The 2D AKLT Model

Numerical Methods In Many Body Physics

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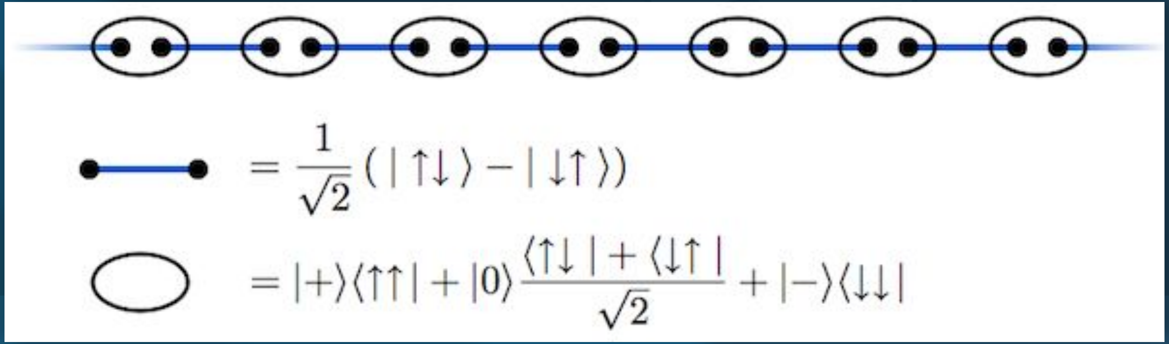
Reminder

- The Affleck-Kennedy-Lieb-Tasaki (AKLT) model is a simple extension of the Heisenberg Model with S=1:

$$H = \sum_j \vec{S}_j \cdot \vec{S}_{j+1} + \frac{1}{3}\left(\vec{S}_j \cdot \vec{S}_{j+1}\right)^2$$

- It’s ground state is exactly solvable!



Reminder: The Sequel

- The ground state can be written symbolically as an MPS:

$$|\psi\rangle = \sum_{\sigma} \mathrm{Tr}\left(A^{\sigma_1} A^{\sigma_2} \cdots A^{\sigma_L}\right) |\sigma\rangle \;\; ; \quad A^{\sigma} = \frac{2}{\sqrt{3}} M^{\sigma} \Sigma$$

$$\Sigma = \begin{pmatrix} 0 & 1/\sqrt{2} \\ -1/\sqrt{2} & 0 \end{pmatrix} \; ; \quad M^{+} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \; ; \quad M^{0} = \begin{pmatrix} 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 \end{pmatrix} \; ; \quad M^{-} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

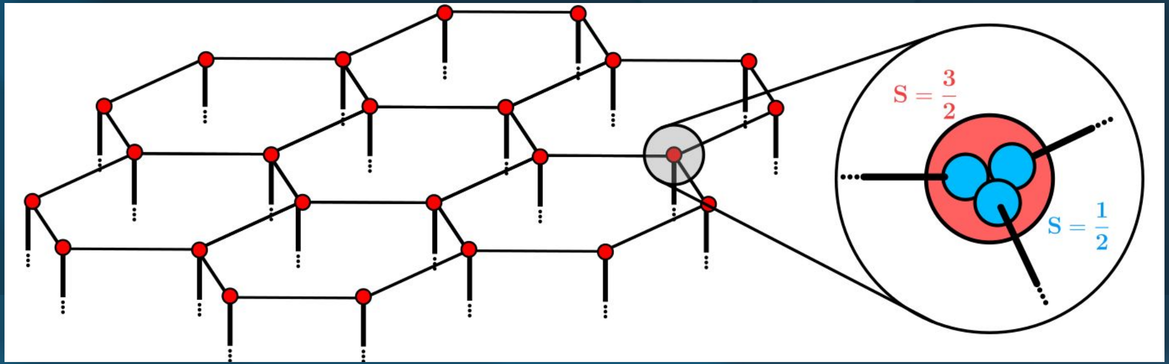
- The A are the site-tensors, Σ are the anti-symmetrization, and M^{σ} are the projection onto spin-1 states.
- Open boundary conditions means a missing Σ, which creates emerging spin-½ particles at the edges.

Into The Second Dimension

- The AKLT model can be extended to 2D on a Honeycomb Lattice of particles with spin $\frac{3}{2}$. The Hamiltonian is:

$$H = \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j + \frac{116}{243}\left(\vec{S}_i \cdot \vec{S}_j\right)^2 + \frac{16}{243}\left(\vec{S}_i \cdot \vec{S}_j\right)^3$$

- The ground state splits each node into three spin-½ virtual particles:



PEPS Is The New MPS

- In 2D the MPS is replaced by Projected Entangled Pair States (PEPS), which are constructed in a similar manner:

$$|\psi\rangle = \sum_{\sigma} \left(\prod_{\mathcal{S} \in \mathbb{S}} \Theta_{\mathcal{S}}^{\sigma} \right) \left(\prod_{\mathcal{B} \in \mathbb{B}} \Sigma_{\mathcal{B}} \right) |\sigma\rangle$$

- Here $\mathcal{S} \in \mathbb{S}$ are the sites and $\mathcal{B} \in \mathbb{B}$ are the bonds. The A^{σ} tensors are formed by contracting each $\Sigma_{\mathcal{B}}$ with a fitting $\Theta_{\mathcal{S}}^{\sigma}$.

- As in the 1D case, edge state correspond to fractional-spin states, of spin-1 and spin-½. This is what we want to find.

The Chamber Of Secrets

I. Contraction as been performed with:

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@tensor ψ[σ1,σ2,a,b,c,d] = A[σ1,a,b,k] * A2[σ2,k,c,d]
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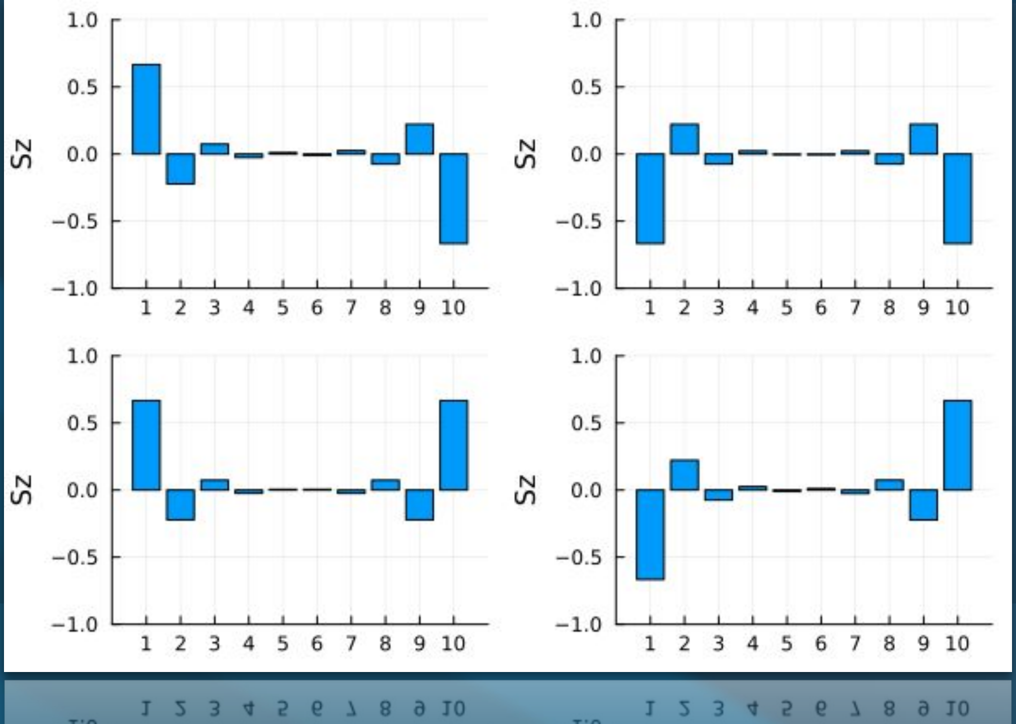
So the number of indices (and the dimension of the wavefunction) is exponential in the number of sites, and PEPS does not help.

II. The projection operators are defined as:

$$\begin{aligned}
\Theta^{3/2}_{abc} &= \delta_{a,\uparrow}\delta_{b,\uparrow}\delta_{c,\uparrow} \\
\Theta^{1/2}_{abc} &= \frac{1}{\sqrt{3}}\left(\delta_{a,}\delta_{b,\uparrow}\delta_{c,\downarrow} + \delta_{a,\uparrow}\delta_{b,\downarrow}\delta_{c,\uparrow} + \delta_{a,\downarrow}\delta_{b,\uparrow}\delta_{c,\uparrow}\right) \\
\Theta^{-1/2}_{abc} &= \frac{1}{\sqrt{3}}\left(\delta_{a,\downarrow}\delta_{b,\downarrow}\delta_{c,\uparrow} + \delta_{a,\downarrow}\delta_{b,\uparrow}\delta_{c,\downarrow} + \delta_{a,\uparrow}\delta_{b,\downarrow}\delta_{c,\downarrow}\right) \\
\Theta^{-3/2}_{abc} &= \delta_{a,\downarrow}\delta_{b,\downarrow}\delta_{c,\downarrow}
\end{aligned}$$

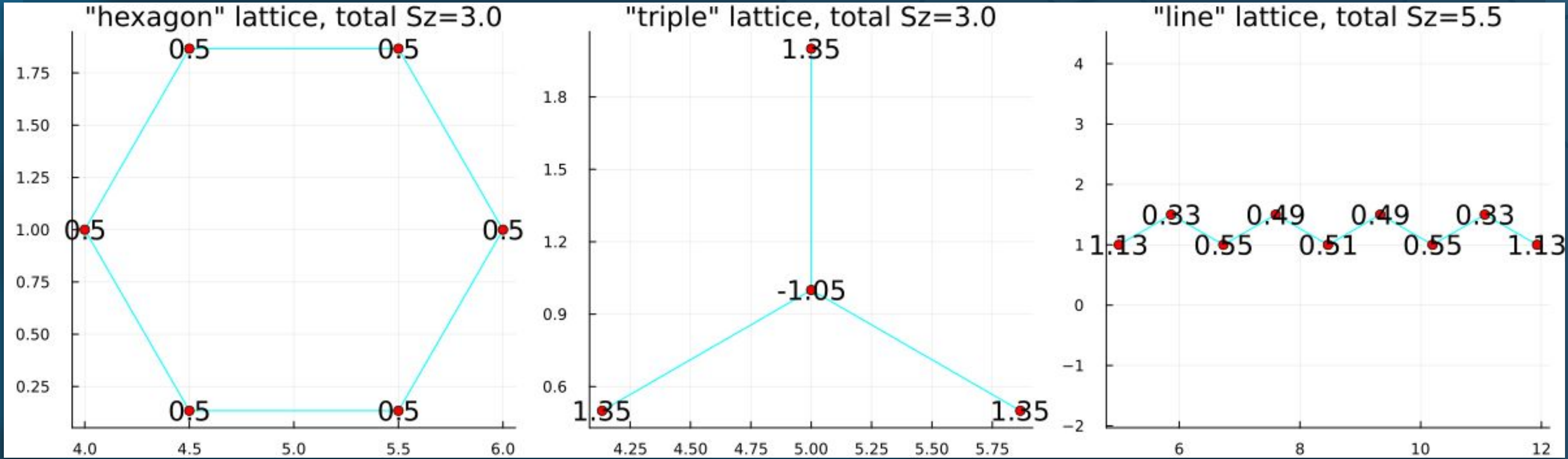
Room of Requirement

- An example of "smeared" spin in 1D MPS of the AKLT model:



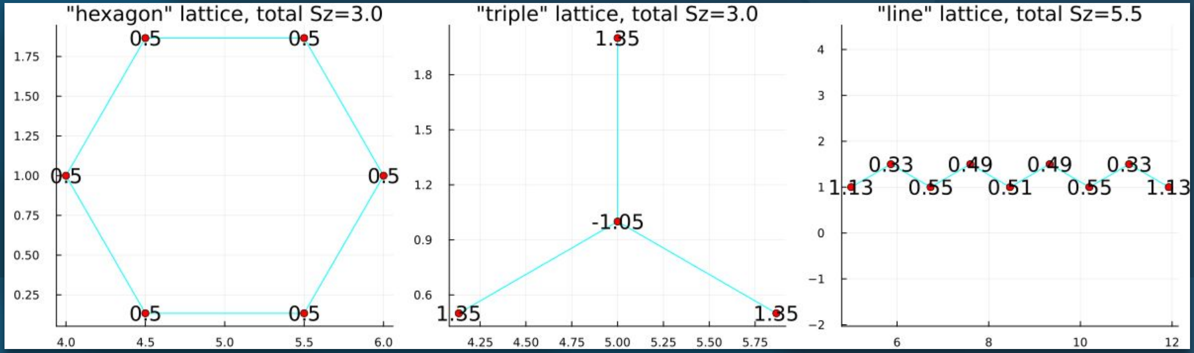
Results

- We simulated several geometries of the 2D AKLT model, and calculated $\langle S_z \rangle$ on each site:



Issues And Future Prospects

- The SVD method for MPS contraction (orthogonality center) cannot be used.
- Therefore only full contraction is possible, and PEPS are still exponential.
- Also, the contraction order is hard coded per geometry, since tracking the indices becomes very hard very quickly for arbitrary geometry.
- Finding methods to circumvent full contraction, and find optimal contraction order per geometry, are the main remaining issues.



Questions?

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