Optimal online Learning using potential functions

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Abstract

We study regret-minimizing online algorithms based on potential functions. First, we show that any algorithm with a regret bound that holds for any ϵ is equivalent to a potential minimizing algorithm and vice versa. Second we should a min-max learning algorithm for known horizon. We show a regret bound that is close to optimal when the horizon is not known. Finally we give an algorithm with second order bounds that characterize easy sequences.

1 Introduction

In this paper we study a popular variant of online learning called *the decision-theoretic online learning game* (DTOL) [10]. DTOL (Figure 1) is a repeated zero sum game between a *learner* and an *adversary*. The adversary controls the losses of N actions, while the learner controls a distribution over the actions.

For $i = 1, \ldots, T$

1. The learner chooses a weight function $P(i, j)$ over the actions $j \in \{1, \ldots, N\}$ such that $\sum_{j=1}^{N} P(i, j) = 1$
2. The adversary chooses an *instantaneous loss* for each of the N actions: $l_j^i \in [-1, +1]$ for $j \in \{1, \ldots, N\}$.
3. The *cumulative loss of action* j is $L_j^i = \sum_{s=1}^{i} l_j^s$.
4. The learner incurs an *instantaneous average loss* defined as $\ell^i = \sum_{j=1}^{N} P(i, j) l_j^i$
5. The *cumulative loss of the learner* is $L_\ell^i = \sum_{s=1}^{i} \ell^s$
6. The *cumulative regret* of the learner with respect to action j is $R_j^i = L_\ell^i - L_j^i$.

Figure 1: Decision theoretic online learning

The work on DTOL is rich and diverse. In order to place our work in this context we identify four axis on which existing work lies.

has been extended and refined in several dimensions. Our work combines these extension into a single simple algorithm. We descrive this dimensions below.

1. **Zero-order vs Second order bounds:** Zero order regret bounds [12] depend only on the number of actions N and the length of the sequence T and have the form

$$\max_j R_j^T < C\sqrt{T \ln N} \tag{1}$$

Where C is a small constant.

Second order bounds take advantage of so-called "easy sequences" where $|l_j^i| < 1$. Roughly speaking, second order bounds replace the length of the sequence T with a sum of squares that is smaller than T for easy sequences. Several definitions for the sum of squares have been analyzed. Cesa-Binachi et al. [6] propose two such quantities. The *Quadratic Variation* $Q_j^T = \sum_{t=1}^T (l_j^t)^2$ measures the variation of the action j, and Q^* is the quadratic variation of the best action at time T. The *cumulative variance* V_T is defined as

$$\mathrm{Var}_i = \sum_{j=1}^N P_j^i (l_j^i)^2 - \left(\sum_{j=1}^N P_j^i l_j^i \right)^2 \quad \text{and} \quad V_T = \sum_{i=1}^T \mathrm{Var}_i$$

Koolen and Van Erven [14] propose a variant of the quadratic variation that uses $(l_j^t - \ell^t)^2$ instead of $(l_j^t)^2$. It remained unclear which of these variations is "best" or, indeed, if the notion of "best" is well defined.

2. **Quantile based bounds** It is possible to give regret bounds where the total loss of the algorithm is compared to the total loss of the best ϵ fraction of the actions. Indeed many regret bounds hold when N is replaced with $1/\epsilon$.

   Using quantiles allows actions sets that are uncountably infinite. For example consider an action set that corresponds to linear functions with real valued parameters.

   We formalize uncountably infinite sets of actions by defining a probability space Ω where $\omega \in \Omega$ is an action and the loss that action ω incurs on iteration t is a random variable l_ω^t. Note that this definition includes the case where the number of actions is finite, as Ω can be a finite space.

3. **Potential based algorithms** Two main paradigms are known for designing online learning algorithm: the *follow the leader* paradigm and the *potential function*paradigm. Our work here uses the potential paradigm. A potential function is a positive increasing function $\phi : \mathbb{R} \to \mathbb{R}$. The main quantity that is analyzed is the average potential the actions, $\Phi = \mathbf{E}_{R \sim \mathbf{\Psi}} [\phi(R)]$. The regret bounds are proved by combining an upper bound on the score with Markov inequality:

$$\mathbf{P}_{\rho \sim \mathbf{\Psi}} [\rho \geq R] \leq \frac{\mathbf{E}_{\rho \sim \mathbf{\Psi}} [\phi(\rho)]}{\phi(R)}$$

4. **parameter-free algorithms** Many DTOL algorithms require setting a *learning rate* parameter. Setting this parameter, in turn, requires having prior information about N, T,or ϵ. The resulting bounds hold only if the sequences of losses conform to the a-priori assumption. In many cases "doubling tricks" can be used to circumvent this dependence at the cost of significant increases in constants. In contrast, several papers [7, 16, **?**, 15] presented so-called "parameter free" algorithms which do not require tuning of parameters.

Using these definitions, we describe the contributions of this paper.

- We provide min-max analysis for potential and quantile based online algorithms with a finite known horizon and a potential function with four strictly positive derivativs. In particular, we show that Brownian motion is the optimal adversarial strategy for any such potential function.
- We show how to remove the finite horizon assumption and derive both exponential weights and NormalHedge potentials.
- We provide second order bounds on the parameter-free normal-Hedge algorithm that depend only on the percentile ϵ and a variant of the cumulative variance. We show that this bound is the optimal bound for any potential based algorithm.
- We construct a lower bound that holds for any online algorithm and is very close to the normal hedge upper bound.

We start with the bounded horizon case. Fixing a potential function at the end of the game $\phi(T, R)$ and the strategies used by the learner and the adversary, we define potential functions $\phi(i, R)$ for iterations $i = T-1, T-2, \ldots, 0$ such that the score $\Phi(t)$ is guaranteed to be equal for all of the iterations.

$$\Phi(T) = \Phi(T-1) = \cdots = \Phi(0)$$

This allows us to analyze the game one iteration at a time and construct good strategies for both sides. We name this potential based game the *Integer Time Game*, the analysis of this game is given in Section 5. The analysis assumes only that the final potential $\phi(T, R)$ is strictly positive and has strictly positive first and second derivatives (We denote the set of functions that have $0, \ldots, k$ strictly positive derivatives by $\mathcal{P}^k$, the formal definition is given in Section 4)

The strategies yielded by the analysis guarantee bounds on the final score. The adversarial strategy guarantees $\Phi^{\uparrow} \leq \Phi(T)$, while the learner's strategy guarantees $\Phi(T) \leq \Phi^{\downarrow}$. Unfortunately, these bounds don't match, i.e. $\Phi^{\uparrow} < \Phi^{\downarrow}$. In other words our proposed strategies are not min-max optimal. The question of whether there exist min/max strategies for the integer time game is open.

To find min/max strategies we expand the game. We call the expanded game the *discrete time game*. The expansion involves giving the more options to the adversary, but not to the learner. As a result, any upper bound $\Phi^{\downarrow}$ that holds for a learner strategy in the discrete time game also holds in the integer time game.

The added option for the adversary is to declare, at the beginning of each iteration, the range of values of the instantanous losses. In the integer time game this range is set to $[-1, +1]$. In the discrete time game the range is chosen by the adversary on iteration i to be $[-s_i, +s_i]$ for $1 \geq s_i > 0$. To keep the game balanced between the adversary and the learner we replace the iteration number i with real valued *time* parameter and let $t_{i+1} = t_i + s_i^2$. this and another necessary adjustment are described in Section 6. Section 6.1 describes strategies used for the discrete game which are scaled versions of the strategies for the integer time game.

We fix the potential at the end of the game $\boldsymbol{\phi}_{\mathcal{T}} \in \mathcal{P}^4$ and consider a sequence of adversarial and learner strategies indexed by k: $Q_{D(k)}, P_{D(k)}$, where $\forall i,\ s_i^k = \frac{\sqrt{\mathcal{T}}}{2^k}$ for some constant $\mathcal{T}$. We prove two facts regarding the limit $k \to \infty$. The first (Thm. 8) is that $\lim_{k\to\infty} \phi^{\downarrow}_{P_{D(k)}} - \phi^{\uparrow}_{Q_{D(k)}} \to 0$. The second (Thm. 9) is that, if $\phi(T, R)$

$$\forall k, \forall 0 \leq i \leq 2^{2k}, t_i = i2^{-2k}\mathcal{T},\ \forall R, \quad , \phi^{\uparrow}_{Q_{D(k+1)}}(t_i, R) > \phi^{\uparrow}_{Q_{D(k)}}(t_i, R)$$

Taken together these facts imply that, if the fixed potential function for the end of the game $\boldsymbol{\phi}_{\mathcal{T}}$ is in $\mathcal{P}^4$, then there exists a potential function $\phi(t, R)$. The adversarial strategy corresponding to this potential function corresponds to Brownian motion. The backwards recursion used to computer the potential for $t \leq \mathcal{T}$ is a partial differential equation known as the Kolmogorov Backward equation.

The main result of this paper is that a *single* adversarial strategy, i.e. Brownian motion, is optimal for any sufficiently convex potential functions.

The discrete time game presents the adversary with a dilemma. On the one hand, The adversary has to declare, on each iteration, an upper bound on the range of the losses $[-s_i, s_i]$ where $s_i > 0$. On the other hand, it wants to set s_i as small as possible. [1]

We introduce a variant of the game called the *continuous time game* to alleviate this dilemma. In this came the adversary does not announce the step size and the learner behaves as if the step size is infinitesimally small. In this case time is advanced according to the variance of the actual losses. This much more natural algorithm yields a regret bound that depends on the cumulative variance and is smaller for easy, low variance sequences.

Until this point our theory holds for any final potential function in $\mathcal{P}^4$. We conclude by analyzing two specific potentials.

1. We derive a potential function and a corresponding learning algorithm that is min/max optimal for a given time horizon $\mathcal{T}$. The optimality is in the sense that the simultanous regret bound for time $\mathcal{T}$ has a matching simultanous lower bound.

[1] The situation is similar to a folklore game in which each player writes down a number on a piece of paper and the player with the largest number wins.

2. By finding solutions to the Kolmogorov Backward equation that hold for all $t > 0$ we eliminate the need to define a final potential. As a result we get an "anytime" learning algorithm that can be stopped at any time. The specific potential we analyze is Normal-Hedge []. NormalHedge is not min/max optimal for any time, but it is almost optimal for all times.

2 related work

Most of the papers on potential based online algorithms consider one or a few potential functions. Most common is the exponential potential, but others have been considered [5]. A natural question is what is the difference between potential functions and whether some potential function is "best".

In this paper we consider a large set of potential functions, specifically, potential functions that are strictly positive and have strictly positive derivatives of orders up to four. The exponential potential and the NormalHedge potential [7, 15] are member of this set.

To analyze these potential functions we define a different game, which we call the "potential game". In this game the primary goal of the learner is not to minimize regret, rather, it is to minimize the final score Φ^T. To do so we define potential functions for intermediate steps: $0 \leq t < T$.[2]

3 Main Results

1. **Uniform regret bound** There exists an online learning algorithm such that for any $\nu > 0$ (set in advance) and any t, ϵ (holds uniformly) the following regret bound holds.

$$R_\epsilon \leq \sqrt{(t+\nu)\left(\ln(t+\nu) + 2\ln\frac{1}{\epsilon}\right)} \tag{2}$$

2. **Second order bound**

3. **optimality of Brownian motion** For any potential function in $\mathcal{P}^4$ the min/max value of any state (t, R) is attained by Brownian motion on the part of the adversary for any $s \geq t$.

4 Preliminaries

We define some terms and notationthat will be used in the rest of the paper.

Positivity We require that potential functions have positive derivatives for a range of degree. To that end we use the following definition:

Definition 1 (Strict Positivity of degree k)**.** *A function $f : \mathbb{R} \to \mathbb{R}$ is strictly positive of degree k, denoted $f \in \mathcal{P}^k$ if the derivatives of orders 0 to k: $f(x), \frac{d}{dx}f(x), \ldots, \frac{d^k}{dx^k}f(x)$ exist and are strictly positive.*

The following useful lemma states that $\mathcal{P}^k$ is closed under positive conbinations.

Lemma 1. *Suppose that for $i = 1, \ldots, n$, $f_i \in \mathcal{P}^k$ and $\alpha_i > 0$, Then $\sum_{i=1}^k \alpha_i f_i \in \mathcal{P}^k$*

Divisibility: To reach optimality we need the set of actions to be arbitrarily divisible. Intuitively, We replace the finite set of actions with a continuous mass, so that each set of actions can be partitioned into two parts of equal weight. Formally, we define the set of actions to be a probability space (Ω, σ, μ) such that $\omega \in \Omega$ is a particular action. We require that the space is *arbitrarily divisible*, which means that for any $s \in \sigma$, there exist a partition $u, v \in \sigma$ such that $u \cup v = s, u \cap v = \emptyset$, and $\mathbf{P}[u] = \mathbf{P}[v] = \frac{1}{2}\mathbf{P}[s]$.

[2]The analysis described here builds on a long line of work. Including the Binomial Weights algorithm and it's variants [4, 1, 2] as well as drifting games [18, 9].

State: The *state* of a game at iteration i, denoted $\mathbf{\Psi}(i)$, is a random variable that maps each action $\omega \in \Omega$ to the cumulative regret of ω at time i: R^i_ω. The sequence of cumulative regrets corresponding to action ω is the *path* of ω:

$$S_\omega = (R^1_\omega, R^2_\omega, \ldots, R^N_\omega) \tag{3}$$

Generalized binomial distribution We denote by $\mathbb{B}(n, s)$ the distribution over the reals defined by $\sum_{i=1}^n X_i$ where X_i are iid binary random variables which attain the values $-s, +s$ with equal probabilities.

Expected value shorthand: Suppose P is a distribution over the reals, and $f : \mathbb{R} \to \mathbb{R}$, we use the following short-hand notation for the expected value of f under the distribution P:

$$P \odot f \doteq \mathbf{E}_{x \sim P}\left[f(x)\right]$$

We define the *score* at iteration i as the average potential with respect to the state:

$$\Phi(i) = \mathbf{\Psi}(i) \odot \phi(i) \doteq \mathbf{E}_{R \sim \mathbf{\Psi}(i)}\left[\phi(i, R)\right]$$

Note that in this short-hand notation we suppress the variable with respect to which the integration is defined, which will always be R.

Convolution: Let A, B be two independent random variables. We define the convolution $A \oplus B$ to be the distribution of $x + y$. A constant a corresponds to the point mass distribution concentrated at a. For convenience we define $A \ominus B = A \oplus (-B)$

5 Integer time game

The integer time game is described in Figure 2. The integer time game generalizes the decision theoretic online learning problem [11] in the following ways:

1. The goal of the learner in DTOL is to guarantee an upper bounds on the regret. The learner's goal in the integer time game is to minimize the final score. From theorem **??** we know that if we set the final potential as $\boldsymbol{\phi}_T(R) = \frac{1}{G(R)}$ then the two conditions are equivalent, allowing us to focus on the score.

2. The number of iterations T is given as input, as is the potential function at the end: $\boldsymbol{\phi}_T(R)$.

3. The actions are assumed to be *divisible*. For our purposes it is enough to assume that any action can be split into two equal weight parts.

The key to the potential based analysis is that using the predefined final potential we can define potential functions and scores for all iterations $1, \ldots, T-1$. This is explained in the next subsection.

5.1 Defining potential Functions for all iterations

The potential game defines the *final* potential function $\boldsymbol{\phi}_T$, at the end of the game. We will now show, that we can extend the definition of a potential function to all iterations of the game.

A single action defines a path S_ω (as defined in (3)). Fixing the strategies of the learner and the adversary determines a distribution $\mathcal{D}$ over paths. We describe two equivalent ways to define $\phi_{P,Q}(i, R)$ for $i < T$

1. **Using conditional expectation** We can define the potential on iteration i based on the fixed potential at iteration T.

$$\forall i = 1, \ldots, T,\ \forall R \quad \phi_{P,Q}(i, R) = \mathbf{E}_{\omega \sim \mathcal{D} | R^i_\omega = R}\left[\phi(T, R^T_\omega)\right] \tag{7}$$

2. **Using backward induction** It is sometimes convenient to compute the the potential for time i from the potential at time $i + 1$:

$$\forall i = 1, \ldots, T-1, R \quad \phi_{P,Q}(i, R) = \mathbf{E}_{\omega \sim \mathcal{D} | M^i_\omega = R}\left[\phi_{P,Q}(i+1, R^{i+1}_\omega)\right] \tag{8}$$

Initialization:

- input: T : The number of iterations.
- Final iteration potential function: $\boldsymbol{\phi}_T \in \mathcal{P}^2$
- $\boldsymbol{\Psi}(1) = \delta(0)$ is the initial state of the game which is a point mass distribution at 0.

For $i = 1, 2, \ldots, T$:

1. The learner chooses a non-negative random variable over Ω that is the *weight function* $P(i, R)$ such that $\boldsymbol{\Psi}(i) \odot P(i) = 1$
2. The adversary chooses a function $Q(i, R)$ that maps i, R to a distribution over $[-1, +1]$. This random variable corresponds to the instantaneous loss of each action at time t.
3. We define the *bias* at (i, R) to be
$$B(i, R) \doteq \mathbf{E}_{l \sim Q(i,R)}[l] \tag{4}$$
4. the average loss is
$$\ell(i) = \boldsymbol{\Psi}(i, R) \odot (P(i, R)B(i, R)) \tag{5}$$
5. The state is updated.
$$\boldsymbol{\Psi}(i + 1) = \mathbf{E}_{R \sim \boldsymbol{\Psi}(i)}[R \oplus Q(i, R)] \oplus -\ell(i) \tag{6}$$
Where $Q(i, R)$ is the distribution of the losses of actions with respect to which the regret is R after iteration $i - 1$. $\oplus$ denotes the convolution as defined above.

The final score is calculated: $\Phi(T) = \boldsymbol{\Psi}(T) \odot \boldsymbol{\phi}_T$.
The goal of the learner is to minimize this score, the goal of the adversary is to maximize it.

Figure 2: The integer time game

by using backwards induction: $i = T - 1, T - 2, \ldots, 1$ we can compute the potential for all iterations.

We use Equations (4,5) and marginalizing over R to express Equation (8) in terms of the single step strategies:
$$\forall i = 1, \ldots, T - 1, R \quad \phi_{P,Q}(i, R) \doteq \mathbf{E}_{r \sim [(R - \ell(i)) \oplus Q(i,R)]}[\phi_{P,Q}(i + 1, r)] \tag{9}$$

The score at iteration i is defined as $\Phi(i) = \boldsymbol{\Psi}(i) \odot \phi(i)$. The scores are all different expressions for calculating the expected final potential for the fixed strategies Q, P. Therefor the scores are all equal, as expressed in the following theorem:

Theorem 2. *Assuming $P(i, R), Q(i, R)$ are fixed for all $i = 1, \ldots, T - 1$, then*

$$\boldsymbol{\Psi}(T) \odot \phi(T) = \Phi(T) = \Phi(T - 1) = \cdots = \Phi(1) = \phi_{P,Q}(0, 0)$$

A few things worth noting:

1. $\phi_{P,Q}(i, R)$ is the the final expected potential given that the paths starts at (i, R) and that the strategies used by both players in iterations $i, \ldots, T$ are fixed. Note also that which strategies were used in iterations $1, \ldots, i - 1$ is of no consequence. The effect of past choices is captured by the state $\boldsymbol{\Psi}(i)$.
2. The final expected potential is equal to $\phi(0, 0)$ which is the potential at the common starting point: $i = 1$, $R = 0$.

5.2 Upper and Lower potentials

Next,we vary the strategies of one side or the other to define upper and lower potentials.

$$\exists P, \quad \forall Q, \ \ \forall 1 \le i \le T, \ \ \forall R \in \mathbb{R}, \ \ \phi_P^{\downarrow}(i,R) \ge \phi_{P,Q}(i,R) \tag{10}$$

$$\exists Q, \quad \forall P, \quad \forall 1 \le i \le T, \ \ \forall R \in \mathbb{R}, \ \ \phi_Q^{\uparrow}(i,R) \le \phi_{P,Q}(i,R) \tag{11}$$

In words, $\phi_P^{\downarrow}$ is an upper bound on the potential that is guaranteed by the learner strategy P while $\phi_{Q_D}^{\uparrow}$ is a lower bound that is guaranteed by the adversarial strategy Q.

Following the same argument as the one leading to Theorem 2. We define upper and lower scores $\Phi_P^{\downarrow}(i), \Phi_Q^{\uparrow}(i)$ such that

$$\boldsymbol{\Psi}_P(T) \odot \boldsymbol{\phi}_T = \Phi_P^{\downarrow}(T) = \Phi_P^{\downarrow}(T-1) = \cdots = \Phi_P^{\downarrow}(0) = \phi_P^{\downarrow}(0,0) \tag{12}$$

and

$$\boldsymbol{\Psi}_Q(T) \odot \boldsymbol{\phi}_T = \Phi_Q^{\uparrow}(T) = \Phi_Q^{\uparrow}(T-1) = \cdots = \Phi_Q^{\uparrow}(0) = \phi_Q^{\uparrow}(0,0) \tag{13}$$

Our ultimate goal is to find strategies P and Q such that

$$\forall i, R, \quad \phi_Q^{\uparrow}(i,R) = \phi_P^{\downarrow}(i,R) \tag{14}$$

in particular, $\Phi_Q^{\uparrow}(0) = \phi_Q^{\uparrow}(0,0) = \phi_P^{\downarrow}(0,0) = \Phi_P^{\downarrow}(0)$. This means that Q, P are a min/max pair of strategies and that $\Phi_Q^{\uparrow}(0) = \Phi_P^{\downarrow}(0)$ define the min/max value of the game.

We do not achieve this for the integer game described in the next section. To achieve min/max optimality we extend the integer time game to the discrete time game (section 6) and to the continuous time game (7).

5.3 Strategies for the integer time game

We assume that $\boldsymbol{\phi}_T \in \mathcal{P}^2$, in other words, the final potential is positive, increasing and convex. $\boldsymbol{\phi}_T$ defines the upper and lower potentials at time T:

$$\phi_{Q_I}^{\uparrow}(T,R) = \phi_{P_I}^{\downarrow}(T,R) = \boldsymbol{\phi}_T(R)$$

We define a backwards recursion for the lower potential:

$$\phi_{Q_I}^{\uparrow}(i-1,R) = \frac{\phi_{Q_I}^{\uparrow}(i,R+1) + \phi_{Q_I}^{\uparrow}(i,R-1)}{2} \tag{15}$$

and a backwards recursion for the upper potential:

$$\phi_{P_I}^{\downarrow}(i-1,R) = \frac{\phi_{P_I}^{\downarrow}(i,R+2) + \phi_{P_I}^{\downarrow}(i,R-2)}{2} \tag{16}$$

We define strategies that correspond to these potentials. A strategy for the adversary:

$$Q_I(i,R) = \begin{cases} +1 & \text{w.p. } \frac{1}{2} \\ -1 & \text{w.p. } \frac{1}{2} \end{cases} \tag{17}$$

and a strategy for the learner:

$$P_I(i,R) = \frac{1}{Z} \frac{\phi(i,R+2) - \phi(i,R-2)}{2} \tag{18}$$

Where Z is a normalization factor

$$Z = \mathbf{E}_{R \sim \boldsymbol{\Psi}(i)} \left[\frac{\phi(i,R+2) - \phi(i,R-2)}{2} \right]$$

The following lemma states that these strategies guarantee the corresponding potentials.

Lemma 3.
Let i be an integer between 1 and T
If $\phi^{\uparrow}_{Q_I}(i,R) \in \mathcal{P}^2$

1. **Positivity:** $\phi^{\uparrow}_{Q_I}(i-1,R) \in \mathcal{P}^2$
2. **Adversary:** *The adversarial strategy (17) guarantees the recursion given in Eq. (15)*

If $\phi^{\downarrow}_{P_I}(i,R) \in \mathcal{P}^2$

1. **Positivity:** $\phi^{\downarrow}_{P_I}(i-1,R) \in \mathcal{P}^2$
2. **Learner:** *The learner strategy (18) guarantees the recursion given in Eq. (16)*

Proof. We prove each claim in turn

1. **Positivity:** Follows from Lemma 1.
2. **Adversary:** By symmetry adversarial strategy (17) guarantees that the aggregate loss (5) is zero regardless of the choice of the learner: $\ell(i) = 0$. Therefor the state update (6) is equivalent to the symmetric random walk:
$$\mathbf{\Psi}(i) = \frac{1}{2}((\mathbf{\Psi}(i) \oplus 1) + (\mathbf{\Psi}(i) \ominus 1))$$
Which in turn implies that if the adversary plays Q^* and the learner plays an arbitrary strategy P
$$\phi^{\uparrow}_{Q_I}(i-1,R) = \frac{\phi^{\uparrow}_{Q_I}(i,R-1) + \phi^{\uparrow}_{Q_I}(i,R+1)}{2} \tag{19}$$
As this adversarial strategy is oblivious to the learner's strategy, it guarantees that the average value at iteration i is *equal* to the average of the lower value at iteration i.
3. **Learner:** Plugging learner's strategy (18) into equation (5) we find that
$$\ell(i) = \frac{1}{Z_i}\mathbf{E}_{R\sim\mathbf{\Psi}(i)}\left[\left(\phi^{\downarrow}_{P_I}(i,R+2) - \phi^{\downarrow}_{P_I}(i,R-2)\right)B(i,R)\right] \tag{20}$$
Consider the score at iteration i when the learner's strategy is P^* and the adversarial strategy Q is arbitrary
$$\Phi_{P^*,Q}(i,R) = \mathbf{E}_{R\sim\mathbf{\Psi}(i)}\left[\mathbf{E}_{y\sim Q(i)(R)}\left[\phi(i,R+y-\ell(i))\right]\right] \tag{21}$$
As $\phi(i,\cdot)$ is convex and as $y - \ell(i) \in [-2,2]$,
$$\phi^{\downarrow}_{P_I}(i-1,R+y) \le \frac{\phi^{\downarrow}_{P_I}(i,R+2) + \phi^{\downarrow}_{P_I}(i,R-2)}{2} + (y-\ell(i))\frac{\phi^{\downarrow}_{P_I}(i,R+2) - \phi^{\downarrow}_{P_I}(i,R-2)}{2} \tag{22}$$
Combining the equations (20) and (21) we find that
$$\begin{aligned}
\Phi_{P^*,Q}(i,R) &= \mathbf{E}_{R\sim\mathbf{\Psi}(i)}\left[\mathbf{E}_{y\sim Q(i)(R)}\left[\phi^{\downarrow}_{P_I}(i,R+y-\ell(i))\right]\right] &(23)\\
&\le \mathbf{E}_{R\sim\mathbf{\Psi}(i)}\left[\frac{\phi^{\downarrow}_{P_I}(i,R+2) + \phi^{\downarrow}_{P_I}(i,R-2)}{2}\right] &(24)\\
&+ \mathbf{E}_{R\sim\mathbf{\Psi}(i)}\left[\mathbf{E}_{y\sim Q(i)(R)}\left[(y-\ell(i))\frac{\phi^{\downarrow}_{P_I}(i,R+2) - \phi^{\downarrow}_{P_I}(i,R-2)}{2}\right]\right] &(25)
\end{aligned}$$

The final step is to show that the term (25) is equal to zero. As $\ell(i)$ is a constant with respect to R and y the term (25) can be written as:

$$\mathbf{E}_{R\sim\boldsymbol{\Psi}(i)}\left[\mathbf{E}_{y\sim Q(i)(R)}\left[(y-\ell(i))\frac{\phi^{\downarrow}_{P_I}(i,R+2)-\phi^{\downarrow}_{P_I}(i,R-2)}{2}\right]\right] \tag{26}$$

$$= \mathbf{E}_{R\sim\boldsymbol{\Psi}(i)}\left[B(i,R)\frac{\phi^{\downarrow}_{P_I}(i,R+2)-\phi^{\downarrow}_{P_I}(i,R-2)}{2}\right] \tag{27}$$

$$- \ell(i)\mathbf{E}_{R\sim\boldsymbol{\Psi}(i)}\left[\frac{\phi^{\downarrow}_{P_I}(i,R+2)-\phi^{\downarrow}_{P_I}(i,R-2)}{2}\right] \tag{28}$$

$$= 0 \tag{29}$$

□

Repeating the induction steps of Lemma 3 from $i=T$ to $i=1$ yields the following theorem.

Theorem 4. *Let $\boldsymbol{\phi}_T \in \mathcal{P}^2$, for any iteration $0 \le i \le T$ and regret $R_0 \in \mathbb{R}$*

- *The lower potential guaranteed by Q_I is*

$$\phi^{\uparrow}_{Q_I}(i,R_0) = \mathbf{E}_{R\sim R_0\oplus\mathbb{B}(T-i,1)}\left[\boldsymbol{\phi}_T(R)\right]$$

- *The upper potential guaranteed by P_I is*

$$\phi^{\downarrow}_{P_I}(i,R_0) = \mathbf{E}_{R\sim R_0\oplus\mathbb{B}(T-i,2)}\left[\boldsymbol{\phi}_T(R)\right]$$

Plugging in $i=0$,$R=0$ we get the following Corrolary:

Corollary 5. *if the learner plays P_I on every iteration it guarantees that the final score satisfies*

$$\boldsymbol{\Psi}(T)\odot\boldsymbol{\phi}_T \le \mathbb{B}(T,2)\odot\boldsymbol{\phi}_T$$

If the Adversary plays Q_I on every iteration it guarantees that:

$$\boldsymbol{\Psi}(T)\odot\boldsymbol{\phi}_T = \mathbb{B}(T,1)\odot\boldsymbol{\phi}_T$$

6 From integer to discrete time

The upper and lower bound on the final score given in Theorem 4 do not match. If $\boldsymbol{\phi}_T \in \mathcal{P}^2$ then $\mathbb{B}(T,1)\odot\boldsymbol{\phi}_T < \mathbb{B}(T,2)\odot\boldsymbol{\phi}_T$ In other words, the strategies (17,18) are not a min/max pair.[3]

To close this gap we extend the integer time game into a new game we call the discrete time game (Fig. 3). The discrete time game increases the options available to the adversary, but not to the learner. As the integer step game is a special case of the new game, any upper potential that can be guaranteed by the learner in the discrete time game is also an upper potential for the discrete time game.

In the integer time game the loss of each action is in the range $[-1,+1]$, in the discrete time game the adversary chooses, on iteration i a step size $0 < s_i \le 1$ which restricts the losses to the range $[-s_i,+s_i]$. Note that by always choosing $s_i = 1$, the adversary can choose to play the integer time game.

We make two additional alterations to the integer time game in order to keep the game fair. An unfair game is one where one side always wins. We list the alterations and then justify them.

[3]There might be other (pure) strategies for the integer game that are a min/max pair, we conjecture that is not the case, and seek a extension of the game that would yield min/max strategies.

Initialization: $t_0 = 0$

On iteration $i = 1, 2, \ldots$

1. If $t_i = \mathcal{T}$ the game terminates.
2. The adversary chooses a *step size* $0 < s_i \leq \min(\sqrt{1 - t_i}, 1)$, which advances time by $t_i = t_{i-1} + s_i^2$
3. Given s_i, the learner chooses a distribution $P(i)$ over $\mathbb{R}$.
4. The adversary chooses a mapping from $\mathbb{R}$ to distributions over $[-s_i, +s_i]$: $Q(t, \cdot) : \mathbb{R} \to \Delta^{[-s_i, +s_i]}$
5. The aggregate loss is calculated:

$$\ell(t_i) = \mathbf{E}_{R \sim \mathbf{\Psi}(t_i)} \left[P(t_i, R) B(t_i, R) \right] \text{ where } B(t_i, R) \doteq \mathbf{E}_{y \sim Q(t_i, R)} \left[y \right] \tag{30}$$

   Such that $|\ell(t_i)| \leq s_i^2$

6. The state is updated.

$$\mathbf{\Psi}(t_i) = \mathbf{E}_{R \sim \mathbf{\Psi}(t_i)} \left[Q(t_i, R) \oplus (R - \ell(t_i)) \right]$$

   Where $\oplus$ is a convolution as defined in the preliminaries.

Upon termination, the final value is calculated:

$$\Phi(\mathcal{T}) = \mathbf{\Psi}(\mathcal{T}) \odot \phi(\mathcal{T})$$

Figure 3: The discrete time game

1. **real-valued time** In the integer time game we use an integer to indicate the iteration number: $i = 1, 2, \ldots, T$. In the discrete time game we use an positive real value, which we call "time" and use the update rule $t_{i+1} = t_i + s_i^2$, and define the final time, which is used in the regret bound, to be $\mathcal{T} = \sum_{i=0}^{T} s_i^2$
2. **Bounded average loss** We restrict the average loss to a range much smaller than $[-s_i, +s_i]$, specifically: $|\ell(i)| \leq s_i^2$

Note that both of these conditions hold trivially when $s_i = 1$

1. **Justification of real-valued time** To justify these choices we consider the following adversarial strategy for the discrete time game:

$$Q_D[s, p](t, R) = \begin{cases} +s & \text{w.p. } p \\ -s & \text{w.p. } 1 - p \end{cases} \tag{31}$$

   From Equation (13) we get that the initial score,

$$\Phi^{\uparrow}_{Q_D}(0) = \Phi^{\uparrow}_{Q_D}(T) = \mathbf{\Psi}_{Q_D}(T) \odot \phi(T)$$

   On the other hand, we know that $\mathbf{\Psi}_{Q_D}(T) = \mathbb{B}(T, s)$. Suppose T is large enough that the normal approximation for the binomial can be used. Let $\mathcal{N}(\mu, \sigma^2)$ be the normal distribution with mean μ and variance σ^2.

$$\lim_{T \to \infty} \Phi^{\uparrow}_{Q_D}(0) = \mathcal{N}(0, Ts^2) \odot \phi(T) \tag{32}$$

   Recall that $\phi(T)$ is a fixed strictly convex function. It is not hard to see that if $Ts^2 \to 0$ minimizes $\Phi^{\uparrow}_{Q_D}(0)$ and makes it equal to to $\phi(T, 0)$, which means that the learner wins, while if $Ts^2 \to \infty$,

$\Phi^{\uparrow}_{Q_D}(0) \to \infty$ which means that the adversary wins. In order to keep the game balanced keep Ts^2 constant as we let $s \to 0$. We achieve that by defining the real-valued discrete time as $t_j = \sum_{i=0}^{j-1} s_i^2$.

2. **Justification of bounding average loss** Suppose the game is played for T iterations and that the adversary uses the strategy $Q_D\left[s, \frac{1}{2} + \epsilon\right](t, R)$ and that $s = \frac{1}{\sqrt{T}}$. In this case the loss of the learner in iteration i is $\ell(i) = 2s\epsilon$ and the total loss is

$$L_\ell^T = \sum_{i=0}^{T-1} \ell(i) = T2\epsilon s = \frac{2\epsilon}{s}$$

.

If ϵ is kept constant as $s \to 0$ then $\lim_{T\to\infty} L_\ell^T = \infty$, biasing the game towards the adversary. On the other hand, if $\epsilon = s^\alpha$ for $\alpha < 1$ then $L_\ell^T \to 0$, biasing the game towards the learner. To keep the game balanced we have to set $\epsilon = cs$ for some constant c. Without loss of generality we set $c = 1$.

Generalizing this to the game where the adversary can choose a different s_i in each iteration we get the constraint $|\ell(i)| \leq s_i^2$

6.1 Strategies for discrete time

We fix a real number $\mathcal{T}$ as the real length of the game.

We define a sequence of adversarial strategies, indexed by k, where the step size of $Q_{D(k)}$ is $s_k = 2^{-2k}\sqrt{\mathcal{T}}$.

We define a sequence of adversarial strategies $Q_{D(k)}$ and matching learner strategies $P_{D(k)}$ for $k = 0, 1, 2, \ldots$. The adversarial strategies are designed so that the upper and lower potentials converge to a limit as $k \to \infty$.

We set the time points $t_i = is_k^2$ for $i = 0, 1, \ldots, 2^{2k}$. We call the resulting games k-discrete and denote them as $D(k)$.

For a given k we define upper and lower potentials for each t_i. This is done by induction starting with the final potential function $\boldsymbol{\phi}_{\mathcal{T}}(R) = \phi^{\uparrow}_{Q_{D(k)}}(\mathcal{T}, R) = \phi^{\downarrow}_{P_{D(k)}}(\mathcal{T}, R)$ and iterating backwards for $i = T, T-1, \ldots, 0$, $t_i = is_k^2$

$$\phi^{\uparrow}_{Q_{D(k)}}(t_{i-1}, R) = \frac{\phi^{\uparrow}_{Q_{D(k)}}(t_i, R + s_k) + \phi^{\uparrow}_{Q_{D(k)}}(t_i, R - s_k)}{2} \tag{33}$$

$$\phi^{\downarrow}_{P_{D(k)}}(t_{i-1}, R) = \frac{\phi^{\downarrow}_{P_{D(k)}}(t_i, R + s_k(1 + s_k)) + \phi^{\downarrow}_{P_{D(k)}}(t_i, R - s_k(1 + s_k))}{2} \tag{34}$$

These upper and lower potentials correspond to strategies for the adversary and the learner. The adversarial strategy is

$$Q_{D(k)} = \begin{cases} +s_k & \text{w.p. } \frac{1}{2} \\ -s_k & \text{w.p. } \frac{1}{2} \end{cases} \tag{35}$$

The learner's strategy is:

$$P_{D(k)}(t_i, R) = \frac{1}{Z} \frac{\phi^{\downarrow}_{P_{D(k)}}(t_{i+1}, R + s_k(1 + s_k)) - \phi^{\downarrow}_{P_{D(k)}}(t_{i+1}, R - s_k(1 + s_k))}{2} \tag{36}$$

$$\text{where } Z = \mathbf{E}_{R\sim\boldsymbol{\Psi}(t_{i+1})}\left[\frac{\phi^{\downarrow}_{P_{D(k)}}(t_{i+1}, R + s_k(1 + s_k)) - \phi^{\downarrow}_{P_{D(k)}}(t_{i+1}, R - s_k(1 + s_k)}{2}\right]$$

The potentials and strategies defined above are scaled versions of the integer time potential recursions defined in Equations (15,16) and the strategies defined in Equations (17,18). Specifically, the games operate on lattices that we will now describe.

The adversarial strategy Q_I defines the following lattice over i and R:

$$\boldsymbol{I}_T = \{(i, 2j-i) \,|0 \le i \le T, 0 \le j \le i\}$$

The k'th adversarial strategy $Q_{D(k)}$ uses step size $s_k = \sqrt{\mathcal{T}}2^{-k}$ and time increments $s_k^2 = \mathcal{T}2^{-2k}$. We define the *game lattice* for k as the set of (t, R) pairs that are reached by $Q_{D(k)}$.

$$\boldsymbol{K}_{\mathcal{T},k} = \left\{(t, R) \,\middle|\, t = is_k^2, 0 \le i \le 2^{2k}, R = (2j-i)s_k, 0 \le j \le i\right\}$$

$\boldsymbol{I}_T$ is a special case of $\boldsymbol{K}_{\mathcal{T},k}$ because setting $\mathcal{T} = T = 2^{2k}$ we get that $s_k = s_k^2 = 1$ and $\boldsymbol{K}_{\mathcal{T},k} = \boldsymbol{I}_T$.

It is not hard to show that the lattices get finer with k, i.e. if $j \le k$, $\boldsymbol{K}_{\mathcal{T},j} \subseteq \boldsymbol{K}_{\mathcal{T},k}$.

The following Lemma parallels Lemma 3 for the integer time game.

Lemma 6.
Let i be an integer between 1 and T
If $\phi^{\uparrow}_{Q_{D(k)}}(t_i, R) \in \mathcal{P}^2$

1. *$\phi^{\uparrow}_{Q_{D(k)}}(t_{i-1}, R) \in \mathcal{P}^2$*
2. *The adversarial strategy (35) guarantees the recursion given in Eq. (33)*

If $\phi^{\downarrow}_{P_{D(k)}}(t_i, R) \in \mathcal{P}^2$

1. *$\phi^{\downarrow}_{P_{D(k)}}(t_{i-1}, R) \in \mathcal{P}^2$*
2. *The learner strategy (36) guarantees the recursion given in Eq. (34)*

Proof. The statement of the Lemma and the proof are scaled versions of Lemma 3 and its proof. The iteration step is s_k^2 instead of 1 while the loss/gain of an action in a single step is $[-s_k, s_k]$ instead of $[-1, +1]$.

One change worth noting is at the step from Equation (21) and Equation (22), where the bound $y - \ell(i) \in [-2, 2]$ is replace by $y - \ell(i) \in [-s_k - s_k^2, s_k + s_k^2]$. This follows from the bound $|\ell(i)| \le s_k^2$ which is discussed in Section 6. □

Theorem 7. *Let $\boldsymbol{\phi}_{\mathcal{T}} \in \mathcal{P}^2$ be the final potential in the discrete time game. Fix k and the step size $s_k = \sqrt{\mathcal{T}}2^{-k}$, and let $t_i = is_k^2$ for $i = 0, 1, \ldots, 2^{2k}$ and let R_0 be a real value, then*

- *The lower potential guaranteed by $Q_{D(k)}$ is*

$$\phi^{\uparrow}_{Q_{D(k)}}(t_i, R_0) = \mathbf{E}_{R \sim R_0 \oplus \mathbb{B}(2^{2k}-i, s_k)}\left[\boldsymbol{\phi}_{\mathcal{T}}(R)\right] \tag{37}$$

- *The upper potential guaranteed by $P_{D(k)}$ is*

$$\phi^{\downarrow}_{P_{D(k)}}(t_i, R_0) = \mathbf{E}_{R \sim R_0 \oplus \mathbb{B}(2^{2k}-i, s_k(1+s_k))}\left[\boldsymbol{\phi}_{\mathcal{T}}(R)\right] \tag{38}$$

Using Theorem 7 we can show that, as $k \to \infty$, the upper lower potential converge to the same limit.

Theorem 8.
Fix $\mathcal{T}$ and assume $\boldsymbol{\phi}_{\mathcal{T}} \in \mathcal{P}^2$. Consider the sequence of upper and lower potentials $\phi^{\downarrow}_{P_{D(k)}}, \phi^{\uparrow}_{Q_{D(k)}}$ for $k = 0, 1, 2, \ldots$.

Then for any $0 < t \le \mathcal{T}$ and any R_0:

$$\lim_{k\to\infty} \phi^{\downarrow}_{P_{D(k)}}(t, R_0) = \lim_{k\to\infty} \phi^{\uparrow}_{Q_{D(k)}}(t, R_0) = \mathcal{N}(R_0, \mathcal{T} - t) \odot \boldsymbol{\phi}_{\mathcal{T}} \tag{39}$$

Proof. We first assume that $(t, R_0) \in \boldsymbol{K}_{j,\mathcal{T}}$ and that $k \geq j$. We later expand to any $0 < t \leq \mathcal{T}$ and any $R_0 \in \mathbb{R}$. Consider Equation 38 for $P_{D(k)}$ and $P_{D(j)}$, keeping t and j constant and letting $k \to \infty$.

$$\phi^{\downarrow}_{P_{D(j)}}(t, R_0) = \mathbf{E}_{R \sim R_0 \oplus \mathbb{B}(2^{2j} - i_j, s_j(1+s_j))} \left[\boldsymbol{\phi}_{\mathcal{T}}(R) \right] \tag{40}$$

$$\phi^{\downarrow}_{P_{D(k)}}(t, R_0) = \mathbf{E}_{R \sim R_0 \oplus \mathbb{B}(2^{2k} - i_k, s_k(1+s_k))} \left[\boldsymbol{\phi}_{\mathcal{T}}(R) \right] \tag{41}$$

We rewrite the binomial factor in Eq (41)

$$\mathbb{B}\big(2^{2k} - i_k, s_k(1+s_k)\big) = \mathbb{B}\Big(2^{2(k-j)}\big(2^{2j} - i_j\big), 2^{j-k} s_j (1 + 2^{j-k} s_j)\Big)$$

As j is constant, s_j is constant and so is $a_j \doteq 2^{2j} - i_j$. Multiplying the number of steps by the variance per step we get

$$Var(\mathbb{B}_k) = 2^{2(k-j)} a_j \big(2^{j-k} s_j (1 + (2^{j-k} s_j)\big)^2 = a_j s_j^2 (1 + (2^{j-k} s_j))^2$$

As s_j, a_j are constants we get that $\lim_{k\to\infty} Var(\mathbb{B}_k) = a_j s_j$. From the central limit theorem we get that for any $(t, R_0) \in \boldsymbol{K}_{j,\mathcal{T}}$

$$\lim_{k\to\infty} \phi^{\downarrow}_{P_{D(k)}}(t, R_0) \odot \boldsymbol{\phi}_{\mathcal{T}} = \mathcal{N}(R_0, \mathcal{T} - t) \odot \boldsymbol{\phi}_{\mathcal{T}} \tag{42}$$

Our argument hold for all $(t, R_0) \in \bigcup_{k=0}^{\infty} \boldsymbol{K}_{k,\mathcal{T}}$ which is dense in the set $0 < t \leq \mathcal{T}, R_0 \in \mathbb{R}$. On the other hand, $\phi^{\downarrow}_{P_{D(k)}}(t, R) \odot \boldsymbol{\phi}_{\mathcal{T}}$ is continuous in both t and R, therefor Equation (42) holds for all t and R.

As similar (slightly simpler) argsument holds for the lower potential limit $\lim_{k\to\infty} \phi^{\uparrow}_{Q_{D(k)}}(t, R_0)$

□

We have shown that in the limit $s \to 0$ the learner and the adversary converge to the same potential function. In the next section we show that this limit is the min/max solution by describing conditions under which the adversary prefers using ever smaller steps size.

6.2 The adversary prefers smaller steps

Theorem 42 characterizes the limits of the upper and lower potentials, as $k \to \infty$ are equal to each other and to $\mathcal{N}(R_0, \mathcal{T} - t) \odot \boldsymbol{\phi}_{\mathcal{T}}$ To show that this limit corresponds to the min/max slolution of the game we need to show that the adversary perfers smaller steps. In other words, that for any t, R, $\phi^{\uparrow}_{Q_{D(k)}}(t, R)$ increases with k.

To prove this claim we strengther the condition $\boldsymbol{\phi}_{\mathcal{T}} \in \mathcal{P}^2$ used above to $\boldsymbol{\phi}_{\mathcal{T}} \in^4$. In words, we assume that the function $\boldsymbol{\phi}_{\mathcal{T}}(R)$ is continuous and strictly positive and it's first four derivatives are continuous and strictly positive.

We use the sequence of discrete adversarial strategies $Q_{D(k)}, k = 1, 2, \ldots$ defined in Section 6.1.

Theorem 9. *If $\boldsymbol{\phi}_{\mathcal{T}} \in \mathcal{P}^4$, and $\mathcal{T} > 0$ then for any $k > 0$, any $t \in [0, \mathcal{T}]$ and any R*

$$\phi^{\uparrow}_{Q_{D(k+1)}}(t, R) > \phi^{\uparrow}_{Q_{D(k)}}(t, R)$$

The proof of the theorem relies on a reduction to a simpler case: dividing a single time step of duration τ into four time steps of duration $\tau/4$

Lemma 10. *If $\boldsymbol{\phi}_{\tau} \in \mathcal{P}^4$, and $\tau > 0$ then for any R*

$$\phi^{\uparrow}_{Q_{D(1)}}(0, R) > \phi^{\uparrow}_{Q_{D(0)}}(0, R)$$

Proof. The step size is $s_k = 2^{-k}\sqrt{\tau}$, therefor $s_0 = \sqrt{\tau}, s_1 = \frac{\sqrt{\tau}}{2}$. The time icrement is $\Delta t_k = s_k^2$, therefor $\Delta t_0 = \tau, \Delta t_1 = \frac{\tau}{4}$. In other words, $k = 0$ corresponds to a single step of size , while $k = 1$ corresponds to four steps of size $\frac{1}{2}$.

By definition $\boldsymbol{\phi}_\tau(R) = \phi^{\uparrow}_{Q_{D(0)}}(\tau, R) = \phi^{\uparrow}_{Q_{D(1)}}(\tau, R)$
For $k = 0$ we we get the recursion

$$\phi^{\uparrow}_{Q_{D(0)}}(0, R) = \frac{\phi^{\uparrow}_{Q_{D(0)}}(\tau, R - \sqrt{\tau}) + \phi^{\uparrow}_{Q_{D(0)}}(\tau, R + \sqrt{\tau})}{2} = \frac{\boldsymbol{\phi}_\tau(R - \sqrt{\tau}) + \boldsymbol{\phi}_\tau(R + \sqrt{\tau})}{2} \tag{43}$$

For $k = 1$ we we have for $i = 0, 1, 2, 3$:

$$\phi^{\uparrow}_{Q_{D(0)}}\left(\frac{i}{4}\tau, R\right) = \frac{\phi^{\uparrow}_{Q_{D(0)}}\left(\frac{i+1}{4}\tau, R - \frac{1}{2}\sqrt{\tau}\right) + \phi^{\uparrow}_{Q_{D(0)}}\left(\frac{i+1}{4}\tau, R + \frac{1}{2}\sqrt{\tau}\right)}{2} \tag{44}$$

Combining Equation (44) for $k = 0, 1, 2, 3$ we get

$$\begin{aligned} \phi^{\uparrow}_{Q_{D(1)}}(0, R) &= \frac{1}{16}\Big[\phi^{\uparrow}_{Q_{D(1)}}\big(\tau, R{-}2\sqrt{\tau}\big) + 4\phi^{\uparrow}_{Q_{D(1)}}\big(\tau, R{-}\sqrt{\tau}\big) \\ &\quad + 6\phi^{\uparrow}_{Q_{D(1)}}(\tau, R) + 4\phi^{\uparrow}_{Q_{D(1)}}\big(\tau, R{+}\sqrt{\tau}\big) + \phi^{\uparrow}_{Q_{D(1)}}\big(\tau, R{+}2\sqrt{\tau}\big)\Big] \\ &= \frac{1}{16}\left[\boldsymbol{\phi}_\tau\big(R{-}2\sqrt{\tau}\big) + 4\boldsymbol{\phi}_\tau\big(R{-}\sqrt{\tau}\big) + 6\boldsymbol{\phi}_\tau(R) + 4\boldsymbol{\phi}_\tau\big(R{+}\sqrt{\tau}\big) + \boldsymbol{\phi}_\tau\big(R{+}2\sqrt{\tau}\big)\right] \end{aligned} \tag{45}$$

the difference between Equations (45) and (43) is

$$\begin{aligned} &\phi^{\uparrow}_{Q_{D(1)}}(0, R) - \phi^{\uparrow}_{Q_{D(0)}}(0, R) \\ &= \frac{1}{16}\left[\boldsymbol{\phi}_\tau\big(R{-}2\sqrt{\tau}\big) - 4\boldsymbol{\phi}_\tau\big(R{-}\sqrt{\tau}\big) + 6\boldsymbol{\phi}_\tau(R) - 4\boldsymbol{\phi}_\tau\big(R{+}\sqrt{\tau}\big) + \boldsymbol{\phi}_\tau\big(R{+}2\sqrt{\tau}\big)\right] \end{aligned} \tag{46}$$

Our goal is to show that the LHS of Eqn. 46 is positive. This is equivalent to proving positivity of

$$\begin{aligned} g_a(R) &= \frac{2}{3a^2}\Big(\phi^{\uparrow}_{Q_{D(1)}}(0, R) - \phi^{\uparrow}_{Q_{D(0)}}(0, R)\Big) \\ &= \frac{1}{24a^4}\left[\boldsymbol{\phi}_\tau(R{-}2a) - 4\boldsymbol{\phi}_\tau(R{-}a) + 6\boldsymbol{\phi}_\tau(R) - 4\boldsymbol{\phi}_\tau(R{+}a) + \boldsymbol{\phi}_\tau(R{+}2a)\right] \end{aligned} \tag{47}$$

where $a = 2s_1 = \sqrt{\tau}$

The function $g_a(R)$ has a special form called "divided differences". The proof of the following lemma uses this fact to show that Eqn (47) is strictly postive.

Lemma 11. *If $\boldsymbol{\phi}_\tau \in \mathcal{P}^4$ and $\tau > 0$, then $\forall R, g_s(R) > 0$*

The proof of Lemma 11 is given in appendix B

Proof. of Theorem 9
The proof is by double induction over k and over t_i. For $k = 1, 2, \ldots$ we consider the the loss step $s = 2^{-k-1}\sqrt{\mathcal{T}}$ and the time step $\Delta t = s^2 = 2^{-2k-2}\mathcal{T}$. For each game iteration $i = 0, \ldots, 2^{2k} - 1$ we fix the potential at time $t_1 = (i{+}1)2^{-2k}\mathcal{T}$ and We consider the difference between the potential at $t_0 = (i{+}1)2^{-2k}\mathcal{T}$

we take a finite backward induction over $t = T - 2^{-2k}, T - 2 \times 2^{-2k}, T - 3 \times 2^{-2k}, \cdots, 0$. Our inductive claims are that $\phi_{k+1}(t, R) > \phi_k(t, R)$ and $\phi_{k+1}(t, R)$,$\phi_k(t, R)$ are continuous, strongly convex and have a strongly positive fourth derivative. That these claims carry over from $t = T{-}i{\times}2^{-2k}$ to $t = T{-}(i{+}1){\times}2^{-2k}$ follows directly from Lemma **??**.

The theorem follows by forward induction on k.

□

Theorem 8 characterizes the limit

$$\lim_{k\to\infty} \phi^{\downarrow}_{P_{D(k)}}(t, R_0) = \lim_{k\to\infty} \phi^{\uparrow}_{Q_{D(k)}}(t, R_0) = \mathcal{N}(R_0, \mathcal{T} - t) \odot \boldsymbol{\phi}_{\mathcal{T}} \tag{48}$$

Theorem 9 states that increasing k is always advantageous to the adversary.
Together these theorems show that the the min/max optimal potential function is $\mathcal{N}(R_0, \mathcal{T} - t) \odot \boldsymbol{\phi}_{\mathcal{T}}$.

7 Brownian motion

There seems to be a paradox: the adversary prefers to set $s_i > 0$ as small as possible. On the other hand, there is no minimal strictly positive number, so whatever the adversary chooses has to be suboptimal. In other words, time is not continuous, it increases in discrete steps. As that is the case, why is brownian motion still the correct way to compute the potential?

One can use the following argument: the learner knows the range $[-s_i, +s_i]$ for the next instantanous losses before it has to choose the weights he places on the actions. On the other hand, it does not know the range of the following losses, but he knows that the adversary always prefers small ranges. The safe thing for the learner to do is to assume that the following steps will be infinitesmaly small, i.e. that the future losses form a brownian process

It is well known that the limit of random walks where $s \to 0$ and $\Delta t = s^2$ is the the Brownian or Wiener process (see [13]).

An alternative characterization of Brownian Process is

$$\mathbf{P}\left[X_{t+\Delta t} = x_1 | X_t = x_0\right] = e^{-\frac{(x_1 - x_0)^2}{2\Delta t}}$$

The backwards recursion that defines the value function is the celebrated Backwards Kolmogorov Equation with no drift and unit variance

$$\frac{\partial}{\partial t}\phi(t, R) + \frac{1}{2}\frac{\partial^2}{\partial R^2}\phi(t, R) = 0 \tag{49}$$

Given a final value function with a strictly positive fourth derivative we can use Equation (49) to compute the value function for all $0 \leq t \leq T$. We will do so in he next section.

8 The continuous time game and bounds for easy sequences

In Section 6 we have shown that the integer time game has a natural extension to a setting where $\Delta t_i = s_i^2$. We also demonstrated sequences of adversarial strategies $S_1, S_2, \ldots$ such that $\sup_{k\to\infty} \phi^{\uparrow}_{Q_k}(0, R) =$

We characterized the optimal adversarial strategy for the discrete time game (Section **??**), which corresponds to the adversary choosing the loss to be s_i or $-s_i$ with equal probabilities. A natural question at this point is to characterize the regret when the adversary is not optimal, or the sequences are "easy".

To see that such an improvement is possible, consider the following *constant* adversary. This adversary associates the same loss to all actions on iteration i, formally, $Q(i, R) = l$. In this case the average loss is also equal to l, $\ell(i) = l$ which means that all of the instantaneous regrets are $r = l - \ell(t_i) = 0$, which, in turn, implies that $\mathbf{\Psi}(i) = \mathbf{\Psi}(i+1)$. As the state did not change, it makes sense to set $t_{i+1} = t_i$, rather than $t_{i+1} = t_i + s_i^2$.

We observe two extremes for the adversarial behavior. The constant adversary described above for which $t_{i+1} = t_i$, and the random walk adversary described earlier, in which each action is split into two, one half with loss $-s_i$ and the other with loss $+s_i$. In which case $t_{i+1} = t_i + s_i^2$ which is the maximal increase in t that the adversary can guarantee. The analysis below shows that these are two extremes on a spectrum and that intermediate cases can be characterized using a variance-like quantity.

We define a variant of the discrete time game (**??**) For concreteness we include the learner's strategy, which is the limit of the strategy in the discrete game when $s_i \to 0$.

Our characterization applies to the limit where the s_i are small. Formally, we define

Definition 2. *We say that an instance of the discrete time game is (n, s, τ)-bounded if it consists of n iterations and $\forall\ 0 < i \leq n,\ \ s_i < s$ and $\sum_{j=1}^{n} s_j^2 = \tau$*

Note that $\tau > t_n$ and that τ depends only on the ranges s_i while t_n depends on the variance. $t_n = T$ is the dominant term in the regret bound, while τ controls the error term.

Set $t_1 = 0$
Fix maximal step $0 < s < 1$
On iteration $i = 1, 2, \ldots$

1. If $t_i = T$ the game terminates.
2. Given t_i, the learner chooses a distribution $P(i)$ over $\mathbb{R}$:
$$P^{cc}(t,R) = \frac{1}{Z^{cc}} \left.\frac{\partial}{\partial r}\right|_{r=R} \phi(t,r) \text{ where } Z^{cc} = \mathbf{E}_{R\sim\mathbf{\Psi}(t_i)} \left[\left.\frac{\partial}{\partial r}\right|_{r=R} \phi(t,r) \right] \tag{50}$$
3. The adversary chooses a *step size* $0 < s_i \leq s$ and a mapping from $\mathbb{R}$ to distributions over $[-s_i, +s_i]$: $Q(t) : \mathbb{R} \to \Delta^{[-s_i,+s_i]}$
4. The aggregate loss is calculated:
$$\ell(t_i) = \mathbf{E}_{R\sim\mathbf{\Psi}(t_i)} \left[P^{cc}(t_i,R) B(t_i,R) \right], \text{ where } B(t_i,R) \doteq \mathbf{E}_{y\sim Q(t_i,R)} [y] \tag{51}$$
the aggregate loss is restricted to $|\ell(t_i)| \leq c s_i^2$.
5. Increment $t_{i+1} = t_i + \Delta t_i$ where
$$\Delta t_i = \mathbf{E}_{R\sim\mathbf{\Psi}(t_i)} \left[H(t_i,R) \ \ \mathbf{E}_{y\sim Q(t_i,R)} \left[(y - \ell(t_i))^2 \right] \right] \tag{52}$$
Where
$$H(t_i,R) = \frac{1}{Z^H} \left.\frac{\partial^2}{\partial r^2}\right|_{r=R} \phi(t_i,r) \text{ and } Z^H = \mathbf{E}_{R\sim\mathbf{\Psi}(t_i)} \left[\left.\frac{\partial^2}{\partial r^2}\right|_{r=R} \phi(t_i,r) \right] \tag{53}$$
6. The state is updated.
$$\mathbf{\Psi}(t_{i+1}) = \mathbf{E}_{R\sim\mathbf{\Psi}(t_i)} \left[Q(t_i)(R) \oplus (R - \ell(t_i)) \right]$$

Figure 4: The continuous time game and learner strategy

Theorem 12. *Let $\phi \in \mathcal{P}^\infty$ be a potential function that satisfies the Kolmogorov backward equation (49). Fix the total time τ and let G_n be an $(n, \sqrt{\frac{\tau}{n}}, \tau)$-bounded game. Let $n \to \infty$.*

Then
$$\Phi(\mathbf{\Psi}(\tau)) \leq \Phi(\mathbf{\Psi}(0)) + O\left(\frac{1}{\sqrt{n}}\right)$$

The proof is given in appendix C
If we define
$$V_n = t_n = \sum_{i=1}^{n} \Delta t_i = \sum_{i=1}^{n} \mathbf{E}_{R\sim\mathbf{\Psi}(i)} \left[\mathbf{E}_{y\sim Q(t_i,R)} \left[H(t_i,R)((y-\ell_i)^2) \right] \right] \tag{59}$$
We can use V_n instead of T giving us a variance based bound.

9 Anytime potential functions

The results up to this point hold for any potential function in $\mathcal{P}^4$. Given a final potential function $\boldsymbol{\phi}_\mathcal{T} \in \mathcal{P}^4$ we can compute the potential for any $0 \leq t \leq \mathcal{T}$ and any R using the equation
$$\phi(t,R) = \mathcal{N}(R_0, \mathcal{T} - t) \odot \boldsymbol{\phi}_\mathcal{T} \tag{60}$$

Set $V_0 = 1$
For $i = 0, 1, 2, \ldots$

1. The learner chooses a distribution

$$P^{NH2}(V_i, R) = \frac{1}{Z} R e^{\frac{R^2}{2V_i}} \text{ where } Z = \mathbf{E}_{R \sim \mathbf{\Psi}_i} \left[R e^{\frac{R^2}{2V_i}} \right] \tag{54}$$

2. The aggregate loss is calculated:

$$\ell_i = \mathbf{E}_{R \sim \mathbf{\Psi}(i)} \left[P^{NH2}(V_i, R) B(V_i, R) \right], \quad \text{where } B(V_i, R) \doteq \mathbf{E}_{y \sim Q(V_i, R)} [y] \tag{55}$$

3. Increment $V_{i+1} = V_i + Var_i$ where

$$Var_i = \mathbf{E}_{R \sim \mathbf{\Psi}(i)} \left[H(t_i, R) \ \mathbf{E}_{y \sim Q(i,R)} \left[(y - \ell_i)^2 \right] \right] \tag{56}$$

$$\text{where } W(t_i, R) = e^{\frac{(R+1)^2}{2t_i}} \left(\frac{1}{t^{3/2}} + \frac{(R+1)^2}{t^{5/2}} \right) \tag{57}$$

$$H(t_i, R) = \frac{W(t_i, R)}{\mathbf{E}_{\rho \sim \mathbf{\Psi}(i)} [W(t_i, \rho)]} \tag{58}$$

4. The state is updated.

$$\mathbf{\Psi}(i+1) = \mathbf{E}_{R \sim \mathbf{\Psi}(i)} [Q(i, R) \oplus (R - \ell_i)]$$

Figure 5: NormalHedge.2

By using uses the R-derivative of this potential function to define the weights the learner guarantees that the final average score is at most $\phi(0, 0)$.

A major limition of this result is that the horizon $\mathcal{T}$ is set in advance. It is desirable that the potential is defined without knowledge of the horizon. In what follows we show that Hedge and NormalHedge can both be used in such "anytime" algorithms.

Our solution is based on the observation that a potential function satisfies Eqn (60) if and only if it satisfies the Kolmogorov backwards PDE (49):

$$\frac{\partial}{\partial t} \phi(t, R) + \frac{1}{2} \frac{\partial^2}{\partial r^2} \phi(t, R) = 0 \tag{61}$$

The potential $\boldsymbol{\phi}_{\mathcal{T}} \in \mathcal{P}^4$ defines a boundary condition of the PDE.

We derive our anytime algorithm by finding solutions to the Kolmogorov PDE that are not restricted in time, and that have a fixed parametric form. In other words, the evolution of the potential with time is defined by changing the parameter values, without changing the form.

We describe two potential functions that are solutions of PDE. In the following section we use our general results to prove simultanous regret bounds

The exponential potential function which corresponds to exponential weights algorithm corresponds to the following equation

$$\phi_{\exp}(R, t) = e^{\sqrt{2}\eta R - \eta^2 t} \tag{62}$$

Equation 62.

For the standard (non simultanous) bound we fix ϵ and t, choose $\eta = \sqrt{\frac{\ln(1/\epsilon)}{t}}$ and get a bound of the form

$$R_\epsilon \leq \sqrt{2t \ln \frac{1}{\epsilon}} \tag{63}$$

To derive a simultanous regret bound we fix the learning rate η and take the reciprocal of the potential:

$$G(R,t) \leq e^{\eta^2 t - \sqrt{2}\eta R}$$

Which holds for any t and R. The bound $G(R,t)$ depends on η.

The NormalHedge potential function parametrized by $\nu > 0$ is:

$$\phi_{\mathrm{NH}(\nu)}(R,t) = \begin{cases} \frac{1}{\sqrt{t+\nu}} \exp\left(\frac{R^2}{2(t+\nu)}\right) & \text{if } R \geq 0 \\ \frac{1}{\sqrt{t+\nu}} & \text{if } R < 0 \end{cases} \tag{64}$$

The function $\phi_{\mathrm{NH}}(R,t)$ is not in $\mathcal{P}^4$, however, the positive part $R \geq 0$ is in $\mathcal{P}^4$ while the negative part $R \leq 0$ is a constant. A constant potential corresponds to zero weight which means that actions whose regret is negative are ignored by the learner. In this case the optimal adversarial is not unconstrained brownian motion, instead it is brownian motion with a reflective boundary at $R = 0$.

10 Upper and lower bounds on the simultanous regret

Plan: (1) first order bound with Variable ν to show optimality. (2) Second order bound with $\nu = 1$

We now combine Theorems **??** and 12 with the Normal-Hedge potential (64) to derive a second order bound on the regret of NormalHedge.

$$R_{\mathrm{NH}(\nu)}(\epsilon) \leq \sqrt{(t_i + \nu)\left(2\ln\frac{1}{2\epsilon} + \ln(t_i + \nu)\right)} \tag{65}$$

Where $t_i = \nu + \sum_{j=1}^{i} \Delta t_j$

$$\Delta t_i = \mathbf{E}_{R\sim\mathbf{\Psi}(i)}\left[H(t_i,R)\ \ \mathbf{E}_{y\sim Q(t_i,R)}\left[(y-\ell_i)^2\right]\right] \tag{66}$$

and

$$W(t_i,R) = e^{\frac{R^2}{2t_i}}\left(\frac{1}{t^{3/2}} + \frac{R^2}{t^{5/2}}\right) \tag{67}$$

$$H(t_i,R) = \frac{W(t_i,R)}{\mathbf{E}_{\rho\sim\mathbf{\Psi}(i)}\left[W(t_i,\rho)\right]} \tag{68}$$

The initial potential is $1/\sqrt{\nu}$ and it remains this way in the continuous case. In the discrete case it is $1/\nu + O(1/n)$ where n is the number of steps We can upper bound the potential by $1/\nu + 1$

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A Proof of Theorem ??

Proof.

- **A distribution $\mathbf{\Psi}$ satisfies SRB for G if it satisfies APB for $\phi(R) = G(R)^{-1}$**
  Assume by contradiction that $\mathbf{\Psi}$ does not satisfy the simultaneous bound. In other words there exists $a \in \mathbb{R}$ such that $\mathbf{P}_{R\sim\mathbf{\Psi}}[R > a] > B(a)$. From Markov inequality and the fact that ϕ is non decreasing we get
  $$\mathbf{E}_{R\sim\mathbf{\Psi}}[\phi(R)] \geq \phi(a)\mathbf{P}_{R\sim\mathbf{\Psi}}[R > a] > \phi(a)B(a) = \frac{B(a)}{B(a)} = 1$$
  but $\mathbf{E}_{R\sim\mathbf{\Psi}}[\phi(R)] > 1$ contradicts the average potential assumption for the potential $\phi(R) = B(R)^{-1}$

- **A distribution Ψ satisfies an APB for ϕ if it satisfies SRB for G and $\int_{-\infty}^{\infty}\phi(R)G(R)dR \leq 1$**
  From the SRB condition we have

$$\forall R, \mathbf{P}_{\rho\sim\mathbf{\Psi}}\left[\rho \geq R\right] \leq G(R)$$

  From the condition $\int_{-\infty}^{\infty}\phi(R)G(R)dR \leq 1$ we get

$$1 \geq \int_{-\infty}^{\infty}\phi(R)G(R)dR \geq \int_{-\infty}^{\infty}\phi(R)\mathbf{P}_{\rho\sim\mathbf{\Psi}}\left[\rho \geq R\right]dR = \mathbf{E}_{R\sim\mathbf{\Psi}}\left[\phi(R)\right]$$

□

B Divided differences of a function

The function $g_s(R)$ has a special form called "divided difference" that has been extensively studied [17, 3, 8]. and is closely related to to derivatives of different orders. Using this connection and the fact that $\phi(\cdot, R) \in \mathcal{P}^4$ we prove the following lemma:

We conclude that if $\phi(t', R)$ has a strictly positive fourth derivative then $\phi_{k+1}(t, R) > \phi_k(t, R)$ for all R, proving the first part of the lemma.

The second part of the lemma follows from the fact that both $\phi_{k+1}(t, R)$ and $\phi_k(t, R)$ are convex combinations of $\phi(t, R)$ and therefor retain their continuity and convexity properties.

A function $\boldsymbol{\phi}$ that satisfies inequality **??** is said to be *4'th order convex* (see details in in [3]).

Following[3] we give a brief review of divided differences and of n-convexity.

Let $f : [a, b] \to \mathbb{R}$ be a function from the segment $[a, b]$ to the reals.

Definition 3 (n'th order divided difference of a function)**.** *The n'th order divided different of a function $f : [a, b] \to \mathbb{R}$ at mutually distinct and ordered points $a \leq x_0 < x_1 < \cdots < x_n \leq b$ defined recursively by*

$$[x_i; f] = f(x_i),\ i \in 0, \ldots n,$$

$$[x_0, \ldots, x_n; f] = \frac{[x_1, \ldots, x_n; f] - [x_0, \ldots, x_{n-1}; f]}{x_n - x_0}$$

Definition 4 (n-convexity)**.** *A function $f : [a, b] \to \mathbb{R}$ is said to be n-convex $n \geq 0$ if and only if for all choices of $n + 1$ distinct points: $a \leq x_0 < x_1 < \cdots < x_n \leq b$, $[x_0, \ldots, x_n; f] \geq 0$ holds.*

n-convexity is has a close connection to the sign of $f^{(n)}$ - the n'th derivative of f, this connection was proved in 1965 by popoviciu [17].

Theorem 13. *If $f^{(n)}$ exists then f is n-convex if and only if $f^{(n)} \geq 0$.*

The next lemma states that the function $g(R) > 0$ as defined in Equation (**??**).

Proof. **of Lemma (11)**

Fix t and define $f(x) = \phi(t, x)$. Let $(x_0, x_1, x_2, x_3, x_4) = (R - 2s, R - s, R, R + s, R + 2s)$

Using this notation we can rewrite $g(R)$ in the form

$$h(x_0, x_1, x_2, x_3, x_4) = \frac{1}{24s^4}(f(x_4) - 4f(x_3) + 6f(x_2) - 4f(x_1) + f(x_0)) \tag{69}$$

Is the 4-th order divided difference of $\phi(t, \cdot)$

1.

$$[x_i; f] = f(x_i)$$

2.

$$[x_i, x_{i+1}; f] = \frac{f(x_{i+1}) - f(x_i)}{s}$$

3.

$$[x_i, x_{i+1}, x_{i+2}; f] = \frac{\frac{f(x_{i+2})-f(x_{i+1})}{s} - \frac{f(x_{i+1})-f(x_i)}{s}}{2s} = \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{2s^2}$$

4.

$$\begin{aligned}[x_i, x_{i+1}, x_{i+2}, x_{i+3}; f] &= \frac{\frac{f(x_{i+3})-2f(x_{i+2})+f(x_{i+1})}{2s^2} - \frac{f(x_{i+2})-2f(x_{i+1})+f(x_i)}{2s^2}}{3s} \\ &= \frac{f(x_{i+3}) - 3f(x_{i+2}) + 3f(x_{i+1}) - f(x_i)}{6s^3}\end{aligned}$$

5.

$$\begin{aligned}[x_i, x_{i+1}, x_{i+2}, x_{i+3}, x_{i+4}; f] &= \frac{\frac{f(x_{i+4})-3f(x_{i+3})+3f(x_{i+2})-f(x_{i+1})}{6s^3} - \frac{f(x_{i+3})-3f(x_{i+2})+3f(x_{i+1})-f(x_i)}{6s^3}}{4s} \\ &= \frac{f(x_{i+4}) - 4f(x_{i+3}) + 6f(x_{i+2}) - 4f(x_{i+1}) + f(x_i)}{24s^4}\end{aligned}$$

□

C Proof of Theorem 12

We start with two technical lemmas

Lemma 14. *Let $f(x) \in \mathcal{P}^2$, i.e. $f(x), f'(x), f''(x) > 0$ for all $x \in \mathbb{R}$, let $h(x)$ be a uniformly bounded function: $\forall x,\ |h(x)| < 1$. Let $\mathbf{\Psi}$ be a distribution over $\mathbb{R}$. If $\mathbf{E}_{x\sim\mathbf{\Psi}}[f(x)]$ is well-defined (and finite) , then $\mathbf{E}_{x\sim\mathbf{\Psi}}[h(x)f'(x)]$ is well defined (and finite) as well.*

Proof. Assume by contradiction that $\mathbf{E}_{x\sim\mathbf{\Psi}}[h(x)f'(x)]$ is undefined. Define $h^+(x) = \max(0, h(x))$. As $f'(x) > 0$, this implies that either $\mathbf{E}_{x\sim\mathbf{\Psi}}[h^+(x)f'(x)] = \infty$ or $\mathbf{E}_{x\sim\mathbf{\Psi}}[(-h)^+(x)f'(x)] = \infty$ (or both).

Assume wlog that $\mathbf{E}_{x\sim\mathbf{\Psi}}[h^+(x)f'(x)] = \infty$. As $f'(x) > 0$ and $0 \le h^+(x) \le 1$ we get that $\mathbf{E}_{x\sim\mathbf{\Psi}}[f'(x)] = \infty$. As $f(x+1) \ge f'(x)$ we get that $\mathbf{E}_{x\sim\mathbf{\Psi}}[f(x)] = \infty$ which is a contradiction. □

Lemma 15. *Let $f(x, y)$ be a differentiable function with continuous derivatives up to degree three. Then*

$$f(x_0 + \Delta x, y_0 + \Delta y) = f(x_0, y_0) + \left\{ \frac{\partial}{\partial x}\Big|_{\substack{x,\ y = \\ x_0,\ y_0}} f(x, y) \right\} \Delta x + \left\{ \frac{\partial}{\partial y}\Big|_{\substack{x,\ y = \\ x_0,\ y_0}} f(x, y) \right\} \Delta y \tag{70}$$

$$+ \quad \frac{1}{2}\left\{ \frac{\partial^2}{\partial x^2}\Big|_{\substack{x,\ y = \\ x_0,\ y_0}} f(x, y) \right\} \Delta x^2 + \left\{ \frac{\partial^2}{\partial x \partial y}\Big|_{\substack{x,\ y = \\ x_0,\ y_0}} f(x, y) \right\} \Delta x \Delta y + \frac{1}{2}\left\{ \frac{\partial^2}{\partial y^2}\Big|_{\substack{x,\ y = \\ x_0,\ y_0}} f(x, y) \right\} \Delta y^2 \tag{71}$$

$$+ \quad \frac{1}{6}\left\{ \frac{\partial^3}{\partial x^3}\Big|_{\substack{x,\ y = \\ x_0 + t\Delta x,\ y_0 + t\Delta y}} f(x, y) \right\} \Delta x^3 + \frac{1}{2}\left\{ \frac{\partial^3}{\partial x^2 \partial y}\Big|_{\substack{x,\ y = \\ x_0 + t\Delta x,\ y_0 + t\Delta y}} f(x, y) \right\} \Delta x^2 \Delta y \tag{72}$$

$$+ \frac{1}{2}\left\{ \frac{\partial^3}{\partial x \partial y^2}\Big|_{\substack{x,\ y = \\ x_0 + t\Delta x,\ y_0 + t\Delta y}} f(x, y) \right\} \Delta x \Delta y^2 + \frac{1}{6}\left\{ \frac{\partial^3}{\partial y^3}\Big|_{\substack{x,\ y = \\ x_0 + t\Delta x,\ y_0 + t\Delta y}} f(x, y) \right\} \Delta y^3 \tag{73}$$

for some $0 \le t \le 1$.

Proof. of Lemma 15 Let $F : [0,1] \to \mathbb{R}$ be defined as $F(t) = f(x(t), y(t))$ where $x(t) = x_0 + t\Delta x$ and $y(t) = y_0 + t\Delta y$. Then $F(0) = f(x_0, y_0)$ and $F(1) = f(x_0 + \Delta x, y_0 + \Delta y)$. It is easy to verify that

$$\frac{d}{dt}F(t) = \frac{\partial}{\partial x}f(x(t), y(t))\Delta x + \frac{\partial}{\partial y}f(x(t), y(t))\Delta y$$

and that in general:

$$\frac{d^n}{dt^n}F(t) = \sum_{m=1}^{n} \binom{n}{m} \frac{\partial^n}{\partial x^m \partial y^{n-m}} f(x_0 + t\Delta x, y_0 + t\Delta y)\Delta x^m \Delta y^{n-m} \tag{74}$$

As f has partial derivatives up to degree 3, so does F. Using the Taylor expansion of F and the intermediate point theorem we get that

$$f(x_0 + \Delta x, y_0 + \Delta y) = F(1) = F(0) + \frac{d}{dt}F(0) + \frac{1}{2}\frac{d^2}{dt^2}F(0) + \frac{1}{6}\frac{d^3}{dt^3}F(t') \tag{75}$$

Where $0 \leq t' \leq 1$. Using Eq (74) to expand each term in Eq. (75) completes the proof. □

Proof. of Theorem 12
We prove the claim by an upper bound on the increase of potential that holds for any iteration $1 \leq i \leq n$:

$$\Phi(\mathbf{\Psi}(t_{i+1})) \leq \Phi(\mathbf{\Psi}(i)) + as_i^3 \text{ for some constant } a > 0 \tag{76}$$

Summing inequality (76) over all iterations we get that

$$\Phi(\mathbf{\Psi}(T)) \leq \Phi(\mathbf{\Psi}(0)) + c\sum_{i=1}^{n} s_i^3 \leq \Phi(\mathbf{\Psi}(0)) + as\sum_{i=1}^{n} s_i^2 = \Phi(\mathbf{\Psi}(0)) + asT \tag{77}$$

From which the statement of the theorem follows.

We now prove inequality (76). We use the notation $r = y - \ell(i)$ to denote the instantaneous regret at iteration i.

Applying Lemma 15 to $\phi(t_{i+1}, R_{i+1}) = \phi(t_i + \Delta t_i, R_i + r_i)$ we get

$$\begin{aligned}
\phi(t_i + \Delta t_i, R_i + r_i) &= \phi(t_i, R_i) && (78)\\
&+ \left\{ \frac{\partial}{\partial \rho}\Big|_{\substack{\tau,\ \rho = \\ t_i,\ R}} \phi(\tau, \rho) \right\} r_i && (79)\\
&+ \left\{ \frac{\partial}{\partial \tau}\Big|_{\substack{\tau,\ \rho = \\ t_i,\ R}} \phi(\tau, \rho) \right\} \Delta t_i && (80)\\
&+ \frac{1}{2} \left\{ \frac{\partial^2}{\partial \rho^2}\Big|_{\substack{\tau,\ \rho = \\ t_i,\ R}} \phi(\tau, \rho) \right\} r_i^2 && (81)\\
&+ \left\{ \frac{\partial^2}{\partial r \partial \tau}\Big|_{\substack{\tau,\ \rho = \\ t_i,\ R}} \phi(\tau, \rho) \right\} r_i \Delta t_i && (82)\\
&+ \frac{1}{2} \left\{ \frac{\partial^2}{\partial \tau^2}\Big|_{\substack{\tau,\ \rho = \\ t_i,\ R}} \phi(\tau, \rho) \right\} \Delta t_i^2 && (83)\\
&+ \frac{1}{6} \left\{ \frac{\partial^3}{\partial \rho^3}\Big|_{\substack{\tau,\ \rho = \\ t_i + g\Delta t_i,\ R_i + g r_i}} \phi(\tau, \rho) \right\} r_i^3 && (84)\\
&+ \frac{1}{2} \left\{ \frac{\partial^3}{\partial \rho^2 \partial \tau}\Big|_{\substack{\tau,\ \rho = \\ t_i + g\Delta t_i,\ R_i + g r_i}} \phi(\tau, \rho) \right\} r_i^2 \Delta t_i && (85)\\
&+ \frac{1}{2} \left\{ \frac{\partial^3}{\partial \rho \partial \tau^2}\Big|_{\substack{\tau,\ \rho = \\ t_i + g\Delta t_i,\ R_i + g r_i}} \phi(\tau, \rho) \right\} r_i \Delta t_i^2 && (86)\\
&+ \frac{1}{6} \left\{ \frac{\partial^3}{\partial \tau^3}\Big|_{\substack{\tau,\ \rho = \\ t_i + g\Delta t_i,\ R_i + g r_i}} \phi(\tau, \rho) \right\} \Delta t_i^3 && (87)
\end{aligned}$$

for some $0 \leq g \leq 1$.

By assumption ϕ satisfies the Kolmogorov backward equation:

$$\frac{\partial}{\partial \tau}\phi(\tau, \rho) = -\frac{1}{2}\frac{\partial^2}{\partial r^2}\phi(\tau, \rho)$$

Combining this equation with the exchangeability of the order of partial derivative (Clairiaut's Theorem) we can substitute all partial derivatives with respect to τ with partial derivatives with respect to ρ using the following equation.

$$\frac{\partial^{n+m}}{\partial \rho^n \partial \tau^m}\phi(\tau, \rho) = (-1)^m \frac{\partial^{n+2m}}{\partial \rho^{n+2m}}\phi(\tau, \rho)$$

Which yields

$$\begin{aligned}
\phi(t_i + \Delta t_i, R_i + r_i) &= \phi(t_i, R_i) && (88)\\
&+ \left\{ \frac{\partial}{\partial \rho}\Big|_{\substack{\tau,\,\rho = \\ t_i,\,R}} \phi(\tau,\rho) \right\} r_i && (89)\\
&+ \left\{ \frac{\partial^2}{\partial \rho^2}\Big|_{\substack{\tau,\,\rho = \\ t_i,\,R}} \phi(\tau,\rho) \right\} \left(\frac{r_i^2}{2} - \Delta t_i \right) && (90)\\
&- \left\{ \frac{\partial^3}{\partial \rho^3}\Big|_{\substack{\tau,\,\rho = \\ t_i,\,R}} \phi(\tau,\rho) \right\} r_i \Delta t_i && (91)\\
&+ \frac{1}{2} \left\{ \frac{\partial^4}{\partial \rho^4}\Big|_{\substack{\tau,\,\rho = \\ t_i,\,R}} \phi(\tau,\rho) \right\} \Delta t_i^2 && (92)\\
&+ \frac{1}{6} \left\{ \frac{\partial^3}{\partial \rho^3}\Big|_{\substack{\tau,\,\rho = \\ t_i + g\Delta t_i,\,R_i + g r_i}} \phi(\tau,\rho) \right\} r_i^3 && (93)\\
&- \frac{1}{2} \left\{ \frac{\partial^4}{\partial \rho^4}\Big|_{\substack{\tau,\,\rho = \\ t_i + g\Delta t_i,\,R_i + g r_i}} \phi(\tau,\rho) \right\} r_i^2 \Delta t_i && (94)\\
&+ \frac{1}{2} \left\{ \frac{\partial^5}{\partial \rho^5}\Big|_{\substack{\tau,\,\rho = \\ t_i + g\Delta t_i,\,R_i + g r_i}} \phi(\tau,\rho) \right\} r_i \Delta t_i^2 && (95)\\
&- \frac{1}{6} \left\{ \frac{\partial^6}{\partial \rho^6}\Big|_{\substack{\tau,\,\rho = \\ t_i + g\Delta t_i,\,R_i + g r_i}} \phi(\tau,\rho) \right\} \Delta t_i^3 && (96)
\end{aligned}$$

From the assumption that the game is (n, s, T)-bounded we get that

1. $|r_i| \leq s_i + c s_i^2 \leq 2 s_i$
2. $\Delta t_i \leq s_i^2 \leq s^2$

given these inequalities we can rewrite the second factor in each term as follows, where $|h_i(\cdot)| \leq 1$

- **For (89):** $r_i = 2 s_i \frac{r_i}{2 s_i} = 2 s_i h_1(r_i)$.
- **For (90):** $r_i^2 - \frac{1}{2}\Delta t_i = 4 s_i^2 \frac{r_i^2 - \frac{1}{2}\Delta t_i}{4 s_i^2} = 4 s_i^2 h_2(r_i, \Delta t_i)$
- **For (91):** $r_i \Delta t_i = 2 s_i^3 \frac{r_i \Delta t_i}{2 s_i^3} = 2 s_i^3 h_3(r_i, \Delta t_i)$
- **For (92):** $\Delta t_i^2 = s_i^4 \frac{\Delta t_i^2}{s_i^4} = s_i^3 h_4(\Delta t_i)$
- **For (93):** $r_i^3 = 8 s_i^3 \frac{r_i^3}{8 s_i^3} = 8 s_i^3 h_5(r_i, \Delta t_i)$
- **For (94):** $r_i^2 \Delta t_i = 4 s_i^4 \frac{r_i^2 \Delta t_i}{4 s_i^4} = 4 s_i^3 h_6(r_i, \Delta t_i)$
- **For (95):** $r_i \Delta t_i^2 = 2 s_i^5 \frac{r_i \Delta t_i^2}{2 s_i^5}$

- **For (96):** $\Delta t_i^3 = s_i^6 \frac{\Delta t_i^3}{s_i^6}$

We therefor get the simplified equation

$$\begin{aligned}
\phi(t_i + \Delta t_i, R_i + r_i) &= \phi(t, R) + \left\{ \frac{\partial}{\partial r}\Big|_{\substack{\tau,\,\rho = \\ t_i,\,R}} \phi(\tau, \rho) \right\} r + \left\{ \frac{\partial}{\partial t}\Big|_{\substack{\tau,\,\rho = \\ t_i,\,R}} \phi(\tau, \rho) \right\} \Delta t \\
&+ \frac{1}{2} \left\{ \frac{\partial^2}{\partial r^2}\Big|_{\substack{\tau,\,\rho = \\ t_i,\,R}} \phi(\tau, \rho) \right\} r^2 \\
&+ \left\{ \frac{\partial^2}{\partial r \partial t}\Big|_{\substack{\tau,\,\rho = \\ t_i,\,R}} \phi(\tau, \rho) \right\} r_i \Delta t_i \\
&+ \frac{1}{6} \left\{ \frac{\partial^3}{\partial r^3}\Big|_{\substack{\tau,\,\rho = \\ t_i,\,R}} \phi(\tau, \rho) \right\} r_i^3 + O(s^4)
\end{aligned}$$

and therefor

$$\begin{aligned}
\phi(t_i + \Delta t_i, R + r) &= \phi(t_i, R) + \left\{ \frac{\partial}{\partial r}\Big|_{\substack{\tau,\,\rho = \\ t_i,\,R}} \phi(\tau, \rho) \right\} r \\
&+ \left\{ \frac{\partial^2}{\partial r^2}\Big|_{\substack{\tau,\,\rho = \\ t_i,\,R}} \phi(\tau, \rho) \right\} (r^2 - \Delta t_i) + O(s^3) \qquad (97)
\end{aligned}$$

Our next step is to consider the expected value of (97) wrt $R \sim \mathbf{\Psi}(i)$, $y \sim Q(t_i, R)$ for an arbitrary adversarial strategy Q.

We will show that the expected potential does not increase:

$$\mathbf{E}_{R \sim \mathbf{\Psi}(i)} \left[\mathbf{E}_{y \sim Q(t_i, R)} \left[\phi(t_i + \Delta t_i, R + y - \ell_i) \right] \right] \leq \mathbf{E}_{R \sim \mathbf{\Psi}(t_i)} \left[\phi(t_i, R) \right] \qquad (98)$$

Plugging Eq (97) into the LHS of Eq (98) we get

$$\begin{aligned}
&\mathbf{E}_{R \sim \mathbf{\Psi}(t_i)} \left[\mathbf{E}_{y \sim Q(t_i, R)} \left[\phi(t_i + \Delta t_i, R + y - \ell_i) \right] \right] && (99) \\
&= \mathbf{E}_{R \sim \mathbf{\Psi}(t_i)} \left[\phi(t_i, R) \right] && (100) \\
&+ \mathbf{E}_{R \sim \mathbf{\Psi}(t_i)} \left[\mathbf{E}_{y \sim Q(t_i, R)} \left[\left\{ \frac{\partial}{\partial r}\Big|_{\substack{\tau,\,\rho = \\ t_i,\,R}} \phi(\tau, \rho) \right\} (y - \ell_i) \right] \right] && (101) \\
&+ \mathbf{E}_{R \sim \mathbf{\Psi}(t_i)} \left[\mathbf{E}_{y \sim Q(t_i, R)} \left[\left\{ \frac{\partial^2}{\partial r^2}\Big|_{\substack{\tau,\,\rho = \\ t_i,\,R}} \phi(\tau, \rho) \right\} ((y - \ell_i)^2 - \Delta t_i) \right] \right] && (102) \\
&+ O(s^3) && (103)
\end{aligned}$$

Some care is needed here. we need to show that the expected value are all finite. We assume that the expected potential (Eq (100) is finite. Using Lemma 14 this implies that the expected value of higher derivatives of $\frac{\partial}{\partial R}\phi(R)$ are also finite.[4]

To prove inequality (76), we need to show that the terms 101 and 102 are smaller or equal to zero.

[4] I need to clean this up and find an argument that the expected value for mixed derivatives is also finite.

Term (101) is equal to zero:

As ℓ_i is a constant relative to R and y, and $\left\{ \frac{\partial}{\partial r}\Big|_{\substack{\tau,\ \rho = \\ t_i,\ R}} \phi(\tau,\rho) \right\}$ is a constant with respect to y we can rewrite (101) as

$$\mathbf{E}_{R\sim\mathbf{\Psi}(t_i)}\left[\left\{ \frac{\partial}{\partial r}\Big|_{\substack{\tau,\ \rho = \\ t_i,\ R}} \phi(\tau,\rho) \right\} \mathbf{E}_{y\sim Q(t_i,R)}\left[y\right]\right] - \ell_i \mathbf{E}_{R\sim\mathbf{\Psi}(t_i)}\left[\left\{ \frac{\partial}{\partial r}\Big|_{\substack{\tau,\ \rho = \\ t_i,\ R}} \phi(\tau,\rho) \right\}\right] \tag{104}$$

Combining the definitions of $\ell(t)$ (30) and and the learner's strategy P^{cc} (54) we get that

$$\ell(t_i) \quad = \quad \mathbf{E}_{R\sim\mathbf{\Psi}t_i}\left[\frac{1}{Z}\left\{ \frac{\partial}{\partial r}\Big|_{\substack{\tau,\ \rho = \\ t_i,\ R}} \phi(\tau,\rho) \right\} \mathbf{E}_{y\sim Q(i,R)}\left[y\right]\right] \text{ where } Z = \mathbf{E}_{R\sim\mathbf{\Psi}t_i}\left[\frac{1}{Z}\left\{ \frac{\partial}{\partial r}\Big|_{\substack{\tau,\ \rho = \\ t_i,\ R}} \phi(\tau,\rho) \right\}\right] \tag{105}$$

Plugging (105) into (104) and recalling the requirement that $\ell(t_i) < \infty$ we find that term (101) is equal to zero.

Term (102) is equal to zero:

As Δt_i is a constant relative to y, we can take it outside the expectation and plug in the definition of Δt_i (56)

$$\mathbf{E}_{R\sim\mathbf{\Psi}(t_i)}\left[\mathbf{E}_{y\sim Q(t_i,R)}\left[Q(t_i,R)\right]\left\{ \frac{\partial^2}{\partial r^2}\Big|_{\substack{\tau,\ \rho = \\ t_i,\ R}} \phi(\tau,\rho) \right\}(y-\ell(t_i))^2 - \Delta t_i\right] = \Delta t_i - \Delta t_i = 0 \tag{106}$$

Where $G(t_i, R)$ is defined in Equation (**??**) We find that (102) is zero.

Finally (103) is negligible relative to the other terms as $s \to 0$. □