CS446: Machine Learning, Fall 2018, Homework 1

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Problem (2)

(a) Multiply all nontrivial inequalities and the primal optimization problem by -1 to, firstly, turn this into a minimization problem and, secondly, get inequalities that follow the conventions for the Lagrangian. Explicitly, we are trying to now solve

$$\begin{aligned} &\min_{\{w_1,w_2,w_3,w_4\}} -w_1 + -2w_2 + -3w_3 + -4w_4, \\ s.t. \quad & w_1 \geq 0, w_2 \geq 0, w_3 \geq 0, w_4 \geq 0, \\ & -4w_1 + -3w_2 + -2w_3 + -w_4 + 10 \leq 0, \\ & -2w_1 + -w_2 + -w_3 + -2w_4 + 4 \leq 0. \end{aligned}$$

Here λ is the vector $[\lambda_1, \lambda_2]$ and each $\lambda_i \geq 0$. Set W to be the constraint region given by $\{w_1 \geq 0, w_2 \geq 0, w_3 \geq 0, w_4 \geq 0\}$. The Lagrangian is

$$\begin{aligned} L(w,\lambda)_{\{w \in W\}} = \\ & -w_1 + -2w_2 + -3w_3 + -4w_4 + \lambda_1(-4w_1 + -3w_2 + -2w_3 + -w_4 + 10) \\ & \qquad + \lambda_2(-2w_1 + -w_2 + -w_3 + -2w_4 + 4) \end{aligned}$$

Now we re-arrange the objective function to express it as

$$\sum_{i=1}^{4}(w_i)(\text{Some expression involving } \lambda_1 \text{ and } \lambda_2).$$

$$\begin{aligned} L(w,\lambda)_{\{w \in W\}} = \\ & w_1(-1 - 4\lambda_1 - 2\lambda_2) + w_2(-2 + -3\lambda_1 - 2\lambda_2) \\ & \qquad + w_3(-3 - 2\lambda_1 - \lambda_2) + w_4(-4 - \lambda_1 - 2\lambda_2) + 10\lambda_1 + 4\lambda_2 \end{aligned}$$

Let $\Lambda = \{(\lambda_1, \lambda_2) | \lambda_i \geq 0\}$. The dual problem is

$$\max_{\lambda \in \Lambda} g(\lambda) = \max_{\lambda \in \Lambda} \inf_{w \in W} L(w, \lambda)$$

$$\begin{aligned}
&\max_{\lambda\in\Lambda} 10\lambda_1 + 4\lambda_2, \\
s.t. \quad &\lambda_1 \geq 0, \lambda_2 \geq 0, \\
&-1 - 4\lambda_1 - 2\lambda_2 \geq 0, \\
&-2 + -3\lambda_1 - 2\lambda_2 \geq 0, \\
&-3 - 2\lambda_1 - \lambda_2 \geq 0, \\
&-4 - \lambda_1 - 2\lambda_2 \geq 0.
\end{aligned}$$

or, equivalently:

$$\begin{aligned}
&\max_{\lambda\in\Lambda} 10\lambda_1 + 4\lambda_2, \\
s.t. \quad &\lambda_1 \geq 0, \lambda_2 \geq 0, \\
&1 + 4\lambda_1 + 2\lambda_2 \leq 0, \\
&2 + 3\lambda_1 + 2\lambda_2 \leq 0, \\
&3 + 2\lambda_1 + \lambda_2 \leq 0, \\
&4 + \lambda_1 + 2\lambda_2 \leq 0.
\end{aligned}$$

This problem is infeasible.

(b)
Let $w = [w_1, w_2]$. The Lagrangian is

$$\frac{1}{2}w^T w + v(2w_1 + w_2) = \frac{1}{2}w^T w + v(w)^T(a)$$

where

$$a = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Let $\mathrm{N} = \{v | v \in \mathbb{R}\}$. The dual problem is

$$\max_{v\in\mathrm{N}} g(v) = \max_{v\in\mathrm{N}} \inf_{w\in W} \left(\frac{1}{2}w^T w + v(w^T)(a)\right)$$

To minimize this, find the gradient with respect to w and set that to 0.

$$\begin{aligned}
\nabla_w L &= w^T + a^T v \\
\implies w^T + a^T v &= 0 \\
\implies w &= (-va^T)^T \\
&= -va
\end{aligned}$$

Thus, the dual problem is

$$\max_{v\in\mathrm{N}} g(v) = \max_{v\in\mathrm{N}} \frac{1}{2}(v^2)a^T a - v(v)(a^T a) = \max_{v\in\mathrm{N}} \frac{-(v^2)a^T a}{2}$$