# Online Mirrored Descent

Based on Elad Hazan's Text

Section 5.3 Overview

# Introduction to Online Mirrored Descent

**Definition:** Online Mirrored Descent (OMD) is a generalization of gradient descent that applies transformations using a regularization function.

- Extends gradient descent by performing updates in a dual space.
- Regularization controls stability and enables better bounds.
- Two versions: **Lazy OMD** (projects at decision time) and **Agile OMD** (maintains feasibility at all times).

# Mathematical Formulation of OMD

**Decision Protocol:**

- At iteration $t$, the learner selects $x_t \in K$.
- The adversary reveals a loss function $f_t$.
- The learner updates $x_t$ using a regularized optimization step.

# Algorithm: Online Mirrored Descent (OMD)

**Algorithm 1** Online Mirrored Descent

**Input:** Learning rate $\eta > 0$, regularization function $R(x)$.
Initialize $y_1$ such that $\nabla R(y_1) = 0$ and $x_1 = \arg\min_{x \in K} B_R(x \| y_1)$.
**for** $t = 1$ to $T$ **do**
  Play $x_t$.
  Observe payoff function $f_t$ and compute $\nabla_t = \nabla f_t(x_t)$.
  Update $y_t$:
  - **Lazy:** $\nabla R(y_{t+1}) = \nabla R(y_t) - \eta \nabla_t$
  - **Agile:** $\nabla R(y_{t+1}) = \nabla R(x_t) - \eta \nabla_t$

  Project: $x_{t+1} = \arg\min_{x \in K} B_R(x \| y_{t+1})$.
**end for**

# Equivalence of Lazy OMD and RFTL

**Lemma:** When cost functions $f_1, \ldots, f_T$ are linear, Lazy OMD and Regularized Follow-The-Leader (RFTL) produce identical predictions.

$$\arg\min_{x \in K} B_R(x||y_t) = \arg\min_{x \in K} \left( \sum_{s=1}^{t-1} \eta \nabla_s^T x + R(x) \right) \tag{1}$$

**Proof:** This follows from the uniqueness of the solution for strictly convex $R(x)$ and the definition of the Bregman divergence.

# Regret Bounds for OMD

**Theorem:** The regret of OMD for any $u \in K$ satisfies:

$$\text{regret}_T \leq \frac{\eta}{4} \sum_{t=1}^{T} \|\nabla_t\|_{*t}^2 + \frac{R(u) - R(x_1)}{2\eta}. \tag{2}$$

**Corollary:** If $\|\nabla_t\|_{*t} \leq G_R$ for all $t$, then optimal tuning of $\eta$ gives:

$$\text{regret}_T \leq D_R G_R \sqrt{T}. \tag{3}$$

# Proof of Regret Bound

**Step 1: Bregman Divergence Expansion**

$$B_R(x||y) = R(x) - R(y) - \nabla R(y)^T (x - y). \tag{4}$$

**Step 2: Expanding the Recursion for $y_t$**

$$\nabla R(y_{t+1}) = \nabla R(y_t) - \eta \nabla_t. \tag{5}$$

**Step 3: Bounding the Sum of Divergences**

$$\sum_{t=1}^{T} B_R(x||y_t) \leq \frac{1}{2\eta}(R(x) - R(x_1)) + \frac{\eta}{4} \sum_{t=1}^{T} \|\nabla_t\|_{*t}^2. \tag{6}$$

# Conclusion

- Online Mirrored Descent generalizes online gradient descent using regularization.
- Lazy OMD and RFTL are equivalent for linear functions.
- The regret bound is dependent on the choice of $R(x)$ and learning rate $\eta$.
- The proofs rely on properties of Bregman divergence and convexity arguments.