

# Chapter 4

# The Fourier Series and

# Fourier Transform

# Representation of Signals in Terms of Frequency Components

- Consider the CT signal defined by

$$x(t) = \sum_{k=1}^N A_k \cos(\omega_k t + \theta_k), \quad t \in \mathbb{R}$$

- The frequencies 'present in the signal' are the frequency  $\omega_k$  of the component sinusoids
- The signal  $x(t)$  is completely characterized by the set of frequencies  $\omega_k$ , the set of amplitudes  $A_k$  and the set of phases  $\theta_k$

# Example: Sum of Sinusoids

- Consider the CT signal given by

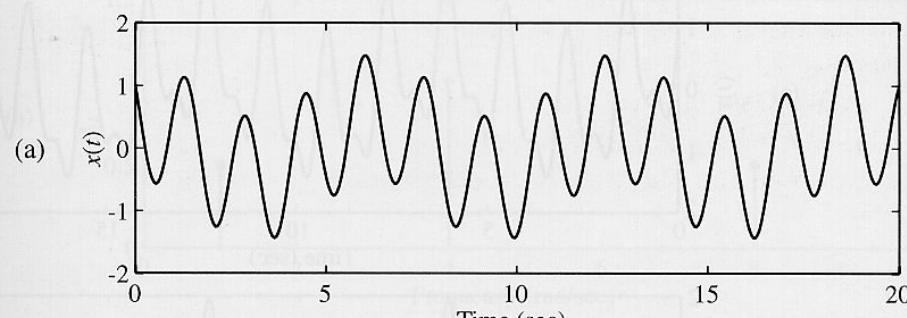
$$x(t) = A_1 \cos(t) + A_2 \cos(4t + \pi/3) + A_3 \cos(8t + \pi/2),$$

$$t \in \mathbb{R}$$

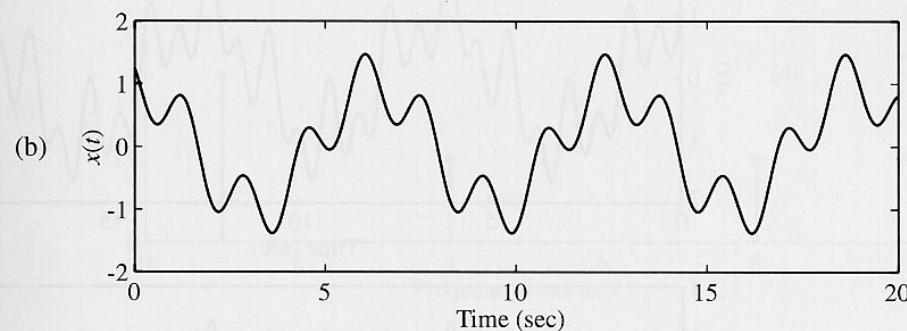
- The signal has only three frequency components at 1, 4, and 8 rad/sec, amplitudes  $A_1, A_2, A_3$  and phases  $0, \pi/3, \pi/2$
- The shape of the signal  $x(t)$  depends on the relative magnitudes of the frequency components, specified in terms of the amplitudes  $A_1, A_2, A_3$

# Example: Sum of Sinusoids -Cont'd

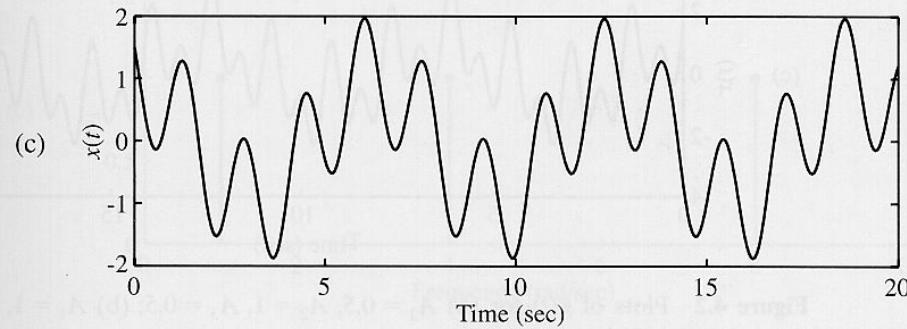
$$\begin{cases} A_1 = 0.5 \\ A_2 = 1 \\ A_3 = 0 \end{cases}$$



$$\begin{cases} A_1 = 1 \\ A_2 = 0.5 \\ A_3 = 0 \end{cases}$$



$$\begin{cases} A_1 = 1 \\ A_2 = 1 \\ A_3 = 0 \end{cases}$$



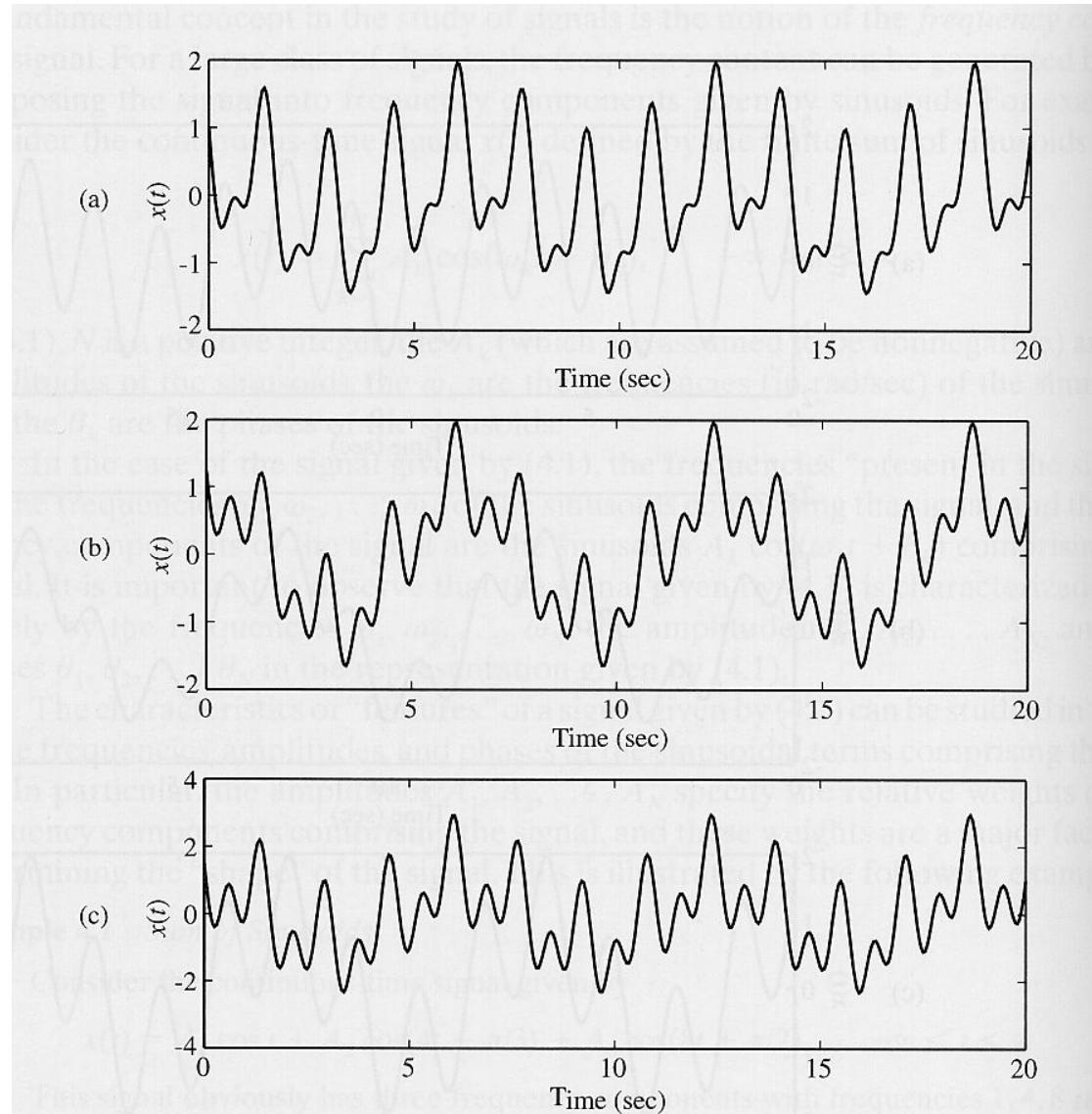
**Figure 4.1** Plots of  $x(t)$  for (a)  $A_1 = 0.5, A_2 = 1, A_3 = 0$ ; (b)  $A_1 = 1, A_2 = 0.5, A_3 = 0$ ; and (c)  $A_1 = 1, A_2 = 1, A_3 = 0$ .

# Example: Sum of Sinusoids -Cont'd

$$\begin{cases} A_1 = 0.5 \\ A_2 = 1 \\ A_3 = 0.5 \end{cases}$$

$$\begin{cases} A_1 = 1 \\ A_2 = 0.5 \\ A_3 = 0.5 \end{cases}$$

$$\begin{cases} A_1 = 1 \\ A_2 = 1 \\ A_3 = 1 \end{cases}$$



**Figure 4.2** Plots of  $x(t)$  for (a)  $A_1 = 0.5, A_2 = 1, A_3 = 0.5$ ; (b)  $A_1 = 1, A_2 = 0.5$ ,

# Amplitude Spectrum

- Plot of the amplitudes  $A_k$  of the sinusoids making up  $x(t)$  vs.  $\omega$

- Example:

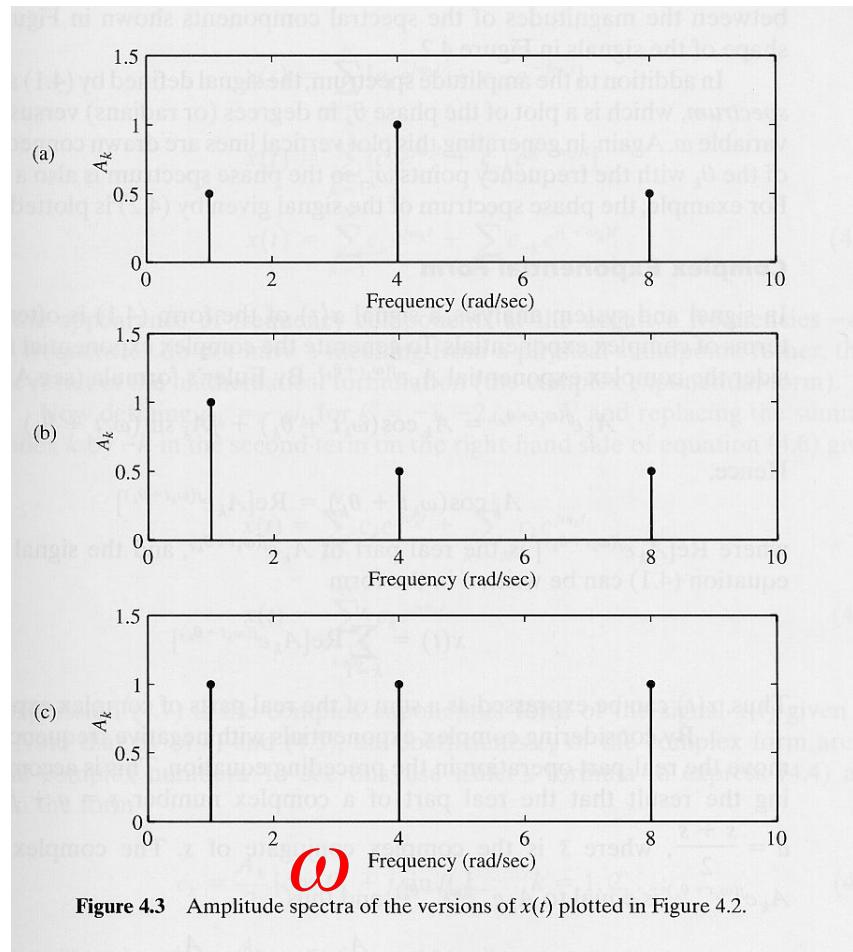
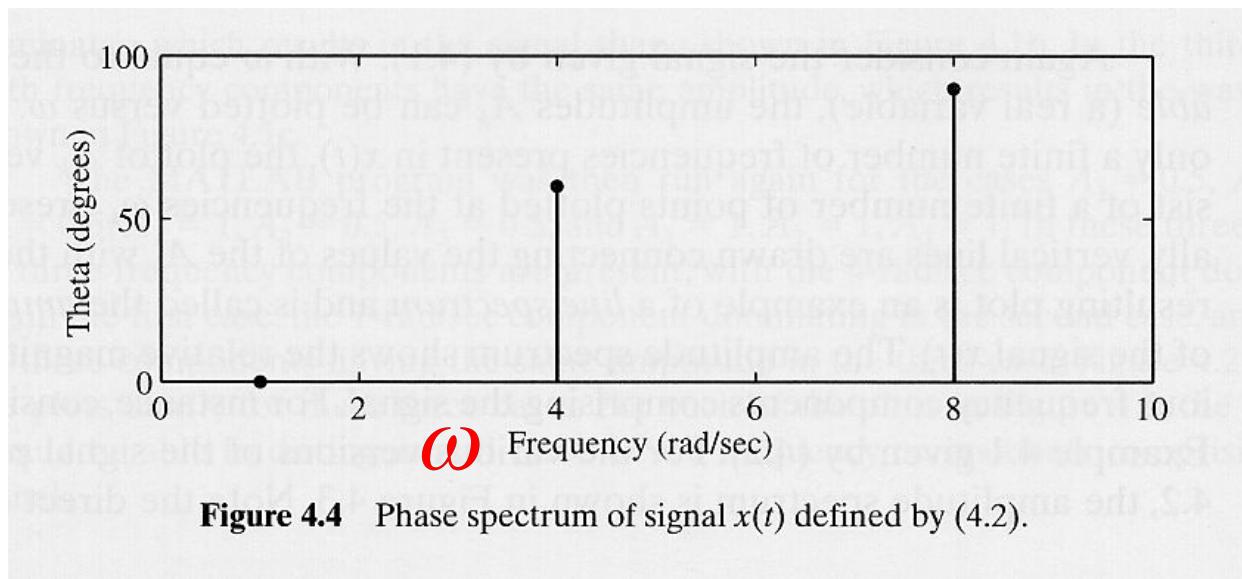


Figure 4.3 Amplitude spectra of the versions of  $x(t)$  plotted in Figure 4.2.

# Phase Spectrum

- Plot of the phases  $\theta_k$  of the sinusoids making up  $x(t)$  vs.  $\omega$
- Example:



# Complex Exponential Form

- Euler formula:  $e^{j\alpha} = \cos(\alpha) + j \sin(\alpha)$
- Thus

$$A_k \cos(\omega_k t + \theta_k) = \Re \left[ A_k e^{j(\omega_k t + \theta_k)} \right]$$

↑  
real part

whence

$$x(t) = \sum_{k=1}^N \Re \left[ A_k e^{j(\omega_k t + \theta_k)} \right], \quad t \in \mathbb{R}$$

## Complex Exponential Form – Cont'd

- And, recalling that  $\Re(z) = (z + z^*)/2$  where  $z = a + jb$  we can also write

$$x(t) = \sum_{k=1}^N \frac{1}{2} \left[ A_k e^{j(\omega_k t + \theta_k)} + A_k e^{-j(\omega_k t + \theta_k)} \right], \quad t \in \mathbb{R}$$

- This signal contains both positive and negative frequencies
- The *negative frequencies*  $-\omega_k$  stem from writing the *cosine* in terms of complex exponentials and have no physical meaning

## Complex Exponential Form – Cont'd

- By defining

$$c_k = \frac{A_k}{2} e^{j\theta_k} \quad c_{-k} = \frac{A_k}{2} e^{-j\theta_k}$$

it is also

$$x(t) = \sum_{k=1}^N \left[ c_k e^{j\omega_k t} + c_{-k} e^{-j\omega_k t} \right] = \sum_{k=-N}^N c_k e^{j\omega_k t}, \quad t \in \mathbb{R}$$

$$\bigoplus_{k \neq 0} \bigotimes \quad \bigotimes \bigotimes$$

complex exponential form  
of the signal  $x(t)$

# Line Spectra

- The *amplitude spectrum* of  $x(t)$  is defined as the plot of the magnitudes  $|c_k|$  versus  $\omega$
- The *phase spectrum* of  $x(t)$  is defined as the plot of the angles  $\angle c_k = \arg(c_k)$  versus  $\omega$
- This results in *line spectra* which are defined for both positive and negative frequencies
- Notice: for  $k = 1, 2, \dots$

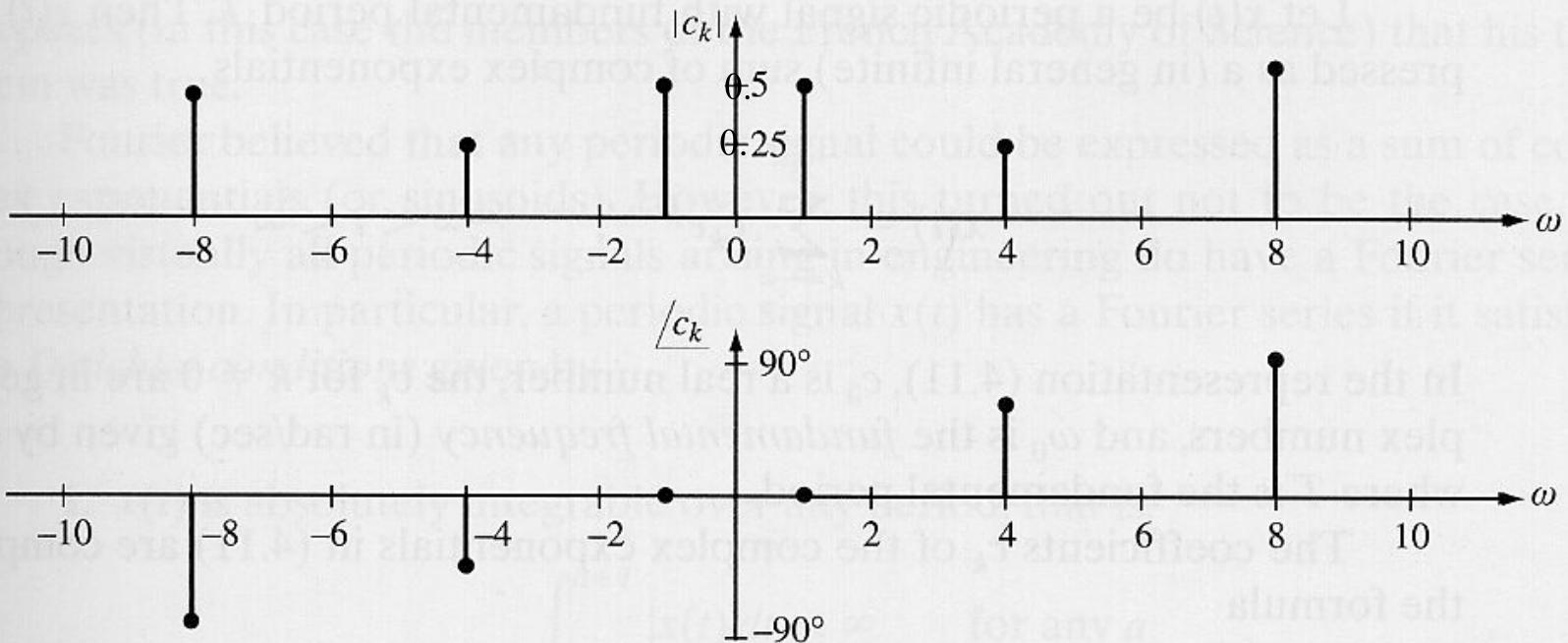
$$|c_k| = |c_{-k}|$$

$$\angle c_k = -\angle c_{-k}$$

$$\arg(c_k) = -\arg(c_{-k})$$

## Example: Line Spectra

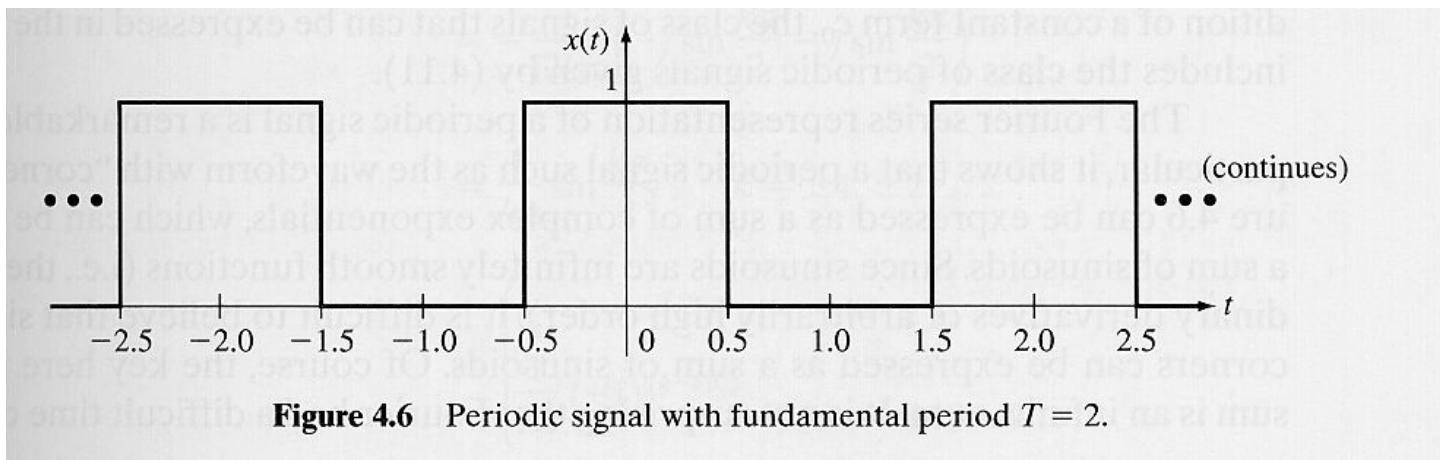
$$x(t) = \cos(t) + 0.5 \cos(4t + \pi/3) + \cos(8t + \pi/2)$$



**Figure 4.5** Line spectra for the signal in Example 4.2.

# Fourier Series Representation of Periodic Signals

- Let  $x(t)$  be a CT periodic signal with period  $T$ , i.e.,  $x(t + T) = x(t), \quad \forall t \in R$
- Example: the rectangular pulse train



# The Fourier Series

- Then,  $x(t)$  can be expressed as

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}, \quad t \in \mathbb{R}$$

where  $\omega_0 = 2\pi / T$  is the *fundamental frequency* (rad/sec) of the signal and

$$c_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} dt, \quad k = 0, \pm 1, \pm 2, \dots$$

$c_0$  is called the *constant or dc component* of  $x(t)$

## The Fourier Series – Cont'd

- The frequencies  $k\omega_0$  present in  $x(t)$  are integer multiples of the fundamental frequency  $\omega_0$
- Notice that, if the dc term  $c_0$  is added to

$$x(t) = \sum_{k=-N}^{N, k \neq 0} c_k e^{j\omega_k t}$$

and we set  $N = \infty$ , the Fourier series is a special case of the above equation where all the frequencies are integer multiples of  $\omega_0$

# Dirichlet Conditions

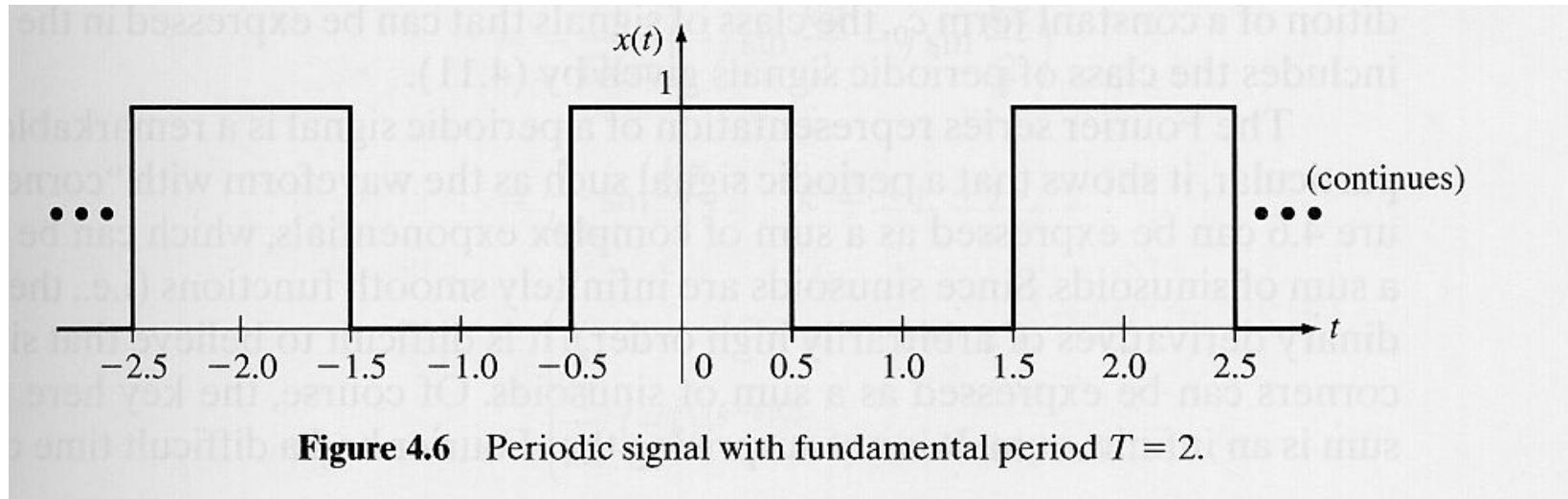
- A periodic signal  $x(t)$ , has a Fourier series if it satisfies the following conditions:

1.  $x(t)$  is **absolutely integrable** over any period, namely

$$\int_a^{a+T} |x(t)| dt < \infty, \quad \forall a \in \mathbb{R}$$

2.  $x(t)$  has only a **finite number of maxima and minima** over any period
3.  $x(t)$  has only a **finite number of discontinuities** over any period

# Example: The Rectangular Pulse Train

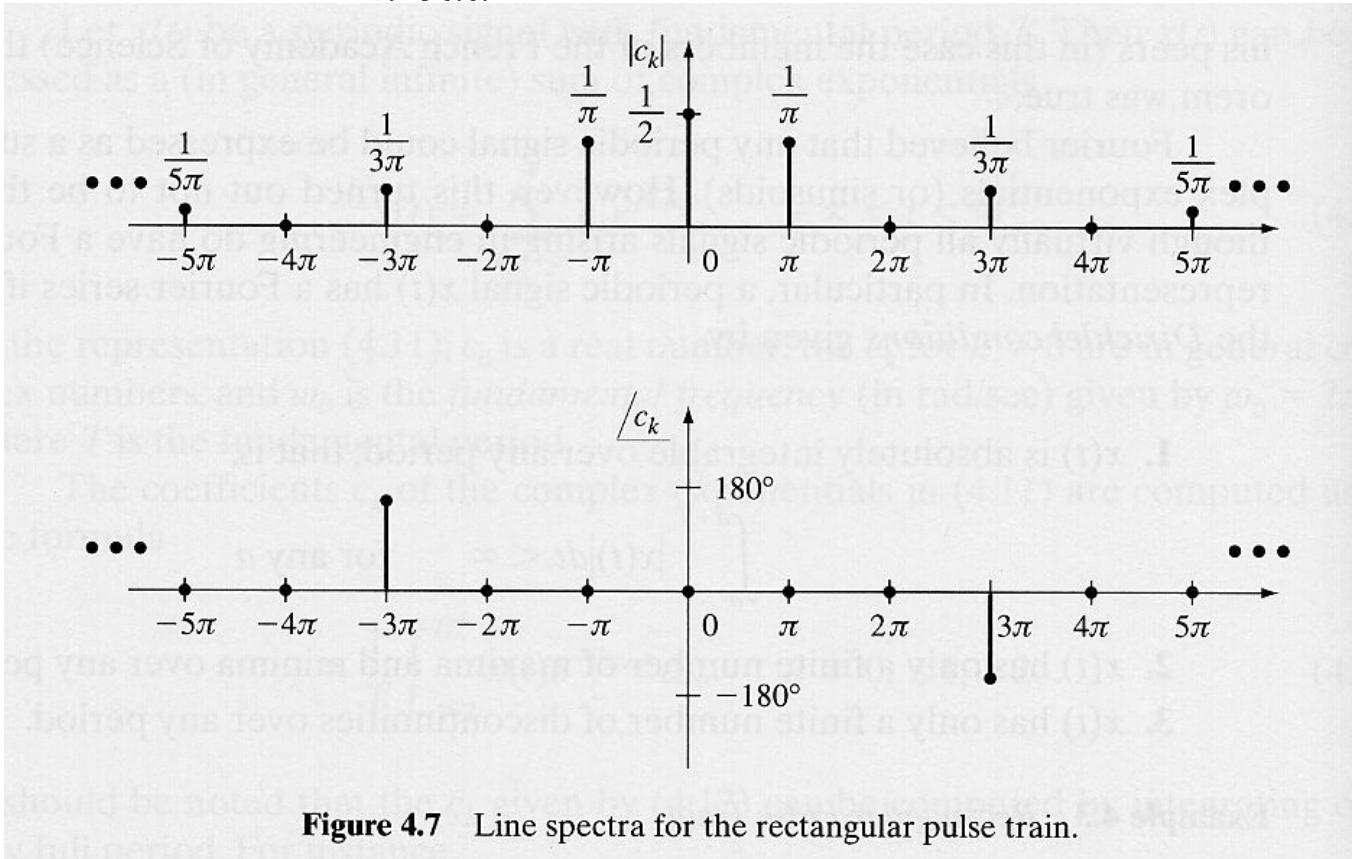


- From figure,  $T = 2$  whence  $\omega_0 = 2\pi / 2 = \pi$
- Clearly  $x(t)$  satisfies the Dirichlet conditions and thus has a Fourier series representation

# Example: The Rectangular Pulse Train – Cont'd

$$x(t) = \frac{1}{2} + \sum_{k=-\infty}^{\infty} \frac{1}{k\pi} (-1)^{|(k-1)/2|} e^{jk\pi t}, \quad t \in \mathbb{R}$$

*k odd*



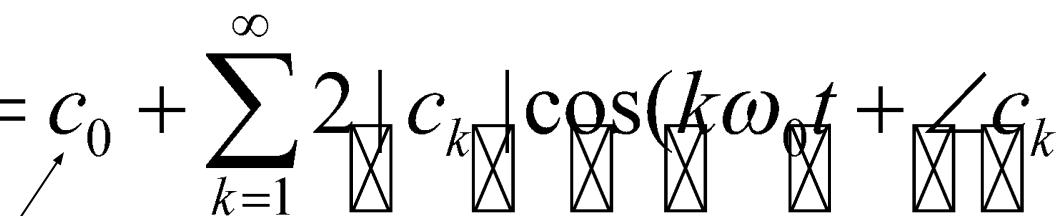
# Trigonometric Fourier Series

- By using Euler's formula, we can rewrite

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}, \quad t \in \mathbb{R}$$

as

$$x(t) = c_0 + \sum_{k=1}^{\infty} 2|c_k| \cos(k\omega_0 t + \angle c_k), \quad t \in \mathbb{R}$$

  
dc component                             $k$ -th harmonic

- This expression is called the **trigonometric Fourier series** of  $x(t)$

# Example: Trigonometric Fourier Series of the Rectangular Pulse Train

- The expression

$$x(t) = \frac{1}{2} + \sum_{k=-\infty}^{\infty} \frac{1}{k\pi} (-1)^{|(k-1)/2|} e^{jk\pi t}, \quad t \in \mathbb{R}$$

can be rewritten as

$$x(t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{2}{k\pi} \cos\left(k\pi t + \left[(-1)^{(k-1)/2} - 1\right] \frac{\pi}{2}\right), \quad t \in \mathbb{R}$$

# Gibbs Phenomenon

- Given an odd positive integer  $N$ , define the  $N$ -th partial sum of the previous series

$$x_N(t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^N \frac{2}{k\pi} \cos\left(k\pi t + \left[(-1)^{(k-1)/2} - 1\right] \frac{\pi}{2}\right), \quad t \in \mathbb{R}$$

- According to Fourier's theorem, it should be

$$\lim_{N \rightarrow \infty} |x_N(t) - x(t)| = 0$$

# Gibbs Phenomenon – Cont'd

$x_3(t)$

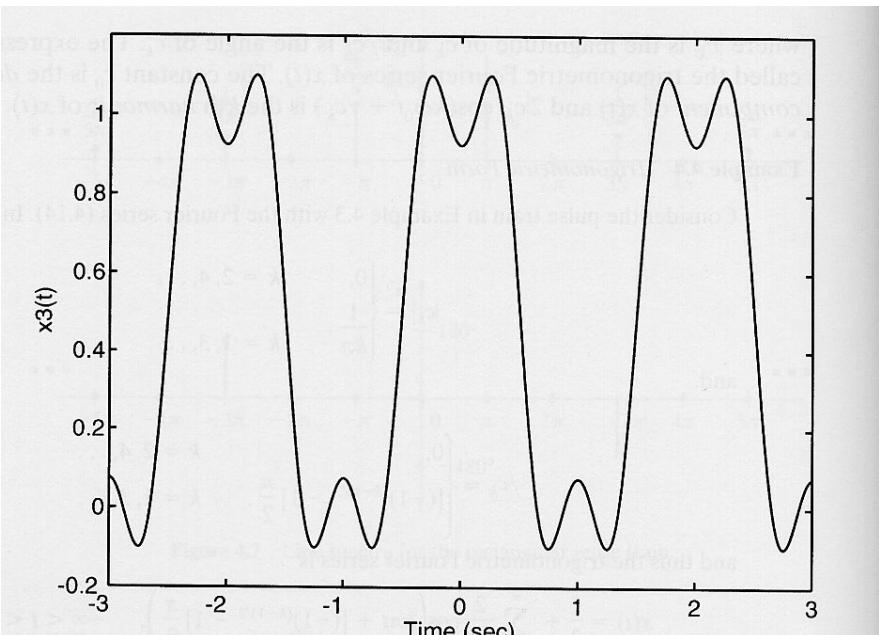


Figure 4.8 Plot of  $x_N(t)$  when  $N = 3$ .

$x_9(t)$

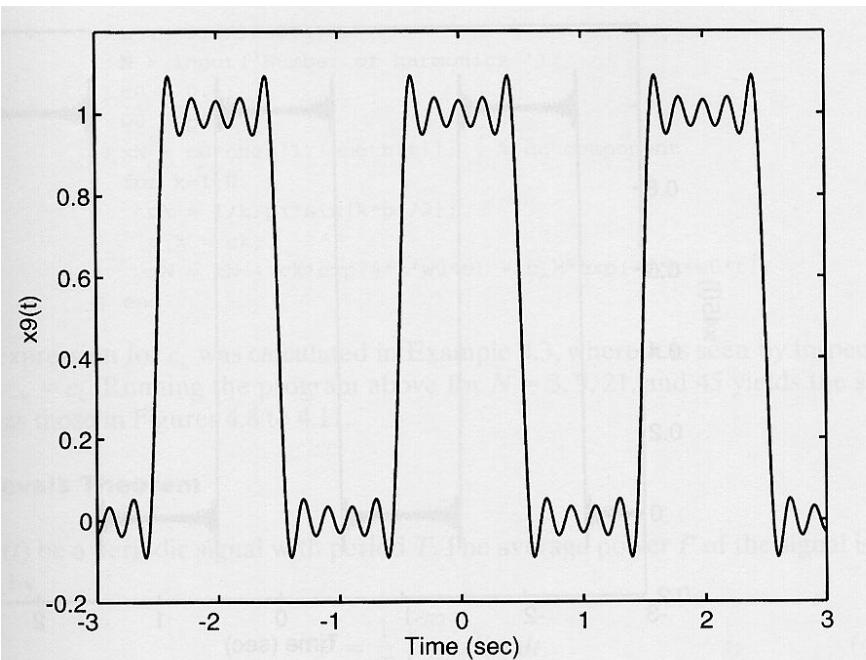


Figure 4.9 Approximation  $x_9(t)$ .

# Gibbs Phenomenon – Cont'd

$x_{21}(t)$

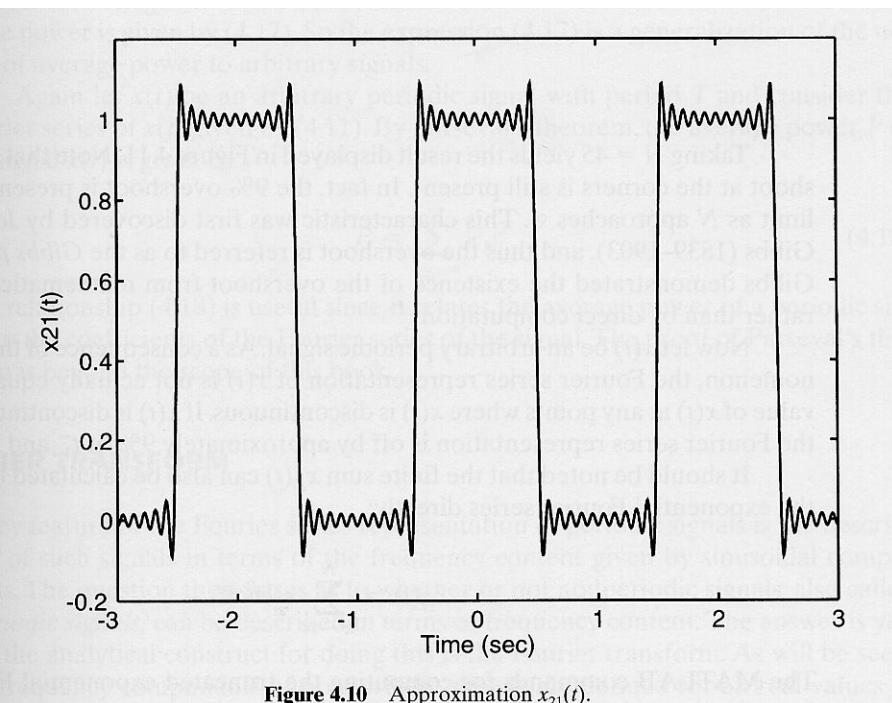


Figure 4.10 Approximation  $x_{21}(t)$ .

$x_{45}(t)$

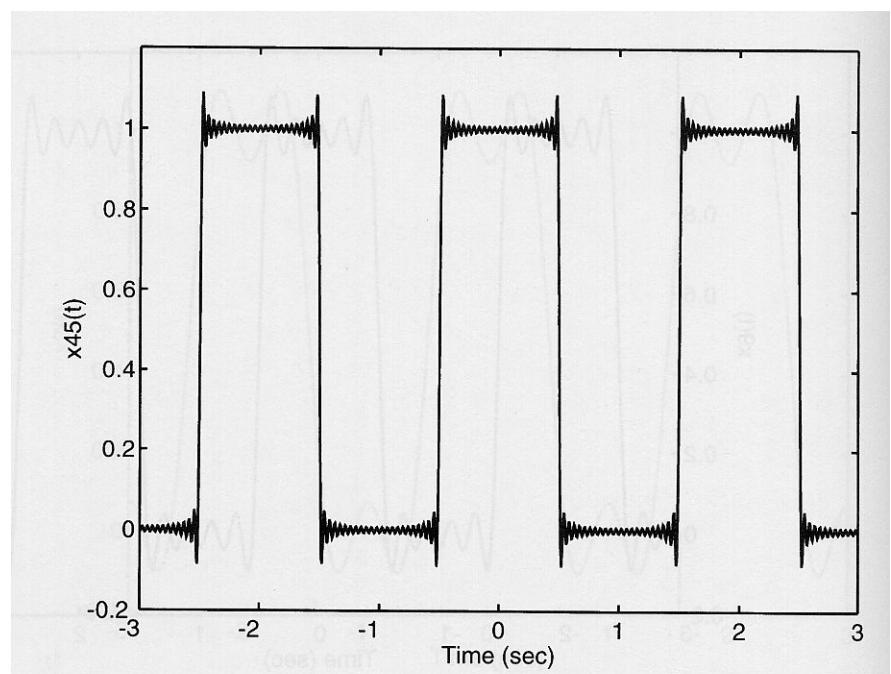


Figure 4.11 The signal  $x_{45}(t)$ .

**overshoot:** about 9 % of the signal magnitude  
(present even if )  $N \rightarrow \infty$

# Parseval's Theorem

- Let  $x(t)$  be a periodic signal with period  $T$
- The *average power*  $P$  of the signal is defined as

$$P = \frac{1}{T} \int_{-T/2}^{T/2} x^2(t) dt$$

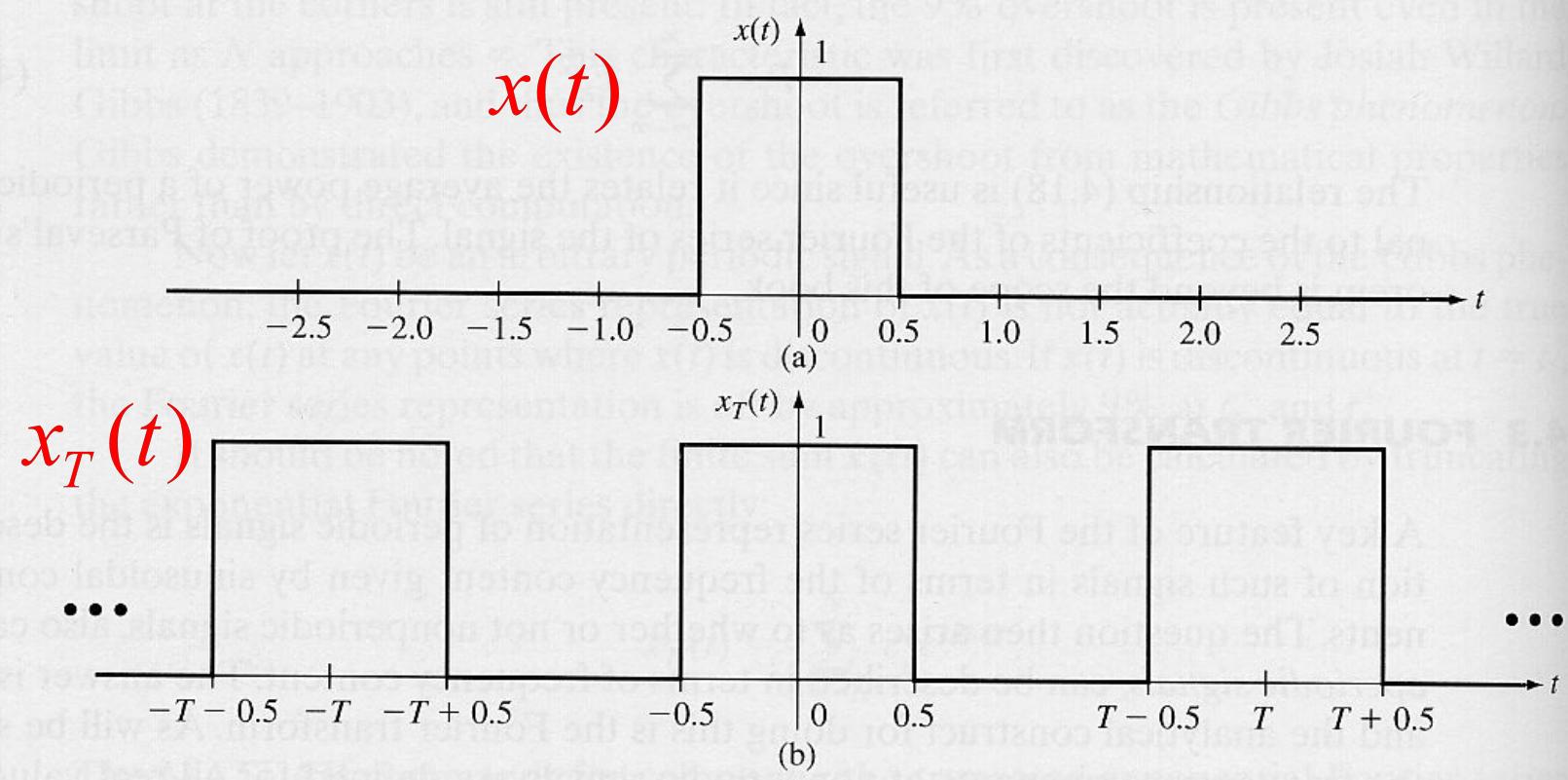
- Expressing the signal as  $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$ ,  $t \in \mathbb{R}$  it is also

$$P = \sum_{k=-\infty}^{\infty} |c_k|^2$$

# Fourier Transform

- We have seen that periodic signals can be represented with the Fourier series
- Can **aperiodic signals** be analyzed in terms of frequency components?
- Yes, and the Fourier transform provides the tool for this analysis
- The major difference w.r.t. the line spectra of periodic signals is that the **spectra of aperiodic signals** are defined for all real values of the frequency variable  $\omega$  not just for a discrete set of values

# Frequency Content of the Rectangular Pulse



**Figure 4.12** Plots of the (a) one-second rectangular pulse and (b) pulse train.

$$x(t) = \lim_{T \rightarrow \infty} x_T(t)$$

# Frequency Content of the Rectangular Pulse – Cont'd

- Since  $x_T(t)$  is periodic with period  $T$ , we can write

$$x_T(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}, \quad t \in \mathbb{R}$$

where

$$c_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk\omega_o t} dt, \quad k = 0, \pm 1, \pm 2, \dots$$

# Frequency Content of the Rectangular Pulse – Cont'd

- What happens to the frequency components of  $x_T(t)$  as  $T \rightarrow \infty$ ?
- For  $k = 0$

$$c_0 = \frac{1}{T}$$

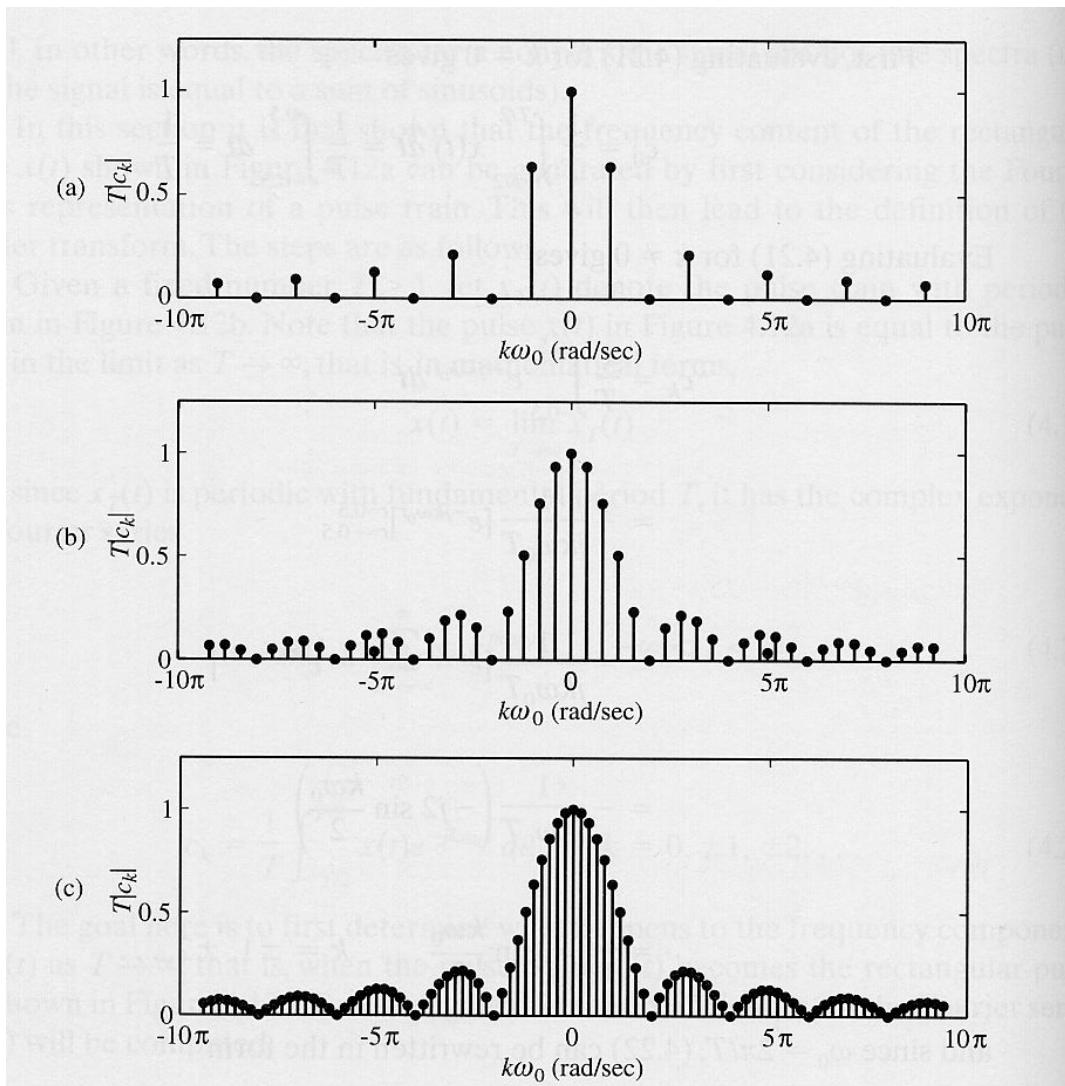
- For  $k \neq 0$

$$c_k = \frac{2}{k\omega_0 T} \sin\left(\frac{k\omega_0}{2}\right) = \frac{1}{k\pi} \sin\left(\frac{k\omega_0}{2}\right), \quad k = \pm 1, \pm 2, \dots$$

$\omega_0 = 2\pi / T$

# Frequency Content of the Rectangular Pulse – Cont'd

plots of  $T |c_k|$  vs.  $\omega = k\omega_0$  for  $T = 2, 5, 10$



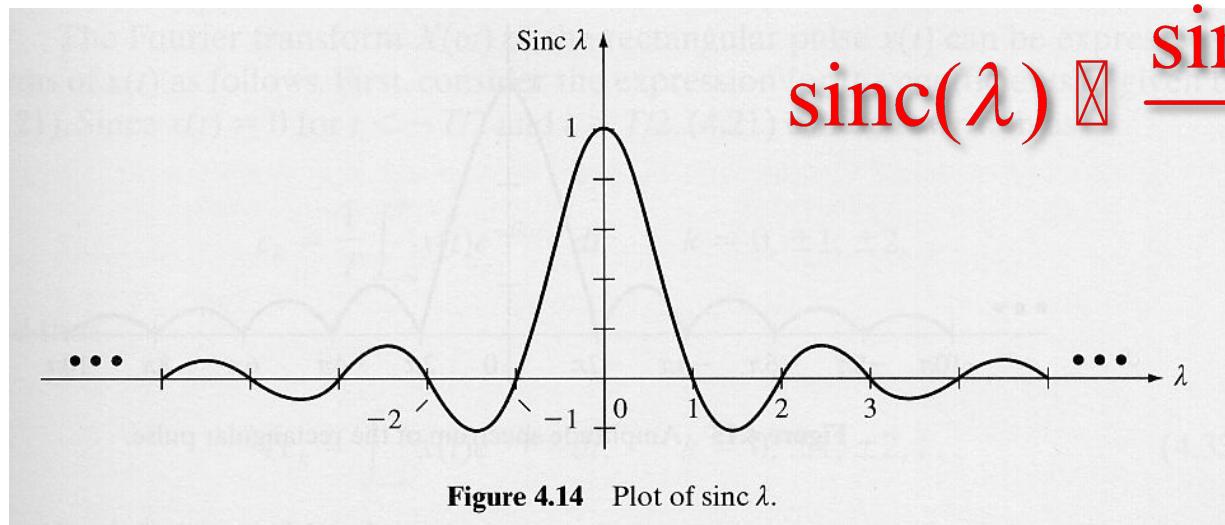
**Figure 4.13** Plot of scaled spectrum of  $x_T(t)$  for (a)  $T = 2$ , (b)  $T = 5$ , and (c)  $T = 10$ .

# Frequency Content of the Rectangular Pulse – Cont'd

- It can be easily shown that

$$\lim_{T \rightarrow \infty} T c_k = \text{sinc}\left(\frac{\omega}{2\pi}\right), \quad \omega \in \mathbb{X}$$

where



# Fourier Transform of the Rectangular Pulse

- The Fourier transform of the rectangular pulse  $x(t)$  is defined to be the limit of  $Tc_k$  as  $T \rightarrow \infty$ , i.e.,

$$X(\omega) = \lim_{T \rightarrow \infty} Tc_k = \text{sinc}\left(\frac{\omega}{2\pi}\right), \quad \omega \in \mathbb{R}$$

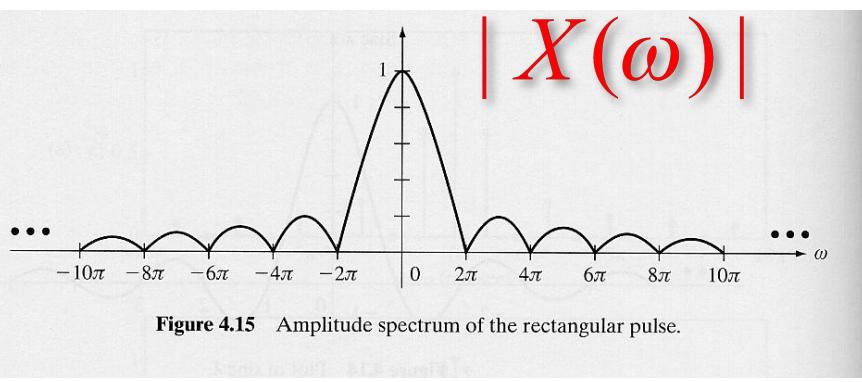


Figure 4.15 Amplitude spectrum of the rectangular pulse.

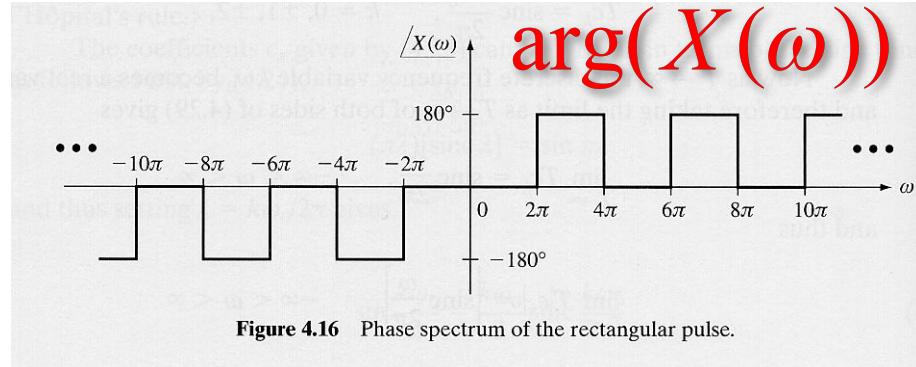


Figure 4.16 Phase spectrum of the rectangular pulse.

# Fourier Transform of the Rectangular Pulse – Cont'd

- The Fourier transform  $X(\omega)$  of the rectangular pulse  $x(t)$  can be expressed in terms of  $x(t)$  as follows:

$$c_k = \frac{1}{T} \int_{-\infty}^{\infty} x(t) e^{-jk\omega_o t} dt, \quad k = 0, \pm 1, \pm 2, \dots$$

whence

$$x(t) = 0 \text{ for } t < -T/2 \text{ and } t > T/2$$

$$Tc_k = \int_{-\infty}^{\infty} x(t) e^{-jk\omega_o t} dt, \quad k = 0, \pm 1, \pm 2, \dots$$

# Fourier Transform of the Rectangular Pulse – Cont'd

- Now, by definition  $X(\omega) = \lim_{T \rightarrow \infty} T c_k$  and, since  $k\omega_0 \rightarrow \omega$  as  $T \rightarrow \infty$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt, \quad \omega \in \mathbb{R}$$

- The *inverse Fourier transform* of  $X(\omega)$  is

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega, \quad t \in \mathbb{R}$$

# The Fourier Transform in the General Case

- Given a signal  $x(t)$ , its *Fourier transform*  $X(\omega)$  defined as

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt, \quad \omega \in \mathbb{R}$$

- A signal  $x(t)$  is said to have a *Fourier transform in the ordinary sense* if the above integral converges

# The Fourier Transform in the General Case – Cont'd

- The integral does converge if
  1. the signal  $x(t)$  is “*well-behaved*”
  2. and  $x(t)$  is *absolutely integrable*, namely,

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty$$

- Note: *well behaved* means that the signal has a finite number of discontinuities, maxima, and minima within any finite time interval

# Example: The DC or Constant Signal

- Consider the signal  $x(t) = 1, \quad t \in \mathbb{R}$
- Clearly  $x(t)$  does not satisfy the first requirement since

$$\int_{-\infty}^{\infty} |x(t)| dt = \int_{-\infty}^{\infty} dt = \infty$$

- Therefore, the constant signal does not have a *Fourier transform in the ordinary sense*
- Later on, we'll see that it has however a *Fourier transform in a generalized sense*

# Example: The Exponential Signal

- Consider the signal  $x(T) = e^{-bt}u(t)$ ,  $b \in \mathbb{R}$
- Its Fourier transform is given by

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} e^{-bt} u(t) e^{-j\omega t} dt \\ &= \int_0^{\infty} e^{-(b+j\omega)t} dt = -\frac{1}{b+j\omega} \left[ e^{-(b+j\omega)t} \right]_{t=0}^{t=\infty} \end{aligned}$$

## Example: The Exponential Signal – Cont'd

- If  $b < 0$  ,  $X(\omega)$  does not exist
- If  $b = 0$  ,  $x(t) = u(t)$  and  $X(\omega)$  does not exist either in the ordinary sense
- If  $b > 0$  , it is

$$X(\omega) = \frac{1}{b + j\omega}$$

amplitude spectrum

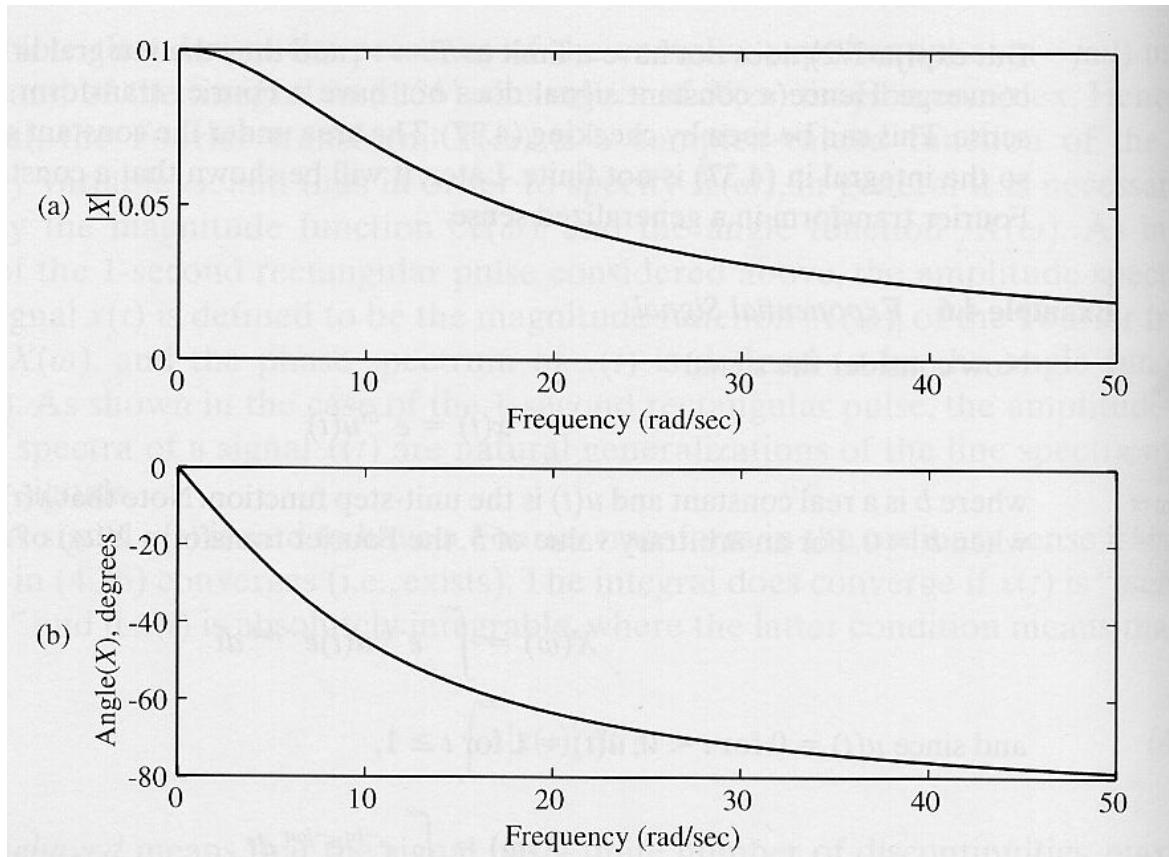
$$|X(\omega)| = \frac{1}{\sqrt{b^2 + \omega^2}}$$

phase spectrum

$$\arg(X(\omega)) = -\arctan\left(\frac{\omega}{b}\right)$$

# Example: Amplitude and Phase Spectra of the Exponential Signal

$$x(t) = e^{-10t} u(t)$$



**Figure 4.17** Plots of the (a) amplitude and (b) phase spectra of  $x(t) = \exp(-10t)u(t)$ .

# Rectangular Form of the Fourier Transform

- Consider

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt, \quad \omega \in \mathbb{R}$$

- Since  $X(\omega)$  in general is a complex function, by using Euler's formula

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \cos(\omega t) dt + j \left( - \int_{-\infty}^{\infty} x(t) \sin(\omega t) dt \right)$$

$$X(\omega) = R(\omega) + jI(\omega)$$

# Polar Form of the Fourier Transform

- $X(\omega) = R(\omega) + jI(\omega)$  can be expressed in a polar form as

$$X(\omega) = |X(\omega)| \exp(j \arg(X(\omega)))$$

where

$$|X(\omega)| = \sqrt{R^2(\omega) + I^2(\omega)}$$

$$\arg(X(\omega)) = \arctan\left(\frac{I(\omega)}{R(\omega)}\right)$$

# Fourier Transform of Real-Valued Signals

- If  $x(t)$  is real-valued, it is

$$X(-\omega) = X^*(\omega) \quad \text{Hermitian symmetry}$$

- Moreover

$$X^*(\omega) = |X(\omega)| \exp(-j \arg(X(\omega)))$$

whence

$$|X(-\omega)| = |X(\omega)| \quad \text{and}$$

$$\arg(X(-\omega)) = -\arg(X(\omega))$$

# Fourier Transforms of Signals with Even or Odd Symmetry

- Even signal:  $x(t) = x(-t)$

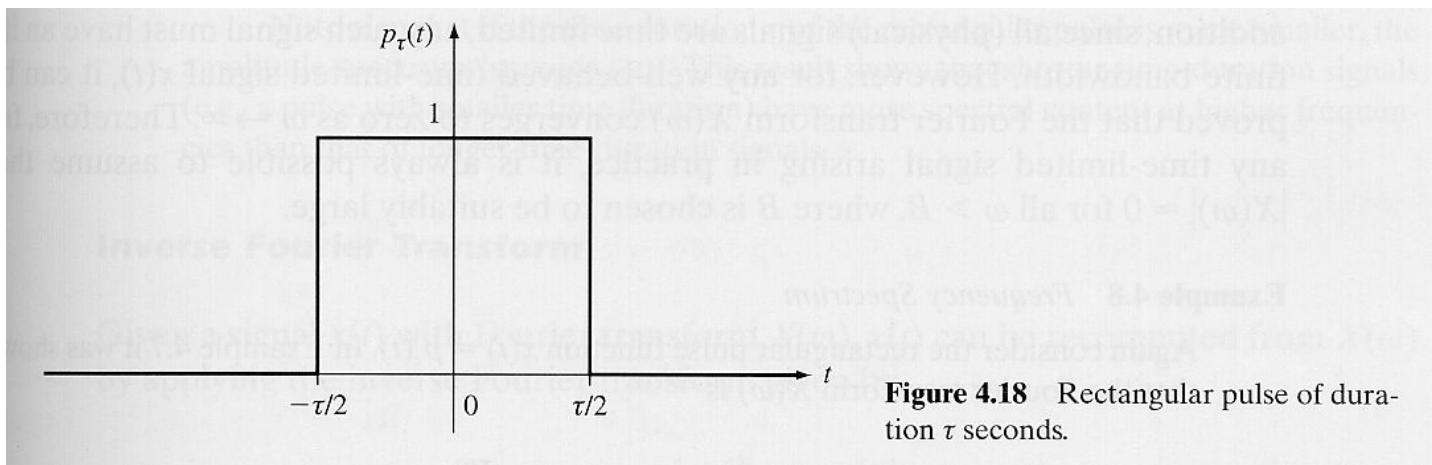
$$X(\omega) = 2 \int_0^{\infty} x(t) \cos(\omega t) dt$$

- Odd signal:  $x(t) = -x(-t)$

$$X(\omega) = -j 2 \int_0^{\infty} x(t) \sin(\omega t) dt$$

# Example: Fourier Transform of the Rectangular Pulse

- Consider the even signal



**Figure 4.18** Rectangular pulse of duration  $\tau$  seconds.

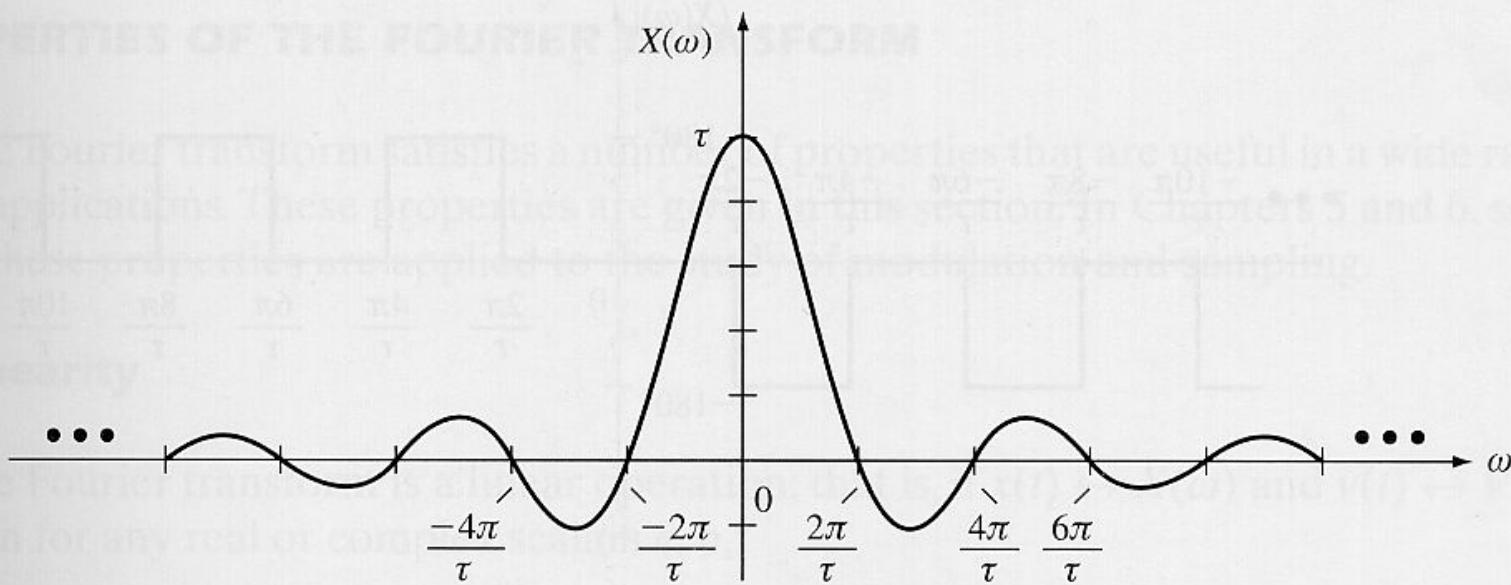
- It is

$$X(\omega) = 2 \int_0^{\tau/2} (1) \cos(\omega t) dt = \frac{2}{\omega} [\sin(\omega t)]_{t=0}^{t=\tau/2} = \frac{2}{\omega} \sin\left(\frac{\omega\tau}{2}\right)$$

$$= \tau \operatorname{sinc}\left(\frac{\omega\tau}{2\pi}\right)$$

# Example: Fourier Transform of the Rectangular Pulse – Cont'd

$$X(\omega) = \tau \operatorname{sinc}\left(\frac{\omega\tau}{2\pi}\right)$$



**Figure 4.19** Fourier transform of the  $\tau$ -second rectangular pulse.

# Example: Fourier Transform of the Rectangular Pulse – Cont'd

amplitude spectrum

phase spectrum

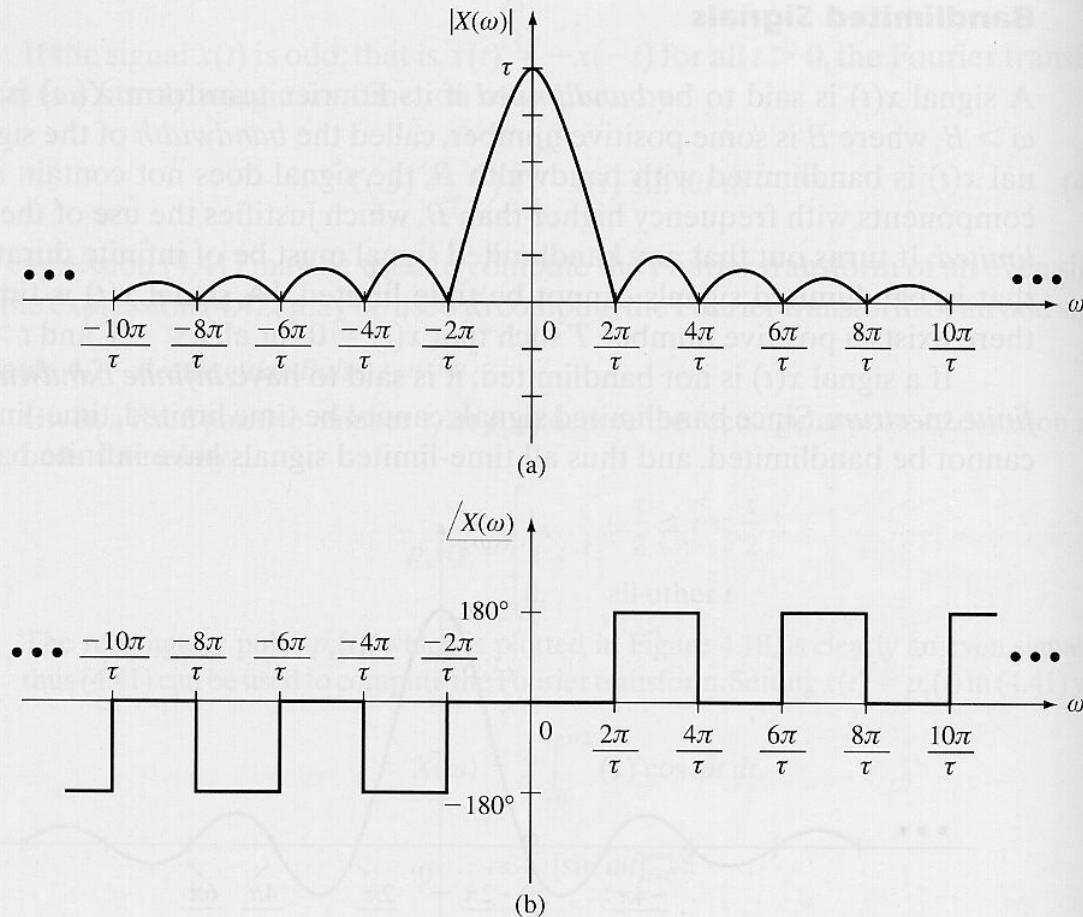


Figure 4.20 (a) Amplitude and (b) phase spectra of the rectangular pulse.

# Bandlimited Signals

- A signal  $x(t)$  is said to be *bandlimited* if its Fourier transform  $X(\omega)$  is zero for all  $\omega > B$  where  $B$  is some positive number, called the *bandwidth of the signal*
- It turns out that any bandlimited signal must have an infinite duration in time, i.e., bandlimited signals cannot be time limited

# Bandlimited Signals – Cont'd

- If a signal  $x(t)$  is not bandlimited, it is said to have *infinite bandwidth* or an *infinite spectrum*
- Time-limited signals cannot be bandlimited and thus all time-limited signals have infinite bandwidth
- However, for any well-behaved signal  $x(t)$  it can be proven that  $\lim_{\omega \rightarrow \infty} X(\omega) = 0$  whence it can be assumed that

$$|X(\omega)| \approx 0 \quad \forall \omega > B$$

$B$  being a convenient large number

# Inverse Fourier Transform

- Given a signal  $x(t)$  with Fourier transform  $X(\omega)$ ,  $x(t)$  can be recomputed from  $X(\omega)$  applying the inverse Fourier transform given by

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega, \quad t \in \mathbb{R}$$

- Transform pair

$$x(t) \leftrightarrow X(\omega)$$

# Properties of the Fourier Transform

$$x(t) \leftrightarrow X(\omega) \quad y(t) \leftrightarrow Y(\omega)$$

- *Linearity:*

$$\alpha x(t) + \beta y(t) \leftrightarrow \alpha X(\omega) + \beta Y(\omega)$$

- *Left or Right Shift in Time:*

$$x(t - t_0) \leftrightarrow X(\omega)e^{-j\omega t_0}$$

- *Time Scaling:*

$$x(at) \leftrightarrow \frac{1}{a} X\left(\frac{\omega}{a}\right)$$

# Properties of the Fourier Transform

- *Time Reversal:*

$$x(-t) \leftrightarrow X(-\omega)$$

- *Multiplication by a Power of t:*

$$t^n x(t) \leftrightarrow (j)^n \frac{d^n}{d\omega^n} X(\omega)$$

- *Multiplication by a Complex Exponential:*

$$x(t)e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0)$$

# Properties of the Fourier Transform

- *Multiplication by a Sinusoid (Modulation):*

$$x(t) \sin(\omega_0 t) \leftrightarrow \frac{j}{2} [X(\omega + \omega_0) - X(\omega - \omega_0)]$$

$$x(t) \cos(\omega_0 t) \leftrightarrow \frac{1}{2} [X(\omega + \omega_0) + X(\omega - \omega_0)]$$

- *Differentiation in the Time Domain:*

$$\frac{d^n}{dt^n} x(t) \leftrightarrow (j\omega)^n X(\omega)$$

# Properties of the Fourier Transform

- *Integration in the Time Domain:*

$$\int_{-\infty}^t x(\tau) d\tau \leftrightarrow \frac{1}{j\omega} X(\omega) + \pi X(0) \delta(\omega)$$

- *Convolution in the Time Domain:*

$$x(t) * y(t) \leftrightarrow X(\omega)Y(\omega)$$

- *Multiplication in the Time Domain:*

$$x(t)y(t) \leftrightarrow X(\omega) * Y(\omega)$$

# Properties of the Fourier Transform

- *Parseval's Theorem:*

$$\int_{\mathbb{R}} x(t)y(t)dt \leftrightarrow \frac{1}{2\pi} \int_{\mathbb{R}} X^*(\omega)Y(\omega)d\omega$$

$$\text{if } y(t) = x(t) \quad \int_{\mathbb{R}} |x(t)|^2 dt \leftrightarrow \frac{1}{2\pi} \int_{\mathbb{R}} |X(\omega)|^2 d\omega$$

- *Duality:*

$$X(t) \leftrightarrow 2\pi x(-\omega)$$

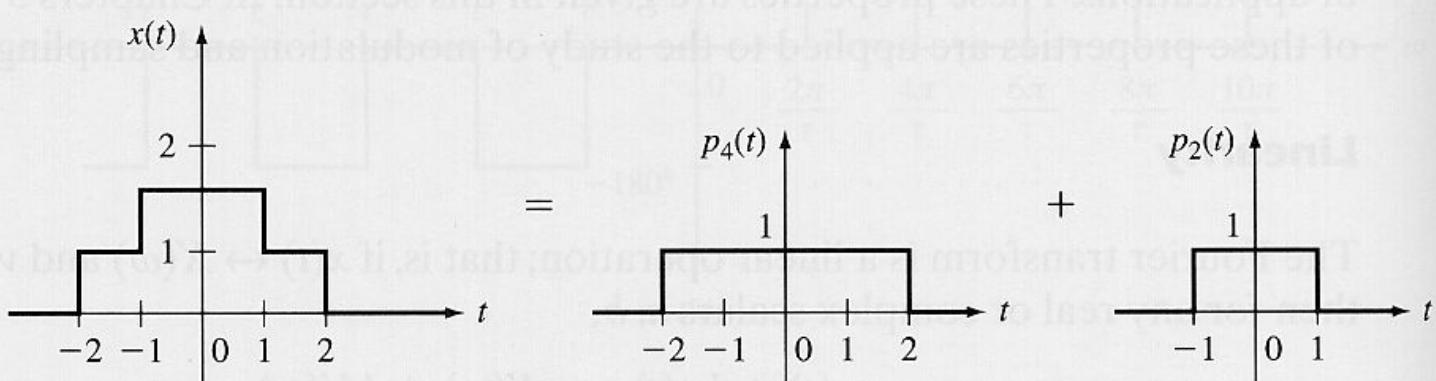
# Properties of the Fourier Transform - Summary

**TABLE 4.1 PROPERTIES OF THE FOURIER TRANSFORM**

Property	Transform Pair/Property
Linearity	$ax(t) + bv(t) \leftrightarrow aX(\omega) + bV(\omega)$
Right or left shift in time	$x(t - c) \leftrightarrow X(\omega)e^{-j\omega c}$
Time scaling	$x(at) \leftrightarrow \frac{1}{a} X\left(\frac{\omega}{a}\right) \quad a > 0$
Time reversal	$x(-t) \leftrightarrow X(-\omega) = \overline{X(\omega)}$
Multiplication by a power of t	$t^n x(t) \leftrightarrow j^n \frac{d^n}{d\omega^n} X(\omega) \quad n = 1, 2, \dots$
Multiplication by a complex exponential	$x(t)e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0) \quad \omega_0 \text{ real}$
Multiplication by $\sin \omega_0 t$	$x(t) \sin \omega_0 t \leftrightarrow \frac{j}{2} [X(\omega + \omega_0) - X(\omega - \omega_0)]$
Multiplication by $\cos \omega_0 t$	$x(t) \cos \omega_0 t \leftrightarrow \frac{1}{2} [X(\omega + \omega_0) + X(\omega - \omega_0)]$
Differentiation in the time domain	$\frac{d^n}{dt^n} x(t) \leftrightarrow (j\omega)^n X(\omega) \quad n = 1, 2, \dots$
Integration	$\int_{-\infty}^t x(\lambda) d\lambda \leftrightarrow \frac{1}{j\omega} X(\omega) + \pi X(0)\delta(\omega)$
Convolution in the time domain	$x(t) * v(t) \leftrightarrow X(\omega)V(\omega)$
Multiplication in the time domain	$x(t)v(t) \leftrightarrow \frac{1}{2\pi} X(\omega) * V(\omega)$
Parseval's theorem	$\int_{-\infty}^{\infty} x(t)v(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{X(\omega)}V(\omega) d\omega$
Special case of Parseval's theorem	$\int_{-\infty}^{\infty} x^2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty}  X(\omega) ^2 d\omega$
Duality	$X(t) \leftrightarrow 2\pi x(-\omega)$

# Example: Linearity

$$x(t) = p_4(t) + p_2(t)$$

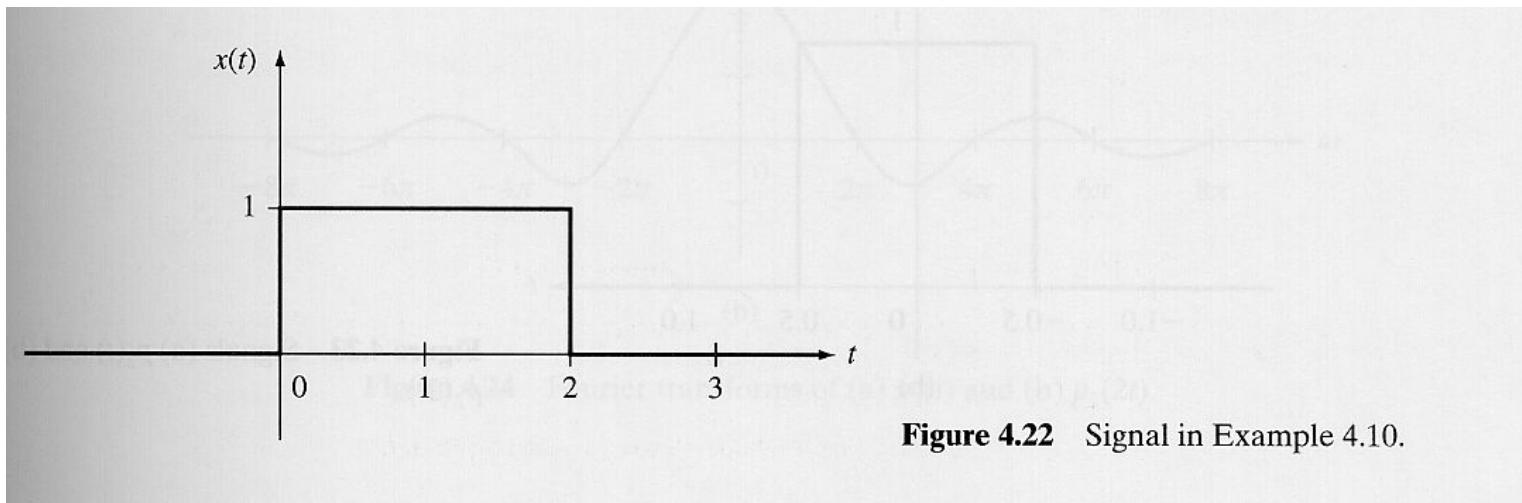


**Figure 4.21** Signal in Example 4.9.

$$X(\omega) = 4\text{sinc}\left(\frac{2\omega}{\pi}\right) + 2\text{sinc}\left(\frac{\omega}{\pi}\right)$$

## Example: Time Shift

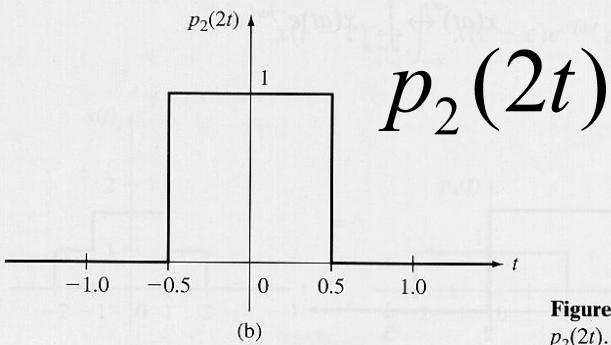
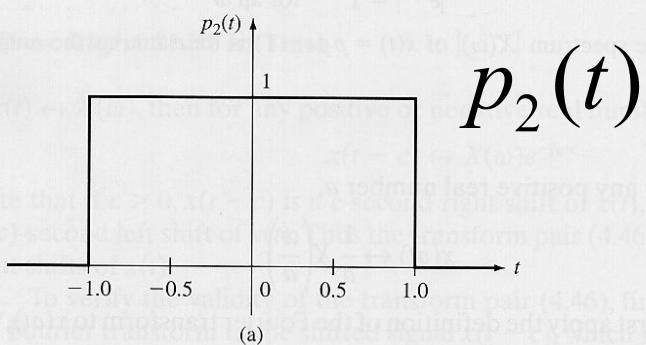
$$x(t) = p_2(t - 1)$$



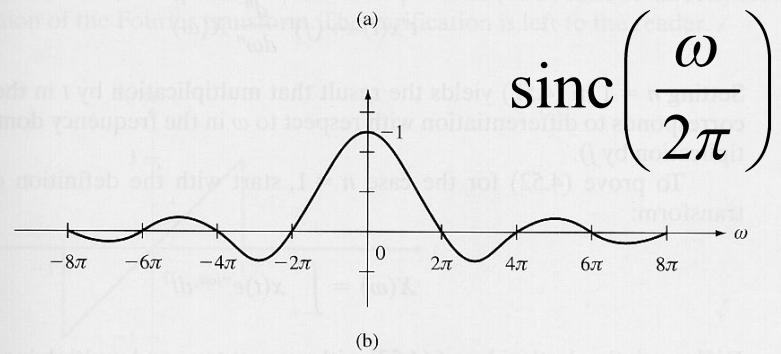
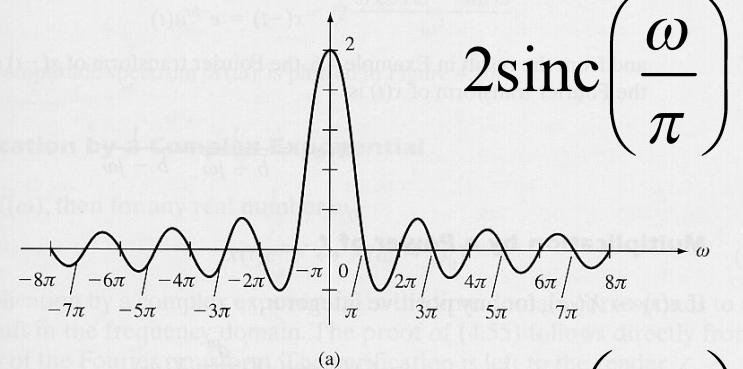
**Figure 4.22** Signal in Example 4.10.

$$X(\omega) = 2\text{sinc}\left(\frac{\omega}{\pi}\right)e^{-j\omega}$$

# Example: Time Scaling



**Figure 4.23** Signals (a)  $p_2(t)$  and (b)  $p_2(2t)$ .



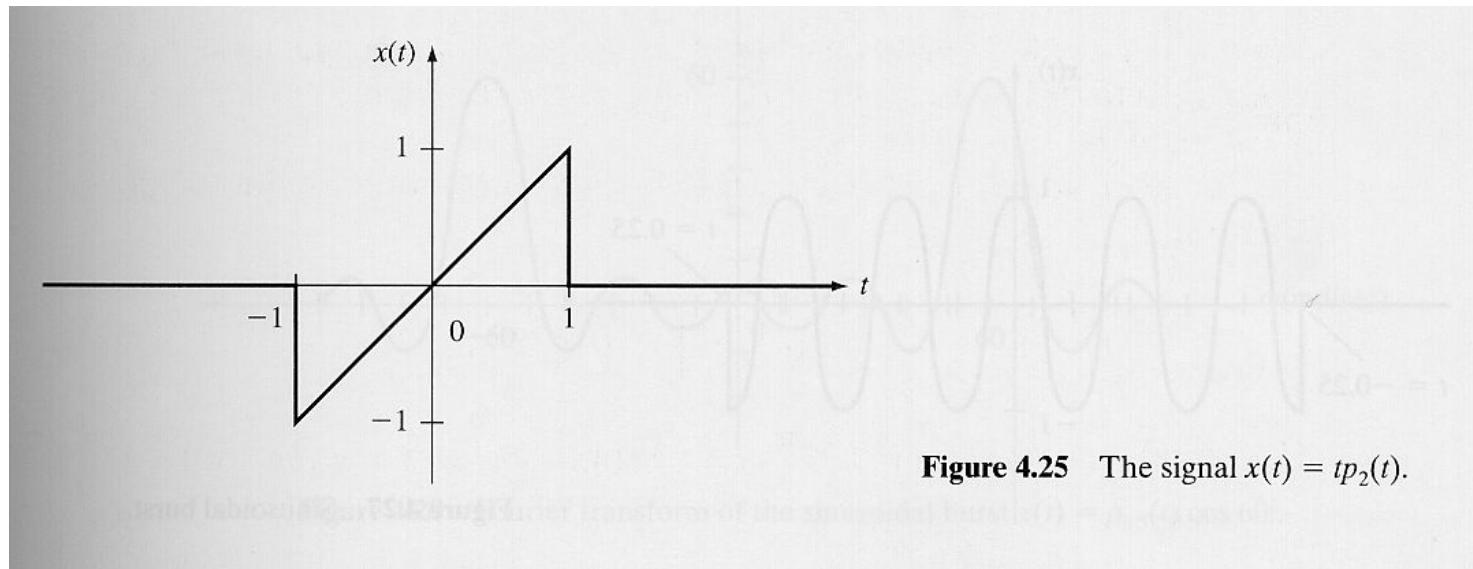
**Figure 4.24** Fourier transforms of (a)  $p_2(t)$  and (b)  $p_2(2t)$ .

$a > 1$  time compression  $\leftrightarrow$  frequency expansion

$0 < a < 1$  time expansion  $\leftrightarrow$  frequency compression

# Example: Multiplication in Time

$$x(t) = tp_2(t)$$

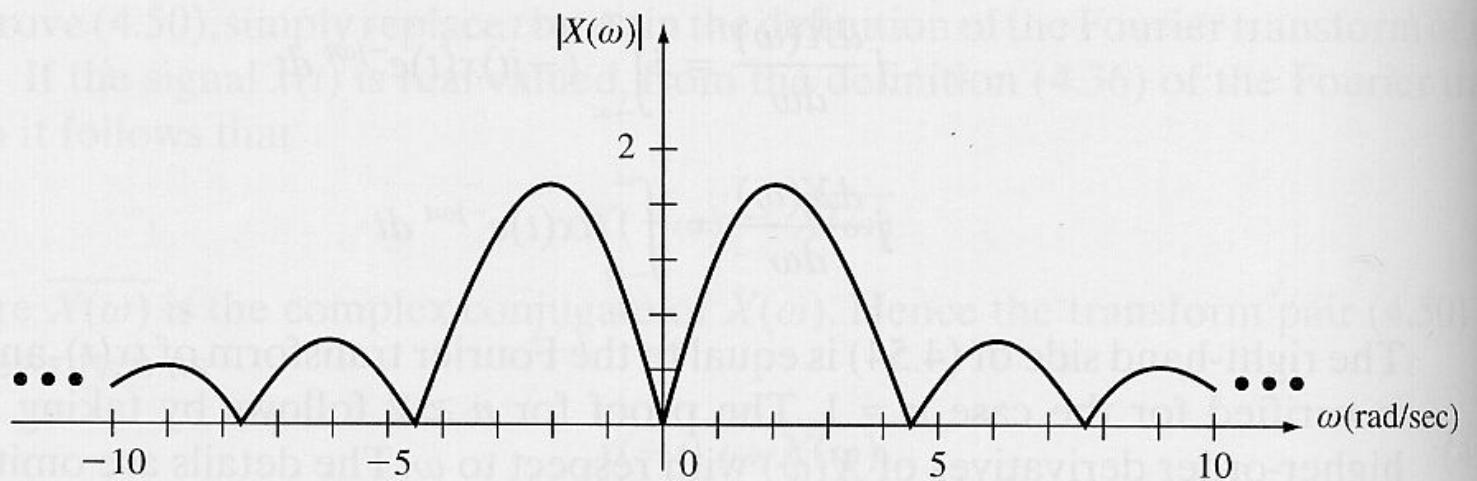


**Figure 4.25** The signal  $x(t) = tp_2(t)$ .

$$X(\omega) = j \frac{d}{d\omega} \left( 2 \operatorname{sinc} \left( \frac{\omega}{\pi} \right) \right) = j 2 \frac{d}{d\omega} \left( \frac{\sin \omega}{\omega} \right) = j 2 \frac{\omega \cos \omega - \sin \omega}{\omega^2}$$

## Example: Multiplication in Time – Cont'd

$$X(\omega) = j2 \frac{\omega \cos \omega - \sin \omega}{\omega^2}$$

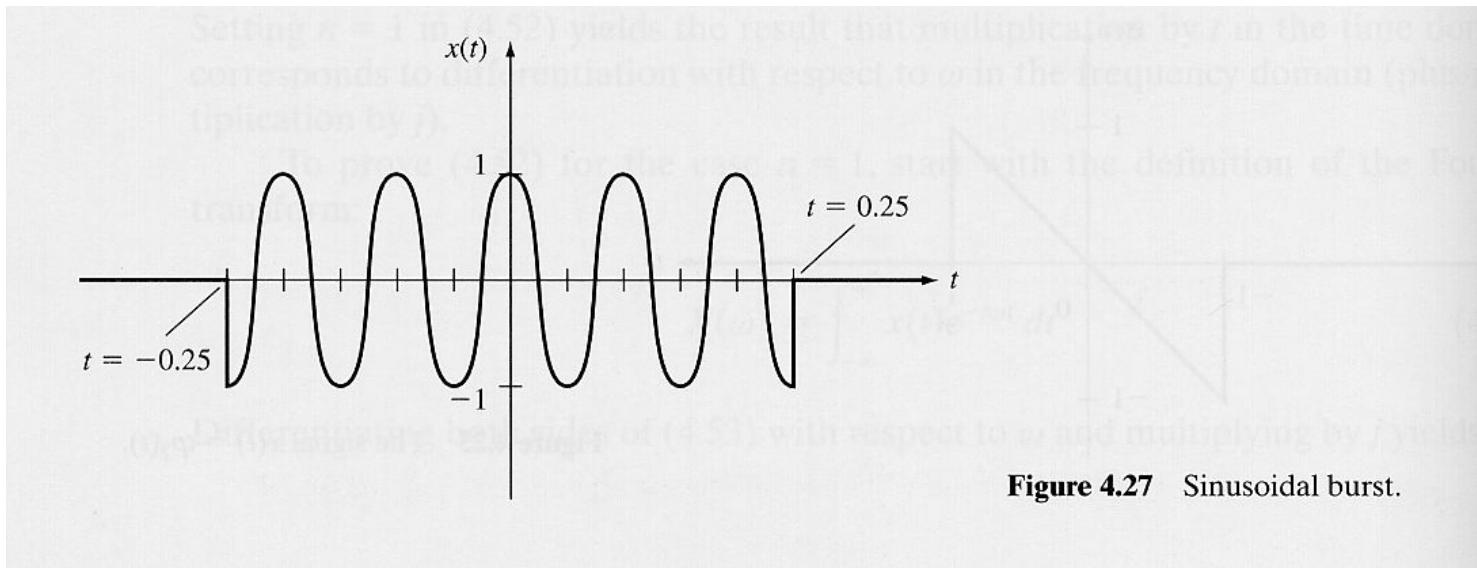


**Figure 4.26** Amplitude spectrum of the signal in Figure 4.25.

# Example: Multiplication by a Sinusoid

$$x(t) = p_\tau(t) \cos(\omega_0 t)$$

sinusoidal  
burst

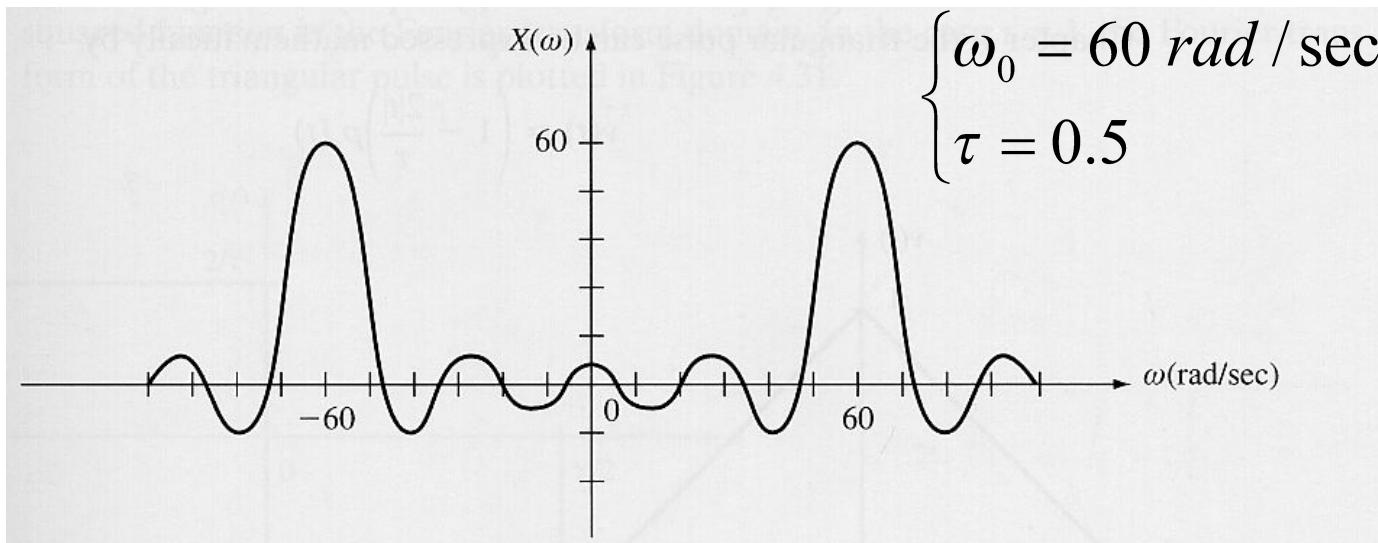


**Figure 4.27** Sinusoidal burst.

$$X(\omega) = \frac{1}{2} \left[ \tau \operatorname{sinc}\left(\frac{\tau(\omega + \omega_0)}{2\pi}\right) + \tau \operatorname{sinc}\left(\frac{\tau(\omega - \omega_0)}{2\pi}\right) \right]$$

## Example: Multiplication by a Sinusoid – Cont'd

$$X(\omega) = \frac{1}{2} \left[ \tau \operatorname{sinc}\left(\frac{\tau(\omega + \omega_0)}{2\pi}\right) + \tau \operatorname{sinc}\left(\frac{\tau(\omega - \omega_0)}{2\pi}\right) \right]$$



**Figure 4.28** Fourier transform of the sinusoidal burst  $x(t) = p_{0.5}(t) \cos 60t$ .

# Example: Integration in the Time Domain

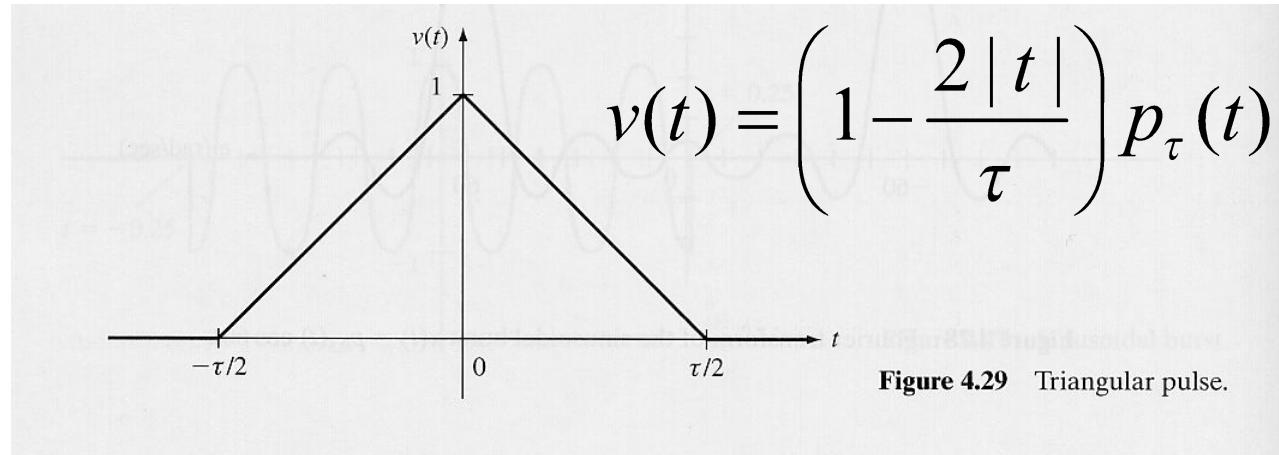


Figure 4.29 Triangular pulse.

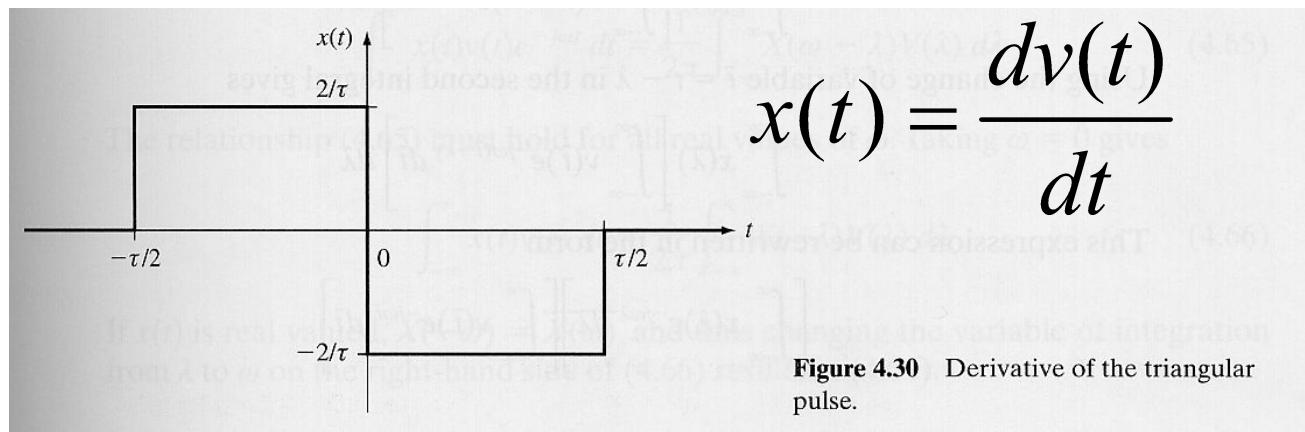


Figure 4.30 Derivative of the triangular pulse.

## Example: Integration in the Time Domain – Cont'd

- The Fourier transform of  $x(t)$  can be easily found to be

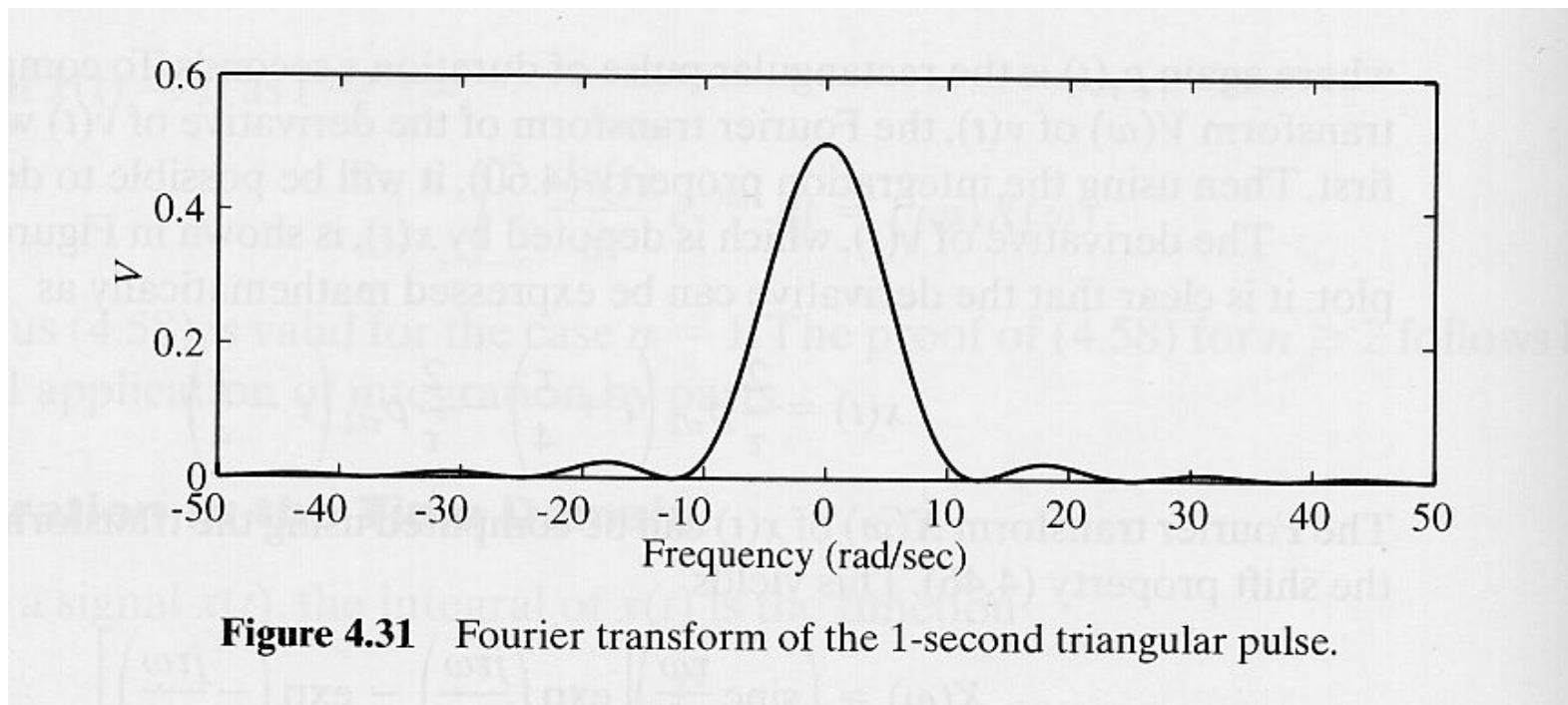
$$X(\omega) = \left( \text{sinc}\left(\frac{\tau\omega}{4\pi}\right) \right) \left( j2 \sin\left(\frac{\tau\omega}{4}\right) \right)$$

- Now, by using the integration property, it is

$$V(\omega) = \frac{1}{j\omega} X(\omega) + \pi X(0)\delta(\omega) = \frac{\tau}{2} \text{sinc}^2\left(\frac{\tau\omega}{4\pi}\right)$$

## Example: Integration in the Time Domain – Cont'd

$$V(\omega) = \frac{\tau}{2} \operatorname{sinc}^2\left(\frac{\tau\omega}{4\pi}\right)$$



**Figure 4.31** Fourier transform of the 1-second triangular pulse.

# Generalized Fourier Transform

- Fourier transform of  $\delta(t)$

$$\int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1 \quad \Rightarrow \quad \delta(t) \leftrightarrow 1$$

- Applying the duality property

$$x(t) = 1, t \in \mathbb{R} \leftrightarrow 2\pi \delta(\omega)$$



*generalized Fourier transform*  
of the constant signal  $x(t) = 1, t \in \mathbb{R}$

# Generalized Fourier Transform of Sinusoidal Signals

$$\cos(\omega_0 t) \leftrightarrow \pi [\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$$

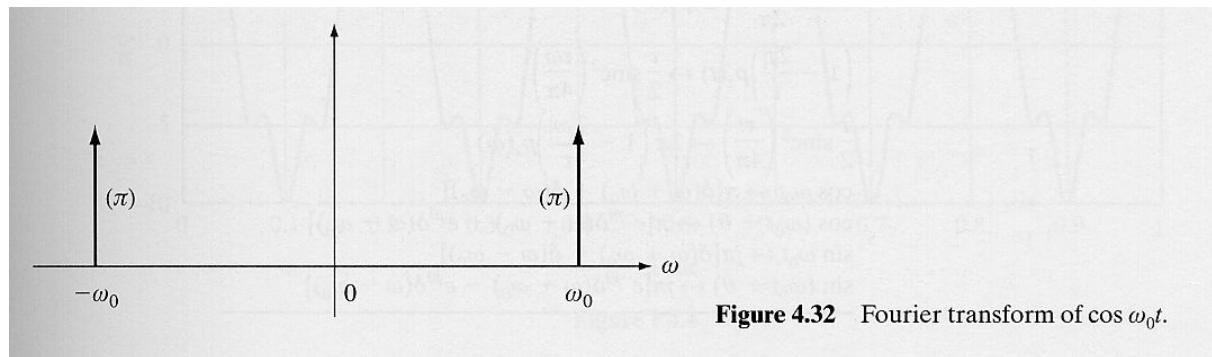


Figure 4.32 Fourier transform of  $\cos \omega_0 t$ .

$$\sin(\omega_0 t) \leftrightarrow j\pi [\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$$

# Fourier Transform of Periodic Signals

- Let  $x(t)$  be a periodic signal with period  $T$ ; as such, it can be represented with its Fourier transform

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \quad \omega_0 = 2\pi / T$$

- Since  $e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0)$ , it is

$$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi c_k \delta(\omega - k\omega_0)$$

# Fourier Transform of the Unit-Step Function

- Since

$$u(t) = \int_{-\infty}^t \delta(\tau) d\tau$$

using the integration property, it is

$$u(t) = \int_{-\infty}^t \delta(\tau) d\tau \leftrightarrow \frac{1}{j\omega} + \pi\delta(\omega)$$

# Common Fourier Transform Pairs

**TABLE 4.2 COMMON FOURIER TRANSFORM PAIRS**

$$1, \quad -\infty < t < \infty \leftrightarrow 2\pi\delta(\omega)$$

$$-0.5 + u(t) \leftrightarrow \frac{1}{j\omega}$$

$$u(t) \leftrightarrow \pi\delta(\omega) + \frac{1}{j\omega}$$

$$\delta(t) \leftrightarrow 1$$

$$\delta(t - c) \leftrightarrow e^{-j\omega c}, \quad c \text{ any real number}$$

$$e^{-bt}u(t) \leftrightarrow \frac{1}{j\omega + b}, \quad b > 0$$

$$e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0), \omega_0 \text{ any real number}$$

$$p_\tau(t) \leftrightarrow \tau \operatorname{sinc} \frac{\tau\omega}{2\pi}$$

$$\tau \operatorname{sinc} \frac{\tau t}{2\pi} \leftrightarrow 2\pi p_\tau(\omega)$$

$$\left(1 - \frac{2|t|}{\tau}\right)p_\tau(t) \leftrightarrow \frac{\tau}{2} \operatorname{sinc}^2 \left(\frac{\tau\omega}{4\pi}\right)$$

$$\frac{\tau}{2} \operatorname{sinc}^2 \left(\frac{\tau t}{4\pi}\right) \leftrightarrow 2\pi \left(1 - \frac{2|\omega|}{\tau}\right)p_\tau(\omega)$$

$$\cos \omega_0 t \leftrightarrow \pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$$

$$\cos(\omega_0 t + \theta) \leftrightarrow \pi[e^{-j\theta}\delta(\omega + \omega_0) + e^{j\theta}\delta(\omega - \omega_0)]$$

$$\sin \omega_0 t \leftrightarrow j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$$

$$\sin(\omega_0 t + \theta) \leftrightarrow j\pi[e^{-j\theta}\delta(\omega + \omega_0) - e^{j\theta}\delta(\omega - \omega_0)]$$