CS446: Machine Learning, Fall 2018, Homework 0

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Worked individually

Problem (1)

Solution:

Problem 1.1.

Find those λ so that $\det(A - \lambda I) = 0$; then those $\vec{x}$ such that $(A - \lambda I)\vec{x} = 0$ (ie the null space of $(A - \lambda I)$) are the eigenvectors corresponding to λ.

$$\begin{aligned}
\det\left(\begin{bmatrix} 3-\lambda & 1 \\ 8 & 1-\lambda \end{bmatrix}\right) &= 0 \qquad (1) \\
(3-\lambda)(1-\lambda) - 8 &= 0 \\
3 - 4\lambda + \lambda^2 - 8 &= 0 \\
\lambda^2 - 4\lambda - 5 &= 0 \\
(\lambda - 5)(\lambda + 1) &= 0 \\
\implies \lambda &= 5, -1
\end{aligned}$$

Now let us find the eigenvalue corresponding to $\lambda = 5$; substitute $\lambda = 5$ into (1)

$$\begin{bmatrix} -2 & 1 \\ 8 & -4 \end{bmatrix}$$

We see that the null space of this matrix is exactly

$$\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\} \qquad (2)$$

Now let us find the eigenvalue corresponding to $\lambda = -1$; substitute $\lambda = -1$ into (1)

$$\begin{bmatrix} 4 & 1 \\ 8 & 2 \end{bmatrix}$$

We see that the null space of this matrix is exactly

$$\left\{\begin{bmatrix} 1 \\ -4 \end{bmatrix}\right\} \qquad (3)$$

Thus $\lambda = 5$ has eigenvector in expression (2) and $\lambda = -1$ has eigenvector in expression (3)

Problem 1.2.

Suppose that $\lambda = 0$; then Ax is the zero vector. If $\lambda > 0$, then Ax points in the direction of x; if $\lambda < 0$, then Ax points in the direction opposite of x. In either case, whether $\lambda > 0$ or $\lambda < 0$, the magnitude of Ax is the same as x scaled by $|\lambda|$.

Problem 1.3.

Proof. Suppose that (λ, x) is an eigenvalue-eigenvector pair of A. By definition $Ax = \lambda x$, whence

$$\begin{aligned} x^T A x &= x^T \lambda x \\ &= \lambda x^T x \\ &= \lambda \|x\|_2^2 \end{aligned}$$

Since $x^T A x > 0$, we must have:

$$\begin{aligned} \lambda \|x\|_2^2 > 0 \implies \\ \lambda > 0 \end{aligned}$$

$\square$