Vovk's aggregating algorithm
Mixable and unmixable loss functions

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Section 3.5 in "Prediction, Learning and Games"

Outline

Log Loss and Absolute loss

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- The general prediction game
- Vovk's algorithm
- mixable loss functions

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- The goal is to minimize the regret.

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  - Is there an advantage to the algorithm relative to DTOL?

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4. Each expert incurs loss $\ell_t(i) = \lambda(\omega^t, \gamma_i^t)$
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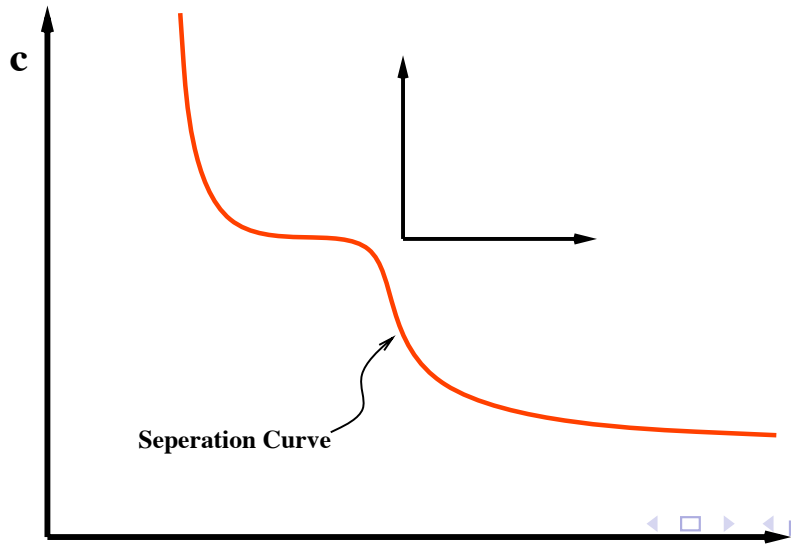

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- Corresponds to regression.

Square loss (Breier Loss)

- $$\lambda_{\text{sq}}(\omega, \gamma) = (\omega - \gamma)^2$$
- $P[\omega^t = 1] = q, \quad P[\omega^t = 0] = 1 - q,$ optimal prediction $\gamma^t = q$
- Bounded loss.
- Defines a metric (symmetric and triangle ineq.)
- Corresponds to regression.
- $$L_A^T \leq \min_i L_i^T + \frac{1}{2} \ln N$$

Hellinger Loss

- $$\lambda_{\mathrm{hel}}(\omega, \gamma) = \frac{1}{2}\left(\left(\sqrt{\omega} + \sqrt{\gamma}\right)^2 + \left(\sqrt{1-\omega} + \sqrt{1-\gamma}\right)^2\right)$$

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- Probability of making a mistake if predicting 0 or 1 using a biased coin
- If $P[\omega^t = 1] = q$, $P[\omega^t = 0] = 1 - q$, then the optimal prediction is

$$\gamma^t = \begin{cases} 1 & \text{if } q > 1/2, \\ 0 & \text{otherwise} \end{cases}$$

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- **Which losses behave like entropy loss and which behave like hedge loss?**

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- There is no universally optimal prediction
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Vovk's meta-algorithm

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- Vovk's result: *yes!* a good choice for γ_t always exists!

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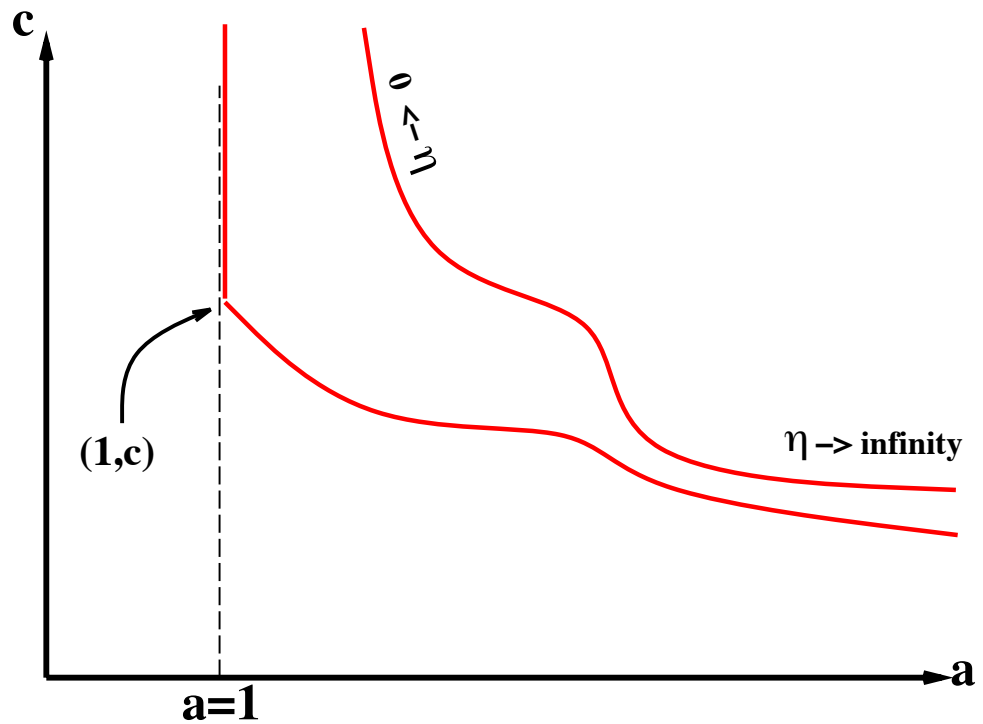

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- Equivalently - the image of the set Γ under the mapping $F(\gamma) = \left\langle e^{-\eta \lambda(\omega, \gamma)} \right\rangle_{\omega \in \Omega}$ is concave.

convexity condition: Pictorially

- **Example:** Suppose $\Omega = \{0, 1\}$, $\Gamma = [0, 1]$. then

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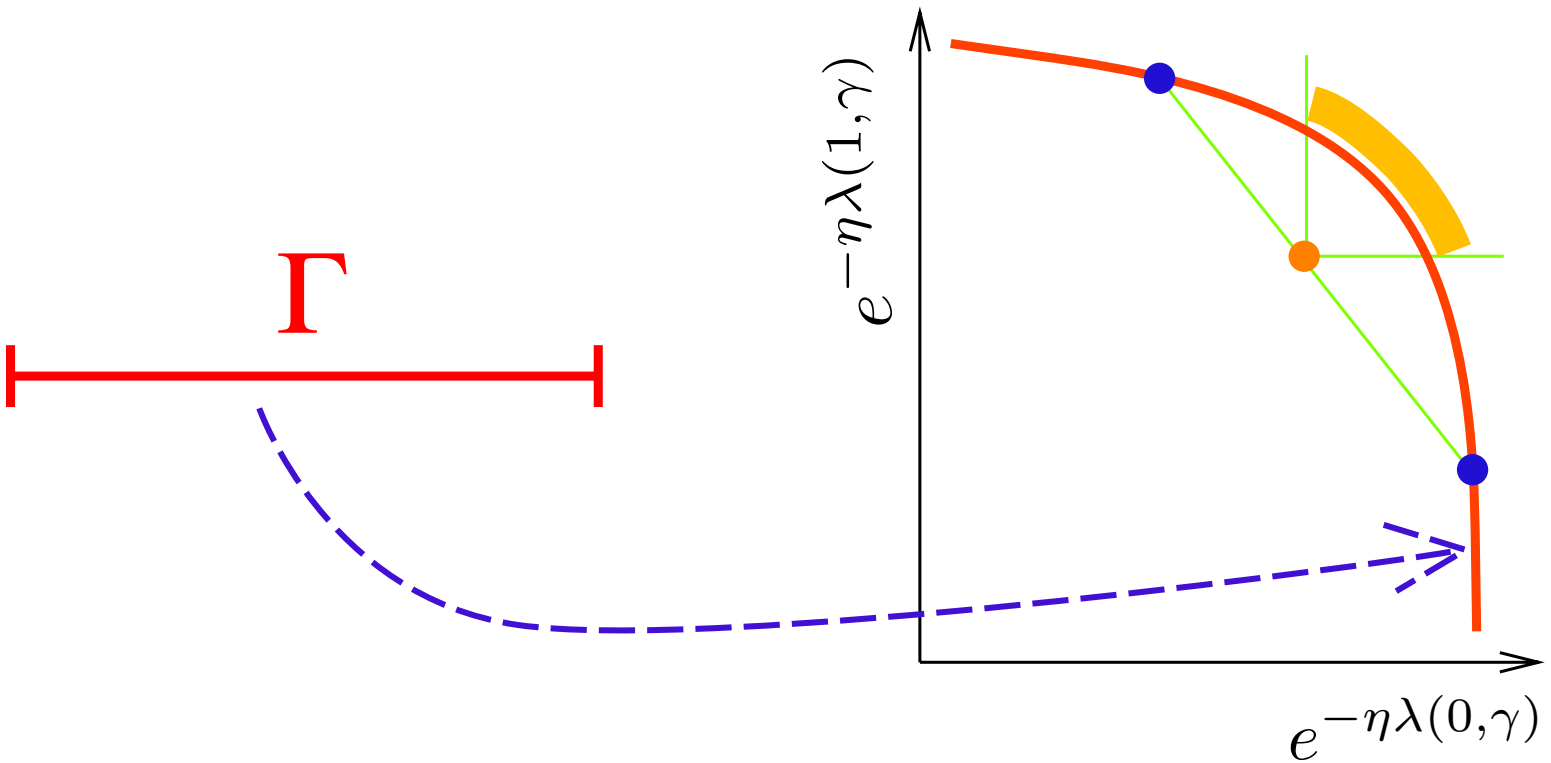

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- We are back to the online Bayes algorithm.

Vovk algorithm for square loss

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- $$L_A \le L_{\min} + \frac{1}{2} \ln N$$

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- Which yields the bound

$$L_A \leq L_{\min} + 2 \ln N$$

Summary of bounds for mixable losses

Loss Functions:	c values: $(\eta = 1/c)$	
	$\mathbf{pred}_{\mathrm{wmean}}(v, x)$	$\mathbf{pred}_{\mathrm{Vovk}}(v,$
$L_{\mathrm{sq}}(p, q)$	2	$1/2$
$L_{\mathrm{ent}}(p, q)$	1	1
$L_{\mathrm{hel}}(p, q)$	1	$1/\sqrt{2}$

Figure 2. $(c, 1/c)$-realizability: c values for loss and prediction