Cart and Pole Problem

a) Describe the equations of motion that govern this system.

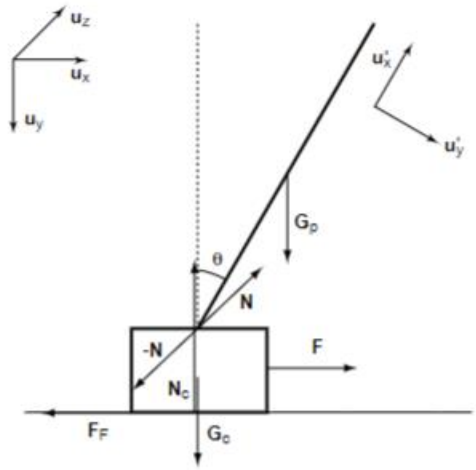

Figure 1: The cart-pole system.

The cart is assumed to be on a one-dimensional track, where the pole can move in the vertical plane parallel to the track. The system deals with a multitude of forces that act on the cart and pole. We'll first establish those forces.

** We will assume it is a frictionless system*

Forces acting on the System

- **F** represents the force parallel to the track.
- $\mathbf{m_c}$ represents the mass of the cart.
- $\mathbf{m_p}$ represents the mass of the pole.
- **2l** represents the length of the pole.
- **x** represents the position of the cart.
- **Θ** represents the angle between the pole and the vertical axis.
- $\mathbf{a_c}$ is the acceleration of the cart (can also be represented as the second derivative of the cart's position, $\ddot{x}$)
- **N** represents the reaction force that the cart has on the pole.
- **-N** represents the force the pole will act on the cart.
- **G** represents gravity

Now that our forces have been established, we can now discuss the equations that govern the system.

Forces on the Cart

Newton's Second Law

The cart can be governed by Newton's second law, which says that force = mass * acceleration. Using this law, we can determine that the following equation represents the forces acting on the cart:

$$\mathbf{F} + \mathbf{G_c} - \mathbf{N} + \mathbf{N_c} = \mathbf{m_c}\, \mathbf{a_c} \qquad (1)$$

This equation is stating that the sum of all forces on the cart is equal to the mass * the acceleration of the carts. The forces acting on the cart are:

- Force parallel to the track
- The force of gravity acting on the cart

- The reaction force of the pole on the cart
- The reaction force of the cart on the pole

The equation can be broken up to represent the x and y axis.

x-axis

$$\mathbf{F} - \mathbf{N_x} = \mathbf{m_c}\ddot{\mathbf{x}} \tag{2}$$

Here we have the force parallel to the track acting against the pole's force on the x-axis.

y-axis

$$\mathbf{m_c g} + \mathbf{N_y} - \mathbf{N_c} = 0 \tag{3}$$

In this equation, we can see that the only forces acting on the cart are the mass of the cart * gravity and the pole's force on the y-axis working against the force of the cart. The equation is set to 0 because there is no acceleration on the y-axis as the cart can only move in the horizontal plane.

Forces on the Pole

Linear Movement of the Pole

To find the linear movement of the pole, we will use Newton's second law (F=ma).

$$\mathbf{N} + \mathbf{G_p} = \mathbf{m_p a_p} \tag{4}$$

Where:

$$\mathbf{G_p} = \mathbf{m_p}\ \mathbf{g}\ \mathbf{u_y}$$

Here, we have the force of the cart acting on the pole and the gravity acting on the pole equal to the mass of the pole * the acceleration of the pole's center of mass.

Acceleration of the Pole's Center of Mass

The acceleration of the pole's center of mass is represented by the following equation.

$$\mathbf{a_p} = \mathbf{a_c} + \boldsymbol{\varepsilon} \times \mathbf{r_p} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r_p}) \tag{5}$$

Where:

Vector representing to position of the center of mass relative to the articulation point:

$$\mathbf{r_p} = \mathbf{l}\ (\sin\boldsymbol{\theta}\ \mathbf{u_x} - \cos\boldsymbol{\theta}\ \mathbf{u_y})$$

Rotation of the pole with angular velocity:

$$\boldsymbol{\omega} = \dot{\boldsymbol{\theta}}\ \mathbf{u_z}$$

Angular acceleration:

$$\boldsymbol{\varepsilon} = \ddot{\boldsymbol{\theta}}\ \mathbf{u_z}$$

Adding the Pole's Center of Mass Acceleration to the Linear Movement of the Pole Equation

We can now introduce equation 5 into equation 4. By decomposing the equation to look at the x-axis and y-axis independently, we get:

x-axis

$$\mathbf{N_x} = \mathbf{m_p}\ (\ddot{\mathbf{x}} + \mathbf{l}\ddot{\boldsymbol{\theta}}\cos\boldsymbol{\theta} - \mathbf{l}\dot{\boldsymbol{\theta}}^2\sin\boldsymbol{\theta}) \tag{6}$$

y-axis

$$\mathbf{m_p\, g - N_y = m_p\, (l\ddot{\theta}} \sin\boldsymbol{\theta} + \mathbf{l\dot{\theta}}^2 \cos\boldsymbol{\theta}) \tag{7}$$

Newton’s Second Law to the Rotational Movement of the pole around the articulation point

$$\mathbf{M = I\, \varepsilon + r_p \times a_c} \tag{8}$$

Where:

Sum of all non-inertial torques acting on the pole relative to the articulation (Note: Neglecting Friction):

$$\mathbf{M = r_p \times G_p - \dot{\theta}\, u_z}$$

The moment of inertia of the pole relative to the articulation:

$$\mathbf{I} = 4/3\, \mathbf{m_p\, l}^2$$

Torque generated by the inertial force caused by the acceleration of the cart:

$$\mathbf{-r_p \times a_c}$$

Taking all of this into account, we get:

$$\mathbf{m_p\, g\, l} \sin\boldsymbol{\theta} - \dot{\boldsymbol{\theta}} = 4/3\, \mathbf{m_p\, l}^2 \ddot{\boldsymbol{\theta}} + \mathbf{m_p \ddot{x}\, l} \cos\boldsymbol{\theta} \tag{9}$$

Adding the rotational movement to forces on the x-axis and Linear movement of the pole

By combining equations 8, 6, and 2 we get:

$$\ddot{\mathbf{x}} = \frac{\mathbf{F + m_p\, l}\, (\dot{\boldsymbol{\theta}}^2 \sin\boldsymbol{\theta} - \ddot{\boldsymbol{\theta}} \cos\boldsymbol{\theta}) - \mathbf{F_f}}{\mathbf{m_c + m_p}} \tag{10}$$

Final Equations

$$\ddot{\theta} = \frac{g \sin\theta + \cos\theta \left(\frac{-F - m_p\, l\, \dot{\theta}^2 \sin\theta}{m_c + m_p} \right)}{l \left(\frac{4}{3} - \frac{m_p \cos^2\theta}{m_c + m_p} \right)}$$

$$\ddot{x} = \frac{F + m_p\, l\, (\dot{\theta}^2 \sin\theta - \ddot{\theta} \cos\theta)}{m_c + m_p}.$$

$$N_c = (m_c + m_p)\, g - m_p\, l\, (\ddot{\theta} \sin\theta + \dot{\theta}^2 \cos\theta)$$

To sum it up, $\ddot{\boldsymbol{\theta}}$ is the angular acceleration of the pole between the angle of the pole and the vertical axis. $\ddot{\mathbf{x}}$ is the acceleration of the cart. $\mathbf{N_c}$ is the force of the cart on the pole. It is important to note that **g** is positive in these equations.

b) What is the maximum angle that my pole can fall to before it cannot recover if max Force $F = 7N$?

To solve this, we need to use the restoring torque equation.

$T = r * F * sin(\theta)$

Where:

- r = the distance from the pivot point to the point where the force is applied
- F = the force applied
- Θ = angle of deflection

$T_{gravity} = m * g * r * sin(\theta_{max})$

Where:

- m = mass of the pole
- g = gravity ($9.81\ m/s^2$)

If we set the two torques equal to each other, we get:

$r * F * sin(\theta) = m * g * r * sin(\theta)$

Solving for θ:

$Sin(\theta) = m*g/F$

$\Theta = sin^{-1}(m*g/F)$

Substitute our values:

$\Theta = sin^{-1}(0.5 * 9.81 / 7)$

Θ = 44.427 degrees OR 0.775 radians