The Context Algorithm

Yoav Freund

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Outline

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Fixed Length Markov Models

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Variable Length Markov Model (VMM)

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Slides from Frans Willems

The online Bayes Algorithm

- Total loss of expert i

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- Prediction of algorithm A

$$\mathbf{p}_A^t = \frac{\sum_{i=1}^{N} w_i^t \mathbf{p}_i^t}{\sum_{i=1}^{N} w_i^t}$$

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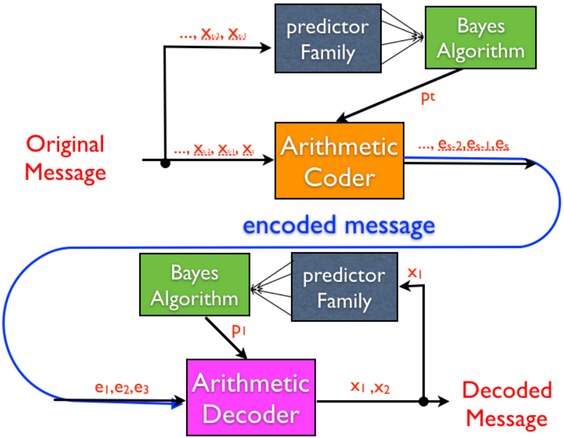
Universal Online coding



Combining large predictor families

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- We talked about the KT preictor.
- Today we consider the much richer set of variable length markov models.
- The set of predictors is of exponential size, but the algorithm is efficient.

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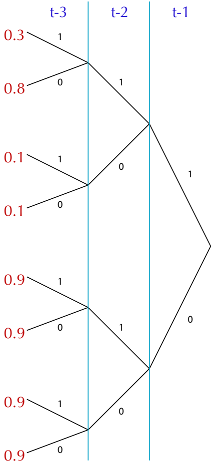

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- Total regret is at most $2^{k-1} \log T$

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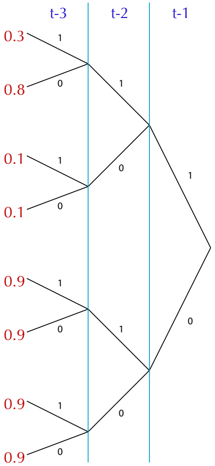

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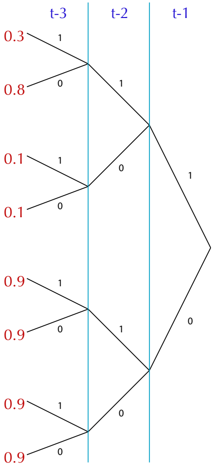

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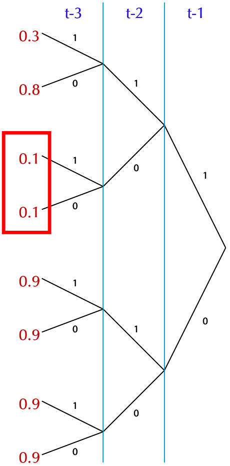

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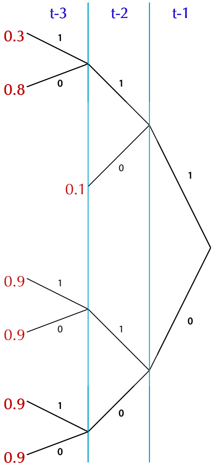

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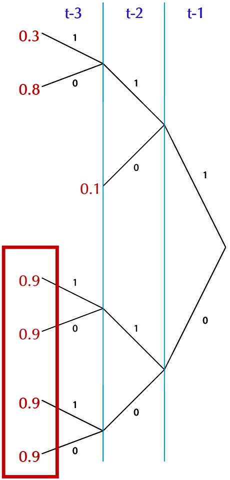

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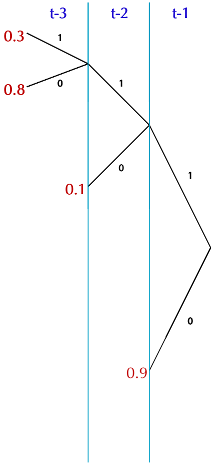

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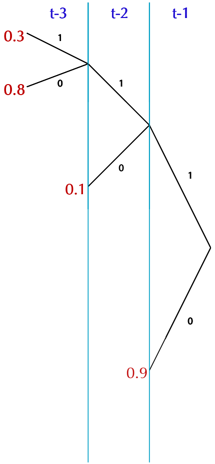

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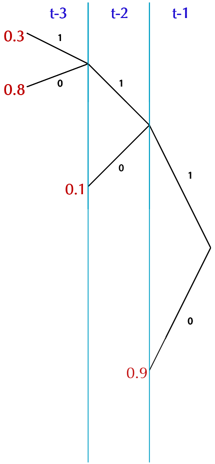

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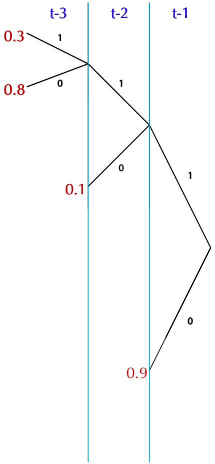

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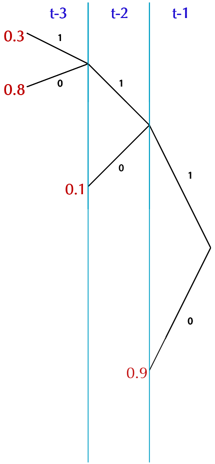

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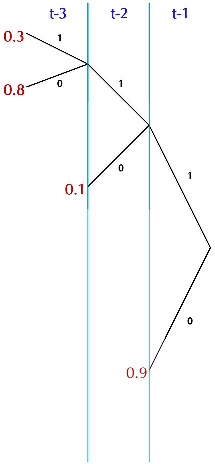

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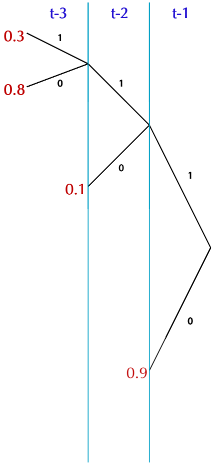

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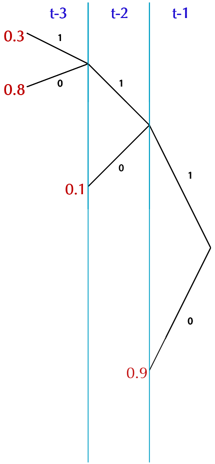

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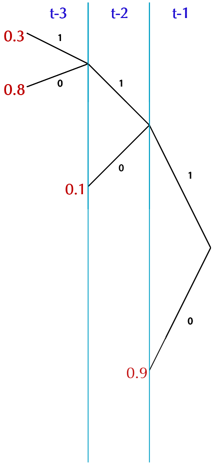

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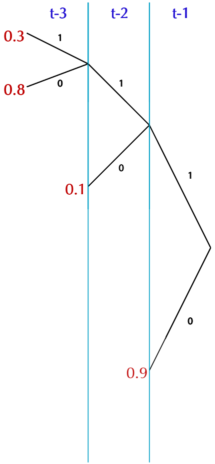

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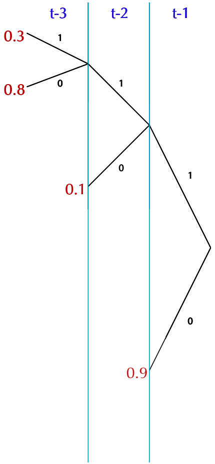

- Reducing number of leaves from 8 to 4 means
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- When we have little data, we can get better prediction even if the children are not Exactly the same

Prefix trees / Tries

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- First - simple but inefficient algorithm, Second - efficient algorithms.

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- The number of prunings of a full tree of depth k is $O(2^{2^k})$ while maintaining all of the counts requires $O(2^k)$.

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Efficient averaging over the prior (procedure)

- This is not the method used in the original paper, it appears in

  Willems, Frans MJ, Ali Nowbakht, and Paul AJ Volf. "Maximum a posteriori probability tree models." (2002)

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Conditioning and defining the mixing factor

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- Complexity: each forward and backwards takes O(depth of tree)

Slides from Frans Willems

Implementation

Assume that in node s the counts $a_s(x_1^{t-1})$ and $b_s(x_1^{t-1})$ are stored, as well as $\beta^s(x_1^{t-1})$. We then get the following sequence of operations:

1. Node $0s$ delivers cond. wei. probability $P_w^{0s}(X_t = 1|x_1^{t-1})$ to node s.
2. Cond. est. probability $P_e^s(X_t = 1|x_1^{t-1})$ is determined as follows:
$$P_e^s(X_t = 1|x_1^{t-1}) = \frac{b_s(x_1^{t-1}) + 1/2}{a_s(x_1^{t-1}) + b_s(x_1^{t-1}) + 1}. \tag{3}$$
3. Now $P_w^s(X_t = 1|x_1^{t-1})$ can be computed as in (1).
4. The ratio $\beta^s(\cdot)$ is then updated with symbol x_t as follows:
$$\beta^s(x_1^{t-1}, x_t) = \beta^s(x_1^{t-1}) \cdot \frac{P_e^s(X_t = x_t|x_1^{t-1})}{P_w^{0s}(X_t = x_t|x_1^{t-1})}. \tag{4}$$
5. Finally, depending on the value x_t, either count $a_s(x_1^{t-1})$ or $b_s(x_1^{t-1})$ is incremented.