

Evolution with Positive Feedback between Phenotype and Inheritance

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Abstract

TODO

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Model

Wright-Fisher model

We model evolution of a constant, finite size population with non-overlapping generations using a Wright-Fisher model with natural selection, inheritance, and random genetic drift. This model is based on the simulation description in the final paragraph of the *Materials & Methods* section of Xue and Leibler (2016). Indeed, a simulation based on this model (<https://github.com/yoavram/Milpitas>) allowed us to reproduce Figure 2 of Xue and Leibler (2016) (Fig. 1).

Definitions

- N : constant population size.
- ϕ_i : phenotype of individual i , $1 \leq i \leq N$, $\phi_i \in \{A, B\}$.
- ϵ_t : the environment at generation t , $\epsilon_t \in \{A, B\}$.
- W : individual fitness when phenotype and environment match, $\phi_i = \epsilon_t$.
- w : individual fitness when phenotype and environment do not match, $\phi_i \neq \epsilon_t$.
- ω_i : fitness of individual i at time t , $\omega_i = W \cdot 1_{\phi_i = \epsilon_t} + w \cdot 1_{\phi_i \neq \epsilon_t}$.
- $\bar{\omega}$: population mean fitness.
- π_i : phenotype probability, the probability that individual i becomes phenotype A , $1 \leq i \leq N$.
- Π : set of phenotype probabilities in the population, $\Pi = \{\pi_i\}_{1 \leq i \leq N}$
- η : phenotypic inheritance rate, $0 \leq \eta \leq 1$.

Reproduction

For each offspring in the population of generation $t+1$ we choose a parent from the population of generation t and this choice depends on the parent relative fitness: the probability that individual i is the parent is relative to its fitness, ω_i . Therefore, reproduction is modeled by a multinomial distribution. Therefore, reproduction includes the effects of natural selection and random genetic drift.

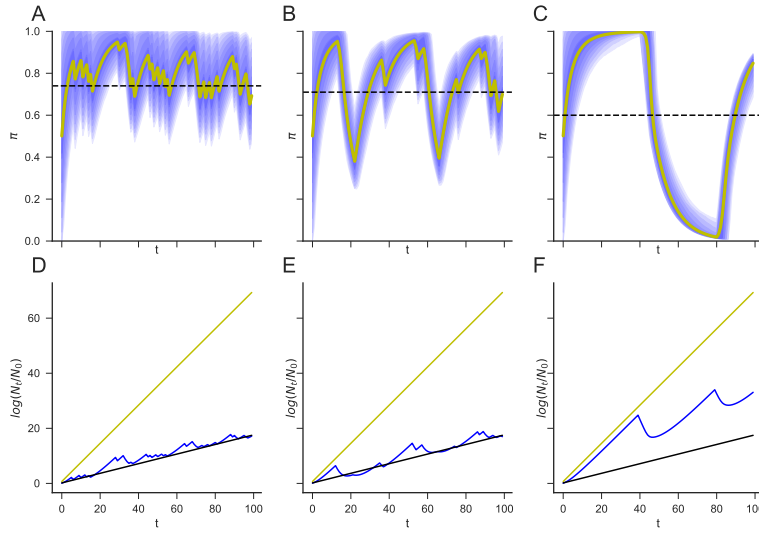


Figure 1: Reproduction of Figure 2 from Xue and Leibler (2016). **A-C**: the average π in the population over time and the distribution in blue; **D-F**: the effective population growth rate. **A,D**: the environment is randomly chosen at each generation such that $P(A)=0.7$ and $P(B)=0.3$; **B,E**: the environment flips between A and B, the duration of each is geometrically distributed with $p=1/10$ for A and $p=1/5$ for B; **C,F**: the environments flips every 40 generations. Parameters: $N=100,000$, # generations = 100, $\eta=0.1$, $W=2$, $w=0.2$.

Inheritance

The offspring inherits the phenotype probability of the parent with a modification – if the parent became A , then the offspring is even more likely to be A ; if the parent was B , then the offspring is less likely to be A . Specifically, for parent k and offspring i :

$$\pi_i = \pi_k \cdot (1 - \eta) + \eta \cdot 1_{\phi_k=A} \quad (1)$$

Note

The notation in Eq. 1 is different from Eq. 1 in Xue and Leibler (2016), as i denotes individual, rather than phenotype. But the process is the same.

Note

The expected difference between parent and offspring phenotype probability is $E[\pi_i - \pi_k | \pi_k] = 0$.

Iteration

At each generation t , Π is updated according to the following steps. Initial values can be determined (*i.e.*, $\forall i, \pi_i = 0.5$), or values can be drawn from an initial distribution (*i.e.*, $\pi_i \sim TN(0.5, 0.05)$, TN is the truncated normal distribution). In addition, the sequence ϵ_t is independent of the iteration on Pi .

At each generation t :

1. **Development:** the phenotypes of all individuals are drawn from corresponding Bernoulli distributions depending on their phenotype probabilities: $P(\phi_i = A) = \pi_i$.
2. **Fitness:** the fitness of all individuals is set: $\omega_i = W \cdot 1_{\phi_i=\epsilon_t} + w \cdot 1_{\phi_i \neq \epsilon_t}$.
3. **Reproduction:** the number of offspring of each individual, b_i , is drawn from a multinomial distribution $MN(N, \{\frac{\omega_i}{\sum_i \omega_i}\}_{1 \leq i \leq N})$, such that

$$P(b_1 = x_1, \dots, b_N = x_N) = \frac{N!}{x_1! \cdot \dots \cdot x_N!} \cdot \left(\frac{\omega_1}{\sum_i \omega_i}\right)^{x_1} \cdot \dots \cdot \left(\frac{\omega_N}{\sum_i \omega_i}\right)^{x_N}$$

4. **Inheritance:** the set of phenotype probabilities of the offspring generation is updated using Eq. 1 such that for each i , Π includes exactly b_i copies of $(\pi_i \cdot (1 - \eta) + \eta \cdot 1_{\phi_i=A})$.

Note

Only development and reproduction are stochastic; natural selection and drift occur at the reproduction step.

Note

If $\eta = 0$, $\pi_i \in \{0, 1\}$, and $\epsilon_t \equiv A$, then we have a standard single locus bi-allelic selection-drift Wright-Fisher model.

Recurrence equation

We approximate the Wright-Fisher model using a recurrence equation. This approximation is most suitable when the population is composed of a single lineage – for example, when selection is extreme and there is a common ancestor.

Define x to be the probability that a random individual in the population is A . What is x' , the probability that a random offspring of that individual is A ?

Assuming an “infinite” population undergoing exponential growth, this depends on (i) if the parent was A or B , with probabilities x and $1-x$, (ii) on the relative contribution of A and B phenotypes to the next generation in terms of fitness, and (iii) on the probability that offspring of A or B phenotypes eventually become A , according to the inheritance process (Eq. 1):

$$x' = x \cdot \frac{\omega_A}{\bar{\omega}} \cdot ((1 - \eta)x + \eta) + (1 - x) \cdot \frac{\omega_B}{\bar{\omega}} \cdot (1 - \eta)x \quad (2)$$

Where ω_A and ω_B are the fitness of phenotypes A and B in the current generation (depending on ϵ_t).

When we write a similar recurrence for the probability that an individual is B ($(1-x)' = F(1-x)$) and sum the two equations, we find that $\bar{\omega} = x\omega_A + (1-x)\omega_B$ is the mean fitness.

This recurrence equation can be reorganized to:

$$x' = x \frac{x(1 - \eta)(\omega_A - \omega_B) + \eta\omega_A + (1 - \eta)\omega_B}{x(\omega_A - \omega_B) + \omega_B} \quad (3)$$

Results

Constant environment

We start with a constant environment $\epsilon_t = A \forall t$, such that $\omega_A = W > w = \omega_B$.

Proposition for $\eta = 0$

If $\eta = 0$ then any $x \in [0, 1]$ solves $x' = x$.

Proof

Set $x' = x, \eta = 0$ in Eq. 3:

$$x = x \frac{x(\omega_A - \omega_B) + \omega_B}{x(\omega_A - \omega_B) + \omega_B}$$

First, $x = 0$ solves this equality. Otherwise, for $x > 0$:

$$\Leftrightarrow x(\omega_A - \omega_B) + \omega_B = x(\omega_A - \omega_B) + \omega_B$$

which is an identity ■.

Proposition for $W = w$

If $W = w$ then any $x \in [0, 1]$ solves $x' = x$.

Proof

Set $W = w > 0, x' = x \neq 0$ in Eq. 3:

$$\begin{aligned} x &= x\eta W + (1 - \eta)WW \Leftrightarrow \\ &1 = 1\eta + (1 - \eta)1 \Leftrightarrow \\ &1 = 1 \end{aligned}$$

■

Proposition for general case

If $\eta > 0$ and $\omega_A > \omega_B$, then $x^* = 1$ is the only solution for $x' = x$.

Proof

First, we check that $x^* = 1$ solves $x' = x$. Set $x = 1$ in Eq. 3:

$$x' = \frac{(1-\eta)(\omega_A - \omega_B) + \eta\omega_A + (1-\eta)\omega_B}{\omega_A} = \frac{\omega_A}{\omega_A} = 1$$

Next, we check that $x = 0$ doesn't solve $x' = x$ by setting $x = 0$ in Eq. 3:

$$x' = \frac{\eta\omega_A + (1-\eta)\omega_B}{\omega_B} > 0$$

since both the denominator and the numerator are positive.

Next, we check that any $0 < x < 1$ doesn't solve $x' = x$. Set $x' = x$ in Eq. 2 and substituting $\bar{\omega} = x\omega_A + (1-x)\omega_B$:

$$\begin{aligned} x &= x \cdot \frac{\omega_A}{\bar{\omega}} \cdot ((1-\eta)x + \eta) + (1-x) \cdot \frac{\omega_B}{\bar{\omega}} \cdot (1-\eta)x \Leftrightarrow \\ \bar{\omega} &= \omega_A \cdot ((1-\eta)x + \eta) + (1-x) \cdot \omega_B \cdot (1-\eta) \Leftrightarrow \\ x\omega_A + (1-x)\omega_B &= (1-\eta)x\omega_A + \eta\omega_A + (1-\eta)(1-x)\omega_B \Leftrightarrow \\ \eta(x\omega_A + (1-x)\omega_B) &= \eta\omega_A \Leftrightarrow \\ x\omega_A + (1-x)\omega_B &= \omega_A \Leftrightarrow \\ (1-x)\omega_A &= (1-x)\omega_B \Leftrightarrow \\ x &= 1 \end{aligned}$$

■

Periodic environment

We concentrate on periodic environments in which both environments occur exactly the same number of generations in each “period”. A simple example is $A1B1=ABABABAB\dots$, in which the environment switches every generation every generation from A to B and vice versa, or $A2B1=AABAABAAB\dots$ in which the every two $_A_$ s are followed by a single B . In general, $AkBl$ denotes an environmental regime in which the period is of length $k+l$ and composed of exactly k $_A_$ s and l $_B_$ s.

We simulated evolution in such environments, and Fig. 2 shows the evolution of the distribution of π in a population evolving in three such environmental regimes.

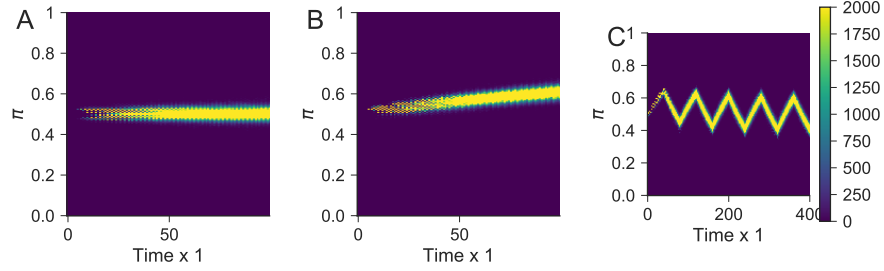


Figure 2: Distribution of π in populations evolving in periodic environments. (A) A1B1, (B) A2B1, (C) A40B40. Parameters: $N=100,000$, $\eta=0.01$, $W=1$, $w=0.1$.

A1B1 regime

When the environment changes every generation, we can write the following recursion, which sets $\omega_A = W, \omega_B = w$ in Eq. 3 to determine x' and then sets $\omega_A = w, \omega_B = W$ to determine x'' :

$$\begin{aligned} x' &= x \frac{x(1-\eta)(W-w) + \eta W + (1-\eta)w}{x(W-w) + w} \\ x'' &= x' \frac{x(1-\eta)(w-W) + \eta w + (1-\eta)W}{x'(w-W) + W} \end{aligned} \quad (4)$$

We are looking for solutions for $x'' = x$, which evaluates to a quartic polynomial. Two solutions are $x = 0, 1$ (assign to Eq. 7 to check), but there are two more potential solutions such that

$$\begin{aligned} x'' - x &= x(1-x)G(x) = 0 \\ G(x) &= Ax^2 + Bx + C \end{aligned}$$

Using SymPy, a Python library for symbolic mathematics, a free alternative to Wolfram Mathematica™ (Team 2014), we find all four solution of $x'' - x = 0$:

$$G(x) = x^2 - \frac{W(1-\eta) - w(3-\eta)}{(2-\eta)(W-w)}x - \frac{w}{(2-\eta)(W-w)} \quad (5)$$

To find the roots of $G(x)$, recall that $W > w \geq 0, 1 \geq \eta \geq 0$, so

$$G(0) = \frac{-w}{(2-\eta)(W-w)} < 0$$

and

$$G(1) = 1 - \frac{W(1-\eta) - w(3-\eta)}{(2-\eta)(W-w)} - \frac{w}{(2-\eta)(W-w)} = \frac{W}{(2-\eta)(W-w)} > 0$$

and $\lim_{x \rightarrow \pm\infty} G(x) = +\infty$.

Therefore, one root of $G(x)$ is negative and one, \tilde{x} , is positive and below 1. Let $\delta = \frac{-B - \sqrt{B^2 - 4AC}}{2A} - \frac{-B + \sqrt{B^2 - 4AC}}{2A}$ (where A, B, C are the coefficients of $G(x)$, defined in Eq. 5). Then, $\delta = \frac{\sqrt{(W+w)^2 - \eta(2-\eta)(W-w)^2}}{(2-\eta)(W-w)}$. Because $\eta(2-\eta)$ is maximized at 1,

$$(W+w)^2 - \eta(2-\eta)(W-w)^2 > (W+w)^2 - (W-w)^2 = 4Ww > 0$$

so $\delta > 0$. Therefore, the positive root is:

$$\tilde{x} = \frac{-B - \sqrt{B^2 - 4AC}}{2A} \quad (6)$$

Fig. 3 shows \tilde{x} (dashed green) compared with x from iteration of Eq. 7 (blue) and with the population mean π ($\bar{\pi}$) in Wright-Fisher simulations (orange) for several combinations of η, W, w . All iterations started with $\bar{\pi} = 0.5$; in the WF simulations, population size N is 100,000, the initial population is drawn from $N(0.5, 0.05)$, and the results are based on 50 simulations per parameter set. Note that the x-axis shows every other generation* (end of each period). The analytic approximation is good when selection is extreme ($w = 0$), but overestimates $\bar{\pi}$ when selection is not extreme ($w = 0.1$). In both cases the initial population distribution did not affect the results (as long as it wasn't trivial, *i.e.* $\pi = 0$, see Fig. 5).

A2B1 regime

In the *A2B1* regime (every two generations in the *A* environment are followed by a generation in environment *B*), an analytic approximation is not possible, as solving $x''' - x = 0$ requires solving a polynomial of degree 6. However, iterating the relevant recurrence equation:

$$\begin{aligned} x' &= x \frac{x(1-\eta)(W-w) + \eta W + (1-\eta)w}{x(W-w) + w} \\ x'' &= x' \frac{x'(1-\eta)(W-w) + \eta W + (1-\eta)w}{x'(W-w) + w} \\ x''' &= x'' \frac{x''(1-\eta)(w-W) + \eta w + (1-\eta)W}{x''(w-W) + W} \end{aligned} \quad (7)$$

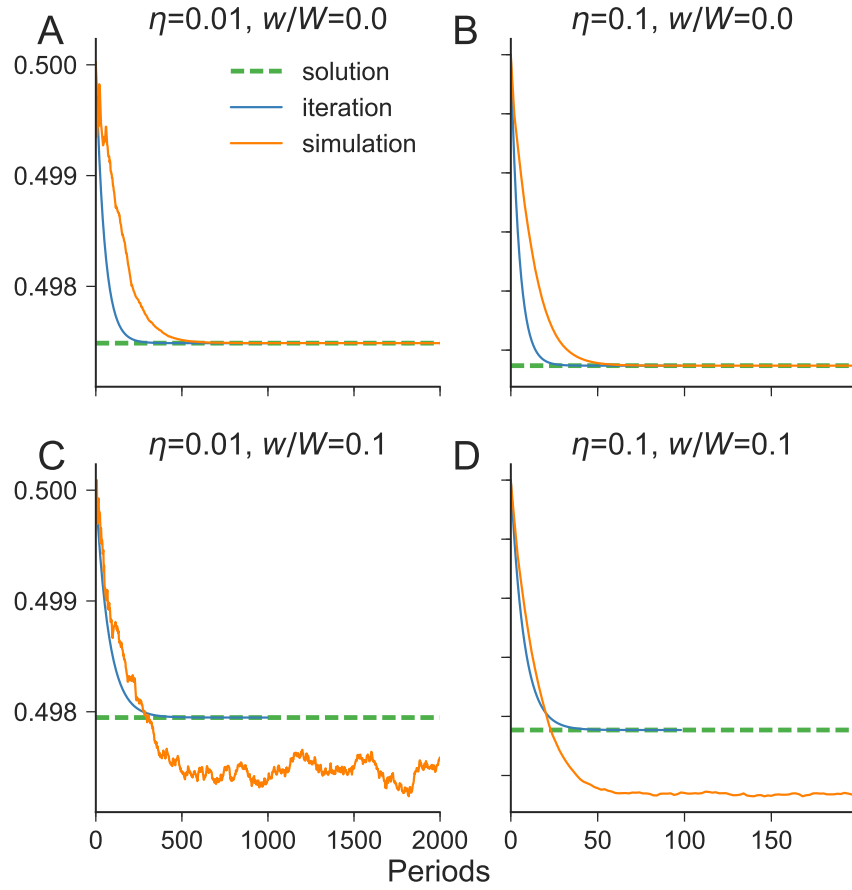


Figure 3: Population mean π in environment regime $A1B1$. $N=100,000$.

provides similar results: the equilibrium value is in good fit with Wright-Fisher simulations for extreme selection ($w = 0$) but over estimates the equilibrium otherwise (Fig. 4).

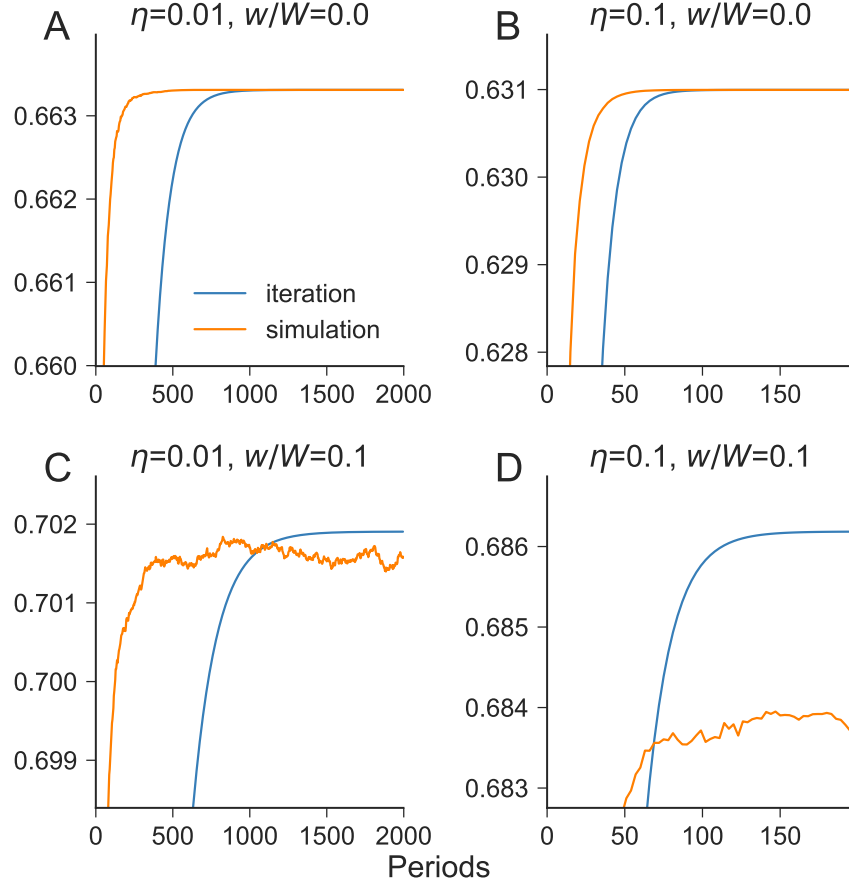


Figure 4: Population mean π in environment regime $A2B1$. $N=100,000$.

Protected polymorphisms in $AkBl$ regime

What can we say about the more general case of k generations in environment A and l generations in B ? We examine the existence of a *protected polymorphism* (Prout 1968), which means that none of the phenotypes become extinct even when initially rare. Environments A and B select for $\pi = 1$ and $\pi = 0$, respectively, and these are absorbing states: if all individuals are, for example, $\pi = 0$, then they are all of phenotype B and all offspring will be $\pi = 0$, too. Mathematically, we examine the stability of $x = 0$ and $x = 1$; if both are unstable,

then a protected polymorphism occurs. Intuitively, this will happen if neither environment occurs enough to fix its preferred state.

We rewrite Eq. 3 as $x' = x \cdot f_A(x)$ in environment A and $x' = x \cdot f_B(x)$ in environment B , where:

$$f_A(x) = \frac{x(1-\eta)(W-w) + \eta W + (1-\eta)w}{x(W-w) + w}$$

$$f_B(x) = \frac{x(1-\eta)(w-W) + \eta w + (1-\eta)W}{x(w-W) + W}$$

We concentrate on $l \geq k$ and check if $x = 0$ is stable, because (i) if $x = 0$ is not stable when $l \geq k$ then $x = 1$ cannot be stable either, as selection is stronger, on the whole, towards 0; and (ii) checking the other case (stability of $x = 1$ when $k \geq l$) is symmetric, and can be done in the same way by writing a recurrence equation for the frequency y of phenotype B rather than A and studying the case of $y = 0$.

To check if $x = 0$ is stable, we start with a value very close to 0 and check if after a period of $k+l$ generations the population is closer or farther from 0 compared to where it started.

For $x_0 = x(t=0) \sim 0$, we can use a linear approximation of the form $f_A(x_0) = f_A(0) + o(x_0)$ and $f_B(x_0) = f_B(0) + o(x_0)$, where:

$$f_A(0) = 1 + \eta\left(\frac{W-w}{w}\right)$$

$$f_B(0) = 1 + \eta\left(\frac{w-W}{W}\right)$$

For k generations in environment A , and l generations with environment B , in any given order, we can write:

$$x_{k+l} = x(t = k+l) \approx x_0 f_A^k(0) f_B^l(0) \Rightarrow$$

$$\frac{x_{k+l}}{x_0} \approx f_A^k(0) f_B^l(0)$$

so that if we start very close to zero ($x_0 \sim 0$), the multiplicative change over the $k+l$ generations can be approximated by $f_A^k(0) f_B^l(0)$.

If $f_A^k(0) f_B^l(0) > 1$, then $x = 0$ is not stable; since $x = 1$ is not stable either (due to $l \geq k$), then we have a *protected polymorphism* somewhere ($0 < x(t) < 1$ for any generation t). In contrast, if $f_A^k(0) f_B^l(0) < 1$, then $x = 0$ is stable and the *protected polymorphism* disappears.

Following we examine the protected polymorphism in several special and general cases.

$$W = w$$

In this case, fitness in both environments is equivalent, there is no selection, and therefore evolution is neutral. Indeed, we find that $f_A(x) = f_B(x) \equiv 1$, without an approximation.

$$\eta = 0$$

In this case, there is no feedback between phenotype and inheritance, and only drift generates genetic variance, and evolution is neutral. Indeed, we get $f_A(x) = f_B(x) \equiv 1$.

$$\eta = 1$$

In this case, development is not stochastic, and after one generation the model becomes a standard two-type genetic model. Only genetic drift generates genetic variance, but natural selection does play a role. Indeed, we get $f_A^k(0)f_B^l(0) = \left(\frac{W}{w}\right)^{k-l}$. Since $W > w$, we find that $\frac{x_{k+l}}{x_0}$ is

$$\begin{cases} < 1 & , k < l \\ = 1 & , k = l \\ > 1 & , k > l \end{cases}$$

Proposition for $k = l$

If $k = l, W > w > 0, 1 > \eta > 0$, then $f_A^k(0)f_B^l(0) > 1$.

Proof

First, $f_A^k(0)f_B^l(0) = (f_A(0)f_B(0))^k > 1$ iff $f_A(0)f_B(0) > 1$.

To show the latter,

$$f_A(0)f_B(0) =$$

$$\begin{aligned}
& (1 + \eta \frac{W-w}{w}) \cdot (1 + \eta \frac{w-W}{W}) = \\
& (1 - \eta + \eta \frac{W}{w}) \cdot (1 - \eta + \eta \frac{w}{W}) = \\
& (1 - \eta)^2 + \eta^2 + \eta(1 - \eta)(\frac{W}{w} + \frac{w}{W}) = \\
& 1 - 2\eta(1 - \eta) + \eta(1 - \eta)(\frac{W^2 + w^2}{Ww}) = \\
& 1 + \eta(1 - \eta) \frac{W^2 - 2Ww + w^2}{Ww} = \\
& 1 + \eta(1 - \eta) \frac{(W - w)^2}{Ww}
\end{aligned}$$

which, under the proposition conditions, is > 1 . ■

Proposition for $l > k = 1$

If $l > 1 + (1 - \eta) \frac{W-w}{w}$ then $f_A(0)f_B^l(0) < 1$.

Proof

Set $n = l - 1$. Then,

$$\begin{aligned}
n > (1 - \eta) \frac{W-w}{w} & \Leftrightarrow \\
n\eta \frac{W-w}{W} & > \eta(1 - \eta) \frac{(W-w)^2}{Ww} \Leftrightarrow \\
1 - n\eta \frac{w-W}{W} & > 1 + \eta(1 - \eta) \frac{(W-w)^2}{Ww} \Leftrightarrow \\
1 & > \frac{1 + \eta(1 - \eta) \frac{(W-w)^2}{Ww}}{1 - n\eta \frac{w-W}{W}}
\end{aligned} \tag{8}$$

Now, $W > w \Rightarrow 1 \geq \frac{W-w}{W} \geq 0$, and together with $1 \geq \eta \geq 0$ we get $0 \geq \eta \frac{w-W}{W} \geq -1$. These conditions allow us to use the following Bernoulli inequality (proof with induction):

$$(1 + x)^n \leq \frac{1}{1 - nx}, \quad \forall x \in [-1, 0], \forall n \in \mathbb{N}. \tag{9}$$

From the Bernoulli inequality we have:

$$\left(1 + \eta \frac{w-W}{W}\right)^n \leq \frac{1}{1 - n\eta \frac{w-W}{W}} \quad (10)$$

Taken together, Eq. 8 and Eq. 10 imply that:

$$\begin{aligned} f_A(0)f_B^{n+1}(0) &= \\ &\left(1 + \eta \frac{W-w}{w}\right) \left(1 + \eta \frac{w-W}{W}\right) \left(1 + \eta \frac{w-W}{W}\right)^n = \\ &\left(1 + \eta(1-\eta) \frac{(W-w)^2}{Ww}\right) \left(1 + \eta \frac{w-W}{W}\right)^n \leq \\ &\frac{1 + \eta(1-\eta) \frac{(W-w)^2}{Ww}}{1 - n\eta \frac{w-W}{W}} < 1 \end{aligned}$$

■

Proposition for general case: $l > k \geq 1$

If $l > k \left(1 + (1-\eta) \frac{W-w}{w}\right)$, then $f_A^k(0)f_B^l(0) < 1$.

Proof

First, assume $\frac{l-k}{k} \in \mathbb{N}$ and set $n = \frac{l-k}{k} \Rightarrow n > (1-\eta) \frac{W-w}{w}$.

Now, using the previous proposition,

$$f_A^k(0)f_B^l(0) = f_A^k(0)f_B^{(n+1)k}(0) = (f_A(0)f_B^{n+1}(0))^k < 1$$

because $\forall y > 0, k > 0 \ y < 1 \Rightarrow y^k < 1$.

Next, relax the assumption $\frac{l-k}{k} \in \mathbb{N}$; set $n = \lceil \frac{l-k}{k} \rceil > \frac{l-k}{k} > (1-\eta) \frac{W-w}{w}$, then

$$f_A^k(0)f_B^l(0) < f_A^k(0)f_B^{(n+1)k}(0) = (f_A(0)f_B^{n+1}(0))^k < 1$$

and again, the previous proposition provides the last inequality. ■

Proposition

If $l < k \left(1 + \frac{(1-\eta) \frac{W-w}{w}}{1 + \eta(1-\eta) \frac{(W-w)^2}{Ww}}\right)$ then $f_A^k(0)f_B^l(0) > 1$.

Proof

Similar to previous proposition, but using a different Bernoulli inequality:

$$(1+x)^n \geq 1+nx, \quad \forall x > -1, \forall n \in \mathbb{R} \setminus (0,1).$$

■

Random environments

TODO: - local stochastic stability - see Levikson & Karlin, 1970's; Liberman & Karlin, 1970's

Supporting figures

- Fig. 5

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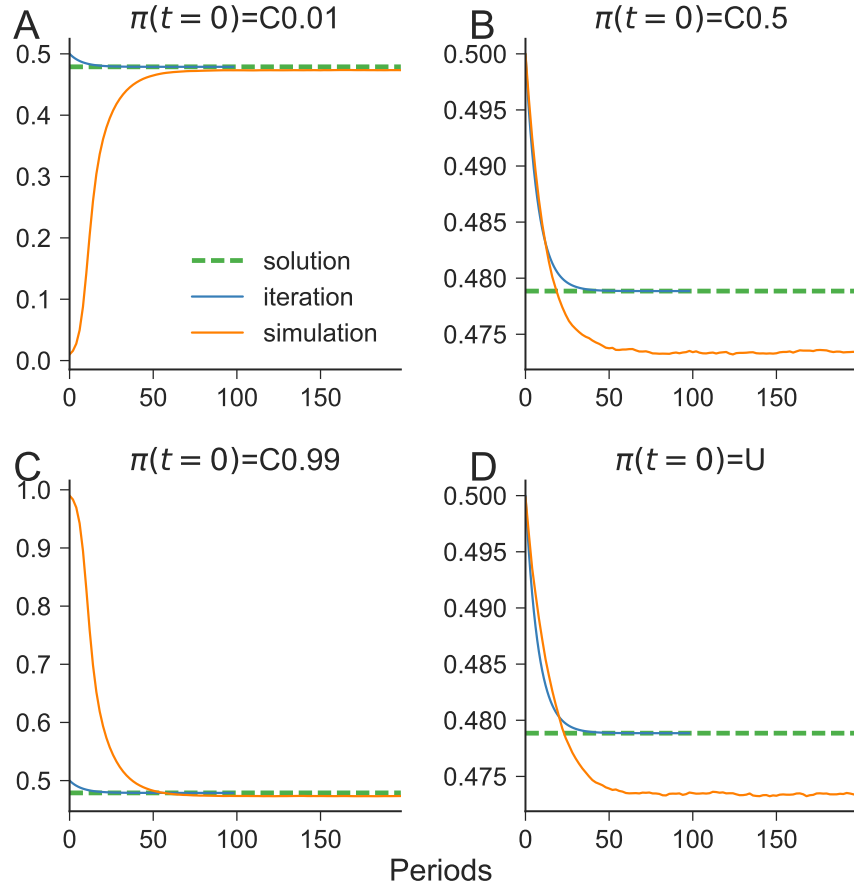


Figure 5: Population mean π in environment regime *A1B1*. Initial population distribution: (A) $\pi_i=0.01$; (B) $\pi_i=0.5$; (C) $\pi_i=0.99$; (D) $\pi_i \sim Uniform(0, 1)$. $N=100,000$.