CS446: Machine Learning, Fall 2018, Homework 2

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Problem (2)

(a)

Since A is symmetric, A is unitarily diagonalizable, meaning that

$$A = P^T D P$$

Where $P^T = P^{-1}$ and D is diagonal

Since D is diagonal and A positive semi-definite, the eigenvalues of A occupy the diagonal entries of D and all eigenvalues are non-negative. As a consequence, we can define a square root for D, which is the matrix obtained by taking the square root of each diagonal entry in D. We call this matrix E

$$\begin{aligned} A &= P^T E E P \\ \implies x^T A x' &= x^T P^T E E P x' \\ &= (EPx)^T (EPx') \end{aligned}$$

Thus, we see that a feature transformation ϕ exists defined by $\phi(x) = EPx$ such that $x^T A x' = k(x, x')\phi(x)^T \phi(x')$.

(b)

Since k is a valid kernel, $k(x, x')$ can be decomposed into the inner product of some feature transformation ϕ. That is, $k(x, x') = \phi(x)^T \phi(x')$.

Define a new feature transformation $\psi(x) = f(x)\phi(x)$. Then observe that

$$\psi(x)^T \psi(x^*) = f(x)\phi(x)^T \phi(x^*) f(x^*)$$

(c)

We show that $x^T K x \geq 0$ for all $x \in \mathbb{R}^n$. Recall that inner products produce non-negative values in $\mathbb{R}$ and that they are symmetric. Thus K is a symmetric matrix with no negative entries. It follows that

$$\begin{aligned} x^T A x &= \sum_{i,j} x_i x_j A_{ij} \\ &= 2 \sum_{i,j>i} x_i x_j A_{ij} \end{aligned}$$

Since $x_i x_j A_{ij} = x_j x_i A_{ij}$

Now suppose that arbitrary x is given. x can be decomposed as the sum of two vectors x_1 and x_2 such that every entry in x_1 is non-negative and every entry in x_2 is non-positive. It follows that

$$\begin{aligned} x^T A x &= (x_1 + x_2)^T A (x_1 + x_2) \\ &= x_1^T A x_1 + x_1^T A x_2 + x_2^T A x_2 + x_2^T A x_2 \end{aligned}$$

From (1), we know that:

$$x_1^T A x_1 = 2 \sum_{i,j>i} x_i^i x_j^i A_{ij}$$

From how we defined x_1, we conclude that this foregoing expression is non-negative. By similar reasoning, we can conclude that $x_2^T A x_2$ is non-negative

By construction, $x_1^T A x_2$ and $x_2^T A x_1$ are both zero, since wherever x_1 is not zero, x_2 is zero and vice versa. Hence

$$x_1^T A x_1 + x_1^T A x_2 + x_2^T A x_2 + x_2^T A x_2 \geq 0$$

This completes the proof and the problem.

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