Online Learning and Online Convex Optimization

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Online Convex Optimization (OCO)

Algorithm

Input: A convex set S

For $t = 1, 2, \ldots$

- Predict a vector $w_t \in S$
- Receive a convex loss function $f_t : S \to \mathbb{R}$
- Suffer loss $f_t(w_t)$

Regret Definition

Regret of the Algorithm:

$$\text{Regret}_T(u) = \sum_{t=1}^{T} f_t(w_t) - \sum_{t=1}^{T} f_t(u). \tag{1}$$

Regret relative to a set of vectors U:

$$\text{Regret}_T(U) = \max_{u \in U} \text{Regret}_T(u). \tag{2}$$

Follow-the-Leader Algorithm

FTL Strategy

At round t, select:

$$w_t = \text{argmin}_{w \in S} \sum_{i=1}^{t-1} f_i(w)$$

- Natural approach: Choose best performer on past data
- Simple but can be unstable
- Requires solving optimization problem each round

FTL Regret Analysis

Theorem (Lemma 2.1)

For any $u \in S$:

$$Regret_T(u) \leq \sum_{t=1}^{T} [f_t(w_t) - f_t(w_{t+1})]$$

Complete Proof.

By induction on T:

- Base case: $T = 1$ trivial as $f_1(w_1) - f_1(u) \leq 0$
- Inductive step: Assume holds for $T - 1$, then

$$\sum_{t=1}^{T} [f_t(w_t) - f_t(u)]$$

$$= \sum_{t=1}^{T-1} [f_t(w_t) - f_t(u)] + [f_T(w_T) - f_T(u)]$$

Quadratic Optimization Example

Example (Quadratic Loss)

For $f_t(w) = \frac{1}{2}\|w - z_t\|_2^2$:

- FTL update: $w_t = \frac{1}{t-1}\sum_{i=1}^{t-1} z_i$
- Regret bound: $O(\log T)$

Regret Calculation.

$$\begin{aligned}\text{Regret}_T(u) &\leq \sum_{t=1}^{T} \frac{1}{t}\|w_t - z_t\|^2 \\ &\leq \sum_{t=1}^{T} \frac{(2L)^2}{t} = 4L^2(\log T + 1)\end{aligned}$$

where $L = \max_t \|z_t\|$ $\square$

FTRL Regret Bound

Theorem (Theorem 2.4)

For linear $f_t(w) = \langle w, z_t \rangle$ *and* $R(w) = \frac{1}{2\eta}\|w\|_2^2$*:*

$$Regret_T(U) \le \frac{B^2}{2\eta} + \eta T L^2$$

Proof.

Using lemma 3.3 and strong convexity:

$$\begin{aligned}\sum_{t=1}^{T} \langle w_t - u, z_t \rangle &\le \frac{1}{2\eta}\|u\|^2 + \eta \sum_{t=1}^{T} \|z_t\|^2 \\ &\le \frac{B^2}{2\eta} + \eta T L^2\end{aligned}$$

Minimizing over η gives $O(\sqrt{T})$ bound $\square$

Online Gradient Descent Example

Example (OGD from FTRL)

Update rule:

$$w_{t+1} = w_t - \eta z_t$$

Special case of FTRL with $R(w) = \frac{1}{2\eta}\|w\|_2^2$

Regret Bound.

From FTRL theorem:

$$\begin{aligned}\text{Regret} &\leq \frac{\|u\|^2}{2\eta} + \eta \sum_{t=1}^{T} \|z_t\|^2 \\ &\leq \frac{B^2}{2\eta} + \eta T L^2\end{aligned}$$

$\square$

Practical Considerations

Doubling Trick

- Removes need to know time horizon T
- Divide time into epochs $2^m, 2^{m+1} - 1$
- Regret increases by constant factor:

$$\sum_{m=0}^{\log T} \sqrt{2^m} = O(\sqrt{T})$$

Example (Optimal η)

Setting $\eta = \frac{B}{L}\sqrt{\frac{2}{T}}$ gives:

$$BL\sqrt{2T}$$