Tracking the best Expert

Yoav Freund

February 25, 2025

Based on “Tracking the best linear predictor” and “Tracking the best expert” by Herbster and Warmuth. Also, section 11.5 in Prediction learning and Games.

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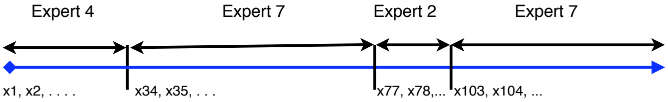
- 

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- Requires maintaining $O\left(n^{k+1}\left(\frac{el}{k}\right)^k\right)$ weights.

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- Then using the partition-expert algorithm for the switching-experts case we get a bound on the regret $\frac{1}{\eta}\left((k+1)\log n + k \log \frac{l}{k} + k\right)$

Weight sharing algorithms

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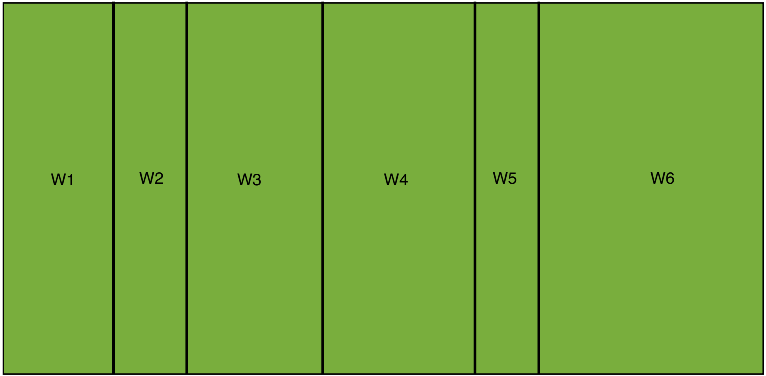
The fixed-share algorithm



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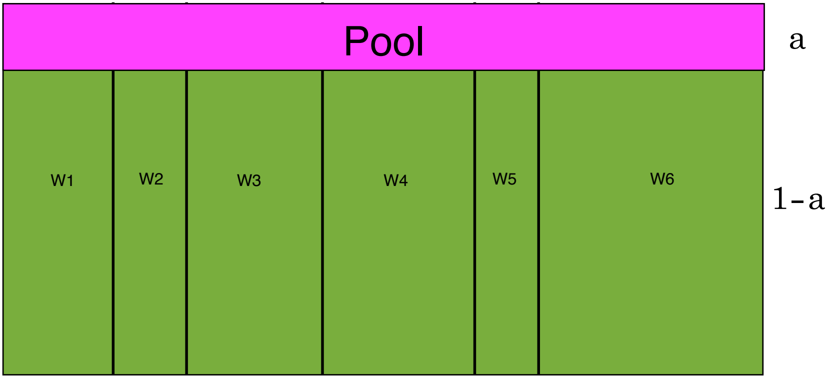

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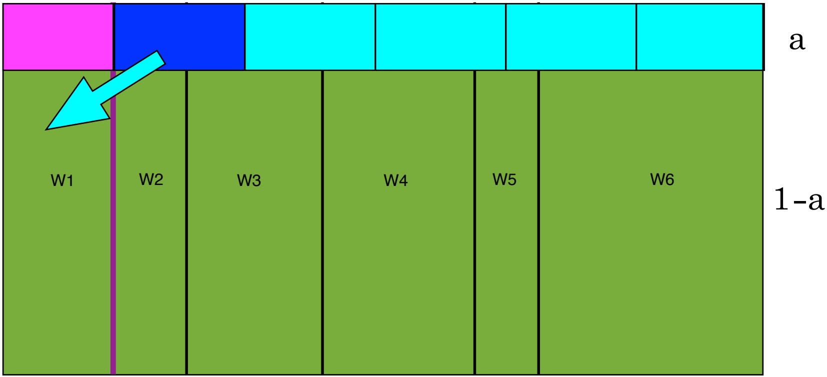

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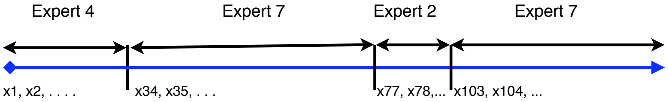
- The harder question is how to lower bound $\sum_{i=1}^{n} w^s_{l+1,i}$

Lower bounding the final total weight

- Fix some switching experts sequence:



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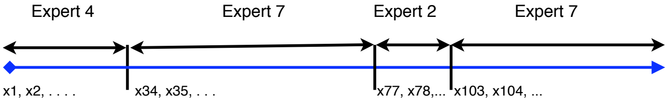

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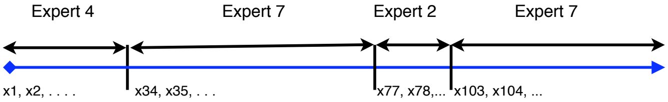

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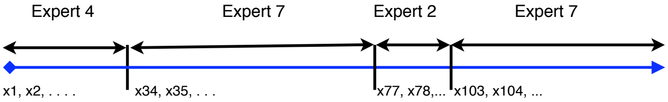

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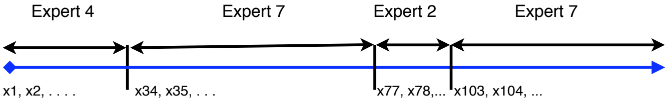

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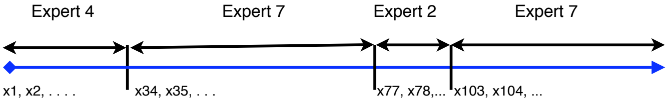

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  - $1 - \alpha$ on iterations with no switch.
  - $\frac{\alpha}{n-1}$ on iterations where a switch occurs.

Bound for arbitrary α

- Combining we lower bound the final weight of the last expert in the sequence

$$w^s_{l+1,e_k} \geq \frac{1}{n} e^{-\eta L_*} (1-\alpha)^{l-k-1} \left(\frac{\alpha}{n-1}\right)^k$$

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- Combining the upper and lower bounds we get that for any sequence

$$L_A \leq L_* + \frac{1}{\eta}\left(\ln n + (l-k-1)\ln\frac{1}{1-\alpha} + k\left(\ln\frac{1}{\alpha} + \ln(n-1)\right)\right)$$

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$$H(\alpha^*) = -\alpha^* \ln \alpha^* - (1-\alpha^*)\ln(1-\alpha^*)$$

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- Not so for square loss!

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- The algorithm does not concentrate only on the best expert, even if the last switch is in the distant past.

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- Requires that the loss be bounded.
- Works for square loss, but not for log loss!

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If $\ell_{t,i} = 1$, then expert i contributes like fixed share.
Expert can get fraction of the total weight arbitrarily close to 1.
Shares the weight quickly if $\ell_{t,i} > 0$

Bound for variable share

$$L_A - L_* \leq \frac{1}{\eta} \ln n + \left(1 + \frac{1}{(1-\alpha)\eta}\right) L_* + k\left(1 + \frac{1}{\eta}\left(\ln n - 1 + \ln \frac{1}{\alpha} + \ln \frac{1}{1-\alpha}\right)\right)$$

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Some Experiments

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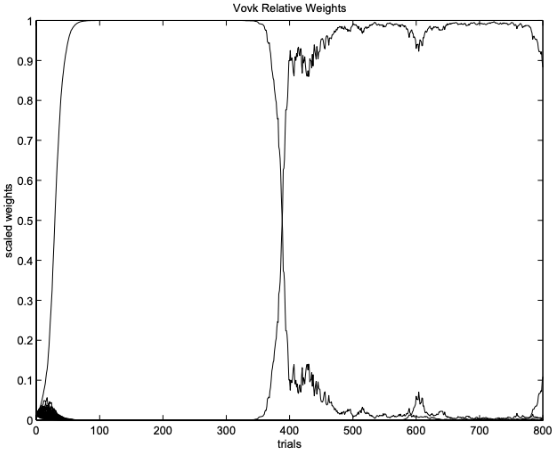
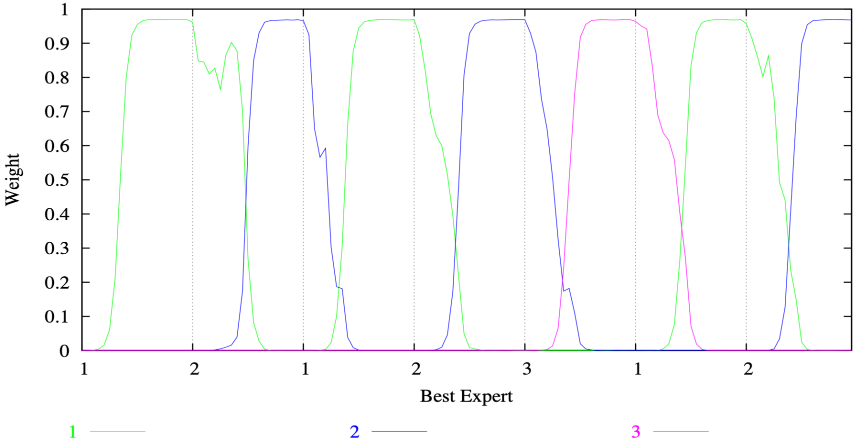
- time is divided into equal length segments
- In each segment a different expert is good.

An experiment using static experts



An experiment using fixed share



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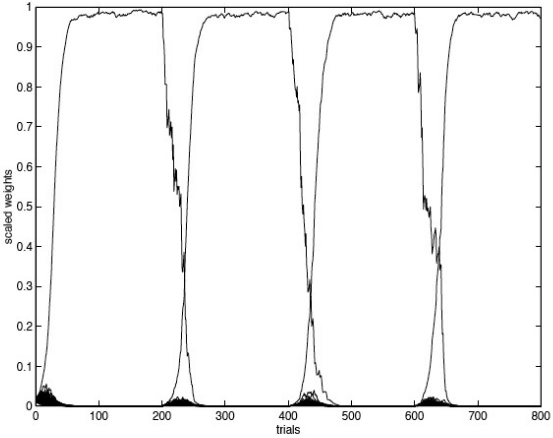


Analysis using Bregman divergences

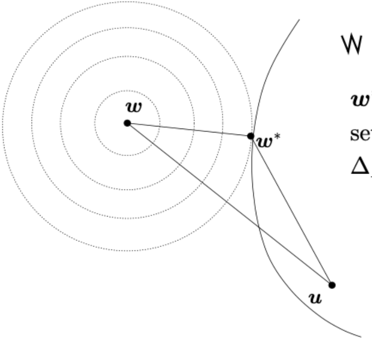
Mirror Descent (non-switching)

$$w_{t+1} = \arg\min_w \left[\eta \sum_{s=1}^{t} \langle w, z_s \rangle + D_R(w \| w_1) \right]$$

Regret Bound:

$$\sum_{t=1}^{T} \langle w_t - w^*, z_t \rangle \leq D_R(w^* \| w_1) + \sum_{t=1}^{T} D_R(w_t \| w_{t+1}).$$

A Pythagorean Theorem [Br,Cs,A,HW]



w^* is projection of w onto convex set $\mathcal{W}$ w.r.t. Bregman divergence Δ_F:

$$w^* = \operatorname*{argmin}_{u \in \mathcal{W}} \Delta_F(u, w)$$

Theorem:

$$\Delta_F(u, w) \geq \Delta_F(u, w^*) + \Delta_F(w^*, w)$$

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plus three-point identity:

$$D_R(a \,\|\, b) \;+\; D_R(b \,\|\, c) \;-\; D_R(a \,\|\, c) \;=\; \langle a - b,\; \nabla R(c) - \nabla R(b)\rangle.$$

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Incorporating Switching:

- Switching is controlled by $D_R\big(w_{t+1} \,\|\, w_t\big)$.

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$$\sum_{t=1}^{T} \langle w_t - w^*,\, z_t\rangle \;\leq\; D_R\big(w^* \,\|\, w_1\big) \;+\; \sum_{t=1}^{T} D_R\big(w_{t+1} \,\|\, w_t\big).$$

Incorporating Switching:

- Switching is controlled by $D_R\big(w_{t+1} \,\|\, w_t\big)$.
- The total regret depends on the regularizer $R(w)$.

Fixed Share Algorithm

Fixed Share Update:

$$w_{t+1}^i = (1-\alpha)\frac{w_t^i e^{-\eta z_t^i}}{\sum_j w_t^j e^{-\eta z_t^j}} + \frac{\alpha}{N}.$$

Impact on Pythagorean Inequality:

$$D_R(w^* \| w_t) \geq D_R(w^* \| w_{t+1}) + D_R(w_{t+1} \| w_t).$$

Modification: The divergence $D_R(w_{t+1} \| w_t)$ increases due to the uniform mixing factor α.

Regret Bound:

$$\sum_{t=1}^{T} \langle w_t - w^*, z_t \rangle \leq D_R(w^* \| w_1) + \sum_{t=1}^{T} \left[D_R(w_t \| w_{t+1}) + \alpha D_{KL}(w_t \| u) \right].$$

Variable Share Algorithm

Variable Share Update:

$$w_{t+1}^i = (1 - \alpha_t)\frac{w_t^i e^{-\eta z_t^i}}{\sum_j w_t^j e^{-\eta z_t^j}} + \alpha_t S_t^i.$$

Impact on Pythagorean Inequality:

$$D_R(w^* \| w_t) \geq D_R(w^* \| w_{t+1}) + D_R(w_{t+1} \| w_t) + \alpha_t D_{KL}(w_t \| u).$$

Modification: The divergence term now depends on α_t, making it adaptive rather than constant.

Regret Bound:

$$\sum_{t=1}^{T} \langle w_t - w^*, z_t \rangle \leq D_R(w^* \| w_1) + \sum_{t=1}^{T} \left[D_R(w_t \| w_{t+1}) + \alpha_t D_{KL}(w_t \| u) \right].$$