chapter 1

jovyntls

solution to a linear equation

- inconsistent → has no solution
- solution → a point of intersection
- solution set of the equation → set of all solutions to the equation
    - $\{(1+s, 2s, s) | s \in \mathbb{R}\}$
- general solution of the equation → expression that gives us all the solutions to the equation
    - $\begin{cases} x = t \\ y = 2t - 1 \end{cases}$

homogenous linear systems

- homogenous → rightmost column is all zeros
    - has at least one solution (the trivial solution)
- trivial solution → $x_1, x_2, \ldots, x_n = 0$
- non-trivial solution → any other solution

> 💡 a homogenous system of linear equations has either
> * **only the trivial solution**, or
> * **infinitely many solutions AND trivial solution**

elementary row operations

1. multiply equation by a non-zero constant
    - $cR_i, c \neq 0$
2. interchange 2 equations
    - $R_i \leftrightarrow R_j$
3. add a multiple of one equation to another equation
    - $R_i + cR_j, c \in \mathbb{R}$
    - convention: R_i (first row written) changes

> 📌 TAKE NOTE: cannot multiply by zero or divide by zero ⇒ split cases if you want to multiply/divide by a variable!!

(R)REF

> 💡 every matrix has a **unique RREF** but can have multiple REF.

- **no solution** if last column is a pivot column
- **unique solution** if every column is a pivot column
- **infinite solutions** if there is a non-pivot column (besides last column)
    - non pivot column = arbitrary parameter

chapter 2

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inverse

- **UNIQUENESS OF INVERSES** → if B and C are inverses of A, then $B = C$.
- **CANCELLATION LAWS** → applies if A is invertible
    - if B_1 and B_2 are $m \times n$ matrices such that $AB_1 = AB_2$, then $B_1 = B_2$.
    - if C_1 and C_2 are $m \times n$ matrices such that $C_1 A = C_2 A$, then $C_1 = C_2$.
- **2x2 INVERSE** → if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

properties

if A, B are invertible matrices and c is a nonzero scalar,

- cA is invertible; $(cA)^{-1} = \frac{1}{c} A^{-1}$
- A^T is invertible; $(A^T)^{-1} = (A^{-1})^T$
- A^{-1} is invertible; $(A^{-1})^{-1} = A$
- AB is invertible; $(AB)^{-1} = B^{-1} A^{-1}$
- $(A^n)^{-1} = (A^{-1})^n$

if A, B are square matrices of the same size and $AB = I$, then

- A and B are invertible
- $A^{-1} = B; \quad B^{-1} = A$
- $BA = I$

singular matrices

let A, B be square matrices of the same size.

- if A is singular, then AB and BA are singular
- if AB is singular, then A or B is singular. (or both)

transpose

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- if c is a scalar, then $(cA)^T = cA^T$
- $(AB)^T = B^T A^T$

conditions for invertibility

let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

- A is invertible $\iff$ $\det(A) = ad - bc \neq 0$
- A is invertible $\iff$ RREF is the identity matrix

📌 if the REF of A has at least one **singular** (zero) row, then A is NOT invertible

equivalent statements for invertibility

let A be a square matrix. then the following statements are equivalent:

1. A is invertible
2. the linear system $Ax = 0$ has only the trivial solution
3. the RREF of A is the identity matrix
4. A can be expressed as a product of elementary matrices

adjoints

$$\text{if } A \text{ is an invertible matrix, then} \\ A^{-1} = \frac{1}{\det(A)} adj(A)$$

cramer's rule

- to solve linear systems ⇒ $x_i = \frac{\det(A_i)}{\det(A)}$
    - where $\det(A_i)$ is obtained from replacing the i^{th} column of A by b.

elementary row operations

- $E_3 E_2 E_1 A = B$
    - $A = E_1^{-1} E_2^{-1} E_3^{-1} B$

📌 post-multiplication: becomes an elementary column operation ⇒ produces column equivalent matrix

determinant of elementary row operations

$$\text{if } E \text{ is an elementary matrix of the same size as } A, \\ \det(B) = \det(E)\det(A) = \det(EA)$$

- $A \xrightarrow{kR_n} B \quad \Rightarrow \quad \det(B) = k\det(A) \quad ; \quad \det(E) = k$
- $A \xrightarrow{R_n \leftrightarrow R_m} B \quad \Rightarrow \quad \det(B) = -\det(A) \quad ; \quad \det(E) = -1$
- $A \xrightarrow{R_n + kR_m} B \quad \Rightarrow \quad \det(B) = \det(A) \quad ; \quad \det(E) = 1$

operations on determinant

let A, B be square matrices of order n and let c be a scalar.

- $\det(cA) = c^n \det(A)$
- $\det(AB) = \det(A)\det(B)$
- $\det(A^{-1}) = \frac{1}{\det(A)}$

💡 SHOELACE METHOD for 3x3 matrix

common determinants

- triangular matrix → product of diagonal entries
- square matrix → $\det(A) = \det(A^T)$
- two identical rows/columns → $\det(A) = 0$

chapter 3

solution sets

> 💡 if a system of linear equation has *n* variables, then its solution set is a subset of $\mathbb{R}^n$.

$$\text{the general solution to the linear system } \begin{cases} x+y+z=0 \\ x-y+2z=1 \end{cases}$$

- vector form → $(x,y,z) = (\frac{1}{2} - \frac{3}{2}t, -\frac{1}{2} + \frac{1}{2}t, t)$ where $t \in \mathbb{R}$
- implicit form → $\{(x,y,z) \mid x+y+z=0 \text{ and } x-y+2z=1\}$
- explicit form → $\{(\frac{1}{2} - \frac{3}{2}t, -\frac{1}{2} + \frac{1}{2}t, t) \mid t \in \mathbb{R}\}$
  - (solution set)

terminology: vector spaces and subspaces

- a set V is a vector space → if $V = \mathbb{R}^n$ or V is a **subspace** of $\mathbb{R}^n$.
- a set W is a subspace of V → if W is a vector space and $W \subseteq V$.
  - W is a subspace of $\mathbb{R}^n$ which lies completely inside V.
  - e.g. a line overlapping with a plane is a subspace of the plane

linear span: basic properties

Let $S = \{u_1, u_2, \cdots, u_n\} \subseteq \mathbb{R}^n$.

1. $\mathbf{0} \in \text{span}(S)$
2. $\forall v_1, v_2, \ldots, v_r \in \text{span}(S)$ and $c_1, c_2, \ldots, c_r \in \mathbb{R}$,

   $c_1v_1 + c_2v_2 + \cdots + c_rv_r \in \text{span}(S)$

consistent linear systems

$$\text{let } S = \{u_1, u_2, \ldots, u_n\}$$
$$span(S) = \mathbb{R}^n \iff \text{the linear system}$$
$$\left[\begin{array}{c|c} u_1 & k_1 \\ u_2 & k_2 \\ \vdots & \vdots \\ u_n & k_n \end{array}\right] \text{ is consistent } \forall k_1, k_2, \ldots, k_n \in \mathbb{R}$$

bases

S is a basis (plural bases) for V if

1. S is linearly independent
2. S spans V.

> 💡 basis of V → set of the smallest size that can span V

- basis of the zero space = $\emptyset$
- every other space has infinite bases.

coordinate systems

$$\text{the coordinate vector of } V \text{ relative to } S,$$
$$(v)_s = (c_1, c_2, \ldots, c_k) \in \mathbb{R}^k$$

- $(v)_s$ → row vector
- $[v]_s$ → column vector

> ❗ for $v \in V \subseteq \mathbb{R}^n$ and $(v)_s \in \mathbb{R}^k$, it is possible that $n \neq k$

- standard basis $E = \{e_1, e_2, \ldots, e_n\}$ where $e_1 = (1, 0, \ldots, 0), e_2 = (0, 1, \ldots, 0), e_n = (0, 0, \ldots, 1)$

properties

- any vector in $\mathbb{R}^n$ can be expressed uniquely in the standard basis
  - $(u)_E = (u_1, u_2, \ldots, u_n) = u.$
- two vectors are equal $\iff$ their coordinates are equal (in any basis)
  - For any $u, v \in V, u = v \iff (u)_S = (v)_S$
- linear combination
  - For any $v_1, v_2, \ldots, v_r \in V$ and $c_1, c_2, \ldots, c_r \in \mathbb{R}$,

    $(c_1v_1 + c_2v_2 + \cdots + c_rv_r)_S = c_1(v_1)_S + c_2(v_2)_S + \cdots + c_r(v_r)_S.$

subspaces

- subspace → the span of a set of vectors in $\mathbb{R}^n$

$$\text{Let } V \text{ be a subset of } \mathbb{R}^n. \, V \text{ is a subspace of } \mathbb{R}^n \text{ if}$$
$$V = \text{span}(S) \text{ for some vectors } u_1, u_2, \ldots, u_k \in \mathbb{R}^n.$$

A subspace $V \subseteq \mathbb{R}^n$

(i) (Contains the origin) $O \in V$

(ii) (Closed under linear combinations) $\forall u, v \in V, \alpha, \beta \in \mathbb{R}, \alpha u + \beta v \in V$

- V is a subspace spanned by S
  - V is a subspace spanned by $u_1, u_2, \ldots, u_k$
- S spans V
  - $u_1, u_2, \ldots, u_k$ spans V

dimensions

- $\dim(V)$, dimension of a vector space V → number of vectors in a basis for V.
  - dimension of zero space = 0
  - $\dim(\mathbb{R}^n) = n$

> 💡 dimension of solution space = # of non-pivot columns

equivalent statements

Let V be a vector space of dimension k and S is a subset of V.

1. S is a basis for V
   - i.e. S is linearly independent and S spans V
2. S is linearly independent and $|S| = k$.
3. S spans V and $|S| = k$.

any 2 of 3 conditions: S is a basis of V

1. S is linearly independent
2. S spans V

important properties

- $\text{REF has no zero row} \Rightarrow span(S) = \mathbb{R}^n$

$$\text{Let } S = \{u_1, u_2, \cdots, u_k\} \subseteq \mathbb{R}^n. \\ k < n \Rightarrow span(S) \neq \mathbb{R}^n$$

- one vector cannot span $\mathbb{R}^2$;
- one vector or two vectors cannot span $\mathbb{R}^3$

subsets

- to show $\text{span}\{u_1, u_2, u_3\} \subseteq \text{span}\{v_1, v_2\}$:
  - show that u_1, u_2, u_3 are **linear combinations** of v_1, v_2
  - RREF of $\left[\begin{array}{cc|c|c|c} v_1 & v_2 & u_1 & u_2 & u_3 \end{array}\right]$ is consistent
- to show $\text{span}\{u_1, u_2, u_3\} \subseteq V$:
  - show that u_1, u_2 can be subbed into V (implicit form)
  - if $v_1, v_2, \ldots, v_m \in span(S) \Rightarrow span\{v_1, v_2, \ldots, v_m\} \subseteq span(S)$
- to show $A = B$:
  - show that $A \subseteq B \land B \subseteq A$

linear independence

$$c_1u_1 + c_2u_2 + \cdots + c_ku_k = 0 \quad (*)$$

- $S = \{0\}$ is linearly dependent!
- if (*) only has the trivial solution, then S is a linearly independent set
- if (*) has non-trivial solutions, S is a linearly dependent set

- a set of vectors is linearly (in)dependent $\iff$ they are linearly (in)dependent in the other basis
  - $v_1, v_2, \ldots, v_r$ are *linearly (in)dependent* in V $\iff$ $(v_1)_S, (v_2)_S, \ldots, (v_r)_S$ are *linearly (in)dependent* vectors in $\mathbb{R}^k$.
- a set of vectors spans V $\iff$ their coordinate vectors relative to S span $\mathbb{R}^k$.
  - $\text{span}\{v_1, v_2, \ldots, v_r\} = V \iff \text{span}\{(v_1)_S, (v_2)_S, \ldots, (v_r)_S\} = \mathbb{R}^k$

invertible matrices

let A be a square matrix. the following statements are equivalent:

1. A is invertible
2. the linear system $Ax = 0$ has only the trivial solution
3. RREF of A is the identity matrix
4. A can be expressed as a product of elementary matrices
5. $\det(A) \neq 0$
6. The rows of A form a basis for $\mathbb{R}^n$.
7. The columns of A form a basis for $\mathbb{R}^n$.

redundant vectors

- is a linear combination of the rest
- if u_k is a linear combination of $u_1, u_2, \ldots, u_{k-1}$, then $span\{u_1, u_2, \ldots, u_{k-1}\} = span\{u_1, u_2, \ldots, u_{k-1}, u_k\}$

3. $|S| = k$

dimensions of subspaces

$$\text{Let } U \text{ be a subspace of vector space } V. \\ \text{Then } \dim(U) \leq \dim(V). \\ \text{If } \dim(U) = \dim(V) \text{ then } U = V.$$

- a subset T of V with $|T| > \dim(V)$ must be linearly dependent.

transition matrix

- $P = \begin{bmatrix} [u_1]_T & [u_2]_T & \cdots & [u_k]_T \end{bmatrix}$ for $S = \{u_1, u_2, \ldots, u_k\}$

$$\left[\begin{array}{c|c} T & S \end{array}\right] \xrightarrow{\text{G-J Elimination}} \left[\begin{array}{c|c} I & P \end{array}\right] \\ [w]_T = P[w]_S$$

- P is the transition matrix from S to T
- P^{-1} is the transition matrix from T to S.

chapter 4

row & column space

- row space → the subspace of $\mathbb{R}^n$ spanned by rows of A

  $= \text{span}\{r_1, r_2, \ldots, r_m\} \subseteq \mathbb{R}^n$
  $= \text{ column space of } A^T$

  $$\text{where } A = \begin{bmatrix} r_1 \\ \vdots \\ r_m \end{bmatrix}, \; r_i = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix},$$

  $$\text{or } A = \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix}, \; c = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix}$$

- column space → the subspace of $\mathbb{R}^n$ spanned by the **columns** of A

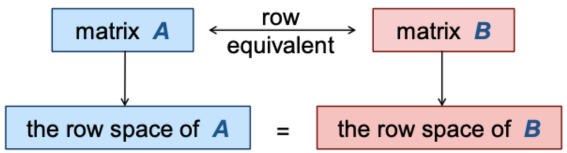
  $= \text{span}\{c_1, c_2, \ldots, c_n\} \subseteq \mathbb{R}^n$

  $= \text{ row space of } A^T$

  $= \{Au \mid u \in \mathbb{R}^n\}$

  - basis of column space of A is obtained by the columns of A that correspond to pivot columns of the REF

row equivalence



- matrices are row-equivalent $\iff$ they have the same **RREF**.
- ✓ reflexive, symmetric and transitive
- elementary operations preserve row space

ranks

- rank of a matrix → the dimension of its row space (and column space).
  - the row space and column space of a matrix has the same dimension. For REF: # of nonzero rows = # of pivot columns
- full rank → $\text{rank}(A) = \min\{m, n\}$ for a matrix A of size $m \times n$
  - **square matrix** has full rank $\iff \det(A) \neq 0$
- properties
  - $\text{rank}(0) = 0, \qquad \text{rank}(I_n) = n, \qquad \text{rank}(A) = A^T$
  - $\text{rank}(A) \leq \min\{m, n\}$ for a $m \times n$ matrix A
  - $\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$

linear systems

- a linear system $Ax = b$ is consistent

  $\iff b$ lies in the column space of A

  $\iff A$ and $(A \mid b)$ have the **same rank**.
- a consistent linear system $Ax = b$ has only one solution

  $\iff$ the nullspace of A is $\{0\}$
- suppose v is a solution of the linear system $Ax = b$.

  solution set of the system

  $= \{u + v \mid u \text{ is an element of the nullspace of } A\}.$

nullspace & nullites

- nullspace (of A) → the solution space of the homogenous linear system $Ax = 0$
- nullity (of A) → dimension of the nullspace of A
  - $\text{nullity}(A) = \dim(\text{nullspace of } A)$
  - $\text{nullity}(A) \leq \dim(\mathbb{R}^n) = n$

dimension theorem

- $\text{rank}(A^T) + \text{nullity}(A^T) = \text{number of rows in } A$
- $\text{rank}(A) + \text{nullity}(A) = \text{number of columns in } A$

chapter 5.1-5.2

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dot product

- distance → $d(u,v) = \|u-v\|$
- norm/length → $\| u \| = \sqrt{u \cdot u} = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$
  - unit vector → vectors of norm 1
- dot product → $u \cdot v = uv^T = \sum_{i=1}^n u_i v_i$

  $= u_1v_1 + u_2v_2 + \cdots + u_nv_n$
- angle between u and v →

  $\theta = \cos^{-1}(\frac{u \cdot v}{\|u\|\|v\|}) = \cos^{-1}(\frac{\|u\|^2+\|v\|^2-\|u-v\|^2}{2\|u\|\|v\|})$

  in $\mathbb{R}^n$: $\theta = \cos^{-1}(\frac{u_1v_1+u_2v_2+\cdots+u_nv_n}{\|u\|\|v\|})$
- cosine rule:

  $\|u-v\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\|\cos\theta$

basic properties

- symmetric → $u \cdot v = v \cdot u$
- distributivity → $w \cdot (u+v) = w \cdot u + w \cdot v$
- scalar multiplication → $(cu) \cdot v = u \cdot (cv) = c(u \cdot v)$
  - vectors are NOT associative - $(u \cdot v) \cdot w \neq u \cdot (v \cdot w)$
- scalar multiplication for length → $\|cu\| = |c|\|u\|$
- positive definite → $u \cdot u \geq 0$
  - $u \cdot u = 0 \iff u = 0$
- cauchy-schwarz inequality → $|u \cdot v| \leq \|u\|\|v\|$
- triangle inequality → $\|u+v\| \leq \|u\| + \|v\|$
- distance between vectors → $d(u,w) \leq d(u,v) + d(v,w)$

orthogonality

- orthogonal → $u \cdot v = 0, \quad \theta = \frac{\pi}{2}$
  - 0 is orthogonal to every subspace and the whole $\mathbb{R}^n$
- orthogonal set → every pair of distinct vectors are orthogonal
  - a set containing only one (non-zero) vector is always an orthogonal set
  - orthogonal $\Rightarrow$ linearly independent
    - but linear independence $\not\Rightarrow$ orthogonality
- orthonormal set → orthogonal set; every vector is a unit vector
  - e.g. standard basis $E = \{e_1, e_2, \ldots, e_n\}$ for $\mathbb{R}^n$
  - 0 cannot be normalised ⇒ a set containing a zero vector cannot be orthonormal

orthogonal/orthonormal bases

- to show that S is an orthogonal/orthonormal basis for V:
  1. S is orthogonal/orthonormal ($\Rightarrow$ linear independence)
  2. $|S| = \dim(V)$ or $span(S) = V$

orthogonal bases

$$\text{Let } S = \{u_1, u_2, \ldots, u_n\} \text{ be an orthogonal basis for } V.$$
$$\text{Then for any } w \in V,$$
$$w = \frac{w \cdot u_1}{u_1 \cdot u_1}u_1 + \frac{w \cdot u_2}{u_2 \cdot u_2}u_2 + \cdots + \frac{w \cdot u_k}{u_k \cdot u_k}u_k$$
$$(w)_S = \left(\frac{w \cdot u_1}{u_1 \cdot u_1}, \frac{w \cdot u_2}{u_2 \cdot u_2}, \ldots, \frac{w \cdot u_k}{u_k \cdot u_k}\right)$$

orthonormal bases

$$\text{Let } S = \{u_1, u_2, \ldots, u_n\} \text{ be an orthonormal basis for } V.$$
$$\text{Then for any } w \in V,$$
$$w = (w \cdot v_1)v_1 + (w \cdot v_2)v_2 + \cdots + (w \cdot v_k)v_k$$
$$(w)_S = (w \cdot v_1,\ w \cdot v_2,\ \ldots,\ w \cdot v_k)$$

💡 the solution space of a matrix is **orthogonal** to its row space.

projections

Let V be a subspace of $\mathbb{R}^n$.

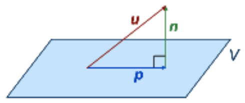
Every $u \in \mathbb{R}^n$ can be written uniquely as

$u = n + p$

where p is a vector in V

and n is a vector orthogonal to V.

The vector p is called the (orthogonal) projection of u onto V.



A vector $u \in \mathbb{R}^n$ is orthogonal to V is u is orthogonal to all vectors in V.

orthogonal bases & projections

let V be a subspace for $\mathbb{R}^n$ and $\{u_1, u_2, \ldots, u_k\}$ an *orthogonal basis* for V.

$$\text{for any } \mathbf{w} \in \mathbb{R}^n,$$
$$\frac{w \cdot u_1}{u_1 \cdot u_1}u_1 + \frac{w \cdot u_2}{u_2 \cdot u_2}u_2 + \cdots + \frac{w \cdot u_k}{u_k \cdot u_k}u_k$$
$$\text{is the projection of } \mathbf{w} \text{ onto } V.$$

orthonormal bases & projections

let V be a subspace for $\mathbb{R}^n$ and $\{v_1, v_2, \ldots, v_k\}$ an *orthonormal basis* for V.

$$\text{for any } \mathbf{w} \in \mathbb{R}^n,$$
$$(w \cdot u_1)u_1 + (w \cdot u_2)u_2 + \cdots + (w \cdot u_k)u_k$$
$$\text{is the projection of } \mathbf{w} \text{ onto } V.$$

Gram-Schmidt Process

Let $\{u_1, u_2, \ldots, u_k\}$ be a basis for a vector space V.

Let $v_1 = u_1$,

$$v_2 = u_2 - \frac{u_2 \cdot v_1}{v_1 \cdot v_1}v_1,$$
$$v_3 = u_3 - \frac{u_3 \cdot v_1}{v_1 \cdot v_1}v_1 - \frac{u_3 \cdot v_2}{v_2 \cdot v_2}v_2,$$
$$\vdots$$
$$v_k = u_k - \frac{u_k \cdot v_1}{v_1 \cdot v_1}v_1 - \frac{u_k \cdot v_2}{v_2 \cdot v_2}v_2 - \cdots - \frac{u_k \cdot v_{k-1}}{v_{k-1} \cdot v_{k-1}}v_{k-1}.$$

Then $\{v_1, v_2, \ldots, v_k\}$ is an orthogonal basis for V.

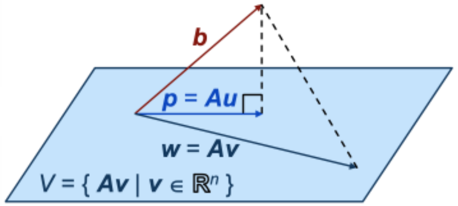
chapter 5.3-5.4

best approximations

a vector $u \in \mathbb{R}^n$ is a **least squares solution** to the linear system $Ax = b$

$\iff$ $p = Au$ is the **best approximation** of b onto the column space of A

$\iff$ $p = Au$ is the **projection** of b onto the column space of A.



p is the best approximation of u in V.

$$d(u,p) \le d(u,v) \quad \text{for all } v \in V$$

$$\| b - Au \| \le \| b - Av \| \quad \text{for all } v \in \mathbb{R}^n$$

least squares solution

- u is the **least squares solution** to the system $Ax = b$

  $\iff$ $b - Au$ is orthogonal to $a_1, a_2, \dots, a_n$ ($A = [a_1\ a_2\ \dots\ a_n]$)

  $\iff$ u is a solution to $A^T Ax = A^T b$

finding least squares solution

- *using projection*: x is a least squares solution $\iff$ $Ax = p$, where p is the projection of b on the column space of A (using Gram-Schmidt)
- *without projection*: use $A^T Ax = A^T b$
- find projection of a vector onto a span using least squares solution:
  - let the span be the column space of matrix A. let the vector be b.
  - let u be the solution to the linear system $A^T Ax = A^T b$
  - projection = Au (u is any least squares solution)

orthogonal matrices

- orthogonal → $A^{-1} = A^T$ (a square matrix)

transition matrices

let S and T form two **orthonormal bases** for a vector space;

let P be the transition matrix from S to T.

- P is an orthogonal matrix.
- $P^T = P^{-1} =$ transition matrix from T to S.

rotation of xy-coordinates

let $E = \{e_1, e_2\}$ and $S = \{u_1, u_2\}$ where e_1, e_2, u_1, u_2 are unit vectors along the x, y, x', y' axes

- $u_1 = (\cos\theta, \sin\theta) = e_1 \cos\theta + e_2 \sin\theta$
- $u_2 = (-\sin\theta, \cos\theta) = -e_1 \sin\theta + e_2 \cos\theta$
- transition matrix from S to E,

  $$P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

  - P^T = transition matrix from E to S

conversion from xy to $x'y'$

Let $v = (x, y) \in \mathbb{R}^2, \quad (v)_S = (x', y')$.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = [v]_S = P^T [v]_E = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x' = x\cos\theta + y\sin\theta$$
$$y' = -x\sin\theta + y\cos\theta$$

equivalent statements

1. A is orthogonal
2. the rows of A form an orthonormal basis for $\mathbb{R}^n$
3. the columns of A form an orthonormal basis for $\mathbb{R}^n$

chapter 6

jovyntls

eigenvalues & eigenvectors

let A be a square matrix of order n.

- eigenvector → a *nonzero* column vector $u \in \mathbb{R}^n$ such that $Au = \lambda u$ for a scalar λ (eigenvalue)
  - u is an eigenvector of A associated with λ
  - $Au \in \text{span}\{u\}$
- for eigenvectors u, v, w, $\begin{bmatrix} u & v & w \end{bmatrix}^{-1} A \begin{bmatrix} u & v & w \end{bmatrix} = [\lambda_u \ \lambda_v \ \lambda_w]$
- triangular matrix → eigenvalues are the diagonal entries
- row operations DO NOT preserve eigenvalues
- transpose preserves eigenvalues!!

characteristic polynomials

- λ is an eigenvalue for A

  $\iff \exists u \in \mathbb{R}^n \backslash \{0\} \mid (\lambda I - A)u = 0$
  $\iff \det(\lambda I - A) = 0$
- characteristic equation of A → $\det(\lambda I - A) = 0$
- characteristic polynomial of A → $\det(\lambda I - A)$
  - eigenvalue $\iff$ it is a root of the polynomial

> 💡 every odd degree polynomial has at least one real root

eigenspaces

- E_λ or $E_\lambda(A)$ → eigenspace of A associated with the eigenvalue λ
- eigenspace → all eigenvectors of A associated with λ
  - all vectors u such that $Au = \lambda u$
  - **solution space** of the linear system $(\lambda I - A)x = 0$
  - always a subspace of $\mathbb{R}^n$
- if u is a nonzero vector in E_λ, u is an eigenvector of A associated with λ

diagonalization

- diagonalizable → there exists an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.
  - P diagonalizes A.
  - $n \times n$ square matrix A is diagonalisable $\iff$ A has n linearly independent eigenvectors

diagonalizing a matrix

Let A be a square matrix of order n.

Step 1: Find all distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ (say, by solving the characteristic equation $\det(\lambda I - A) = 0$).

Step 2: For each eigenvalue λ_i, find a basis S_{λ_i} for the eigenspace E_{λ_i}.

Step 3: Let $S = S_{\lambda_1} \cup S_{\lambda_2} \cup \cdots \cup S_{\lambda_k}$.
(a) If $|S| < n$, then A is not diagonalizable.
(b) If $|S| = n$, say, $S = \{u_1, u_2, \ldots, u_n\}$, then A is diagonalizable and $P = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}$ is an invertible matrix that diagonalizes A.

power of matrices

suppose A is invertible (i.e. $\lambda_i \neq 0$ for all i). Then

$$A^{-1} = P \begin{bmatrix} \lambda_1^{-1} & 0 & \cdots & 0 \\ 0 & \lambda_2^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^{-1} \end{bmatrix} P^{-1}$$

For any $m \in \mathbb{Z}^+$,

$$A^{-m} = P \begin{bmatrix} \lambda_1^{-m} & 0 & \cdots & 0 \\ 0 & \lambda_2^{-m} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^{-m} \end{bmatrix} P^{-m}$$

orthogonal diagonalization

- orthogonally diagonalizable → there exists an **orthogonal matrix** P such that $P^TAP = D$ (D is a **diagonal matrix**)
  - orthogonally diagonalizable $\iff$ symmetric ($A^T = A$)
  - P orthogonally diagonalizes A.
  - uses orthonormal bases for diagonalisation

how to orthogonally diagonalize

Let A be a symmetric matrix of order n.

Step 1: Find all distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ (by solving the characteristic equation $\det(\lambda I - A) = 0$).

Step 2: Find each eigenvalue λ_i,
(a) find a basis S_{λ_i} for the eigenspace E_{λ_i}, and then
(b) use the Gram-Schmidt Process (Theorem 5.2.19) to transform S_{λ_i} to an orthonormal basis T_{λ_i}.

Step 3: Let $T = T_{\lambda_1} \cup T_{\lambda_2} \cup \cdots \cup T_{\lambda_k}$, say, $T = \{v_1, v_2, \ldots, v_n\}$.
Then $P = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$ is an orthogonal matrix that orthogonally diagonalizes A.

equivalent statements

let A be a square matrix. the following statements are equivalent:

1. A is invertible
2. the linear system $Ax = 0$ has only the trivial solution
3. RREF of A is the identity matrix
4. A can be expressed as a product of elementary matrices
5. $\det(A) \neq 0$
6. The rows of A form a basis for $\mathbb{R}^n$.
7. The columns of A form a basis for $\mathbb{R}^n$.
8. $\text{rank}(A) = n$
9. 0 is not an eigenvalue of A.

checking if a matrix is diagonalizable

suppose the **characteristic polynomial** of A is factorised as

$$\begin{gathered}\det(\lambda I - A) = (\lambda - \lambda_1)^{r_1}(\lambda - \lambda_2)^{r_2} \cdots (\lambda - \lambda_k)^{r_k} \\ \text{where } \lambda_1, \ldots, \lambda^k \text{ are distinct eigenvalues of } A. \\ A \text{ is diagonalizable} \\ \iff \dim(E_{\lambda_i}) = r_i \quad \text{for each eigenvalue } \lambda_i \\ \iff |\ S_{\lambda_i}\ | = r_i\end{gathered}$$

- $r_1 + r_2 + \cdots + r_k = n$
- if any one of the eigenspaces has dimensions less than r_i, then the matrix is not diagonalizable
- If A has n distinct eigenvalues, then A is diagonalisable.

chapter 7

linear tranformations from $\mathbb{R}^n \to \mathbb{R}^m$

- linear transformation → a **mapping** : $\mathbb{R}^n \to \mathbb{R}^m$ of the form
  - if $n = m$, then T is a linear operator on $\mathbb{R}^n$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 & a_{12}x_2 & \cdots & a_{1n}x_n \\ a_{21}x_1 & a_{22}x_2 & \cdots & a_{2n}x_n \\ \vdots & \vdots & & \vdots \\ a_{m1}x_1 & a_{m2}x_2 & \cdots & a_{mn}x_n \end{bmatrix}$$

$$\text{for } (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$$

- the matrix $(a_{ij})_{m \times n}$ is the **standard matrix** for T.
  - linear transformation = multiplication by the standard matrix

alternative definition

(respects linear combinations)

let V and W be vector spaces.

a mapping $T : V \to W$ is a linear transformation $\iff$

$$T(cu + dv) = cT(u) + dT(v) \quad \forall u, v \in V \text{ and } c, d \in \mathbb{R}$$

common mappings

- identity mapping, $I : \mathbb{R}^n \to \mathbb{R}^n$
  - standard matrix for I is the **identity matrix** I_n
  - I is a **linear operator** on $\mathbb{R}^n$
- zero mapping, $O : \mathbb{R}^n \to \mathbb{R}^m$
  - standard matrix for O is the **zero matrix** $0_{m \times n}$

basic properties

let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

- $T(0) = 0$
- if $u_1, u_2, \ldots, u_k \in \mathbb{R}^n$ and $c_1, c_2, \ldots, c_k \in \mathbb{R}$, then
  $T(c_1u_1 + c_2u_2 + \cdots + c_ku_k) = c_1T(u_1) + c_2T(u_2) + \cdots + c_kT(u_k)$

standard matrices

for $T : \mathbb{R}^n \to \mathbb{R}^m$,

- standard matrix, A → $\begin{bmatrix} T(e_1) & T(e_2) & \cdots & T(e_n) \end{bmatrix}$

  $$T(e_i) = Ae_i = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mi} \end{bmatrix} = \text{the } i^{th} \text{ column of } A$$

  - image of basis vectors of the standard basis

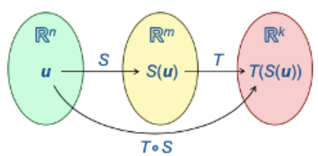
bases for $\mathbb{R}^n$

$$\text{let } \{u_1, u_2, \ldots, u_n\} \text{ be a basis for } \mathbb{R}^n. \\ \text{for any vector } v \in \mathbb{R}^n,\ v = c_1u_1 + c_2u_2 + \cdots + c_nu_n \\ \text{for some } c_1, \ldots, c_n \in \mathbb{R}^n$$

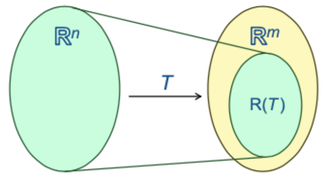
- $\{u_1, u_2, \ldots, u_n\}$ are the basis vectors
- the image $T(v)$ of v is completely determined by the images $T(u_1), T(u_2), \ldots, T(u_n)$ of the basis vectors

compositions of mappings

- composition of T with S → a mapping from $\mathbb{R}^n$ to $\mathbb{R}^k$ defined by $(T \circ S)(u) = T(S(u))$ for $u \in \mathbb{R}^n$
- for all $u \in \mathbb{R}^n$, $(T \circ S)(u) = T(S(u)) = T(Au) = BAu$
  - BA is the standard matrix of $T \circ S$

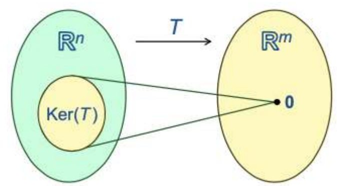


range



- range of T, $R(T)$ → the set of images of T
  - $R(T) = \{T(u) \mid u \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$
  - $R(T) = \text{span}\{T(u_1), T(u_2), \ldots, T(u_n)\}$
  - $R(T) =$ the column space of the standard matrix A
- rank of T → the dimension of $R(T)$
  - $\text{rank}(T) = \dim(R(T)) = \dim(\text{column space of } A) = \text{rank}(A)$

kernel



- kernel of T, $\ker(T)$ → the set of vectors in $\mathbb{R}^n$ whose image is the **zero vector** in $\mathbb{R}^m$
  - $\ker(T) = \{u \mid T(u) = 0\} \subseteq \mathbb{R}^n$
  - $\ker(T) =$ the nullspace of the standard matrix A
- the **nullity** of T is the dimension of $\ker(T)$.
  - $\text{nullity}(T) = \dim(\ker(T)) = \text{nullity}(A)$

dimension theorem for linear transformation

$$\begin{aligned} \text{rank}(T) + \text{nullity}(T) &= n \\ = \text{rank}(A) + \text{nullity}(A) \\ = \text{number of columns in } A \end{aligned}$$