

# Introduction to Online Learning Algorithms

Yoav Freund

January 7, 2025

# Outline

About this Course

Halving Algorithm

Perceptron

Estimating the mean

## Class web site

- ▶ All of the class material is available from the github repository  
<https://github.com/yoavfreund/2025-online-learning>

## Class web site

- ▶ All of the class material is available from the github repository  
<https://github.com/yoavfreund/2025-online-learning>



## Class web site

- ▶ All of the class material is available from the github repository

<https://github.com/yoavfreund/2025-online-learning>



- ▶ Instructor: Yoav Freund: [yfreund@ucsd.edu](mailto:yfreund@ucsd.edu)

## Class web site

- ▶ All of the class material is available from the github repository

<https://github.com/yoavfreund/2025-online-learning>



- ▶
- ▶ Instructor: Yoav Freund: [yfreund@ucsd.edu](mailto:yfreund@ucsd.edu)
- ▶ TA: Parsa Mirtaheri: [smirtaheri@ucsd.edu](mailto:smirtaheri@ucsd.edu)

## Class web site

- ▶ All of the class material is available from the github repository

<https://github.com/yoavfreund/2025-online-learning>



- ▶
- ▶ Instructor: Yoav Freund: [yfreund@ucsd.edu](mailto:yfreund@ucsd.edu)
- ▶ TA: Parsa Mirtaheri: [smirtaheri@ucsd.edu](mailto:smirtaheri@ucsd.edu)
- ▶ Office Hours: TBD

## HW / Evaluation

- ▶ 5 HW assignments for  $5 \cdot 15 = 75$  opints



## HW / Evaluation

- ▶ 5 HW assignments for  $5 \cdot 15 = 75$  opints
- ▶ A final for 25 points.

# Example trace for Halving Algorithm

## Example trace for Halving Algorithm

expert1  
expert2  
expert3  
expert4  
expert5  
expert6  
expert7  
expert8

alg.

## Example trace for Halving Algorithm

expert1  
expert2  
expert3  
expert4  
expert5  
expert6  
expert7  
expert8

alg.

outcome

## Example trace for Halving Algorithm

	$t = 1$
expert1	1
expert2	1
expert3	0
expert4	1
expert5	1
expert6	0
expert7	1
expert8	1

alg.  
outcome

## Example trace for Halving Algorithm

	$t = 1$
expert1	1
expert2	1
expert3	0
expert4	1
expert5	1
expert6	0
expert7	1
expert8	1
alg.	1
outcome	

## Example trace for Halving Algorithm

	$t = 1$
expert1	1
expert2	1
expert3	0
expert4	1
expert5	1
expert6	0
expert7	1
expert8	1
alg.	1
outcome	1

## Example trace for Halving Algorithm

	$t = 1$	$t = 2$
expert1	1	1
expert2	1	0
expert3	0	-
expert4	1	0
expert5	1	0
expert6	0	-
expert7	1	1
expert8	1	1
alg.	1	
outcome	1	



## Example trace for Halving Algorithm

	$t = 1$	$t = 2$
expert1	1	1
expert2	1	0
expert3	0	-
expert4	1	0
expert5	1	0
expert6	0	-
expert7	1	1
expert8	1	1
alg.	1	0
outcome	1	

## Example trace for Halving Algorithm

	$t = 1$	$t = 2$
expert1	1	1
expert2	1	0
expert3	0	-
expert4	1	0
expert5	1	0
expert6	0	-
expert7	1	1
expert8	1	1
alg.	1	0
outcome	1	1

## Example trace for Halving Algorithm

	$t = 1$	$t = 2$	$t = 3$
expert1	1	1	1
expert2	1	0	-
expert3	0	-	-
expert4	1	0	-
expert5	1	0	-
expert6	0	-	-
expert7	1	1	1
expert8	1	1	1
alg.	1	0	
outcome	1	1	

## Example trace for Halving Algorithm

	$t = 1$	$t = 2$	$t = 3$
expert1	1	1	1
expert2	1	0	-
expert3	0	-	-
expert4	1	0	-
expert5	1	0	-
expert6	0	-	-
expert7	1	1	1
expert8	1	1	1
alg.	1	0	1
outcome	1	1	

## Example trace for Halving Algorithm

	$t = 1$	$t = 2$	$t = 3$
expert1	1	1	1
expert2	1	0	-
expert3	0	-	-
expert4	1	0	-
expert5	1	0	-
expert6	0	-	-
expert7	1	1	1
expert8	1	1	1
alg.	1	0	1
outcome	1	1	1

## Example trace for Halving Algorithm

	$t = 1$	$t = 2$	$t = 3$	$t = 4$
expert1	1	1	1	1
expert2	1	0	-	-
expert3	0	-	-	-
expert4	1	0	-	-
expert5	1	0	-	-
expert6	0	-	-	-
expert7	1	1	1	1
expert8	1	1	1	0
alg.	1	0	1	
outcome	1	1	1	

## Example trace for Halving Algorithm

	$t = 1$	$t = 2$	$t = 3$	$t = 4$
expert1	1	1	1	1
expert2	1	0	-	-
expert3	0	-	-	-
expert4	1	0	-	-
expert5	1	0	-	-
expert6	0	-	-	-
expert7	1	1	1	1
expert8	1	1	1	0
alg.	1	0	1	1
outcome	1	1	1	

## Example trace for Halving Algorithm

	$t = 1$	$t = 2$	$t = 3$	$t = 4$
expert1	1	1	1	1
expert2	1	0	-	-
expert3	0	-	-	-
expert4	1	0	-	-
expert5	1	0	-	-
expert6	0	-	-	-
expert7	1	1	1	1
expert8	1	1	1	0
alg.	1	0	1	1
outcome	1	1	1	0



## Example trace for Halving Algorithm

	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
expert1	1	1	1	1	-
expert2	1	0	-	-	-
expert3	0	-	-	-	-
expert4	1	0	-	-	-
expert5	1	0	-	-	-
expert6	0	-	-	-	-
expert7	1	1	1	1	-
expert8	1	1	1	0	0
alg.	1	0	1	1	
outcome	1	1	1	0	

## Example trace for Halving Algorithm

	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
expert1	1	1	1	1	-
expert2	1	0	-	-	-
expert3	0	-	-	-	-
expert4	1	0	-	-	-
expert5	1	0	-	-	-
expert6	0	-	-	-	-
expert7	1	1	1	1	-
expert8	1	1	1	0	0
alg.	1	0	1	1	0
outcome	1	1	1	0	

## Example trace for Halving Algorithm

	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
expert1	1	1	1	1	-
expert2	1	0	-	-	-
expert3	0	-	-	-	-
expert4	1	0	-	-	-
expert5	1	0	-	-	-
expert6	0	-	-	-	-
expert7	1	1	1	1	-
expert8	1	1	1	0	0
alg.	1	0	1	1	0
outcome	1	1	1	0	0

## Mistake bound for Halving algorithm

- ▶ Each time algorithm makes a mistakes, the pool of perfect experts is halved (at least).

## Mistake bound for Halving algorithm

- ▶ Each time algorithm makes a mistakes, the pool of perfect experts is halved (at least).
- ▶ We assume that at least one expert is perfect.

## Mistake bound for Halving algorithm

- ▶ Each time algorithm makes a mistakes, the pool of perfect experts is halved (at least).
- ▶ We assume that at least one expert is perfect.
- ▶ Number of mistakes is at most  $\log_2 N$ .

## Mistake bound for Halving algorithm

- ▶ Each time algorithm makes a mistake, the pool of perfect experts is halved (at least).
- ▶ We assume that at least one expert is perfect.
- ▶ Number of mistakes is at most  $\log_2 N$ .
- ▶ No stochastic assumptions whatsoever.

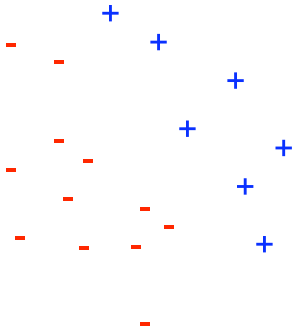
## Mistake bound for Halving algorithm

- ▶ Each time algorithm makes a mistake, the pool of perfect experts is halved (at least).
- ▶ We assume that at least one expert is perfect.
- ▶ Number of mistakes is at most  $\log_2 N$ .
- ▶ No stochastic assumptions whatsoever.
- ▶ Proof is based on combining a lower and upper bounds on the number of perfect experts.

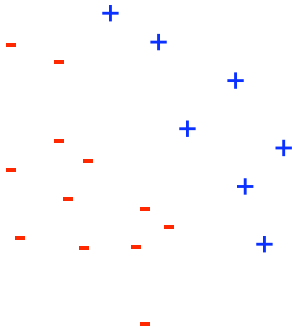


# The Perceptron Problem

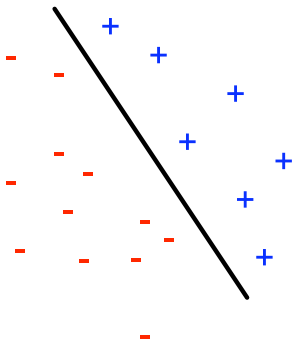
# The Perceptron Problem



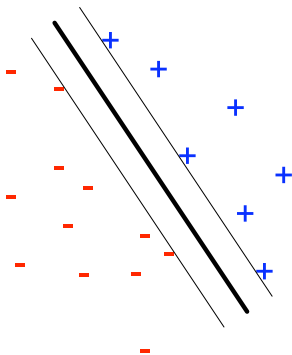
# The Perceptron Problem



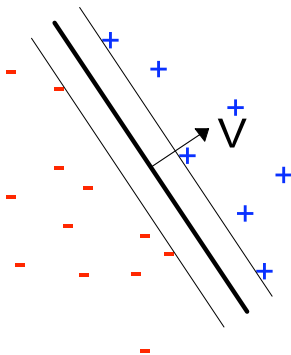
# The Perceptron Problem



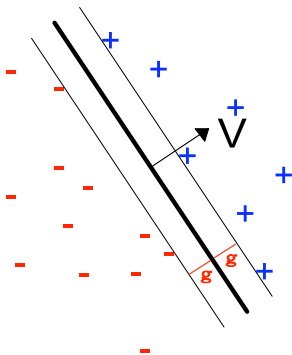
# The Perceptron Problem



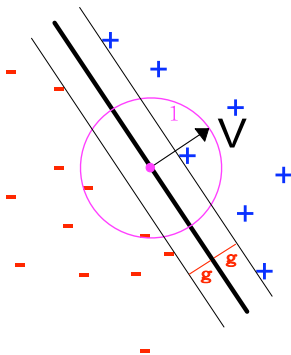
# The Perceptron Problem



# The Perceptron Problem

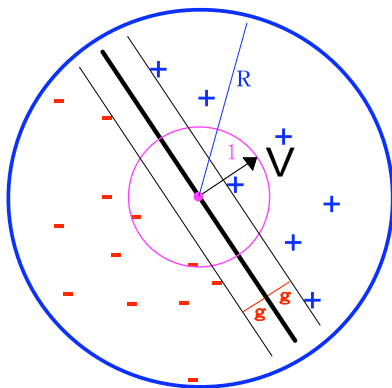


# The Perceptron Problem

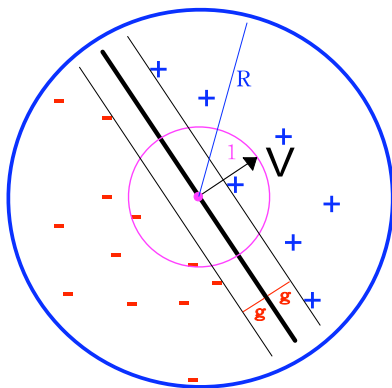




# The Perceptron Problem

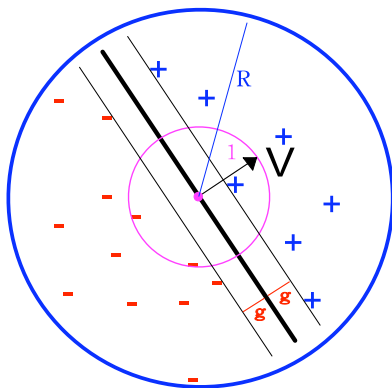


# The Perceptron Problem



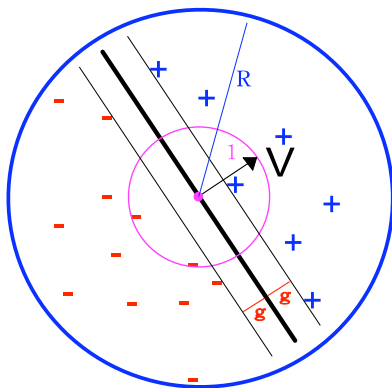
►  $\|\vec{V}\| = 1$

# The Perceptron Problem



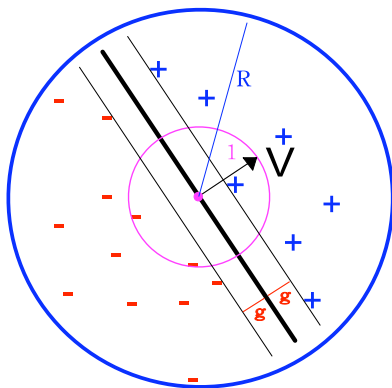
- ▶  $\|\vec{V}\| = 1$
- ▶ Example =  $(\vec{X}, y)$ ,  
 $y \in \{-1, +1\}$ .

# The Perceptron Problem



- ▶  $\|\vec{V}\| = 1$
- ▶ Example =  $(\vec{X}, y)$ ,  
 $y \in \{-1, +1\}$ .
- ▶  $\forall \vec{X}, \|\vec{X}\| \leq R$ .

# The Perceptron Problem



- ▶  $\|\vec{V}\| = 1$
- ▶ Example =  $(\vec{X}, y)$ ,  
 $y \in \{-1, +1\}$ .
- ▶  $\forall \vec{X}, \|\vec{X}\| \leq R$ .
- ▶  $\forall (\vec{X}, y),$   
 $y(\vec{X} \cdot \vec{V}) \geq g$

# The Perceptron learning algorithm

- ▶ An online algorithm. Examples presented one by one.

# The Perceptron learning algorithm

- ▶ An online algorithm. Examples presented one by one.
- ▶ start with  $\vec{W}_0 = \vec{0}$ .

# The Perceptron learning algorithm

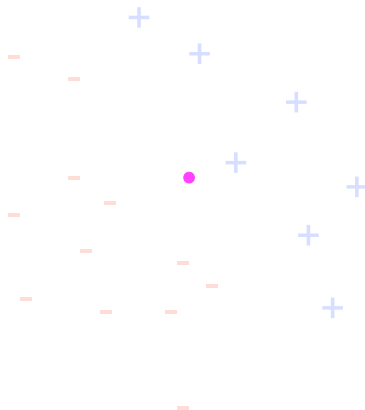
- ▶ An online algorithm. Examples presented one by one.
- ▶ start with  $\vec{W}_0 = \vec{0}$ .
- ▶ If mistake:  $(\vec{W}_i \cdot \vec{X}_i)y_i \leq 0$



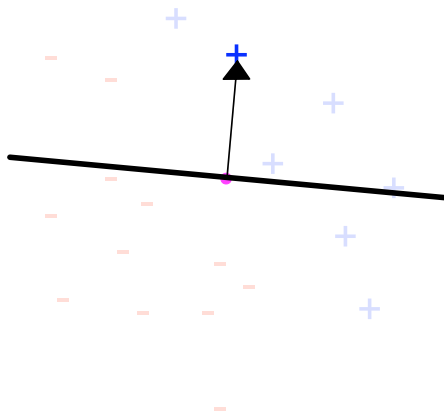
# The Perceptron learning algorithm

- ▶ An online algorithm. Examples presented one by one.
- ▶ start with  $\vec{W}_0 = \vec{0}$ .
- ▶ If mistake:  $(\vec{W}_i \cdot \vec{X}_i)y_i \leq 0$ 
  - ▶ Update  $\vec{W}_{i+1} = \vec{W}_i + y_i \vec{X}_i$ .

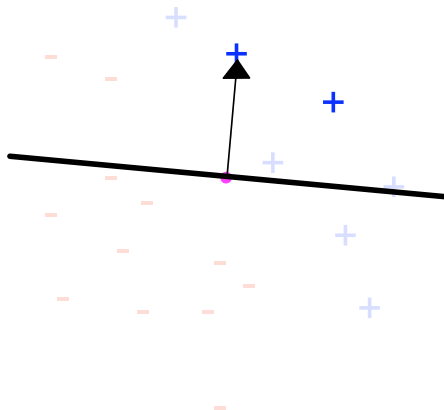
## Example trace for the perceptron algorithm



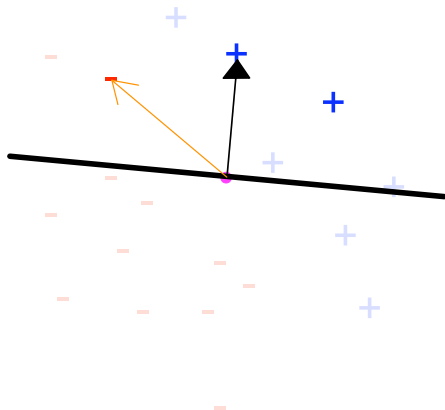
## Example trace for the perceptron algorithm



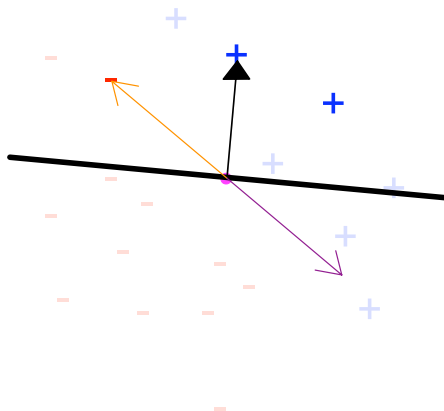
## Example trace for the perceptron algorithm



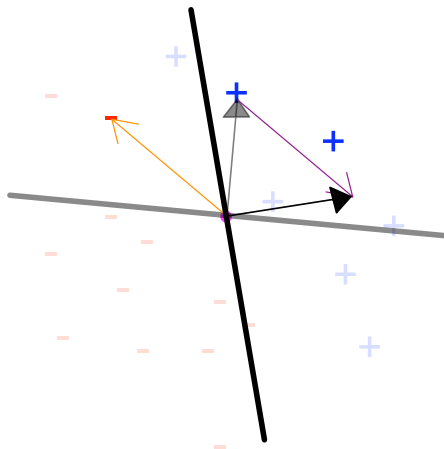
## Example trace for the perceptron algorithm



## Example trace for the perceptron algorithm



## Example trace for the perceptron algorithm



## Bound on number of mistakes

- ▶ The number of mistakes that the perceptron algorithm can make is at most  $\left(\frac{R}{g}\right)^2$ .



## Bound on number of mistakes

- ▶ The number of mistakes that the perceptron algorithm can make is at most  $\left(\frac{R}{g}\right)^2$ .
- ▶ Proof by combining upper and lower bounds on  $\|\vec{W}\|$ .

## Pythagorean Lemma

If  $(\vec{W}_i \cdot X_i)y < 0$  then

## Pythagorean Lemma

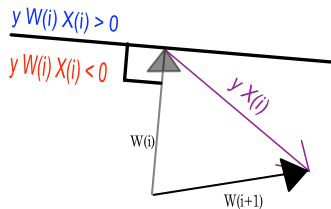
If  $(\vec{W}_i \cdot X_i)y < 0$  then

$$\|\vec{W}_{i+1}\|^2 = \|\vec{W}_i + y_i \vec{X}_i\|^2 \leq \|\vec{W}_i\|^2 + \|\vec{X}_i\|^2$$

## Pythagorean Lemma

If  $(\vec{W}_i \cdot \vec{X}_i)y < 0$  then

$$\|\vec{W}_{i+1}\|^2 = \|\vec{W}_i + y_i \vec{X}_i\|^2 \leq \|\vec{W}_i\|^2 + \|\vec{X}_i\|^2$$



# Upper bound on $\|\vec{W}_i\|$

## Upper bound on $\|\vec{W}_i\|$

Proof by induction

- ▶ Claim:  $\|\vec{W}_i\|^2 \leq iR^2$

## Upper bound on $\|\vec{W}_i\|$

Proof by induction

- ▶ Claim:  $\|\vec{W}_i\|^2 \leq iR^2$
- ▶ Base:  $i = 0, \|\vec{W}_0\|^2 = 0$

## Upper bound on $\|\vec{W}_i\|$

Proof by induction

- ▶ Claim:  $\|\vec{W}_i\|^2 \leq iR^2$
- ▶ Base:  $i = 0$ ,  $\|\vec{W}_0\|^2 = 0$
- ▶ Induction step (assume for  $i$  and prove for  $i + 1$ ):  
$$\begin{aligned}\|\vec{W}_{i+1}\|^2 &\leq \|\vec{W}_i\|^2 + \|\vec{X}_i\|^2 \\ &\leq \|\vec{W}_i\|^2 + R^2 \leq (i + 1)R^2\end{aligned}$$



# Lower bound on $\|\vec{W}_i\|$

## Lower bound on $\|\vec{W}_i\|$

$$\|\vec{W}_i\| \geq \vec{W}_i \cdot \vec{V} \text{ because } \|\vec{V}\| = 1.$$

## Lower bound on $\|\vec{W}_i\|$

$\|\vec{W}_i\| \geq \vec{W}_i \cdot \vec{V}$  because  $\|\vec{V}\| = 1$ .

Let  $i$  denote the number of mistakes made so far.

## Lower bound on $\|\vec{W}_i\|$

$\|\vec{W}_i\| \geq \vec{W}_i \cdot \vec{V}$  because  $\|\vec{V}\| = 1$ .

Let  $i$  denote the number of mistakes made so far.

We prove a lower bound on  $\vec{W}_i \cdot \vec{V}$  by induction over  $i$

- Claim:  $\vec{W}_i \cdot \vec{V} \geq ig$

## Lower bound on $\|\vec{W}_i\|$

$\|\vec{W}_i\| \geq \vec{W}_i \cdot \vec{V}$  because  $\|\vec{V}\| = 1$ .

Let  $i$  denote the number of mistakes made so far.

We prove a lower bound on  $\vec{W}_i \cdot \vec{V}$  by induction over  $i$

- ▶ Claim:  $\vec{W}_i \cdot \vec{V} \geq ig$
- ▶ Base:  $i = 0$ ,  $\vec{W}_0 \cdot \vec{V} = 0$

## Lower bound on $\|\vec{W}_i\|$

$\|\vec{W}_i\| \geq \vec{W}_i \cdot \vec{V}$  because  $\|\vec{V}\| = 1$ .

Let  $i$  denote the number of mistakes made so far.

We prove a lower bound on  $\vec{W}_i \cdot \vec{V}$  by induction over  $i$

- ▶ Claim:  $\vec{W}_i \cdot \vec{V} \geq ig$
- ▶ Base:  $i = 0$ ,  $\vec{W}_0 \cdot \vec{V} = 0$
- ▶ Induction step (assume for  $i$  and prove for  $i + 1$ ):  
$$\vec{W}_{i+1} \cdot \vec{V} = \left( \vec{W}_i + \vec{X}_i y_i \right) \vec{V}$$

## Lower bound on $\|\vec{W}_i\|$

$\|\vec{W}_i\| \geq \vec{W}_i \cdot \vec{V}$  because  $\|\vec{V}\| = 1$ .

Let  $i$  denote the number of mistakes made so far.

We prove a lower bound on  $\vec{W}_i \cdot \vec{V}$  by induction over  $i$

- ▶ Claim:  $\vec{W}_i \cdot \vec{V} \geq ig$
- ▶ Base:  $i = 0$ ,  $\vec{W}_0 \cdot \vec{V} = 0$
- ▶ Induction step (assume for  $i$  and prove for  $i + 1$ ):  
$$\vec{W}_{i+1} \cdot \vec{V} = \left( \vec{W}_i + \vec{X}_i y_i \right) \vec{V}$$

## Lower bound on $\|\vec{W}_i\|$

$\|\vec{W}_i\| \geq \vec{W}_i \cdot \vec{V}$  because  $\|\vec{V}\| = 1$ .

Let  $i$  denote the number of mistakes made so far.

We prove a lower bound on  $\vec{W}_i \cdot \vec{V}$  by induction over  $i$

- ▶ Claim:  $\vec{W}_i \cdot \vec{V} \geq ig$
- ▶ Base:  $i = 0$ ,  $\vec{W}_0 \cdot \vec{V} = 0$
- ▶ Induction step (assume for  $i$  and prove for  $i + 1$ ):  
$$\begin{aligned}\vec{W}_{i+1} \cdot \vec{V} &= (\vec{W}_i + \vec{X}_i y_i) \cdot \vec{V} = \vec{W}_i \cdot \vec{V} + y_i \vec{X}_i \cdot \vec{V} \\ &\geq ig + g = (i + 1)g\end{aligned}$$



# Combining the upper and lower bounds

## Combining the upper and lower bounds

$$(ig)^2 \leq \|\vec{W}_i\|^2 \leq iR^2$$

## Combining the upper and lower bounds

$$(ig)^2 \leq \|\vec{W}_i\|^2 \leq iR^2$$

Thus:

$$i \leq \left(\frac{R}{g}\right)^2$$

# The mean estimation game

- ▶ An adversary chooses a real number  $y_t \in [0, 1]$  and keeps it secret.

# The mean estimation game

- ▶ An adversary chooses a real number  $y_t \in [0, 1]$  and keeps it secret.
- ▶ You make a guess of the secret number  $x_t$

# The mean estimation game

- ▶ An adversary chooses a real number  $y_t \in [0, 1]$  and keeps it secret.
- ▶ You make a guess of the secret number  $x_t$
- ▶ The adversary reveals the secret and you pay  $(x_t - y_t)^2$

# The mean estimation game

- ▶ An adversary chooses a real number  $y_t \in [0, 1]$  and keeps it secret.
- ▶ You make a guess of the secret number  $x_t$
- ▶ The adversary reveals the secret and you pay  $(x_t - y_t)^2$
- ▶ You want to minimize  $\frac{1}{T} \sum_{t=1}^T (x_t - y_t)^2$

# The mean estimation game

- ▶ An adversary chooses a real number  $y_t \in [0, 1]$  and keeps it secret.
- ▶ You make a guess of the secret number  $x_t$
- ▶ The adversary reveals the secret and you pay  $(x_t - y_t)^2$
- ▶ You want to minimize  $\frac{1}{T} \sum_{t=1}^T (x_t - y_t)^2$
- ▶ Impossible without additional constraints.



# Adversary is a fixed distribution

- ▶ Suppose that the adversary draws  $y_1, y_2, \dots, y_T$  IID from a fixed distribution over  $[0, 1]$  with mean  $\mu$  and std  $\sigma$ .

# Adversary is a fixed distribution

- ▶ Suppose that the adversary draws  $y_1, y_2, \dots, y_T$  IID from a fixed distribution over  $[0, 1]$  with mean  $\mu$  and std  $\sigma$ .
- ▶ Optimal prediction  $x_t = \mu$

# Adversary is a fixed distribution

- ▶ Suppose that the adversary draws  $y_1, y_2, \dots, y_T$  IID from a fixed distribution over  $[0, 1]$  with mean  $\mu$  and std  $\sigma$ .
- ▶ Optimal prediction  $x_t = \mu$
- ▶  $E_Y [(\mu - Y)^2] = \sigma^2$

# Adversary is a fixed distribution

- ▶ Suppose that the adversary draws  $y_1, y_2, \dots, y_T$  IID from a fixed distribution over  $[0, 1]$  with mean  $\mu$  and std  $\sigma$ .
- ▶ Optimal prediction  $x_t = \mu$
- ▶  $E_Y [(\mu - Y)^2] = \sigma^2$
- ▶ Online prediction: predict  $x_{t+1}$  from  $Y^t = \langle Y_1, Y_2, \dots, Y_t \rangle$ .

## Adversary is a fixed distribution

- ▶ Suppose that the adversary draws  $y_1, y_2, \dots, y_T$  IID from a fixed distribution over  $[0, 1]$  with mean  $\mu$  and std  $\sigma$ .
- ▶ Optimal prediction  $x_t = \mu$
- ▶  $E_Y [(\mu - Y)^2] = \sigma^2$
- ▶ Online prediction: predict  $x_{t+1}$  from  $Y^t = \langle Y_1, Y_2, \dots, Y_t \rangle$ .
- ▶ **Expected regret**: compare performance of algorithm to  $\text{Regret} = E_{Y^T} [(x_t - Y_t)^2] - \sigma^2$

## Individual sequence bounds

- ▶ Make no assumption about how the sequence is generated.

# Individual sequence bounds

- ▶ Make no assumption about how the sequence is generated.
- ▶ The best constant value for  $x$  in hind-sight:

$$x_T^* \doteq \operatorname{argmin}_{x \in [0,1]} \sum_{t=1}^T (x - y_t)^2, \quad x_T^* = \frac{1}{T} \sum_{t=1}^T y_t$$

## Individual sequence bounds

- ▶ Make no assumption about how the sequence is generated.
- ▶ The best constant value for  $x$  in hind-sight:

$$x_T^* \doteq \operatorname{argmin}_{x \in [0,1]} \sum_{t=1}^T (x - y_t)^2, \quad x_T^* = \frac{1}{T} \sum_{t=1}^T y_t$$

- ▶ Regret: the loss over and above the loss of  $x_T^*$ . **for the worst-case sequence**

$$\operatorname{Regret}_T = \sum_{t=1}^T (x_t - y_t)^2 - \sum_{t=1}^T (x_T^* - y_t)^2$$



## Individual sequence bounds

- ▶ Make no assumption about how the sequence is generated.
- ▶ The best constant value for  $x$  in hind-sight:

$$x_T^* \doteq \operatorname{argmin}_{x \in [0,1]} \sum_{t=1}^T (x - y_t)^2, \quad x_T^* = \frac{1}{T} \sum_{t=1}^T y_t$$

- ▶ **Regret:** the loss over and above the loss of  $x_T^*$ . **for the worst-case sequence**

$$\operatorname{Regret}_T = \sum_{t=1}^T (x_t - y_t)^2 - \sum_{t=1}^T (x_T^* - y_t)^2$$

- ▶ **Goal:** sublinear regret  $\lim_{T \rightarrow \infty} \frac{\operatorname{Regret}_T}{T} = 0$

# Follow the Leader

- ▶ Idea: set  $x_{t+1}$  to be the best constant prediction on  $y_1, \dots, y_t$

# Follow the Leader

- ▶ Idea: set  $x_{t+1}$  to be the best constant prediction on  $y_1, \dots, y_t$
- ▶  $x_{t+1} = \operatorname{argmin}_{x \in [0,1]} \sum_{i=1}^t (x - y_i)^2 = x_t^*$

# Follow the Leader

- ▶ Idea: set  $x_{t+1}$  to be the best constant prediction on  $y_1, \dots, y_t$
- ▶  $x_{t+1} = \operatorname{argmin}_{x \in [0,1]} \sum_{i=1}^t (x - y_i)^2 = x_t^*$
- ▶ We will prove that the regret of this algorithm is upper bound by  $2 + 2 \ln T$

# Regret Bound

## Theorem

*Let  $y_t \in [0, 1]$  for  $t = 1, \dots, T$  an arbitrary sequence of numbers.*

*Let the algorithm output be  $x_t = x_{t-1}^* = \frac{1}{t-1} \sum_{i=1}^{t-1} y_i$ , then*

$$\text{Regret}_T = \sum_{t=1}^T (x_t - y_t)^2 - \sum_{t=1}^T (x_T^* - y_t)^2 \leq 2(1 + \ln T)$$

## Lemma

Let  $x_1, x_2, \dots$  be the sequence of predictions produced by FTL.  
Then for all  $u \in R$  (In particular, for  $u = x_T^*$ ):

$$\sum_{t=1}^T \left( (x_t - y_t)^2 - (u - y_t)^2 \right) \leq \sum_{t=1}^T \left( (x_t - y_t)^2 - (x_t^* - y_t)^2 \right)$$

### Proof Sketch:

Subtract  $\sum_{t=1}^T (x_t - y_t)^2$  from both sides to get an equivalent claim:

$$\sum_{t=1}^T (x_t^* - y_t)^2 \leq \sum_{t=1}^T (u - y_t)^2$$

The inequality is proven by induction on  $T$ .

Proof.

## Proof.

- ▶ Base case ( $T = 1$ ):  $(x_1^* - y_1)^2 = (y_1 - y_1)^2 = 0 \leq (u - y_1)^2$



## Proof.

- ▶ Base case ( $T = 1$ ):  $(x_1^* - y_1)^2 = (y_1 - y_1)^2 = 0 \leq (u - y_1)^2$
- ▶ Induction hypothesis:  $\sum_{t=1}^{T-1} (x_t^* - y_t)^2 \leq \sum_{t=1}^{T-1} (u - y_t)^2$

## Proof.

- ▶ Base case ( $T = 1$ ):  $(x_1^* - y_1)^2 = (y_1 - y_1)^2 = 0 \leq (u - y_1)^2$
- ▶ Induction hypothesis:  $\sum_{t=1}^{T-1} (x_t^* - y_t)^2 \leq \sum_{t=1}^{T-1} (u - y_t)^2$
- ▶ Induction step:

$$\sum_{t=1}^{T-1} (x_t^* - y_t)^2 \leq \sum_{t=1}^{T-1} (x_{T-1}^* - y_t)^2 \leq \sum_{t=1}^{T-1} (x_T^* - y_t)^2$$

Adding  $(x_T^* - y_T)^2$  to both sides gives:

$$\sum_{t=1}^T (x_t^* - y_t)^2 \leq \sum_{t=1}^T (x_T^* - y_t)^2 \leq \sum_{t=1}^T (u - y_t)^2$$

## Proof of the theorem

First, note that in FTL we have:

$$x_t = x_{t-1}^* = \frac{1}{t-1} \sum_{i=1}^{t-1} y_i = \frac{t}{t-1} \cdot \left( \frac{1}{t} \sum_{i=1}^t y_i - \frac{y_t}{t} \right) = \frac{t}{t-1} \cdot \left( x_t^* - \frac{y_t}{t} \right)$$

Subtracting  $x_t^*$  from both sides, we get  $x_t - x_t^* = \frac{x_t^* - y_t}{t-1}$ . Then:

$$\begin{aligned} \text{Regret}_T &= \sum_{t=1}^T (x_t - y_t)^2 - \sum_{t=1}^T (x_t^* - y_t)^2 \\ &\leq \sum_{t=1}^T (x_t - y_t)^2 - (x_t^* - y_t)^2 \quad (\text{Lemma}) \\ &= \sum_{t=1}^T (x_t + x_t^* - 2y_t)(x_t - x_t^*) \leq \sum_{t=1}^T \frac{2}{t-1} \leq 2(1 + \ln T) \end{aligned}$$

# What the class will cover

# What the class will cover

# What the class will cover

- ▶ Introduction (with Mean)

# What the class will cover

- ▶ Introduction (with Mean)
- ▶ Exponential weights algorithms

# What the class will cover

- ▶ Introduction (with Mean)
- ▶ Exponential weights algorithms
  - ▶ Hedge



# What the class will cover

- ▶ Introduction (with Mean)
- ▶ Exponential weights algorithms
  - ▶ Hedge
  - ▶ Mixability

# What the class will cover

- ▶ Introduction (with Mean)
- ▶ Exponential weights algorithms
  - ▶ Hedge
  - ▶ Mixability
  - ▶ Bregman Divergences

# What the class will cover

- ▶ Introduction (with Mean)
- ▶ Exponential weights algorithms
  - ▶ Hedge
  - ▶ Mixability
  - ▶ Bregman Divergences

## What the class will cover

- ▶ Introduction (with Mean)
- ▶ Exponential weights algorithms
  - ▶ Hedge
  - ▶ Mixability
  - ▶ Bregman Divergences
- ▶ Online learning and Coding

## What the class will cover

- ▶ Introduction (with Mean)
- ▶ Exponential weights algorithms
  - ▶ Hedge
  - ▶ Mixability
  - ▶ Bregman Divergences
- ▶ Online learning and Coding
  - ▶ Universal Coding

# What the class will cover

- ▶ Introduction (with Mean)
- ▶ Exponential weights algorithms
  - ▶ Hedge
  - ▶ Mixability
  - ▶ Bregman Divergences
- ▶ Online learning and Coding
  - ▶ Universal Coding
  - ▶ Continuous Experts

# What the class will cover

- ▶ Introduction (with Mean)
- ▶ Exponential weights algorithms
  - ▶ Hedge
  - ▶ Mixability
  - ▶ Bregman Divergences
- ▶ Online learning and Coding
  - ▶ Universal Coding
  - ▶ Continuous Experts
  - ▶ The Context Algorithm

# What the class will cover

- ▶ Introduction (with Mean)
- ▶ Exponential weights algorithms
  - ▶ Hedge
  - ▶ Mixability
  - ▶ Bregman Divergences
- ▶ Online learning and Coding
  - ▶ Universal Coding
  - ▶ Continuous Experts
  - ▶ The Context Algorithm
- ▶ Multiple arm Bandit



# What the class will cover

- ▶ Introduction (with Mean)
- ▶ Exponential weights algorithms
  - ▶ Hedge
  - ▶ Mixability
  - ▶ Bregman Divergences
- ▶ Online learning and Coding
  - ▶ Universal Coding
  - ▶ Continuous Experts
  - ▶ The Context Algorithm
- ▶ Multiple arm Bandit
- ▶ Tracking

# What the class will cover

- ▶ Introduction (with Mean)
- ▶ Exponential weights algorithms
  - ▶ Hedge
  - ▶ Mixability
  - ▶ Bregman Divergences
- ▶ Online learning and Coding
  - ▶ Universal Coding
  - ▶ Continuous Experts
  - ▶ The Context Algorithm
- ▶ Multiple arm Bandit
- ▶ Tracking
  - ▶ Tracking

# What the class will cover

- ▶ Introduction (with Mean)
- ▶ Exponential weights algorithms
  - ▶ Hedge
  - ▶ Mixability
  - ▶ Bregman Divergences
- ▶ Online learning and Coding
  - ▶ Universal Coding
  - ▶ Continuous Experts
  - ▶ The Context Algorithm
- ▶ Multiple arm Bandit
- ▶ Tracking
  - ▶ Tracking
  - ▶ Tracking within a small set of experts

# What the class will cover

- ▶ Introduction (with Mean)
- ▶ Exponential weights algorithms
  - ▶ Hedge
  - ▶ Mixability
  - ▶ Bregman Divergences
- ▶ Online learning and Coding
  - ▶ Universal Coding
  - ▶ Continuous Experts
  - ▶ The Context Algorithm
- ▶ Multiple arm Bandit
- ▶ Tracking
  - ▶ Tracking
  - ▶ Tracking within a small set of experts

# What the class will cover

- ▶ Introduction (with Mean)
- ▶ Exponential weights algorithms
  - ▶ Hedge
  - ▶ Mixability
  - ▶ Bregman Divergences
- ▶ Online learning and Coding
  - ▶ Universal Coding
  - ▶ Continuous Experts
  - ▶ The Context Algorithm
- ▶ Multiple arm Bandit
- ▶ Tracking
  - ▶ Tracking
  - ▶ Tracking within a small set of experts
- ▶ Online learning and game theory

# What the class will cover

- ▶ Introduction (with Mean)
- ▶ Exponential weights algorithms
  - ▶ Hedge
  - ▶ Mixability
  - ▶ Bregman Divergences
- ▶ Online learning and Coding
  - ▶ Universal Coding
  - ▶ Continuous Experts
  - ▶ The Context Algorithm
- ▶ Multiple arm Bandit
- ▶ Tracking
  - ▶ Tracking
  - ▶ Tracking within a small set of experts
- ▶ Online learning and game theory
  - ▶ Repeated Matrix Games

# What the class will cover

- ▶ Introduction (with Mean)
- ▶ Exponential weights algorithms
  - ▶ Hedge
  - ▶ Mixability
  - ▶ Bregman Divergences
- ▶ Online learning and Coding
  - ▶ Universal Coding
  - ▶ Continuous Experts
  - ▶ The Context Algorithm
- ▶ Multiple arm Bandit
- ▶ Tracking
  - ▶ Tracking
  - ▶ Tracking within a small set of experts
- ▶ Online learning and game theory
  - ▶ Repeated Matrix Games
  - ▶ Internal regret.

# What the class will cover

- ▶ Introduction (with Mean)
- ▶ Exponential weights algorithms
  - ▶ Hedge
  - ▶ Mixability
  - ▶ Bregman Divergences
- ▶ Online learning and Coding
  - ▶ Universal Coding
  - ▶ Continuous Experts
  - ▶ The Context Algorithm
- ▶ Multiple arm Bandit
- ▶ Tracking
  - ▶ Tracking
  - ▶ Tracking within a small set of experts
- ▶ Online learning and game theory
  - ▶ Repeated Matrix Games
  - ▶ Internal regret.
  - ▶ Drifting games



# What the class will cover

- ▶ Introduction (with Mean)
- ▶ Exponential weights algorithms
  - ▶ Hedge
  - ▶ Mixability
  - ▶ Bregman Divergences
- ▶ Online learning and Coding
  - ▶ Universal Coding
  - ▶ Continuous Experts
  - ▶ The Context Algorithm
- ▶ Multiple arm Bandit
- ▶ Tracking
  - ▶ Tracking
  - ▶ Tracking within a small set of experts
- ▶ Online learning and game theory
  - ▶ Repeated Matrix Games
  - ▶ Internal regret.
  - ▶ Drifting games
  - ▶ NormalHedge

# What the class will cover

- ▶ Introduction (with Mean)
- ▶ Exponential weights algorithms
  - ▶ Hedge
  - ▶ Mixability
  - ▶ Bregman Divergences
- ▶ Online learning and Coding
  - ▶ Universal Coding
  - ▶ Continuous Experts
  - ▶ The Context Algorithm
- ▶ Multiple arm Bandit
- ▶ Tracking
  - ▶ Tracking
  - ▶ Tracking within a small set of experts
- ▶ Online learning and game theory
  - ▶ Repeated Matrix Games
  - ▶ Internal regret.
  - ▶ Drifting games
  - ▶ NormalHedge
- ▶ Online Convex Optimizatio

# What the class will cover

- ▶ Introduction (with Mean)
- ▶ Exponential weights algorithms
  - ▶ Hedge
  - ▶ Mixability
  - ▶ Bregman Divergences
- ▶ Online learning and Coding
  - ▶ Universal Coding
  - ▶ Continuous Experts
  - ▶ The Context Algorithm
- ▶ Multiple arm Bandit
- ▶ Tracking
  - ▶ Tracking
  - ▶ Tracking within a small set of experts
- ▶ Online learning and game theory
  - ▶ Repeated Matrix Games
  - ▶ Internal regret.
  - ▶ Drifting games
  - ▶ NormalHedge
- ▶ Online Convex Optimizatio
  - ▶ Follow the regularized leader

# What the class will cover

- ▶ Introduction (with Mean)
- ▶ Exponential weights algorithms
  - ▶ Hedge
  - ▶ Mixability
  - ▶ Bregman Divergences
- ▶ Online learning and Coding
  - ▶ Universal Coding
  - ▶ Continuous Experts
  - ▶ The Context Algorithm
- ▶ Multiple arm Bandit
- ▶ Tracking
  - ▶ Tracking
  - ▶ Tracking within a small set of experts
- ▶ Online learning and game theory
  - ▶ Repeated Matrix Games
  - ▶ Internal regret.
  - ▶ Drifting games
  - ▶ NormalHedge
- ▶ Online Convex Optimizatio
  - ▶ Follow the regularized leader
  - ▶ Dual Descent

# What the class will cover

- ▶ Introduction (with Mean)
- ▶ Exponential weights algorithms
  - ▶ Hedge
  - ▶ Mixability
  - ▶ Bregman Divergences
- ▶ Online learning and Coding
  - ▶ Universal Coding
  - ▶ Continuous Experts
  - ▶ The Context Algorithm
- ▶ Multiple arm Bandit
- ▶ Tracking
  - ▶ Tracking
  - ▶ Tracking within a small set of experts
- ▶ Online learning and game theory
  - ▶ Repeated Matrix Games
  - ▶ Internal regret.
  - ▶ Drifting games
  - ▶ NormalHedge
- ▶ Online Convex Optimizatio
  - ▶ Follow the regularized leader
  - ▶ Dual Descent
  - ▶ AdaGrad