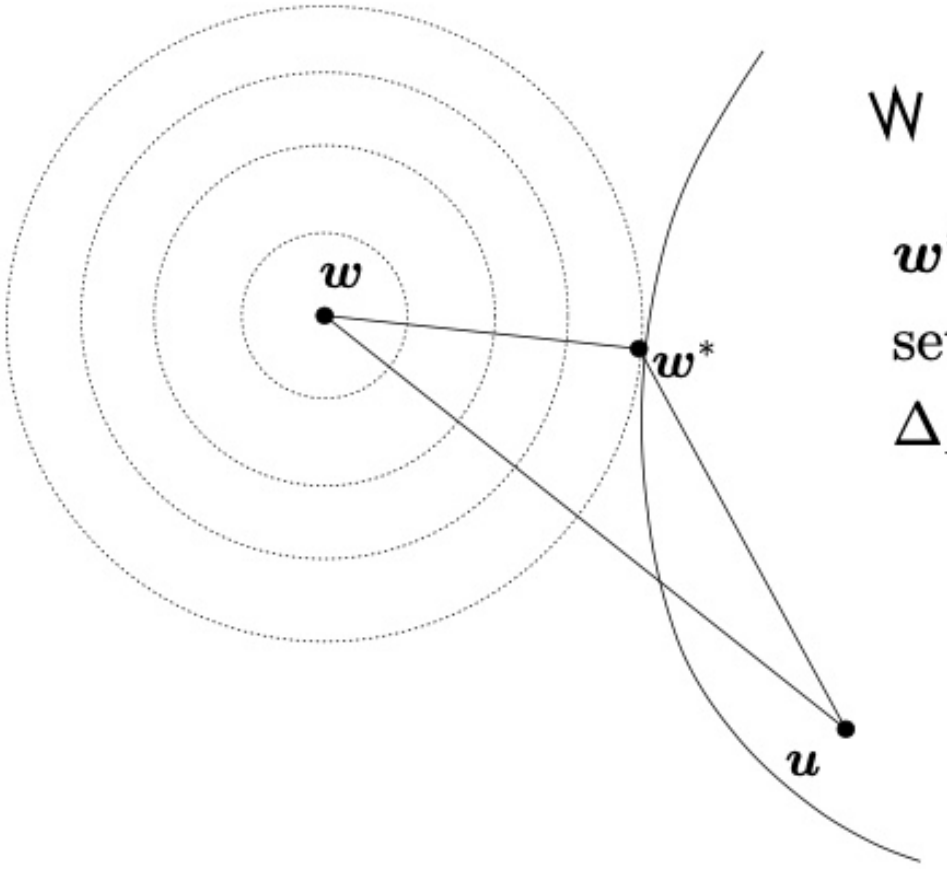

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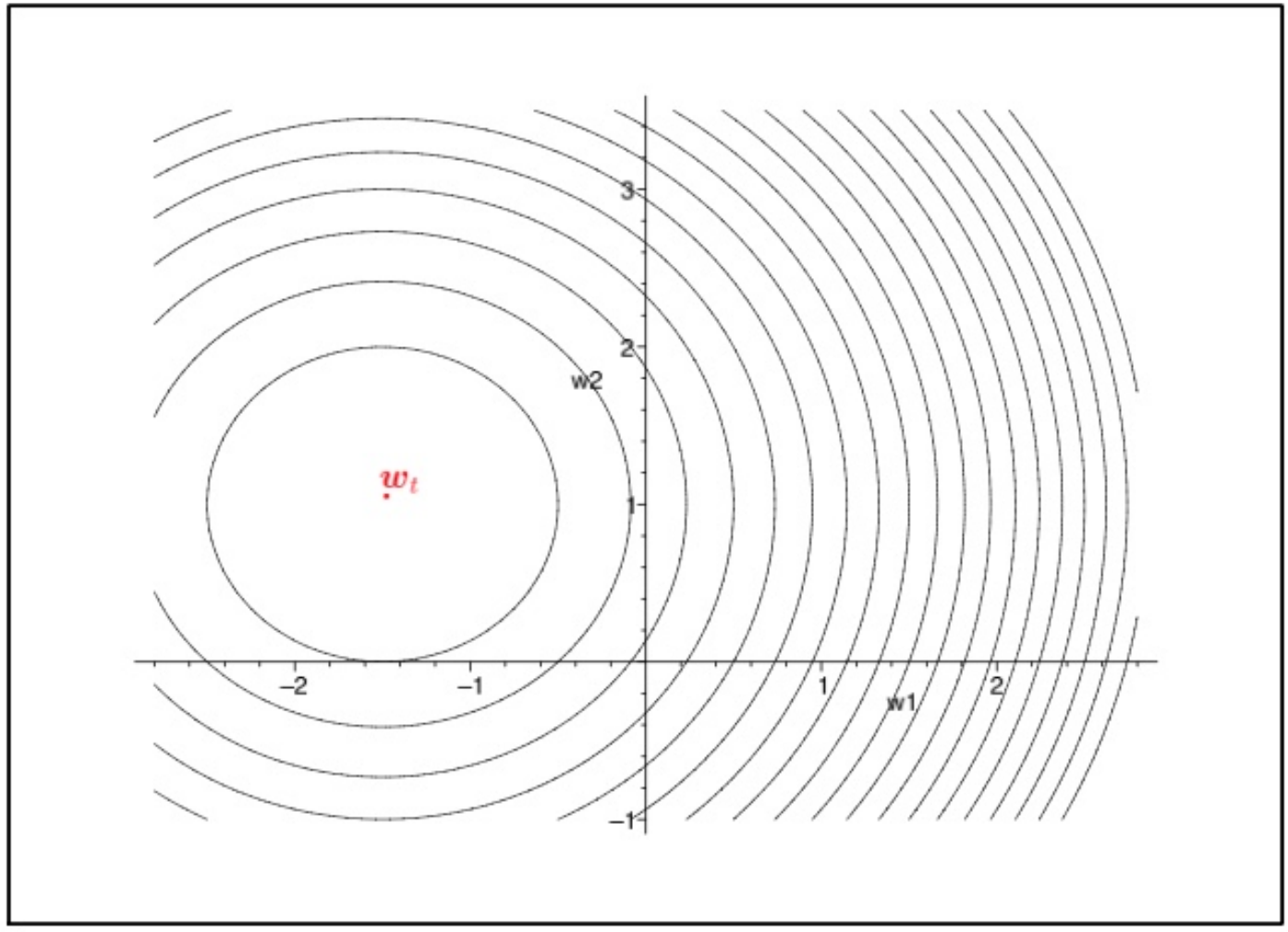

Second step: Relate $\Delta_F(\boldsymbol{w}_t, \boldsymbol{w}_{t+1})$ to loss $L_t(\boldsymbol{w}_t)$

Loss & divergence are dependent

Get $\Delta_F(\boldsymbol{w}_t, \boldsymbol{w}_{t+1}) \leq$ const. $L_t(\boldsymbol{w}_t)$

Then solve for $\sum_t L_t(\boldsymbol{w}_t)$

Yield bounds of the form

$$\sum_t L_t(\boldsymbol{w}_t) \leq a \sum_t L_t(\boldsymbol{u}) + b\, \Delta_F(\boldsymbol{u}, \boldsymbol{w}_1)$$

a, b constants, $a > 1$.

Regret bounds ($a = 1$):

time changing η, subtler analysis [AG]

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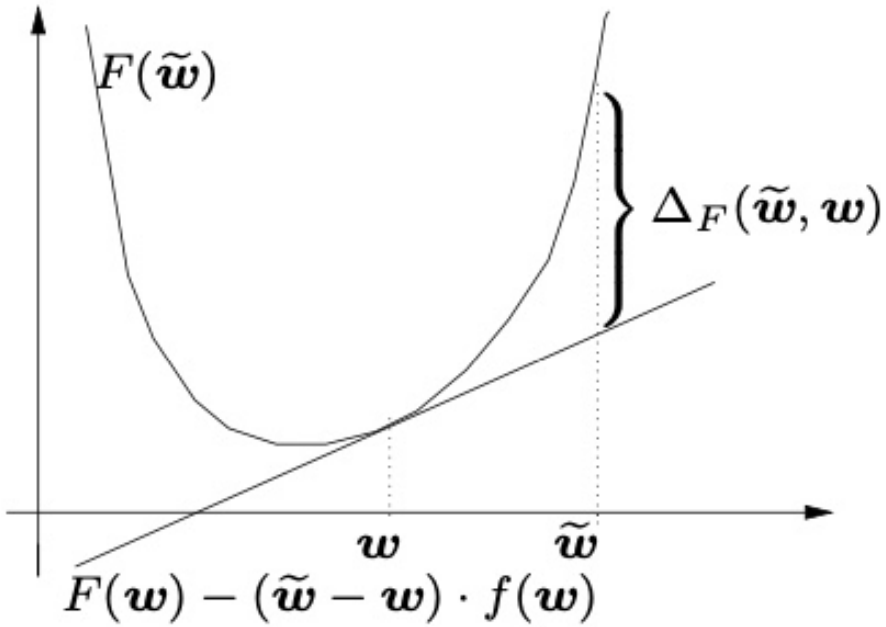

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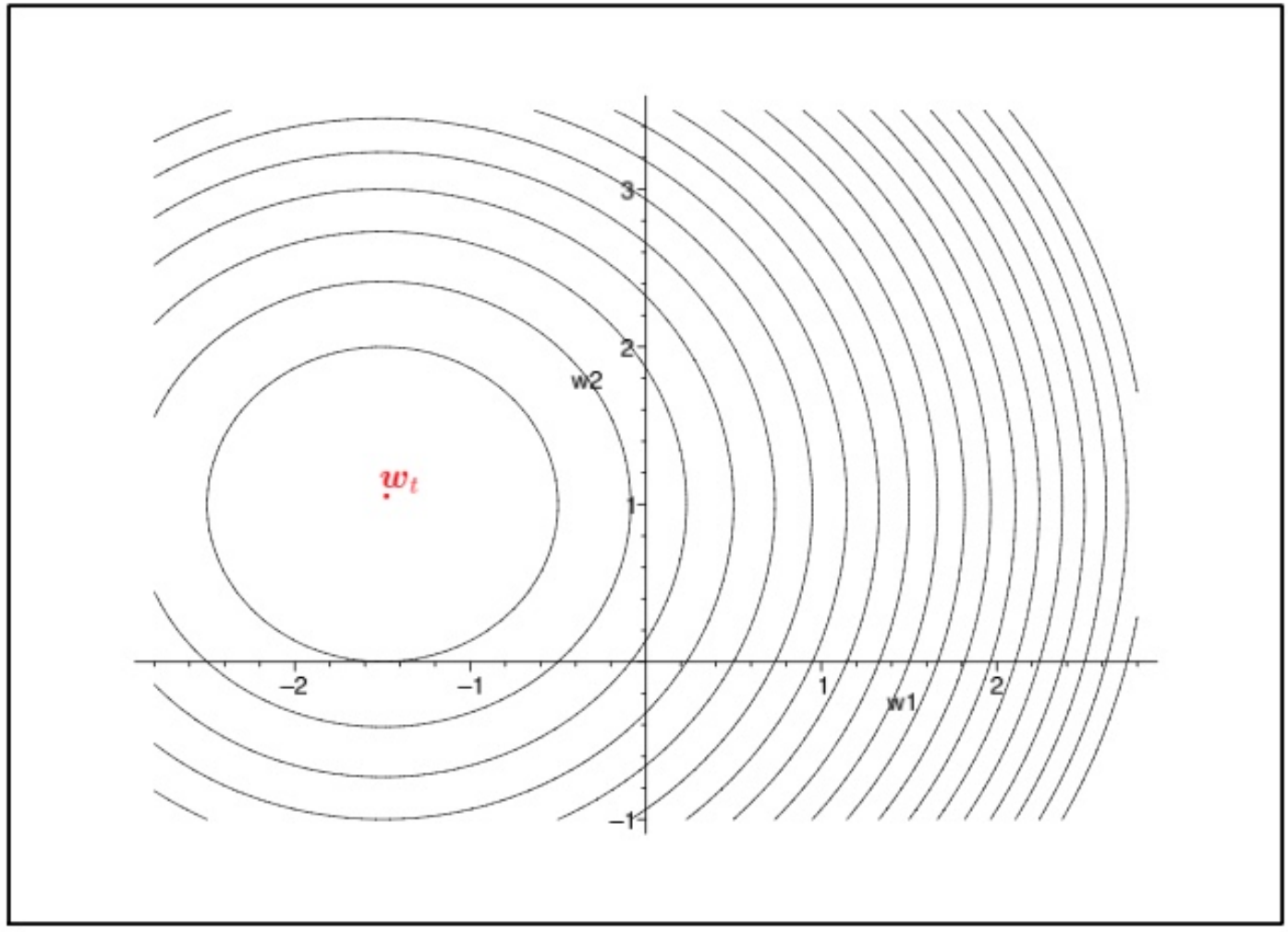

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$$\Delta_F(\widetilde{\boldsymbol{w}}, \boldsymbol{w}) = \frac{1}{2}||\widetilde{\boldsymbol{w}}||_q^2 + \frac{1}{2}||\boldsymbol{w}||_q^2 - \widetilde{\boldsymbol{w}} \cdot f(\boldsymbol{w})$$

When $p = q = 2$ this reduces to squared Euclidean distance (Widrow-Hoff).

Examples

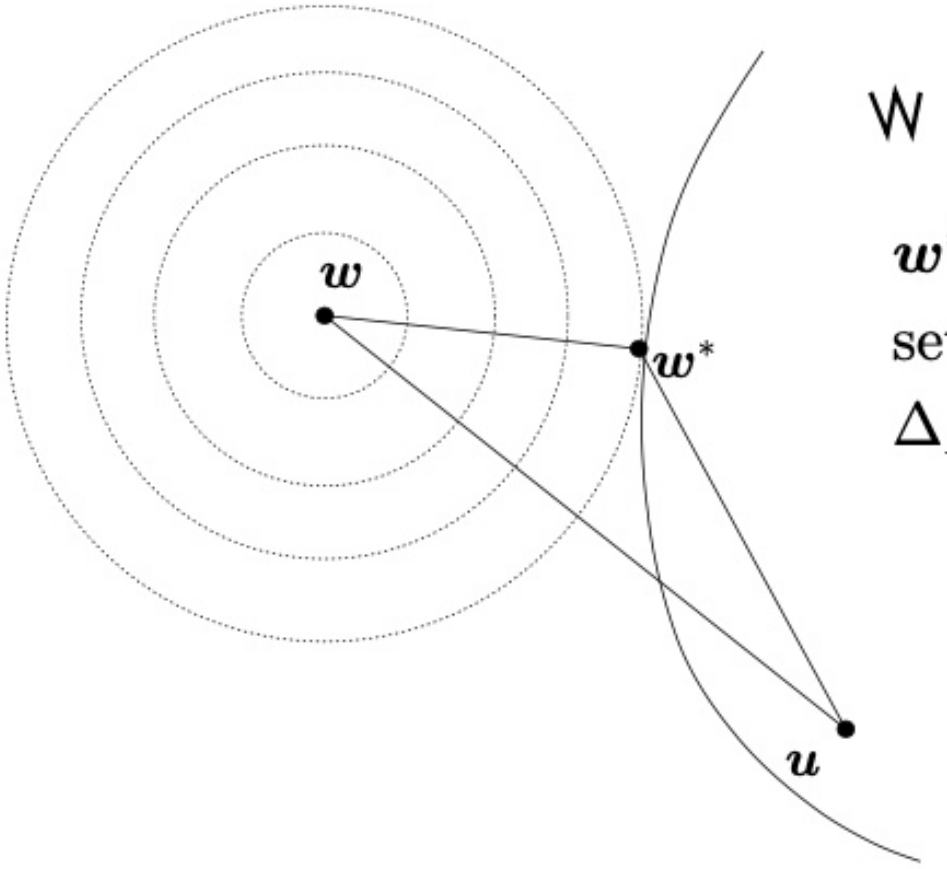
Squared Euclidean Distance

$$\begin{aligned} F(\boldsymbol{w}) &= ||\boldsymbol{w}||_2^2/2 \\ f(\boldsymbol{w}) &= \boldsymbol{w} \\ \Delta_F(\widetilde{\boldsymbol{w}}, \boldsymbol{w}) &= ||\widetilde{\boldsymbol{w}}||_2^2/2 - ||\boldsymbol{w}||_2^2/2 - (\widetilde{\boldsymbol{w}} - \boldsymbol{w}) \cdot \boldsymbol{w} \\ &= ||\widetilde{\boldsymbol{w}} - \boldsymbol{w}||_2^2/2 \end{aligned}$$

(Unnormalized) Relative Entropy

$$\begin{aligned} F(\boldsymbol{w}) &= \sum_i (w_i \ln w_i - w_i) \\ f(\boldsymbol{w}) &= \ln \boldsymbol{w} \\ \Delta_F(\widetilde{\boldsymbol{w}}, \boldsymbol{w}) &= \sum_i \left(\widetilde{w_i} \ln \frac{\widetilde{w_i}}{w_i} + w_i - \widetilde{w_i} \right) \end{aligned}$$

A Pythagorean Theorem [Br,Cs,A,HW]



$\mathcal{W}$

$\boldsymbol{w}^*$ is projection of $\boldsymbol{w}$ onto convex set $\mathcal{W}$ w.r.t. Bregman divergence Δ_F:

$$\boldsymbol{w}^* = \operatorname*{argmin}_{\boldsymbol{u} \in \mathcal{W}} \Delta_F(\boldsymbol{u}, \boldsymbol{w})$$

Theorem:

$$\Delta_F(\boldsymbol{u}, w) \geq \Delta_F(\boldsymbol{u}, \boldsymbol{w}^*) + \Delta_F(\boldsymbol{w}^*, \boldsymbol{w})$$

Bregman Divergences: more properties

6. $\nabla_{\widetilde{\boldsymbol{w}}} \Delta_F(\widetilde{\boldsymbol{w}}, \boldsymbol{w})$

$$\begin{aligned} &= \nabla F(\widetilde{\boldsymbol{w}}) - \nabla_{\widetilde{\boldsymbol{w}}}(\widetilde{\boldsymbol{w}} \nabla_{\boldsymbol{w}} F(\boldsymbol{w})) \\ &= f(\widetilde{\boldsymbol{w}}) - f(\boldsymbol{w}) \end{aligned}$$

7. $\Delta_F(\boldsymbol{w}_1, \boldsymbol{w}_2) + \Delta_F(\boldsymbol{w}_2, \boldsymbol{w}_3)$

$$\begin{aligned} &= F(\boldsymbol{w}_1) - F(\boldsymbol{w}_2) - (\boldsymbol{w}_1 - \boldsymbol{w}_2) f(\boldsymbol{w}_2) \\ &\quad F(\boldsymbol{w}_2) - F(\boldsymbol{w}_3) - (\boldsymbol{w}_2 - \boldsymbol{w}_3) f(\boldsymbol{w}_3) \\ &= \Delta_F(\boldsymbol{w}_1, \boldsymbol{w}_3) + (\boldsymbol{w}_1 - \boldsymbol{w}_2) \cdot (f(\boldsymbol{w}_3) - f(\boldsymbol{w}_2)) \end{aligned}$$

Bregman Divergences: Simple Properties

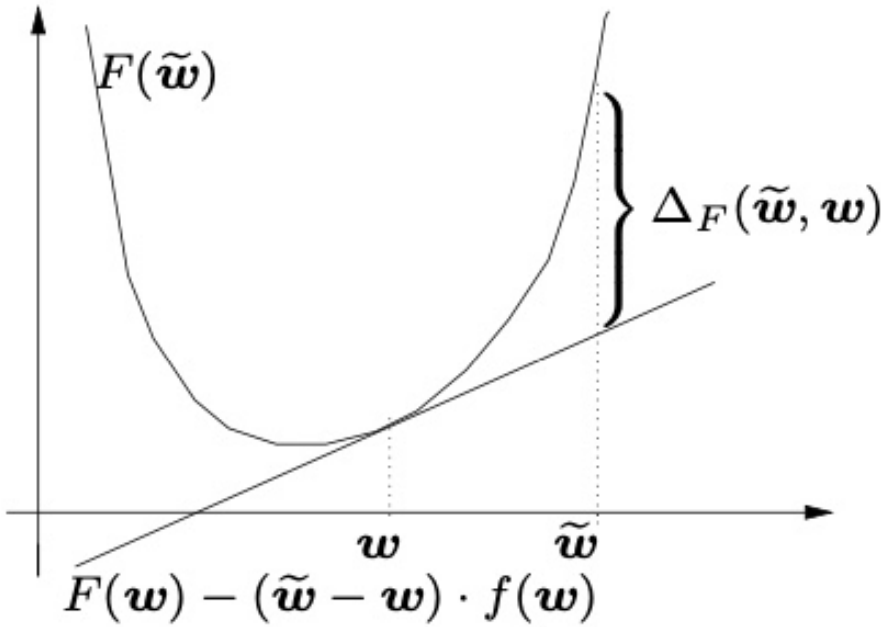
1. $\Delta_F(\widetilde{\boldsymbol{w}}, \boldsymbol{w})$ is convex in $\widetilde{\boldsymbol{w}}$
2. $\Delta_F(\widetilde{\boldsymbol{w}}, \boldsymbol{w}) \geq 0$
   If F convex equality holds iff $\widetilde{\boldsymbol{w}} = \boldsymbol{w}$
3. Usually not symmetric: $\Delta_F(\widetilde{\boldsymbol{w}}, \boldsymbol{w}) \neq \Delta_F(\boldsymbol{w}, \widetilde{\boldsymbol{w}})$
4. Linearity (for $a \geq 0$):
   $\Delta_{F+a\,H}(\widetilde{\boldsymbol{w}}, \boldsymbol{w}) = \Delta_F(\widetilde{\boldsymbol{w}}, \boldsymbol{w}) + a\,\Delta_H(\widetilde{\boldsymbol{w}}, \boldsymbol{w})$
5. Unaffected by linear terms ($a \in \mathbf{R}$, $\boldsymbol{b} \in \mathbf{R}^n$):
   $\Delta_{H+a\widetilde{\boldsymbol{w}}+\boldsymbol{b}}(\widetilde{\boldsymbol{w}}, \boldsymbol{w}) = \Delta_H(\widetilde{\boldsymbol{w}}, \boldsymbol{w})$

Bregman Divergences [Br,CL,Cs]

For any differentiable convex function F

$$\Delta_F(\widetilde{\boldsymbol{w}}, \boldsymbol{w}) = F(\widetilde{\boldsymbol{w}}) - F(\boldsymbol{w}) - (\widetilde{\boldsymbol{w}} - \boldsymbol{w}) \cdot \underbrace{\nabla_{\boldsymbol{w}} F(\boldsymbol{w})}_{f(\boldsymbol{w})}$$

$$= F(\widetilde{\boldsymbol{w}}) - \begin{array}{c}\text{supporting hyperplane} \\ \text{through } (\boldsymbol{w}, F(\boldsymbol{w}))\end{array}$$



Bregman Divergences [Br,CL,Cs]

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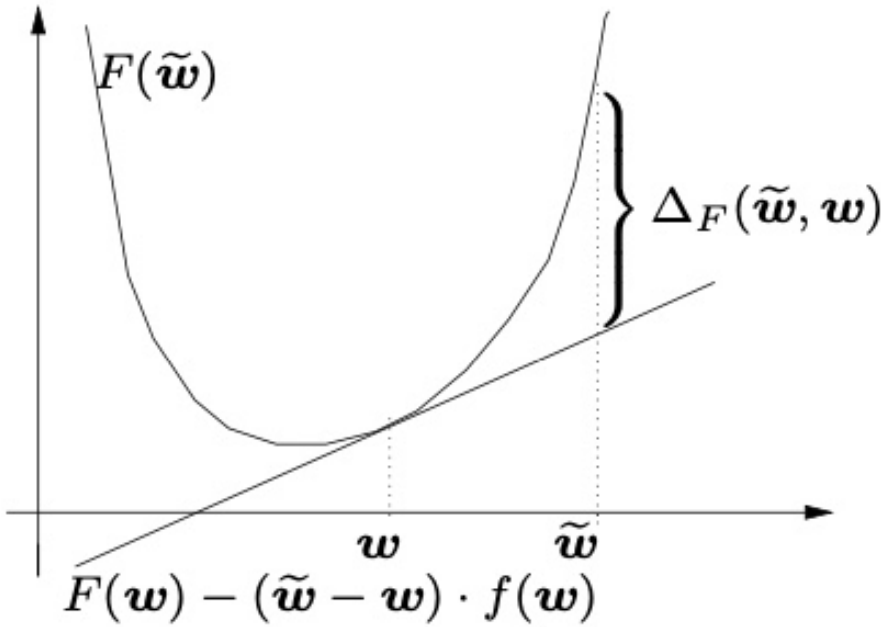

Divergence as Regularization

- Use a divergence term $D(\omega_t, \omega_{t+1})$ to measure the size of the update.
- Find ω_{t+1} to balance loss reduction and divergence:

$$\text{argmin}_\omega = L(\omega, x_t) + \eta D(\omega, \omega_t)$$

- If update is small, we can think of it in terms of gradients:

$$\bigtriangledown_\omega (L(\omega, x_t) + \eta D(\omega, \omega_t))$$