Vovk's aggregating algorithm
Mixable and unmixable loss functions

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Section 3.5 in "Prediction, Learning and Games"

Outline

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- The goal is to minimize the regret.

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  - Is there an advantage to the algorithm relative to DTOL?

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4. Each expert incurs loss $\ell_t(i) = \lambda(\omega^t, \gamma_i^t)$
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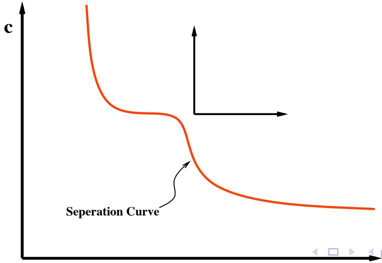

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$$\gamma^t = \begin{cases} 1 & \text{if } q > 1/2, \\ 0 & \text{otherwise} \end{cases}$$

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- **Which losses behave like entropy loss and which behave like hedge loss?**

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- There is no universally optimal prediction
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- Vovk's result: *yes!* a good choice for γ_t always exists!

Vovk's algorithm is the the highest achiever [Vovk95]

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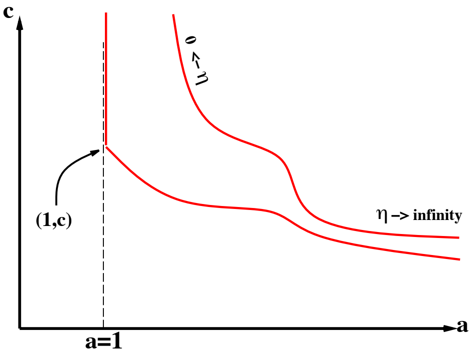

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- A Loss function is mixable if a pair of the form $(1, c),\ c < \infty$ is achievable.

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- Equivalently - the image of the set Γ under the mapping $F(\gamma) = \left\langle e^{-\eta\lambda(\omega,\gamma)}\right\rangle_{\omega\in\Omega}$ is concave.

convexity condition: Pictorially

- **Example:** Suppose $\Omega = \{0, 1\}$, $\Gamma = [0, 1]$. then

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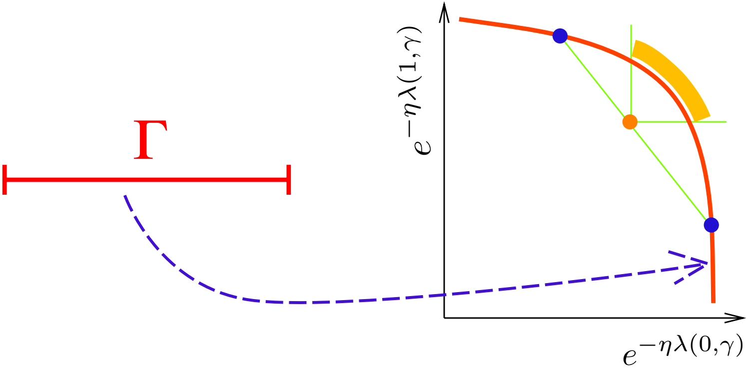

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- We are back to the online Bayes algorithm.

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$$1 - \sqrt{-\frac{1}{2}\ln\sum_i V_i^t e^{-2(1-p_i^t)^2}} \le p^t \le \sqrt{-\frac{1}{2}\ln\sum_i V_i^t e^{-2(p_i^t)^2}}$$

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- Which yields the bound

$$L_A \leq L_{\min} + 2 \ln N$$

Summary of bounds for mixable losses

Loss Functions:	c values: $(\eta = 1/c)$	
	$\mathbf{pred}_{\text{wmean}}(v, x)$	$\mathbf{pred}_{\text{Vovk}}(v,$
$L_{\text{sq}}(p, q)$	2	$1/2$
$L_{\text{ent}}(p, q)$	1	1
$L_{\text{hel}}(p, q)$	1	$1/\sqrt{2}$

Figure 2. $(c, 1/c)$-realizability: c values for loss and prediction