Tracking the best Expert

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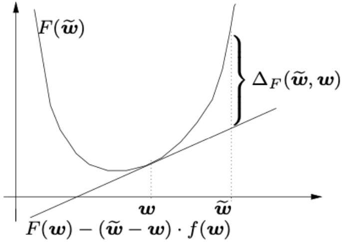
Based on “Tracking the best linear predictor” and “Tracking the best expert” by Herbster and Warmuth. Also, section 11.5 in Prediction learning and Games.

Bregman Divergences [Br,CL,Cs]

For any differentiable convex function F

$$\Delta_F(\tilde{\boldsymbol{w}}, \boldsymbol{w}) = F(\tilde{\boldsymbol{w}}) - F(\boldsymbol{w}) - (\tilde{\boldsymbol{w}} - \boldsymbol{w}) \cdot \underbrace{\nabla_{\boldsymbol{w}} F(\boldsymbol{w})}_{f(\boldsymbol{w})}$$

$$= F(\tilde{\boldsymbol{w}}) - \begin{array}{c}\text{supporting hyperplane}\\ \text{through } (\boldsymbol{w}, F(\boldsymbol{w}))\end{array}$$



Bregman Divergences: Simple Properties

1. $\Delta_F(\widetilde{\boldsymbol{w}}, \boldsymbol{w})$ is convex in $\widetilde{\boldsymbol{w}}$
2. $\Delta_F(\widetilde{\boldsymbol{w}}, \boldsymbol{w}) \geq 0$
   If F convex equality holds iff $\widetilde{\boldsymbol{w}} = \boldsymbol{w}$
3. Usually not symmetric: $\Delta_F(\widetilde{\boldsymbol{w}}, \boldsymbol{w}) \neq \Delta_F(\boldsymbol{w}, \widetilde{\boldsymbol{w}})$
4. Linearity (for $a \geq 0$):
   $\Delta_{F+a\,H}(\widetilde{\boldsymbol{w}}, \boldsymbol{w}) = \Delta_F(\widetilde{\boldsymbol{w}}, \boldsymbol{w}) + a\,\Delta_H(\widetilde{\boldsymbol{w}}, \boldsymbol{w})$
5. Unaffected by linear terms ($a \in \mathbf{R}$, $\boldsymbol{b} \in \mathbf{R}^n$):
   $\Delta_{H+a\widetilde{\boldsymbol{w}}+\boldsymbol{b}}(\widetilde{\boldsymbol{w}}, \boldsymbol{w}) = \Delta_H(\widetilde{\boldsymbol{w}}, \boldsymbol{w})$

Bregman Divergences: more properties

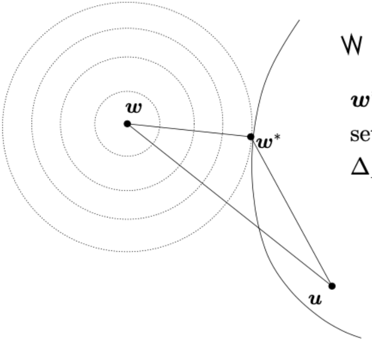
6. $\nabla_{\widetilde{\boldsymbol{w}}} \Delta_F(\widetilde{\boldsymbol{w}}, \boldsymbol{w})$

$$\begin{aligned} &= \nabla F(\widetilde{\boldsymbol{w}}) - \nabla_{\widetilde{\boldsymbol{w}}}(\widetilde{\boldsymbol{w}} \nabla_{\boldsymbol{w}} F(\boldsymbol{w})) \\ &= f(\widetilde{\boldsymbol{w}}) - f(\boldsymbol{w}) \end{aligned}$$

7. $\Delta_F(\boldsymbol{w}_1, \boldsymbol{w}_2) + \Delta_F(\boldsymbol{w}_2, \boldsymbol{w}_3)$

$$\begin{aligned} = \;& F(\boldsymbol{w}_1) - F(\boldsymbol{w}_2) - (\boldsymbol{w}_1 - \boldsymbol{w}_2) f(\boldsymbol{w}_2) \\ & F(\boldsymbol{w}_2) - F(\boldsymbol{w}_3) - (\boldsymbol{w}_2 - \boldsymbol{w}_3) f(\boldsymbol{w}_3) \\ = \;& \Delta_F(\boldsymbol{w}_1, \boldsymbol{w}_3) + (\boldsymbol{w}_1 - \boldsymbol{w}_2) \cdot (f(\boldsymbol{w}_3) - f(\boldsymbol{w}_2)) \end{aligned}$$

A Pythagorean Theorem [Br,Cs,A,HW]



$\boldsymbol{w}^*$ is projection of $\boldsymbol{w}$ onto convex set $\mathcal{W}$ w.r.t. Bregman divergence Δ_F:

$$\boldsymbol{w}^* = \operatorname*{argmin}_{\boldsymbol{u} \in \mathcal{W}} \Delta_F(\boldsymbol{u}, \boldsymbol{w})$$

Theorem:

$$\Delta_F(\boldsymbol{u}, w) \geq \Delta_F(\boldsymbol{u}, \boldsymbol{w}^*) + \Delta_F(\boldsymbol{w}^*, \boldsymbol{w})$$

Unnormalized Relative entropy

- prediction, outcome $\mathbf{p}, \mathbf{q}$ are n dimensional vectors with non-negative coordinates.
- Loss is RE extended to non-negative vectors:

$$\mathrm{RE}\left(\mathbf{p} \ \| \ \mathbf{q}\right) = \sum_{i=1}^{n} p_i \log \frac{p_i}{q_i} - \sum_{i=1}^{n} (q_i - p_i)$$

Coincides with RE when $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = 1$

- Unnormalized RE is the Bregman divergence corresponding to the unnormalized entropy:

$$F(\mathbf{p}) = \sum_{i=1}^{n} p_i \log p_i - \sum_{i=1}^{n} p_i$$

Inequalities for Unnormalized Relative entropy

- No triangle inequality
  $\exists \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 \;\; \mathrm{RE}(\mathbf{p}_1 \parallel \mathbf{p}_3) > \mathrm{RE}(\mathbf{p}_1 \parallel \mathbf{p}_2) + \mathrm{RE}(\mathbf{p}_2 \parallel \mathbf{p}_3)$
- Generalized Pythagorean inequality For any closed convex set S and any point $\mathbf{p}_1 \notin S$, define the projection of $\mathbf{p}_1$ on S to be $\mathbf{p}_2 = \mathrm{argmin}_{\mathbf{u} \in S} \mathrm{RE}(\mathbf{p}_1 \parallel \mathbf{u})$, then:

$$\forall \mathbf{p}_3 \in S; \;\; \mathrm{RE}(\mathbf{p}_1 \parallel \mathbf{p}_3) \geq \mathrm{RE}(\mathbf{p}_1 \parallel \mathbf{p}_2) + \mathrm{RE}(\mathbf{p}_2 \parallel \mathbf{p}_3)$$

half squared euclidean distance

- prediction, outcome $\mathbf{u}, \mathbf{v} \in R^n$
- $$\lambda_{\mathrm{sq}}(\mathbf{u}, \mathbf{v}) = \frac{1}{2}\|\mathbf{u} - \mathbf{v}\|^2 = \frac{1}{2}\sum_{i=1}^{n}(u_i - v_i)^2$$
- Bregman divergence with respect to the square euclidean norm
  $$\|\mathbf{v}\|_2$$
- Triangle inequality does not hold.
- Pythagoras inequality : For any closed convex set S and any point $\mathbf{v}_1 \notin S$, define the projection of $\mathbf{v}_1$ on S to be $\mathbf{v}_2 = \mathrm{argmin}_{\mathbf{u} \in S}\|\mathbf{v}_1 - \mathbf{u}\|^2$, then:
  $$\forall \mathbf{v}_3 \in S; \quad \|\mathbf{v}_1 - \mathbf{v}_3\|^2 \geq \|\mathbf{v}_1 - \mathbf{v}_2\|^2 + \|\mathbf{v}_2 - \mathbf{v}_3\|^2$$

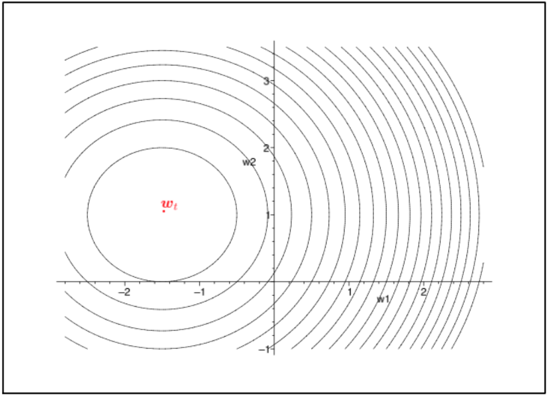
Divergence: Euclidean Distance Squared

$$\Delta_F(\boldsymbol{w}, \boldsymbol{w}_t) = \|\boldsymbol{w} - \boldsymbol{w}_t\|_2^2/2$$

$\boldsymbol{w}_t = (-3/2, 1)$

$\boldsymbol{x}_t = (1, -0.5)$

$y_t = 1$



Bregman divergence regularization

- Idea: Set $\mathbf{w}_{t+1}$ to be $\mathbf{u}$ that minimizes:

$$\Delta_F(\mathbf{w}_t, \mathbf{u}) + \alpha \ell_t(\mathbf{u})$$

- In general, hard to compute the minimum.
- Efficient approximation Mirror Descent. Will be covered later.

General Motivation of Updates [KW]

Trade-off between two term:

$$\boldsymbol{w}_{t+1} = \underset{\boldsymbol{w}}{\operatorname{argmin}}\Big(\underbrace{\Delta_F(\boldsymbol{w}, \boldsymbol{w}_t)}_{weight\ domain} + \eta_t \underbrace{L_t(\boldsymbol{w})}_{label\ domain} \Big)$$

$\Delta_F(\boldsymbol{w}, \boldsymbol{w}_t)$ is "regularization term" and serves as measure of progress in the analysis.

When loss L is convex (in $\boldsymbol{w}$)

$$\nabla_{\boldsymbol{w}}(\Delta_F(\boldsymbol{w}, \boldsymbol{w}_t) + \eta_t L_t(\boldsymbol{w})) = 0$$

iff

$$f(\boldsymbol{w}) - f(\boldsymbol{w}_t) + \eta_t \underbrace{\nabla L_t(\boldsymbol{w})}_{\approx \nabla L_t(\boldsymbol{w}_t)} = 0$$

$$\Rightarrow \quad \boldsymbol{w}_{t+1} = f^{-1}\left(f(\boldsymbol{w}_t) - \eta_t \nabla L_t(\boldsymbol{w}_t)\right)$$

How to prove relative loss bounds?

Loss: $L_t(\boldsymbol{w}) = L((\boldsymbol{x}_t, y_t), \boldsymbol{w})$ convex in $\boldsymbol{w}$

Divergence: $\Delta_F(\boldsymbol{u}, \boldsymbol{w}) = F(\boldsymbol{u}) - F(\boldsymbol{w}) - (\boldsymbol{u} - \boldsymbol{w}) \cdot f(\boldsymbol{w})$

Update: $f(\boldsymbol{w}_{t+1}) - f(\boldsymbol{w}_t) = -\eta \nabla_{\boldsymbol{w}} L_t(\boldsymbol{w}_t)$

$$
\begin{aligned}
L_t(\boldsymbol{u}) \;&\overbrace{\geq}^{\text{convexity}}\; L_t(\boldsymbol{w}_t) + (\boldsymbol{u} - \boldsymbol{w}_t) \cdot \underbrace{\nabla_{\boldsymbol{w}} L_t(\boldsymbol{w}_t)}_{\text{update}} \\
&= L_t(\boldsymbol{w}_t) - \frac{1}{\eta} \underbrace{(\boldsymbol{u} - \boldsymbol{w}_t) \cdot (f(\boldsymbol{w}_{t+1}) - f(\boldsymbol{w}_t))}_{\text{prop. 7 of } \Delta_F} \\
&= L_t(\boldsymbol{w}_t) + \frac{1}{\eta} (\Delta_F(\boldsymbol{u}, \boldsymbol{w}_{t+1}) - \Delta_F(\boldsymbol{u}, \boldsymbol{w}_t) - \Delta_F(\boldsymbol{w}_t, \boldsymbol{w}_{t+1}))
\end{aligned}
$$

Tracking Linear Experts

Switching experts setup

- Usually: compare algorithm's total loss to total loss of the best expert.
- Switching experts: compare algorithm's total loss to total loss of best expert sequence with k switches.
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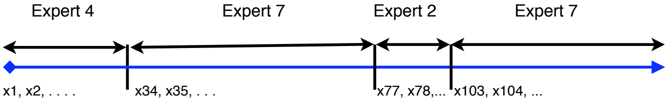


An inefficient algorithm

- Fix:
  - l - sequence length
  - k - number of switches
  - n - number of experts
- Consider one partition-expert per sequence of switching experts.
- No. of partition-experts : $\binom{l}{k-1} n(n-1)^k = O\left(n^{k+1}\left(\frac{el}{k}\right)^k\right)$
- The log-loss regret is at most $(k+1)\log n + k \log \frac{l}{k} + k$
- Requires maintaining $O\left(n^{k+1}\left(\frac{el}{k}\right)^k\right)$ weights.

generalization to mixable losses

- In this lecture we assume loss function is mixable.
- There is an exponential weights algorithm with learning rate η that achieves (in the non-switching case) a bound

$$L_A \leq \min_i L_i + \frac{1}{\eta} \log n$$

- Then using the partition-expert algorithm for the switching-experts case we get a bound on the regret $\frac{1}{\eta}\left((k+1)\log n + k \log \frac{l}{k} + k\right)$

Weight sharing algorithms

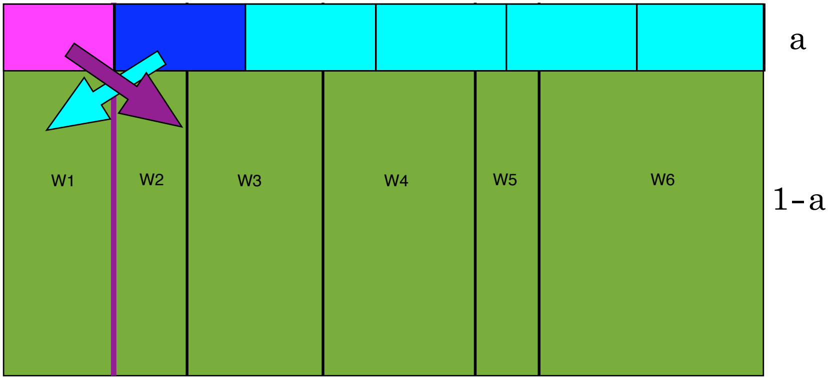
- Update weights in two stages: loss update then share update.
- Prediction uses the normalized s weights $w^s_{t,i} / \sum_j w^s_{t,j}$
- Loss update is the same as always, but defines intermediate m weights:

$$w^m_{t,i} = w^s_{t,i} e^{-\eta L(y_t, x_{t,i})}$$

- Share update: redistribute the weights
- Fixed-share:

$$\begin{aligned} pool &= \alpha \sum_{i=1}^{n} w^m_{t,i} \\ w^s_{t+1,i} &= (1-\alpha) w^m_{t,i} + \frac{1}{n-1} \left(pool - \alpha w^m_{t,i} \right) \end{aligned}$$

The fixed-share algorithm



Proving a bound on the fixed-share

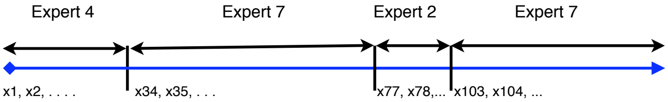
- The relation between algorithm loss and total weight does not change because share update does not change the total weight.
- Thus we still have

$$L_A \leq \frac{1}{\eta} \sum_{i=1}^{n} w^s_{l+1,i}$$

- The harder question is how to lower bound $\sum_{i=1}^{n} w^s_{l+1,i}$

Lower bounding the final total weight

- Fix some switching experts sequence:



- "follow" the weight of the chosen expert i_t.
- The loss update reduces the weight by a factor of $e^{-\eta \ell_{t,i_t}}$.
- The share update reduces the weight by a factor larger than:
  - $1 - \alpha$ on iterations with no switch.
  - $\frac{\alpha}{n-1}$ on iterations where a switch occurs.

Bound for arbitrary α

- Combining we lower bound the final weight of the last expert in the sequence

$$w^s_{l+1,e_k} \geq \frac{1}{n} e^{-\eta L_*} (1-\alpha)^{l-k-1} \left(\frac{\alpha}{n-1} \right)^k$$

Where L_* is the cumulative loss of the switching sequence of experts.

- Combining the upper and lower bounds we get that for any sequence

$$L_A \leq L_* + \frac{1}{\eta} \left(\ln n + (l-k-1) \ln \frac{1}{1-\alpha} + k \left(\ln \frac{1}{\alpha} + \ln(n-1) \right) \right)$$

Tuning α

- let k^* be the best number of switches (in hind sight) and $\alpha^* = k^*/l$
- Suppose we use $\alpha \approx \alpha^*$ then the bound that we get is

$$L_A \leq L_* + \frac{1}{\eta}((k+1)\ln n + (l-1)(H(\alpha^*) + D_{\mathrm{KL}}(\alpha^*||\alpha)))$$

Where

$$H(\alpha^*) = -\alpha^* \ln \alpha^* - (1-\alpha^*)\ln(1-\alpha^*)$$

$$D_{\mathrm{KL}}(\alpha^*||\alpha) = \alpha^* \ln \frac{\alpha^*}{\alpha}(1-\alpha^*)\ln \frac{1-\alpha^*}{1-\alpha}$$

- This is very close to the loss of the computationally inefficient algorithm.
- For the log loss case this is essentially optimal.
- Not so for square loss!

What can we hope to improve?

- In the fixed-share algorithm, the weight of a suboptimal expert never decreases below α/n.
- The algorithm does not concentrate only on the best expert, even if the last switch is in the distant past.
- The regret depends on the length of the sequence.

The idea of variable-share

- Let the fraction of the total weight given to the best expert get arbitrarily close to 1.
- we can get a regret bound that depends only on the number of switches, not on the lenght of the sequence.
- Requires that the loss be bounded.
- Works for square loss, but not for log loss!

Variable-share

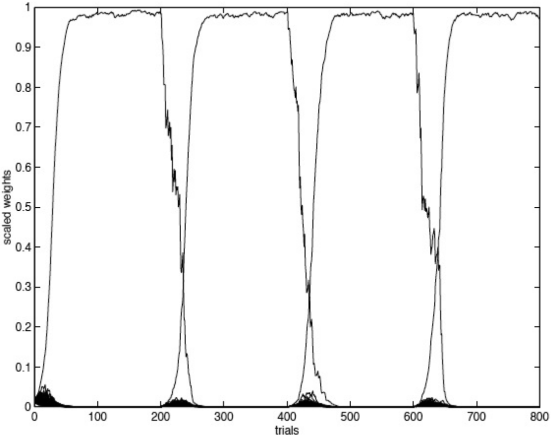
$$
\begin{aligned}
pool &= \sum_{i=1}^{n} \left(1 - (1-\alpha)^{\ell_{t,i}}\right) w^m_{t,i} \\
w^s_{t+1,i} &= (1-\alpha)^{\ell_{t,i}} w^m_{t,i} + \frac{1}{n-1}\left(pool - \left(1 - (1-\alpha)^{\ell_{t,i}}\right) w^m_{t,i}\right)
\end{aligned}
$$

If $\ell_{t,i} = 0$, then expert i does not contribute to the pool.
Expert can get fraction of the total weight arbitrarily close to 1.
Shares the weight quickly if $\ell_{t,i} > 0$

Bound for variable share

- $$\frac{1}{\eta}\ln n + \left(1 + \frac{1}{(1-\alpha)\eta}\right) L_* + k\left(1 + \frac{1}{\eta}\left(\ln n - 1 + \ln\frac{1}{\alpha} + \ln\frac{1}{1-\alpha}\right)\right)$$
- α should be tuned so that it is (close to) $\frac{k}{2k+L_*}$

An experiment using variable share



Switching within a small subset

- Suppose the best switching sequence is repeatedly switching among a small subset of the experts $n' \ll n$
- In the context of speech recognition - the speaker repeatedly uses a small number of phonemes.
- If we know the subset, we can pay $\ln n'$ per switch rather than $\ln n$
- Can track switches much more closely.
- Easy to describe an inefficient algorithm (consider all $\binom{n}{n'}$ subsets.)
- Switching to Slides from Manfred Warmuth.