1 Final Report – Aahan Agrawal

What does this report do?

This report addresses the question (with respect to the corn yield data set that I have been working on): Does introducing varying slopes and/or varying intercepts into a regression model f allow the resultant model to outperform f. A thorough examination of this question would require testing far more models than feasibly test-able. My examination tests models belonging to a wide enough coverage of models, however, that we can (with high confidence) say the following:

An ordinary least squares model f using ...

(a). only *linear* terms can be improved by making f hierarchical.

(b). only *linear and quadratic* can be improved by making f hierarchical.

(c). *splines* is not likely improveable by making f hierarchical.

What does this report not do?

This report does not answer the question *Can a hierarchical linear model (aka a mixed effects model)* beat our current, best model? – which is codenamed *vpd_spline_evi_poly*. This report only constructed hierarchical models using *lmer*; moreover, there are additional features of hierarchical linear models that we can test. For example, does a heterogenous residual covariance structure lead to better model performance, or does a hierarchical linear model obtained by specifying Bayesian priors outperform our model. The former can be assessed using the package *nlme*; the latter can be assessed using packages like *stan*.

1.1 Input Variables

The input variables selected are precisely those input variablesthat Yan Li's model explored different combinations of. These variables were found through some previous analysis to be the most potent ones.

Abbreviation	Expanded Term
VPD	vpdave5 + vpdave6 + vpdave7 + vpdave8
TAVE	tave5 + tave6 + tave7 + tave8
PRECIP	precip5 + precip6 + precip7 + precip8
EVI	evi5 + evi6 + evi7 + evi8
VPD^2	vpdave5 + vpdave6 + vpdave7 + vpdave8 + $vpdave5^2$ + $vpdave6^2$ + $vpdave7^2$ + $vpdave8^2$
$TAVE^2$	tave5 + tave6 + tave7 + tave8 + $tave5^2$ + $tave6^2$ + $tave7^2$ + $tave8^2$
$PRECIP^2$	precip5 + precip6 + precip7 + precip8 + $precip5^2$ + $precip6^2$ + $precip7^2$ + $precip8^2$
EVI^2	evi5 + evi6 + evi7 + evi8 + $evi5^2$ + $evi6^2$ + $evi7^2$ + $evi8^2$
VPD_2	vpdave6 + vpdave7 + vpdave8 + vpdave9 + $vpdave6^2$ + $vpdave7^2$ + $vpdave8^2$ + $vpdave9^2$

2 Model Performances

The tables below list model configurations and their performance(in terms of RMSE and R^2) using leave-one-out cross validation on years from 2003 to 2006. The formula specifications below are specifications that the package *lme4* can fit using the function *lmer*. In each formula, Y represents *rainfed yield anomaly*[1].

2.1 Models with only Linear Terms

For each of the $\binom{4}{2} = 6$ combinations of 2 predictors from the 4, previous predictors – the following tables show that, consistently, a varying slope and varying intercept model produces a non-trivial reduction (the reduction is at least 1) in RMSE.

[1]Note to self: add more details for others, if needbe

$Y \sim \text{VPD} + \text{EVI}$	20.544680	0.690445212535
$Y \sim \text{VPD} + \text{EVI} + (1 \mid \text{State}) + \sum_{j=5}^{8}(0 + \text{vpdave}j)\mid\text{State})$	19.364378	0.712176959856
$Y \sim \text{VPD} + \text{EVI} + (1 \mid \text{State}) + \sum_{j=5}^{8}(0 + \text{evi}j)\mid\text{State})$	19.354932	0.728661647677
$Y \sim \text{VPD} + \text{EVI} + (1 \mid \text{State}) + \sum_{j=5}^{8}(0 + \text{evi}j + \text{vpdave}j)\mid\text{State})$	18.311311	0.735956309523
$Y \sim \text{VPD} + \text{EVI} + \sum_{j=5}^{8}(0 + \text{evi}j + \text{vpdave}j)\mid\text{State})$	18.635979	0.726468705361
Model File: vpd_evi.csv		
$Y \sim \text{VPD} + \text{TAVE}$	25.484906	0.514984759119
$Y \sim \text{VPD} + \text{TAVE} + (1 \mid \text{State}) + \sum_{j=5}^{8}(0 + \text{vpdave}j)\mid\text{State})$	23.141803	0.634082857334
$Y \sim \text{VPD} + \text{TAVE} + (1 \mid \text{State}) + \sum_{j=5}^{8}(0 + \text{tave}j)\mid\text{State})$	25.444579	0.571100953388
$Y \sim \text{VPD} + \text{TAVE} + (1 \mid \text{State}) + \sum_{j=5}^{8}(0 + \text{tave}j + \text{vpdave}j)\mid\text{State})$	22.087563	0.651670587067
$Y \sim \text{VPD} + \text{TAVE} + \sum_{j=5}^{8}(0 + \text{tave}j + \text{vpdave}j)\mid\text{State})$	23.200899	0.596111655752
Model File: vpd_tave.csv		
$Y \sim \text{VPD} + \text{PRECIP}$	27.478782	0.534561106423
$Y \sim \text{VPD} + \text{PRECIP} + (1 \mid \text{State}) + \sum_{j=5}^{8}(0 + \text{vpdave}j)\mid\text{State})$	23.353088	0.616543312825
$Y \sim \text{VPD} + \text{PRECIP} + (1 \mid \text{State}) + \sum_{j=5}^{8}(0 + \text{precip}j)\mid\text{State})$	26.898276	0.420015747551
$Y \sim \text{VPD} + \text{PRECIP} + (1 \mid \text{State}) + \sum_{j=5}^{8}(0 + \text{precip}j + \text{vpdave}j)\mid\text{State})$	22.964402	0.611345395363
$Y \sim \text{VPD} + \text{PRECIP} + \sum_{j=5}^{8}(0 + \text{precip}j + \text{vpdave}j)\mid\text{State})$	24.361307	0.55026376432
Model File: vpd_precip.csv		
$Y \sim \text{TAVE} + \text{PRECIP}$	31.575936	0.282355708807
$Y \sim \text{TAVE} + \text{PRECIP} + (1 \mid \text{State}) + \sum_{j=5}^{8}(0 + \text{tave}j)\mid\text{State})$	25.438664	0.578968289794
$Y \sim \text{TAVE} + \text{PRECIP} + (1 \mid \text{State}) + \sum_{j=5}^{8}(0 + \text{precip}j)\mid\text{State})$	26.421840	0.438308246651
$Y \sim \text{TAVE} + \text{PRECIP} + (1 \mid \text{State}) + \sum_{j=5}^{8}(0 + \text{precip}j + \text{tave}j)\mid\text{State})$	24.579791	0.582927778914
$Y \sim \text{TAVE} + \text{PRECIP} + \sum_{j=5}^{8}(0 + \text{precip}j + \text{tave}j)\mid\text{State})$	28.574156	0.439021248291
Model File: tave_precip.csv		
$Y \sim \text{TAVE} + \text{EVI}$	20.855124	0.669898327672
$Y \sim \text{TAVE} + \text{EVI} + (1 \mid \text{State}) + \sum_{j=5}^{8}(0 + \text{tave}j)\mid\text{State})$	19.993899	0.707643929601
$Y \sim \text{TAVE} + \text{EVI} + (1 \mid \text{State}) + \sum_{j=5}^{8}(0 + \text{evi}j)\mid\text{State})$	18.887613	0.731858082188
$Y \sim \text{TAVE} + \text{EVI} + (1 \mid \text{State}) + \sum_{j=5}^{8}(0 + \text{evi}j + \text{tave}j)\mid\text{State})$	18.970730	0.715730529989
$Y \sim \text{TAVE} + \text{EVI} + \sum_{j=5}^{8}(0 + \text{evi}j + \text{tave}j)\mid\text{State})$	19.477666	0.701345161395
Model File: tave_evi.csv		
$Y \sim \text{PRECIP} + \text{EVI}$	20.753574	0.671635695019
$Y \sim \text{PRECIP} + \text{EVI} + (1 \mid \text{State}) + \sum_{j=5}^{8}(0 + \text{precip}j)\mid\text{State})$	19.765463	0.696725746026
$Y \sim \text{PRECIP} + \text{EVI} + (1 \mid \text{State}) + \sum_{j=5}^{8}(0 + \text{evi}j)\mid\text{State})$	19.023711	0.730069240143
$Y \sim \text{PRECIP} + \text{EVI} + (1 \mid \text{State}) + \sum_{j=5}^{8}(0 + \text{evi}j + \text{precip}j)\mid\text{State})$	19.000063	0.703405604865
$Y \sim \text{PRECIP} + \text{EVI} + \sum_{j=5}^{8}(0 + \text{evi}j + \text{precip}j)\mid\text{State})$	19.141144	0.704596192832
Model File: precip_evi.csv		

$Y \sim \text{VPD} + \text{EVI} + \text{FIPS}$	Too Slow	
$Y \sim \text{VPD} + \text{EVI} + (1 \mid \text{State}) + \sum_{j=5}^{8}(0 + \text{vpdave}j)\mid\text{State}) + \text{FIPS}$	Too Slow	
$Y \sim \text{VPD} + \text{EVI} + (1 \mid \text{State}) + \sum_{j=5}^{8}(0 + \text{evi}j)\mid\text{State}) + \text{FIPS}$	Too Slow	
$Y \sim \text{VPD} + \text{EVI} + (1 \mid \text{State}) + \sum_{j=5}^{8}(0 + \text{evi}j + \text{vpdave}j)\mid\text{State}) + \text{FIPS}$	Too Slow	
$Y \sim \text{VPD} + \text{EVI} + \sum_{j=5}^{8}(0 + \text{evi}j + \text{vpdave}j)\mid\text{State}) + \text{FIPS}$	Too Slow	
$Y \sim \text{VPD} + \text{TAVE} + \text{FIPS}$	Too Slow	
$Y \sim \text{VPD} + \text{TAVE} + (1 \mid \text{State}) + \sum_{j=5}^{8}(0 + \text{vpdave}j)\mid\text{State}) + \text{FIPS}$	Too Slow	
$Y \sim \text{VPD} + \text{TAVE} + (1 \mid \text{State}) + \sum_{j=5}^{8}(0 + \text{tave}j)\mid\text{State}) + \text{FIPS}$	Too Slow	
$Y \sim \text{VPD} + \text{TAVE} + (1 \mid \text{State}) + \sum_{j=5}^{8}(0 + \text{tave}j + \text{vpdave}j)\mid\text{State}) + \text{FIPS}$	Too Slow	
$Y \sim \text{VPD} + \text{TAVE} + \sum_{j=5}^{8}(0 + \text{tave}j + \text{vpdave}j)\mid\text{State}) + \text{FIPS}$	Too Slow	
$Y \sim \text{VPD} + \text{PRECIP} + \text{FIPS}$	Too Slow	
$Y \sim \text{VPD} + \text{PRECIP} + (1 \mid \text{State}) + \sum_{j=5}^{8}(0 + \text{vpdave}j)\mid\text{State}) + \text{FIPS}$	Too Slow	
$Y \sim \text{VPD} + \text{PRECIP} + (1 \mid \text{State}) + \sum_{j=5}^{8}(0 + \text{precip}j)\mid\text{State}) + \text{FIPS}$	Too Slow	
$Y \sim \text{VPD} + \text{PRECIP} + (1 \mid \text{State}) + \sum_{j=5}^{8}(0 + \text{precip}j + \text{vpdave}j)\mid\text{State}) + \text{FIPS}$	Too Slow	
$Y \sim \text{VPD} + \text{PRECIP} + \sum_{j=5}^{8}(0 + \text{precip}j + \text{vpdave}j)\mid\text{State}) + \text{FIPS}$	Too Slow	
$Y \sim \text{TAVE} + \text{PRECIP} + \text{FIPS}$	Too Slow	
$Y \sim \text{TAVE} + \text{PRECIP} + (1 \mid \text{State}) + \sum_{j=5}^{8}(0 + \text{tave}j)\mid\text{State}) + \text{FIPS}$	Too Slow	
$Y \sim \text{TAVE} + \text{PRECIP} + (1 \mid \text{State}) + \sum_{j=5}^{8}(0 + \text{precip}j)\mid\text{State}) + \text{FIPS}$	Too Slow	
$Y \sim \text{TAVE} + \text{PRECIP} + (1 \mid \text{State}) + \sum_{j=5}^{8}(0 + \text{precip}j + \text{tave}j)\mid\text{State}) + \text{FIPS}$	Too Slow	
$Y \sim \text{TAVE} + \text{PRECIP} + \sum_{j=5}^{8}(0 + \text{precip}j + \text{tave}j)\mid\text{State}) + \text{FIPS}$	Too Slow	
$Y \sim \text{TAVE} + \text{EVI} + \text{FIPS}$	Too Slow	
$Y \sim \text{TAVE} + \text{EVI} + (1 \mid \text{State}) + \sum_{j=5}^{8}(0 + \text{tave}j)\mid\text{State}) + \text{FIPS}$	Too Slow	
$Y \sim \text{TAVE} + \text{EVI} + (1 \mid \text{State}) + \sum_{j=5}^{8}(0 + \text{evi}j)\mid\text{State}) + \text{FIPS}$	Too Slow	
$Y \sim \text{TAVE} + \text{EVI} + (1 \mid \text{State}) + \sum_{j=5}^{8}(0 + \text{evi}j + \text{tave}j)\mid\text{State}) + \text{FIPS}$	Too Slow	
$Y \sim \text{TAVE} + \text{EVI} + \sum_{j=5}^{8}(0 + \text{evi}j + \text{tave}j)\mid\text{State}) + \text{FIPS}$	Too Slow	
$Y \sim \text{PRECIP} + \text{EVI} + \text{FIPS}$	Too Slow	
$Y \sim \text{PRECIP} + \text{EVI} + (1 \mid \text{State}) + \sum_{j=5}^{8}(0 + \text{precip}j)\mid\text{State}) + \text{FIPS}$	Too Slow	
$Y \sim \text{PRECIP} + \text{EVI} + (1 \mid \text{State}) + \sum_{j=5}^{8}(0 + \text{evi}j)\mid\text{State}) + \text{FIPS}$	Too Slow	
$Y \sim \text{PRECIP} + \text{EVI} + (1 \mid \text{State}) + \sum_{j=5}^{8}(0 + \text{evi}j + \text{precip}j)\mid\text{State}) + \text{FIPS}$	Too Slow	
$Y \sim \text{PRECIP} + \text{EVI} + \sum_{j=5}^{8}(0 + \text{evi}j + \text{precip}j)\mid\text{State}) + \text{FIPS}$	Too Slow	

Remark 2.1. The only difference between the foregoing table and the one before is the inclusion of the categorical variable FIPS.

2.2 Models with both Linear and Quadratic Terms

A limitation to comparing OLS models with their hierarchical counterparts is the speed of the function *lmer*. I had to judiciously choose which quadratic configurations to test. I tested that two variable configuration that performed best in a hierarchical, linear setting (aka, the best two predictors from the previous section). These predictors were *vpd* and *evi*.

The best hierarchical model obtained, thus far, is the first model below. Yan's model achieves 14.12 on my computer; this model is, thus, ≈ 1.2 from beating it.

0	$Y \sim \text{VPD}^2 + \text{EVI}^2 + \text{FIPS}$	16.215485	0.82670546701
1	$Y \sim (1\|\text{State/FIPS}) + (0 + \text{vpdave}j^2\|\text{State/FIPS}) + \text{VPD}^2 + \text{EVI}^2$	15.3038210479	0.82638076292
2	$Y \sim (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{vpd}j\|\text{State/FIPS}) + \text{VPD}^2 + \text{EVI}^2$	16.8172458781	0.82800441011
3	$Y \sim (1\|\text{State}) + \sum_{j=5}^{8}(0 + \text{vpdave}j^2\|\text{State}) + \text{VPD}^2 + \text{EVI}^2$	18.1374174468	0.73889498198
4	$Y \sim \text{FIPS} + (1\|\text{State}) + \sum_{j=5}^{8}(0 + \text{vpdave}j^2\|\text{State}) + \text{VPD}^2 + \text{EVI}^2$	15.4375549205	0.82252370108
5	$Y \sim \text{FIPS} + (0 + \text{vpdave}j^2\|\text{State}) + \text{VPD}^2 + \text{EVI}^2$	15.4375532927	0.82252371087
6	$Y \sim (1\|\text{State}) + \sum_{j=5}^{8}(0 + \text{vpd}j\|\text{State}) + \text{VPD}^2 + \text{EVI}^2$	18.6300956404	0.73921728429
7	$Y \sim (1\|\text{State}) + \text{FIPS} + \sum_{j=5}^{8}(0 + \text{vpd}j\|\text{State}) + \text{VPD}^2 + \text{EVI}^2$	17.0266628131	0.82194197081
8	$Y \sim \text{FIPS} + \sum_{j=5}^{8}(0 + \text{vpd}j\|\text{State}) + \text{VPD}^2 + \text{EVI}^2$	17.0266636351	0.82194197247
9	$Y \sim (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{vpdave}j^2\|\text{State}) + \sum_{j=5}^{8}(\text{vpd}j) + \text{EVI}^2$	15.41829	Not Recorded
9.5	$Y \sim (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{vpdave}j^2\|\text{State}) + \sum_{j=5}^{8}(\text{vpd}j) + \text{EVI}^2 + \text{PRECIP}^2$	15.15744	0.82342695699
9.75	$Y \sim (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{vpdave}j^2\|\text{State}) + \sum_{j=5}^{8}(\text{vpd}j) + \text{EVI}^2 + \text{PRECIP}$	15.56885	Not Recorded
9.8	$Y \sim (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{vpdave}j^2\|\text{State}) + \sum_{j=5}^{8}(\text{vpd}j) + \text{EVI}^2 + \text{PRECIP} + \text{TAVE}$	15.57028	Not Recorded
9.85	$Y \sim (1\|\text{State/FIPS}) + \text{VPD}^2 + \text{EVI}^2 + \text{PRECIP}^2$	15.57028	0.82745422452
9.9	$Y \sim \text{FIPS} + \text{VPD}^2 + \text{EVI}^2 + \text{PRECIP}^2$	15.2295013561	0.82624534574
10	$Y \sim (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{vpd}j\|\text{State}) + \sum_{j=5}^{8}(\text{vpd}j^2) + \text{EVI}^2$	16.87682	Not Recorded

Remark 2.2. We form conclusions by comparing two models at a time. Given models x and y, we denote a comparison between x and y as $x \to y$ or, equivalently, $y \to x$. These conclusions are not absolute conclusions applicable to all combinations of such predictors; they are specific to VPD and EVI.

- $0 \to 1$
  - A hierarchical model at the nesting level of State/FIPS outperforms its OLS counterpart by only making the quadratic tems of VPD hierarchical.
- $1 \to 2$
  - Using quadratic terms in a hierarchical fashion with VPD produces a lower RMSE than linear terms in a hierarchical fashion.
- $1 \to 3$
  - Using a hierarchical nesting at the level of State/FIPS vastly outperforms one at the level of State.
- $1 \to 4$
  - If we switch to a hierarchical nesting at the level of State, then the resulting model M still performs as well as (1) if we include FIPS as a categorical predictor.
- $1 \to 5$
  - If, in addition to performing the transition from $1 \to 4$, we remove a state varying intercept, then model performance is similar.
- $1 \to 9$
  - If we exclude fixed quadratic terms from (1) to produce (9), then the model is essentially unaffected.
- $1 \to 25$
  - By additionally including vpdave$j$ (in linear form) in a hiearchical form, we actually increase model RMSE.
- **Conclusion:** We conclude that designating exclusive designation of vpdavej^2 with a hierarchical nesting of State/FIPS (whether or not we include vpdavej^2 as fixed effects) accounts for an increase in model performance relative to (0).

10	$Y \sim (1\|\text{State/FIPS}) + (0 + \text{evi}j^2\|\text{State/FIPS}) + \text{VPD}^2 + \text{EVI}^2$	16.19348181	0.81746556962
11	$Y \sim (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{evi}j\|\text{State/FIPS}) + \text{VPD}^2 + \text{EVI}^2$	16.1919480524	0.82051827534
12	$Y \sim (1\|\text{State}) + (0 + \text{evi}j^2\|\text{State}) + \text{VPD}^2 + \text{EVI}^2$	18.1768259011	0.75207577016
13	$Y \sim \text{FIPS} + (1\|\text{State}) + (0 + \text{evi}j^2\|\text{State}) + \text{VPD}^2 + \text{EVI}^2$	16.3115256355	0.82493200093
14	$Y \sim \text{FIPS} + (0 + \text{evi}j^2\|\text{State}) + \text{VPD}^2 + \text{EVI}^2$	16.3115262051	0.82493196759
15	$Y \sim (1\|\text{State}) + \sum_{j=5}^{8}(0 + \text{evi}j\|\text{State}) + \text{VPD}^2 + \text{EVI}^2$	18.0094398131	0.76372365318
16	$Y \sim (1\|\text{State}) + \text{FIPS} + \sum_{j=5}^{8}(0 + \text{evi}j\|\text{State}) + \text{VPD}^2 + \text{EVI}^2$	16.1087687869	0.82473129050
17	$Y \sim \text{FIPS} + \sum_{j=5}^{8}(0 + \text{evi}j\|\text{State}) + \text{VPD}^2 + \text{EVI}^2$	16.10877	
17.5	$Y \sim \sum_{j=5}^{8}(0 + \text{evi}j\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{evi}j^2\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{vpdave}j\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{vpdave}j^2\|\text{State/FIPS})$	18.0485664898	0.7382263473

Remark 2.3. No worthwhile conclusions were derived from the models above.

Remark 2.4. Across the tested models below, we see that OLS outperforms any hierarchical variant. These following models differ from the previous ones in that in the following models, if any type of term (like EVI or VPD) is hierarchical, then both its linear and quadratic terms are hierarchical. In the previous models, only some terms were made hierarchical (for example, in the first model, only the quadratic terms for VPD were made hierarchical).

18	$Y \sim \sum_{j=5}^{8}(0 + \text{evi}j\|\text{State}) + \sum_{j=5}^{8}(0 + \text{evi}j^2\|\text{State}) + \sum_{j=5}^{8}(0 + \text{vpdave}j\|\text{State}) + \sum_{j=5}^{8}(0 + \text{vpdave}j^2\|\text{State})$	16.8330513198	0.77100827729
19	$Y \sim (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{evi}j\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{evi}j^2\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{vpdave}j\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{vpdave}j^2\|\text{State/FIPS})$	Too Slow	
20	$Y \sim (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{evi}j\|\text{State}) + \sum_{j=5}^{8}(0 + \text{evi}j^2\|\text{State}) + \sum_{j=5}^{8}(0 + \text{vpdave}j\|\text{State}) + \sum_{j=5}^{8}(0 + \text{vpdave}j^2\|\text{State})$	16.1995687149	0.81278781759
21	$Y \sim \text{FIPS} + \sum_{j=5}^{8}(0 + \text{evi}j\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{evi}j^2\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{vpdave}j\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{vpdave}j^2\|\text{State/FIPS})$	Too Slow	
22	$Y \sim \text{FIPS} + \sum_{j=5}^{8}(0 + \text{evi}j\|\text{State}) + \sum_{j=5}^{8}(0 + \text{evi}j^2\|\text{State}) + \sum_{j=5}^{8}(0 + \text{vpdave}j\|\text{State}) + \sum_{j=5}^{8}(0 + \text{vpdave}j^2\|\text{State})$	Too Slow	

Remark 2.5. The models above used no fixed slopes. The only worthwhile conclusion is that any attempt to use both predictors (using all terms of either predictor) in a hierarchical fashion at the nesting level of State/FIPS is too slow; one must, in such instances, use a nesting level at State.

23	$Y \sim \text{VPD}^2 + \text{EVI}^2 + \text{FIPS}$	16.215485	0.82670546701
24	$Y \sim (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{vpdave}j + \text{vpdave}j^2\|\text{State/FIPS}) + \text{VPD}^2 + \text{EVI}^2$	16.340984	0.83515008835
25	$Y \sim (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{vpd}j\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{vpd}j^2\|\text{State/FIPS}) + \text{VPD}^2 + \text{EVI}^2$	16.363754	0.83468196762
25.5	$Y \sim (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{vpd}j\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{vpd}j^2\|\text{State/FIPS}) + \sum_{j=5}^{8}\text{evi}j + \sum_{j=5}^{8}\text{evi}j^2$	16.2832078828	0.8368325038
26	$Y \sim (1\|\text{State}) + \sum_{j=5}^{8}(0 + \text{vpdave}j + \text{vpdave}j^2\|\text{State}) + \text{VPD}^2 + \text{EVI}^2$	22.101831	0.6260326316
27	$Y \sim (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{evi}j\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{evi}j^2\|\text{State/FIPS}) + \text{VPD}^2 + \text{EVI}^2$	16.348400	0.81215868291
28	$Y \sim (1\|\text{State}) + \sum_{j=5}^{8}(0 + \text{evi}j\|\text{State}) + \sum_{j=5}^{8}(0 + \text{evi}j^2\|\text{State}) + \text{VPD}^2 + \text{EVI}^2$	18.115606	0.76806716927
29	$Y \sim (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{evi}j\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{evi}j^2\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{vpd}j\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{vpd}j^2\|\text{State/FIPS}) + \text{VPD}^2 + \text{EVI}^2$	16.592315	0.8058721273
30	$Y \sim (1\|\text{State}) + \sum_{j=5}^{8}(0 + \text{evi}j\|\text{State}) + \sum_{j=5}^{8}(0 + \text{evi}j^2\|\text{State}) + \sum_{j=5}^{8}(0 + \text{vpdave}j\|\text{State}) + \sum_{j=5}^{8}(0 + \text{vpdave}j^2\|\text{State}) + \text{VPD}^2 + \text{EVI}^2$	18.101450	0.73820557413
31	$Y \sim (1\|\text{State}) + \sum_{j=5}^{8}(0 + \text{vpdave}j + \text{vpdave}j^2\|\text{State}) + \text{VPD}^2 + \text{EVI}^2 + \text{FIPS}$	Too Slow	
32	$Y \sim (1\|\text{State}) + \sum_{j=5}^{8}(0 + \text{evi}j) + \sum_{j=5}^{8}(0 + \text{evi}j^2\|\text{State}) + \text{VPD}^2 + \text{EVI}^2 + \text{FIPS}$	Too Slow	
33	$Y \sim (1\|\text{State}) + \sum_{j=5}^{8}(0 + \text{evi}j\|\text{State}) + \sum_{j=5}^{8}(0 + \text{evi}j^2\|\text{State}) + \sum_{j=5}^{8}(0 + \text{vpdave}j\|\text{State}) + \sum_{j=5}^{8}(0 + \text{vpdave}j^2\|\text{State}) + \text{VPD}^2 + \text{EVI}^2 + \text{FIPS}$	Too Slow	

Remark 2.6. The following models were mistakenly tested. They may be of potential use, so they have been included.

Formula	RMSE	R^2
$Y \sim \text{FIPS} + \text{VPD}^2 + \text{EVI}^2 + \text{TAVE}^2$	15.897125	
$Y \sim (1\|\text{State/FIPS}) + \text{VPD}^2 + \text{EVI}^2 + \text{TAVE}^2$	15.613374	
$Y \sim (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(\text{vpd}j\|\text{State/FIPS}) + \text{VPD}^2 + \text{EVI}^2 + \text{TAVE}^2$	41.027641	
$Y \sim (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{vpdave}j\|\text{State/FIPS}) + \text{VPD}^2 + \text{EVI}^2 + \text{TAVE}^2$	15.579032	
$Y \sim (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{vpdave}j + \text{vpdave}j^2\|\text{State/FIPS}) + \text{VPD}^2 + \text{EVI}^2 + \text{TAVE}^2$	15.576379	

$Y \sim \text{FIPS} + \text{VPD}_2^2 + \text{EVI}^2 + \text{TAVE}^2$	15.648232	0.817781770286
$Y \sim (1\|\text{State/FIPS}) + \text{VPD}_2^2 + \text{EVI}^2 + \text{TAVE}^2$	16.081548	0.810822930257
$Y \sim (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(\text{tave}j\|\text{State/FIPS}) + \text{VPD}_2^2 + \text{EVI}^2 + \text{TAVE}^2$	26.064453	0.565483242945
$Y \sim (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{tave}j\|\text{State/FIPS}) + \text{VPD}_2^2 + \text{EVI}^2 + \text{TAVE}^2$	16.373051	0.807700513379
$Y \sim (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{tave}j + \text{tave}j^2\|\text{State/FIPS}) + \text{VPD}_2^2 + \text{EVI}^2 + \text{TAVE}^2$	16.377212	0.800983591304

2.3 Models with Splines

It is far too slow to make spline terms hierarchical in lmer. The following table tests other configurations in which *evi* and *vpd* have been set to be hierarchical. These configurations are feasibly testable.

34	$Y \sim$ bs(vpdave6) + bs(vpdave7) + bs(vpdave8) + bs(precip6) + bs(precip7) + bs(precip8) + bs(precip9) + (1|State/FIPS) + (0 + evi5 + evi6 + evi7 + evi8 + evi9 | State/FIPS) + I(evi5^2) + I(evi6^2) + I(evi7^2) + I(evi8^2) + I(evi9^2)	15.681009	
35	$Y \sim$ bs(vpdave6) + bs(vpdave7) + bs(vpdave8) + bs(precip6) + bs(precip7) + bs(precip8) + bs(precip9) + (1|State/FIPS) + (0 + evi5 + evi6 + evi7 + evi8 + evi9 + evi5^2 + evi6^2 + evi7^2 + evi8^2 + evi9^2|State/FIPS)	16.690290	
36	$Y \sim$ bs(vpdave6) + bs(vpdave7) + bs(vpdave8) + bs(precip6) + bs(precip7) + bs(precip8) + bs(precip9) + (1|State/FIPS) + $\sum_{j=5}^{8}$ (0 + evi$j$ | State/FIPS) + evi5 + evi6 + evi7 + evi8 + evi9 + evi5^2 + evi6^2 + evi7^2 + evi8^2 + evi9^2	15.7699073035	0.82156127563
	Model File: vpd_spline_evi_mixed_fixed: vpd_spline_evi_poly_3_levels_linear		
37	$Y \sim$ bs(vpdave6) + bs(vpdave7) + bs(vpdave8) + bs(precip6) + bs(precip7) + bs(precip8) + bs(precip9) + FIPS + $\sum_{j=5}^{8}$ (0 + evi$j$ | State/FIPS) + evi5 + evi6 + evi7 + evi8 + evi9 + evi5^2 + evi6^2 + evi7^2 + evi8^2 + evi9^2	Too Slow	
	Model File: vpd_spline_evi_mixed_fixed: vpd_spline_evi_poly_3_levels_linear		
38	$Y \sim$ bs(vpdave6) + bs(vpdave7) + bs(vpdave8) + bs(precip6) + bs(precip7) + bs(precip8) + bs(precip9) + (1|State/FIPS) + $\sum_{j=5}^{8}$ (0 + evi$j$ | State/FIPS + $\sum_{j=5}^{8}$ (0 + evij^2 | State/FIPS + evi5 + evi6 + evi7 + evi8 + evi9 + evi5^2 + evi6^2 + evi7^2 + evi8^2 + evi9^2	16.1487508508	0.81503536719
	Model File: vpd_spline_evi_mixed_fixed: vpd_spline_evi_poly_3_levels_quadratic		
39	$Y \sim$ bs(vpdave6) + bs(vpdave7) + bs(vpdave8) + bs(precip6) + bs(precip7) + bs(precip8) + bs(precip9) + FIPS + $\sum_{j=5}^{8}$ (0 + evi$j$ | State/FIPS + $\sum_{j=5}^{8}$ (0 + evij^2 | State/FIPS + evi5 + evi6 + evi7 + evi8 + evi9 + evi5^2 + evi6^2 + evi7^2 + evi8^2 + evi9^2	Too Slow	
	Model File: vpd_spline_evi_mixed_fixed: vpd_spline_evi_poly_3_levels_quadratic		
40	$Y \sim$ bs(vpdave6) + bs(vpdave7) + bs(vpdave8) + bs(precip6) + bs(precip7) + bs(precip8) + bs(precip9) + (1|State) + $\sum_{j=5}^{8}$ (0 + evi$j$ | State + evi5 + evi6 + evi7 + evi8 + evi9 + evi5^2 + evi6^2 + evi7^2 + evi8^2 + evi9^2	15.7699073035	0.82156127563
	Model File: vpd_spline_evi_mixed_fixed: vpd_spline_evi_poly_2_levels_linear		
41	$Y \sim$ bs(vpdave6) + bs(vpdave7) + bs(vpdave8) + bs(precip6) + bs(precip7) + bs(precip8) + bs(precip9) + (1|State/FIPS) + $\sum_{j=5}^{8}$ (0 + evi$j$ | State/FIPS + $\sum_{j=5}^{8}$ (0 + evij^2 | State/FIPS + evi5 + evi6 + evi7 + evi8 + evi9 + evi5^2 + evi6^2 + evi7^2 + evi8^2 + evi9^2	16.4815882954	0.81903032958
	Model File: vpd_spline_evi_mixed_fixed: vpd_spline_evi_poly_2_levels_quadratic		
42	$Y \sim$ (bs(vpdave6) | State) + (bs(vpdave7) | State) + (bs(vpdave8) | State) + (bs(precip6) | State) + (bs(precip7) | State) + (bs(precip8) | State) + (bs(precip9) | State) + (1|State) + $\sum_{j=5}^{8}$ (0 + evi$j$ | State + $\sum_{j=5}^{8}$ (0 + evij^2 | State + evi5 + evi6 + evi7 + evi8 + evi9 + evi5^2 + evi6^2 + evi7^2 + evi8^2 + evi9^2	Too Slow	
	Model File: vpd_spline_evi_mixed_fixed: vpd_spline_level_evi_poly_2_levels_quadratic		

2.4 July 26 Models

43	$Y \sim \text{VPD} + \text{EVI} + (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{vpdave}j^2\|\text{State})$	15.9939086929	0.81204656700
44	$Y \sim \text{VPD}^2 + \text{EVI} + \text{FIPS}$	16.0647094694	0.80083571922
45	$Y \sim \text{VPD} + \text{EVI} + (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{vpdave}j\|\text{State})$	16.4501305392	0.79071819885
45.5	$Y \sim \text{EVI} + (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{vpdave}j\|\text{State})$	16.4657997817	0.79079442688
46	$Y \sim \text{VPD} + \text{EVI} + \text{FIPS}$	16.2808292464	0.806938161
47	$Y \sim \text{VPD} + \text{PRECIP}$	27.4787816043	0.53456110642
48	$Y \sim \sum_{j=5}^{8}(0 + \text{vpdave}j\|\text{State}) + \text{PRECIP}$	24.6774265412	0.53578949152

2.5 August 6 Models

0	$Y \sim \text{FIPS} + \text{VPD}^2 + \text{PRECIP}^2$	17.304	0.772
1	$Y \sim (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(\text{vpdave}j^2) + \sum_{j=5}^{8}(\text{vpdave}j) + \sum_{j=5}^{8}(\text{precip}j^2) + \sum_{j=5}^{8}(0 + \text{precip}j\|\text{State})$	18.409	0.742
2	$Y \sim (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(\text{vpdave}j^2) + \sum_{j=5}^{8}(\text{vpdave}j) + \sum_{j=5}^{8}(0 + \text{precip}j^2\|\text{State}) + \sum_{j=5}^{8}(\text{precip}j)$	17.456	0.753
3	$Y \sim (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(\text{vpdave}j^2) + \sum_{j=5}^{8}(\text{vpdave}j) + \sum_{j=5}^{8}(0 + \text{precip}j^2\|\text{State}) + \sum_{j=5}^{8}(0 + \text{precip}j\|\text{State})$	18.155	0.737
4	$Y \sim (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(\text{vpdave}j^2) + \sum_{j=5}^{8}(0 + \text{vpdave}j\|\text{State}) + \sum_{j=5}^{8}(\text{precip}j^2) + \sum_{j=5}^{8}(\text{precip}j)$	17.782	0.781
5	$Y \sim (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(\text{vpdave}j^2) + \sum_{j=5}^{8}(0 + \text{vpdave}j\|\text{State}) + \sum_{j=5}^{8}(\text{precip}j^2) + \sum_{j=5}^{8}(0 + \text{precip}j\|\text{State})$	17.631	0.773
6	$Y \sim (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(\text{vpdave}j^2) + \sum_{j=5}^{8}(0 + \text{vpdave}j\|\text{State}) + \sum_{j=5}^{8}(0 + \text{precip}j^2\|\text{State}) + \sum_{j=5}^{8}(\text{precip}j)$	17.580	0.770
7	$Y \sim (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(\text{vpdave}j^2) + \sum_{j=5}^{8}(0 + \text{vpdave}j\|\text{State}) + \sum_{j=5}^{8}(0 + \text{precip}j^2\|\text{State}) + \sum_{j=5}^{8}(0 + \text{precip}j\|\text{State})$	17.918	0.756
8	$Y \sim (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{vpdave}j^2\|\text{State}) + \sum_{j=5}^{8}(\text{vpdave}j) + \sum_{j=5}^{8}(\text{precip}j^2) + \sum_{j=5}^{8}(\text{precip}j)$	17.658	0.783
9	$Y \sim (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{vpdave}j^2\|\text{State}) + \sum_{j=5}^{8}(\text{vpdave}j) + \sum_{j=5}^{8}(\text{precip}j^2) + \sum_{j=5}^{8}(0 + \text{precip}j\|\text{State})$	17.552	0.767
10	$Y \sim (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{vpdave}j^2\|\text{State}) + \sum_{j=5}^{8}(\text{vpdave}j) + \sum_{j=5}^{8}(0 + \text{precip}j^2\|\text{State}) + \sum_{j=5}^{8}(\text{precip}j)$	17.338	0.785
11	$Y \sim (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{vpdave}j^2\|\text{State}) + \sum_{j=5}^{8}(\text{vpdave}j) + \sum_{j=5}^{8}(0 + \text{precip}j^2\|\text{State}) + \sum_{j=5}^{8}(0 + \text{precip}j\|\text{State})$	17.531	0.768
12	$Y \sim (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{vpdave}j^2\|\text{State}) + \sum_{j=5}^{8}(0 + \text{vpdave}j\|\text{State}) + \sum_{j=5}^{8}(\text{precip}j^2) + \sum_{j=5}^{8}(\text{precip}j)$	17.724	0.777
13	$Y \sim (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{vpdave}j^2\|\text{State}) + \sum_{j=5}^{8}(0 + \text{vpdave}j\|\text{State}) + \sum_{j=5}^{8}(\text{precip}j^2) + \sum_{j=5}^{8}(0 + \text{precip}j\|\text{State})$	17.767	0.771
14	$Y \sim (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{vpdave}j^2\|\text{State}) + \sum_{j=5}^{8}(0 + \text{vpdave}j\|\text{State}) + \sum_{j=5}^{8}(0 + \text{precip}j^2\|\text{State}) + \sum_{j=5}^{8}(\text{precip}j)$	17.826	0.774
15	$Y \sim (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{vpdave}j^2\|\text{State}) + \sum_{j=5}^{8}(0 + \text{vpdave}j\|\text{State}) + \sum_{j=5}^{8}(0 + \text{precip}j^2\|\text{State}) + \sum_{j=5}^{8}(0 + \text{precip}j\|\text{State})$	18.028	0.773

0	$Y \sim \text{FIPS} + \text{TAVE}^2 + \text{PRECIP}^2$	17.922	
1	$Y \sim (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(\text{tave}j^2) + \sum_{j=5}^{8}(\text{tave}j) + \sum_{j=5}^{8}(\text{precip}j^2) + \sum_{j=5}^{8}(0 + \text{precip}j\|\text{State})$	17.862	
2	$Y \sim (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(\text{tave}j^2) + \sum_{j=5}^{8}(\text{tave}j) + \sum_{j=5}^{8}(0 + \text{precip}j^2\|\text{State}) + \sum_{j=5}^{8}(\text{precip}j)$	18.058	
3	$Y \sim (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(\text{tave}j^2) + \sum_{j=5}^{8}(\text{tave}j) + \sum_{j=5}^{8}(0 + \text{precip}j^2\|\text{State}) + \sum_{j=5}^{8}(0 + \text{precip}j\|\text{State})$	18.438	
4	$Y \sim (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(\text{tave}j^2) + \sum_{j=5}^{8}(0 + \text{tave}j\|\text{State}) + \sum_{j=5}^{8}(\text{precip}j^2) + \sum_{j=5}^{8}(\text{precip}j)$	18.940	
5	$Y \sim (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(\text{tave}j^2) + \sum_{j=5}^{8}(0 + \text{tave}j\|\text{State}) + \sum_{j=5}^{8}(\text{precip}j^2) + \sum_{j=5}^{8}(0 + \text{precip}j\|\text{State})$	19.076	
6	$Y \sim (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(\text{tave}j^2) + \sum_{j=5}^{8}(0 + \text{tave}j\|\text{State}) + \sum_{j=5}^{8}(0 + \text{precip}j^2\|\text{State}) + \sum_{j=5}^{8}(\text{precip}j)$	18.766	
7	$Y \sim (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(\text{tave}j^2) + \sum_{j=5}^{8}(0 + \text{tave}j\|\text{State}) + \sum_{j=5}^{8}(0 + \text{precip}j^2\|\text{State}) + \sum_{j=5}^{8}(0 + \text{precip}j\|\text{State})$	19.625	
8	$Y \sim (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{tave}j^2\|\text{State}) + \sum_{j=5}^{8}(\text{tave}j) + \sum_{j=5}^{8}(\text{precip}j^2) + \sum_{j=5}^{8}(\text{precip}j)$	18.627	
9	$Y \sim (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{tave}j^2\|\text{State}) + \sum_{j=5}^{8}(\text{tave}j) + \sum_{j=5}^{8}(\text{precip}j^2) + \sum_{j=5}^{8}(0 + \text{precip}j\|\text{State})$	18.508	
10	$Y \sim (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{tave}j^2\|\text{State}) + \sum_{j=5}^{8}(\text{tave}j) + \sum_{j=5}^{8}(0 + \text{precip}j^2\|\text{State}) + \sum_{j=5}^{8}(\text{precip}j)$	18.632	
11	$Y \sim (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{tave}j^2\|\text{State}) + \sum_{j=5}^{8}(\text{tave}j) + \sum_{j=5}^{8}(0 + \text{precip}j^2\|\text{State}) + \sum_{j=5}^{8}(0 + \text{precip}j\|\text{State})$	18.658	
12	$Y \sim (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{tave}j^2\|\text{State}) + \sum_{j=5}^{8}(0 + \text{tave}j\|\text{State}) + \sum_{j=5}^{8}(\text{precip}j^2) + \sum_{j=5}^{8}(\text{precip}j)$	18.981	
13	$Y \sim (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{tave}j^2\|\text{State}) + \sum_{j=5}^{8}(0 + \text{tave}j\|\text{State}) + \sum_{j=5}^{8}(\text{precip}j^2) + \sum_{j=5}^{8}(0 + \text{precip}j\|\text{State})$	18.781	
14	$Y \sim (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{tave}j^2\|\text{State}) + \sum_{j=5}^{8}(0 + \text{tave}j\|\text{State}) + \sum_{j=5}^{8}(0 + \text{precip}j^2\|\text{State}) + \sum_{j=5}^{8}(\text{precip}j)$	18.918	
15	$Y \sim (1\|\text{State/FIPS}) + \sum_{j=5}^{8}(0 + \text{tave}j^2\|\text{State}) + \sum_{j=5}^{8}(0 + \text{tave}j\|\text{State}) + \sum_{j=5}^{8}(0 + \text{precip}j^2\|\text{State}) + \sum_{j=5}^{8}(0 + \text{precip}j\|\text{State})$	19.095	