

# Exponential Weights Algorithms for Online Learning

Yoav Freund

► slides in

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- ▶ In PLG: pages 12-25

# Outline

## Decision Theoretic Online learning

Hedging vs. Halving

Failure of Follow the leader

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**Hedge( $\eta$ )**Algorithm

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Upper bound on  $\sum_{i=1}^N w_i^{T+1}$

Lower bound on  $\sum_{i=1}^N w_i^{T+1}$

Combining Upper and Lower bounds

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Lower Bounds



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- ▶ Fits nicely in game theory

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- ▶ Basic idea - reduce probability of lossy actions, but **not all the way to zero**.
- ▶ **Modified Goal:** minimize **difference between** expected total loss and minimal total loss of repeating one action.

$$\sum_{t=1}^T \mathbf{p}^t \cdot \ell_t - \min_i \left( \sum_{t=1}^T \ell_i^t \right)$$

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  - ▶ Algorithm predicts **1** with probability  $\sum_{i: e_i^t=1} p_i^t$ .
  - ▶ outcome  $o_i^t$  is revealed.  $\ell_i^t = 0$  if  $e_i^t = o_i^t$ ,  $\ell_i^t = 1$  otherwise.

- └ Decision Theoretic Online learning
  - └ Failure of Follow the leader

## Failure of follow the leader

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expert1 loss  
expert cumul

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FTL cumul

## Failure of follow the leader

 $t = 1$ 

expert1 loss 0.5

expert cumul 0.5

expert2 loss 0.0

expert2 cumul 0.0

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## Failure of follow the leader

	$t = 1$	$t = 2$
expert1 loss	0.5	0.0
expert1 cumul	0.5	0.5
expert2 loss	0.0	1.0
expert2 cumul	0.0	1.0
FTL cumul	0.0	1.0

## Failure of follow the leader

	$t = 1$	$t = 2$	$t = 3$
expert1 loss	0.5	0.0	1.0
expert1 cumul	0.5	0.5	1.5
expert2 loss	0.0	1.0	0.0
expert2 cumul	0.0	1.0	1.0
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	$t = 1$	$t = 2$	$t = 3$	$t = 4$
expert1 loss	0.5	0.0	1.0	0.0
expert1 cumul	0.5	0.5	1.5	1.5
expert2 loss	0.0	1.0	0.0	1.0
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	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
expert1 loss	0.5	0.0	1.0	0.0	1.0
expert1 cumul	0.5	0.5	1.5	1.5	2.5
expert2 loss	0.0	1.0	0.0	1.0	0.0
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expert2 loss	0.0	1.0	0.0	1.0	0.0	1.0
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expert1 loss	0.5	0.0	1.0	0.0	1.0	0.0	1.0
expert1 cumul	0.5	0.5	1.5	1.5	2.5	2.5	3.5
expert2 loss	0.0	1.0	0.0	1.0	0.0	1.0	0.0
expert2 cumul	0.0	1.0	1.0	2.0	2.0	3.0	3.0
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Note freedom to choose initial weight ( $w_i^1$ )  $\sum_{i=1}^n w_i^1 = 1$ .

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## Choosing the initial weights

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- ▶ Plays a similar role to prior distribution in Bayesian algorithms.

## Bound on the loss of **Hedge**( $\eta$ ) Algorithm

### Theorem (main theorem)

For any sequence of loss vectors  $\ell_1, \dots, \ell_T$ , and for any  $i \in \{1, \dots, N\}$ , we have

$$L_{\text{Hedge}(\eta)} \leq \frac{-\ln(w_i^1) + \eta L_i}{1 - e^{-\eta}}.$$

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- ▶ Note effect of the limits  $\eta \rightarrow 0$  and  $\eta \rightarrow \infty$
- ▶ **Proof:** by combining upper and lower bounds on  $\sum_{i=1}^N w_i^{T+1}$

## Hedge( $\eta$ )

└ Bound on total loss

└ Upper bound on  $\sum_{i=1}^N w_i^{T+1}$

Upper bound on  $\sum_{i=1}^N w_i^{T+1}$

Lemma (upper bound)

For any sequence of loss vectors  $\ell_1, \dots, \ell_T$  we have

$$\ln \left( \sum_{i=1}^N w_i^{T+1} \right) \leq -(1 - e^{-\eta}) L_{\text{Hedge}(\eta)}.$$

## Hedge( $\eta$ )

└ Bound on total loss

└ Upper bound on  $\sum_{i=1}^N w_i^{T+1}$

## Proof of upper bound (slide 1)

► If  $a \geq 0$  then  $a^r$  is convex.

## Hedge( $\eta$ )

└ Bound on total loss

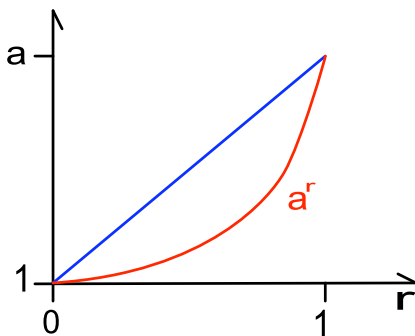
└ Upper bound on  $\sum_{i=1}^N w_i^{T+1}$

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## Hedge( $\eta$ )

└ Bound on total loss

└ Upper bound on  $\sum_{i=1}^N w_i^{T+1}$

## Proof of upper bound (slide 2)

Applying  $a^r \leq 1 - (1 - a)^r$  where  $a = e^{-\eta}$ ,  $r = \ell_i^t$

$$\sum_{i=1}^N w_i^{t+1} = \sum_{i=1}^N w_i^t e^{-\eta \ell_i^t}$$

## Hedge( $\eta$ )

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Applying  $a^r \leq 1 - (1 - a)^r$  where  $a = e^{-\eta}, r = \ell_i^t$

$$\begin{aligned}\sum_{i=1}^N w_i^{t+1} &= \sum_{i=1}^N w_i^t e^{-\eta \ell_i^t} \\ &\leq \sum_{i=1}^N w_i^t (1 - (1 - e^{-\eta}) \ell_i^t)\end{aligned}$$

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## Hedge( $\eta$ )

└ Bound on total loss

└ Upper bound on  $\sum_{i=1}^N w_i^{T+1}$

## Proof of upper bound (slide 3)

### ► Combining

$$\sum_{i=1}^N w_i^{t+1} \leq \left( \sum_{i=1}^N w_i^t \right) (1 - (1 - e^{-\eta}) \mathbf{p}^t \cdot \ell_t)$$

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### ► for $t = 1, \dots, T$

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► for  $t = 1, \dots, T$

► yields

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► for  $t = 1, \dots, T$

► yields

$$\begin{aligned} \sum_{i=1}^N w_i^{T+1} &\leq \prod_{t=1}^T (1 - (1 - e^{-\eta}) \mathbf{p}^t \cdot \ell_t) \\ &\leq \exp \left( -(1 - e^{-\eta}) \sum_{t=1}^T \mathbf{p}^t \cdot \ell_t \right) \end{aligned}$$

since  $1 + x \leq e^x$  for  $x = -(1 - e^{-\eta})$ .

## Hedge( $\eta$ )

└ Bound on total loss

└ Lower bound on  $\sum_{i=1}^N w_i^{T+1}$

## Lower bound on $\sum_{i=1}^N w_i^{T+1}$

For any  $j = 1, \dots, N$ :

$$\sum_{i=1}^N w_i^{T+1} \geq w_j^{T+1} = w_j^1 e^{-\eta L_j}$$

## Combining Upper and Lower bounds

- Combining bounds on  $\ln \left( \sum_{i=1}^N w_i^{T+1} \right)$

$$\ln w_j^1 - \eta L_j \leq \ln \sum_{i=1}^N w_i^{T+1} \leq -(1 - e^{-\eta}) \sum_{t=1}^T \mathbf{p}^t \cdot \ell_t$$



## Combining Upper and Lower bounds

- ▶ Combining bounds on  $\ln \left( \sum_{i=1}^N w_i^{T+1} \right)$

$$\ln w_j^1 - \eta L_j \leq \ln \sum_{i=1}^N w_i^{T+1} \leq -(1 - e^{-\eta}) \sum_{t=1}^T \mathbf{p}^t \cdot \ell_t$$

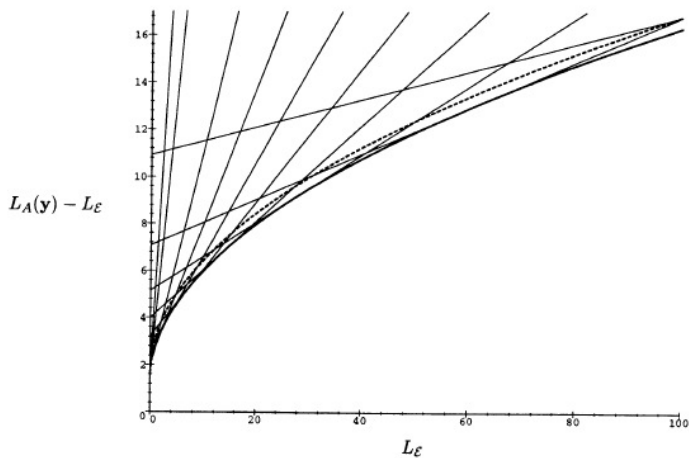
- ▶ Reversing signs, using  $L_{\text{Hedge}(\eta)} = \sum_{t=1}^T \mathbf{p}^t \cdot \ell_t$  and reorganizing we get

$$L_{\text{Hedge}(\eta)} \leq \frac{-\ln(w_i^1) + \eta L_i}{1 - e^{-\eta}}$$

# Tuning $\eta$

*How to Use Expert Advice*

451



## Tuning $\eta$

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- ▶ use uniform initial weights  $\mathbf{w}^1 = \langle 1/N, \dots, 1/N \rangle$

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- ▶ set

$$\eta = \ln \left( 1 + \sqrt{\frac{2 \ln N}{\tilde{L}}} \right) \approx \sqrt{\frac{2 \ln N}{\tilde{L}}}$$

- ▶ use uniform initial weights  $\mathbf{w}^1 = \langle 1/N, \dots, 1/N \rangle$
- ▶ Then

$$L_{\text{Hedge}(\eta)} \leq \frac{-\ln(w_i^1) + \eta L_i}{1 - e^{-\eta}} \leq \min_i L_i + \sqrt{2\tilde{L} \ln N} + \ln N$$

# Exact tuning of $\eta$

## 2.2 How to choose $\beta$

So far, we have analyzed **Hedge**( $\beta$ ) for a given choice of  $\beta$ , and we have proved reasonable bounds for any choice of  $\beta$ . In practice, we will often want to choose  $\beta$  so as to maximally exploit any prior knowledge we may have about the specific problem at hand.

The following lemma will be helpful for choosing  $\beta$  using the bounds derived above.

**Lemma 4** Suppose  $0 \leq L \leq \tilde{L}$  and  $0 < R \leq \tilde{R}$ . Let  $\beta = g(\tilde{L}/\tilde{R})$  where  $g(z) = 1/(1 + \sqrt{2/z})$ . Then

$$\frac{-L \ln \beta + R}{1 - \beta} \leq L + \sqrt{2\tilde{L}\tilde{R}} + R.$$

**Proof:** (Sketch) It can be shown that  $-\ln \beta \leq (1 - \beta^2)/(2\beta)$  for  $\beta \in (0, 1]$ . Applying this approximation and the given choice of  $\beta$  yields the result. ■

Lemma 4 can be applied to any of the bounds above since all of these bounds have the form given in the lemma. For example, suppose we have  $N$  strategies, and we also know a prior bound  $\tilde{L}$  on the loss of the best strategy. Then, combining Equation (9) and Lemma 4, we have

$$L_{\text{Hedge}(\beta)} \leq \min_i L_i + \sqrt{2\tilde{L} \ln N} + \ln N \quad (11)$$

## Tuning $\eta$ as a function of $T$

- trivially  $\min_i L_i \leq T$ , yielding

$$L_{\text{Hedge}(\eta)} \leq \min_i L_i + \sqrt{2T \ln N} + \ln N$$



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- ▶ per iteration we get:

$$\frac{L_{\text{Hedge}(\eta)}}{T} \leq \min_i \frac{L_i}{T} + \sqrt{\frac{2 \ln N}{T}} + \frac{\ln N}{T}$$

## How good is this bound?

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- ▶ The adversarial strategy is random, extremely simple, and does not depend on the hedging strategy!

## The adversarial strategy

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- ▶ Detailed proof quite involved. See section 3.7 in PLG.

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- ▶ A trivial random data, in which there is nothing to be learned forces **any** algorithm to suffer this total loss

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- ▶ Observing only the loss of chosen action - the multi-armed bandit problem. Will get to that later in the course.

# Homework

- ▶ Due thursday January 16, 2025



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**2.1** Assume that you have to predict a sequence  $Y_1, Y_2, \dots \in \{0, 1\}$  of i.i.d. random variables with unknown distribution, your decision space is  $[0, 1]$ , and the loss function is  $\ell(\hat{p}, y) = |\hat{p} - y|$ . How would you proceed? Try to estimate the cumulative loss of your forecaster and compare it to the cumulative loss of the best of the two experts, one of which always predicts 1 and the other always predicts 0. Which are the most “difficult” distributions? How does your (expected) regret compare to that of the weighted average algorithm (which does not “know” that the outcome sequence is i.i.d.)?