Chapter 2

Finite Element Spaces - General Theory

2.1 FEM

Trinagluation

Consider a variational formulation for second elliptic p.d.e

$$a(u,v) = f(v), \quad \forall v \in V. \tag{2.1}$$

Let Ω be an open bounded set in $\mathbb{R}^d$ with Lipschitz-continuous boundary and let $\mathcal{T}_h$ be a *triangulation* of Ω, $\mathcal{T}_h = \{K : \text{ element}\}$. Let V_h be a certain approximate subspace of V, usually a space of piecewise polynomials such that for each $K \in \mathcal{T}_h$,

$$P_K = \{v_h|_K : v_h \in V_h\}$$

consists of polynomials on K. There exists a basis for V_h whose functions have small support. We write $X_h = X_h(\Omega, \mathcal{T}_h, V_h)$ and call it the *finite element space*. We shall usually use X_h to mean the space V_h.

Three basic ingredients of finite element space are.

(FEM 1)[Triangulation] Ω is subdivided into a finite number of subsets K(diam $(K) \leq h$), called finite element in such a way that

($\mathcal{T}_h$1) $\bar{\Omega} = \cup_{K\in\mathcal{T}_h} K$

($\mathcal{T}_h$2) Each $K \in \mathcal{T}_h$ is a closed polyhedron and $\overset{\circ}{K}$ is nonempty

($\mathcal{T}_h$3) For any two elements K_1, K_2, we have either $K_1 = K_2$ or $\mathring{K}_1 \cap \mathring{K}_2 = \emptyset$

($\mathcal{T}_h$4) For each $K \in \mathcal{T}_h$. the boundary ∂K is Lipschitz continuous

($\mathcal{T}_h$5) If $f = K_1 \cap K_2 \neq \emptyset$ then f is either a common face, side, or vertex of K_1 and K_2.

(FEM 2) The functions in P_K are polynomials or close to polynomials so that the resulting linear system is sparse or structured(to insure linear system is easily solvable).

(FEM 3) There exists a canonical basis for V_h whose functions have small support and can be easily described.

We usually write $H^m(K)$ for $H^m(\mathring{K})$.

We assume each element K is obtained as $K = F_K(\hat{K})$ where $\hat{K}$ is a *reference element* and F_K is an invertible affine map: $F_K(\hat{x}) = B_K\hat{x} + b_K$, B_K being a nonsingular matrix. (When F_K is not affine, we have more general shape, but we do not consider them here). We consider two cases:

(Simplex) The reference polyhedron $\hat{K}$ is the unit d-simplex, i.e, the triangle with vertices $(0,0), (1,0), (0,1)$ (when $d = 2$) or tetrahedron with vertices $(0,0,0), (1,0,0), (0,1,0), (0,0,1)$ (when $d = 3$).

(Unit cube) The reference polyhedron $\hat{K}$ is the unit d-cube, i.e, the rectangle $[0,1]^d$. As a consequence, K is parallelogram (when $d = 2$) or parallelepiped.(when $d = 3$)

2.2 Piecewise Polynomial spaces

Recall: For $\alpha = (\alpha_1, \cdots, \alpha_d), (\alpha_i \in \mathbb{Z}^+)$, let $|\alpha| = \sum_{i=1}^d \alpha_i$ and

$$\mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}, \; \partial^\alpha u = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d} u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}.$$

Now we define X_h which approximate the infinite dimensional space X and satisfies above conditions. Let P_k be the set of all polynomials of degree less than equal to k in variables $x_1, \cdots, x_d$ and Q_k be the set of all polynomials of degree less than equal to k in each variable $x_1, \cdots, x_d$. Then for any $p \in P_k$, we see

$$p(x_1, \cdots, x_d) = \sum C_{\boldsymbol{\alpha}} x_1^{\alpha_1} \cdots x_d^{\alpha_d}, \; \alpha_1 + \cdots + \alpha_d \leq k.$$

The multi-index $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_d)$ satisfies $\alpha_0 + \alpha_1 + \cdots + \alpha_d = k$ for some nonnegative integer α_0. Thus the number of distinct terms are the same as the number of choosing k elements from the set $R = \{1, x_1, x_2, \cdots, x_d\}$ allowing repetition. So we have

$$\dim P_k = {}_{d+1}H_k = \binom{d+k}{k} = {}_{d+k}C_k, \quad \dim Q_k = (k+1)^d. \tag{2.2}$$

Set

$$P_K = \{v_h|_K : v_h \in X_h\}.$$

We define the most commonly used spaces X_h as

$$X_h = X_h^k := \{v_h \subset C^0(\bar{\Omega}), v_h|_K \in P_k, \forall K \in \mathcal{T}_h\}, \quad K \text{ triangular} \tag{2.3}$$

$$X_h = X_h^k := \{v_h \subset C^0(\bar{\Omega}), v_h|_K \in Q_k, \forall K \in \mathcal{T}_h\}, \quad K \text{ rectangular} \tag{2.4}$$

Proposition 2.2.1. A function $v : \Omega \to \mathbb{R}$ belongs to $H^1(\Omega)$ iff

(1) $v|_K \in H^1(K)$ for each $K \in \mathcal{T}_h$;

(2) for each common face $f = K_1 \cap K_2$, the trace on f of $v|_{K_1}$ and $v|_{K_2}$ coincides. In other words, $v \in C^0(\Omega)$.

Proof. Note that $\Omega = \bigcup K$. Let $v \in X_h$. Suppose conditions (1), (2) holds. We need to show that for each $i = 1, \cdots, d$, the derivatives $\partial v/\partial x_i$ exists and belongs to $L^2(\Omega)$. By definition of weak derivative, we must find functions $v_i \in L^2(\Omega)$ such that

$$\int_\Omega v_i \phi = -\int_\Omega v \partial_i \phi, \quad \forall \phi \in \mathcal{D} = C_0^\infty(\Omega).$$

A natural candidate is v_i defined by $v_i|_K = \partial_i(v|_K)$ on each K. Indeed, for each K with Lipschitz continuous boundary ∂K we have by Green's formula,

$$\int_K \partial_i(v|_K)\phi dx = -\int_K v|_K \partial_i \phi dx + \int_{\partial K} v|_K \phi n_{i,K} ds,$$

where $n_{i,K}$ is the i-th component of the unit outward normal vector along ∂K.

Summing over all finite elements,

$$\begin{aligned}
\sum_K \int_K \partial_i(v|_K)\phi dx &= -\sum_K \int_K v|_K \partial_i \phi dx + \sum_K \int_{\partial K} v|_K \phi n_{i,K} ds && (2.5)\\
&= -\sum_K \int_K v\partial_i \phi dx = -\int_\Omega v\partial_i\phi dx && (2.6)\\
&= \int_\Omega \sum_K \partial_i(v|_K)\chi_K \phi dx \equiv \int_\Omega v_i \phi dx. && (2.7)
\end{aligned}$$

The second sum on the rhs of first equation vanishes since either ∂K is a portion of $\partial\Omega$, or ∂K is adjacent to some other triangle so that the contributions from the adjacent elements cancel each other by (2). Thus the functions defined by $v_i := \sum_K \partial_i(v|_K)\chi_K \in L^2(\Omega)$ are the desired function. Conversely, if $v \in H^1(\Omega)$ then (1) holds trivially. Moreover,

$$\partial_i(v|_K) = (\partial_i v)|_K,\ i = 1, \cdots, d.$$

Now for $\forall \phi \in \mathcal{D}$

$$\begin{aligned}
\int_\Omega (\partial_i v)\phi dx &= -\int_\Omega v\partial_i \phi\, dx = -\sum_K \int_K v|_K \partial_i \phi\, dx\\
&= -\sum_K \int_{\partial K} v|_K \phi n_{i,K} ds + \sum_K \int_K \partial_i(v|_K)\phi\, dx\\
&= -\sum_K \int_{\partial K} v|_K \phi n_{i,K} ds + \sum_K \int_K (\partial_i v)|_K \phi\, dx\\
&= -\sum_K \int_{\partial K} v|_K \phi n_{i,K} ds + \int_\Omega (\partial_i v)\phi\, dx.
\end{aligned}$$

Hence we get $\sum_K \int_{\partial K} v|_K \phi n_{i,K} ds = 0$. Let K_1 and K_2 be any two elements having f as the common edge. If we restrict ϕ to have support on a neighborhood of the common edge, then we have

$$\int_f (v|_{K_1} - v|_{K_2})\phi n_{i,f} ds = 0,\ i = 1, \cdots, d,$$

where $n_{i,f}$ is the common unit normal vector to f. From this we see (2) is satisfied. $\square$

Remark 2.2.1. (1) V_h may not be a subspace of $V = H_0^1(\Omega)$, say in the case of curved boundary.

(2) The Bilinear form and linear form in the discrete problem are usually replaced by some approximation. This is the case when numerical integration is used.

By a *conforming finite element method*, we mean the finite element method for which V_h is a subspace of V and the bilinear form of the discrete problem are identical to the original one.

2.3 Degrees of freedom, Shape functions of finite elements

A d-simplex in $\mathbb{R}^d$ is the convex hull K of $(d+1)$ points $\mathbf{a}_i \in \mathbb{R}^d$, which are called vertices. We assume that they do not degenerate, i.e.,

$$K = \left\{ \mathbf{x} = \sum_{i=1}^{d+1} \lambda_i \mathbf{a}_i, 0 \le \lambda_i \le 1, \sum_{i=1}^{d+1} \lambda_i = 1 \right\}$$

where

$$A\boldsymbol{\lambda} = \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix}, \quad A = \begin{pmatrix} \mathbf{a}_1, & \mathbf{a}_2, & \cdots, & \mathbf{a}_{d+1} \\ 1, & 1, & \cdots, & 1 \end{pmatrix}$$

is a nonsingular system.

For $d = 2, K$ is a triangle and for $d = 3$, K is a tetrahedron. The unique solution $\lambda_i, (1 \le i \le d+1)$ of

$$\begin{cases} \sum_{j=1}^{d+1} \mathbf{a}_{ij} \lambda_j = \mathbf{x}_i \\ \sum_{j=1}^{d+1} \lambda_j = 1 \end{cases} \tag{2.8}$$

are called the *barycentric coordinates* of $\mathbf{x} \in \mathbb{R}^d$. The *barycenter* or *center of gravity* of a simplex K is the point of K whose all barycentric coordinates are $\frac{1}{d+1}$. Let P_i be the set of all polynomials of total degree i.

Example 2.3.1. Each $p \in P_1$ is completely determined by its values at $\mathbf{a}_i, 1 \le i \le d+1$.

We say the parameters that uniquely determines the function in the space P_K are called *degrees of freedom* and use Σ_K to denote the set of degrees of freedom.

Example 2.3.2. Refer to figure 2.3.

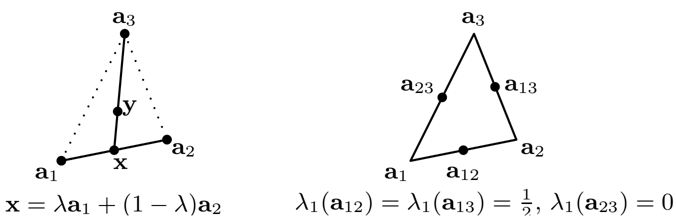

Figure 2.1: barycentric coordinate of 1 dim and 2 dim

(1) For a point $\mathbf{y}$ on the bisecting line, we have

$$\mathbf{y} = \mu\mathbf{x}+(1-\mu)\mathbf{a}_3 = \mu(\lambda\mathbf{a}_1+(1-\lambda)\mathbf{a}_2)+(1-\mu)\mathbf{a}_3 := \lambda_1\mathbf{a}_1+\lambda_2\mathbf{a}_2+\lambda_3\mathbf{a}_3.$$

(2) The barycentric coordinate λ_1 of $\mathbf{a}_{12}$ is 1/2 and that of any point on the line segment $\overline{\mathbf{a}_2\mathbf{a}_3}$ is zero!

d-simplex of type 1-linear functions

$$dimP_1(K) = d+1, \tag{2.9}$$
$$\Sigma_K = \{p(\mathbf{a}_i), 1 \le i \le d+1\}, \tag{2.10}$$
$$p = \sum_{i=1}^{d+1} p(\mathbf{a}_i)\lambda_i, \quad \forall p \in P_1, \tag{2.11}$$

where λ_i is the barycentric coordinates and in this case satisfy $\lambda_i(\mathbf{a}_j) = \delta_{ij}$. Hence $\{\lambda_i\}_{i=1}^3$ is a basis for $P_1(K)$. For the reference element $\hat{K}$, we have

$$\lambda_1(x,y) = 1-x-y, \quad \lambda_2(x,y) = x, \quad \lambda_3(x,y) = y. \tag{2.12}$$

d-simplex of type 2-quadratic functions

Define $\mathbf{a}_{ij} := \frac{1}{2}(\mathbf{a}_i + \mathbf{a}_j)$, $i < j$.

$$dimP_2(K) = \frac{(d+1)(d+2)}{2}, \tag{2.13}$$
$$\Sigma_K = \{p(\mathbf{a}_i), p(\mathbf{a}_{ij}), 1 \le i < j \le d+1\}, \tag{2.14}$$
$$p = \sum_{i=1}^{d+1} \lambda_i(2\lambda_i - 1)p(a_i) + \sum_{i<j} 4\lambda_i\lambda_j p(\mathbf{a}_{ij}), \tag{2.15}$$

where λ_k satisfy $\lambda_k(\mathbf{a}_{ij}) = \frac{1}{2}(\delta_{ik} + \delta_{kj}), 1 \le i < j \le d+1$.

d-simplex of type 3-cubic functions

Define $\mathbf{a}_{iij} := \frac{1}{3}(2\mathbf{a}_i + \mathbf{a}_j)$ for $i \neq j$, and $\mathbf{a}_{ijk} := \frac{1}{3}(\mathbf{a}_i + \mathbf{a}_j + \mathbf{a}_k)$ for $i < j < k$.

$$\begin{aligned} dim P_3(K) &= \frac{(d+1)(d+2)(d+3)}{6}, \qquad (2.16) \\ \Sigma_K &= \{p(\mathbf{a}_i), p(\mathbf{a}_{iij}), 1 \le i \neq j \le d+1, p(\mathbf{a}_{ijk}), 1 \le i < j < k \le d+1\} \\ p &= \sum_{i=1}^{d+1} \frac{\lambda_i(3\lambda_i - 1)(3\lambda_i - 2)}{2} p(\mathbf{a}_i) + \sum_{i \neq j} \frac{9\lambda_i\lambda_j(3\lambda_i - 1)}{2} p(\mathbf{a}_{iij}) \\ &\quad + \sum_{i<j<k} 27\lambda_i\lambda_j\lambda_k p(\mathbf{a}_{ijk}). \qquad (2.17) \end{aligned}$$

In general, $\dim P_k(K) = \begin{pmatrix} d+k \\ k \end{pmatrix} = {}_{d+k}C_k$.

Associated finite element space

Impose a condition on the Triangulation ($\mathcal{T}_h$ 5): Any face of any d-simplex K_1 in the triangulation is either a subset of $\partial\Omega$ or a face of another d-simplex K_2 in the triangulation.

Given a triangulation $\mathcal{T}_h$, we can associate a natural finite element space X_h satisfying for $v \in X_h$ in type (1)

(1) the restriction v_K is in $P_K = P_1(K)$ for each $K \in \mathcal{T}_h$.

(2) v is completely determined by its values at all vertices of the triangulation.

For $v \in X_h$ in type (2)

(1) the restriction v_K is in $P_K = P_2(K)$ for each $K \in \mathcal{T}_h$.

(2) v is completely determined by its values at all vertices and all the midpoints of the edges of the triangulation.

In all cases, a function v in X_h is determined by the *degrees of freedom*

$$\Sigma_h = \{p(\mathbf{a}_i), \mathbf{a}_i \in N_h\} \qquad (2.18)$$

Here N_h is certain finite set of points of $\bar{\Omega}$. Now consider canonical basis functions satisfying

$$\phi_i(\mathbf{a}_j) = \delta_{ij},$$

then such functions form a basis and has small support.

- First the linear basis functions are (cf. figure 2.3)

$$p_1 = 1 - x - y, \quad p_2 = x, \quad p_3 = y.$$

- The quadratic functions on triangle are (cf. figure 2.3):

$$\begin{aligned} p_1 &= (1-2x-2y)(1-x-y) && (2.19) \\ p_2 &= 4x(1-x-y) && (2.20) \\ p_3 &= x(2x-1) && (2.21) \\ p_4 &= 4y(1-x-y), \quad \text{etc.} && (2.22) \end{aligned}$$

What are p_5, p_6 ?

Now d-rectangles, say unit square(or cubes).

- Rectangles of type 1, $P_K = P_1[0,1] \otimes P_1[0,1]$, dim $P_K = 2^d$. Its elements are bilinear; $p_1 = x(1-y), p_2 = (1-x)(1-y), p_3 = (1-x)y, p_4 = xy$. Notice that they are constructed so that the nodal values are either zero or one.

- Rectangles of type 2, $P_K = P_2[0,1] \otimes P_2[0,1]$, dim $P_K = 3^d$. Its elements are biquadratic;

$$\begin{aligned} p_1 &= (1-x)(1-y)(1-2x)(1-2y) && (2.23) \\ p_2 &= x(1-y)(1-2x)(1-2y) && (2.24) \\ p_3 &= xy(1-2x)(1-2y) && (2.25) \\ p_4 &= y(1-x)(1-2x)(1-2y), \quad \text{etc.} && (2.26) \end{aligned}$$

One can also consider rectangles of type 3 or type 3' for which interior nodes all deleted.

Figure 2.2: Linear and quadratic element on reference triangle

Figure 2.3: Linear and quadratic element on reference rectangle

Interpolation operator

Let us denote a finite element (K, P, Σ) as triples where Σ is a set of linearly independent linear forms ϕ_i, say point evaluation at a_i or derivatives at a_i. Such $\{\phi_i\}$ are called the *dual basis.* With N= degrees of freedom, we assume for each ϕ_i, there exists a unique $p_j \in P_K$ such that $\phi_i(p_j) = \delta_{ij}$.

Given a function $v \in K \to \mathbb{R}$ sufficiently smooth, we let

$$\Pi_K v = \sum_{i=1}^{N} \phi_i(v) p_i.$$

This equals $\sum_{i=1}^{N} v(\mathbf{a}_i) p_i$ if ϕ_i is the dual basis corresponding to the nodal values. Note that $\Pi_h p = p$, $\forall p \in P$. This is called the P*-interpolant* of the function v. The reference version $\hat{\Pi}_{\hat{K}}$ is similarly defined.

The global interpolation $\Pi_h u$ is similarly defined, and called the X_h-*interpolation.*

Affine families of finite elements.

A family of finite elements is called an *affine family* if all its elements are affine equivalent to a single reference element. The concept of an affine family of finite element is important because

(1) In practical computations, most of work involved in the computation of the coefficients of linear system is performed on a reference finite element, not on a generic finite element.

(2) For such affine families, an elegant interpolation theory can be developed, which is the basis of the most convergence theorems.

Notation :$\widehat{\Pi_K(v)} = \Pi_K(v) \circ F_K$.

Example 2.3.3. (1) Assume $P_K = P_1(K)$, the linear element. The basis functions are

$$p_1 = 1 - x - y, \quad p_2 = x, \quad p_3 = y.$$

Since the nodal basis functions satisfy $\hat{p}_i = p_i \circ F_K$ we obtain

$$\widehat{\Pi_K(v)} = \sum v(\mathbf{a}_i)(p_i \circ F_K) = \sum v(F_K(\hat{\mathbf{a}}_i))\hat{p}_i = \sum \hat{v}(\hat{\mathbf{a}}_i)\hat{p}_i = \hat{\Pi}_{\hat{K}}(\hat{v}). \tag{2.27}$$

(2) Suppose we are given a triple (K, P, Σ) of triangle of type (2). Let $\hat{K}$ be a triangle(called a reference triangle) with vertices $\hat{a}_i$ and midpoint $\hat{a}_{ij} = (\hat{a}_i + \hat{a}_j)/2$. Let

$$\hat{\Sigma} = \{p(\hat{a}_i), i = 1, 2, 3,\ p(\hat{a}_{ij}), 1 \le i < j \le 3\}.$$

They are given so that $(\hat{K}, \hat{P}, \hat{\Sigma})$, $\hat{P} = P_2(\hat{K})$ is also a triangle of type 2. Given $K \in \mathcal{T}_h$ let $F_K = B_K\hat{\mathbf{x}} + \mathbf{b}_K : \hat{K} \to K$ be the unique affine mapping such that

$$F_K(\hat{a}_i) = a_i, \quad 1 \le i \le 3.$$

Then automatically it follows that

$$F_K(\hat{a}_{ij}) = a_{ij}, \quad 1 \le i < j \le 3.$$

Thus rather than prescribing such a family by the data K, P_K and Σ_K, we give just one reference element $(\hat{K}, \hat{P}, \hat{\Sigma})$ and the affine mapping F_K.

Then the generic element (K, P, Σ) is given by :

$$\begin{aligned} K &= F_K(\hat{K}) && (2.28)\\ P_K &= \{p : K \to \mathbb{R} : p = \hat{p} \circ F_K^{-1}, \hat{p} \in \hat{P}\} && (2.29)\\ \Sigma_K &= \{p(F_K(\hat{\mathbf{a}}_i)), 1 \le i \le 3,\ p(F_K(\hat{\mathbf{a}}_{ij})), 1 \le i < j \le 3\}. && (2.30) \end{aligned}$$

In fact any two Lagrangian finite elements of the same type are affine equivalent. In this example $P_K = P_2(K)$ because F_K is affine.

2.4 Interpolation error

Let $\bar{\Omega} = \cup K_h$ be a polygonal and let $V_h \subset V (= C^0(\Omega))$.

Theorem 2.4.1 (Cea's lemma)**.** *The solution u_h of the variational problem $a(u_h, v) = (f, v), \quad \forall v \in V_h$ satisfies*

$$\|u - u_h\|_{1,\Omega} \le C \inf_{\chi \in V_h} \|u - \chi\|_{1,\Omega}.$$

Proof. By Poincaré inequality, there is a constant α such that

$$\alpha \|v\|_{1,\Omega}^2 \le a(v, v), \ \ v \in H_0^1(\Omega).$$

Thus we have, for any $v_h \in V_h$

$$\begin{aligned} \alpha \|u - u_h\|_{1,\Omega}^2 &\le a(u - u_h, u - u_h)\\ &= a(u - u_h, u - v_h)\\ &\le M \|u - u_h\|_{1,\Omega} \|u - v_h\|_{1,\Omega}, \end{aligned}$$

where we used the orthogonality of FEM solution u_h:

$$a(u_h - u, v_h) = 0, \forall v_h \in V_h.$$

Canceling the factor $\|u - u_h\|_{1,\Omega}$, we obtain the result. □

In one dimensional case with piecewise linear elements, it is known that the infimum is attained when $\chi = \Pi_h u$, the X_h-interpolation. But in general, it is hard to find such χ. Instead Cea' lemma shows

$$\|u - u_h\|_{1,\Omega} \le C \|u - \Pi_h u\|_{1,\Omega}.$$

We shall show $\|u-\Pi_h u\|_{1,\Omega} \leq O(h^s)$ for some s. Taking into account that we are using the $\|\cdot\|_{1,\Omega}$ norm and that $(\Pi_h u)|_K = \Pi_K u$, we have

$$\|u-\Pi_h u\|_{1,\Omega} = (\sum_K \|u-\Pi_h u\|_{1,K}^2)^{1/2}.$$

Thus the estimate of the global error is reduced to the estimate of the local error $\|u-\Pi_h u\|_{1,K}$.

A typical result we will prove is : For a finite element which can be embedded in an affine family and whose P_K-interpolation leaves the polynomials of degree k invariant, (equiv., $P_k(K) \subset P_K$), there exists a C independent of K and v such that

$$|v-\Pi_K v|_{m,K} \leq C\frac{h_K^{k+1}}{\rho_K^m}|v|_{k+1,K}, \quad 0 \leq m \leq k+1,$$

where h_K is diameter of K and ρ_K is the maximum of diameters of spheres inscribed in K.

Proposition 2.4.1. Let Ω and $\hat{\Omega}$ be any affine equivalent open set. For $v \in H^m(\Omega)$ define $\hat{v} = v \circ F_\Omega$. Then $\hat{v} \in H^m(\hat{\Omega})$ and there is a constant C such that

$$|\hat{v}|_{m,\hat{\Omega}} \leq C\|B\|^m |det\, B|^{-1/2}|v|_{m,\Omega}, \quad \forall v \in H^m(\Omega), \tag{2.31}$$

and

$$|v|_{m,\Omega} \leq C\|B^{-1}\|^m |det\, B|^{1/2}|\hat{v}|_{m,\hat{\Omega}}, \quad \forall \hat{v} \in H^m(\hat{\Omega}). \tag{2.32}$$

Proof.

$$\frac{\partial \hat{v}}{\partial \hat{x}_i} = \sum_k \frac{\partial v}{\partial x_k}\frac{\partial x_k}{\partial \hat{x}_i}, \quad \frac{\partial^2 \hat{v}}{\partial \hat{x}_i \partial \hat{x}_j} = \sum_{k,\ell} \frac{\partial^2 v}{\partial x_k \partial x_\ell}\frac{\partial x_k}{\partial \hat{x}_i}\frac{\partial x_\ell}{\partial \hat{x}_j}.$$

In other words,

$$\hat{v}_{\hat{x}_i} = x_{k,i}v_{x_k}, \quad \hat{v}_{\hat{x}_i\hat{x}_j} = x_{k,i}v_{x_k x_\ell}x_{\ell,j}.$$

Since $(x_{k,i})_{k,i} = B$ and $(v_{x_k})_k = \operatorname{grad} v$, $(v_{x_k x_\ell})_{k,\ell} = D^2 v, \cdots$, we have

$$\hat{\operatorname{grad}}\, \hat{v} = B^t \operatorname{grad} v, \text{ and } \hat{D}^2\hat{v} = B^t D^2 v B, \cdots.$$

For each α with $|\alpha| = m$

$$\int |\partial^\alpha \hat{v}|^2 d\hat{x} \leq C\|B\|^{2m}|J|^{-1}\int |\partial^\alpha v|^2 dx.$$

Summing over all $|\alpha| = m$ we get (2.31). □

Proposition 2.4.2. The following hold:

$$C\|B\| \leq \frac{h_\Omega}{\hat{\rho}}, \quad \|B^{-1}\| \leq \frac{\hat{h}}{\rho_\Omega}. \tag{2.33}$$

Proof. Note that $\|B\| = \frac{1}{\hat{\rho}} \sup_{\|\boldsymbol{\xi}\|=\hat{\rho}} \|B\boldsymbol{\xi}\|$. For each $\boldsymbol{\xi}$ with $\|\boldsymbol{\xi}\| = \hat{\rho}$, find two points $\hat{x}, \hat{y} \in \hat{\Omega}$ such that $\hat{x} - \hat{y} = \boldsymbol{\xi}$. Since $B_\Omega \boldsymbol{\xi} = F_\Omega(\hat{x}) - F_\Omega(\hat{y})$ we have $\|B_\Omega \boldsymbol{\xi}\| \leq h_\Omega$ and the first estimate follows. The second inequality is similar. □

Corollary 2.4.2.

$$|\hat{v}|_{m,\hat{K}} \approx Ch^{m-1}|v|_{m,K}, \quad 0 \leq m \leq k+1, \quad \forall v \in H^m(\Omega). \tag{2.34}$$

Now we need to estimate the semi norm of $(v - \Pi_\Omega v)$ in $H^m(\Omega)$.

Proposition 2.4.3 (Deny-Lions Lemma, Cialet. p115)**.** For $k \geq 0$ we have a const $C(\Omega)$ such that

$$\inf_{p \in P_k} \|v + p\|_{k+1,\Omega} \leq C(\Omega)|v|_{k+1,\Omega}, \ \forall v \in H^{k+1}(\Omega). \tag{2.35}$$

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2.4.1 Polynomial preserving operators

Theorem 2.4.3. *Let $0 \leq m \leq k+1$, $k \geq 0$. Let $W^{k+1,p}(\hat{\Omega}) \hookrightarrow W^{m,q}(\hat{\Omega})$ and $\hat{\Pi} : W^{k+1,p}(\hat{\Omega}) \to W^{m,q}(\hat{\Omega})$ be a linear mapping such that*

$$\hat{\Pi}\hat{p} = \hat{p}, \quad \forall \hat{p} \in P_k(\hat{\Omega}). \tag{2.36}$$

For any open set Ω affine equivalent to $\hat{\Omega}$, define $\Pi_\Omega v$ through the relation:

$$\widehat{\Pi_\Omega v} = \hat{\Pi}\hat{v}, \quad \forall \hat{v} \in W^{k+1,p}(\hat{\Omega}), \forall v \in W^{k+1,p}(\Omega). \tag{2.37}$$

Then there exists a constant $C(\hat{\Pi}, \hat{\Omega})$ such that

$$|v - \Pi_\Omega v|_{m,\Omega} \leq C(\hat{\Pi}, \hat{\Omega}) m(\Omega)^{1/q-1/p} \frac{h^{k+1}}{\rho^m} |v|_{k+1,p,\Omega}, \ v \in W^{k+1,p}(\Omega). \tag{2.38}$$

Proof. Using polynomial invariance, we have

$$\hat{v} - \hat{\Pi}\hat{v} = (I - \hat{\Pi})(\hat{v} + \hat{p}), \ \forall \hat{v} \in W^{k+1,p}(\hat{\Omega}), \ \forall \hat{p} \in P_k(\hat{\Omega}).$$

From which we have that

$$|\hat{v} - \hat{\Pi}\hat{v}|_{m,q,\hat{\Omega}} \leq \|(I - \hat{\Pi})\|_{\mathcal{L}} \inf_{\hat{p}\in\hat{\Omega}} \|\hat{v} + \hat{p}\|_{k+1,p,\hat{\Omega}} \tag{2.39}$$

$$\leq C(\hat{\Pi}, \hat{\Omega})|\hat{v}|_{k+1,p,\hat{\Omega}} \tag{2.40}$$

by Proposition 2.4.3. Here $\|(I - \hat{\Pi})\|_{\mathcal{L}}$ denotes the operator norm-we assume it is bounded- see p.123 of Ciarlet. From (2.32) we have

$$|v - \Pi v|_{m,\Omega} \leq C\|B^{-1}\|^m |\det(B)|^{1/2}|\hat{v} - \hat{\Pi}\hat{v}|_{m,\hat{\Omega}}. \tag{2.41}$$

Thus combining this with (2.40), (2.33) and using $\|B\| \leq h/\hat{\rho}$, $\|B^{-1}\| \leq \hat{h}/\rho$ and the fact that $\hat{\rho}$ and $\hat{h}$ are independent of h, we obtain (2.38). □

2.4.2 Interpolation errors $|v - \Pi_h v|_{m,p,K}$ for affine families

Throughout this section we assume the following (H1), (H2) and (H3).

Definition 2.4.4. (H1) (p. 124) A family of triangulation $\mathcal{T}_h$ is *regular* if there is $\sigma > 1$ such that

(i) $\max_K \frac{h_K}{\rho_K} \leq \sigma$ and

(ii) h_K approaches zero.

In other words, the family of elements $(K, P_K, \Sigma_K), K \in \mathcal{T}_h$ is a *regular family of elements.*

(H2) All finite elements $(K, P_K, \Sigma_K), K \in \cup\mathcal{T}_h$ are affine equivalent to a single reference element $(\hat{K}, \hat{P}, \hat{\Sigma})$.

(H3) All finite elements $(K, P_K, \Sigma_K), K \in \cup\mathcal{T}_h$ are class C^0.

Specializing the above results to finite elements, we obtain estimates of the interpolation errors $|v - \Pi_K v|_{m,p,K}$. For simplicity, we take $p = q = 2$ below.

For regular families, i.e, $h_K \leq \sigma\rho_K$, we have for $0 \leq m \leq k+1$:

Theorem 2.4.5. *In addition to (H1), (H2) and (H3) assume there are integers $0 \leq s \leq k$ such that*

$$P_k(\hat{K}) \subset \hat{P} \subset H^1(\hat{K}), H^{k+1}(\hat{K}) \hookrightarrow C^s(\hat{K}), \tag{2.42}$$

where s is the maximal order of partial derivatives appearing in the definition of the set $\hat{\Sigma}$. Then there exists a const independent of h such that

$$|v-\Pi_K v|_{m,K} \le Ch^{k+1-m}|v|_{k+1,K}, \ m=0,1, \tag{2.43}$$

$$\left(\sum_K \|v-\Pi_h v\|_{m,K}^2\right)^{1/2} \le Ch^{k+1-m}|v|_{k+1,\Omega}, \ m=0,1. \tag{2.44}$$

Proof. Note the boundedness of $\|(I-\hat{\Pi})\|_{\mathcal{L}}$(independent of K), i.e,

$$\|\hat{\Pi}\hat{v}\|_{m,q,\hat{K}} \ \le \ C(\hat{K},\hat{P},\hat{\Sigma})\|\hat{v}\|_{k+1,p,\hat{K}}. \tag{2.45}$$

Use theorem 2.4.3 for $\hat{K}=\hat{\Omega}$, $K=\Omega$. The result is a restatement of (2.44). □

2.5 Interpolation theory-Bramble Hilbert lemma

We let $W^{m,p}(\Omega)$ the space of all functions $u\in L^p(\Omega)$ for which all partial derivatives of u up to order m belong to $L^p(\Omega)$, equipped with the norm

$$\begin{cases} \|u\|_{m,p,\Omega} &= \left(\sum_{|\alpha|\le m}\int_\Omega |\partial^\alpha u|^p dx\right)^{1/p}, \text{ if } 1\le p<\infty, \\ \|u\|_{m,\infty,\Omega} &= \max_{|\alpha|\le m}\{\|\partial^\alpha u\|_\infty\} \text{ if } p=\infty. \end{cases}$$

The space $W^{m,p}(\Omega)$ is a Banach space. We shall also consider the semi-norms

$$\begin{cases} |u|_{m,p,\Omega} &= \left(\sum_{|\alpha|=m}\int_\Omega |\partial^\alpha u|^p dx\right)^{1/p}, \text{ if } 1\le p<\infty, \\ |u|_{m,\infty,\Omega} &= \max_{|\alpha|=m}\{\|\partial^\alpha u\|_\infty\}, \text{ if } p=\infty. \end{cases}$$

The Sobolev space $W_0^{m,p}(\Omega)$ is the closure of the space $\mathcal{D}(\Omega)$ in the space $W^{m,p}(\Omega)$. We let

$$H^m(\Omega)=W^{m,2}(\Omega) \text{ and } H_0^m(\Omega)=W_0^{m,2}(\Omega), \tag{2.46}$$

$$\|u\|_{m,2,\Omega}=\|u\|_{m,\Omega}, \ \ |u|_{m,2,\Omega}=|u|_{m,\Omega}. \tag{2.47}$$

The following proposition is almost the same as Deny-Lion Lemma. But in this note it will be used to estimate the consistency error (e.g., estimate the quadrature error).

Proposition 2.5.1. [Bramble-Hilbert lemma, p192 Ciarlet] Let Ω be an open subset of $\mathbb{R}^d$ with Lipschitz-continuous boundary. For some integer $m, k \geq 0$ and let ℓ be a continuous linear form on the space $W^{k+1,p}(\Omega)$ such that

$$\ell(p) = 0, \quad \forall p \in P_k(\Omega). \tag{2.48}$$

Then for $v \in W^{k+1,p}(\Omega)$, we have

$$|\ell(v)| \leq C(\Omega)\|\ell\|^*_{k+1,p,\Omega} \inf_{p\in P_k} \|v+p\|_{k+1,p,\Omega} \leq C|v|_{k+1,p,\Omega}, \, \forall v \in W^{k+1,p}(\Omega), \tag{2.49}$$

where $\|\cdot\|^*_{k+1,p,\Omega}$ is the norm of the dual space of $W^{k+1,p}(\Omega)$.

Proof. Let v be a function in the space $W^{k+1,p}(\Omega)$. We have

$$|\ell(v)| = |\ell(v+p)| \leq \|\ell\|^*_{k+1,p,\Omega}\|v+p\|_{k+1,p,\Omega} \text{ for any } p \in P_k(\Omega),$$

and the result follows by proposition 2.4.3(Deny-Lion). □

In particular, if we let $\ell(v) = |(I - \Pi_\Omega)(v)|_{m,p,\Omega}$, then we have

$$|v - \Pi_\Omega v|_{m,p,\Omega} \leq \|I - \Pi_\Omega\|^* \inf_{p\in P_k} \|v+p\|_{k+1,p,\Omega}.$$

Notice the difference between B-H and Deny-Lion lemma and subsequent argument. The Bramble-Hilbert lemma is more general.

Definition 2.5.1. Let $0 < \alpha \leq 1$. We say f is Hölder continuous (order α) if

$$|f(x) - f(y)| \leq C|x-y|^\alpha$$

for all x, y in the domain. When $\alpha = 1$ it is called Lipschitz continuous.

Define $\mathcal{C}^{m,\alpha}(\bar{\Omega})$ to be the space of all functions in $\mathcal{C}^m(\bar{\Omega})$ whose m-th derivatives satisfy the Hölder continuity. We equip it with the norm

$$\|v\|_{\mathcal{C}^{m,\alpha}(\bar{\Omega})} := \|v\|_{m,\infty\bar{\Omega}} + \max_{|\beta|=m} \sup_{x\neq y\in\bar{\Omega}} \frac{|\partial^\beta v(x) - \partial^\beta v(y)|}{\|x-y\|^\alpha}$$

and we call it the Hölder spaces of order $0 < \alpha \leq 1$.

Theorem 2.5.2. *(Sobolev Imbedding Theorem) For all integers $m \geq 0$ and* *all* $1 \leq p \leq \infty$,

$$W^{m,p}(\Omega) \hookrightarrow L^q(\Omega) \qquad \text{with } \frac{1}{q} = \frac{1}{p} - \frac{m}{n}, \text{ if } m < \frac{n}{p}, \tag{2.50}$$

$$W^{m,p}(\Omega) \hookrightarrow L^q(\Omega) \qquad \text{for all } q \in [1, \infty), \text{ if } m = \frac{n}{p}, \tag{2.51}$$

$$W^{m,p}(\Omega) \hookrightarrow \mathcal{C}^{0,m-n/p}(\bar{\Omega}), \text{ if } \frac{n}{p} < m < \frac{n}{p} + 1, \tag{2.52}$$

$$W^{m,p}(\Omega) \hookrightarrow \mathcal{C}^{k,m-n/p}(\bar{\Omega}), \text{ if } \frac{n}{p} + k < m < \frac{n}{p} + k + 1, \tag{2.53}$$

$$W^{m,p}(\Omega) \hookrightarrow \mathcal{C}^{0,\alpha}(\bar{\Omega}) \qquad \text{for all } 0 < \alpha < 1, \text{ if } m = \frac{n}{p} + 1, \tag{2.54}$$

$$W^{m,p}(\Omega) \hookrightarrow \mathcal{C}^{0,1}(\bar{\Omega}), \qquad \text{if } \frac{n}{p} + 1 < m. \tag{2.55}$$

Theorem 2.5.3. *(Kondrasov theorems) We have the compact injections*

$$W^{m,p}(\Omega) \overset{c}{\hookrightarrow} L^q(\Omega) \text{ for all } 1 \leq q < p^* \text{ with } \frac{1}{p^*} = \frac{1}{p} - \frac{m}{n}, \text{ if } m < \frac{n}{p}, \tag{2.56}$$

$$W^{m,p}(\Omega) \overset{c}{\hookrightarrow} L^q(\Omega) \text{ for all } q \in [1, \infty), \text{ if } m = \frac{n}{p}, \tag{2.57}$$

$$W^{m,p}(\Omega) \overset{c}{\hookrightarrow} \mathcal{C}^0(\bar{\Omega}), \qquad \text{if } m > \frac{n}{p}. \tag{2.58}$$

The compact injection $H^1(\Omega) \overset{c}{\hookrightarrow} L^2(\Omega)$ *is called the Rellich theorem.*

2.6 Estimate in H^1 error : $\|u - u_h\|_{1,\Omega}$

Theorem 2.6.1. *Let* u *be the solution of variational problem belong to* $H^{k+1}(\Omega)$ *and* u_h *be the finite element solution. Under the same assumption as Theorem 2.4.5, we have*

$$\|u - u_h\|_{1,\Omega} \leq Ch^k |u|_{k+1,\Omega}. \tag{2.59}$$

Proof. Use Cea's Lemma and the estimate in the interpolation error. □

2.7 Estimate of the L^2 error : $\|u - u_h\|_{0,\Omega}$ - Aubin Nitsche lemma

We shall derive L^2 error estimate from H^1 error estimate (Theorem 3.2.2). We get a pickup of h. For this, we note that $H_0^1 \hookrightarrow L^2$. We show

Theorem 2.7.1. *(Aubin-Nitsche lemma) We have*

$$\|u - u_h\|_0 \leq M\|u - u_h\|_1 \sup_{g \in L^2} \left\{ \frac{1}{\|g\|_0} \inf_{\phi_h} \|\phi_g - \phi_h\|_1 \right\}, \tag{2.60}$$

where for any $g \in L^2, \phi_g \in H_0^1$ is the unique solution of the variational problem:

$$a(v, \phi_g) = (g, v), \forall v \in H_0^1. \tag{2.61}$$

Proof. First of all, notice that

$$\|u - u_h\|_0 = \sup_{g \in L^2} \frac{|(g, u - u_h)|}{\|g\|_0}. \tag{2.62}$$

The solution of (2.61) satisfy

$$a(u - u_h, \phi_g) = (g, u - u_h),$$

while

$$a(u - u_h, \phi_h) = 0, \ \forall \phi_h \in V_h.$$

Thus

$$a(u - u_h, \phi_g - \phi_h) = (g, u - u_h), \ \forall \phi_h \in V_h,$$

and therefore,

$$|(g, u - u_h)| \leq M\|u - u_h\|_1 \inf_{\phi_h} \|\phi_g - \phi_h\|_1. \tag{2.63}$$

The conclusion now follows from (2.62). □

Note that in (2.61) the order of arguments are interchanged. Problem (2.61) is a special case of the general problem: Given any element $g \in V$, find $\phi \in V$ such that

$$a(v, \phi) = g(v), \ \forall v \in V.$$

Such a problem is called the adjoint problem of (2.1).

A second order boundary value problem whose variational formulation is (2.1), resp. (2.61) is said to be regular if the following conditions holds:

(1) For any $f \in L^2$, resp. any $g \in L^2$, the corresponding solution u_f, resp. u_g, is in $H^2 \cap V$.

(2) There exists a constant C such that

$$\|u_f\|_{2,\Omega} \leq C\|f\|_{0,\Omega},\ \forall f \in L^2(\Omega), \tag{2.64}$$

$$\|\phi_g\|_{2,\Omega} \leq C\|g\|_{0,\Omega},\ \forall g \in L^2(\Omega). \tag{2.65}$$

Remark 2.7.2. Consider (2.1). Then without the regularity assumption we only know that

$$\alpha\|u_f\|_{1,\Omega} \leq \|f\|^* = \sup_{v\in V} \frac{|f(v)|}{\|v\|_{1,\Omega}} \tag{2.66}$$

$$= \sup_{v\in V} \frac{|\int fv dx|}{\|v\|_{1,\Omega}} \leq \|f\|_{0,\Omega},\ \forall f \in L^2(\Omega). \tag{2.67}$$

Theorem 2.7.3. *In addition to (H1), (H2), and (H3), assume $s = 0, d \leq 3$, and that for some $k \geq 1$ the solution u is in the space $H^{k+1}(\Omega)$ and the inclusion*

$$P_k(\hat{K}) \subset \hat{P} \subset H^1(\hat{K}) \tag{2.68}$$

hold. Then if the adjoint problem is regular, there exists a constant C independent of h such that

$$\|u - u_h\|_{0,\Omega} \leq Ch^{k+1}|u|_{k+1,\Omega}. \tag{2.69}$$

Proof. Since $d \leq 3$, the inclusion $H^2(\hat{K}) \hookrightarrow \mathcal{C}(\hat{K})$ holds. Applying Theorem 2.7.1 and inequality (2.65), we obtain, for each $g \in L^2(\Omega)$,

$$\inf_{\phi_h\in V_h} \|\phi_g - \phi_h\|_{1,\Omega} \leq \|\phi_g - \Pi_h\phi_g\|_{1,\Omega} \leq Ch\|\phi_g\|_{2,\Omega} \leq Ch\|g\|_{0,\Omega}.$$

Combining this with (2.60) yields

$$\|u - u_h\|_{0,\Omega} \leq Ch\|u - u_h\|_{1,\Omega}.$$

□

2.8 Noncoercive forms

Let V and H be Hilbert spaces with $V \subset H$ and

$$\|u\|_H \leq \|u\|_V, \quad u \in V. \tag{2.70}$$

Let $A(\cdot,\cdot)$ be a bounded bilinear form on $V \times V$, i.e,

$$|A(u,v)| \leq \beta \|u\|_V \|v\|_V, \quad u,v \in V. \tag{2.71}$$

Let $V_n, = 1, 2, \cdots$, be a sequence of finite dimensional subspace of V and suppose that there exist positive constants ρ and γ such that

$$\rho \|u\|_V - \gamma \|u\|_H \leq \sup_{v \in V_n} \frac{|A(u,v)|}{\|v\|_V}, \quad u \in V_n. \tag{2.72}$$

Finally, suppose there exists a sequence of positive numbers $\{\delta_n\}$ with $\lim_{n\to\infty} \delta_n = 0$, and such that for every $e_n \in V$ satisfying

$$A(e_n, \phi) = 0, \quad \forall \phi \in V_n,$$

then it is true that

$$\|e_n\|_H \leq \delta_n \|e_n\|_V. \tag{2.73}$$

Theorem 2.8.1. *Let $u \in V$ be given and consider the problem of finding $u_n \in V_n$ such that*

$$A(u - u_n, \phi) = 0, \quad \phi \in V_n. \tag{2.74}$$

If conditions (2.70)-(2.73) hold, then there exists an integer N_0, independent of u, such that (2.74) has a unique solution u_n for all $n \geq N_0$. Moreover, there exist a constant C such that

$$\|u - u_n\|_V \quad \leq C \min_{\chi \in V_n} \|u - \chi\|_V \tag{2.75}$$

$$\|u - u_n\|_H \quad \leq C\delta_n \min_{\chi \in V_n} \|u - \chi\|_V. \tag{2.76}$$

Proof. Assume $u_n \in V_n$ is a solution of (2.74). Then

$$A(u_n - \chi, v) = A(u - \chi, v), \quad \forall \chi, v \in V_n.$$

Hence from (2.71) and (2.72),

$$\begin{aligned} \rho \|u_n - \chi\|_V - \gamma \|u_n - \chi\|_H \quad &\leq \sup_{\substack{\phi \in V_n \\ \|\phi\|_V = 1}} |A(u_n - \chi, \phi)| \\ &= \sup_{\substack{\phi \in V_n \\ \|\phi\|_V = 1}} |A(u - \chi, \phi)| \\ &\leq \beta \|u - \chi\|_V, \quad \forall \chi \in V_n. \end{aligned} \tag{2.77}$$

We may assume $\gamma \geq 0$. By (2.73) with $e_n = u - u_n$, we get

$$(\rho - \gamma\delta_n)\|u - u_n\|_V \leq \rho\|u - u_n\|_V - \gamma\|u - u_n\|_H.$$

By triangle inequality

$$\begin{aligned} \rho\|u - u_n\|_V - \gamma\|u - u_n\|_H \leq \quad & \rho\|u - \chi\|_V + \gamma\|u - \chi\|_H & (2.78) \\ & +(\rho\|\chi - u_n\|_V - \gamma\|\chi - u_n\|_H). & (2.79) \end{aligned}$$

Combining, using (2.77), we have

$$\begin{aligned} (\rho - \gamma\delta_n)\|u - u_n\|_V &\leq \rho\|u - \chi\|_V + \gamma\|u - \chi\|_H + \beta\|u - \chi\|_V & (2.80) \\ &\leq (\rho + \gamma + \beta)\|u - \chi\|_V, \quad \chi \in V_n. & (2.81) \end{aligned}$$

The estimate $\|u - \chi\|_H \leq \|u - \chi\|_V$ comes from (2.70). Since $\lim \delta_n = 0$, there exists an integer N_0 such that $\delta_n \leq \rho/(2\gamma)$ for $n \geq N_0$. Then

$$\|u - u_n\|_V \leq C\|u - \chi\|_V, \quad \chi \in V_n,$$

where $C = 2\frac{(\rho+\gamma+\beta)}{\rho}$. Thus (2.75) holds. (2.76) follows immediately from (2.73).

So far we have shown that if $u_n \in V_n$ is a solution of (2.74), then there exists N_0 such that (2.75) and (2.76) holds. Now we shall show existence and uniqueness by proving uniqueness.

We now show uniqueness:

Assume u_n and v_n are two solutions of (2.74), $w_n = u_n - v_n$ satisfies

$$A(w_n, \phi) = 0, \quad \phi \in V_n.$$

Then w_n is a solution of (2.74) for the case $u = 0$. Then from (2.75),

$$\|w_n\|_V \leq C \min_{\chi \in V_n} \|0 - \chi\|_V = 0.$$

Thus $u_n = v_n$, when $n > N_0$. Now we need to show the existence of u_n. We rewrite (2.74) as

$$A(u_n, \phi) = G(\phi), \quad \forall \phi \in V_n,$$

where $G(\phi) = A(u, \phi)$. But in the case of finite dimension, existence is equivalent to uniqueness. □

(1) An Observation Concerning Ritz-Galerkin Methods with Indefinite Bilinear Forms, Alfred H. Schatz, Math. comp. Vol. 28, No. 128, 1974, 959-962.

(2) Some new error estimates for RITZ-GALERKIN methods with minimal regularity assumptions, A H. SCHATZ and J. WANG, Math. comp. Vol. 65, 1996, Pages 19-27.

Remark 2.8.2. I. In applications, V is usually taken as $H^1(\Omega)$ and H is $L^2(\Omega)$. Then (2.73) implies that the L^2-error goes to zero faster than the $H^1(\Omega)$-error. Note that assumption (2.72) is implied by either one of the following:
(2.72)' $A(\cdot,\cdot)$ is coercive;
(2.72)" there exist constants $\rho > 0$ and γ such that

$$\rho\|u\|_V^2 - \gamma\|u\|_H^2 \leq A(u,u), \quad u \in V. \tag{2.82}$$

Example 2.8.3. Let $V = H_0^1(\Omega)$, and

$$A(u,v) = (\mathcal{L}u, v) + (\mathbf{b}^T\nabla u, v) + (cu, v),$$

with $G(v) = (g,v), \quad v \in V$. With the assumption that $b_i \in C^1(\bar{\Omega})$, we can show that

$$\left|\int_\Omega \mathbf{b}^T\nabla uvdxdy\right| \leq \sum_{i=1}^n \int_\Omega \left|b_i\frac{\partial u}{\partial x_i}v\right| dxdy \tag{2.83}$$

$$\leq b_1\|v\|\sum_{i=1}^n \|\frac{\partial u}{\partial x_i}\| \leq b_1 n\|u\|_1\|v\|_1. \tag{2.84}$$

and hence $A(u,v)$ is bounded. Further we can show that

$$\int_\Omega (\mathbf{b}^T\nabla u)udxdy = -\frac{1}{2}\int_\Omega (\nabla\cdot\mathbf{b})u^2dxdy$$

and it follows that

$$A(u,u) = (\mathcal{L}u,u) + (\phi,u^2)$$

where $\phi = c - \frac{1}{2}\,\mathrm{div}\cdot \mathbf{b}$. Hence with $c_0 = \max|\phi|$, we have

$$A(u,u) \geq \rho\|u\|_1^2 - c_0\|u\|^2, \quad u \in H_0^1(\Omega). \tag{2.85}$$

This is a special case of *Gårding's inequality.*

Let $H = L^2(\Omega)$. Then assumption (2.70) is satisfied and (2.85) yields (3)". Hence (2.72) is also satisfied. We conclude from Theorem 2.8.1 and remark that if there exists $\hat{u} \in H_0^1(\Omega)$ such that

$$A(u,v) = G(v), \quad \forall v \in H_0^1(\Omega),$$

then for any family of subspaces $\{V_N\} \subset H_0^1(\Omega)$ satisfying assumption (2.73), the Galerkin solution u_N exists and has the properties

$$\|u-u_N\|_1 \leq C \min_{\chi\in V_N} \|u-\chi\|_1, \quad \|u-u_N\| \leq \delta_N C \min_{\chi\in V_N}\|u-\chi\|_1,$$

where u is the generalized solution of the boundary value problem

$$-\nabla\mathcal{K}\nabla u + \mathbf{b}^T\nabla u + cu = g \text{ in } \Omega, \tag{2.86}$$
$$u = 0 \text{ on } \partial\Omega. \tag{2.87}$$

Note that we do not assume that $c(x) \geq 0$. Note that if $c_0 < \rho$, then $A(\cdot,\cdot)$ is coercive, the Lax-Milgram lemma is applicable, and the existence of u is guaranteed.

Exercise 2.8.4. (1) (10pts) Show that if either (2.72)' or (2.72)" holds then (2.73) holds.

(2) (10pts) Prove Gårding's inequality without the assumption on the smoothness of $\mathbf{b}$. (Estimate the first order term directly and arithmetic-geometric inequality.)

(3) (10pts) Show that (2.72) holds for the above example directly.

2.9 Eigenvalues and miscellany

This part of note is from Quarteroni and Valli. p.195. Here $d = 2,3$ is the dimension of Ω.

Definition 2.9.1. A family of triangulation $\mathcal{T}_h$ is *quasi-uniform* if it is regular and there is $\tau > 0$ such that

$$\min_K h_K \geq \tau h. \tag{2.88}$$

Here $h = \max h_K, K \in \mathcal{T}_h$.

Proposition 2.9.2. *Let $\mathcal{T}_h$ be quasi-uniform family of triangulation of Ω. There exists constants C_1, C_2 such that for $v_h \in V_h$, $v_h = \sum \eta_i \phi_i$,*

$$C_1 h^d |\boldsymbol{\eta}|^2 \leq \|v_h\|_0^2 \leq C_2 h^d |\boldsymbol{\eta}|^2. \tag{2.89}$$

Proof. Since $\mathcal{T}_h$ is regular, for any given finite element node, the number of elements sharing the node is bounded uniformly with resp. to h. Hence it suffices to show that

$$C_1^* h^d \sum_{i=1}^{T} \eta_i^2 \leq \int_K v_h^2 \leq C_2^* h^d \sum_{i=1}^{T} \eta_i^2. \tag{2.90}$$

Here T is the number of degrees of freedom associated with K. First we show it for reference element and then use $\hat{v} = v_h \circ F_K$, where F_K is the affine map from $\hat{K}$ to K. Thus

$$\hat{v} = \sum_{i=1}^{T} \eta_i \hat{\phi}_i.$$

Define for $\hat{v} \neq 0$,

$$\psi(\hat{v}) := \frac{\int_{\hat{K}} \hat{v}^2}{\sum_{i=1}^{T} \eta_i^2}.$$

This function is clearly positive and continuous and hence $\psi(\hat{v})$ has positive minimum and maximum (C_1^*, C_2^*) on the unit sphere: $S^1 = \{\hat{v} \in V : \|\hat{v}\|_0 = 1\}$. Since it is homogeneous of zero degree, i.e, $\psi(t\boldsymbol{\eta}) = \psi(\boldsymbol{\eta})$ for $t > 0$, we have for any $\hat{v} \neq 0$, the scaled function $\frac{\hat{v}}{\|\hat{v}\|_0}$ belongs to the unit sphere, and hence we have

$$0 < C_1^* \leq \psi(\frac{\hat{v}}{\|\hat{v}\|_0}) = \psi(\hat{v}) \leq C_2^*.$$

Hence

$$C_1^* \sum_{i=1}^{T} \eta_i^2 \leq \int_{\hat{K}} \hat{v}^2 \leq C_2^* \sum_{i=1}^{T} \eta_i^2, \ \forall \hat{v} \neq 0. \tag{2.91}$$

This clearly holds for $\hat{v} = 0$. An alternative proof maybe:

$$\begin{aligned}(\hat{v}, \hat{v}) &= (\sum_{i=1}^{T} \eta_i \hat{\phi}_i, \sum_{i=1}^{T} \eta_i \hat{\phi}_i) \\ &= \boldsymbol{\eta}^T M \boldsymbol{\eta},\ M_{ij} = (\hat{\phi}_i, \hat{\phi}_j).\end{aligned}$$

Since M is nonsingular, the function $\boldsymbol{\eta}^T M \boldsymbol{\eta}$ is continuous on $\mathbb{R}^T \backslash \{0\}$. Considering on the unit sphere, we deduce there are positive constants μ_m, μ_M independent of h such that

$$\mu_m |\boldsymbol{\eta}|^2 \leq \boldsymbol{\eta}^T M \boldsymbol{\eta} \leq \mu_M |\boldsymbol{\eta}|^2.$$

Thus, we obtain (2.91). Considering the integral $\int_K v_h^2$, we see

$$\int_K v_h^2 \, dx = \int_K (\hat{v} \circ F_K^{-1})^2 \, dx = \int_{\hat{K}} \hat{v}^2 |det\, B_K| \, d\hat{x}. \tag{2.92}$$

Choosing $v_h = 1$ we have

$$|det\ B_K| = \frac{meas(K)}{meas(\hat{K})} \leq C h_K^d.$$

On the other hand, since the family $\mathcal{T}_h$ is regular, we have

$$|det\ B_K| \geq C h_K^d.$$

This together with (2.91), (2.92), we obtain (2.90). $\square$

Proposition 2.9.3 (Inverse inequality)**.** *Let $\mathcal{T}_h$ be quasi-uniform family of triangulation of Ω. There exists constants such that for $v_h \in V_h$,*

$$\|\nabla v_h\|_0^2 \leq C h^{-2} \|v_h\|_0^2. \tag{2.93}$$

Proof. It suffices to prove

$$\int_K |\nabla v_h|^2 \leq C h^{-2} \int_K v_h^2. \tag{2.94}$$

Again on the reference element, we consider

$$\psi^*(\hat{v}) := \frac{\int_{\hat{K}} |\nabla \hat{v}|^2}{\int_{\hat{K}} |\hat{v}|^2}.$$

Since it is homogeneous of zero degree, bounded, hence by the same argument as before,

$$\int_K |\nabla v_h|^2 \leq C\|B_K^{-1}\|^2 \int_K v_h^2 \leq \frac{C}{\rho_K^2}\int_K v_h^2.$$

Now the regularity of triangulation and (2.90) gives the result. □

Now we turn to the estimate the spectral condition number of A. Writing $v_h = \sum \eta_i \phi_i$, we have

$$\frac{(A\boldsymbol{\eta},\boldsymbol{\eta})}{|\boldsymbol{\eta}|^2} = \frac{a(v_h, v_h)}{|\boldsymbol{\eta}|^2}. \tag{2.95}$$

Since $\mathcal{A}(\cdot,\cdot)^{1/2}$ is equiv to H^1-norm, we have by (2.89) and (2.92),

$$\alpha C_1 h^d \leq \frac{(A\boldsymbol{\eta},\boldsymbol{\eta})}{|\boldsymbol{\eta}|^2} \leq \gamma C_2 h^d(1 + C_3 h^{-2}). \tag{2.96}$$

Hence

$$\frac{\lambda_M}{\lambda_m} \leq C(1 + C_3 h^{-2}) = O(h^{-2}).$$

More precisely, we have shown that any eigenvalue of A satisfies

$$\alpha C_1 h^d \leq \lambda \leq \gamma C_2 h^d(1 + C_3 h^{-2}).$$

Now we compare the spectrum of A and the spectrum of bilinear form $a(\cdot,\cdot)$. Since

$$a(w_h, v_h) = \lambda(w_h, v_h), \quad v_h \in V_h.$$

Thus (2.96) is equivalent to

$$\alpha \leq \frac{a(w_h, w_h)}{\|w_h\|_0^2} = \lambda \leq \gamma \frac{\|w_h\|_1^2}{\|w_h\|_0^2} \leq \gamma C_2(1 + C_3 h^{-2}). \tag{2.97}$$

Hence the eigenvalues of a satisfy $\alpha \leq \lambda \leq \gamma C_2(1 + C_3 h^{-2})$. Notice the extra factor h^d appearing in the spectrum of the stiffness matrix A. For this reason, sometimes A is scaled by h^{-d} so that the spectrum is equivalent to $a(\cdot,\cdot)$. This is a correct finite dimensional approximation of elliptic operator, which has eigenvalues in (α, ∞).

Example 2.9.4. We consider

$$\begin{aligned} -\Delta u &= f \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega. \end{aligned}$$

When the unit square is divided by n equal intervals along x-axis and y-axis, then the corresponding matrix A scaled by h^{-2} is $(n-1) \times (n-1)$block-diagonal matrix of the form:

$$A = \frac{1}{h^2} \begin{bmatrix} B & -I & 0 & \cdots & \\ -I & B & -I & 0 & \\ & -I & \ddots & \ddots & \\ & & \ddots & B & -I \\ & \cdots & 0 & -I & B \end{bmatrix} \tag{2.98}$$

where

$$B = \begin{bmatrix} 4 & -1 & 0 & \cdots & \\ -1 & 4 & -1 & 0 & \\ & -1 & \ddots & \ddots & \\ & & \ddots & 4 & -1 \\ & \cdots & 0 & -1 & 4 \end{bmatrix}$$

is $(n-1) \times (n-1)$ matrix. In fact this A is the representation w.r.t to the discrete L^2 inner product $(\cdot,\cdot)_h := \sum_i h^2 u_i v_i$. The eigenvectors (up to constant) of $(n-1)^2 \times (n-1)^2$ matrix A are

$$\mathbf{x}_{\nu\mu}(x, y) = \sin(\nu\pi x)\sin(\mu\pi y), \tag{2.99}$$

with the corresponding eigenvalues

$$\lambda_{\nu\mu} = 4h^{-2}(\sin^2(\nu\pi h/2) + \sin^2(\mu\pi h/2)), \quad 1 \le \nu, \mu \le n-1. \tag{2.100}$$

2.10 Inverse inequalities

In this section, in addition to regularity of $\mathcal{T}_h$, assume that it is quasi-uniform, i.e, there is a positive number $\tau > 0$ such that

$$\min_K h_K \ge \tau h, \quad \forall h > 0.$$

Theorem 2.10.1. *Let $\mathcal{T}_h$ satisfy the hypothesis (H1) and (H2) and let*

$$l \le m \text{ and } \hat{P} \subset W^{m,p}(\hat{K}).$$

Then we have

$$\left(\sum_K |v_h|^p_{m,p,K}\right)^{1/p} \leq Ch^{l-m}\left(\sum_K |v_h|^p_{l,p,K}\right)^{1/p}, \quad \forall v_h \in X_h. \tag{2.101}$$

Proof. Given $v_h \in X_h$, we have by Proposition 2.4.1,

$$\begin{aligned} |\hat{v}_K|_{l,p,\hat{K}} &\leq C\|B_K\|^l |\det(B)|^{-1/p}|v_h|_{l,p,K}, & (2.102)\\ |v_h|_{m,p,\hat{K}} &\leq C\|B_K^{-1}\|^l |\det(B)|^{1/p}|\hat{v}_K|_{m,p,K}, & (2.103)\end{aligned}$$

where the function $\hat{v}_K$ is the standard correspondence with the function $v_h|_K$.

Define the space

$$\hat{N} = \{\hat{p} \in \hat{P}; |\hat{p}|_{l,p,\hat{K}} = 0\} = \begin{cases} 0 & \text{if } l = 0,\\ \hat{P} \cap P_{l-1}(\hat{K}) & \text{if } l \geq 1.\end{cases}$$

Since $l \leq m$, $|\hat{p}|_{m,p,\hat{K}} = 0$ for $\hat{p} \in \hat{N}$ and hence

$$\|\dot{\hat{p}}\|_{m,p,K} = \inf_{\hat{s}\in\hat{N}} |\hat{p} - \hat{s}|_{m,p,K}$$

is a norm over the quotient space $\hat{P}/\hat{N}$. Since this quotient space is finite dimensional, this norm is equivalent to the quotient norm $\|\cdot\|_{l,p,\hat{K}}$ therefore there exists a constant C such that

$$|\hat{p}|_{m,p,\hat{K}} = \|\dot{\hat{p}}\|_{m,p,\hat{K}} \leq C\|\hat{p}\|_{l,p,\hat{K}}. \tag{2.104}$$

By regularity and inverse property, we obtain from (2.103) and (2.104) and Theorem 3.1.3,

$$|v_h|_{m,p,K} \leq Ch^{l-m}|v_h|_{l,p,K}. \tag{2.105}$$

Summing over all elements,

$$\left(\sum_K |v_h|^p_{m,p,K}\right)^{1/p} \leq Ch^{l-m}\left(\sum_K |v_h|^p_{l,p,K}\right)^{1/p}.$$

□

2.11 Fractional order interpolation

See Hitchhiker's Guid to fractional Sobolev space.

Define

$$\overset{\circ}{W}{}^{k}_{p}(\Omega) = \overline{C_0^\infty(\Omega)},$$

where the closure is taken w.r.t $W_p^k(\Omega)$ norm.

Definition 2.11.1. For $s < 0$ and $1 < p < \infty$, define $W_p^s(\Omega) := (\overset{\circ}{W}{}^{-s}_{q}(\Omega))'$ where $1/p + 1/q = 1$. The norm is

$$|u|^p_{W_p^s(\Omega)} = \sup_{v \neq 0} \frac{< u, v >_\Omega}{\|v\|_{W_q^{-s}(\Omega)}}$$

Definition 2.11.2. For $0 < s < 1$, define

$$|u|^p_{W_p^s(\Omega)} = \int\int \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dx\, dy.$$

This is a semi norm, together L^2 norm it makes a norm on $W_p^s(\Omega)$.

$$[f]_{\theta,p,\Omega} := \left(\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x - y|^{\theta p + n}}\, dx\, dy \right)^{\frac{1}{p}}.$$

Let $s > 0$ be not an integer and set $\theta = s - \lfloor s \rfloor \in (0, 1)$. Using the same idea as for the Holder spaces, the Sobolev-Slobodeckij space[7] $W^{s,p}(\Omega)$ is defined as

$$W^{s,p}(\Omega) := \left\{ f \in W^{\lfloor s \rfloor, p}(\Omega) : \sup_{|\alpha| = \lfloor s \rfloor} [D^\alpha f]_{\theta,p,\Omega} < \infty \right\}.$$

It is a Banach space for the norm

$$\|f\|_{W^{s,p}(\Omega)} := \|f\|_{W^{\lfloor s \rfloor, p}(\Omega)} + \sup_{|\alpha| = \lfloor s \rfloor} [D^\alpha f]_{\theta,p,\Omega}.$$

If Ω is suitably regular in the sense that there exist certain extension operators, then also the Sobolev-Slobodeckij spaces form a scale of Banach spaces, i.e. one has the continuous injections or embeddings

$$W^{k+1,p}(\Omega) \hookrightarrow W^{s',p}(\Omega) \hookrightarrow W^{s,p}(\Omega) \hookrightarrow W^{k,p}(\Omega), \quad k \leqslant s \leqslant s' \leqslant k + 1.$$

There are examples of irregular Ω such that $W^{1,p}(\Omega)$ is not even a vector subspace of $W^{s,p}(\Omega)$ for $0 < s < 1$.

From an abstract point of view, the spaces $W^{s,p}(\Omega)$ coincide with the real interpolation spaces of Sobolev spaces, i.e. in the sense of equivalent norms the following holds:

$$W^{s,p}(\Omega) = \Big(W^{k,p}(\Omega), W^{k+1,p}(\Omega)\Big)_{\theta,p}, \quad k \in \mathbb{N}, s \in (k, k+1), \theta = s - \lfloor s \rfloor$$

Theorem 2.11.3.

$$\inf \|f - c\|_{\alpha,T} \leq Ch^{1-\alpha}\|f\|_{1,T}, \;\; 0 < \alpha < 1.$$

Lemma 2.11.4. *For g in $H^1(K)$*

$$|g|_{\alpha,T} \leq Ch^{n-1-\alpha}|g|_{1,T}, \;\; 0 < \alpha < 1.$$

Proof. Let $\eta = x/h, \xi = y/h$. Then with $p = 2$ in the definition

$$\begin{aligned} |g|^2_{\alpha,T} &= \int_T\int_T \frac{|g(x) - g(y)|^2}{|x-y|^{2+2\alpha}} dxdy \\ &= h^{2n-n-2\alpha}\int_{\hat{T}}\int_{\hat{T}} \frac{|\hat{g}(x) - \hat{g}(y)|^2}{|\eta - \xi|^{n+2\alpha}} d\eta d\xi \\ &= h^{n-2\alpha}|\hat{g}|^2_{\alpha,\hat{T}} \leq Ch^{n-2\alpha}|\hat{g}|^2_{1,\hat{T}} = Ch^{2n-2-2\alpha}|g|^2_{1,T}. \end{aligned}$$

□

Remark: This is fractional Poincaré inequality with average zero.

2.11.1 Trace theorem

Theorem 2.11.5 (Trace theorem-Orane Jecker ppt)**.** *Let Ω be $C^{k-1,1}$ domain. For $\frac{1}{2} < s \leq k$ the trace operator*

$$\gamma : H^s(\Omega) \to H^{s-\frac{1}{2}}(\Gamma)$$

is bounded. There exists $C > 0$ s.t

$$\|\gamma v\|_{H^{s-\frac{1}{2}}(\Gamma)} \leq C\|v\|_{H^s(\Omega)}. \tag{2.106}$$

Theorem 2.11.6 (Inverse trace theorem). *The trace operator γ has a right inverse:*

$$\mathcal{E} : H^{s-\frac{1}{2}}(\Gamma) \to H^s(\Omega)$$

satisfying $(\gamma \circ \mathcal{E})w = w$ for all $w \in H^{s-\frac{1}{2}}(\Gamma)$. There exists $C > 0$ s.t

$$\|\mathcal{E}w\|_{H^s(\Omega)} \le C\|w\|_{H^{s-\frac{1}{2}}(\Gamma)} \tag{2.107}$$

for all $w \in H^s(\Omega)$.

Remark: γ is surjective and has $\mathcal{E}$ is injective.

Lemma 2.11.7. *Let $\phi \in H^1(\Omega)$. Then there exists a constant $C > 0$ such that*

$$\|\phi\|_{L^2(\partial\Omega)} \le C(\Omega)\|\phi\|^{1/2}_{L^2(\Omega)}\|\phi\|^{1/2}_{H^1(\Omega)}. \tag{2.108}$$

Lemma 2.11.8. *Let $\phi \in H^1(T)$ and $T^e \subset \partial T$. Then there exists a constant $C > 0$ such that*

$$\|\phi\|_{0,T^e} \le C\left\{\|\phi\|_{0,T}\left(h^{-1}\|\phi\|_{0,T} + \|\nabla\phi\|_{0,T}\right)\right\}^{1/2} \le C\left(h^{-1}\|\phi\|^2_{0,T} + h\|\nabla\phi\|^2_{0,T}\right)^{1/2}.$$

Proof. Standard trace theorem and scaling argument give the result. □

The followings hold by a slight modification.

Lemma 2.11.9. *There exist positive constants C_0, C_1, C_2 independent of the function v such that for all $v \in P_k(T)$,*

$$\|v\|^2_{1,T} \le C_0 h^{-2}\|v\|^2_{0,T}, \quad \|v\|^2_{0,\partial T} \le C_1 h^{-1}\|v\|^2_{0,T} \tag{2.109}$$

and for all $v \in H^1(T)$

$$\|v\|^2_{0,e} \le C_2(h^{-1}\|v\|^2_{0,T} + h|v|^2_{1,T}). \tag{2.110}$$

2.12 Nonconforming Finite element method

One basic assumption on finite element space is

$$V_h \subset V = H^1_0(\Omega). \tag{2.111}$$

We consider two cases where this condition is violated. First case arises when we approximate smooth domain by triangles. In this case boundary condition cannot be met exactly; $V_h \subset H^1(\Omega)$ but $V_h \not\subset H^1_0(\Omega)$.

The other case (2.111) is violated arises when we use "nonconforming" fem of Crouzeix-Raviart. This can happen on a polygonal domain where boundary conditions are exactly satisfied.

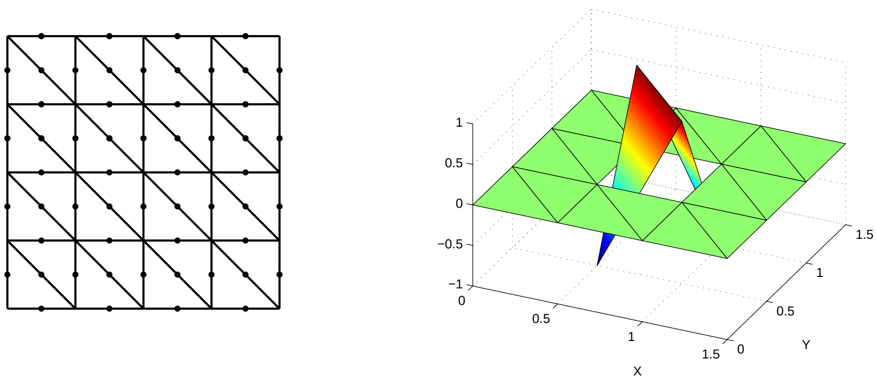
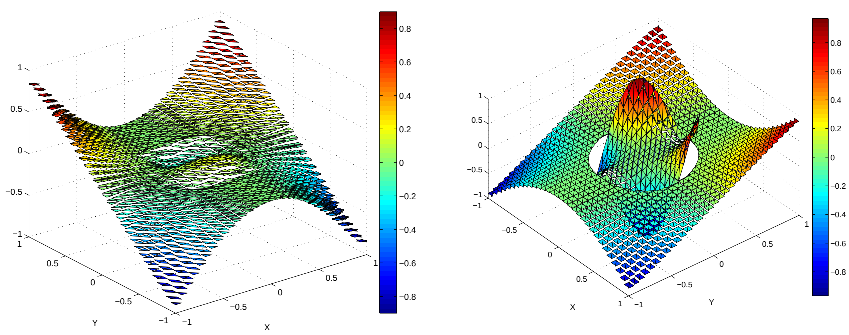


Figure 2.4: Crouzeix-Raviart nonconforming basis



(a) p_h^0 in Ω (b) $p_h = p_h^0 + p_h^*$ in Ω

Figure 2.5: some functions in CR space

2.12.1 Nonconforming FEM of Crouzeix-Raviart

With the usual triangulation $\mathcal{T}_h$, we define the space of piecewise linear finite element space (whose element is not necessarily continuous)

$$V_h = \begin{pmatrix} v : v|_K \text{ in linear on } K \text{ for all } K \\ v \text{ is continuous at midpoint of edges and} \\ v = 0 \text{ at the mid points on boundary edges} \end{pmatrix}.$$

Define bilinear form on $V_h + V$

$$a_h(v, w) = \sum_K \int_K \nabla v \cdot \nabla w \, dx, \tag{2.112}$$

and for $v \in V_h$ we define the equivalent energy norm

$$\|v\|_{a_h} = \sqrt{a_h(v, v)}.$$

Then the discrete problem is : Find $u_h \in V_h$ satisfying

$$a_h(u_h, v) = F(v), \quad v \in V_h. \tag{2.113}$$

Note that $a_h(\cdot, \cdot)$ reduces to $a(\cdot, \cdot)$ form on V. To check the consistency error we see for $w \in V_h$

$$\begin{aligned} a_h(u, w) - f(w) &= \sum_K \int_K \nabla u \cdot \nabla w \, dx - \int_K f w \, dx \\ &= \sum_K \left[\int_{\partial K} \frac{\partial u}{\partial n} w \, ds - \int_K \Delta u w \, dx \right] - \int_K f w \, dx \\ &= \sum_K \int_{\partial K} \frac{\partial u}{\partial n} w \, ds = \sum_K \sum_{e \subset \partial K} \int_e \frac{\partial u}{\partial n} [w] \, ds. \end{aligned}$$

Thus the consistency error is

$$a_h(u - u_h, v_h) = \sum_K \sum_{e \subset \partial K} \int_e \frac{\partial u}{\partial n} [v_h] \, ds. \tag{2.114}$$

Let

(1) h_K: the diameter of K

(2) ρ_K: the diameter of inscribed sphere of K

(3) $\sigma(K) = \frac{h_K}{\rho_K}$.

Let $u \in V$ satisfy

$$a(u, v) = F(v), \quad v \in V \tag{2.115}$$

and $u \in V_h$ satisfy

$$a_h(u_h, v) = F(v), \quad v \in V_h. \tag{2.116}$$

Lemma 2.12.1 (Poincaré inequality)**.** *There exists a constant $C > 0$ s.t. for all $v_h \in V_h$*

$$\|v_h\|_{L^2(\Omega)} \leq C a_h(v_h, v_h).$$

Lemma 2.12.2 (Second Strang lemma)**.** *Let $u \in V$ and $u_h \in V_h$ be arbitrary. Then*

$$\|u - u_h\|_{a_h} \leq \inf \|u - v_h\|_{a_h} + \sup_{v_h} \frac{a_h(u - u_h, v_h)}{\|v_h\|_{a_h}}.$$

Proof. For any $w \in V_h$

$$\begin{aligned} \|u - u_h\|_{a_h} &\leq \|u - w\|_{a_h} + \|w - u_h\|_{a_h} && (2.117) \\ &\leq \|u - w\|_{a_h} + \sup_{v_h} \frac{a_h(w - u_h, v_h)}{\|v_h\|_{a_h}}. && (2.118) \end{aligned}$$

Choose $\tilde{u} \in V_h$ satisfying

$$a_h(u - \tilde{u}, v_h) = 0, \quad \forall v_h \in V_h.$$

A consequence is that(an orthogonal projection)

$$\|u - \tilde{u}\|_{a_h} = \inf_{v_h} \|u - v_h\|_{a_h}. \tag{2.119}$$

Proof of (2.119) For any $\chi \in V_h$ we have

$$\begin{aligned} \|u - \tilde{u} - \chi\|_{a_h}^2 &= \|u - \tilde{u}\|_{a_h}^2 + \|\chi\|_{a_h}^2 - 2a_h(u - \tilde{u}, \chi) \\ &= \|u - \tilde{u}\|_{a_h}^2 + \|\chi\|_{a_h}^2 \\ &\geq \|u - \tilde{u}\|_{a_h}^2. \end{aligned}$$

So for any $v_h \in V_h$

$$\|u - v_h\|_{a_h} \geq \|u - \tilde{u}\|_{a_h}.$$

Now

$$a_h(\tilde{u} - u_h, v_h) = a_h(\tilde{u} - u + u - u_h, v_h) = a_h(u - u_h, v_h).$$

Combining this with $w = \tilde{u}$ in (2.118) we obtain the result. $\square$

Remark 2.12.3. In most applications we take u to be the solution and u_h be its fem solution. But the lemma holds for arbitrary pair u, u_h.

Next we estimate the second term of (2.118).

Lemma 2.12.4. *Let m, μ be integers with $0 \le m \le \mu$. Let $P^\mu \hat{v}$ be a polynomial of degree of freedom as we shall se. Then*

$$\left| \int_e \phi(v - P^\mu v)\, ds \right| \le C\sigma(K) h^{m+1} |\phi|_{1,K} |v|_{m+1,K} \tag{2.120}$$

for all $\phi \in H^1(K)$ and $v \in H^{m+1}(K)$.

Proof. Let us use reference element $\hat{K}$. Assume

$$F : \hat{\mathbf{x}} \to F(\hat{\mathbf{x}}) = B\hat{\mathbf{x}} + \mathbf{b}.$$

We can see

$$\int_e \phi(v - P^\mu v)\, ds = |B'| \int_{\hat{e}} \hat{\phi}(\hat{v} - P^\mu \hat{v})\, d\hat{s}, \tag{2.121}$$

where B' is the matrix by crossing out the n-th row and column from B. So consider the functional

$$\hat{\phi} \to \int_{\hat{e}} \hat{\phi}(\hat{v} - P^\mu \hat{v})\, d\hat{s}$$

which is continuous over $H^1(\hat{K})$ whose norm is less than

$$\|\hat{v} - P^\mu \hat{v}\|_{\hat{e}}$$

and vanishes on P_m. Then

$$\begin{aligned}
\left| \int_{\hat{e}} \hat{\phi}(\hat{v} - P^\mu \hat{v})\, d\hat{s} \right| &= \left| \int_{\hat{e}} (\hat{\phi} - P^0 \hat{\phi})(\hat{v} - P^\mu \hat{v})\, d\hat{s} \right| && (2.122)\\
&\le c_1 \|\hat{\phi} - P^0 \hat{\phi}\|_{\hat{e}} \|\hat{v} - P^\mu \hat{v}\|_{\hat{e}} && (2.123)\\
&\le c_2 \|\hat{\phi} - P^0 \hat{\phi}\|_{1,\hat{K}} \|\hat{v} - P^\mu \hat{v}\|_{1,\hat{K}} && (2.124)\\
&\le C_2 |\hat{\phi}|_{1,\hat{K}} |\hat{v}|_{m+1,\hat{K}}, && (2.125)
\end{aligned}$$

where the last inequality follows from Bramble-Hilbert lemma. ((2.123) maybe skipped.) So

$$\left|\int_e \phi(v - P^\mu v)\, ds\right| \leq C_3 |\det(B')| \cdot |\hat{\phi}|_{1,\hat{K}} |\hat{v}|_{m+1,\hat{K}}. \tag{2.126}$$

Recall the scaling argument

$$|\hat{v}|_{\ell,\hat{K}} \leq |\det(B')|^{-1/2} \|B'\| \cdot |v|_{\ell,K} \text{ for all } v \in H^1(K). \tag{2.127}$$

So

$$\left|\int_e \phi(v - P^\mu v)\, ds\right| \leq C_3 |\det(B')| \cdot |\det(B)|^{-1} \|B\|^{m+2} |\phi|_{1,K} |v|_{m+1,K}. \tag{2.128}$$

Check that

$$|\det(B')| \leq |\det(B)| \cdot \|B^{-1}\|.$$

Combine all of above,

$$\left|\int_e \phi(v - P^\mu v)\, ds\right| \leq C_3 |\det(B)|^{-1} \|B\|^{m+2} |\phi|_{1,K} |v|_{m+1,K}, \tag{2.129}$$

and noting

$$\|B\| \leq \frac{h_K}{\rho_{\hat{K}}}, \quad \|B^{-1}\| \leq \frac{h_{\hat{K}}}{\rho_K},$$

we get the result. □

Applying this to consistency error term (2.114) with $\phi = \frac{\partial u}{\partial n}$ we obtain

Theorem 2.12.5. *We have*

$$\|u - u_h\|_{a_h} \leq Ch|u|_{2,\Omega}. \tag{2.130}$$

Proof. For the consistency error, we have from (2.114)

$$a_h(u - u_h, v_h) = \sum_{K \in \mathcal{K}_h} < \frac{\partial u}{\partial n}, v_h >_{\partial K} = \sum_K \sum_{e \subset \partial K} \int_e \frac{\partial u}{\partial n} [v_h]\, ds, \tag{2.131}$$

where $v_h \in V_h(\Omega)$ and n is a unit outward normal vector on each ∂K. Since u belongs to $H^2(\Omega)$, and $v_h \in V_h$ has well-defined average value on the interior edges, and vanishing average on the boundary, we have, by Lemma 2.120

$$
\begin{aligned}
\sum_{K\in\mathcal{K}_h} < \frac{\partial u}{\partial n}, v_h >_{\partial K} &= \sum_{K\in\mathcal{K}_h}\sum_{e\subset\partial K} < \frac{\partial u}{\partial n} - (\overline{\frac{\partial u}{\partial n}})_e, v_h >_e \\
&\leq \sum_{K\in\mathcal{K}_h} Ch \left|\frac{\partial u}{\partial n}\right|_{1,K} |v_h|_{1,K} \\
&\leq Ch\|u\|_{H^2(\Omega)}\|v_h\|_{1,h}. \qquad (2.132)
\end{aligned}
$$

Combining this with Lemma 2.12.1, 2.12.2 and the approximation property of the space V_h we obtain the result. □

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