Dual Descent

February 19, 2025

Chapter 2 in Shai Shalev Shwartz / Online Learning and Online convex Optimization

Review: Property of FoRel Algorithm

Lemma 2.3:
Let $w_1, w_2, \ldots$ be the sequence of vectors produced by the FoReL algorithm. Then, for all $u \in S$, we have:

$$\sum_{t=1}^{T}(f_t(w_t) - f_t(u)) \leq R(u) - R(w_1) + \sum_{t=1}^{T}(f_t(w_t) - f_t(w_{t+1}))$$

Review: One step of Gradient Descent using strongly convex regularizer

Lemma 2.10:
Let $R : S \to \mathbb{R}$ be a σ-strongly-convex function over S with respect to a norm $\|\cdot\|$. Let $w_1, w_2, \ldots$ be the predictions of the FoReL algorithm. Then, for all t, if f_t is L_t-Lipschitz with respect to $\|\cdot\|$, we have:

$$f_t(w_t) - f_t(w_{t+1}) \leq L_t\|w_t - w_{t+1}\| \leq \frac{L_t^2}{\sigma}$$

Main Theorem regarding FoReL using stongly convex regularizer

Let $f_1, \dots, f_T$ be a sequence of convex functions with the following conditions:

- f_t is L_t-Lipschitz with respect to some norm $\| \cdot \|$.

Then, for all $u \in S$,

$$\text{Regret}_T(u) \leq R(u) - \min_{v \in S} R(v) + \frac{TL^2}{\sigma}$$

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- FoReL is run on the sequence with a regularization function that is σ-strongly-convex with respect to the same norm.

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- If Experts correspond to exponential distributions and loss is log loss- we can use conjugate priors. (recall: biased coins).
- We need a new trick to compute $\mathrm{w}_t = \nabla R(\text{Regret}_t)$ efficiently.

FoReL Update Rule for linear cost function

Define $z_{1:t} = \sum_{i=1}^{t} z_i$, the FoReL update rule can be written as

$$
\begin{aligned}
w_{t+1} &= \arg\min_{w} R(w) + \sum_{i=1}^{t} \langle w, z_i \rangle \\
&= \arg\min_{w} R(w) + \langle w, z_{1:t} \rangle \\
&= \arg\max_{w} \langle w, -z_{1:t} \rangle - R(w).
\end{aligned}
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Mirror Descent Update for linear functions

Update rule

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Link Function:

$$g(\theta) = \arg\max_{w} \langle w, \theta \rangle - R(w),$$

Update rule can be re-written as

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3. $w_{t+1} = g(\theta_{t+1})$

Identical update to FTRL for linear loss functions.
What about general convex loss functions?

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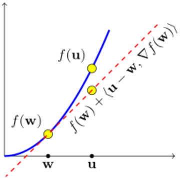
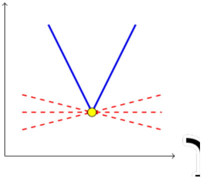
- if gradient $\nabla f(x)$ exists, then $\partial f(x) = \{\nabla f(x)\}$





Example Generalized Online Gradient Descent

Consider the ℓ_2 setup where the functions $f_1, f_2, \ldots$ are convex (but not necessarily differentiable). Let η be the learning rate.

$$w_{t+1} = w_t - \eta z_t, \quad z_t \in \partial f_t(w_t)$$

Identical to FTRL with regularization: $R(w) = \frac{1}{2\eta}\|w\|_2^2$
Regret bound on OGD: From FTRL theorem:

$$\begin{aligned} \text{Regret} &\leq \frac{\|u\|^2}{2\eta} + \eta \sum_{t=1}^{T} \|z_t\|^2 \\ &\leq \frac{B^2}{2\eta} + \eta T L^2 \end{aligned}$$

Gradient based Online Mirror Descent (OMD)

parameter: a link function $g : \mathbb{R}^d \to S$
initialize: $\theta_1 = 0$
for $t = 1, 2, \ldots$

- **predict** $w_t = g(\theta_t)$

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- Using Duality Gives better intuition, more general analysis, tighter bounds.

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- U and V are dual vector spaces, with dual norms.

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- L_1, L_∞ are dual.
- L_2 is self-dual.

Lipschitz condition and the dual norm

Lemma 2.6:
Let $f : S \to \mathbb{R}$ be a convex function. Then, f is L-Lipschitz over S with respect to a norm $\|\cdot\|$ if and only if for all $w \in S$ and $z \in \partial f(w)$ we have:

$$\|z\|_* \leq L$$

where $\|\cdot\|_*$ denotes the dual norm.

Proof of Lemma 2.6

Proof:

Assume that f is L-Lipschitz. For any $w \in S$ and $z \in \partial f(w)$, choose u such that $u - w = \arg\max_{\|v\|=1} \langle v, z \rangle$. Then,

$$\langle z, u - w \rangle = \|z\|_*$$

By the sub-gradient definition,

$$f(u) - f(w) \geq \langle z, u - w \rangle = \|z\|_*$$

Since f is L-Lipschitz,

$$f(u) - f(w) \leq L\|u - w\| = L$$

Combining the inequalities:

$$\|z\|_* \leq L$$

For the converse, assume $\|z\|_* \leq L$ for all $z \in \partial f(w)$. Then,

$$f(u) - f(w) \leq \langle z, u - w \rangle \leq \|z\|_* \|u - w\| \leq L\|u - w\|$$

Hence, f is L-Lipschitz.

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- Fenchel duality Equivalent to Legendre duality for differentiable functions.

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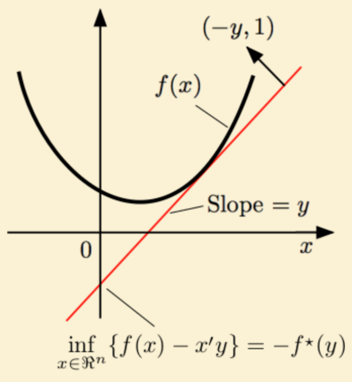

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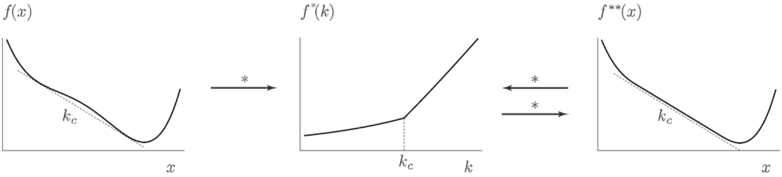
- The dual of any function is convex.



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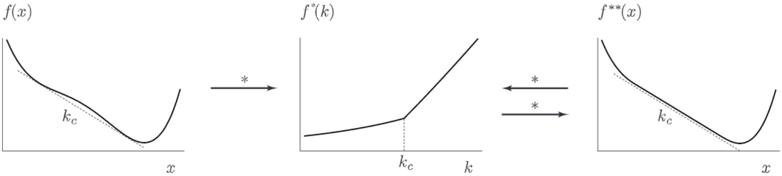
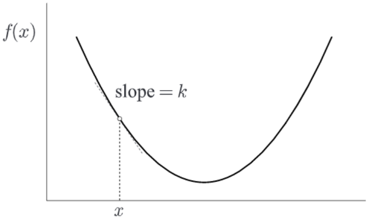
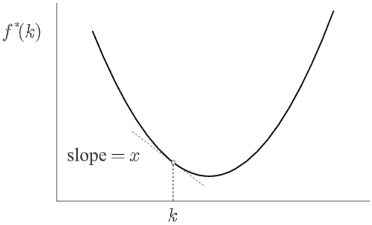
- if F is convex then $F^{**} = F$



Gradient Duality (legendre only)

- If the gradient of f at x is k then the gradient of f^* at k is x





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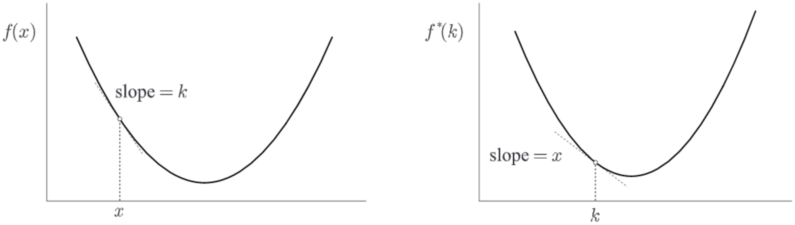
- If the gradient of f at x is k then the gradient of f^* at k is x
- In general:

$$\nabla F^* = (\nabla F)^{-1}$$



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- Note $(\nabla F)^{-1} = \nabla F^*$

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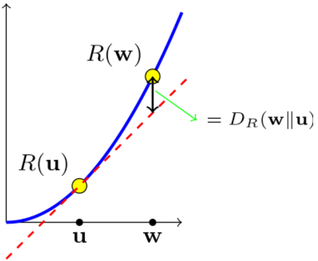

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- $\mathrm{u}' = \nabla F(\mathrm{u})$ and $\mathrm{v}' = \nabla F(\mathrm{v})$
- $D_F(\mathrm{u}, \mathrm{v}) = D_{F^*}(\mathrm{u}', \mathrm{v}')$

Mirror Descent - Step 1

Gradient Step in Dual Space:

$$z_{t+1} = \nabla R(w_t) - \eta \nabla f_t(w_t)$$

Here, $\nabla R(w_t)$ maps the point into the dual space.

Mirror Descent - Step 2

Projection Back to Primal Space:

$$w_{t+1} = \arg\min_{w \in S} D_R(w, z_{t+1})$$

Where $D_R(w, z)$ is the Bregman divergence:

$$D_R(w, z) = R(w) - R(z) - \langle \nabla R(z), w - z \rangle$$

This projection ensures w_{t+1} stays within the feasible set S.

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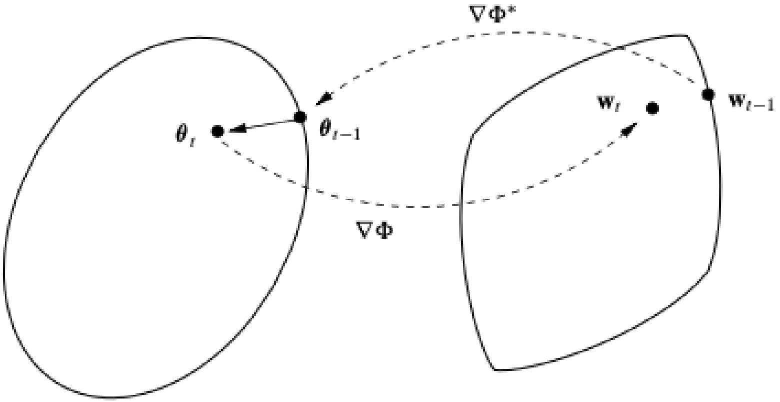
- Projection: As ∇R is the inverse of ∇R^* we get

$$\mathrm{w}_t = \nabla R(\nabla R^*(\mathrm{w}_{t-1}) - \lambda \nabla \ell_t(\mathrm{w}_{t-1}))$$

A picture of mirror descent

$$w_t = \nabla R(\nabla R^*(w_{t-1}) - \lambda \nabla \ell_t(w_{t-1}))$$



Regret Bound for OMD

Lemma 2.20. Suppose that OMD is run with a link function $g = \nabla R^*$. Then, its regret is upper bounded by:

$$\sum_{t=1}^{T} \langle \mathrm{w}_t - \mathrm{u}, \mathrm{z}_t \rangle \leq R(\mathrm{u}) - R(\mathrm{w}_1) + \sum_{t=1}^{T} D_{R^*}(-\mathrm{z}_{1:t} \| - \mathrm{z}_{1:t-1})$$

Furthermore, equality holds for the vector u that minimizes $R(\mathrm{u}) + \sum_t \langle \mathrm{u}, \mathrm{z}_t \rangle$.

Proof: Step 1 - Fenchel–Young Inequality

Using the **Fenchel–Young inequality**, we have:

$$R(\mathrm{u}) + \sum_{t=1}^{T} \langle \mathrm{u}, \mathrm{z}_t \rangle = R(\mathrm{u}) - \langle \mathrm{u}, -\mathrm{z}_{1:T} \rangle \geq -R^*(-\mathrm{z}_{1:T}).$$

Equality holds for u that maximizes $\langle \mathrm{u}, -\mathrm{z}_{1:T} \rangle - R(\mathrm{u})$, hence minimizing $R(\mathrm{u}) + \langle \mathrm{u}, \mathrm{z}_{1:T} \rangle$.

Proof: Step 2 - Bregman Divergence

Since $w_t = \nabla R^*(-z_{1:t-1})$ and using the definition of the Bregman divergence, we rewrite:

$$-R^*(-z_{1:T}) = -R^*(0) - \sum_{t=1}^{T} \left(R^*(-z_{1:t}) - R^*(-z_{1:t-1})\right).$$

Rearranging, we get:

$$= -R^*(0) + \sum_{t=1}^{T} \left(\langle w_t, z_t \rangle - D_{R^*}(-z_{1:t} \| - z_{1:t-1})\right).$$

Final Step

Note: Since

$$R^*(0) = \max_{\mathrm{w}} \langle 0, \mathrm{w} \rangle - R(\mathrm{w}) = -\min_{\mathrm{w}} R(\mathrm{w}) = -R(\mathrm{w}_1),$$

combining all the above, we conclude the proof. $\square$

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- Gradient step: $z_{t+1} = w_t - \eta \nabla f_t(w_t)$

Projection Step for ℓ_2 Norm

Bregman Divergence:

$$D_R(w, z) = \frac{1}{2}\|w - z\|_2^2$$

Projection Back to Primal Space:

$$w_{t+1} = \Pi_S(z_{t+1}) = \arg\min_{w \in S} \frac{1}{2}\|w - z_{t+1}\|_2^2$$

Where Π_S denotes the Euclidean projection onto the feasible set S.

Final Update Rule for ℓ_2 Norm

Combining both steps, the final update rule becomes:

$$w_{t+1} = \Pi_S \left(w_t - \eta \nabla f_t(w_t) \right)$$

This is equivalent to the standard **Projected Gradient Descent** for the ℓ_2 norm.

Optimal Tuning for η and Regret Bound

Regret Bound:

$$\text{Regret}_T(u) \leq \frac{\|u\|_2^2}{2\eta} + \frac{\eta}{2} \sum_{t=1}^{T} \|\nabla f_t(w_t)\|_2^2$$

Assuming $\|u\|_2 \leq B$ and $\|\nabla f_t(w_t)\|_2 \leq L$, this simplifies to:

$$\text{Regret}_T(u) \leq \frac{B^2}{2\eta} + \frac{\eta L^2 T}{2}$$

Optimal η:

$$\eta^* = \frac{B}{L\sqrt{T}}$$

Resulting Regret Bound:

$$\text{Regret}_T(u) \leq BL\sqrt{T}$$

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- Legendre Dual Regularizer $R^*(\mathrm{u}) = \sum_{i=1}^{d} u_i(\ln u_i - 1)$

- Weight update:

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- Normalized exponentiated gradient:

$$w_{i,t} = \frac{w_{i,t-1} e^{-\lambda \nabla \ell_t (w_i - 1)}}{\sum_{j=1}^{d} w_{j,t-1} e^{-\lambda \nabla \ell_t (w_j - 1)}}$$

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- The dual descend algorithm for the exponential regularizer function R and the learning rate $\lambda = \frac{2\epsilon}{X_\infty^2}$ for some $0 < \epsilon < 1$
- yields Loss Bound:

$$L_{A,T} \leq \frac{L_T(\mathrm{u})}{1-\epsilon} + \frac{X_\infty^2 \ln d}{2\epsilon(1-\epsilon)}$$