Exponential Weights Algorithms for Online Learning

Yoav Freund

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- In PLG: pages 12-25

Outline

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- Decision Theoretic Online learning
  - Hedging vs. Halving
  - Failure of Follow the leader
- **Hedge**(η)Algorithm
- Bound on total loss
  - Upper bound on $\sum_{i=1}^{N} w_i^{T+1}$
  - Lower bound on $\sum_{i=1}^{N} w_i^{T+1}$
  - Combining Upper and Lower bounds
- tuning η

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Lower Bounds

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- Fits nicely in game theory

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- Basic idea - reduce probability of lossy actions, but not all the way to zero.
- **Modified Goal:** minimize difference between expected total loss and minimal total loss of repeating one action.

$$\sum_{t=1}^{T} \mathbf{p}^t \cdot \ell_t - \min_i \left(\sum_{t=1}^{T} \ell_i^t \right)$$

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  - Algorithm predicts 1 with probability $\sum_{i:e_i^t=1} p_i^t$.
  - outcome o_i^t is revealed. $\ell_i^t = 0$ if $e_i^t = o_i^t$, $\ell_i^t = 1$ otherwise.

Failure of follow the leader

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expert1 loss
expert cumul

expert2 loss
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FTL cumul

Failure of follow the leader

	$t = 1$
expert1 loss	0.5
expert cumul	0.5
expert2 loss	0.0
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Failure of follow the leader

	$t = 1$	$t = 2$
expert1 loss	0.5	0.0
expert cumul	0.5	0.5
expert2 loss	0.0	1.0
expert2 cumul	0.0	1.0
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Failure of follow the leader

	$t = 1$	$t = 2$	$t = 3$
expert1 loss	0.5	0.0	1.0
expert cumul	0.5	0.5	1.5
expert2 loss	0.0	1.0	0.0
expert2 cumul	0.0	1.0	1.0
FTL cumul	0.0	1.0	2.0

Failure of follow the leader

	$t=1$	$t=2$	$t=3$	$t=4$
expert1 loss	0.5	0.0	1.0	0.0
expert cumul	0.5	0.5	1.5	1.5
expert2 loss	0.0	1.0	0.0	1.0
expert2 cumul	0.0	1.0	1.0	2.0
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	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
expert1 loss	0.5	0.0	1.0	0.0	1.0
expert cumul	0.5	0.5	1.5	1.5	2.5
expert2 loss	0.0	1.0	0.0	1.0	0.0
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	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$	$t = 6$	$t = 7$
expert1 loss	0.5	0.0	1.0	0.0	1.0	0.0	1.0
expert cumul	0.5	0.5	1.5	1.5	2.5	2.5	3.5
expert2 loss	0.0	1.0	0.0	1.0	0.0	1.0	0.0
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$$p_i^t = \frac{w_i^t}{\sum_{j=1}^N w_i^t}, \quad \mathbf{p}^t = \frac{\mathbf{w}^t}{\sum_{j=1}^N w_i^t}$$

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- As $\sum_{i=1}^{n} w_i^1 = 1$ increasing one weight implies decreasing some others.
- Plays a similar role to prior distribution in Bayesian algorithms.

Bound on the loss of **Hedge**(η)Algorithm

Theorem (main theorem)

For any sequence of loss vectors $\ell_1, \ldots, \ell_T$, *and for any* $i \in \{1, \ldots, N\}$, *we have*

$$L_{\mathbf{Hedge}(\eta)} \leq \frac{-\ln(w_i^1) + \eta L_i}{1 - e^{-\eta}}.$$

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- Note effect of the limits $\eta \to 0$ and $\eta \to \infty$
- Proof: by combining upper and lower bounds on $\sum_{i=1}^{N} w_i^{T+1}$

Upper bound on $\sum_{i=1}^{N} w_i^{T+1}$

Lemma (upper bound)

For any sequence of loss vectors $\ell_1, \ldots, \ell_T$ *we have*

$$\ln\left(\sum_{i=1}^{N} w_i^{T+1}\right) \leq -(1-e^{-\eta})L_{\textbf{Hedge}(\eta)}.$$

Proof of upper bound (slide 1)

- If $a \geq 0$ then a^r is convex.

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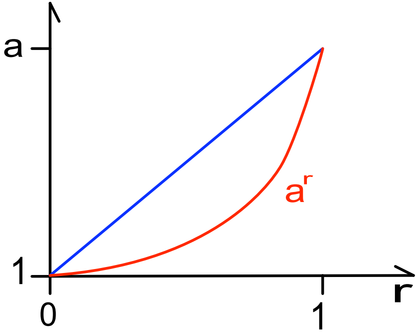

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Proof of upper bound (slide 2)

Applying $a^r \leq 1 - (1-a)^r$ where $a = e^{-\eta}, r = \ell_i^t$

$$\sum_{i=1}^{N} w_i^{t+1} = \sum_{i=1}^{N} w_i^t e^{-\eta \ell_i^t}$$

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$$\begin{aligned}
\sum_{i=1}^{N} w_i^{t+1} &= \sum_{i=1}^{N} w_i^t e^{-\eta \ell_i^t} \\
&\leq \sum_{i=1}^{N} w_i^t \left(1 - (1 - e^{-\eta}) \ell_i^t\right)
\end{aligned}$$

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&\leq \sum_{i=1}^{N} w_i^t \left(1 - (1 - e^{-\eta})\ell_i^t\right) \\
&= \left(\sum_{i=1}^{N} w_i^t\right)\left(1 - (1 - e^{-\eta}) \frac{\mathbf{w}^t}{\sum_{i=1}^{N} w_i^t} \cdot \ell_t\right)
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\sum_{i=1}^N w_i^{t+1} &= \sum_{i=1}^N w_i^t e^{-\eta \ell_i^t} \\
&\le \sum_{i=1}^N w_i^t \left(1-(1-e^{-\eta})\ell_i^t\right) \\
&= \left(\sum_{i=1}^N w_i^t\right)\left(1-(1-e^{-\eta})\frac{\mathbf{w}^t}{\sum_{i=1}^N w_i^t}\cdot \ell_t\right) \\
&= \left(\sum_{i=1}^N w_i^t\right)\left(1-(1-e^{-\eta})\mathbf{p}^t\cdot \ell_t\right)
\end{aligned}$$

Proof of upper bound (slide 3)

- Combining

$$\sum_{i=1}^{N} w_i^{t+1} \leq \left(\sum_{i=1}^{N} w_i^t\right) (1 - (1 - e^{-\eta})\mathbf{p}^t \cdot \ell_t)$$

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- for $t = 1, \ldots, T$

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- yields

$$\sum_{i=1}^{N} w_i^{T+1} \quad \leq \quad \prod_{t=1}^{T}(1-(1-e^{-\eta})\mathbf{p}^t \cdot \ell_t)$$

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- for $t = 1, \dots, T$
- yields

$$\begin{aligned} \sum_{i=1}^{N} w_i^{T+1} &\leq \prod_{t=1}^{T} \left(1 - (1 - e^{-\eta})\mathbf{p}^t \cdot \ell_t\right) \\ &\leq \exp\left(-(1 - e^{-\eta}) \sum_{t=1}^{T} \mathbf{p}^t \cdot \ell_t\right) \end{aligned}$$

since $1 + x \leq e^x$ for $x = -(1 - e^{-\eta})$.

Lower bound on $\sum_{i=1}^{N} w_i^{T+1}$

For any $j = 1, \ldots, N$:

$$\sum_{i=1}^{N} w_i^{T+1} \geq w_j^{T+1} = w_j^1 e^{-\eta L_j}$$

Combining Upper and Lower bounds

- Combining bounds on $\ln\left(\sum_{i=1}^{N} w_i^{T+1}\right)$

$$\ln w_j^1 - \eta L_j \leq \ln \sum_{i=1}^{N} w_i^{T+1} \leq -(1 - e^{-\eta}) \sum_{t=1}^{T} \mathbf{p}^t \cdot \ell_t$$

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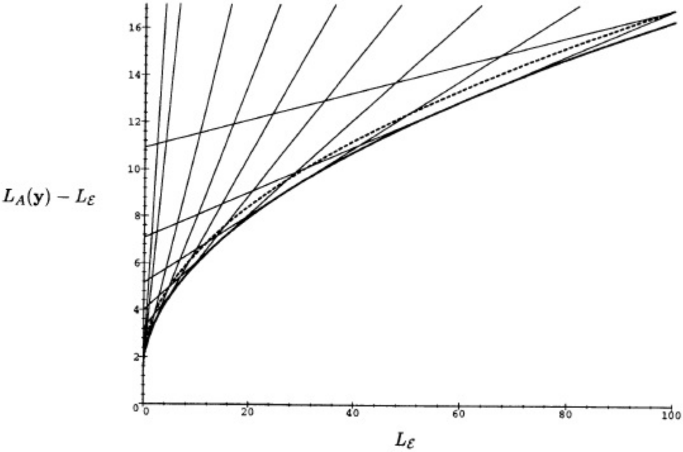
- Reversing signs, using $L_{\mathbf{Hedge}(\eta)} = \sum_{t=1}^{T} \mathbf{p}^t \cdot \ell_t$ and reorganizing we get

$$L_{\mathbf{Hedge}(\eta)} \leq \frac{-\ln(w_i^1) + \eta L_i}{1-e^{-\eta}}$$

Tuning η


$L_A(\mathbf{y}) - L_\mathcal{E}$
0
2
4
6
8
10
12
14
16
0
20
40
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- Then
$$L_{\mathbf{Hedge}(\eta)} \leq \frac{-\ln(w_i^1) + \eta L_i}{1 - e^{-\eta}} \leq \min_i L_i + \sqrt{2\tilde{L}\ln N} + \ln N$$

Tuning η as a function of T

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- per iteration we get:

$$\frac{L_{\textbf{Hedge}(\eta)}}{T} \leq \min_i \frac{L_i}{T} + \sqrt{\frac{2 \ln N}{T}} + \frac{\ln N}{T}$$

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- The adversarial strategy is random, extremely simple, and does not depend on the hedging strategy!

The adversarial strategy

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- Detailed proof quite involved. See section 3.7 in PLG.

Summary

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- Setting $\eta \approx \sqrt{\frac{2 \ln N}{T}}$ guarantees

$$L_{\textbf{Hedge}(\eta)} \leq \min_i L_i + \sqrt{2T \ln N} + \ln N$$

Summary

- Given learning rate η the **Hedge**(η)algorithm satisfies

$$L_{\textbf{Hedge}(\eta)} \leq \frac{\ln N + \eta L_i}{1 - e^{-\eta}}$$

- Setting $\eta \approx \sqrt{\frac{2 \ln N}{T}}$ guarantees

$$L_{\textbf{Hedge}(\eta)} \leq \min_i L_i + \sqrt{2T \ln N} + \ln N$$

- A trivial random data, in which there is nothing to be learned forces any algorithm to suffer this total loss

Some loose threads

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- T is tight only when the loss of experts at each iteration is either 0 or 1. If the loss of the best expert is $o(T)$ then we would like to have a tighter bound.
- Observing only the loss of chosen action - the multi-armed bandit problem. Will get to that later in the course.

Homework

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- **2.1** Assume that you have to predict a sequence $Y_1, Y_2, \ldots \in \{0, 1\}$ of i.i.d. random variables with unknown distribution, your decision space is $[0, 1]$, and the loss function is $\ell(\widehat{p}, y) = |\widehat{p} - y|$. How would you proceed? Try to estimate the cumulative loss of your forecaster and compare it to the cumulative loss of the best of the two experts, one of which always predicts 1 and the other always predicts 0. Which are the most "difficult" distributions? How does your (expected) regret compare to that of the weighted average algorithm (which does not "know" that the outcome sequence is i.i.d.)?