Universal Portfolios

Yoav Freund

March 10, 2025

Based On “Universal Portfolios” by Cover and “Universal Portfolios with Side Information” by Cover and Ordentlich.

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- If $Z < 1$ this guarantees $S(i) > 1$.

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- minus-log-wealth is equal to the sum of log losses.

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- Our goal is to design portfolio strategies with competitive doubling rate.

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- Can we compete against stronger comparators?

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- Market Makers.

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- What if we don't know the distribution?
- We will give an algorithm that performs almost as well as the best constant rebalanced portfolio in hindsight.

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- **Key Idea:** Use a buy and hold mixture over all constant rebalanced portfolios.
- **Bound:** $\forall \mathbf{x}^n : \widehat{W}_n(\mathbf{x}^n) \geq W_n^*(\mathbf{x}^n) - O(m\frac{\log n}{n})$

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$$\hat{\mathbf{b}}_i = \hat{\mathbf{b}}_i(\mathbf{x}^{i-1}) = \frac{\int_{\mathcal{B}} \mathbf{b}\, S_{i-1}(\mathbf{b}, \mathbf{x}^{i-1})\, \mathrm{d}\mu(\mathbf{b})}{\int_{\mathcal{B}} S_{i-1}(\mathbf{b}, \mathbf{x}^{i-1})\, \mathrm{d}\mu(\mathbf{b})}, \quad i = 1, 2, \ldots$$

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- We assume the μ is symmetric therefor $\mathbf{b}_1 = (1/m, \cdots, 1/m)$

Main Theorems

- Theorem 1 For the Uniform distribution.

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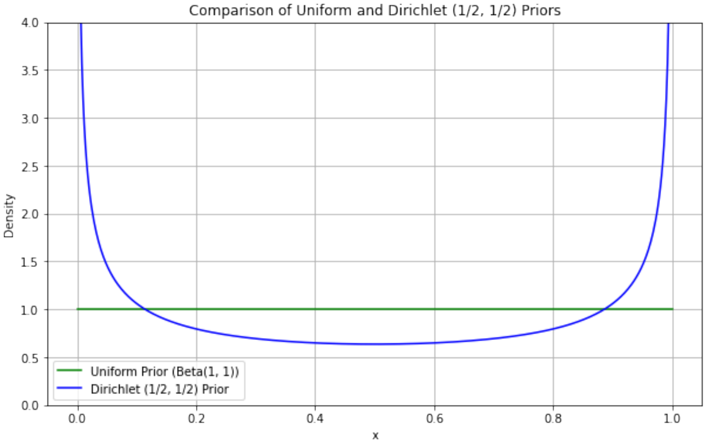
- Theorem 2 For the Dirichlet-$(1/2, \ldots, 1/2)$ distribution.

$$\frac{S_n^*(\mathbf{x}^n)}{\widehat{S}_n(\mathbf{x}^n)} \leq 2(n+1)^{(m-1)/2}$$

Comparing the priors for two stocks



Lemma 2: The μ-Weighted Universal Portfolio

For the μ-weighted universal portfolio:

$$\frac{S_n^*(x^n)}{\hat{S}_n(x^n)} \leq \max_{j^n} \frac{\prod_{i=1}^{n} b_{j_i}^*}{\int_{\mathcal{B}} \prod_{i=1}^{n} b_{j_i} \, d\mu(b)}$$

where the maximum is over the set of sequences of indices

$$j^n \in \{1, \ldots, m\}^n,$$

and

$$\mathbf{b}^* = (b_1^*, \ldots, b_m^*)^t$$

is the best constant rebalanced portfolio for the sequence x^n.

Proof of Theorems

By Upper bounding

$$\max_{j^n} \frac{\prod_{i=1}^{n} b^*_{j_i}}{\int_{\mathcal{B}} \prod_{i=1}^{n} b_{j_i} \, d\mu(b)}$$

For each prior distribution.

Proof of Lemma 2

Recall the Definitions

First, recall the definitions:

$$S_n^*(x^n) = \prod_{i=1}^{n} b^{*t} x_i$$

and

$$\hat{S}_n(x^n) = \int_{\mathcal{B}} \prod_{i=1}^{n} b^t x_i \, d\mu(b).$$

Rewriting $S_n^*(x^n)$

We rewrite the product of sums $S_n^*(x^n)$ as a sum of products.

$$S_n^*(x^n) = \prod_{i=1}^{n} b^{*t} x_i = \prod_{i=1}^{n} \left(\sum_{j=1}^{m} b_j^* x_{ij} \right) = \sum_{j^n \in \{1,\ldots,m\}^n} \prod_{i=1}^{n} b_{j_i}^* x_{j_i}$$

Where

$$j^n = (j_1, j_2, \ldots, j_n) \in \{1, \ldots, m\}^n.$$

Rewriting $\hat{S}_n(x^n)$

Similarly, we rewrite $\hat{S}_n(x^n)$ as:

$$\hat{S}_n(x^n) = \int_{\mathcal{B}} \prod_{i=1}^{n} b^t x_i \, d\mu(b)$$

$$= \sum_{j^n \in \{1,\ldots,m\}^n} \int_{\mathcal{B}} \prod_{i=1}^{n} b_{j_i} x_{ij_i} \, d\mu(b).$$

Ratio of Wealths

The ratio of wealths can now be written as:

$$\frac{S_n^*(x^n)}{\hat{S}_n(x^n)} = \frac{\sum_{j^n \in \{1,\ldots,m\}^n} \prod_{i=1}^{n} b_{j_i}^* x_{j_i}}{\sum_{j^n \in \{1,\ldots,m\}^n} \int_{\mathcal{B}} \prod_{i=1}^{n} b_{j_i} x_{ij_i} \, d\mu(b)}$$

Alternative Formulation of Ratio

$$\frac{S_n^*(x^n)}{\hat{S}_n(x^n)} = \frac{\displaystyle\sum_{j^n : \prod_{i=1}^{n} x_{ij_i} > 0} \prod_{i=1}^{n} b_{j_i}^* x_{j_i}}{\displaystyle\sum_{j^n : \prod_{i=1}^{n} x_{ij_i} > 0} \int_{\mathcal{B}} \prod_{i=1}^{n} b_{j_i} x_{ij_i} \, d\mu(b)}$$

Lemma 1

If $\alpha_1, \ldots, \alpha_n \geq 0$, and $\beta_1, \ldots, \beta_n \geq 0$, then

$$\frac{\sum_{i=1}^{n} \alpha_i}{\sum_{i=1}^{n} \beta_i} \leq \max_j \frac{\alpha_j}{\beta_j}.$$

Applying Lemma 1

We apply Lemma 1 with:

$$\alpha_{(j^n)} \triangleq \prod_{i=1}^{n} b^*_{j_i} x_{ij_i}$$

and

$$\beta_{(j^n)} \triangleq \int_{\mathcal{B}} \prod_{i=1}^{n} b_{j_i} x_{ij_i} \, d\mu(b)$$

for

$$j^n \in \left\{ j^n : \prod_{i=1}^{n} x_{ij_i} > 0 \right\}.$$

Obtaining the Bound

Using Lemma 1, we obtain:

$$\frac{S_n^*(x^n)}{\hat{S}_n(x^n)} \leq \max_{j^n : \prod_{i=1}^{n} x_{ij_i} > 0} \frac{\prod_{i=1}^{n} b_{j_i}^* x_{ij_i}}{\int_{\mathcal{B}} \prod_{i=1}^{n} b_{j_i} x_{ij_i} \, d\mu(b)}$$

Simplifying the Expression

The fraction simplifies to:

$$= \max_{j^n : \prod_{i=1}^{n} x_{ij_i} > 0} \frac{\prod_{i=1}^{n} b^*_{j_i}}{\int_{\mathcal{B}} \prod_{i=1}^{n} b_{j_i} \, d\mu(b)}$$

$$\leq \max_{j^n} \frac{\prod_{i=1}^{n} b^*_{j_i}}{\int_{\mathcal{B}} \prod_{i=1}^{n} b_{j_i} \, d\mu(b)}$$

Completing the Proof

Since the product of x_{ij_i}'s factors out of the numerator and denominator, we conclude:

$$\frac{S_n^*(x^n)}{\hat{S}_n(x^n)} \leq \max_{j^n} \frac{\prod_{i=1}^{n} b_{j_i}^*}{\int_{\mathcal{B}} \prod_{i=1}^{n} b_{j_i} \, d\mu(b)}$$

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- Iroquis was the best performing stock for this period (791% profit)

Iroqu vs. Kinar

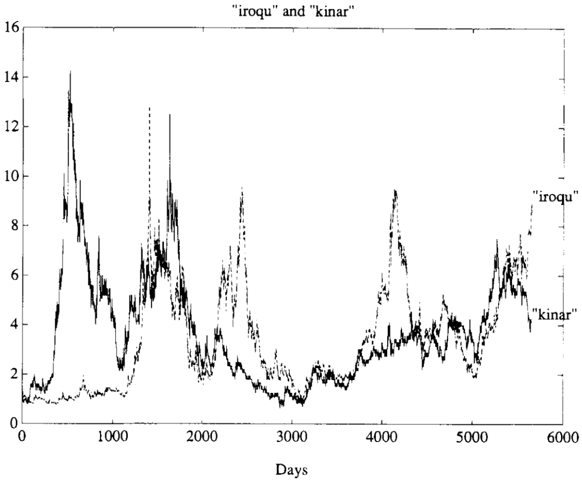
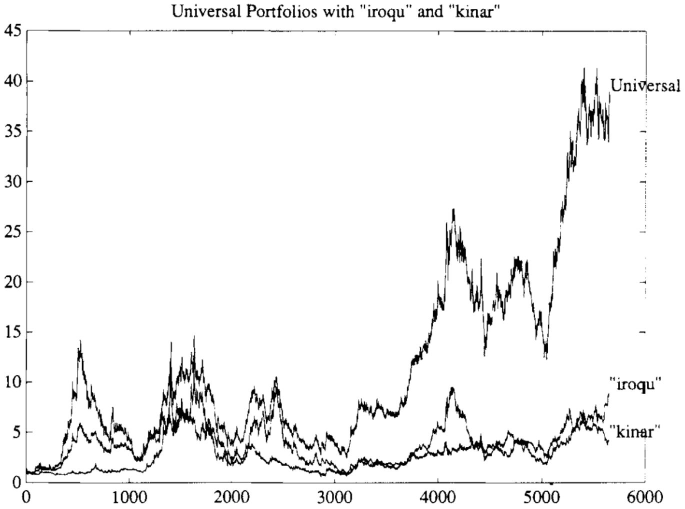


FIGURE 8.1. Performance of Iroquois brands and Kin Ark.

Iroqu vs. Kinar vs Universal



Iroqu vs. Kinar 20yr return of different fixed portfolios

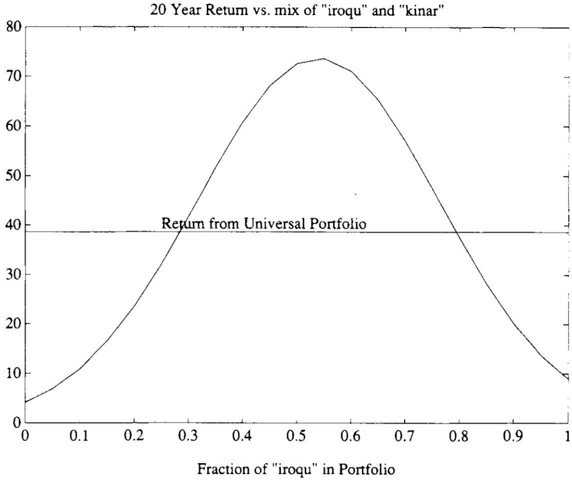


FIGURE 8.2. Performance of rebalanced portfolio.

Iroqu vs. Kinar mix in universal portfolio

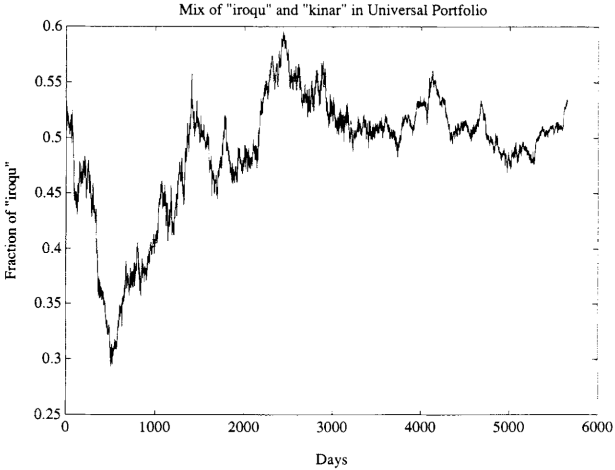
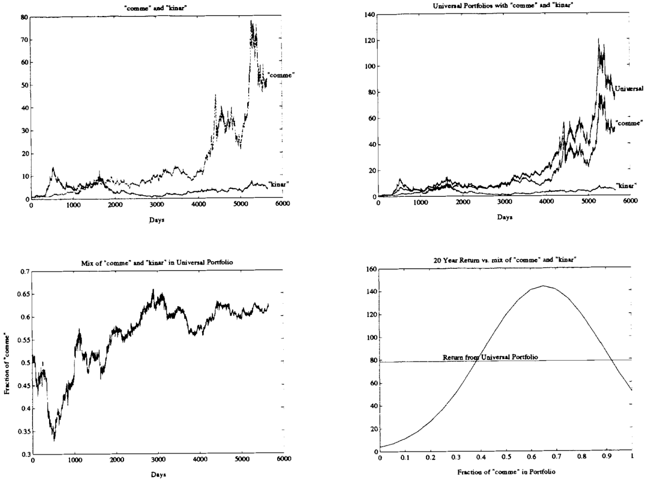


FIGURE 8.4. The portfolio $\hat{b}_k$.

Commercial Metals and Kin Arc



8.5. Commercial Metals and Kin Ark; Performance of Universal Portfolio; Universal Portfolio; Performance of Rebalanced F

Commercial Metals and Mei Corp

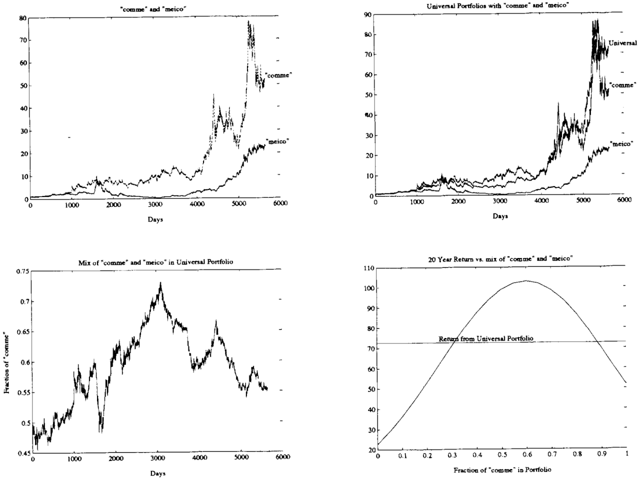


FIGURE 8.6. Commercial Metals and Mei Corp.

IBM and CocaCola

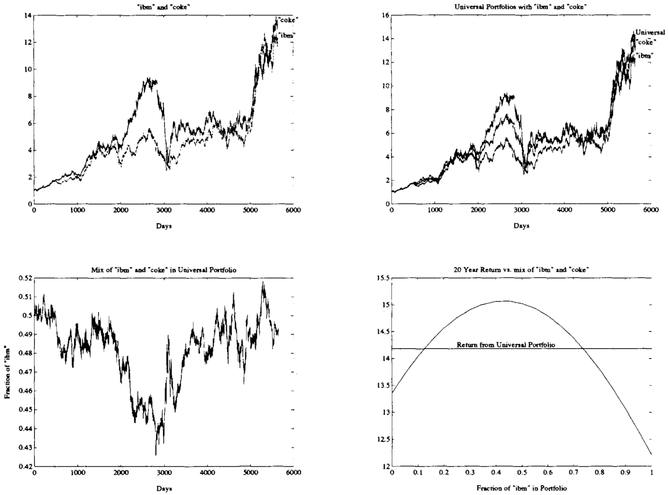


FIGURE 8.7. IBM and Coca-Cola.

Some issues

- Transaction costs are ignored.

Some issues

- Transaction costs are ignored.
- Stocks selected in hind-sight.

Some issues

- Transaction costs are ignored.
- Stocks selected in hind-sight.
- Volatile stocks are sensitive to exit time.

Some issues

- Transaction costs are ignored.
- Stocks selected in hind-sight.
- Volatile stocks are sensitive to exit time.
- Ignores mergers bankruptcies and acquisitions.