Lossless compression and cumulative log loss

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Outline

Lossless data compression

- The guessing game
- Arithmetic coding
- The performance of arithmetic coding

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log loss

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Other properties of log loss

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  - The performance of arithmetic coding
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  - Other properties of log loss
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  - Combining expert advice for cumulative log loss
- Combining experts in the log loss framework
  - The online Bayes Algorithm
  - The performance bound

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- A natural way for describing a distribution.

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- To decode use the same prediction algorithm

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- Widely used in practice.

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- Code = discriminating binary expansion of a point in $[l_t, u_t)$.

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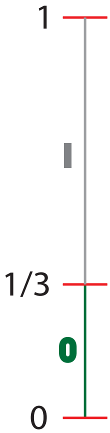

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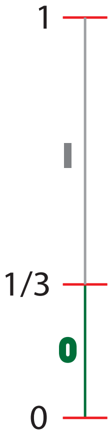

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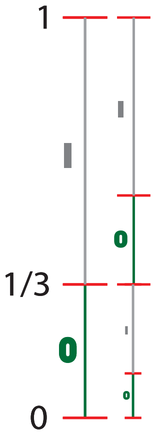

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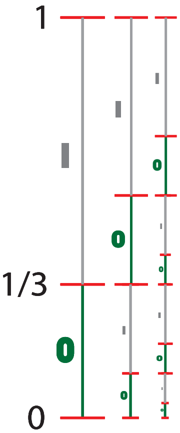

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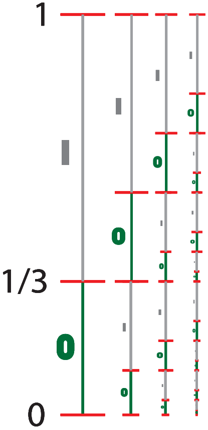

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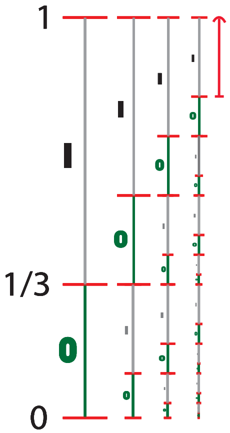

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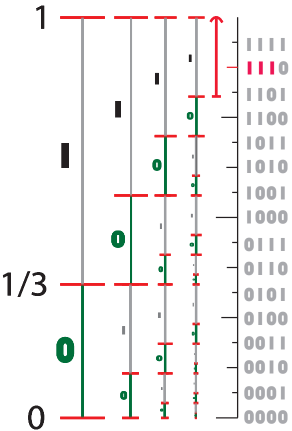
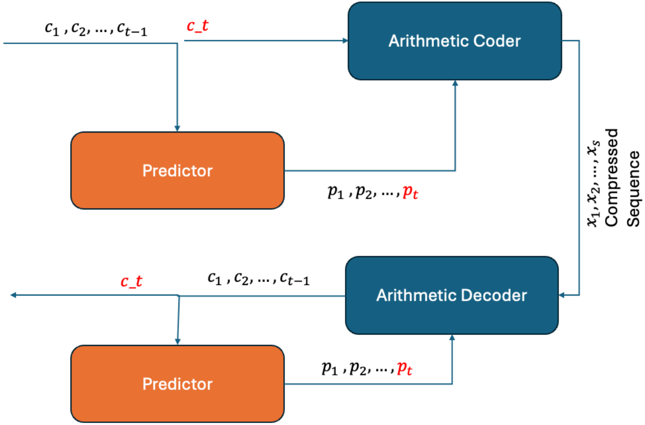


Arithmetic coding (coding/decoding)



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- We call $-\sum_{t=1}^{T} \log_2 p_t(c_t) = -log_2 p(c_1, \ldots c_T)$ the Cumulative log loss

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- We call $-\sum_{t=1}^{T} \log_2 p_t(c_t) = -log_2 p(c_1, \ldots c_T)$ the Cumulative log loss
- Holds for all sequences.

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- Entropy is the expected value of the cumulative log loss

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- The proof of Shannon's lower bound is not trivial (Can be a student lecture).

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- There are other losses with this property, for example, square loss.

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- High risk prediction: Setting $p_t \in \{0, 1\}$ results in Bonus $b_T = 2^T$ if always correct, zero otherwise.
- If forecaster predicts with the true probabilities then

$$E(\log b_T) = T - H(p_T)$$

and that is the maximal expected value for $E(\log b_T)$

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- Taking logs, we get cumulative log loss.

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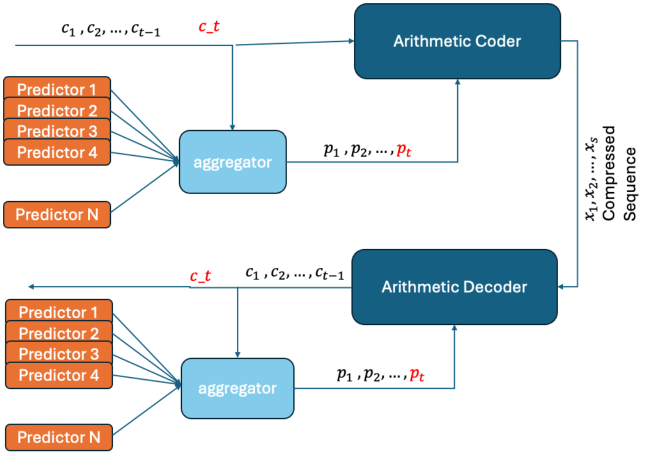
- In horse race: We would like to make almost as much money as the best expert in hind-site.

Universal arithmetic coding



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- Is the key idea of MDL (Minimal Description Length) modeling.
  - Good prediction model = model that minimizes the total code length
- Often inappropriate because based on lossless coding. **Lossy** coding often more appropriate.

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- **Goal:** Total loss of algorithm minus loss of best predictor should be at most $\log_2 N$

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- $\lceil L_A{}^T \rceil$ is the code length if A is combined with arithmetic coding.

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  - c^t is revealed.
- **Goal:** minimize regret:

$$-\sum_{t=1}^{T} \log p_A^t(c^t) + \min_{i=1,\ldots,N} \left(-\sum_{t=1}^{T} \log p_i^t(c^t) \right)$$

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$$\begin{aligned} L_A^T &= -\log W^{T+1} = -\log \sum_{i=1}^{N} w_i^{T+1} \\ &= -\log \sum_{i=1}^{N} \frac{1}{N} e^{-L_i^T} \end{aligned}$$

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- Total Weight is at least the weight of the best expert.

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- Dividing by T we get $\frac{L_A^T}{T} = \min_i \frac{L_i^T}{T} + \frac{\log N}{T}$

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- We don't pay a penalty for copies.
- More generally, the regret is smaller if many of the experts perform well.

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