6.1 The Horse Race

Assumption: Let m horses run in a race. Let the ith horse win with probability p_i.

Payoff:

- If horse i wins, the payoff is o_i for 1.
- An investment of one dollar on horse i yields:

$$\begin{cases} o_i \text{ dollars}, & \text{if horse } i \text{ wins}, \\ 0 \text{ dollars}, & \text{if horse } i \text{ loses}. \end{cases}$$

- The gambler distributes **all** her money on the m possible bets: $b_1, \ldots, b_m$, $b_i \geq 0$, $\sum_i b_i = 1$

Some intuitive possibilities

- Put all of the money on the horse with the highest return.
- Put all of the money on the horse with the highest probability.
- Risky: most probable horse might still lose.
- Better to hedge.

Gambler's Wealth After n Races

Let S_n be the gambler's wealth after n races. Then

$$S_n = \prod_{i=1}^{n} S(X_i),$$

where $S(X) = b(X)\,o(X)$ is the factor by which the gambler's wealth is multiplied when horse X wins.

Constant rebalanced porfolios

- Use a fixed distribution of the stocks $\vec{b}$
- *Not* the same as buy and hold.
- **Example:** $\vec{b} = (1/2, 1/2)$
  Iter: 1,2,3,4,5,...
  Cash: 1,1,1,1,1,...
  Stock: 1,2,1,2,1,...
- **Wealth:** $1, \frac{3}{2}, \frac{3}{4} + \frac{3}{4}\frac{1}{2} = \frac{9}{8}, \frac{9}{8}\frac{3}{2}, \ldots$
- Wealth increases by a factor of $\frac{9}{8}$ every two iterations.

Kelly's rule

- Suppose that the returns are drawn from a fixed and known distribution.
- The optimal strategy, in terms of rate of increase of log wealth, is a constant rebalanced portfolio.
- But which portfolio?

What Are Universal Portfolios?

- **Concept:** Introduced by Thomas M. Cover, a universal portfolio is an investment strategy that asymptotically achieves the same growth rate of wealth as the best rebalanced portfolio in hindsight—*without knowing the future in advance*.
- **Key Idea:** Rather than fix a single strategy, the universal portfolio effectively averages over all possible rebalanced portfolios and updates weights based on observed performance.
- **Goal:** Leverage the *law of large numbers*–type property so that, over time, the universal portfolio tracks the growth rate of the best static rebalancing strategy.

Model Setup

- Consider a market with m assets (e.g. stocks).
- Trading occurs in **discrete time**: $t = 1, 2, \ldots, n$.
- Let $x_t \in \mathbb{R}^m$ be the gross returns of the m assets between time $t-1$ and t.
  - For example, if the j-th asset goes up by 2%, then $x_t^{(j)} = 1.02$.
- A portfolio vector $b_t \in \mathbb{R}^m$ specifies how one's capital is allocated among the m assets at time t.
  - The entries of b_t sum to 1 and are nonnegative: $\sum_{j=1}^{m} b_t^{(j)} = 1$, $b_t^{(j)} \geq 0$.

Wealth Evolution

- Let S_t denote the wealth at time t.
- Given a portfolio b_t at time t, the wealth is updated by

$$S_{t+1} = S_t \left(b_t \cdot x_{t+1} \right),$$

  where $b_t \cdot x_{t+1}$ is the dot product of b_t and x_{t+1}.
- The goal is to choose $\{b_t\}_{t=1}^{n}$ to maximize the final wealth S_{n+1} or equivalently $\log S_{n+1}$.

Cover's Universal Portfolio (Informal Definition)

1. **Consider all constant-rebalanced portfolios.** A constant-rebalanced portfolio (CRP) is one that keeps the same fraction in each asset at every time step.
2. **Assign a prior.** Treat each CRP as an element in the simplex of possible weights $b \in \Delta^m$. Typically, one uses the *Dirichlet* (or uniform) prior over the simplex.
3. **Update posterior.** After each period, update this distribution (the "mixture") over all CRPs based on how well each CRP performed.
4. **Form the next investment by averaging.** Allocate the portfolio b_{t+1} as a weighted average of all possible CRPs, weighted by their posterior performance.

$$b_{t+1} = \int b \, d\mu_t(b),$$

where μ_t is the posterior over the simplex after observing t periods.

Key Result

- **Asymptotic Optimality:** Cover showed that the growth rate of the *universal portfolio* will, in the limit, approach the growth rate of the best *single* constant-rebalanced portfolio in hindsight.
- Formally, let b^* be the CRP that maximizes $\log S_n$ in hindsight. Then the ratio of the universal portfolio's wealth U_n to the wealth of b^* (both starting at 1) grows sub-exponentially in n.

$$\frac{U_n}{S_n(b^*)} \geq \exp(-o(n)) \quad \text{as } n \to \infty.$$

- This means that the universal portfolio is *universally* good, without prior knowledge of which CRP is best.

Practical Construction

1. **Discretize the Simplex:**
   - In practice, the integral over all b in the simplex is approximated by a finite grid or sampling.
2. **Recompute Weights:**

$$w_{t+1}(b) = \frac{w_t(b) \cdot \big(b \cdot x_{t+1}\big)}{\int w_t(u) \cdot \big(u \cdot x_{t+1}\big)\, du}$$

   where $w_t(b)$ is the "weight" or "posterior" for CRP b.
3. **Compute New Investment:**

$$b_{t+1} = \int b\, w_{t+1}(b)\, db \quad \approx \sum_{b \in \mathcal{B}} b\, w_{t+1}(b).$$

4. **Rebalance Accordingly:** Actually execute b_{t+1} in the market at time $t+1$.

Simple Example (2 Assets)

- Suppose there are 2 assets, so $b \in [0,1]$ with $b_1 = b$, $b_2 = 1 - b$.
- **Uniform Prior:** Start with $w_0(b) = 1$ for $b \in [0,1]$.
- **Observations:** If the asset returns over first period are (x_1, x_2), then after seeing that outcome, the weight function updates:

$$w_1(b) \propto b\,x_1 + (1-b)\,x_2.$$

- **Next Step:** The universal strategy at $t = 1$ invests

$$b_1 = \int_0^1 b\,\frac{b\,x_1 + (1-b)\,x_2}{Z}\,db,$$

where Z is a normalization constant to ensure the posterior integrates to 1.

Numerical Illustration

- By discretizing $b \in [0, 1]$ into many small steps, you can numerically approximate the integrals.
- Over time, the algorithm will put more weight on the “best” fraction b^* that maximizes growth, but it still accounts for uncertainty and adapts as the environment changes.

Summary

- **Cover's Universal Portfolio** is a powerful idea that uses mixture methods over all possible constant-rebalanced portfolios.
- **Guarantees:** It asymptotically matches the best constant-rebalanced strategy in hindsight.
- **Implementation:** Although conceptually elegant, the naive integral approach can be computationally expensive. Practical approximations are used (discretization, sampling, etc.).
- **Significance:** This method bridges information theory and portfolio choice, illustrating how "universal" strategies can learn from the market without predictions.

References

T. M. Cover, *Universal Portfolios*, Math. Finance, Vol. 1, No. 1 (1991), pp. 1–29.

T. M. Cover and J. A. Thomas, *Elements of Information Theory*, Wiley, 2nd edition, 2006.

T. M. Cover, *Universal Portfolios*, in The Kelly Capital Growth Investment Criterion: Theory and Practice (2011), World Scientific.

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