Newton iteration.

• Framework:

- Newton 法求方程的根，一般称作牛顿迭代法. 目标是要求 $f(x)=0$ 的根，即求 x^*, s.t $f(x^*)=0$.
- 假设 $f(x)$ 在 x_k 处一阶可导，并在 x_k 利用 Taylor 展开

$$f_k(x) = f(x_k) + (x-x_k)g_k + o(x).$$

其中 $g_k = \nabla f(x_k)$: the gradient of f at x_k.

Let $f_k(x)=0$, we have $x = x_k - g_k^{-1}f(x_k)$, so $x_{k+1} = x_k + d_k$, where d_k is the solution of $\boxed{g_k d_k = -f(x_k)}$

• One scheme:

In the Navier Stokes system, we set

$$F(u,p) = \begin{pmatrix} \int_\Omega \nu \nabla u : \nabla v + (u\cdot\nabla)u\cdot v - p\nabla\cdot v - f\cdot v \\ \int_\Omega -q\nabla\cdot u \end{pmatrix}$$

We set $x=(u,p)$. $g_k = \nabla F(u_k,p_k)$ $d_k = (du_k, dp_k)$.

So, $\nabla F(u_k,p_k)(du_k,dp_k) = -F(u_k,p_k)$

where,

$$\boxed{\begin{aligned} &\nabla F(u_k,p_k)(du_k,dp_k) \\ &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(F(u_k+\epsilon du_k, p_k+\epsilon dp_k) - F(u_k,p_k)\right) \\ &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}(\quad \cdots \quad) \\ &= \begin{pmatrix} \int_\Omega \nu\nabla du_k : \nabla v + (u_k\cdot\nabla)du_k\cdot v + (du_k\cdot\nabla)u_k\cdot v - dp_k\nabla\cdot v \\ \int_\Omega -q\nabla\cdot du_k \end{pmatrix} \end{aligned}}$$

→ ref: 2018 (dealII step-57) Newton iteration for stationary Navier Stokes.pdf.

Therefore, we arrive at the linearized system:

(*1)

$$\left.\begin{aligned} \int_\Omega \nu\nabla du_k : \nabla v + (u_k\cdot\nabla)du_k\cdot v + (du_k\cdot\nabla)u_k\cdot v - dp_k\nabla\cdot v &= \\ \int_\Omega -q\nabla\cdot du_k &= \end{aligned}\right\} = -F(u_k,p_k)$$

where, $-F(u_k,p_k) = \begin{pmatrix} \int_\Omega f\cdot v - \nu\nabla u_k : \nabla v + (u_k\cdot\nabla)u_k\cdot v - p_k\nabla\cdot v \\ \int_\Omega q\nabla\cdot u_k \end{pmatrix}$

In the system (*1), we need to solve du_k and dp_k.

Then, we get

$$u_{k+1} = u_k + du_k$$
$$p_{k+1} = p_k + dp_k$$

Summary:

上面利用 Newton iteration 来求解 N-S 方程，需要分两步:

①: 在 (*1) 系统中求解 du_k, dp_k.

②: 更新 $u_{k+1} = u_k + du_k$; $p_{k+1} = p_k + dp_k$

③: 设计适当的迭代终止条件.

● Another scheme of Newton iteration.

前面所给出的是从 Newton iteration 最基本的定义出发，来求解.

下面从双线性形式直接给出 u_{k+1}, p_{k+1} 的求解.

推导过程见: 2016 (Volker John) Finite element methods for incompressible flow problems.pdf

Sec: 6.3 Iteration schemes for solving the nonlinear problem

Remark: 6.43

直接给出格式:

$$(*2) \quad \begin{pmatrix} a(u_{k+1}, v) + n(u_k, u_{k+1}, v) + n(u_{k+1}, u_k, v) + b(v, p_{k+1}) \\ b(u_{k+1}, q) \end{pmatrix} = \begin{pmatrix} \langle f, v\rangle_{V',V} + n(u_k, u_k, v) \\ 0 \end{pmatrix}$$

其中, $a(w,v) = (\nabla w, \nabla v)$; $b(v,q) = -(\nabla \cdot v, q)$

$n(w,v,z) = \int_\Omega (w\cdot\nabla)v \cdot z$

相比 (*1) 系统, (*2) 在实际应用中更有优势, 可以直接得到 u_{k+1}, p_{k+1}.

☆ 实际上, 在 (*1) 中, 将 $du_k = u_{k+1} - u_k$, $dp_k = p_{k+1} - p_k$ 代入, 即可得 (*2)