Chapter

2 SAMPLING THEORY

Introduction

In any communication system, the message signal originating from a source may be in digital or analog form. If the message signal is derived from a digital source (i.e., digital computer), the signal does not need any form translation for transmission through digital system. However, if the message signal is analog in nature, as in speech and video signal, then it has to be converted into digital form before it can be transmitted through digital system. The process of transforming an analog signal to digital signal is known as sampling and is done by analog-to-digital converter. After sampling process the signal is further processed by quantization and encoding.

In sampling process, the continuous time signal is converted to a discrete time signal by measuring the signal at periodic instant of time. The periodic instant of time or sampling rate must be large enough to permit the analog signal to be reconstructed from the samples with sufficient accuracy. The sampling theorem provides the basis for determining the proper sampling rate for a given signal.

2.1 Sampling Theorem

In general, the sampling theorem states that "*Analog signal can be reproduced from an appropriate set of its samples taken at some fixed interval of time*". This theorem has made it possible to transmit only the samples of analog signal by encoding these samples into block of code words suitable for digital communication.

In broad sense the sampling theorem can be stated into two parts.

1. A finite energy, strictly band limited signal (i.e., containing no frequencies higher than f_m hertz) is completely described by the samples (values) of the signal at instants of time separated by 1/2fm seconds apart (for transmitter end).
2. A finite energy, strictly band limited signal (i.e., containing no frequencies higher than W hertz), may be completely recovered from its samples (values) taken at rate of $2f_m$ per second (for receiving end).

Proof of Sampling Theorem

To prove the sampling theorem, consider an analog signal $x(t)$ (Fig.2.1(a)) which is continuous in both time and amplitude. (The spectrum of signal $x(t)$ is band-limited to f_m Hz as shown in Fig.2.1(b).) Suppose that we sample the signal $x(t)$ at a uniform interval of every T_s seconds. This uniform sampling can be accomplished by multiplying $x(t)$ by an impulse train $d_{T_s}(t)$ of Fig.2.1(c), which results in a sampled signal $x_\delta(t)$ shown in Fig.2.1(e). The sampled signal consists of impulses spaced every T_s seconds (the sample interval). Then, we obtain an infinite sequence of samples spaced T_s seconds apart with amplitude represented by $x(nT_s)$, where $n = 0, \pm 1, \pm 2, \ldots$. We refer to T_s as the sampling period, and its reciprocal i.e., $f_s = 1/T_s$ as the sampling rate. Thus, the mathematical relation between the sampled signal $x_\delta(t)$ and the original analog signal $x(t)$ is

$$x_\delta(t) = x(t)\delta_{T_s}(t)$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t - nT_s) \tag{2.1}$$

Where $\delta(t - nT_s)$ is Dirac delta function located at time $t = nT_s$. In Eq.(2.1), each delta function in the series is weighted by the corresponding sample value of the input signal $x(t)$ i.e., $x(nT_s)$.

As the impulse train $\delta_{T_s}(t)$ is a periodic signal with period T_s, it can be expressed as an exponential Fourier series as shown by Eq. (2.2),

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn2\pi f_s t}$$

Where, $C_n = 1/T_s$,

$$\delta_{T_s}(t) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} e^{jn2\pi f_s t} \tag{2.2}$$

Therefore,

$$x_\delta(t) = x(t)\delta_{T_s}(t)$$

$$= \frac{1}{T_s}\sum_{n=-\infty}^{\infty} x(t)e^{jn2\pi f_s t} \tag{2.3}$$

To find $X_\delta(f)$, the Fourier transform of $x_\delta(t)$, we take the Fourier transform of the summation in Eq.(2.3). Based on the frequency-shifting property, the transform of the n^{th} term is shifted by nf_s.

$$x(t) \xleftrightarrow{FT} X(f)$$

$$x(t)e^{jn2\pi f_s t} \xleftrightarrow{FT} X(f - nf_s)$$

Therefore.

$$X_\delta(f) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} X(f - nf_s) \tag{2.4}$$

This means that the spectrum $X_\delta(f)$ consists of $X(f)$ (spectrum of signal $x(t)$) scaled by a constant $1/T_s$, repeating periodically with period $f_s = 1/T_s$ Hz, as shown in Fig. 2.1(f).

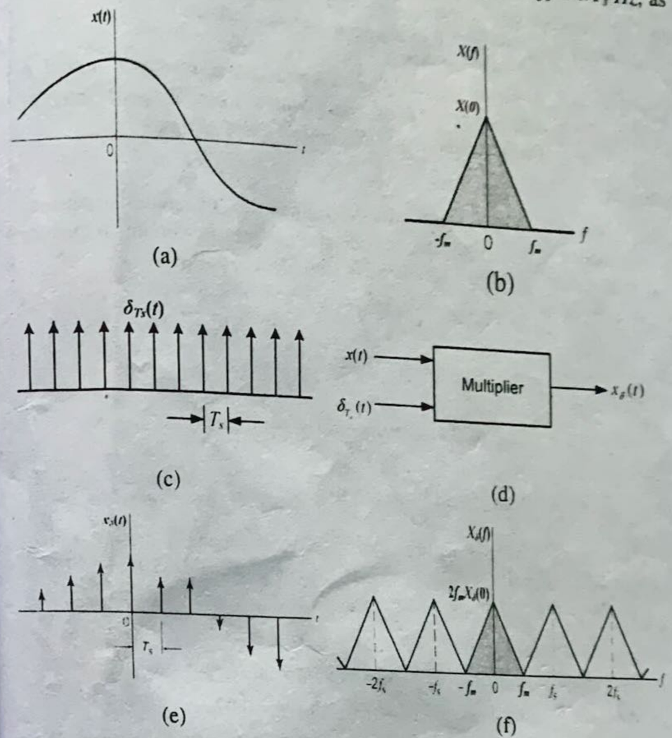


Fig. 2.1.(a) A continuous time signal. (b) Spectrum of continuous time signal. (c) Impulse train as sampling function. (d) Multiplier. (e) Sampled signal. (f) Spectrum of sampled signal.

If we are to reconstruct $x(t)$ from $x_\delta(t)$, we should be able to recover $X(f)$ from $X_\delta(f)$. This is possible if there is no overlap between successive cycles of $X_\delta(f)$. Fig.2.1(f) shows that this requires

$$f_s \geq 2f_m \tag{2.5}$$

Also, the sampling interval $T_s = 1/f_s$. Therefore,

$$T_s \leq \frac{1}{2f_m} \tag{2.6}$$

Thus, as long as the sampling frequency f_s is greater than twice the signal bandwidth f_m (in hertz), $X_\delta(f)$ will consist of non-overlapping repetitions of $X(f)$. When this is true, Fig. 2.1(f) shows that $x(t)$ can be recovered from its samples by passing the sampled signal $x_\delta(t)$ through an ideal low-pass filter of bandwidth f_m Hz. The minimum sampling rate $f_s = 2f_m$ required to recover $x(t)$ from its samples $x_\delta(t)$ is called the Nyquist rate for $x(t)$, and the corresponding sampling interval $T_s = 1/2f_m$ is called the Nyquist interval for $x(t)$ given by Eq. (2.5) and (2.6) respectively.

Example 2.1: ***A signal $x(t)$=Acos(20πt)cos(30πt) is ideally sampled at Nyquist rate. Find the Nyquist Rate, Nyquist interval. Draw the spectra of $x(t)$ and sampled signal.***

Solution:

Solve the given signal to determine the maximum frequency component f_m. Note that to determine the maximum frequency component, the signal should be in sum form as in Eq.(4).

$$x(t) = A\cos(20\pi t)\cos(30\pi t) \tag{1}$$

$$x(t) = \frac{A}{2}[2\cos(20\pi t)\cos(30\pi t) \tag{2}$$

$$x(t) = \frac{A}{2}[\cos((20+30)\pi t) + \cos((20-30)\pi t)] \tag{3}$$

$$x(t) = \frac{A}{2}[\cos(50\pi t) + \cos(10\pi t)] \tag{4}$$

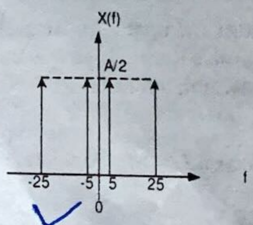
From Eq.(4) we can see that there are two frequencies present which are $W_1=2\pi F_1=50\pi$ and $W_2=2\pi F_2=10\pi$. Therefore, $F_1=25$ *Hz* and $F_2=5$ *Hz*. Now the maximum frequency is F_1, so,

f_m (i.e., maximum frequency present)=F_1=25 *Hz*

Therefore, Nyquist rate $(f_s)=2f_m=2\times25$ *Hz* = 50 *Hz*

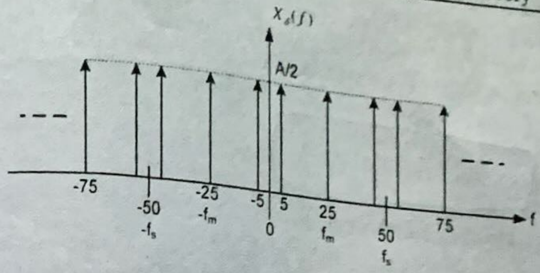
And, the Nyquist interval $(T_s)=1/f_s=1/50$ *sec* = 0.02 *sec*

The spectra of the signal is obtained by taking Fourier Transform. We known that the spectra of cosine is given by



(a)

i think we also look '-' of 5 & 25 because cos(-25) = cos 25



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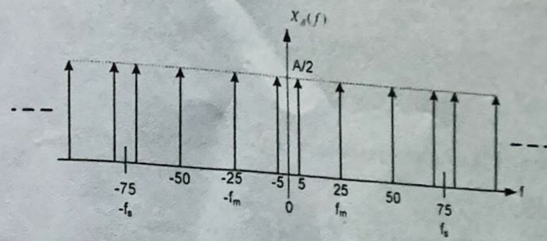


(c)

***Fig.1** (a) Spectrum of $x(t)$. (b) Spectrum of $X_\delta(f)$ for f_s=2f_m.*
(c) Spectrum of $X_\delta(f)$ for f_s>2f_m (i.e., fs=3f_m)

Fig.1(a) is the original spectrum of the signal $X(f)$. When sampled exactly at the Nyquist rate the signals overlap at f = 25 Hz as shown in Fig.1(b). Again when the samples are taken at the rate higher than the Nyquist rate (i.e.,f_s=3f_m), the spectrum is considerably spaced as shown in Fig.1(c).

2.2 Signal Reconstruction

The process of reconstructing a continuous time signal $x(t)$ from its samples is known as interpolation. Fig. 2.2. shows the constructive proof that a signal $x(t)$ band-limited to f_m Hz, can be reconstructed (interpolated) exactly from its samples. This is done by passing the sampled signal through an ideal low-pass filter of bandwidth f_m Hz.

The expression for sampled signal is written as

$$x_\delta(t) = x(t)\,\delta_{T_s}(t) \tag{2.7}$$

As seen from Eq. (2.3), the sampled signal contains a component $(1/T_s)x(t)$ when $n = 0$, and to recover $x(t)$ [or $X(f)$] , the sampled signal given by Eq (2.1) must be sent through an ideal low-pass filter of bandwidth f_m Hz and gain T_s . Such an ideal filter response has the transfer function

$$\boxed{H(f) = T_s \times rect\left(\frac{w}{4\pi f_m}\right)} \quad (2.8)$$

2.2.1 Ideal Reconstruction

To recover the analog signal from its uniform samples, the ideal interpolation filter transfer function found in Eq.(2.8) is shown in Fig.2.2(a). The impulse response of this filter i.e., the inverse Fourier Transform of $H(f)$, is

$$h(t) = F^{-1}\left[T_s \times rect\left(\frac{w}{4\pi f_m}\right)\right] \quad (2.9)$$

$$= 2f_m T_s \text{ sinc } (2\pi f_m t)$$

Assuming that sampling is done at Nyquist rate, then $T_s = \dfrac{1}{2f_m}$

So that, $2f_m T_s = 1$.

Substituting the value of $2f_m T_s$ in Eq.(2.9). We have

$$\boxed{h(t) = \text{sinc}(2\pi f_m t)} \quad (2.10)$$

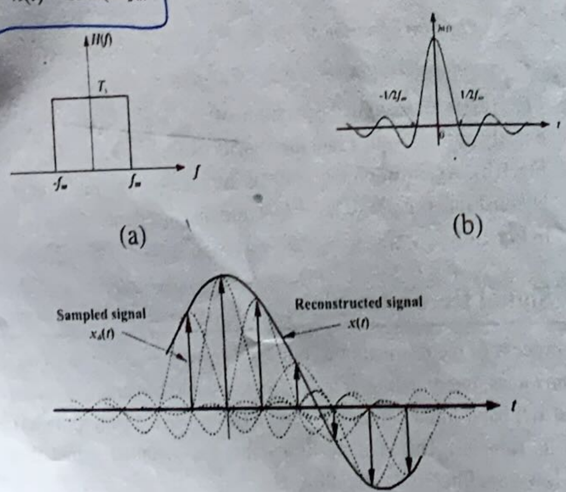


Fig.2.2. (a) Ideal reconstruction filter. (b) Impulse response of ideal reconstruction filter. (c) Reconstructed signal.

The impulse response $h(t)$ given in Eq.(2.10) is shown in Fig.2.2(b). It can be seen from Fig.2.2(b) that $h(t)= 0$ at all Nyquist sampling instants i.e., at $t = \pm n/2f_m$, except at $t = 0$. Now when the sampled signal $x_\delta(t)$ is applied at the input of the filter, the output will be $x(t)$. Each sample in $x_\delta(t)$, being an impulse, produced a sinc pulse of height equal to the strength of the sample. Addition of the sinc pulses produced by all the samples results in $x(t)$. For instant, the k^{th} sample of the input $x_\delta(t)$ is the impulse $x(kT_s)\,\delta(t - kT_s)$. The filter output of this impulse will be $x(kT_s)h(t - kT_s)$. Therefore, the filter output to $x_\delta(t)$ i.e., $x(t)$, may be expressed as a sum given in Eq.(2.11).

$$x(t) = \sum_k x(kT_s)h(t - kT_s) \quad (2.11)$$

$$x(t) = \sum_k x(kT_s) \text{ sinc}[2\pi f_m(t - kT_s)] \quad (2.12)$$

$$x(t) = \sum_k x(kT_s) \text{ sinc}[2\pi f_m t - k\pi)] \quad (2.13)$$

Eq.(2.13) is the interpolation formula, which yields values of $x(t)$ between samples as a weighted sum of all the sample values as shown in Fig.2.2(c).

2.3 Sampling Techniques

In section 2.2, we discussed the sampling technique using impulse train as the sampling function. However, sampling of continuous time signal can be done is several ways. In this section we will discuss different types of sampling techniques which are.

2.3.1 Ideal or instantaneous sampling

Ideal sampling has already been discussed in section 2.2. Ideal sampling uses train of impulse as a sampling function and is also known as instantaneous sampling. Other sampling techniques such as natural sampling and flat top sampling are called practical sampling methods.

2.3.2 Flat top sampling

Flat top sampling is one of the practical sampling methods. Unlike impulse train used in ideal sampling, this method uses train of rectangular pulses with constant width. The top of the rectangular samples remains constant (i.e., flat) and is equal to the instantaneous value of the baseband signal at the start of the sample. Practically, a sample and hold circuit, shown in Fig.2.3, is used to generate the flat top samples. Each flat top samples have duration or width of τ and the sampling rate $f_s = 1/T_s$ as shown in Fig.2.4(c).

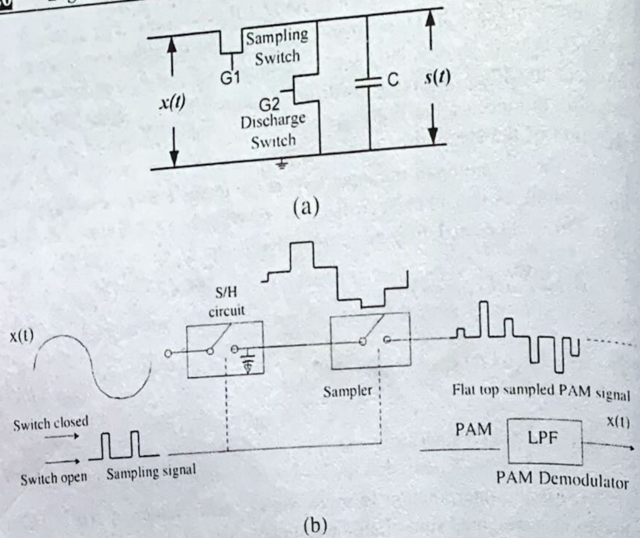


Fig.2.3: (a) Sample and Hold Circuit. (b) Step by step process of flat top sampling

Fig 2.3(b) shows the steps followed in order to obtain flat top sampled signal. Initially, sample and hold circuit is used to sample the input signal at every T_s seconds and hold that sampled voltage for the entire τ seconds. The capacitor charges to the level of the input when the switch is closed and holds that value when the switch opens and until the switch closes again to take a new sample. The result is a staircase signal. This is the input to the sampler. The switch closes for the duration of each sampling signal pulse allowing a portion of each step to become a part of the output. The switch is open for the remainder of each sampling period making the output zero. The resulting output is a flat-top pulse amplitude modulated (PAM) signal.

In the sample and hold circuit shown in Fig.2.3(a), if the switch is closed for more time (pulse width τ is large), input to the holding system is not a single amplitude and will generate a distortion known as aperture effect distortion. Aperture effect distortion can be reduced by reducing the width of the pulses.

Fig.2.4(c) shows the general waveform of flat top samples, in which only the starting of the samples follow the instantaneous value of the baseband signal $x(t)$.The flat top samples $s(t)$ is obtained mathematically from ideal sampled pulses i.e., by convolution of instantaneous samples of $x_\delta(t)$ and a pulse $h(t)$ as shown in Fig.2.4(a) and (b).

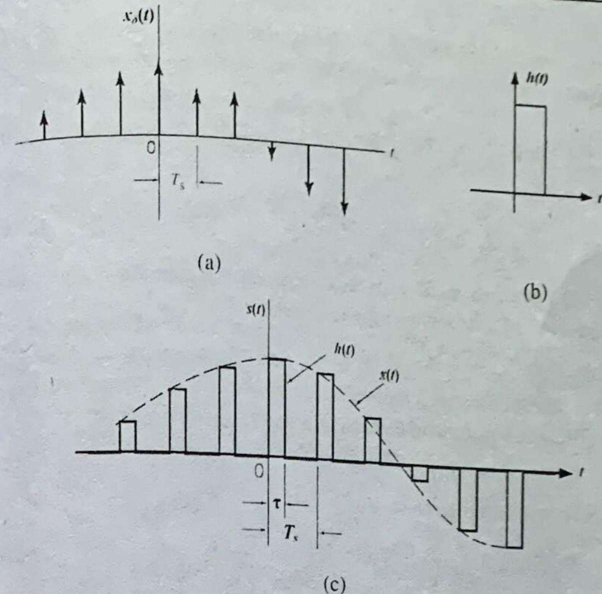


Fig.2.4: Mathematical generation of flat top sampling.

Therefore $s(t)$ will be expressed as

$$s(t) = x_\delta(t) \otimes h(t) \tag{2.14}$$

Where $x_\delta(t)$ is the ideally sampled signal, shown in Fig.2.4(a) and is represented by

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t - nT_s) \tag{2.15}$$

Now, from the property of delta function, we know that for any function $f(t)$

$$f(t) = f(t) \otimes \delta(t) \tag{2.16}$$

Also, according to shifting property of delta function, we know that

$$\int_{-\infty}^{\infty} f(t)\,\delta(t - t_o) = f(t_o) \tag{2.17}$$

Substituting Eq.(2.15) into Eq.(2.14) an using the property of Eq.(2.16) and Eq.(2.17) we get,

$$s(t) = \int_{-\infty}^{\infty} x_\delta(\tau)\,h(t - \tau)\,d\tau \tag{2.18}$$

$$s(t) = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(\tau - nT_s)\,h(t-\tau)\,d\tau \tag{2.19}$$

$$s(t) = \sum_{n=-\infty}^{\infty} x(nTs) \int_{-\infty}^{\infty} \delta(\tau - nT_s)\,h(t-\tau)d\tau \tag{2.20}$$

$$s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,h(\tau - nT_s) \tag{2.21}$$

Eq.(2.21) represents value of $s(t)$ in terms of sampled value $x(nT_s)$ and function $h(\tau - nT_s)$ for flat top sample signal.

Now, again from Eq.(2.14), we have (2.22)

Taking Fourier transform of both sides of Eq.(2.22), we get

$$S(f) = X_\delta(f).H(f) \tag{2.23}$$

We know that $X_\delta(f)$ is given as

$$X_\delta(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s) \tag{2.24}$$

Therefore,

$$S(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s).H(f) \tag{2.25}$$

Aperature Effect

Eq.(2.25) shows that the signal $s(t)$ is obtained by convolution of the signal $x_\delta(t)$ with the filter impulse response $h(t)$, which is a rectangular pulse of unit amplitude and duration τ, as shown in Fig 2.4(b). Each sample of $x(t)$ [i.e., $x_\delta(t)$] is convolved with this rectangular pulse.

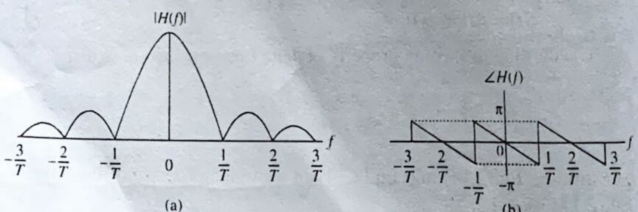


Fig.2.5: (a). Magnitude response $|H(f)|$ of transfer function $H(f)$. (b) Phase response $\angle H(f)$.

Fig.2.5 shows the magnitude and phase spectrum of rectangular pulse $h(t)$. After taking Fourier transform, it is expressed in terms of sinc function as

$$H(f) = F[h(t)] = \tau\ \text{sinc}\ (f\tau)e^{j\pi f t} \tag{2.26}$$

Now, substituting Eq.(2.26) in Eq.(2.25) we obtain

$$S(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s)\ .\ H(f)$$

$$S(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s)\ .\ t\ \text{sinc}\ (f\tau)\ e^{j\pi f t} \tag{2.27}$$

Hence from Eq.(2.27), it may be observed that by using flat top samples an amplitude distortion as well as a delay of $\tau/2$ is introduced in the reconstructed signal $x(t)$ from $s(t)$. The high frequency roll-off of $H(f)$ acts like a low pass filter and thus attenuates the high frequencies of $x(t)$. This type of effect is known as aperture effect shown in Fig.2.6.

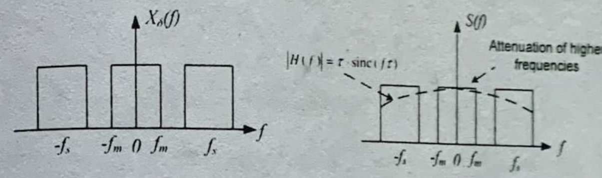


Fig.2.6: Aperature effect

Now as the duration τ of the pulse increases, the aperture effect is more prominent. To correct this distortion, an equalizer is connected in cascade with the low-pass reconstruction filter. As shown in Fig.2.7, the receiver contains a low-pass reconstruction filter with cut off frequency slightly higher than the maximum frequency present in the message signal. The equalizer compensates for the aperture effect. It also compensates for the attenuation by a low-pass reconstruction filter.

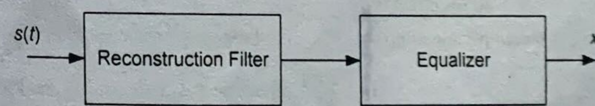


Fig.2.7: System for recovering input signal x(t) from s(t)

The transform function of the equalizer is expressed as

$$H_{eq} = \frac{K.e^{-j2\pi f t_d}}{H(f)} \tag{2.28}$$

Here 't_d' is known as the delay introduced by low-pass filter which is equal to $\tau/2$.

Therefore,

$$H_{eq} = \frac{K.e^{-j\pi f\tau}}{\tau.\text{sinc}(f\tau)\, e^{-j\pi f\tau}} \tag{2.29}$$

$$\text{Or,} \qquad H_{eq} = \frac{K}{\tau.\text{sinc}(f\tau)} \tag{2.30}$$

which is the transfer function of an equalizer.

2.3.3 Natural Sampling

Natural sampling is a practical method. As in flat top sampling, this sampling technique also considers that the pulse has a finite width equal to τ. Let us consider an analog continuous-time signal $x(t)$ which is applied to the input of a switching circuit generating sampling function $c(t)$ which is a train of periodic pulses of amplitude A, width τ and frequency equal to f_s Hz. Here, it is assumed that f_s is higher than Nyquist rate such that sampling theorem is satisfied. Fig.2.8 shows a functional diagram of a natural sampler. With the help of this natural sampler, a sampled signal $x_s(t)$ is obtained by multiplication of sampling function $c(t)$ and input signal $x(t)$.

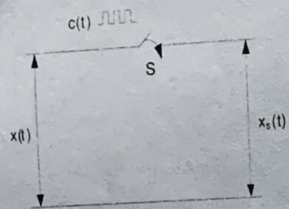
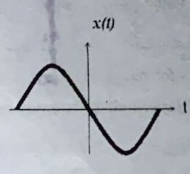


Fig.2.8: Functional diagram of Natural sampling

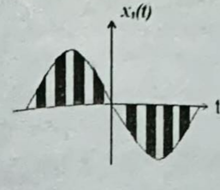
According to Fig.2.8, when the switch is closed the sampling function $c(t)$ goes high and signal $x_s(t)$ follows the amplitude of the input signal $x(t)$ and when the switch is closed the signal $x_s(t)$ goes to zero. i.e.,

$$x_s(t) = \begin{cases} x(t) \text{ for } c(t) = A \\ 0 \text{ for } c(t) = 0 \end{cases} \tag{2.31}$$

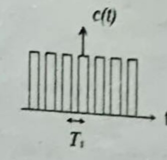
where A is the amplitude of $c(t)$



(a)



(b)



(c)

Fig.2.9: Mathematical generation of Natural sampling.

The waveform of the signals $x(t)$, $x_s(t)$, and $c(t)$ is shown in Fig.2.9(a), (b) and (c) respectively. Now the sampled signal $x_s(t)$ may also be described mathematically as

$$x_s(t) = c(t)\,.\,x(t) \tag{2.32}$$

However, $c(t)$ may be expressed in the form of a complex exponential Fourier series for any periodic wave as

$$c(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi nt/T_o} \tag{2.33}$$

For the periodic pulse train $c(t)$ we have, $f_s = f_o = \frac{1}{T_o} = \frac{1}{T_s}$,

where f_s and T_s is the frequency and period of $c(t)$ respectively. Now, Eq.(2.33) can be written as,

$$c(t) = \sum_{n=-\infty}^{\infty} c_n\, e^{j2\pi nf_s t} \tag{2.34}$$

Since $c(t)$ is a rectangular pulse train, therefore c_n of this wave is a sinc function and is expressed as

$$c_n = \frac{\tau A}{T_s} \text{sinc}\,(f_n\,.\,\tau) \tag{2.35}$$

Where, f_n = harmonic frequency

But here $f_n = nf_s$, or $f_n = \frac{n}{T_s} = \frac{n}{T_o} = nf_o$

Thus by substituting Eq.(2.35) to Eq.(2.34) the Fourier series representation of $c(t)$ will be given as

$$c(t) = \frac{\tau A}{T_s} \sum_{n=-\infty}^{\infty} \text{sinc}\,(f_n\,.\,\tau)\, e^{j2\pi nf_s t} \tag{2.36}$$

This is the required time-domain representation of the naturally sampled signal $x_s(t)$, let us take Fourier transform as

$$X_s(f) = F[x_\delta(t)]$$

$$X_s(f) = \frac{\tau A}{T_s}\sum_{n=-\infty}^{\infty} \text{sinc}\,(f_n.\tau)\; F[e^{j2\pi nf_s t}\,.\,x(t)] \qquad (2.37)$$

Recalling the frequency-shifting property of Fourier transform which states that

$$e^{j2\pi f_s nt}\,.\,x(t) \leftrightarrow X(f-nf_s)$$

Therefore,

$$X_s(f) = \frac{\tau A}{T_s}\sum_{n=-\infty}^{\infty} \text{sinc}\,(f_n\,.\,\tau)\; X(f-nf_s) \qquad (2.38)$$

Since $f_n = nf_s$ = harmonic frequency

Therefore,

$$X_s(f) = \frac{\tau A}{T_s}\sum_{n=-\infty}^{\infty} \text{sinc}\,(nf_s.\tau)\; X(f-nf_s) \qquad (2.39)$$

Eq.(2.39) shows that the spectrum of $x(t)$ i.e., $X(f)$ are periodic in f_s and are weighted by the sinc function. The relation between the spectra $X(f)$ and $X_s(f)$ is illustrated in Fig.2.10. We see that the effect of the finite duration of the sampling pulses is to multiply the n[th] lobe of the spectrum $X_s(f)$ by $\frac{\tau A}{T}$ $\sum_{n=-\infty}^{\infty}$ sinc $(nf_s.\tau)$. The original signal $x(t)$ can be recovered from $x_s(t)$ with no distortion by passing $x_s(t)$ through an ideal low-pass filter whose bandwidth B, which satisfies the condition $f_m < B < f_s - f_m$.

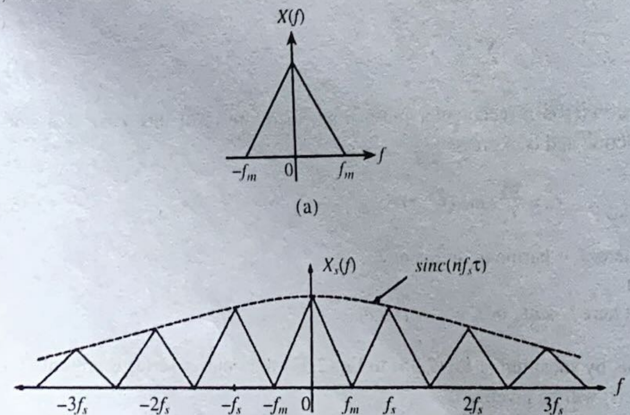


Fig.2.10: (a) Spectrum of continuous time signal. (b) Spectrum of naturally sampled signal.

2.4 Practical considerations

There are various practical issues that arise during conversion of an analog signal to digital form using sampling. Some of the issues occurring during practical implementations are

1. Unrealizable Ideal Reconstruction Filters

If the signal is sampled at the Nyquist rate (i.e., $f_s = 2f_m$), the spectrum $X_\delta(f)$ consists of repetitions of $X(f)$ without any gap in between successive cycles, as shown in Fig. 2.11(a). To recover $x(t)$ from , we need to pass the sampled signal through an ideal low-pass filter. However, ideal low-pass filter is unrealizable in practice. Thus, filter with certain amount of roll-off is used. In doing so, during recovery the portion of the side band of the nearby spectrum in $X_\delta(f)$ will also be filtered out along with the required message signal spectrum.

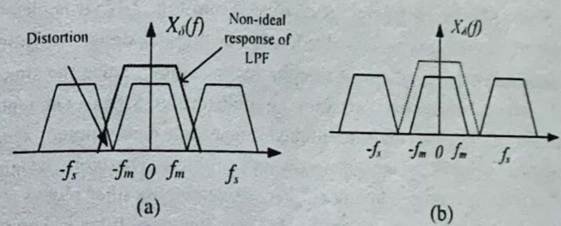


Fig.2.11: *(a) Distortion due to non-ideal reconstruction filter.*

(b) Sampling at rate higher than Nyquist to compensate for non-ideal reconstruction filter.

One of the ways to minimize this effect is to sample the signal at a rate higher than the Nyquist rate (i.e., $f_s > 2f_m$). This results in $X_\delta(f)$, consisting of $X(f)$ with finite band gap between successive cycles, as shown in Fig. 2.11(b). We can now recover $X(f)$ from $X_\delta(f)$ by using a low-pass filter with a gradual cutoff characteristic as shown in Fig. 2.11(b) by the dotted line.

2. Aliasing

According to Nyquist criteria the original signal can be recovered from its samples only if signal is sampled at a rate greater or equal to Nyquist rate (i.e.,$f_s \geq 2f_m$). If the signal is sampled at a rate lower than Nyquist rate (i.e., $f_s < 2f_m$), then the successive cycles of the spectrum $X_\delta(f)$ overlap with each other as shown in Fig. 2.12.

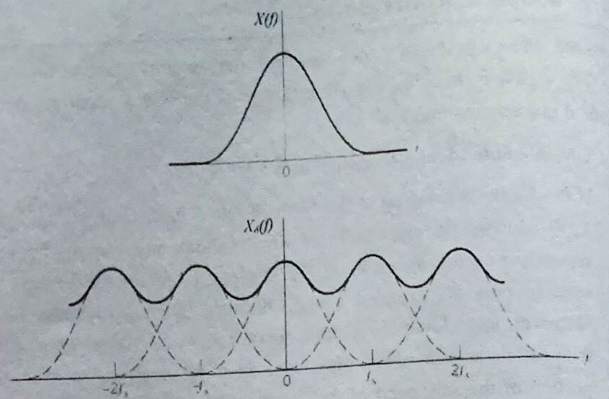


Fig. 2.12: Aliasing Effect

Hence, the signal is under-sampled (i.e., $f_s < 2f_m$), resulting in some amount of spectral folding or aliasing. From Fig. 2.12, it is clear that due to aliasing it is not possible to recover original signal $x(t)$ from the sampled signal $x_\delta(t)$ by just using a low-pass filter, since the signal in the overlap region overlap, resulting in distortion. Signals encountered in real life are usually time limited rather than band-limited. Thus to decide a sampling frequency is always a problem. Therefore, a signal is first passed through a low-pass filter. This low-pass filter blocks all the frequencies which are above f_m Hz. This low-pass filter is known as anti-aliasing or pre-aliasing filter which band-limiting the signal to f_m Hz, making it easier to decide sampling frequency.

2.5 Subsampling Theorem

Continuous band-pass signal that is centered about some frequency other than zero Hz can be sampled using a technique known as band-pass sampling. Band-pass sampling reduces the speed requirement of A/D converters compared to low-pass sampling.

For example, consider sampling the band-limited signal shown in Fig.2.13(a) centered at f_c = 20 MHz, and bandwidth B = 5 MHz. We use the term band-pass sampling for the process of sampling continuous signals whose center frequencies have been translated up from zero Hz. Band-pass sampling is also known by IF sampling, harmonic sampling, sub-Nyquist sampling, and under sampling.

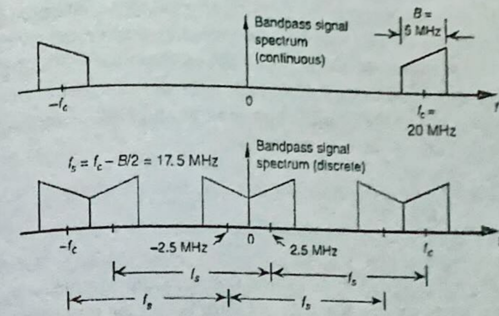


Fig.2.13: Band-pass signal sampling (a) original continuous signal spectrum. (b) sampled signal spectrum replications when sample rate is 17.5 MHz.

In band-pass sampling, we consider the signal's bandwidth rather than its highest frequency component. The negative frequency portion of the signal, centered at $-f_c$, is the mirror image of the positive frequency portion as for real signals. The band-pass signal's highest frequency component is 22.5 MHz (i.e., f_c+B/2). According to the Nyquist criterion (sampling at twice the highest frequency content of the signal) implies that the sampling frequency must be a minimum of 45 MHz (i.e., 2 × 22.5). Consider the effect if the sample rate is 17.5 MHz (i.e., $f_c - B/2$) shown in Fig.2.13(b). Note that the original spectral components remain located at $\pm f_c$, and spectral replications are located exactly at base-band, i.e., joint up against each other at zero Hz. Fig.2.13(b) shows that sampling at 45 MHz was unnecessary to avoid aliasing. Instead spectral replicating effects can be used to our advantage.

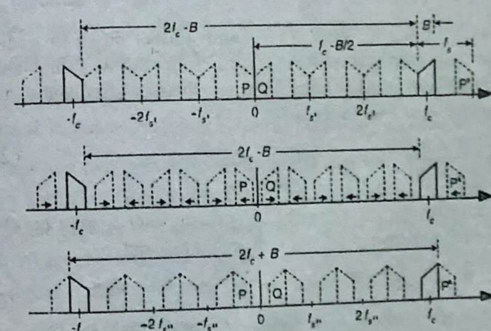


Fig.2.14: Band-pass sampling frequency limits (a) sample rate $f_s=(2f_c-B)/6$. (b) sample rate is less than f'_s. (c) minimum sample rate $f''_s < f'_s$.

Let us consider a signal in Fig.2.13(a) and its sampled value spectrum is that shown in Fig.2.14(a). We can sample continuous signal at a rate, say f'_s. Hz, so the spectral replications of the positive and negative bands, P and Q, just join up against each other exactly at zero Hz. This situation, depicted in Fig 2.14(a), is similar of Fig.2.13(b). With an arbitrary number of replications, suppose m, in the range of $2f_c - B$, we see that

$$mf_s' = 2f_c - B \tag{2.40}$$

$$f_s' = \frac{2f_c - B}{m} \tag{2.41}$$

Fig. 2.14(a) is plotted for m = 6. In this figure, if the sample rate f'_s is increased, the original spectra at $\pm f_c$ do not shift, but all the replications will shift. At zero Hz, the P band will shift to the right, and the Q band will shift to the left. These replications will overlap and aliasing occurs. Thus, from Eq.(2.41), for an arbitrary m, there is a frequency that the sample rate must not exceed, or

$$f_s' \leq \frac{2f_c - B}{m} \tag{2.42}$$

If we reduce the sample rate below the f_s. value shown in Fig 2.14(a), the spacing between replications will decrease in the direction of the arrows in Fig 2.14(b). Again, the original spectra do not shift when the sample rate is changed. At some new sample rate f_s'' where $f_s'' < f_s''$, the replication P' will just join up against the positive original spectrum centered at f_c as shown in Fig 2.14(c). In this condition, we know that

$$(m+1)fs'' = 2f_c - B \tag{2.43}$$

$$f_s'' = \frac{2f_c - B}{(m+1)} \tag{2.44}$$

If f_s'' is decreased further in value, P' will shift further down in frequency and start to overlap with the positive original spectrum at f_c and aliasing occurs. Therefore, from Eq.(2.44) and for $m + 1$, there is a frequency that the sample rate must always exceed, or

$$f_s'' \geq \frac{2f_c - B}{(m+1)} \tag{2.45}$$

We can now combine Eq. (2.42) and (2.43) to say that f_s may be chosen anywhere in the range between f_s'' and f_s' to avoid aliasing, or

$$\frac{2f_c - B}{m} \geq f_s \geq \frac{2f_c - B}{(m+1)} \tag{2.46}$$

where m is an arbitrary, positive integer ensuring that $f_s > 2B$ (For this type of periodic sampling of real signals, known as real or first-order sampling, the Nyquist criterion $f_s > 2B$ must still be satisfied.)

2.6 Some applications of the sampling theorem

The sampling theorem is very important in signal analysis, processing, and transmission because it allows us to replace a continuous time signal by a discrete sequence of numbers. Various application of sampling is thus highlighted as

1. Due to sampling the application of digital filters can be implemented.
2. Sampling makes it possible to implement various pulse modulation schemes. The continuous time signal $x(t)$ is sampled, and sample values are used to modify amplitude, width or position of a periodic pulse train in proportion to the sample value of the signal $x(t)$ resulting in Pulse Amplitude Modulation (PAM), Pulse Width Modulation (PWM) and Pulse Position Modulation (PPM) respectively which will be studied in details in later chapters.
3. Due to sampling, one of the most important pulse modulation scheme used is Pulse Code Modulation (PCM)
4. Frequency Division Multiplexing (FDM) and Time Division Multiplexing (TDM) can be used.

Previous Exam Questions

1. With the mathematical derivation show that original band limited signal reconstructed from its samples taken at Nyquist Rate. What is Aliasing Effect and how it can be minimized.
2. State Nyquist Sampling Theory? Why Sub-Sampling is done in Digital Communication? Explain the effects of deviation that arises because of practical Sampling as compared with Ideal Sampling.
3. A band-pass signal with the spectrum in the range of 80-115 KHz is to be digitized, calculate minimum Sampling Frequency required for the signal.
4. Explain any two issues (considerations) that have to be taken care of while Sampling Continuous Time Signals. A earthquake data recorder traces the signals that changes its polarity a maximum of thirty times each 10 sec. Estimate the Nyquist Sampling frequency and data rate if this signal is to be converted into a 10 bit PCM Signal.
5. State and explain Nyquist Sampling Theorem. A signal $x(t)$= $10\cos(2\pi 2000t)$+ $4\cos(2\pi 3000t)$ is to be sampled and quantized using 256 levels, calculate the minimum sampling frequency and sampling period.

6. A band limited band pass signal centered at 40MHz and having total bandwidth of 60KHz is to be sampled. calculate the minimum sampling frequency.

7. What are the Aliasing and Aperture Effects in sampling and how they can be minimized?

8. A band pass signal centered at 50MHz and having bandwidth of 40 KHz is to be sampled. for converting into a PCM signal. calculate the optimum sampling frequency.

9. State and explain with examples Sampling Theorem for band limited and band pass Signals.

10. A signal $x(t)=\text{sinc}(5\pi t)$ is sampled (using uniformly spaced impulses) at a rate of 10 Hz.

    i. Sketch the sampled of signal (not to scale);

    ii. Sketch the spectrum of the sampled signal for the range $|f|<30$ Hz.

    iii. Explain whether you can recover the signal $x(t)$ from the sampled signal.

11. A signal $x(t)=A\cos(1000\pi t)\cos(5000\pi t)$ is ideally sampled at Nyquist rate. Find the sampling frequency. draw the spectra of $x(t)$ and sampled signal.

12. State and explain Nyquist-Kotelnikov sampling theorem with time domain and frequency domain analysis.

13. What are the practical factors to be considered while sampling? Explain. if two band-limited signals $X_1(t)$ and $X_2(t)$ having bandwidths of W_1 and W_2 Hertz respectively. estimate the maximum sampling interval required for the signal given by $Y(t)= X_1(t)\, X_2(t)$.

14. A signal $x(t)=3\cos(40\pi t)+10\cos(80\pi t)+20\sin(50\pi t)$ is ideally sampled at Nyquist rate. Find the Nyquist Rate, Nyquist interval. Draw the spectra of $x(t)$ and sampled signal.

15. A signal $x(t)=10\cos(1000\pi t)+6\cos(2000\pi t)$ is to be sampled with sampling period of 1 ms. The sampled signal is then passed through an ideal LPF with cutoff frequency of 4KHz.

    (i) Draw the magnitude spectrum of the sampled signal(up to the frequency components centered around twice the sampling frequency)

    (ii) Determine the frequency components at the output of LPF.

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