CS446: Machine Learning, Fall 2018, Homework 0

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Problem (1)

(a)

Note that both H_1 and H_{-1} are parallel hyperplanes, since their normal vectors are equal. It follows that to find the distance between both planes, given a point x_1 lying on H_1, the closest point on H_{-1} to x_1 is the point x_2 obtained from intersecting the line $\{x_1 + wt | t \in \mathbb{R}\}$ with H_{-1}. That is $x_2 = x_1 + wt$. We seek to find $\|wt\|$. Note that $w^T x_1 = 1$ and $w^T x_2 = -1$. Hence:

$$
\begin{aligned}
x_2 - x_1 &= wt \\
\implies w^T(x_2 - x_1) &= (w^t w)t \\
\implies -2 &= \|w\|^2 t \\
\implies t &= \frac{-2}{\|w\|^2} \\
\implies \|wt\| &= \left\| \left[w \left(\frac{-2}{\|w\|^2} \right) \right] \right\| \\
&= \frac{2}{\|w\|}
\end{aligned}
$$

(b)

We prove this by contradiction. Suppose that there is a better maximum margin classifier $w' \neq w$ for the set (X^*, Y^*). We will show that w' is also the maximum margin classifier for (X, Y) then, which is a contradiction, since we assumed that w was the maximum margin classifier and the maximum margin classifier is unique [1]

By assumption, since w' is a better classifier for (X^*, Y^*),

$$y^i (w') T x^i \geq 1 \qquad \text{for all } i \in \mathcal{N} \text{ and} \tag{γ}$$

$$\frac{2}{\|w'\|} \geq \frac{2}{\|w\|}$$

Thus, to prove that w' is the maximum margin classifier for (X, Y), we need to show that

[1] Solving for the maximum margin classifier in the separable case is a strictly convex problem and, hence, the minimizer to the problem is unique.

$$y^j(w')^T x^j \geq 1 \qquad \text{For all } j \notin \mathcal{N}$$

Let $H_1 = \{(x,1) \in (\mathbb{R}^n, \mathbb{R}^n) | (1)w^T x = 1\}$. Without loss of generality, consider any (x^j, y^j) for $j \notin \mathcal{N}$ such that $y^j = 1$. There exists some $(z, 1) \in H_1$ such that $z + \alpha w = x^j$, where $\alpha > 0$. Thus

$$\begin{aligned} y^j(w')^T x^j &= y^j \alpha (w')^T (z + w) \\ &= y^j (w')^T z + y^j \alpha (w')^T w \\ &\geq 1 + y^j \alpha (w')^T w \\ &= 1 + \alpha (w')^T w \end{aligned}$$

It suffices to show that $(w')^T w \geq 0$, to prove that (x^j, y^j) is correctly classified using w'.

Observe that for any $\beta > 0$, we must have that βw is classified using the maximum margin classifier under w, which we label $\mathcal{A}_w$, as having label 1. For if q is the label of βw, then

$$\mathcal{A}_w(\beta w) = q(w^T(\beta w)) > 0 \implies q = 1$$

Thus, in particular, there exists some $\beta' > 0$ such that $\mathcal{A}_w(\beta' w) = 1$, meaning that $\beta' w$ lies on H_1 and we must have, by construction, $(w')^T(\beta' w) > 0 \implies (w')^T w \geq 0$.

Thus w' correctly classifies even (x^j, y^j). Since $j \notin \mathcal{N}$ was arbitrary, $\mathcal{A}'_w$ correctly classifies all $j \notin \mathcal{N}$. $\mathcal{A}'_w$ already correctly classfied those $(x^i, y^i) \in \mathcal{N}$ and $\frac{2}{\|w'\|} \geq \frac{2}{\|w\|}$, so we conclude that w' is a better classifier than w for (X, Y), which is a contradiction.