Tracking the best Expert

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Based on “Tracking the best linear predictor” and “Tracking the best expert” by Herbster and Warmuth. Also, section 11.5 in Prediction learning and Games.

Switching experts setup

- Usually: compare algorithm's total loss to total loss of the best expert.
- Switching experts: compare algorithm's total loss to total loss of best expert sequence with k switches.
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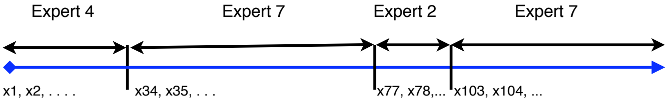


An inefficient algorithm

- Fix:
  - l - sequence length
  - k - number of switches
  - n - number of experts
- Consider one partition-expert per sequence of switching experts.
- No. of partition-experts : $\binom{l}{k-1} n(n-1)^k = O\left(n^{k+1}\left(\frac{el}{k}\right)^k\right)$
- The log-loss regret is at most $(k+1)\log n + k \log \frac{l}{k} + k$
- Requires maintaining $O\left(n^{k+1}\left(\frac{el}{k}\right)^k\right)$ weights.

generalization to mixable losses

- In this lecture we assume loss function is mixable.
- There is an exponential weights algorithm with learning rate η that achieves (in the non-switching case) a bound

$$L_A \leq \min_i L_i + \frac{1}{\eta} \log n$$

- Then using the partition-expert algorithm for the switching-experts case we get a bound on the regret $\frac{1}{\eta}\left((k+1)\log n + k \log \frac{l}{k} + k\right)$

Weight sharing algorithms

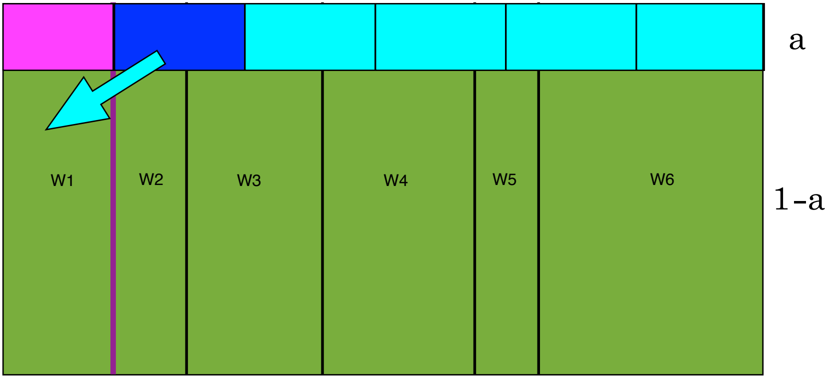
- Update weights in two stages: loss update then share update.
- Prediction uses the normalized s weights $w^s_{t,i}/\sum_j w^s_{t,j}$
- Loss update is the same as always, but defines intermediate m weights:

$$w^m_{t,i} = w^s_{t,i} e^{-\eta L(y_t, x_{t,i})}$$

- Share update: redistribute the weights
- Fixed-share:

$$\begin{aligned} pool &= \alpha \sum_{j=1}^{n} w^m_{t,j} \\ w^s_{t+1,i} &= (1-\alpha) w^m_{t,i} + \frac{1}{n-1}\left(pool - \alpha w^m_{t,i}\right) \end{aligned}$$

The fixed-share algorithm



Proving a bound on the fixed-share

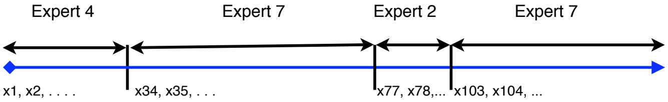
- The relation between algorithm loss and total weight does not change because share update does not change the total weight.
- Thus we still have

$$L_A \leq \frac{1}{\eta} \sum_{i=1}^{n} w^s_{l+1,i}$$

- The harder question is how to lower bound $\sum_{i=1}^{n} w^s_{l+1,i}$

Lower bounding the final total weight

- Fix some switching experts sequence:



- "follow" the weight of the chosen expert i_t.
- The loss update reduces the weight by a factor of $e^{-\eta \ell_{t,i_t}}$.
- The share update reduces the weight by a factor larger than:
  - $1 - \alpha$ on iterations with no switch.
  - $\frac{\alpha}{n-1}$ on iterations where a switch occurs.

Bound for arbitrary α

- Combining we lower bound the final weight of the last expert in the sequence

$$w^s_{l+1,e_k} \geq \frac{1}{n} e^{-\eta L_*} (1-\alpha)^{l-k-1} \left(\frac{\alpha}{n-1}\right)^k$$

Where L_* is the cumulative loss of the switching sequence of experts.

- Combining the upper and lower bounds we get that for any sequence

$$L_A \leq L_* + \frac{1}{\eta}\left(\ln n + (l-k-1)\ln\frac{1}{1-\alpha} + k\left(\ln\frac{1}{\alpha} + \ln(n-1)\right)\right)$$

Tuning α

- let k^* be the best number of switches (in hind sight) and $\alpha^* = k^*/l$
- Suppose we use $\alpha \approx \alpha^*$ then the bound that we get is

$$L_A \leq L_* + \frac{1}{\eta}((k+1)\ln n + (l-1)(H(\alpha^*) + D_{\mathrm{KL}}(\alpha^* || \alpha)))$$

Where

$$H(\alpha^*) = -\alpha^* \ln \alpha^* - (1-\alpha^*)\ln(1-\alpha^*)$$

$$D_{\mathrm{KL}}(\alpha^* || \alpha) = \alpha^* \ln \frac{\alpha^*}{\alpha}(1-\alpha^*)\ln \frac{1-\alpha^*}{1-\alpha}$$

- This is very close to the loss of the computationally inefficient algorithm.
- For the log loss case this is essentially optimal.
- Not so for square loss!

What can we hope to improve?

- In the fixed-share algorithm, the weight of a suboptimal expert never decreases below α/n.
- The regret depends on the length of the sequence.
- The algorithm does not concentrate only on the best expert, even if the last switch is in the distant past.

The idea of variable-share

- Let the fraction of the total weight given to the best expert get arbitrarily close to 1.
- we can get a regret bound that depends only on the number of switches, not on the lenght of the sequence.
- Requires that the loss be bounded.
- Works for square loss, but not for log loss!

Fixed-share:

$$
\begin{aligned}
pool &= \alpha \sum_{j=1}^{n} w_{t,j}^{m} \\
w_{t+1,i}^{s} &= (1-\alpha) w_{t,i}^{m} + \frac{1}{n-1}\left(pool - \alpha w_{t,i}^{m}\right)
\end{aligned}
$$

Variable-share

$$
\begin{aligned}
pool &= \sum_{i=1}^{n} \left(1 - (1-\alpha)^{\ell_{t,i}}\right) w_{t,i}^{m} \\
w_{t+1,i}^{s} &= (1-\alpha)^{\ell_{t,i}} w_{t,i}^{m} + \frac{1}{n-1}\left(pool - \left(1 - (1-\alpha)^{\ell_{t,i}}\right) w_{t,i}^{m}\right)
\end{aligned}
$$

If $\ell_{t,i} = 0$, then expert i does not contribute to the pool.
If $\ell_{t,i} = 1$, then expert i contributes like fixed share.
Expert can get fraction of the total weight arbitrarily close to 1.
Shares the weight quickly if $\ell_{t,i} > 0$

Bound for variable share

$$L_A - L_* \leq \frac{1}{\eta}\ln n + \left(1 + \frac{1}{(1-\alpha)\eta}\right)L_* + k\left(1 + \frac{1}{\eta}\left(\ln n - 1 + \ln\frac{1}{\alpha} + \ln\frac{1}{1-\alpha}\right)\right)$$

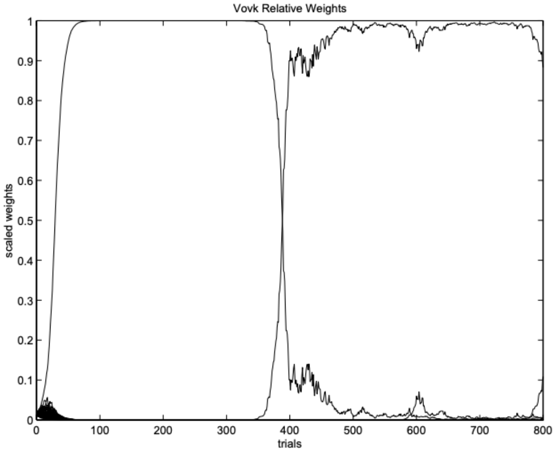
- α should be tuned so that it is (close to) $\frac{k}{2k+L_*}$
- there is no dependence on l the length of the sequence.

Some Experiments

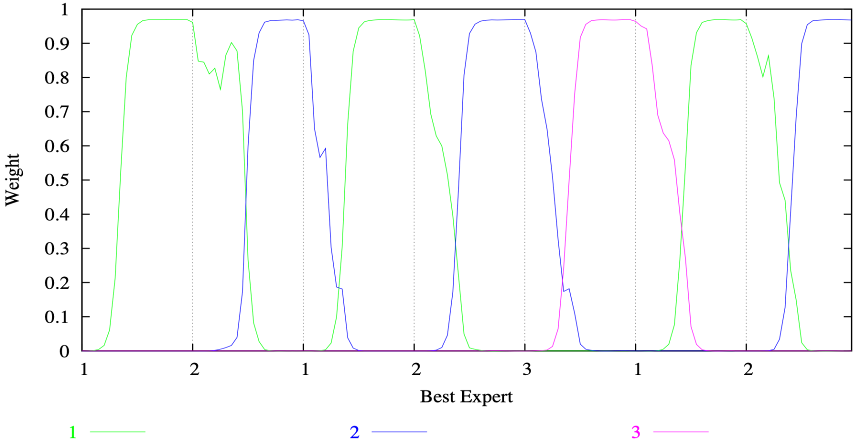
Setup

- 2-3 expeerts
- time is divided into equal length segments
- In each segment a different expert is good.

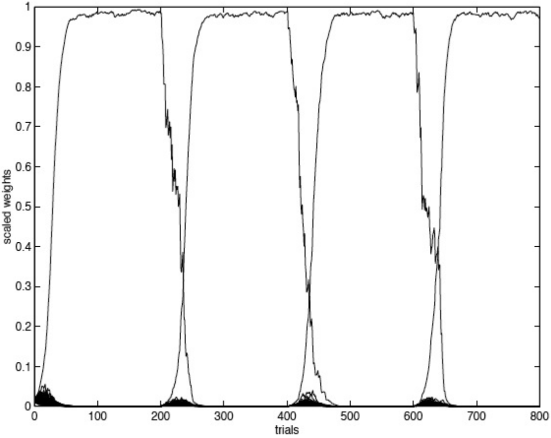
An experiment using static experts



An experiment using fixed share



An experiment using variable share



Bounding Regret Using the Pythagorean Inequality

Pythagorean Inequality:

$$D_R\big(w^* \,\|\, w_t\big) \;\geq\; D_R\big(w^* \,\|\, w_{t+1}\big) \;+\; D_R\big(w_{t+1} \,\|\, w_t\big).$$

plus three-point identity:

$$D_R(a \,\|\, b) \;+\; D_R(b \,\|\, c) \;-\; D_R(a \,\|\, c) \;=\; \langle a-b,\, \nabla R(c) - \nabla R(b)\rangle.$$

yields regret bound:

$$\sum_{t=1}^{T} \langle w_t - w^*,\, z_t \rangle \;\leq\; D_R\big(w^* \,\|\, w_1\big) \;+\; \sum_{t=1}^{T} D_R\big(w_{t+1} \,\|\, w_t\big).$$

Interpretation:

- Switching to a better expert is controlled by $D_R\big(w_{t+1} \,\|\, w_t\big)$.
- The total regret depends on the regularizer $R(w)$.

Effect on Expert Learning

Standard Mirror Descent:

$$w_{t+1} = \arg\min_{w} \left[\eta \sum_{s=1}^{t} \langle w, z_s \rangle + D_R(w \| w_1) \right]$$

Regret Bound:

$$\sum_{t=1}^{T} \langle w_t - w^*, z_t \rangle \leq D_R(w^* \| w_1) + \sum_{t=1}^{T} D_R(w_t \| w_{t+1}).$$

The Pythagorean inequality controls regret by bounding divergence.

Fixed Share Algorithm

Fixed Share Update:

$$w_{t+1}^i = (1-\alpha)\frac{w_t^i e^{-\eta z_t^i}}{\sum_j w_t^j e^{-\eta z_t^j}} + \frac{\alpha}{N}.$$

Impact on Pythagorean Inequality:

$$D_R(w^* \| w_t) \geq D_R(w^* \| w_{t+1}) + D_R(w_{t+1} \| w_t).$$

Modification: The divergence $D_R(w_{t+1} \| w_t)$ increases due to the uniform mixing factor α.

Regret Bound:

$$\sum_{t=1}^{T} \langle w_t - w^*, z_t \rangle \leq D_R(w^* \| w_1) + \sum_{t=1}^{T} \left[D_R(w_t \| w_{t+1}) + \alpha D_{KL}(w_t \| u) \right].$$

Variable Share Algorithm

Variable Share Update:

$$w_{t+1}^i = (1 - \alpha_t) \frac{w_t^i e^{-\eta z_t^i}}{\sum_j w_t^j e^{-\eta z_t^j}} + \alpha_t S_t^i.$$

Impact on Pythagorean Inequality:

$$D_R(w^* \| w_t) \geq D_R(w^* \| w_{t+1}) + D_R(w_{t+1} \| w_t) + \alpha_t D_{KL}(w_t \| u).$$

Modification: The divergence term now depends on α_t, making it adaptive rather than constant.

Regret Bound:

$$\sum_{t=1}^{T} \langle w_t - w^*, z_t \rangle \leq D_R(w^* \| w_1) + \sum_{t=1}^{T} \left[D_R(w_t \| w_{t+1}) + \alpha_t D_{KL}(w_t \| u) \right].$$