1. Derive the following asymptotic moments associated with $m_3(x)$: mean, variance, first order auto-correlation. Furthermore, compute $\nabla_b g(b_0)$. Which moments are informative for estimating b?

   Answer: We obtain the asymptotic moments as follows:

$$\begin{aligned}
\mathbb{E}[x_t] &= \mathbb{E}[\rho_0 x_{t-1} + \epsilon_t] = \rho_0\,\mathbb{E}[x_{t-1}] + \mathbb{E}[\epsilon_t] = \rho_0\,\mathbb{E}[\rho_0 x_{t-2} + \epsilon_{t-1}] = \rho_0^t\,\mathbb{E}[x_0] = 0. \\
\mathbb{E}[(x_t - \mathbb{E}[x_t])^2] &= \mathbb{E}[x_t^2] = \mathbb{E}[(\rho_0 x_{t-1} + \epsilon_t)^2] \\
&= \rho_0^2\,\mathbb{E}[x_{t-1}^2] + 2\rho_0\,\mathbb{E}[x_{t-1}\epsilon_t] + \mathbb{E}[\epsilon_t^2] \\
&= \rho_0^2\,\mathbb{E}[(\rho_0 x_{t-2} + \epsilon_{t-1})^2] + \sigma_0^2 \\
&= \rho_0^{2T}\,\mathbb{E}[x_0^2] + \sigma_0^2 \sum_{i=0}^{T-1} (\rho_0^2)^i \to \frac{\sigma_0^2}{1-\rho_0^2} = \frac{4}{3}. \\
\mathbb{E}[(x_t - \mathbb{E}[x_t])(x_{t-1} - \mathbb{E}[x_{t-1}])] &= \mathbb{E}[x_t x_{t-1}] \\
&= \rho_0\,\mathbb{E}[x_{t-1}^2] + \mathbb{E}[\epsilon_t x_{t-1}] \\
&= \rho_0\,\mathbb{E}[(\rho_0 x_{t-2} + \epsilon_{t-1})^2] \\
&= \rho_0^3\,\mathbb{E}[x_{t-2}^2] + \rho_0\,\mathbb{E}[\epsilon_{t-1}^2] \\
&= \rho_0^{2T-1}\,\mathbb{E}[x_0^2] + \rho_0\sigma_0^2 \sum_{i=0}^{T-2} (\rho_0^2)^i \to \frac{\rho_0\sigma_0^2}{1-\rho_0^2} = \frac{2}{3}
\end{aligned}$$

   Both the variance and the first order correlation are informative for estimating b. ∎

2. Simulate a series of “true” data of length $T = 200$ using (1). We will use this to compute $M_T(x)$.

3. Set H = 10 and simulate H vectors of length T = 200 random variables e_t from $N(0, 1)$. We will use this to compute $M_{TH}(y(b))$. Store these vectors. You will use the same vector of random variables throughout the entire exercise. Since this exercise requires you to estimate σ^2, you want to change the variance of e_t during the estimation. You can simply use σ_{e_t} when the variance is σ^2.

4.

5. Next we estimating the $l = 2$ vector b for the just identified case where m_2 uses the variance and autocorrelation. Given what you found in part (1), do you now think there will be a problem? If not, hopefully the standard error of the estimate of b as well as the J test will tell us something. Let’s see. For this case, perform steps (a)-(d) above.

6. Next, we will consider the overidentified case where m3 uses the mean, variance and autocorrelation. Let’s see. For this case, perform steps (a)-(d) above. Furthermore, bootstrap the the finite sample distribution of the estimators using the following algorithm:

   i. Draw ϵ_t and e_t^h from $N(0, 1)$ for $t = 1, 2, \ldots, T$ and $h = 1, 2, \ldots, H$. Compute $(\hat{b}_{TH}^1), \hat{b}_{TH}^2$ as described.

   ii. Repeat (e) using another seed.