1. Derive the following asymptotic moments associated with $m_3(x)$: mean, variance, first order auto-correlation. Furthermore, compute $\nabla_b g(b_0)$. Which moments are informative for estimating b?

Answer: We obtain the asymptotic moments as follows:

$$\begin{aligned}
\mathbb{E}[x_t] &= \mathbb{E}[\rho_0 x_{t-1} + \epsilon_t] = \rho_0\,\mathbb{E}[x_{t-1}] + \mathbb{E}[\epsilon_t] = \rho_0\,\mathbb{E}[\rho_0 x_{t-2} + \epsilon_{t-1}] = \rho_0^t\,\mathbb{E}[x_0] = 0.\\
\mathbb{E}[(x_t - \mathbb{E}[x_t])^2] &= \mathbb{E}[x_t^2] = \mathbb{E}[(\rho_0 x_{t-1} + \epsilon_t)^2]\\
&= \rho_0^2\,\mathbb{E}[x_{t-1}^2] + 2\rho_0\,\mathbb{E}[x_{t-1}\epsilon_t] + \mathbb{E}[\epsilon_t^2]\\
&= \rho_0^2\,\mathbb{E}[(\rho_0 x_{t-2} + \epsilon_{t-1})^2] + \sigma_0^2\\
&= \rho_0^{2T}\,\mathbb{E}[x_0^2] + \sigma_0^2 \sum_{i=0}^{T-1} (\rho_0^2)^i \to \frac{\sigma_0^2}{1-\rho_0^2} = \frac{4}{3}.\\
\mathbb{E}[(x_t - \mathbb{E}[x_t])(x_{t-1} - \mathbb{E}[x_{t-1}])] &= \mathbb{E}[x_t x_{t-1}]\\
&= \rho_0\,\mathbb{E}[x_{t-1}^2] + \mathbb{E}[\epsilon_t x_{t-1}]\\
&= \rho_0\,\mathbb{E}[(\rho_0 x_{t-2} + \epsilon_{t-1})^2]\\
&= \rho_0^3\,\mathbb{E}[x_{t-2}^2] + \rho_0\,\mathbb{E}[\epsilon_{t-1}^2]\\
&= \rho_0^{2T-1}\,\mathbb{E}[x_0^2] + \rho_0\sigma_0^2 \sum_{i=0}^{T-2} (\rho_0^2)^i \to \frac{\rho_0\sigma_0^2}{1-\rho_0^2} = \frac{2}{3}
\end{aligned}$$

Moreover, we have that $\nabla_b = \begin{pmatrix} \frac{\partial}{\partial\rho} \\ \frac{\partial}{\partial\sigma^2} \end{pmatrix}$. Evaulating the derivative of $g(\cdot)$ at $b = b_0$ gives us

$$\nabla_b g(b_0) = \begin{bmatrix} 0 & 0 \\ \frac{2\rho_0\sigma_0^2}{(1-\rho_0)^2} & \frac{1}{1-\rho_0^2} \\ \frac{\sigma_0^2(1+2\rho_0^2)}{(1-\rho_0^2)^2} & \frac{\rho_0}{1-\rho_0^2} \end{bmatrix} \overset{\text{true values}}{=\!=\!=\!=} \begin{bmatrix} 0 & 0 \\ 4 & \frac{4}{3} \\ \frac{8}{3} & \frac{2}{3} \end{bmatrix}$$

Both the variance and the first order correlation are informative for estimating b. This is because we can estimate the true parameters given the two moments as follows,

$$\rho_0 = \frac{\mathbb{E}[(x_t - \mathbb{E}[x_t])(x_{t-1} - \mathbb{E}[x_{t-1}])]}{\mathbb{E}[(x_t - \mathbb{E}[x_t])^2]} \qquad \sigma^2 = \mathbb{E}[(x_t - \mathbb{E}[x_t])^2] - \frac{\mathbb{E}[(x_t - \mathbb{E}[x_t])(x_{t-1} - \mathbb{E}[x_{t-1}])]}{\mathbb{E}[(x_t - \mathbb{E}[x_t])^2]}$$

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2. Simulate a series of "true" data of length $T = 200$ using (1). We will use this to compute $M_T(x)$.
3. Set H = 10 and simulate H vectors of length T = 200 random variables e_t from $N(0,1)$. We will use this to compute $M_{TH}(y(b))$. Store these vectors. You will use the same vector of random variables throughout the entire exercise. Since this exercise requires you to estimate σ^2, you want to change the variance of e_t during the estimation. You can simply use σ_{e_t} when the variance is σ^2.
4.
5. Next we estimating the $l = 2$ vector b for the just identified case where m_2 uses the variance and autocorrelation. Given what you found in part (1), do you now think there will be a problem? If not, hopefully the standard error of the estimate of b as well as the J test will tell us something. Let's see. For this case, perform steps (a)-(d) above.
6. Next, we will consider the overidentified case where m3 uses the mean, variance and autocorrelation. Let's see. For this case, perform steps (a)-(d) above. Furthermore, bootstrap the the finite sample distribution of the estimators

using the following algorithm:

i. Draw ϵ_t and e_t^h from $N(0,1)$ for $t = 1, 2, \ldots, T$ and $h = 1, 2, \ldots, H$. Compute $(\hat{b}_{TH}^1), \hat{b}_{TH}^2$ as described.

ii. Repeat (e) using another seed.