Question 3

Let S be a set of points in R^d, and let $f: R^d \to R^k$ be an embedding function that satisfies the bounds of the JL-Lemma, meaning that for any two points u, v in S, the following inequality holds:

$$||v - u||^2 <= ||f(v) - f(u)||^2 <= (1 + e)||v - u||^2$$

Now, consider a specific example where S is the set of points in R^2, and the points u, v, and w are defined as:

$u = (0,0)$ $v = (1,0)$ $w = (0,1)$

The area of the triangle $<u,v,w>$ is 0.5.

Now, let's define the function $f(x) = x + (e,0)$ for $x = u, v$ and $f(x) = x + (e,e)$ for $x = w$

The points f(u), f(v), and f(w) are defined as:

$f(u) = (e,0)$ $f(v) = (1+e,0)$ $f(w) = (e,1+e)$

The area of the triangle $<f(u), f(v), f(w)>$ is $(1+e)*(1+e)/2$ which is greater than $0.5 + e/2$

As we can see the area of the triangle $<f(u), f(v), f(w)>$ is not bounded by $(1+c.e) * area(<u,v,w>)$ for any c, it's only bounded by $(1+e/2) * area(<u,v,w>)$

Therefore, it is not true in general that the area of a triangle is always preserved under the embedding function f, and this specific counterexample shows that it is possible for the area of a triangle to be distorted by a factor greater than $1 + c.e$ for any constant c.