# Exponential Weights Algorithms for Online Learning

Yoav Freund

- slides in https://github.com/yoavfreund/2025-online-learning/blob/main/2.Hedge/talk2.handout.pdf

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- In PLG: pages 12-25

# Outline

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## Decision Theoretic Online learning

- Hedging vs. Halving
- Failure of Follow the leader

## **Hedge**$(\eta)$Algorithm

## Bound on total loss

- Upper bound on $\sum_{i=1}^{N} w_i^{T+1}$
- Lower bound on $\sum_{i=1}^{N} w_i^{T+1}$
- Combining Upper and Lower bounds

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## tuning $\eta$

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Lower Bounds

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- Fits nicely in game theory

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- Unlike halving - no action is perfect.
- Basic idea - reduce probability of lossy actions, but not all the way to zero.
- **Modified Goal:** minimize difference between expected total loss and minimal total loss of repeating one action.

$$\sum_{t=1}^{T} \mathbf{p}^t \cdot \ell_t - \min_i \left( \sum_{t=1}^{T} \ell_i^t \right)$$

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  - Algorithm predicts 1 with probability $\sum_{i:e_i^t=1} p_i^t$.
  - outcome $o_i^t$ is revealed. $\ell_i^t = 0$ if $e_i^t = o_i^t$, $\ell_i^t = 1$ otherwise.

# Failure of follow the leader

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expert1 loss
expert1 cumul

expert2 loss
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FTL cumul

# Failure of follow the leader

| | $t = 1$ |
|---|---|
| expert1 loss | 0.5 |
| expert1 cumul | 0.5 |
| | |
| expert2 loss | 0.0 |
| expert2 cumul | 0.0 |
| | |
| FTL cumul | 0.0 |

# Failure of follow the leader

| | $t = 1$ | $t = 2$ |
|---|---|---|
| expert1 loss | 0.5 | 0.0 |
| expert1 cumul | 0.5 | 0.5 |
| expert2 loss | 0.0 | 1.0 |
| expert2 cumul | 0.0 | 1.0 |
| FTL cumul | 0.0 | 1.0 |

# Failure of follow the leader

| | $t = 1$ | $t = 2$ | $t = 3$ |
|---|---|---|---|
| expert1 loss | 0.5 | 0.0 | 1.0 |
| expert1 cumul | 0.5 | 0.5 | 1.5 |
| | | | |
| expert2 loss | 0.0 | 1.0 | 0.0 |
| expert2 cumul | 0.0 | 1.0 | 1.0 |
| | | | |
| FTL cumul | 0.0 | 1.0 | 2.0 |

# Failure of follow the leader

| | $t = 1$ | $t = 2$ | $t = 3$ | $t = 4$ |
|---|---|---|---|---|
| expert1 loss | 0.5 | 0.0 | 1.0 | 0.0 |
| expert1 cumul | 0.5 | 0.5 | 1.5 | 1.5 |
| | | | | |
| expert2 loss | 0.0 | 1.0 | 0.0 | 1.0 |
| expert2 cumul | 0.0 | 1.0 | 1.0 | 2.0 |
| | | | | |
| FTL cumul | 0.0 | 1.0 | 2.0 | 3.0 |

# Failure of follow the leader

| | $t = 1$ | $t = 2$ | $t = 3$ | $t = 4$ | $t = 5$ |
|---|---|---|---|---|---|
| expert1 loss | 0.5 | 0.0 | 1.0 | 0.0 | 1.0 |
| expert1 cumul | 0.5 | 0.5 | 1.5 | 1.5 | 2.5 |
| | | | | | |
| expert2 loss | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 |
| expert2 cumul | 0.0 | 1.0 | 1.0 | 2.0 | 2.0 |
| | | | | | |
| FTL cumul | 0.0 | 1.0 | 2.0 | 3.0 | 4.0 |

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| | $t = 1$ | $t = 2$ | $t = 3$ | $t = 4$ | $t = 5$ | $t = 6$ |
|---|---|---|---|---|---|---|
| expert1 loss | 0.5 | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 |
| expert1 cumul | 0.5 | 0.5 | 1.5 | 1.5 | 2.5 | 2.5 |
| | | | | | | |
| expert2 loss | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 | 1.0 |
| expert2 cumul | 0.0 | 1.0 | 1.0 | 2.0 | 2.0 | 3.0 |
| | | | | | | |
| FTL cumul | 0.0 | 1.0 | 2.0 | 3.0 | 4.0 | 5.0 |

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| | $t=1$ | $t=2$ | $t=3$ | $t=4$ | $t=5$ | $t=6$ | $t=7$ |
|---|---|---|---|---|---|---|---|
| expert1 loss | 0.5 | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 | 1.0 |
| expert1 cumul | 0.5 | 0.5 | 1.5 | 1.5 | 2.5 | 2.5 | 3.5 |
| expert2 loss | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 |
| expert2 cumul | 0.0 | 1.0 | 1.0 | 2.0 | 2.0 | 3.0 | 3.0 |
| FTL cumul | 0.0 | 1.0 | 2.0 | 3.0 | 4.0 | 5.0 | 6.0 |

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Consider action $i$ at time $t$

- Total loss:

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- As $\sum_{i=1}^{n} w_i^1 = 1$ increasing one weight implies decreasing some others.
- Plays a similar role to prior distribution in Bayesian algorithms.

# Bound on the loss of **Hedge**($\eta$)Algorithm

## Theorem (main theorem)

*For any sequence of loss vectors* $\ell_1, \ldots, \ell_T$*, and for any* $i \in \{1, \ldots, N\}$*, we have*

$$L_{\mathbf{Hedge}(\eta)} \leq \frac{-\ln(w_i^1) + \eta L_i}{1 - e^{-\eta}}.$$

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- Note effect of the limits $\eta \to 0$ and $\eta \to \infty$
- Proof: by combining upper and lower bounds on $\sum_{i=1}^{N} w_i^{T+1}$

# Upper bound on $\sum_{i=1}^{N} w_i^{T+1}$

## Lemma (upper bound)

*For any sequence of loss vectors* $\ell_1, \ldots, \ell_T$ *we have*

$$\ln\left(\sum_{i=1}^{N} w_i^{T+1}\right) \leq -(1 - e^{-\eta}) L_{\mathbf{Hedge}(\eta)}.$$

# Proof of upper bound (slide 1)

- If $a \geq 0$ then $a^r$ is convex.

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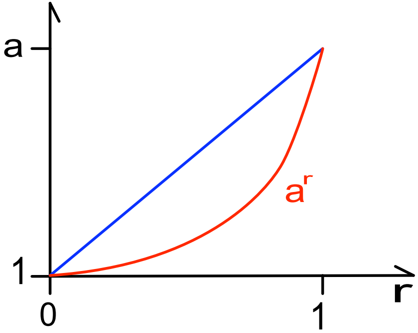


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# Proof of upper bound (slide 2)

Applying $a^r \leq 1 - (1-a)^r$ where $a = e^{-\eta}, r = \ell_i^t$

$$\sum_{i=1}^{N} w_i^{t+1} = \sum_{i=1}^{N} w_i^t e^{-\eta \ell_i^t}$$

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$$\begin{aligned} \sum_{i=1}^{N} w_i^{t+1} &= \sum_{i=1}^{N} w_i^t e^{-\eta \ell_i^t} \\ &\leq \sum_{i=1}^{N} w_i^t \left(1 - (1-e^{-\eta})\ell_i^t\right) \end{aligned}$$

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&\leq \sum_{i=1}^{N} w_i^t \left(1 - (1 - e^{-\eta})\ell_i^t\right) \\
&= \left(\sum_{i=1}^{N} w_i^t\right)\left(1 - (1 - e^{-\eta})\frac{\mathbf{w}^t}{\sum_{i=1}^{N} w_i^t} \cdot \ell_t\right)
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&= \left(\sum_{i=1}^{N} w_i^t\right) \left(1 - (1 - e^{-\eta}) \frac{\mathbf{w}^t}{\sum_{i=1}^{N} w_i^t} \cdot \ell_t\right) \\
&= \left(\sum_{i=1}^{N} w_i^t\right) \left(1 - (1 - e^{-\eta}) \mathbf{p}^t \cdot \ell_t\right)
\end{aligned}
$$

# Proof of upper bound (slide 3)

- Combining

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- for $t = 1, \ldots, T$

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$$\sum_{i=1}^{N} w_i^{T+1} \quad \leq \quad \prod_{t=1}^{T} (1 - (1 - e^{-\eta})\mathbf{p}^t \cdot \ell_t)$$

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- for $t = 1, \ldots, T$
- yields

$$\begin{aligned}\sum_{i=1}^{N} w_i^{T+1} &\leq \prod_{t=1}^{T}(1 - (1 - e^{-\eta})\mathbf{p}^t \cdot \ell_t) \\ &\leq \exp\left(-(1 - e^{-\eta})\sum_{t=1}^{T}\mathbf{p}^t \cdot \ell_t\right)\end{aligned}$$

since $1 + x \leq e^x$ for $x = -(1 - e^{-\eta})$.

# Lower bound on $\sum_{i=1}^{N} w_i^{T+1}$

For any $j = 1, \ldots, N$:

$$\sum_{i=1}^{N} w_i^{T+1} \geq w_j^{T+1} = w_j^1 e^{-\eta L_j}$$

# Combining Upper and Lower bounds

- Combining bounds on $\ln\left(\sum_{i=1}^{N} w_i^{T+1}\right)$

$$\ln w_j^1 - \eta L_j \leq \ln \sum_{i=1}^{N} w_i^{T+1} \leq -(1 - e^{-\eta}) \sum_{t=1}^{T} \mathbf{p}^t \cdot \ell_t$$

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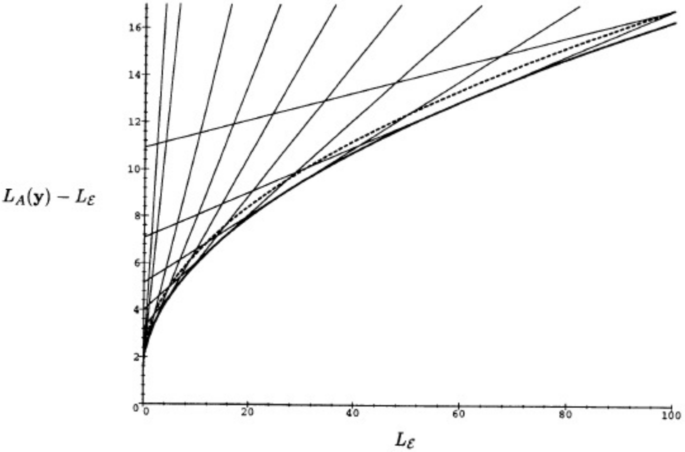
- Reversing signs, using $L_{\mathbf{Hedge}(\eta)} = \sum_{t=1}^{T} \mathbf{p}^t \cdot \ell_t$ and reorganizing we get

$$L_{\mathbf{Hedge}(\eta)} \leq \frac{-\ln(w_i^1) + \eta L_i}{1 - e^{-\eta}}$$

# Tuning $\eta$

*How to Use Expert Advice* 451



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- use uniform initial weights $\mathbf{w}^1 = \langle 1/N, \ldots, 1/N \rangle$
- Then

$$L_{\mathbf{Hedge}(\eta)} \leq \frac{-\ln(w_i^1) + \eta L_i}{1 - e^{-\eta}} \leq \min_i L_i + \sqrt{2\tilde{L}\ln N} + \ln N$$

# Exact tuning of $\eta$

\tilde{R}=R=ln N

### 2.2 How to choose $\beta$

So far, we have analyzed **Hedge**($\beta$) for a given choice of $\beta$, and we have proved reasonable bounds for any choice of $\beta$. In practice, we will often want to choose $\beta$ so as to maximally exploit any prior knowledge we may have about the specific problem at hand.

The following lemma will be helpful for choosing $\beta$ using the bounds derived above.

**Lemma 4** *Suppose* $0 \le L \le \tilde{L}$ *and* $0 < R \le \tilde{R}$. *Let* $\beta = g(\tilde{L}/\tilde{R})$ *where* $g(z) = 1/(1+\sqrt{2/z})$. *Then*

$$\frac{-L\ln\beta + R}{1-\beta} \le L + \sqrt{2\tilde{L}\tilde{R}} + R.$$

**Proof:** (Sketch) It can be shown that $-\ln\beta \le (1-\beta^2)/(2\beta)$ for $\beta \in (0,1]$. Applying this approximation and the given choice of $\beta$ yields the result. ■

Lemma 4 can be applied to any of the bounds above since all of these bounds have the form given in the lemma. For example, suppose we have $N$ strategies, and we also know a prior bound $\tilde{L}$ on the loss of the best strategy. Then, combining Equation (9) and Lemma 4, we have

$$L_{\mathbf{Hedge}(\beta)} \le \min_i L_i + \sqrt{2\tilde{L}\ln N} + \ln N \qquad (11)$$

# Tuning $\eta$ as a function of $T$

- trivially $\min_i L_i \leq T$, yielding

$$L_{\textbf{Hedge}(\eta)} \leq \min_i L_i + \sqrt{2T \ln N} + \ln N$$

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- per iteration we get:

$$\frac{L_{\textbf{Hedge}(\eta)}}{T} \leq \min_i \frac{L_i}{T} + \sqrt{\frac{2 \ln N}{T}} + \frac{\ln N}{T}$$

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- Very good! There is a closely matching lower bound!

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- There exists a stochastic adversarial strategy such that with high probability for any hedging strategy **S** after $T$ trials

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- The adversarial strategy is random, extremely simple, and does not depend on the hedging strategy!

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- Detailed proof quite involved. See section 3.7 in PLG.

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- A trivial random data, in which there is nothing to be learned forces any algorithm to suffer this total loss

# Some loose threads

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- $T$ is tight only when the loss of experts at each iteration is either 0 or 1. If the loss of the best expert is $o(T)$ then we would like to have a tighter bound.
- Observing only the loss of chosen action - the multi-armed bandit problem. Will get to that later in the course.

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- HW1.png