Pawan Khadka

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3. Spectral Analysis

→ It is one of the most widely used methods for data analysis in engineering.

→ Spectral analysis describes a time series function by comparing them to sines and cosines.

3.1. Review of fourier series, fourier Transform,
3.2. Energy and power signal, Parseval's theorem.

• fourier Series.

→ It is the representation of a periodic signal by a linear combination of sines & cosines or complex exponentials. When sines and cosines are combined we have trigonometric fourier series. And when complex exponentials are combined we have the complex exponential fourier series.

⊕ Periodicity : $u(t+T) = u(t)$ for all t.

The smallest time interval for which $u(t)$ is periodic is termed 'fundamental period' T_0, such that $f_0 = \frac{1}{T_0}$, $\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$

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⊕ for odd f^n/signal, $a_n = 0$
for even signal $b_n = 0$.

⊕ Trigonometric fourier series :

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$$

$$t_0 \le t \le t_0 + T \;;\; T = \frac{2\pi}{\omega_0}$$

where,

$$a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\,dt$$

$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cos n\omega_0 t\,dt$$

$$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\sin n\omega_0 t\,dt$$

The above mentioned fourier series can also be written in polar form as,

$$x(t) = D_0 + \sum_{n=1}^{\infty} D_n \cos(n\omega_0 t - \phi_n)$$

where,

$$D_0 = a_0$$

$$D_n = \sqrt{a_n^2 + b_n^2}$$

$$\& \quad \phi_n = \tan^{-1}\frac{b_n}{a_n}$$

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⊕ Complex exponential fourier series.

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi n f_0 t} \qquad \left[f_0 = \frac{1}{T_0} \right]$$

$$= \sum_{n=-\infty}^{\infty} C_n e^{j2\pi n t/T_0}$$

$$= \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

where

$$C_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{-jn\omega_0 t}\,dt$$

Now,

$C_n^* =$ complex conjugate of C_n.

such that,

$$C_n^* = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{jn\omega_0 t}\,dt$$

and,

$$\sum_{n=1}^{\infty} C_n^* e^{-jn\omega_0 t} = \sum_{n=-\infty}^{-1} C_n e^{jn\omega_0 t}$$

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Conversion of Trigonometric F.S. into complex exponential F.S. and vice-versa.

If we know a_0, a_n & b_n, then,

$c_0 = a_0$

$c_n = \frac{1}{2}(a_n - jb_n)$

& $c_n^* = \frac{1}{2}(a_n + jb_n)$

Similarly if we know c_0 &, c_n & c_n^*,

$a_0 = c_0$

$a_n = c_n + c_n^*$

$b_n = j(c_n - c_n^*)$

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$\boxed{\text{function} \rightarrow f^n}$

Symmetry conditions (Helpful in F.S. representation)

i) If $x(t)$ is even, i.e. $x(-t) = x(t)$ then,

$$\int_{-a}^{a} x(t)\,dt = 2\int_{0}^{a} x(t)\,dt$$

ii) If $x(t)$ is odd, i.e. $x(-t) = -x(t)$, then,

$$\int_{-a}^{a} x(t)\,dt = 0.$$

iii) The sum or product of two or more even functions is an even function.

iv) The sum of two or more odd f^{ns} is ~~and~~ an odd f^n.

v) The product of two or ~~and~~ more odd f^{ns} is an even f^n.

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Fourier Transform (F.T)

We define the Fourier transform $X(f)$ for an aperiodic, finite energy waveform $x(t)$ as,

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt \quad , -\infty < f < \infty$$

Basically, the F.T represents the time domain signals that are aperiodic, in frequency domain which makes the signal to be analysed easily.

Similarly, in terms of 'ω',

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \quad -\infty < \omega < \infty$$

And the inverse Fourier transform is,

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$$

or $$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$$\left[\quad X(f) = F[x(t)] \quad \& \quad x(t) = F^{-1}[X(f)] \right.$$

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So, we have $x(t)$ & $X(f)$ as a Fourier Transform pair, symbolically it can be expressed as,

$$x(t) \longleftrightarrow X(f).$$

Dirichlet's conditions.

For a function $x(t)$ to exhibit Fourier series or ~~Four~~ Fourier Transform, it must satisfy following conditions

i) $x(t)$ is well defined and a single valued function.

ii) $x(t)$ must possess only a finite number of discontinuities in the period T.

iii) $x(t)$ must have a finite number of positive and negative maxima (i.e. maxima and minima) in the period T.

iv) The function $x(t)$ is ~~absout~~ absolutely integrable, i.e.

$$\int_{-\infty}^{\infty} x(t) dt < \infty$$

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Properties of Fourier Transform.

i) Time-scaling property.

If $x(t) \longleftrightarrow X(f)$

Then for any real constant 'a',

$$x(at) \longleftrightarrow \frac{1}{|a|} X\left(\frac{f}{a}\right)$$

Proof:

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$$

let $at = y$ then $dt = \frac{dy}{a}$

$$\therefore \text{ F.T. of } x(at) = \int_{-\infty}^{\infty} x(y) e^{-j2\pi f \cdot (y/a)} \cdot \frac{dy}{a}$$

$$= \cancel{\frac{1}{a} \int_{-\infty}^{\infty} x(y) e^{-j(\frac{y}{a})}}$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} x(y) e^{-j2\pi f/a \cdot y} dy$$

$$= \frac{1}{a} X(f/a) \quad \text{For } a = +ve$$

For $a = -ve$, $\text{F.T.}[x(at)] = -\frac{1}{a} X(f/a)$

$$\therefore x(at) \longleftrightarrow \frac{1}{|a|} X(f/a)$$

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ii) Linearity property

If $x_1(t) \longleftrightarrow X_1(f)$
$x_2(t) \longleftrightarrow X_2(f)$

Then,

$$[a_1 x_1(t) + a_2 x_2(t)] \longleftrightarrow [a_1 X_1(f) + a_2 X_2(f)]$$

iii) Duality or symmetry property.

If $x(t) \longleftrightarrow X(f)$
then $X(t) \longleftrightarrow x(-f)$

Proof:

we have, $F^{-1}[X(f)] = x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df$

$$\therefore x(-t) = \int_{-\infty}^{\infty} X(f) e^{-j2\pi f t} df$$

or $x(-f) = \int_{-\infty}^{\infty} X(t) e^{-j2\pi f t} dt$

i.e. just interchanging the variables t & f,

so, $x(-f) = \int_{-\infty}^{\infty} X(t) e^{-j2\pi f t} dt$

or $F[X(t)] = x(-f)$

i.e. $X(t) \longleftrightarrow x(-f)$.

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iv) Time shifting property

If $x(t) \longleftrightarrow X(f)$

then,

$$x(t-b) \longleftrightarrow X(f)\, e^{-j2\pi f b}$$

Proof :

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

for $t = t-b$, let $t\ldots b$, we have,

$$F[x(t-b)] = \int_{-\infty}^{\infty} x(t-b)\, e^{-j2\pi f\, t}\, dt$$

let $t-b = y$, such that $dt = dy$,

Hence,

$$F[x(t-b)] = \int_{-\infty}^{\infty} x(y)\, e^{-j2\pi f(b+y)} \cdot dy$$

$$= \int_{-\infty}^{\infty} x(y)\, e^{-j2\pi f b} \cdot e^{-j2\pi f y}\, dy$$

$$= e^{-j2\pi f b} \cdot \int_{-\infty}^{\infty} x(y)\, e^{-j2\pi f y}\, dy$$

$$F[x(t-b)] = e^{-j2\pi f \cdot b} \cdot X(f)$$

$$\therefore\ x(t-b) \longleftrightarrow X(f)\, e^{-j2\pi f \cdot b}$$

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v) frequency shifting property.

If $x(t) \longleftrightarrow X(f)$,

Then,

$$e^{j2\pi f_0 t} \cdot x(t) \longleftrightarrow X(f-f_0)$$

Proof :

$$X(f) = F[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

Now,

$$F[e^{j2\pi f_0 t} \cdot x(t)] = \int_{-\infty}^{\infty} e^{j2\pi f_0 t} \cdot x(t)\, e^{-j2\pi f t}\, dt$$

or $F[e^{j2\pi f_0 t} \cdot x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi(f-f_0)t}\, dt$

or $F[e^{j2\pi f_0 t} \cdot x(t)] = X(f-f_0)$

$$\therefore\ e^{j2\pi f_0 t} \cdot x(t) \longleftrightarrow X(f-f_0)$$

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vi) Time differentiation property.

If $x(t) \leftrightarrow X(f)$

then,

$$\frac{d\,x(t)}{dt} \leftrightarrow$$

Proof:

We have,

$$F^{-1}[X(f)] = x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t} \cdot df$$

Now, $$\frac{d\,x(t)}{dt} = \frac{d}{dt}\left[\int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t}\, df\right]$$

or $$\frac{d\,x(t)}{dt} = \int_{-\infty}^{\infty} \frac{d}{dt}\left[X(f)\, e^{j2\pi f t}\right] df$$

or $$\frac{d\,x(t)}{dt} = \int_{-\infty}^{\infty} j2\pi f \cdot X(f) \cdot e^{j2\pi f t}\, df$$

or $$\frac{d\,x(t)}{dt} = F^{-1}\left[j2\pi f \cdot X(f)\right]$$

So, $$F\left[\frac{d\,x(t)}{dt}\right] = j2\pi f \cdot X(f)$$

or $$\frac{d\,x(t)}{dt} \leftrightarrow j2\pi f \cdot X(f).$$

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vii) Convolution.

We have,

$$x_1(t) \otimes x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

Now, in Fourier Transform we imply two convolution theorems.

a) Time convolution theorem, where,

if $x_1(t) \leftrightarrow X_1(f)$

$x_2(t) \leftrightarrow X_2(f)$

Then,

$$x_1(t) \otimes x_2(t) \leftrightarrow X_1(f) \cdot X_2(f)$$

b) frequency convolution theorem, where,

if $x_1(t) \leftrightarrow X_1(f)$

$x_2(t) \leftrightarrow X_2(f)$

Then,

$$x_1(t) \cdot x_2(t) \leftrightarrow X_1(f) \otimes X_2(f)$$

Energy Signals.

An energy signal is one which has finite energy and zero average power.

i.e. $x(t)$ is an energy signal if,

$$0 < E < \infty \quad \text{and} \quad P = 0.$$

where,

$$\left.\begin{matrix} E = \text{energy} \\ P = \text{power} \end{matrix}\right\} \text{ of } x(t)$$

For the energy to be finite, the signal amplitude $x(t)$ must tend to zero $[x(t) \to 0]$ as $|t| \to \infty$.

Almost all ~~xxxx~~ practical non-periodic signals which are defined over finite-time [Limited time signals] are energy signals and are expressed as,

$$E = \int_{-\infty}^{\infty} x^2(t)\, dt$$

Parceval's theorem for energy signals.

It states that the energy of a signal may be obtained with the help of its Fourier Transform.

i.e. If we know $X(\omega)$, we can determine the energy of the signal without knowing its time domain.

i.e. $E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = \int_{-\infty}^{\infty} |X(f)|^2 df$

Proof: let $x(t) \longleftrightarrow X(\omega)$.

Now,

$$E = \int_{-\infty}^{\infty} x^2(t)\, dt = \int_{-\infty}^{\infty} x(t) \cdot x(t)\, dt.$$

Also, $x(t) = \cancel{F^{-1}X(\omega)} = F^{-1}[X(\omega)]$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\cancel{2}\omega t}\, d\omega$$

$\therefore$ substituting $x(t)$ in E, we have,

$$E = \int_{-\infty}^{\infty} x(t) \cdot \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \right] dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \left[\int_{-\infty}^{\infty} x(t) e^{j\omega t} dt \right] d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot X(-\omega)\, d\omega$$

$$\left[\because X(-\omega) = \int_{-\infty}^{\infty} x(t) e^{j\omega t} dt \right]$$

$$\therefore E = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\,X(-\omega)\,d\omega .$$

Now for real signal $x(t)$, the F.T. $X(\omega)$ and $X(-\omega)$ are complex conjugates, i.e.

$$X(-\omega) = X^*(\omega)$$

such that,

$$X(\omega)\cdot X(-\omega) = X(\omega)\cdot X^*(\omega) = |X(\omega)|^2$$

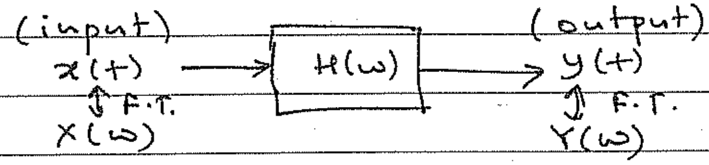
Therefore,

$$\boxed{E = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\,d\omega \quad \text{or} \quad E = \int_{-\infty}^{\infty} |X(f)|^2\,df}$$

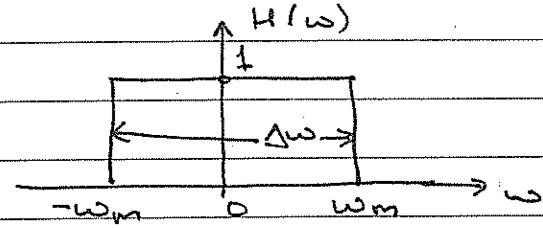
The above equations are also known as Rayleigh energy theorem.

⊛ Energy spectral density (ESD)

Let us apply a signal $x(t)$ to an ideal low pass filter, i.e.



The transfer function of ideal LPF can be shown as,



Now the ~~seq~~ response of a system is expressed as,

$$Y(\omega) = X(\omega)\cdot H(\omega)$$

Now, energy E_0 of output signal $y(t)$ is,

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} |Y(\omega)|^2\,d\omega$$

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or $E_0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega) \cdot H(\omega)|^2 \, d\omega$

Now, $H(\omega) = 1$ for $-\omega_m < \omega < \omega_m$
$= 0$ elsewhere,

$\therefore E_0 = \frac{1}{2\pi} \int_{-\omega_m}^{\omega_m} |X(\omega)|^2 \, d\omega$

considering $X(\omega)$ constant from $-\omega_m$ to ω_m,

$$E_0 = \frac{1}{2\pi} \cdot |X(\omega)|^2 \cdot \int_{-\omega_m}^{\omega_m} 1 \cdot d\omega$$

$$= \frac{1}{2\pi} |X(\omega)|^2 \cdot \omega \Big|_{-\omega_m}^{\omega_m}$$

$$= \frac{1}{2\pi} |X(\omega)|^2 \cdot (\omega_m + \omega_m)$$

$$= \frac{1}{2\pi} |X(\omega)|^2 \cdot 2\omega_m$$

~~Let~~ ~~2ω~~ Putting $2\omega_m = \Delta\omega$, we get,

$$E_0 = \frac{1}{2\pi} |X(\omega)|^2 \Delta\omega = |X(f)|^2 \cdot \Delta f$$

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From the derivation it is clear that E_0 represents the contribution of energy due to bandwidth $\Delta\omega$.

Therefore energy contribution per unit bandwidth will be,

$$\frac{E}{\Delta\omega} \cdot 2\pi = \frac{1}{2\pi} |X(\omega)|^2$$

or $\frac{E}{\Delta f} = |X(\omega)|^2$ or $\frac{E}{\Delta f} = |X(f)|^2$

where, $|X(\omega)|^2$ represents energy per unit bandwidth and is known as Energy spectral density or Energy density spectrum and is denoted by $\Psi(\omega)$

i.e. $\Psi(\omega) = |X(\omega)|^2$
or $\Psi(f) = |X(f)|^2$

So, we can have,

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi(\omega) \, d\omega = \int_{-\infty}^{\infty} \Psi(f) \, df$$

$$= \frac{1}{\pi} \int_{0}^{\infty} \Psi(\omega) \, d\omega = 2 \int_{0}^{\infty} \Psi(f) \, df$$

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⊕ Properties of ESD (Energy spectral density)

i) $E = \int_{-\infty}^{\infty} \Psi(f)\, df$

i.e. total area under the energy spectral density function is equal to the total energy of that signal.

ii) If $x(t)$ is input to a LTI system with transfer function $H(\omega)$, then input and output energy spectral density f^ns are related as,

$$\Psi_o(\omega) = |H(\omega)|^2 \Psi_i(\omega)$$

where,

$\Psi_o(\omega)$: o/p ESD

$\Psi_i(\omega)$: input ESD

$|H(\omega)|^2$: energy gain at 'ω'.

iii) ESD $\Psi(\omega)$ and autocorrelation function $R(\tau)$ form a Fourier transform pair

i.e.

$$R(\tau) \longleftrightarrow \Psi(\omega)$$

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⊕ Power signals:

A power signal is one which has a finite average power and infinite energy. So, a signal $x(t)$ is a power signal if,

$$0 < P < \infty \quad \text{and} \quad E = \infty.$$

where,

E = energy of the signal

P = average power of the signal.

Almost all periodic signals are power signal.

Power (P) of a signal is expressed as,

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)\, dt$$

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(4) Parseval's theorem for power signals.

It states that the power of a signal may be defined in terms of its Fourier series coefficients.

Proof.

For a signal $x(t)$,

$$|x(t)|^2 = x(t) \cdot x^*(t)$$ [x^* = complex conjugate of $x(t)$]

Now,

Power of signal $x(t)$,

$$P = \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 dt = \frac{1}{T}\int_{-T/2}^{T/2} x(t) \cdot x^*(t)\, dt$$

Replacing $x(t)$ by its complex exponential Fourier series, we get,

$$P = \frac{1}{T}\int_{-T/2}^{T/2} x^*(t) \cdot \left[\sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}\right] dt$$

Interchanging the order of integration and summation, we get,

$$P = \frac{1}{T}\sum_{n=-\infty}^{\infty} C_n \int_{-T/2}^{T/2} x^*(t)\, e^{jn\omega_0 t}\, dt$$

Now, since $C_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jn\omega_0 t} dt$

therefore, $C_n^* = \frac{1}{T}\int_{-T/2}^{T/2} x^*(t)\, e^{jn\omega_0 t} dt$

So,

$$P = \cancel{\frac{1}{T}} \sum_{n=-\infty}^{\infty} C_n \cdot \frac{1}{T}\int_{-T/2}^{T/2} x^*(t)\, e^{jn\omega_0 t} dt$$

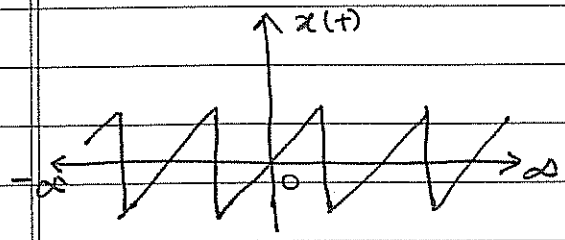
$$= \sum_{n=-\infty}^{\infty} C_n \cdot C_n^*$$

$$= \sum_{n=-\infty}^{\infty} |C_n|^2$$

(15) Power spectral density (PSD).

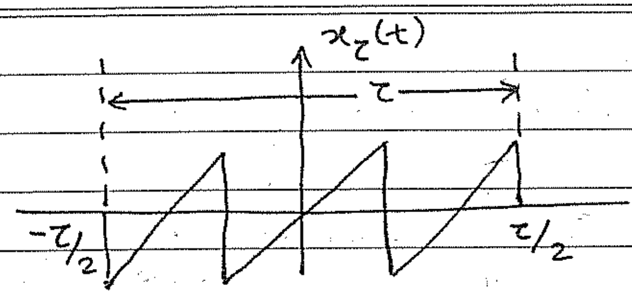
The expression for power spectral density may be derived by assuming the power signal as a limiting case of an energy signal.

Let us consider a power signal $x(t)$ which extends to infinity.



Now let us terminate or limit the signal from $-\tau/2$ to $+\tau/2$ such that it is denoted by $x_\tau(t)$ and expressed as,

$$x_\tau(t) = \begin{cases} x(t) & |t| < \tau/2 \\ 0 & \text{elsewhere} \end{cases}$$



Now since $x_\tau(t)$ is of finite duration, i.e. 'τ', it is an energy signal. Let the energy of this signal be E_τ and may be expressed as,

$$E_\tau = \int_{-\infty}^{\infty} |x_\tau(t)|^2 dt = \int_{-\infty}^{\infty} |X_\tau(f)|^2 df$$

where $X_\tau(f)$ is F.T. of $x_\tau(t)$.

Now,

$$\int_{-\infty}^{\infty} |x_\tau(t)|^2 dt = \int_{-\tau/2}^{\tau/2} |x(t)|^2 dt$$

Therefore,

$$\int_{-\tau/2}^{\tau/2} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X_\tau(f)|^2 df$$

or

$$\frac{1}{\tau} \int_{-\tau/2}^{\tau/2} |x(t)|^2 dt = \frac{1}{\tau} \int_{-\infty}^{\infty} |X_\tau(f)|^2 df$$

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Taking limit $\tau \to \infty$ on both sides,

$$\lim_{\tau\to\infty} \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} |x(t)|^2 dt = \lim_{\tau\to\infty} \frac{1}{\tau} \int_{-\infty}^{\infty} |X_\tau(f)|^2 df$$

The left hand side of the equation above represents average power 'P' of $x(t)$, Therefore,

$$P = \int_{-\infty}^{\infty} \lim_{\tau\to\infty} \frac{|X_\tau(f)|^2}{\tau} df$$

As ~~Lim τ→~~ Limit $\tau \to \infty$, the ratio $\frac{|X_\tau(f)|^2}{\tau}$ may approach some finite value.

Let this finite value be $S(f)$ ~~$S(\omega)$~~, such that

$$S(f) = \lim_{\tau\to\infty} \frac{|X_\tau(f)|^2}{\tau}$$

$$\therefore P = \int_{-\infty}^{\infty} S(f)\, df$$

or $$P = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega)\, d\omega$$

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$$\boxed{P = \int_{-\infty}^{\infty} S(f)\, df = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega)\, d\omega}$$

From the above equation it is clearly seen that the total power of the signal is obtained by multiplying $S(\omega)$ with bandwidth $\Delta\omega$ (i.e. $d\omega$) and integrating over the entire bandwidth.

Hence, $S(\omega)$ or $S(f)$ may be treated as average power per unit bandwidth and is called as power spectral density or power density spectrum.

Also, the term $\frac{|X_\tau(f)|^2}{\tau}$ is called the periodogram of the signal.

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Energy (aperiodic signals)	Power (periodic signals)
$x(t)$ is energy signal if, $0 < E < \infty$ & $P = 0$ and, $E = \int_{-\infty}^{\infty} x^2(t)\,dt$	$x(t)$ is power signal if, $0 < P < \infty$ & $E = \infty$ and $P = \frac{1}{T}\int_{-\infty}^{\infty} x^2(t)\,dt = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x^2(t)\,dt$
Parseval's Theorem: If $x(t) \longleftrightarrow X(\omega)$, $E = \frac{1}{2\pi}\int_{-\infty}^{\infty} (X(\omega))^2\,d\omega$ or $\int_{-\infty}^{\infty} x^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2\,d\omega = \int_{-\infty}^{\infty} \lvert X(f)\rvert^2\,df$	Parseval's theorem. $P = \sum_{n=-\infty}^{\infty} \lvert C_n \rvert^2$
Energy spectral density. $\Psi(\omega) = \lvert X(\omega)\rvert^2$ ∴ $E = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi(\omega)\,d\omega = \int_{-\infty}^{\infty} \Psi(f)\,df$.	Power spectral density, $S(\omega) = \lim_{\tau\to\infty} \frac{(X_\tau(\omega))^2}{\tau}$ ∴ $P = \int_{-\infty}^{\infty} S(\omega)\,df = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\,d\omega$

F.T. Pairs:

$$\delta(t) \longleftrightarrow 1$$

$$e^{j\omega_0 t} \longleftrightarrow 2\pi\,\delta(\omega - \omega_0)$$

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3.3. Power spectral dencity functions of harmonic signal and white noise.

⊞ PSD of harmonic signal.

Let us consider a periodic signal $x(t)$ with period 'T_0' such that it can be expressed in terms of complex exponential fourier series as

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

where $\omega_0 = \frac{2\pi}{T_0}$

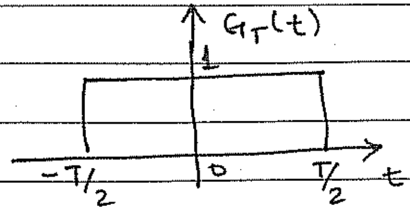
Now,

$$X(\omega) = 2\pi \sum_{n=-\infty}^{\infty} C_n\,\delta(\omega - n\omega_0)$$

Now, we truncate the signal $x(t)$ by introducing a gate function $G_T(t)$ such that,

$$G_T(t) = 1 \quad \text{for } -T/2 < t < T/2$$
$$= 0 \quad \text{elsewhere.}$$

i.e.



The F.T. of $G_T(t)$, i.e.

$$F.T[G_T(t)] = \frac{T}{2\pi} \operatorname{sinc}\left(\frac{\omega T}{2}\right) \quad [\text{sinc } f^n].$$

Now, the truncated periodic signal can be written as,

$$x_T(t) = G_T(t) \cdot x(t)$$

And

$$F.T[x_T(t)] = F.T[G_T(t) \cdot x(t)]$$

Using frequency convolution theorem of F.T.

$$X_T(\omega) = F.T.[G_T(t)] \otimes F.T[x(t)]$$

$$\text{or } X_T(\omega) = \frac{T}{2\pi} \operatorname{Sinc}\left(\frac{\omega T}{2}\right) \otimes 2\pi \sum_{n=-\infty}^{\infty} C_n \cdot \delta(\omega - n\omega_0)$$

$$= T \sum_{n=-\infty}^{\infty} \operatorname{Sinc}\left(\frac{\omega T}{2}\right) \otimes C_n \delta(\omega - n\omega_0)$$

$$= T \sum_{n=-\infty}^{\infty} C_n \operatorname{Sinc}\left[\frac{(\omega - n\omega_0)T}{2}\right]$$

[Σ ~~[illegible]~~

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As per the definition of PSD f^n of $x(t)$,

$$S_x(\omega) = \lim_{T \to \infty} \frac{|X_T(\omega)|^2}{T}$$

$$= \lim_{T \to \infty} \frac{1}{T} \left[\sum_{n=-\infty}^{\infty} |C_n|^2 \cdot T^2 \operatorname{Sinc}^2\left[\frac{(\omega - n\omega_0)T}{2}\right] \right]$$

As the limit $T \to \infty$, the 'sinc²' f^n tends to delta f^n concentrated at $n\omega_0$.

So,

$$S_x(\omega) = 2\pi \sum_{n=-\infty}^{\infty} |C_n|^2 \, \delta(\omega - n\omega_0)$$

Now, the harmonic signal, $u(t) = A\cos(\omega_0 t)$ is a special case of periodic signal for which the value of $n = 1$.

$$\text{So, } U(f) = \frac{A}{2}\delta(\omega - \omega_0) + \frac{A}{2}\delta(\omega + \omega_0)$$

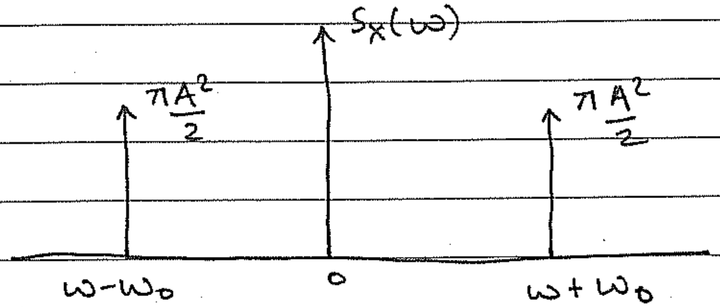
$$\text{Because, } u(t) = A\cos\omega_0 t = \frac{A}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

and $e^{j\omega_0 t} \longleftrightarrow \delta(\omega - \omega_0)$ and $C_n = \frac{A}{2}$

$e^{-j\omega_0 t} \longleftrightarrow \delta(\omega + \omega_0)$

Therefore the psdf of harmonic signal will be,

$$S_X(\omega) = 2\pi\left[\frac{A^2}{4}\delta(\omega-\omega_0) + \frac{A^2}{4}\delta(\omega+\omega_0)\right]$$



And the average power of the harmonic signal is,

$$P = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\left[\frac{A^2}{4}\delta(\omega-\omega_0) + \frac{A^2}{4}\delta(\omega+\omega_0)\right]d\omega$$

$$= \frac{A^2}{4} + \frac{A^2}{4} = \frac{A^2}{2}$$

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Power spectral density f^n of white noise.

White noise is an idealised form of noise. In white noise there is the presence of all frequencies.

White noise can thus be considered as a random signal or process with a flat power spectral density that is independent of frequency. The power spectral density of white noise can now be expressed as,

$$S_{WN}(f) = \frac{N_0}{2} \; ; \quad -\infty < f < \infty \quad [\text{WN - white noise}]$$

Where N_0 : Noise power

and the factor $1/2$ indicates that half of the power is associated with positive frequency and half with negative frequency.

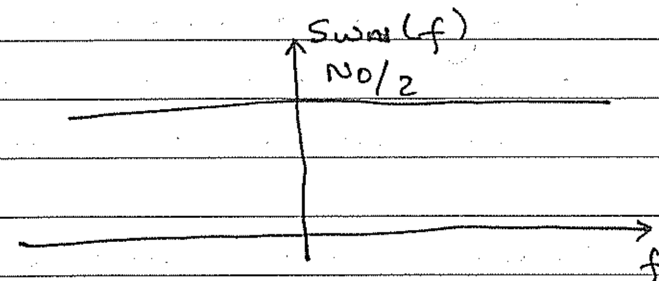

Fig. Psdf of white noise

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3.4 The autocorrelation function (AC), relationship between AC f^n and psdf.

Autocorrelation function gives the measure of similarity, match or coherence between a signal and its delayed replica. Auto-correlation between two signals explains how much a signal is related to its time delayed version.

Let us consider a signal $x(t)$. then the autocorrelation function of this signal with its delayed version will be

$$R(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t) \cdot x^*(t-\tau)\, dt$$

where,

τ = delayed parameter

'*' can be neglected if the signals are real valued signals.

Autocorrelation can be taken as a special case of cross-correlation.

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The autocorrelation function is defined separately for energy signals and for power or periodic signals.

⊞ for energy signals:

Let $x(t)$ be an energy signal, then AC f^n of this signal may be obtained by integrating the product of $x(t)$ and delayed version of its complex conjugate.

i.e.

$$R(\tau) = \int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\, dt$$

where complex valued signal $x(t)$ is delayed in positive time. Now if $x(t)$ is shifted by 'τ' in negative direction,

$$R(\tau) = \int_{-\infty}^{\infty} x(t+\tau)\, x^*(t)\, dt$$

Properties of AC f^n for energy signals.

1. the autocorrelation f^n exhibits conjugate symmetry. i.e.

$$R(\tau) = R^*(-\tau)$$

2. The value of autocorrelation f^n at $\tau = 0$, (i.e. at origin) is equal to the energy of the signal.

$$\text{i.e.} \quad R(0) = \int_{-\infty}^{\infty} |x(t)|^2 dt = E.$$

3. If τ is increased in either direction, the autocorrelation reduces. The AC is maximum at $\tau = 0$.

$$\text{i.e.} \quad |R(\tau)| \leq R(0) \quad \text{for all } \tau.$$

4. The autocorrelation f^n and energy spectrum density f^n of energy signal form a fourier transform pair.

$$\text{i.e.} \quad R(\tau) \longleftrightarrow \Psi(\omega)$$

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for power signals.

Let $x(t)$ be periodic signal with period T_0. The AC f^n of $x(t)$ for one period is,

$$R(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)\, x^*(t-\tau)\, dt$$

And if 'τ' is in negative direction, then,

$$R(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t+\tau)\, x^*(t)\, dt$$

So, for any period T, AC f^n is given as,

$$R(\tau) = \cancel{\frac{1}{T} \int_{-T/2}^{T/2}}$$

$$R(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, x^*(t-\tau)\, dt$$

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Properties of AC. f^n for periodic signals

1. $R(\tau) = R^*(-\tau)$

   i.e. AC f^n exhibit conjugate symmetry.

2. The AC f^n at origin is equal to the average power of the signal,

   i.e.

   $$R(0) = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} |x(t)|^2 dt = P.$$

3. $R(\tau) \le R(0)$

   i.e. AC f^n is maximum at origin.

4. The autocorrelation f^n and power spectral density f^n form a Fourier transform pair.

   i.e.

   $$R(\tau) \longleftrightarrow S(\omega)$$

5. AC. f^n is periodic with the period same as that of the periodic signal.

   i.e.

   $$R(\tau) = R(\tau + nT_0) \;;\; n = 1, 2, 3, \dots$$

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Prove that $A\cos\omega_0 t$ is a power type signal.

$$x(t) = A\cos\omega_0 t$$

$$P = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x^2(t)\,dt$$

$$\Rightarrow x^2(t) = A^2\cos^2\omega_0 t$$

$$= \frac{A^2}{2}[1 + \cos 2\omega_0 t]$$

$$\therefore P = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\frac{A^2}{2}[1+\cos 2\omega_0 t]\,dt$$

$$= \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\frac{A^2}{2}dt + \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\frac{A^2\cos 2\omega_0 t}{2}\,dt.$$

$$= \lim_{T\to\infty}\frac{A^2}{2T}\cdot T = \frac{A^2}{2}$$

$$\text{Power} = (\text{rms})^2 = \frac{A^2}{2}$$

$$\therefore \text{rms} = \frac{A}{\sqrt{2}}$$