Chapter 7.

NOISE PERFORMANCE OF BAND-PASS (MODULATED) COMMUNICATIONS SYSTEMS.

7.1 Effect of noise in envelop and synchronous demodulation of DSB-FC AM, expression for gain parameter, threshold effect in non-linear demodulation of AM.

As the most of the communication systems are bandpass systems, we represent the noise as narrowband bandpass random signal and express mathematically as,

$$n(t) = n_I(t)\cos 2\pi f_c t + n_Q(t)\sin 2\pi f_c t$$

where,

$n_I(t) \rightarrow$ in phase component of noise

$n_Q(t) \rightarrow$ quadrature noise component.

and are baseband signals such that,

$\overline{n^2(t)} = \overline{n_I^2(t)} = \overline{n_Q^2(t)}$ are powers.

Also, the power spectral density, for,

$n(t) = \frac{N_0}{2}$ and i.e. $S_n(f) = \frac{N_0}{2}$ and

for $n_I(t) = n_Q(t) = N_0$.

i.e. $S_{nI}(f) = S_{nQ}(f) = N_0$

Performance evaluation method.

To determine the performance of noise in communication, we need to find the ratio of SNR at output of demodulator to the SNR at input of demodulator. the ratio is known as gain parameter or figure of merit.

i.e.

$$\gamma = \frac{SNR_o}{SNR_i}$$

Receiver Model.

$s(t)$ (AM signal) $\xrightarrow{+}$ Σ (+ $w(t)$) $\rightarrow$ BPF $\xrightarrow{x_i(t)}$ Demodulator $\rightarrow$ o/p signal.

$x_i(t) = s(t) + n(t)$

$w(t) =$ white noise. $\quad w(t) \rightarrow$ BPF $\rightarrow n(t)$

i) For DSB-FC [i.e. DSB with full carrier]

$$x_i(t) = [A_c + x(t)] \cos 2\pi f_c t + n_i(t)$$

Such that signal power at input is,

$$P_{si} = \frac{A_c^2}{2} + \frac{\overline{x^2(t)}}{2} \quad \text{where, } \overline{x^2(t)} = \text{mean square value.}$$

And,

noise power, $P_{Ni} = \overline{n_i^2(t)}$

$$\therefore \ SNR_i = \frac{(A_c^2 + \overline{x^2(t)})/2}{\overline{n_i^2(t)}}$$

Now let us find P_{so} and P_{No} for the cases when the demodulator is

a) Envelope type

b) Synchronous type.

a. Envelope detection

We have,

$$x_i(t) = [A_c + x(t)] \cos 2\pi f_c t + n(t)$$

$$= [A_c + x(t)] \cos 2\pi f_c t + n_I(t) \cos 2\pi f_c t + n_Q(t) \sin 2\pi f_c t$$

$$\sim x_i(t) = [A_c + x(t) + n_I(t)] \cos 2\pi f_c t + n_Q(t) \sin 2\pi f_c t$$

$$= V(t) \cos [2\pi f_c t + \phi(t)]$$

where,

$$V(t) = \sqrt{[A_c + x(t) + n_I(t)]^2 + n_Q^2(t)}$$

$$\& \quad \phi(t) = \tan^{-1}\left[\frac{n_Q(t)}{A_c + x(t) + n_I(t)}\right]$$

Now,

if the average carrier power is large as compared to the average noise power i.e.

$$A_c + x(t) >> n(t), \quad \text{then,}$$

$$[A_c + x(t) + n_I(t)]^2 >> n_Q^2(t)$$

Therefore,

$$V(t) \cong \sqrt{[A_c + x(t) + n_I(t)]^2}$$

$$= A_c + x(t) + n_I(t)$$

and, $\phi(t) \cong$ ~~that~~ 0.

Then the output of an ideal envelope detector

$$V(t) = A_c + x(t) + n_I(t)$$

From $v(t)$, A_c is the DC component which is filtered out by LPF. Finally the noise power and the signal power at the output of demodulator will be,

$$P_{SO} = \overline{x^2(t)}$$

$$P_{NO} = \overline{n_I^2(t)}$$

$\therefore$

$$SNR_0 = \frac{P_{SO}}{P_{NO}} = \frac{\overline{x^2(t)}}{\overline{n_I^2(t)}}$$

And the detection gain,

$$\gamma = \frac{SNR_0}{SNR_i} = \frac{\dfrac{\overline{x^2(t)}}{\overline{n_I^2(t)}}}{\dfrac{A_c^2 + \overline{x^2(t)}}{2\,\overline{n^2(t)}}}$$

But, $\overline{n^2(t)} = \overline{n_I^2(t)}$,

Therefore,

$$\gamma = \frac{2\,\overline{x^2(t)}}{A_c^2 + \overline{x^2(t)}}$$

Such that 'γ' increases as A_c is decreased.

Also, $\gamma = 2 \cdot \eta$ where $\eta = \dfrac{\overline{x^2(t)}}{A_c^2 + \overline{x^2(t)}}$ is the efficiency of DSB-FC AM.

Now, for 100% modulation index and a sinusoidal modulating signal the value of $\eta = 1/3$.

Therefore the maximum system gain provided by the DSB FC is,

$$\gamma = 2 \times \frac{1}{3} < 1.$$

Therefore even for 100% modulation, the demodulator gain is less than 1.

b) Synchronous detection.

In synchronous detection, the received AM signal is multiplied by $\cos 2\pi f_c t$ and then passed to LPF. Considering that the synchronization is ideal and that the transfer function of LPF is flat within the message bandwidth and zero outside it the we get,

$$x_{dem}(t)\big|_{LPF} = x_i(t) \cdot \cos 2\pi f_c t$$

$$= \frac{1}{2} x(t) + \frac{1}{2} n_I(t)$$

Therefore, $P_{SO} = \frac{\overline{x^2(t)}}{4}$ and $P_{NO} = \frac{\overline{n_I^2(t)}}{4}$

and, $SNR_O = \frac{\overline{x^2(t)}}{\overline{n_I^2(t)}}$

So,

$$\gamma = \frac{SNR_O}{SNR_i}$$

$$= \frac{\overline{x^2(t)}}{\overline{n_I^2(t)}} \times \frac{2\overline{n^2(t)}}{A_c^2 + \overline{x^2(t)}}$$

$$\gamma = \frac{2\overline{x^2(t)}}{A_c^2 + x^2(t)}.$$

or $\gamma = 2\eta$

Thus the figure of merit is same for envelope detection as well as synchronous detection.

7.2 (ii) (i) DSB-SC

For DSB-SC, the detection method is always synchronous.

Now, the DSB-SC signal and the noise input to the demodulator can be represented as,

$$x_i(t) = x(t)\cos 2\pi f_c t + n(t)$$

$$\therefore \quad P_{si} = \frac{\overline{x^2(t)}}{2} \quad \& \quad P_{Ni} = \overline{n^2(t)}$$

Now since the demodulator uses synchronous detection, we get the useful signal component as $\frac{x(t)}{2}$. Therefore, we have,

$$x_{dem}(t)\Big|_{LPF} = \frac{x(t)}{2} + \frac{n_I(t)}{2}$$

And,

$$P_{SO} = \frac{\overline{x^2(t)}}{4} \quad \& \quad P_{NO} = \frac{\overline{n_I^2(t)}}{4}$$

$$\therefore \gamma = \frac{SNR_O}{SNR_i} = \frac{\overline{x^2(t)}/4}{\overline{n_I^2(t)}/4} \times \frac{\overline{n^2(t)}}{\overline{x^2(t)}} \times 2 = 2 \quad \left[\because \overline{n^2(t)} = \overline{n_I^2(t)}\right]$$

The gain provided by DSB-SC equalling to '2' comes from the fact that the input signal consisted of two side bands and noise was also distributed in both side bands. After demodulation, the noise bandwidth is reduced to half and hence the noise power is reduced 4 times whereas the signal power is reduced 2 times only.

ii) SSB.

Now if we take the received signal as SSB, then the input to the demodulator can be written as,

$$x_i(t) = x(t)\cos 2\pi f_c t \pm \hat{x}(t)\sin 2\pi f_c t + n(t)$$

So,

$$P_{si} = \frac{\overline{x^2(t)}}{2} + \frac{\overline{\hat{x}^2(t)}}{2} = \overline{x^2(t)}$$

Because, $\overline{x^2(t)} = \overline{\hat{x}^2(t)}$

And

$$P_{Ni} = \overline{n^2(t)}$$

After synchronous demodulation, the output will be

$$x_o(t) = [x(t) \text{ or } \hat{x}(t)] + n_c(t)$$

$$\therefore \quad P_{So} = \overline{x^2(t)}$$

$$\& \quad P_{No} = \overline{n_c^2(t)} = \overline{n_i^2(t)}$$

Therefore,

$$\gamma = \frac{SNR_o}{SNR_i}$$

$$= \frac{\overline{x^2(t)}}{\overline{n_i^2(t)}} \times \frac{\overline{n^2(t)}}{\overline{x^2(t)}} = 1.$$

It may seem that DSB-SC is superior to SSB in terms of noise performance. But it is not true as in case of DSB-SC, the input noise power is twice that of SSB. Therefore performance of DSB-SC & SSB are identical from noise improvement point of view.

Threshold effect in non-linear demodulation of AM.

We have for AM,

$$x_i(t) = V(t)\cos[2\pi f_c t + \phi(t)]$$

where,

$$V(t) = \sqrt{[A_c + x(t) + n_I(t)]^2 + n_Q^2(t)}$$ = envelope of $x_i(t)$.

and

$$\phi(t) = \tan^{-1}\left[\frac{n_Q(t)}{A_c + x(t) + n_I(t)}\right]$$

Now if $n(t) >> A_c + x(t)$ i.e. noise level is comparatively higher than signal level.

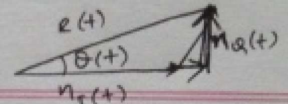
∼ We may deduce that,

$$n_I(t), n_Q(t) >> [A_c + x(t)]$$.

Now, $V(t)$ can be expanded as,

$$V(t) = \sqrt{[A_c + x(t)]^2 + 2[A_c + x(t)]\cdot n_I(t) + n_I^2(t) + n_Q^2(t)}$$

∼ $$V(t) \triangleq \sqrt{n_I^2(t) + n_Q^2(t) + 2[A_c + x(t)]n_I(t)}$$



$$\Rightarrow \frac{n_I(t)}{R(t)} = \cos\theta$$

∼ $$V(t) \triangleq \sqrt{R^2(t) + 2[A_c + x(t)]\cdot n_I(t)}$$

∼ $$V(t) \simeq R(t)\sqrt{1 + \frac{2[A_c + x(t)]\cos\theta(t)}{R(t)}}$$

where, $$R(t) = \sqrt{n_I^2(t) + n_Q^2(t)}$$

$$\theta(t) = \tan^{-1}\left[\frac{n_Q(t)}{n_I(t)}\right]$$

Now,

since $R(t) >> [A_c + x(t)]$,

$$V(t) \simeq R(t)\left[1 + \frac{A_c + x(t)}{R(t)}\cos\theta(t)\right]$$

$$\therefore V(t) = R(t) + [A_c + x(t)]\cos\theta(t).$$

The above equation indicates that the envelope doesnot contain independent message signal $x(t)$ but is rather multiplied by noise term $\cos\theta(t)$.

Thus this loss of message signal in the output of envelope detector due to low carrier to noise ratio (noise >> signal) is called as threshold effect. The threshold effect deteriorates the output SNR more rapidly than the input SNR, when the input SNR is below threshold level.

The 'threshold' can thus be termed as the value of carrier to noise ratio below which the performance of detector deteriorates much rapidly than the proportion decided by carrier to noise ratio.

Alternatively,

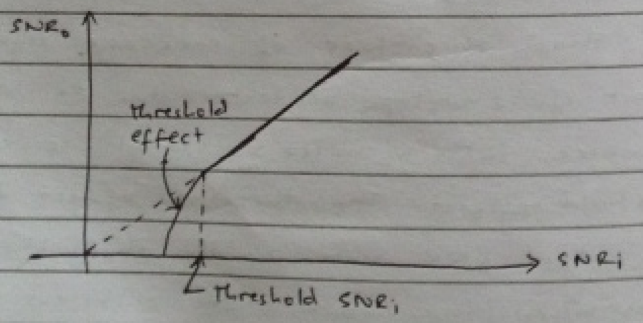
if $\frac{A_c + x(t)}{n(t)} << 1$, $\left[n(t) = n_I(t) \cos 2\pi f_c t + n_Q(t) \sin 2\pi f_c t \right]$

we can represent $n(t)$ ~~as~~. in phasor as,

$$n(t) = r(t) \cos[2\pi f_c t + \theta(t)] \quad \left[r(t) = \sqrt{n_I^2(t) + n_Q^2(t)} \right]$$

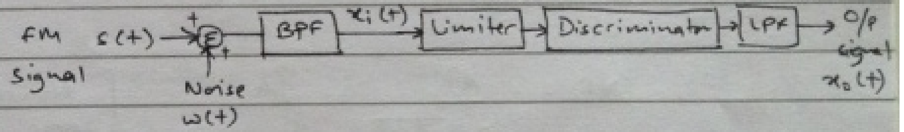
Now, since $n(t)$ is dominant signal, the approximate value of detector output $y(t)$ is given as,

$$y(t) \cong r(t) + A_c \cos\theta(t) + x(t) \cos\theta(t).$$



7.3 Effect of noise (gain parameter) for non-coherent (limiter - discriminator envelop detector) demodulation of FM.

The receiver model for FM demodulation can be shown as,



For, FM,

$$s(t) = A_c \cos[2\pi f_c t + \phi(t)]$$

where, $\phi(t) = 2\pi k_f \int_0^t x(t)\,dt$

k_f = frequency sensitivity.

Now,

$$x_i(t) = s(t) + n(t)$$

where, $n(t) = n_I(t) \cos 2\pi f_c t + n_Q(t) \sin 2\pi f_c t$

$$= R(t) \cos[2\pi f_c t + \theta(t)]$$

where,

$$R(t) = \sqrt{n_I^2(t) + n_Q^2(t)}$$

&

$$\theta(t) = \tan^{-1}\left[\frac{n_Q(t)}{n_I(t)}\right]$$

$$x_i(t) = A_c \cos[2\pi f_c t + \phi(t)] + n(t)$$

$$\therefore P_{si} = \frac{A_c^2}{2}$$

$$\& \; P_{ni} = \overline{n^2(t)}$$

$$\therefore SNR_i = \frac{A_c^2}{2 \cdot \overline{n^2(t)}} = \frac{A_c^2}{4 N_0 \cdot f_m} \quad \left[\begin{array}{l} \because \text{Noise power} = N_0 \cdot BW \\ = N_0 \cdot 2f_m \end{array}\right]$$

Now, to calculate output powers let us consider

$n(t) = 0$ i.e. $n_I(t) \;\&\; n_Q(t) = 0$.

So, $x_i(t) = s(t) = A_c \cos[2\pi f_c t + \phi(t)]$

or $$\frac{d x_i(t)}{dt} = \frac{d}{dt} A_c \cos[2\pi f_c t + \phi(t)]$$

$$= A_c \left\{ \left[-\sin[2\pi f_c t + \phi(t)]\right] \cdot \left[2\pi f_c + \frac{d\phi(t)}{dt}\right] \right\}$$

Now, $\phi(t) = 2\pi k_f \int_0^t x(t)\, dt$

$$\therefore \frac{d\phi(t)}{dt} = 2\pi k_f \cdot x(t)$$

Therefore, the output of discriminator is,

$$\frac{d x_i(t)}{dt} = \underbrace{A_c\,[2\pi f_c + 2\pi k_f \; x(t)]}_{\text{Envelope}} \, [-\sin(2\pi f_c t + \phi(t))]$$

When this output is passed through a LPF that removes DC components as well as components centered around $2\pi f_c$, then, the remainder of envelope,

$$x_o(t)\Big|_{LPF} = A_c \cdot 2\pi k_f \; x(t)$$

$$\therefore P_{so} = A_c^2 \, 4\pi^2 k_f^2 \, \overline{x^2(t)}$$

Now let us consider $x(t) = 0$,

i.e.

$$x_i(t) = A_c \cos 2\pi f_c t + n(t)$$

$$= A_c \cos 2\pi f_c t + n_I \cos 2\pi f_c t + n_Q \sin 2\pi f_c t$$

$$= (A_c + n_I) \cos 2\pi f_c t + n_Q \sin 2\pi f_c t$$

$$= R_n(t) \cos[2\pi f_c t + \theta(t)]$$

where,

$$R_n(t) = \sqrt{(A_c + n_I)^2 + n_Q^2}$$

$$\theta(t) = \tan^{-1}\left[\frac{n_Q}{A_c + n_I}\right]$$

Now,

for $A_c >> n_I(t)$

$$\theta(t) \simeq \tan^{-1}\left[\frac{n_Q(t)}{A_c}\right] \simeq \frac{n_Q(t)}{A_c}$$

Also, the limiter removes the envelope $e_n(t)$ such that the input to the discriminator is,

$$\cos\left[2\pi f_c t + \frac{n_Q(t)}{A_c}\right]$$

$$\therefore \frac{d}{dt}\cos\left[2\pi f_c t + \frac{n_Q(t)}{A_c}\right]$$

$$= \left[2\pi f_c + \frac{d n_Q(t)}{A_c}\right]\left[-\sin\left[2\pi f_c t + \frac{n_Q(t)}{A_c}\right]\right]$$

The above signal when passed through an envelope detector and LPF, we get,

$$x_o(t)\Big|_{LPF} = \frac{1}{A_c}\, d n_Q(t) = n_o(t) \rightarrow \text{noise output.}$$

Now, in terms of spectral densities,

$$S_{no}(f) = S_{nQ}(f) \cdot |H(f)|^2$$

Since we have,

$$\frac{1}{A_c}\frac{d n_Q(t)}{dt} = n_o(t),$$

we have transfer f^n for the discriminator as, $\frac{1}{A_c} j2\pi f$

$$\because \frac{d x(t)}{dt} \xleftrightarrow{F.T} j2\pi f\, X(f).$$

i.e. $n_Q(t) \longrightarrow \boxed{\frac{j2\pi f}{A_c}} \longrightarrow S_{no}(f)$

$S_{nQ}(f)$ $\qquad$ $n_o(t)$

$$\therefore S_{no}(f) = S_{nQ}(f) \cdot |H(f)|^2$$

$$= N_0 \cdot \left|\frac{j2\pi f}{A_c}\right|^2$$

$$= \frac{N_0 \cdot 4\pi^2 f^2}{A_c^2}$$

Now, the output noise power,

$$P_{NO} = \int_{-f_m}^{f_m} S_{no}(f)\, df$$

$$= \int_{-f_m}^{f_m} \frac{N_0 \cdot 4\pi^2 f^2}{A_c^2}\, df$$

$$= \frac{N_0 \cdot 4\pi^2}{A_c^2} \cdot \frac{f^3}{3}\Big|_{-f_m}^{f_m} = \frac{N_0\, 4\pi^2}{A_c} \cdot \frac{2f_m^3}{3}$$

$$= \frac{N_0 8\pi^2 f_m^3}{3A_c}$$

$$\therefore SNR_o = \frac{A_c^2 4\pi^2 k_f^2 \overline{x^2(t)}}{8\pi^2 N_o f_m^3} \times 3A_c^2$$

$$= \frac{3A_c^4 k_f^2 \overline{x^2(t)}}{2 N_o f_m^3}$$

So,

$$\gamma = \frac{SNR_o}{SNR_i}$$

$$= \frac{3A_c^4 k_f^2 \overline{x^2(t)}}{2 N_o f_m^3} \times \frac{4 N_o f_m}{A_c^2}$$

$$\therefore \gamma = \frac{6 A_c^2 k_f^2 \overline{x^2(t)}}{f_m^2}$$

Now,

for single tone FM,

$$s(t) = A_c \cos\left[2\pi f_c t + \frac{\Delta f}{f_m} \sin 2\pi f_m t\right]$$

Also, $s(t) = A_c \cos[2\pi f_c t + \phi(t)]$.

$$\therefore \phi(t) = \frac{\Delta f}{f_m} \sin 2\pi f_m t = 2\pi k_f \int_0^t x(t)\, dt$$

Differentiating both sides we get,

$$\frac{\Delta f}{f_m} \cos 2\pi f_m t \cdot 2\pi f_m = 2\pi k_f \cdot x(t)$$

$$\text{or } x(t) = \frac{\Delta f}{k_f} \cos 2\pi f_m t.$$

$$\text{So, } \overline{x^2(t)} = \left(\frac{\Delta f}{k_f}\right)^2 \cdot \frac{1}{2}$$

So, for single tone FM,

$$\gamma = \frac{6 A_c^2 \cdot k_f^2}{f_m^2} \cdot \left(\frac{\Delta f}{k_f}\right)^2 \cdot \frac{1}{2}$$

$$= \cancel{6A_c} \quad \frac{3 A_c^2 \Delta f^2}{f_m^2}$$

Now, $\frac{\Delta f}{f_m} = \beta$.

$$\therefore \gamma = 3 A_c^2 \cdot \beta^2$$

So we have figure of merit directly proportional to square of modulation index.

Now in narrowband FM, $\beta << 1$, therefore the noise performace deteriorates whereas for wideband FM, $\beta >> 1$ and hence we get better power efficiency and high noise performance.

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Comparison of AM, DSB-SC, SSB and FM.

We can ~~use~~ compare the aforementioned ~~data~~ modulation techniques in terms of.

i) Bandwidth efficiency.

In terms of bandwidth efficiency, SSB-SC shows the best performance as in this case the channel bandwidth is equal to the message bandwidth.

So, when the bandwidth ~~constraint~~ is the major issue, any application uses SSB-SC.

ii) System complexity:

In terms of complexity, DSB-FC reception is the least complicated. Therefore it is used in commercial radio broadcasting. SSB-SC is bit more complex than DSB-SC but due to its bandwidth efficiency, it is preferred over DSB-SC.

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iii) Power efficiency:

In terms of power efficiency, DSB-FC is the least efficient system whereas the wideband FM has the best power efficiency.

Thus in power critical applications FM ~~broad~~ is preferred over other modulation techniques.

threshold effect in FM.

We have figure of merit for FM as,

$\gamma = 3A_c^2\beta^2$

i.e. $\gamma \propto \beta^2$.

Also we have B_{FM} = bandwidth for FM system,

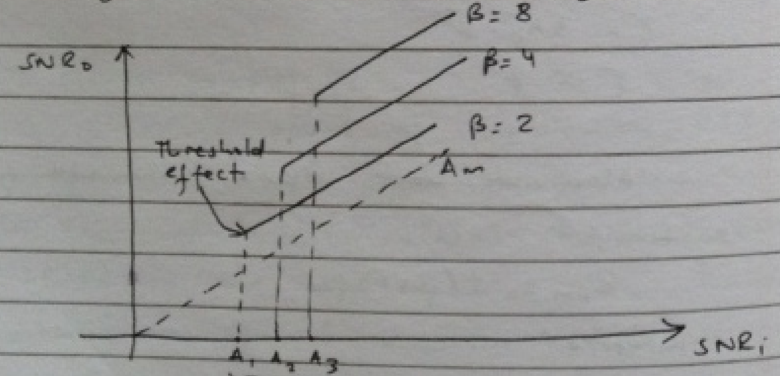
$B_{FM} = 2(\beta+1)f_m$

Therefore increase in β will subsequently increase the input noise power and hence decrease SNR_i.

This will result in noise signal to exceed the input signal level. Thus if we increase the value of β then there will come a point when SNR_o will fall rapidly such that the reception of signal will be impossible.

~~Theref~~ Therefore there will exist a threshold level for SNR_i, below which the FM system will no longer detect the message signal.

General threshold level for FM is 10dB i.e. if ~~SR~~ $SNR_i \le 10dB$, the system will not regenerate the message signal.



Threshold levels for different β.

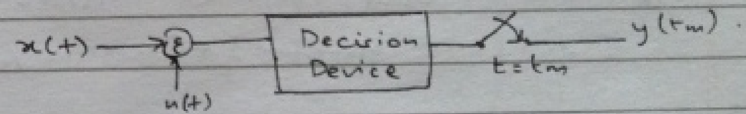
Noise in modulated digital signal.

Noise performance of modulated digital signal is evaluated in terms of error probability. It means the probability ~~of~~ that the detector, in presence of noise, wrongly decodes the received bit.

Let us now deduce the probability of error for binary and M-ary modulated signal.

Binary modulated signal.

Let us consider a PAM signal which doesnot require carrier modulation.



Now for PAM,

$$x(t) = +A \quad \text{if input bit is 1}$$
$$= -A \quad \text{if input bit is 0.}$$

And,

$$y(t_m) = x_o(t) + n(t) = +A + n(t) \quad \text{if 1 is tx.}$$
$$= -A + n(t) \quad \text{if 0 is tx.}$$

Now at the decision device, a threshold

$\lambda = \frac{+A + (-A)}{2} = 0V$ is set.

such that when

$y(t_m) > 0$, it is decided '1' was transmitted

& $y(t_m) < 0$, '0' was transmitted.

Now, the error will occure when any decision is made against the bit transmitted i.e. a '0' is decided when '1' was sent and vice versa.

Therefore for independent bit sequence, the total probability of error can be decided as,

$P_e = P[error|0 sent] \cdot P[0] + P[error|'1' sent] \cdot P[1]$.

$= \frac{1}{2} P[error|0 sent] + \frac{1}{2} P[error|'1' sent]$.

It is evident that the error is due to the noise added in the signal.

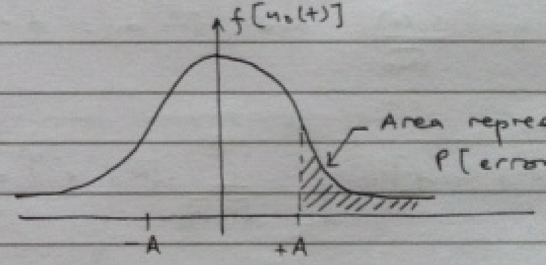
So we can calculate P_e in terms of the pdf of noise $n(t)$.

Now since we assumed $n(t)$ is white Gaussian noise, it follows the Gaussian distribution given by,

$$f[n_0(t)] = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-n_0^2(t)/2\sigma^2}$$

where,

σ^2 = noise variance = $N_0/2$



Now,

$P[error|'0' sent]$ represents that for 0 transmitted, $y(t_m) = \cancel{x_0(t)} -A + n(t) > 0$

$\therefore$ $n(t)$ influences the signal '$-A$' in such a way that the decision boundary for $y(t_m)$ is greater ~~that~~ than 0.

Hence,

$$P[error|'0' sent] = \int_{+A}^{\infty} f[n_0(t)] \, dn_0(t).$$

or $P[\text{error}|0_{sent}] = \int_A^\infty \frac{1}{\sqrt{2\pi\sigma^2}} e^{-n_0^2(t)/2\sigma^2} \, dn_0(t)$,

Now, $\sigma^2 = N_0/2$,

$$\therefore P[\text{error}|0_{sent}] = \int_A^\infty \frac{1}{\sqrt{\pi N_0}} \cdot e^{-n_0^2(t)/N_0} \, dn_0(t).$$

Now, let $\frac{n_0^2(t)}{N_0} = y^2$,

then, $y = \frac{n_0(t)}{\sqrt{N_0}}$ and $\sqrt{N_0}\, dy = d\, n_0(t)$

Therefore,

$$P[\text{error}|0_{sent}] = \int_{A/\sqrt{N_0}}^\infty \frac{1}{\sqrt{\pi}\cdot\sqrt{N_0}} \cdot \sqrt{N_0} \cdot e^{-y^2} dy$$

[$y = \frac{n_0(t)}{\sqrt{N_0}}$

$\therefore$ when $n_0(t) = A$,

$y = \frac{A}{\sqrt{N_0}}$

i.e. Limit

$A \rightarrow A/\sqrt{N_0}$

$\infty \rightarrow \infty$]

$$= \int_{A/\sqrt{N_0}}^\infty \frac{1}{\sqrt{\pi}} e^{-y^2} dy.$$

$$= \frac{1}{2} \cdot \frac{2}{\sqrt{\pi}} \int_{A/\sqrt{N_0}}^\infty e^{-y^2} dy$$

$$= \frac{1}{2} \, erfc\left[\frac{A}{\sqrt{N_0}}\right]$$

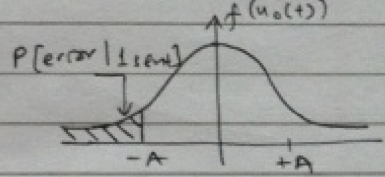
Because $\frac{2}{\sqrt{\pi}} \int_u^\infty e^{-y^2} dy = erfc[u]$

= complimentary error function.

$\therefore P[\text{error}|\text{'0' sent}] = \frac{1}{2} \, erfc\left[\frac{A}{\sqrt{N_0}}\right]$

Similarly,

$P[\text{error}|\text{'1' sent}] = \frac{1}{2} \, erfc\left[\frac{A}{\sqrt{N_0}}\right]$



$\therefore P_e = \frac{1}{2} \cdot \frac{1}{2} \, erfc[A/\sqrt{N_0}]$

$+ \frac{1}{2} \cdot \frac{1}{2} \, erfc\left[\frac{A}{\sqrt{N_0}}\right]$

So, total probability error for binary PAM,

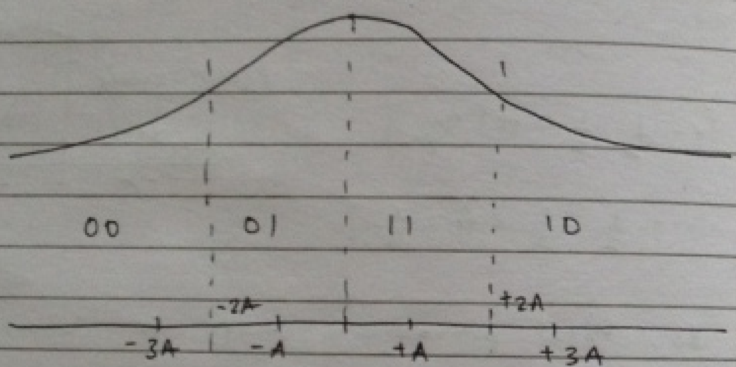
$$P_e = \frac{1}{2} \, erfc\left[\frac{A}{\sqrt{N_0}}\right]$$

Now as the value of $\left(\frac{A}{\sqrt{N_0}}\right)$ increases,

the probability of error decreases.

⊕ M-ary modulated signal.

let us take a case where $M = 4$, i.e. signal is represented in 4 levels. say, $+3A$, $+A$, $-A$ & $-3A$.



Now,

$$P_e = P[\text{error} | 00_{sent}] \cdot P[00] + P[\text{error} | 01_{sent}] \cdot P[01] + P[\text{error} | 11_{sent}] \cdot P[11] + P[\text{error} | 10_{sent}] P[10].$$

Here,

$P[\text{error} | 00_{sent}]$ is contributed in three ways such that,

$$P[\text{error} | 00_{sent}] = P[01 \text{ received} | 00 \text{ sent}] + P[11 \text{ received} | 00 \text{ sent}] + P[10 \text{ received} | 00 \text{ sent}]$$

Now,

$$P[01 | 00] = \int_{+A}^{+3A} f[n_o(t)]\, dn_o(t)$$

$$= \int_{+A}^{\infty} f[n_o(t)]\, dn_o(t) - \int_{3A}^{\infty} f[n_o(t)]\, dn_o(t)$$

$$= \frac{1}{2} \operatorname{erfc}\left[\frac{A}{\sqrt{N_0}}\right] - \frac{1}{2} \operatorname{erfc}\left[\frac{3A}{\sqrt{N_0}}\right]$$

$$P[11 | 00] = \int_{+3A}^{+5A} f[n_o(t)]\, dn_o(t)$$

$$= \frac{1}{2} \operatorname{erfc}\left[\frac{3A}{\sqrt{N_0}}\right] - \frac{1}{2} \operatorname{erfc}\left[\frac{5A}{\sqrt{N_0}}\right]$$

and

$$P[10 | 00] = \int_{+5A}^{\infty} f[n_o(t)]\, dn_o(t)$$

$$= \frac{1}{2} \operatorname{erfc}\left(\frac{5A}{\sqrt{N_0}}\right)$$

Therefore,

$$P[\text{error} | 00_{sent}] = \frac{1}{2} \operatorname{erfc}(A/\sqrt{N_0}) - \frac{1}{2} \operatorname{erfc}[3A/\sqrt{N_0}] + \frac{1}{2} \operatorname{erfc}[3A/\sqrt{N_0}] - \frac{1}{2} \operatorname{erfc}[5A/\sqrt{N_0}] + \frac{1}{2} \operatorname{erfc}[5A/\sqrt{N_0}]$$

$$= \frac{1}{2} \operatorname{erfc}\left[\frac{A}{\sqrt{N_0}}\right]$$

Now,

$$P[\text{error} \mid 01_{\text{sent}}] = P[00 \mid 01_{\text{sent}}] + P[11 \mid 01_{\text{sent}}] + P[10 \mid 01_{\text{sent}}]$$

$$P[00 \mid 01_{\text{sent}}] = \int_{-A}^{-\infty} f[n_o(t)]\, dn_o(t)$$

$$= \int_{A}^{\infty} f[n_o(t)]\, dn_o(t)$$

$$= \frac{1}{2}\, \text{erfc}\left[\frac{A}{\sqrt{N_0}}\right]$$

$$P[11 \mid 01_{\text{sent}}] = \int_{+A}^{+3A} f[n_o(t)]\, dn_o(t)$$

$$= \frac{1}{2}\, \text{erfc}\left[\frac{A}{\sqrt{N_0}}\right] - \frac{1}{2}\, \text{erfc}\left[\frac{3A}{\sqrt{N_0}}\right]$$

$$P[10 \mid 01_{\text{sent}}] = \int_{+3A}^{\infty} f[n_o(t)]\, dn_o(t)$$

$$= \frac{1}{2}\, \text{erfc}\left[\frac{3A}{\sqrt{N_0}}\right]$$

$$\therefore P[\text{error} \mid 01_{\text{sent}}] = \tfrac{1}{2}\, \text{erfc}\left[A/\sqrt{N_0}\right] + \tfrac{1}{2}\, \text{erfc}\left[A/\sqrt{N_0}\right] - \tfrac{1}{2}\, \text{erfc}\left[\frac{3A}{\sqrt{N_0}}\right] + \tfrac{1}{2}\, \text{erfc}\left[3A/\sqrt{N_0}\right]$$

$$= \text{erfc}\left[A/\sqrt{N_0}\right]$$

Similarly,

$$P[\text{error} \mid 11_{\text{sent}}] = \text{erfc}\left[\frac{A}{\sqrt{N_0}}\right]$$

and $$P[\text{error} \mid 10_{\text{sent}}] = \frac{1}{2}\, \text{erfc}\left[\frac{A}{\sqrt{N_0}}\right]$$

Now, $P[00] = P[01] = P[10] = P[11] = 1/4$

$$P_e = \frac{1}{2}\, \text{erfc}\left[\frac{A}{\sqrt{N_0}}\right]\cdot\frac{1}{4} + \cancel{\tfrac{1}{2}}\, \text{erfc}\left[\frac{A}{\sqrt{N_0}}\right]\cdot\frac{1}{4} + \text{erfc}\left[\frac{A}{\sqrt{N_0}}\right]\cdot\frac{1}{4} + \frac{1}{2}\, \text{erfc}\left[\frac{A}{\sqrt{N_0}}\right]\cdot\frac{1}{4}$$

$$\therefore P_e = \frac{3}{4}\, \text{erfc}\left[\frac{A}{\sqrt{N_0}}\right]$$

or $$P_e = \left(\frac{M-1}{M}\right) \text{erfc}\left[\frac{A}{\sqrt{N_0}}\right]$$

and, $$\frac{P_e(M)}{P_e(B)} = \frac{2(M-1)}{M}.$$

(a) Noise performance of modulated digital systems.

Error probability for.

i) BASK.

We have for BASK,

$x_1(t) = A_c \cos 2\pi f_c t$ bit '1'

$x_2(t) = 0$ bit '0'

Now the expression for probability of error for matched filter,

$$P_e = \frac{1}{2} \operatorname{erfc}\left[\frac{x_{01}(T) - x_{02}(T)}{2\sqrt{2}\cdot\sigma}\right]$$

Also we have the relation,

$$\left|\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right|^2_{max} = \frac{2}{N_0}\int_0^T x^2(t)\,dt$$

Here, $x(t) = x_1(t) - x_2(t)$

$\therefore x(t) = A_c \cos 2\pi f_c t$

$= \sqrt{2P} \cos 2\pi f_c t$

$= \sqrt{\frac{2E}{T}} \cos 2\pi f_c t.$

$$\boxed{E = \frac{A_c^2 T}{2}}$$

Therefore,

$$\frac{2}{N_0}\int_0^T x^2(t)\,dt = \frac{2}{N_0}\int_0^T A_c^2 \cos^2 2\pi f_c t\,dt$$

$$= \frac{2}{N_0}\int_0^T A_c^2\left[\frac{1+\cos 2\times 2\pi f_c t}{2}\right]dt$$

$$= \frac{2}{N_0}\left[\int_0^T \frac{A_c^2}{2}\,dt + \underbrace{\int_0^T \frac{A_c^2 \cos 4\pi f_c t}{2}\,dt}_{\to 0}\right]$$

$$= \frac{2}{N_0}\,\frac{A_c^2}{2}\cdot T$$

$$= \frac{2}{N_0}\,\frac{A_c^2 T}{2} = \frac{2E}{N_0}$$

i.e.

$$\left|\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right|^2 = \frac{2E}{N_0}$$

$$\therefore \frac{x_{01}(T) - x_{02}(T)}{\sigma} = \sqrt{\frac{2E}{N_0}}$$

And $$\frac{x_{01}(T) - x_{02}(T)}{2\sqrt{2}\cdot\sigma} = \frac{1}{2\sqrt{2}}\cdot\sqrt{2}\cdot\sqrt{\frac{E}{N_0}} = \frac{1}{2}\sqrt{\frac{E}{N_0}} = \sqrt{\frac{E}{4N_0}}$$

$$\therefore P_e = \frac{1}{2}\operatorname{erfc}\left[\frac{x_{01}(T) - x_{02}(T)}{2\sqrt{2}\cdot\sigma}\right] = \frac{1}{2}\operatorname{erfc}\left[\sqrt{\frac{E}{4N_0}}\right]$$

$\therefore$ Probability of error for BASK, $P_e = \frac{1}{2}\operatorname{erfc}\sqrt{\frac{E}{4N_0}}$

ii) Error probability for BPSK.

We have for BPSK,

$x_1(t) = A_c \cos 2\pi f_c t$ bit '1'

$x_2(t) = -A_c \cos 2\pi f_c t$ bit '0'

$\therefore x(t) = x_1(t) - x_2(t)$

$= 2A_c \cos 2\pi f_c t$

Now,

$$\left(\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right)^2 = \frac{2}{N_0}\int_0^T [2A_c \cos 2\pi f_c t]^2 dt$$

$$= \frac{2}{N_0}\int_0^T 4A_c^2 \cos^2 2\pi f_c t \; dt$$

$$= \frac{8A_c^2}{N_0}\int_0^T \frac{1 + \cos 4\pi f_c t}{2} dt$$

$$= \frac{8A_c^2}{N_0}\left[\int_0^T \frac{1}{2} dt + \underbrace{\int_0^T \frac{\cos 4\pi f_c t}{2} dt}_{\to 0}\right]$$

$$= \frac{8A_c^2 \cdot T}{2N_0}$$

$$= \frac{8A_c^2 \cdot T}{2} \cdot \frac{1}{N_0}$$

$$= \frac{8E}{N_0}$$

$$\therefore \frac{x_{01}(T) - x_{02}(T)}{\sigma} = \sqrt{\frac{8E}{N_0}} = 2\sqrt{\frac{2E}{N_0}}$$

Therefore,

$$\frac{x_{01}(T) - x_{02}(T)}{2\sqrt{2}\cdot\sigma} = \frac{1}{2\sqrt{2}} \cdot 2\sqrt{\frac{2E}{N_0}}$$

$$= \sqrt{\frac{E}{N_0}}$$

$$\therefore P_e = \frac{1}{2}\, \text{erfc}\left[\frac{x_{01}(T) - x_{02}(T)}{2\sqrt{2}\cdot\sigma}\right] = \frac{1}{2}\, \text{erfc}\left[\sqrt{\frac{E}{N_0}}\right]$$

$\therefore$ Probability of error for BPSK,

$$P_e = \frac{1}{2}\, \text{erfc}\left[\sqrt{\frac{E}{N_0}}\right].$$

iii) Error probability of BFSK

We have BFSK signal as,

$x_1(t) = A_c \cos 2\pi f_1 t$

$x_2(t) = A_c \cos 2\pi f_2 t$

or let us have,

$x_1(t) = A_c \cos[2\pi f_c + f]t$

$x_2(t) = A_c \cos[2\pi f_c - f]t$

Therefore,

$$x(t) = x_1(t) - x_2(t)$$
$$= A_c \cos[2\pi f_c + f]t - A_c \cos[2\pi f_c - f]t$$
$$= A_c[\cos[2\pi f_c t + ft] - \cos[2\pi f_c t - ft]]$$
$$= A_c[-2\sin 2\pi f_c t \cdot \sin ft]$$

Now,

$$\left|\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right|^2_{max} = \frac{2}{N_0}\int_0^T A_c^2 \cdot 4\sin^2 2\pi f_c t \cdot \sin^2 ft \; dt$$
$$= \frac{2A_c^2}{N_0}\int_0^T 4\sin^2 2\pi f_c t \cdot \sin^2 ft \; dt.$$

Now,

$$4\sin^2 2\pi f_c t \cdot \sin^2 ft$$
$$= 2\sin^2 2\pi f_c t \cdot 2\sin^2 ft$$
$$= (1 - \cos 4\pi f_c t)(1 - \cos 2ft)$$
$$= 1 - \cos 2\pi ft - \cos 4\pi f_c t + \cos 4\pi f_c t \cdot \cos 2ft$$
$$= \cancel{1 - \cos 2f}$$
$$= 1 - \cos 2\pi ft - \cos 4\pi f_c t + \frac{1}{2}\cos[4\pi f_c t + ft] + \frac{1}{2}\cos[4\pi f_c t - ft]$$

Now for $f_c >> f$, the last three term cancel out each other

i.e. $= 1 - \cos 2\pi ft - \cos 4\pi f_c t + \frac{1}{2}\cos 4\pi f_c t + \frac{1}{2}\cos 4\pi f_c t$

$$= 1 - \cos 2\pi ft$$

Therefore,

$$\frac{2A_c^2}{N_0}\int_0^T 1 - (\cos 2ft)\, dt$$
$$= \frac{2A_c^2}{N_0}\left[\int_0^T 1\, dt - \int_0^T \cos 2ft\, dt\right]$$
$$= \frac{2A_c^2}{N_0}\left[T - \frac{\sin 2fT}{2f}\right]$$
$$= \frac{2A_c^2}{N_0} \cdot T\left[1 - \frac{\sin 2fT}{2fT}\right]$$

i.e.

$$\left.\frac{x_{01}(T)-x_{02}(T)}{\sigma}\right|^2_{max} = \frac{2A_c^2T}{N_0}\left[1-\frac{\sin 2fT}{2fT}\right]$$

The right hand side term is maximum when $2fT = 3\pi/2$,

i.e.

$$\frac{2A_c^2T}{N_0}\left[1-\frac{\sin 2fT}{2fT}\right] = \frac{2A_c^2T}{N_0}\left[1-\frac{\sin 3\pi/2}{3\pi/2}\right]$$

$$= \frac{4\cdot A_c^2T}{2N_0}\left[1-\left(\frac{-1}{3\pi/2}\right)\right]$$

$$= 4.84\frac{E}{N_0}$$

Therefore,

$$\frac{x_{01}(T)-x_{02}(T)}{\sigma} = \sqrt{4.84\frac{E}{N_0}}$$

and, $$\frac{x_{01}(T)-x_{02}(T)}{2\cdot\sqrt{2}\cdot\sigma} = \frac{1}{2\sqrt{2}}\sqrt{\frac{4.84E}{N_0}}$$

$$= \sqrt{\frac{4.84E}{8N_0}}$$

$$= \sqrt{0.65\,E/N_0}$$

$\therefore$

$$P_e = \frac{1}{2}\,\mathrm{erfc}\left[\sqrt{0.65\frac{E}{N_0}}\right]$$ for BFSK.