Combining infinite sets of experts

Yoav Freund

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Freund: Predicting a binary Sequence almost as well the the optimal biased coin.
Risannen: Fisher Information and Stochastic Complexity.

Outline

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Review

Probabilities and codes

- $M_1, \ldots, M_n$ - possible messages

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- Prediction of algorithm A

$$\mathbf{p}_A^t = \frac{\sum_{i=1}^{N} w_i^t \mathbf{p}_i^t}{\sum_{i=1}^{N} w_i^t}$$

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- $V(\vec{b}, \vec{X}, t)$ is computable (recursively enumerable).

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- technical details: On iteration t, $|\vec{X}| = t$. Use the predictions of programs $\vec{b}$ such that $|\vec{b}| \leq t$ and for which $V(\vec{b}, \vec{X}, 2^t) = 1$.
  the unused algorithms predict $1/2$ (insuring a loss of 1)

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- What is the two part code?

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- More generally, the regret is smaller if many of the experts perform well.

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- Can we still get a meaningful bound?

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- We need a new lower bound on the final total weight

Main Idea

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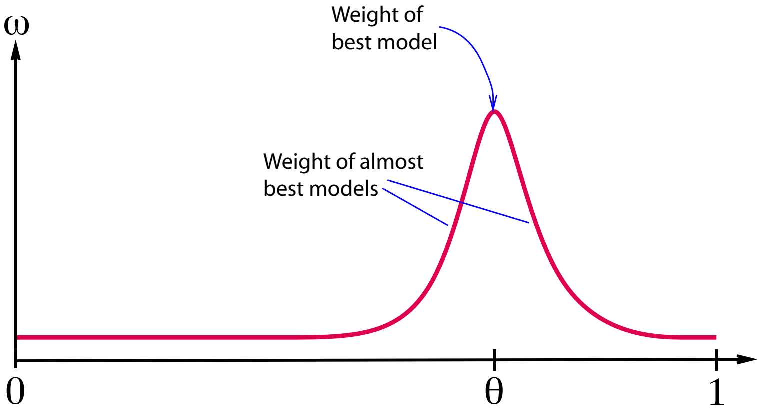

Expanding the exponent around the peak

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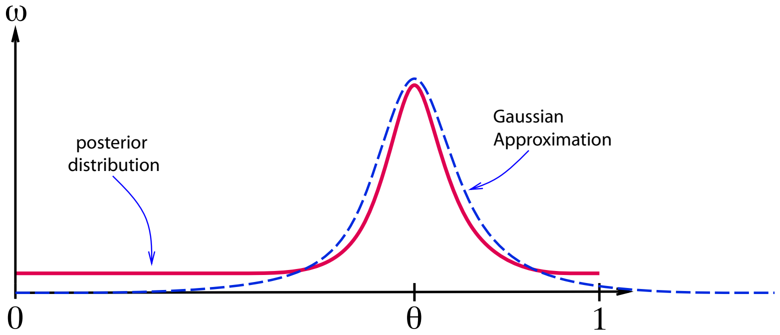

Laplace Approximation, Watson's lemma

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$$\int_0^1 w(\theta) e^{T(g(\hat{\theta},\theta)-g(\hat{\theta},\hat{\theta}))} d\theta$$

$$= \quad w(\hat{\theta}) \sqrt{\frac{-2\pi}{T \left. \frac{d^2}{d\theta^2} \right|_{\theta=\hat{\theta}} (g(\hat{\theta},\theta) - g(\hat{\theta},\hat{\theta}))}} + O(T^{-3/2})$$

Choosing the optimal prior

- Choose $w(\theta)$ to maximize the worst-case final total weight

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- Make bound equal for all $\hat\theta \in [0,1]$ by choosing

$$w^*(\hat\theta) = \frac{1}{Z} \sqrt{\frac{\left.\frac{d^2}{d\theta^2}\right|_{\theta=\hat\theta} (g(\hat\theta,\theta) - g(\hat\theta,\hat\theta))}{-2\pi}} \; ,$$

where Z is the normalization factor:

$$Z = \sqrt{\frac{1}{2\pi}} \int_0^1 \sqrt{\left.\frac{d^2}{d\theta^2}\right|_{\theta=\hat\theta} (g(\hat\theta,\hat\theta) - g(\hat\theta,\theta))} \; d\hat\theta$$

The bound for the optimal prior

- Plugging in we get

$$
\begin{aligned}
L_A - L_{\min} &\leq \ln \int_0^1 w^*(\theta) e^{T(g(\hat{\theta},\theta) - g(\hat{\theta},\hat{\theta}))} d\theta \\
&= \ln \left(\sqrt{\frac{2\pi Z}{T}} + O(T^{-3/2}) \right) \\
&= \frac{1}{2} \ln \frac{T}{2\pi} - \frac{1}{2} \ln Z + O(1/T) \,.
\end{aligned}
$$

Solving for log-loss

- The exponent in the integral is

$$g(\hat{\theta}, \theta) - g(\hat{\theta}, \hat{\theta}) = \hat{\theta} \ln \frac{\hat{\theta}}{\theta} + (1 - \hat{\theta}) \ln \frac{1 - \hat{\theta}}{1 - \theta} = D_{KL}(\hat{\theta} || \theta)$$

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- The optimal prior:

$$w^*(\hat{\theta}) = \frac{1}{\pi\sqrt{\hat{\theta}(1-\hat{\theta})}}$$

  Known in as Jeffrey's prior. And, in this case, the Dirichlet-$(1/2, 1/2)$ prior.

The cumulative log loss of Bayes using Jeffrey's prior

- $$L_A - L_{\min} \leq \frac{1}{2}\ln(T+1) + \frac{1}{2}\ln\frac{\pi}{2} + O(1/T)$$

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- This is called the Trichevsky Trofimov prediction rule.

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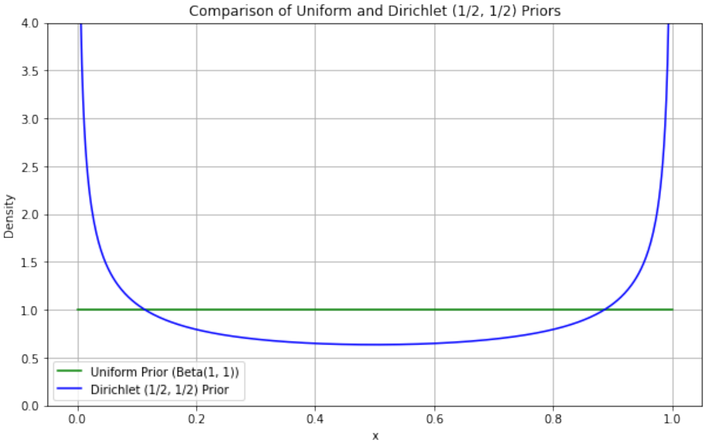
- Suffers larger regret when $\hat{\theta}$ is far from $1/2$

Comparing the priors



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Generalization to larger sets of distributions

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- Many more: 1D: Poisson, Exponential, Gamma ...
  Multi-Variate: Gaussian, Dirichlet, Multivariate t-distribution

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next Class

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