Exponential Weights Algorithms for Online Learning

Yoav Freund

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- In PLG: pages 12-25

Outline

Decision Theoretic Online learning

- Hedging vs. Halving
- Failure of Follow the leader

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Hedge(η)Algorithm

Bound on total loss

- Upper bound on $\sum_{i=1}^{N} w_i^{T+1}$
- Lower bound on $\sum_{i=1}^{N} w_i^{T+1}$
- Combining Upper and Lower bounds

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Lower Bounds

The hedging problem

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- **Goal:** minimize total expected loss
- Here we have stochasticity - but only in **algorithm**, not in **outcome**
- Fits nicely in game theory

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- Unlike halving - no action is perfect.
- Basic idea - reduce probability of lossy actions, but not all the way to zero.
- **Modified Goal:** minimize difference between expected total loss and minimal total loss of repeating one action.

$$\sum_{t=1}^{T} \mathbf{p}^t \cdot \ell_t - \min_i \left(\sum_{t=1}^{T} \ell_i^t \right)$$

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  - Algorithm predicts 1 with probability $\sum_{i:e_i^t=1} p_i^t$.
  - outcome o_i^t is revealed. $\ell_i^t = 0$ if $e_i^t = o_i^t$, $\ell_i^t = 1$ otherwise.

Failure of follow the leader

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expert1 loss
expert1 cumul

expert2 loss
expert2 cumul

FTL cumul

Failure of follow the leader

	$t = 1$
expert1 loss	0.5
expert1 cumul	0.5
expert2 loss	0.0
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FTL cumul	0.0

Failure of follow the leader

	$t = 1$	$t = 2$
expert1 loss	0.5	0.0
expert1 cumul	0.5	0.5
expert2 loss	0.0	1.0
expert2 cumul	0.0	1.0
FTL cumul	0.0	1.0

Failure of follow the leader

	$t = 1$	$t = 2$	$t = 3$
expert1 loss	0.5	0.0	1.0
expert1 cumul	0.5	0.5	1.5
expert2 loss	0.0	1.0	0.0
expert2 cumul	0.0	1.0	1.0
FTL cumul	0.0	1.0	2.0

Failure of follow the leader

	$t = 1$	$t = 2$	$t = 3$	$t = 4$
expert1 loss	0.5	0.0	1.0	0.0
expert1 cumul	0.5	0.5	1.5	1.5
expert2 loss	0.0	1.0	0.0	1.0
expert2 cumul	0.0	1.0	1.0	2.0
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Failure of follow the leader

	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
expert1 loss	0.5	0.0	1.0	0.0	1.0
expert1 cumul	0.5	0.5	1.5	1.5	2.5
expert2 loss	0.0	1.0	0.0	1.0	0.0
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	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$	$t = 6$
expert1 loss	0.5	0.0	1.0	0.0	1.0	0.0
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Failure of follow the leader

	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$	$t=6$	$t=7$
expert1 loss	0.5	0.0	1.0	0.0	1.0	0.0	1.0
expert1 cumul	0.5	0.5	1.5	1.5	2.5	2.5	3.5
expert2 loss	0.0	1.0	0.0	1.0	0.0	1.0	0.0
expert2 cumul	0.0	1.0	1.0	2.0	2.0	3.0	3.0
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Consider action i at time t

- Total loss:

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Note freedom to choose initial weight (w_i^1) $\sum_{i=1}^{n} w_i^1 = 1$.

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$$p_i^t = \frac{w_i^t}{\sum_{j=1}^{N} w_i^t}, \quad \mathbf{p}^t = \frac{\mathbf{w}^t}{\sum_{j=1}^{N} w_i^t}$$

Choosing the initial weights

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- If good action has low initial weight, our total loss will be larger.
- As $\sum_{i=1}^{n} w_i^1 = 1$ increasing one weight implies decreasing some others.
- Plays a similar role to prior distribution in Bayesian algorithms.

Bound on the loss of **Hedge**(η)Algorithm

Theorem (main theorem)

For any sequence of loss vectors $\ell_1, \ldots, \ell_T$*, and for any* $i \in \{1, \ldots, N\}$*, we have*

$$L_{\mathbf{Hedge}(\eta)} \leq \frac{-\ln(w_i^1) + \eta L_i}{1 - e^{-\eta}}.$$

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- Note effect of the limits $\eta \to 0$ and $\eta \to \infty$
- Proof: by combining upper and lower bounds on $\sum_{i=1}^{N} w_i^{T+1}$

Upper bound on $\sum_{i=1}^{N} w_i^{T+1}$

Lemma (upper bound)

For any sequence of loss vectors $\ell_1, \ldots, \ell_T$ *we have*

$$\ln\left(\sum_{i=1}^{N} w_i^{T+1}\right) \le -(1 - e^{-\eta}) L_{\textbf{Hedge}(\eta)}.$$

Proof of upper bound (slide 1)

- If $a \geq 0$ then a^r is convex.

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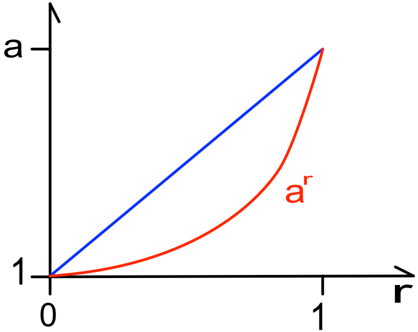

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Proof of upper bound (slide 2)

Applying $a^r \leq 1 - (1-a)^r$ where $a = e^{-\eta}, r = \ell_i^t$

$$\sum_{i=1}^{N} w_i^{t+1} = \sum_{i=1}^{N} w_i^t e^{-\eta \ell_i^t}$$

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$$\begin{aligned}
\sum_{i=1}^{N} w_i^{t+1} &= \sum_{i=1}^{N} w_i^t e^{-\eta \ell_i^t} \\
&\leq \sum_{i=1}^{N} w_i^t \left(1-(1-e^{-\eta})\ell_i^t\right)
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&= \left(\sum_{i=1}^{N} w_i^t\right)\left(1-(1-e^{-\eta})\frac{\mathbf{w}^t}{\sum_{i=1}^{N} w_i^t}\cdot \ell_t\right)
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&= \left(\sum_{i=1}^N w_i^t\right)\left(1-(1-e^{-\eta})\mathbf{p}^t \cdot \ell_t\right)
\end{aligned}
$$

Proof of upper bound (slide 3)

- Combining

$$\sum_{i=1}^{N} w_i^{t+1} \leq \left(\sum_{i=1}^{N} w_i^t\right) \left(1 - (1 - e^{-\eta})\mathbf{p}^t \cdot \ell_t\right)$$

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- for $t = 1, \ldots, T$

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- for $t = 1, \ldots, T$
- yields

$$\sum_{i=1}^{N} w_i^{T+1} \quad \leq \quad \prod_{t=1}^{T}\left(1 - (1 - e^{-\eta})\mathbf{p}^t \cdot \ell_t\right)$$

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- for $t = 1, \ldots, T$
- yields

$$\begin{aligned} \sum_{i=1}^N w_i^{T+1} &\le \prod_{t=1}^T (1-(1-e^{-\eta})\mathbf{p}^t\cdot \ell_t) \\ &\le \exp\left(-(1-e^{-\eta})\sum_{t=1}^T \mathbf{p}^t\cdot \ell_t\right) \end{aligned}$$

since $1 + x \le e^x$ for $x = -(1-e^{-\eta})$.

Lower bound on $\sum_{i=1}^{N} w_i^{T+1}$

For any $j = 1, \ldots, N$:

$$\sum_{i=1}^{N} w_i^{T+1} \geq w_j^{T+1} = w_j^1 e^{-\eta L_j}$$

Combining Upper and Lower bounds

- Combining bounds on $\ln\left(\sum_{i=1}^{N} w_i^{T+1}\right)$

$$\ln w_j^1 - \eta L_j \leq \ln \sum_{i=1}^{N} w_i^{T+1} \leq -(1-e^{-\eta}) \sum_{t=1}^{T} \mathbf{p}^t \cdot \ell_t$$

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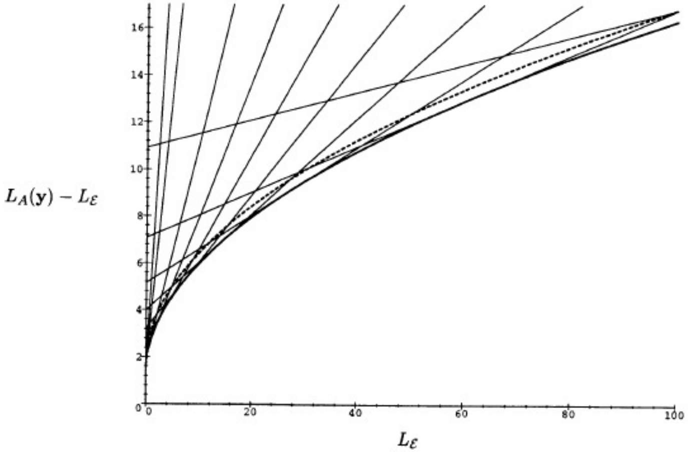
$$\ln w_j^1 - \eta L_j \leq \ln \sum_{i=1}^{N} w_i^{T+1} \leq -(1 - e^{-\eta}) \sum_{t=1}^{T} \mathbf{p}^t \cdot \ell_t$$

- Reversing signs, using $L_{\mathbf{Hedge}(\eta)} = \sum_{t=1}^{T} \mathbf{p}^t \cdot \ell_t$ and reorganizing we get

$$L_{\mathbf{Hedge}(\eta)} \leq \frac{-\ln(w_i^1) + \eta L_i}{1 - e^{-\eta}}$$

Tuning η

How to Use Expert Advice 451



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- use uniform initial weights $\mathbf{w}^1 = \langle 1/N, \ldots, 1/N \rangle$
- Then

$$L_{\mathbf{Hedge}(\eta)} \leq \frac{-\ln(w_i^1) + \eta L_i}{1 - e^{-\eta}} \leq \min_i L_i + \sqrt{2\tilde{L}\ln N} + \ln N$$

Exact tuning of η

2.2 How to choose β

So far, we have analyzed **Hedge**(β) for a given choice of β, and we have proved reasonable bounds for any choice of β. In practice, we will often want to choose β so as to maximally exploit any prior knowledge we may have about the specific problem at hand.

The following lemma will be helpful for choosing β using the bounds derived above.

Lemma 4 *Suppose* $0 \leq L \leq \tilde{L}$ *and* $0 < R \leq \tilde{R}$. *Let* $\beta = g(\tilde{L}/\tilde{R})$ *where* $g(z) = 1/(1 + \sqrt{2/z})$. *Then*

$$\frac{-L \ln \beta + R}{1 - \beta} \leq L + \sqrt{2\tilde{L}\tilde{R}} + R.$$

Proof: (Sketch) It can be shown that $-\ln \beta \leq (1 - \beta^2)/(2\beta)$ for $\beta \in (0, 1]$. Applying this approximation and the given choice of β yields the result. ■

Lemma 4 can be applied to any of the bounds above since all of these bounds have the form given in the lemma. For example, suppose we have N strategies, and we also know a prior bound $\tilde{L}$ on the loss of the best strategy. Then, combining Equation (9) and Lemma 4, we have

$$L_{\mathbf{Hedge}(\beta)} \leq \min_i L_i + \sqrt{2\tilde{L} \ln N} + \ln N \tag{11}$$

Tuning η as a function of T

- trivially $\min_i L_i \leq T$, yielding

$$L_{\textbf{Hedge}(\eta)} \leq \min_i L_i + \sqrt{2T \ln N} + \ln N$$

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- per iteration we get:

$$\frac{L_{\textbf{Hedge}(\eta)}}{T} \leq \min_i \frac{L_i}{T} + \sqrt{\frac{2 \ln N}{T}} + \frac{\ln N}{T}$$

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- The adversarial strategy is random, extremely simple, and does not depend on the hedging strategy!

The adversarial strategy

- Adversary sets each loss ℓ_i^t indepedently at random to 0 or 1 with equal probabilities $(1/2, 1/2)$.

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- Adversary sets each loss ℓ_i^t indepedently at random to 0 or 1 with equal probabilities $(1/2, 1/2)$.
- Obviously, nothing to learn !
  $L_{\mathbf{S}} \approx T/2$.
- On the other hand $\min_i L_i \approx T/2 - \sqrt{2T \ln N}$

The adversarial strategy

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- Detailed proof quite involved. See section 3.7 in PLG.

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- A trivial random data, in which there is nothing to be learned forces any algorithm to suffer this total loss

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- T is tight only when the loss of experts at each iteration is either 0 or 1. If the loss of the best expert is $o(T)$ then we would like to have a tighter bound.
- Observing only the loss of chosen action - the multi-armed bandit problem. Will get to that later in the course.

Homework

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- **2.1** Assume that you have to predict a sequence $Y_1, Y_2, \ldots \in \{0, 1\}$ of i.i.d. random variables with unknown distribution, your decision space is $[0, 1]$, and the loss function is $\ell(\widehat{p}, y) = |\widehat{p} - y|$. How would you proceed? Try to estimate the cumulative loss of your forecaster and compare it to the cumulative loss of the best of the two experts, one of which always predicts 1 and the other always predicts 0. Which are the most "difficult" distributions? How does your (expected) regret compare to that of the weighted average algorithm (which does not "know" that the outcome sequence is i.i.d.)?