Chapter-6 Random signals and noise in Communication System

Date / /

Page No.

6.1 Random variables and processes

The Concept of random process is a logical extension of the random variable. This Extension has been necessary for statistical analysis of random signals encountered in communication system such as voice, video signal, digital data and electrical noise.

Before that we classify variables as

a. Deterministic variable.

b. Random variable

a. Deterministic variable :→

Deterministic variable are those variable which can be exactly predicted or Counted. There are no uncertainty of the value at any instant of time and can be determined with the help of mathematical models.

b. Random variable :→

If the outcome of an experiment cannot be predicted in advance and are represented in terms of probablity. (Chance of occurence) are called Random variables.

To ellaborate it, let us toss a die, Tossing of die is a expeliment or process which has outcomes or result. The set of possible outcomes of an experiment or trial is called the sample space (S) given as $S = [1, 2, 3, 4, 5, 6]$ six outcomes each are called sample points (Number 1 to 6) in the sample space. For every trial the outcomes are random and takes any value of set of sample point.

Therefore Random variable can also be defined as the function which can take any value of sample space. Random variables can be classified as

i) Discrete Random Variable :→ The random variable 'x' is said to be a discrete random variable if 'x' takes on only a finite number of values in any finite observation interval. eg:→ Rolling of Die, Tossing of three Coins simultaneously. There are eight possible outcomes. The sample space is given as

$S = [HHH \ \ HHT \ \ HTH \ \ THH \ \ HTT \ \ THT \ \ TTH \ \ TTT]$

$x =$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
	0	1	1	1	2	2	2	3

Let the number of Tail be represented by Random variable 'x'.

ii) Continuous Random Variable :→

The random variable 'x' is said to be Continuous random variable if 'x' takes any value within the finite observation table eg:→ Amplitude of noise can take to any amount at any instant of time.

Date / /

Page No.

Flair

Flair

* Why: Analysis or study of Random Variable.

Every Communication System encounters Random signal. practical system especially in radio Communication system, the receiver receives incoming signal with a mix of information bearing signal and the interference Component (noise). This makes the received signal random in nature and cannot be predicted precisely. so we take an account of probability theory to Analyze the random variable. This Analysis is prime important to make flawless (error free) System.

* Cumulative Distribution Function (CDF):→

Cumulative Distribution Function (CDF) associated with random variable (X) is defined as the probability that the outcome of an experiment will be one of the outcome for which $\boxed{X \le x}$ where "x is the given number".

There fore Cumulative Density function which is also named Cumulative probability Density Function (CPDF) is a probability description of a random variable which can be both discrete or Continuous Random variable.

CPDF is represented with symbol $F_X(x)$. $\boxed{F_X(x) = P(X \le x)}$

* properties

1. The distribution function $F_X(x)$ is bounded betn 0 and 1. i.e

$$\boxed{0 \le F_X(x) \le 1}$$

2. For no possible events, $x=-\infty$ the distribution function

$$\boxed{F_X(x) = F_X(-\infty) = 0}$$

Again for possible events, $x=\infty$ which includes all possible events $P(X \le \infty)$ is

$$\boxed{F_X(x) = F_X(\infty) = 1}$$

3. The distribution function $F_X(x)$ is a monotonic non decreasing function of x.

For $x_1 < x_2$ the distribution function is

$$\boxed{F_X(x_1) < F_X(x_2)}$$

* Probability Density Function (PDF)

The probability distribution or density function is an alternative description of the probabilistic nature of a random variable. The probability density function (PDF) denoted by $f_X(x)$ is defined in terms of Cumulative distribution function (CDF) $F_X(x)$ as

$$\boxed{f_X(x) = \frac{d}{dx} F_X(x)}$$

The probability of an event $x_1 < X \leq x_2$ can be calculated as

$$p(x_1 < X \leq x_2) = p(X \leq x_2) - p(X \leq x_1)$$
$$= F_X(x_2) - F_X(x_1)$$

$$\boxed{p(x_1 < X \leq x_2) = \int_{x_1}^{x_2} f_X(x)\,dx}$$

The p.D.F $f_X(x)$ denoted by $f(x)$ has properties as

a. $f(x) \geq 0$ for all x

b. $\int_{-\infty}^{\infty} f(x)\,dx = 1$

verified as $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{d F_X(x)}{dx}\,dx$

$$\int_{-\infty}^{\infty} f(x)\,dx = F_X(\infty) - F_X(-\infty) = 1 - 0$$

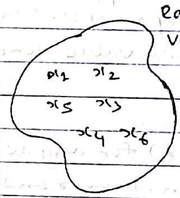
$$\boxed{\int_{-\infty}^{\infty} f(x)\,dx = 1}$$

Flair

* Random process

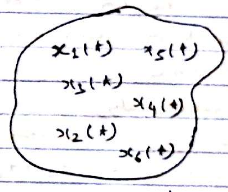
Random process as been discussed is mathematical model: a combined approach of signal analysis and probability theory to interpretate and analyse the communication system.

Random Variable



Independent of time (x_i)

Random process



$x_i(t)$ is a function of time

Above figure shows the clear picture of random variable and process.

For eg: → value of temperature (magnitude or scale) is a random variable (X). The change and fluctuation of temperature with respect to time are the random process specified as $x(t)$. In other words Random or stochastic process is viewed as the outcome of each trial as a function of time.

Random process are described using statistical averages like Mean, Correlation and co-variance.

The Random process are classified as

a. Stationary Random or stochastic process.

b. Non stationary Random process.

Flair

Date / /
Page No.

a. Stationary Random or stochastic process

The statistical characteristics of Random or stochastic process which donot change with time is called Stationary Random or stochastic process. The change of time of origin cannot be detected and the process will be appeared to be same in stationary random process. Statistics like Mean value and Mean square value donot change with time.

Mathematically,
Consider a random process $X(t)$ for any set of time instant $t_1 > t_2 > t_3 \ldots > t_n$. Then random variables $X(t_1)$ $X(t_2)$ $X(t_3) \ldots x(t_n)$ are the random process.

Then

$\boxed{X(t) = X(t+T)}$ where $T \Rightarrow$ Time shift

The probability density function of stationary Random process is invariant under shift of time origin as

$\boxed{f[X(t)] = f[X(t+T)]}$

b. Non stationary Random process :→

The statistical characteristics of Random process which changes with respect to time is called Non stationary random process. Statistics like Mean value, Mean square value, probability density function changes with time and is time variant.

Flair

Date / /
Page No.

For the practical approach we consider stationary Random or stochastic process which are described by the parameters like

* Mean, correlation and co-variance of stochastic process

These parameters are important statistical parameter that provides partial description of the distribution of the process. They play key roles for characterising the random process whose probability density function (pdf) cannot be determined directly.

Consider a strictly random process $X(t)$. Let $X(t_j)$ denote the random variable obtained by observing the stochastic process $X(t)$ at time (t_j).

The mean or average of the random process $X(t)$ is represented as

$\boxed{m_X = E[X(t_j)]}$ for any t_j

Similarly the autocorrelation function of stationary process $X(t)$ is defined as

$\boxed{R_X(t_j - t_k) = E[X(t_j)X(t_k)]}$ for any t_j and t_k.

$X(t_j)$, $X(t_k)$ are the random variable obtained by observing the process $X(t)$ at time t_j and t_k respectively.
As autocorrelation function depends on the time difference

Flair

Scanned by CamScanner

Page No.

$(t_j - t_k)$. If 'τ' represents the time difference, the autocorrelation function can be represented in a more convenient form as

$$\boxed{R_X(\tau) = E[X(t)X(t-\tau)]}$$

* $R_X(\tau)$ is independent of a shift of time origin

Again the Co-variance or Autocovariance which is another important characteristics of stationary random process $X(t)$ is defined as

$$K_X(t_j - t_k) = E[(X(t_j) - m_x)(x(t_k) - m_x)]$$

Therefore

$$\boxed{K_X(\tau) = R_X(\tau) - m_x^2}$$

* WIDE-SENSE STATIONARY PROCESS (WSSP) :→

Strictly saying process is not stationary, yet the process having constant mean value, constant autocorrelation function that is independent of the shift of time origin and a finite autocorrelation function at zero time lag is known as wide-sense stationary process (WSSP). All stationary process is WSSP but all WSSP may not be stationary process.

Properties

Date / / Page No.

1. The autocorrelation function $R_X(\tau)$ of WSSP is an even function of time lag. This means the autocorrelation function $R_X(\tau)$ satisfies the following mathematical relation:

$$\boxed{R_X(\tau) = R_X(-\tau)}$$

2. The mean square value of a WSSP is equal to the autocorrelation function of the process for zero time lag.

Mathematically, $\boxed{R_X(0) = E[X^2(t)]}$

3. The autocorrelation function of WSSP has the maximum magnitude at zero time lag

$$\boxed{|R_X(\tau)| \le R_X(0)}$$

* ERGODIC PROCESS :→

As the mean and the autocorrelation function of a random process is

$$m_x = E[x(t)] \quad \text{and} \quad R_X(\tau) = E[x(t)\,x(t-\tau)].$$

are the ensemble average obtained by computing the average across all the sample function of the process.

Again the time averaged mean of the sample function $x(t)$ of a random process $X(t)$ as

Date / /
Page No.

$$\boxed{\langle x(t)\rangle = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt}$$

The time-averaging autocorrelation function of the sample function $x(t)$ as

$$\boxed{\langle x(t)\,x(t-\tau)\rangle = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x(t)\,x(t-\tau)\,d\tau}$$

are the time-averaging evaluation.

In general, ensemble average and time average are not equal. Therefore special class of random process where the ensemble average is equal to the time average of any sample function is called Ergodic process.

Therefore for random process to be ergodic in mean and autocorrelation function, it must satisfy the following condition

$$\boxed{m_x = E[X(t)] = \langle x(t)\rangle}$$

$$\boxed{R_X(\tau) = E[X(t)\,X(t-\tau)] = \langle x(t)\,x(t-\tau)\rangle}$$

It is very important with the fact that, if the random process is ergodic, then we need only one sample function to compute average. Many stationary process observed in practice are considered as ergodic.

Date / /
Page No.

* psdf and AC function of a ergodic random process

As been discussed to compute ensemble average large number of sample function are required which is not practically feasible. But if the random process is ergodic, then we need only one sample to compute ensemble average.

For random process the fourier transform of its AC function is called psdf. For ergodic process the psDF and autocorrelation function are fourier transform pair.

$$\boxed{S_X(f) = \int_{-\infty}^{\infty} R_X(\tau)\exp(-j2\pi f\tau)\,d\tau} \quad (1)$$

and

$$\boxed{R_{XX}(\tau) = \int_{-\infty}^{\infty} S_X(f)\exp(j2\pi f\tau)\,df} \quad (2)\ [\mathcal{F}^{-1}\{S_X(f)\}]$$

This pair of equation is Einstein-Wiener-Khintchine relation for wide sense stationary process.

The psdf of WSSP has following properties.

Property 1 :→ The zero frequency value of the power spectral density of WSSP is equal to the total area under the graph of the autocorrelation function

$$\boxed{S_Y(0) = \int R_X(\tau)\,d\tau}$$

Property 2 :→ The mean square value of WSSP is equal to the total area under the graph of the power spectral density

$$\boxed{E[x^2(t)] = \int_{-\infty}^{\infty} S_X(f)\,df}$$

Property-3

The power spectral density of WSSP is always non negative.

$$\boxed{S_X(f) \geq 0 \text{ for all } f}$$

Property-4

The PSDF of WSSP is an even function of the frequency

$$\boxed{S_X(-f) = S_X(f)}$$

* White Noise

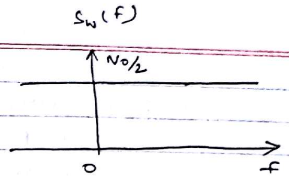
Idealized form of noise is known as white Noise whose power spectral density (PSD) is independent of frequency.

White noise are generally considered for the analysis of communication system which has frequency within the band of electromagnetic radiation. The power spectrum density of white noise process is

$$\boxed{S_W(f) = \frac{N_0}{2}}$$

$N_0 \Rightarrow$ Input Noise at the receiving side measured in watt/Hz.

Factor $\frac{1}{2}$ represents half the power associated with positive frequency and half with negative frequency.



fig(a) :→ power spectral density

The absence of delta function in the power spectral density at the origin depicts that the white noise has no dc power.

Similarly the autocorrelation function is the Inverse Fourier Transform (I.F.T) of power spectral Density given as

$$\boxed{R_W(\tau) = F^{-1}[S_W(f)] = \frac{N_0}{2}\delta(\tau)}$$

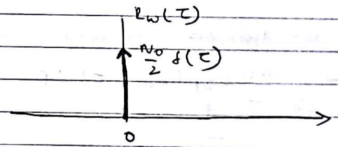


fig :→ Autocorrelation function.

Autocorrelation function of white noise consist of delta function weighted by factor $N_0/2$ occuring at $\tau = 0$.

Therefore white noise has infinite average power and is not physically realizable. Although they are considered for analysis of the system.

Date / /
Page No.

* Ideal low pass Filtered white Noise

Let white noise process $w(t)$ for zero mean and P.S.D $N_0/2$ is applied to Ideal Low pass Filter of bandwidth 'B' and a passband amplitude response of '1'. The power spectral density of the filter output is

$$S_N(f) = \begin{cases} N_0/2 & -B < f < B \\ 0 & |f| > B \end{cases}$$

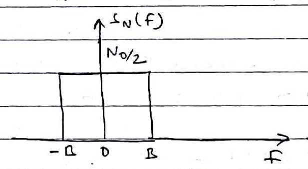


fig:→ Power spectral Density

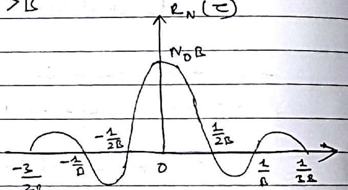


fig:→ AutoCorrelation function

The autocorrelation function is

$$R_N(\tau) = \int_{-B}^{B} \frac{N_0}{2} \exp(j2\pi f\tau)\, df$$

$$R_N(\tau) = \frac{N_0}{2} \int_{-B}^{B} \exp(j2\pi f\tau)\, df$$

$$R_N(\tau) = N_0/2 \left. \frac{\exp(j2\pi f\tau)}{j2\pi\tau} \right|_{-B}^{B} = \frac{N_0}{2} \exp(j2\pi B\tau) - \exp(-j2\pi B\tau)$$

$$R_N(\tau) = \frac{N_0}{2} \times \frac{1}{2j\pi\tau} \times \exp(j2\pi B\tau) - \exp(-j2\pi B\tau) = \frac{N_0}{2} \times \frac{1}{\pi\tau} \times \frac{e^{j2\pi B\tau} - e^{-j2\pi B\tau}}{2j}$$

$$= \frac{N_0}{2} \times \frac{1}{\pi\tau} \times \sin\pi 2B\tau = N_0/\cancel{2} \times \frac{\sin\pi 2B\tau}{\pi 2B\tau} \times \cancel{2}B = BN_0 \operatorname{sinc}(2B\tau)$$

$$\boxed{R_N(\tau) = N_0B \operatorname{sinc}(2B\tau)}$$

Flair

Date / /
Page No.

From above figure, we see the $R_N(\tau)$ has maximum value at origin N_0/B and passes through '0' at $\tau = \pm n/2B$ where $n = 1, 2, 3 \ldots$

* RC Filtering of white Noise

Let white noise process $w(t)$ for zero mean and power spectral density (PSD) '$N_0/2$' be applied to Low pass Filter (LPF). The transfer function of the filter is

$$H(f) = \frac{\text{output}}{\text{Input}}$$

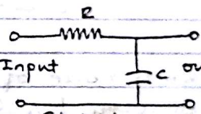


Fig:→ Low pass RC filter

$$H(f) = \frac{1/j\omega C}{R + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega RC} \quad \text{—(1)}$$

As the P.S.D of white noise is $S_W(f) = N_0/2$, the P.S.D of the low pass RC filter is

$$S_{WRC}(f) = \frac{N_0}{2} |H(f)|^2$$

$$\therefore \boxed{S_{WRC}(f) = \frac{N_0}{2} \frac{1}{1 + \omega^2 R^2 C^2}}$$

similarly AutoCorrelation function is

$$R_N(\tau) = IFT\left[S_{WRC}(f)\right]$$

$$\boxed{R_N(\tau) = \frac{N_0}{4RC} \exp\left(-\frac{\tau}{RC}\right)}$$

Flair

Scanned by CamScanner

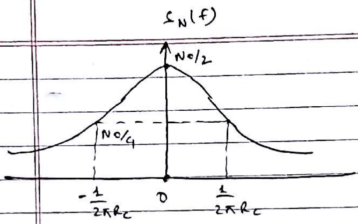


fig:→ P.S.D of filter output

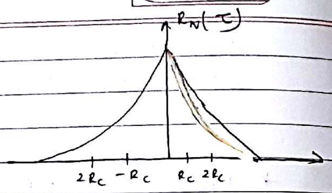


fig:→ Autocorrelation function of filter output

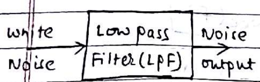
* Noise Equivalent Bandwidth of a Filter

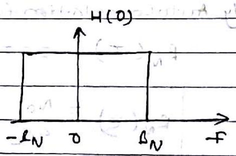
When a source of white noise of zero mean and power spectral density '$N_0/2$' connected to the low pass filter (LPF) of transfer function $H(f)$. The resulting average output noise power is

$$P_N = \frac{N_0}{2}\int_{-\infty}^{\infty}|H(f)|^2\,df$$

$$\boxed{P_N = N_0\int_{0}^{\infty}|H(f)|^2\,df} \quad — (1)$$

Again calculating the average output noise power for ideal low pass filter of zero frequency response $H(0)$ and bandwidth B_N.





$$P_N = N_0 B_N H^2(0)$$

$$\therefore P_N = N_0\int_0^{B_N}|H(f)_{LPF}|^2\,df = N_0 B_N H^2(0)$$

$$\boxed{P_N = N_0 B_N H^2(0)} \quad — (2)$$

From eqn (1) & (2)

$$\cancel{N_0} B_N H^2(0) = \cancel{N_0}\int_0^{\infty}|H(f)|^2\,df$$

$$\boxed{B_N = \frac{\int_0^{B_N}|H(f)|^2}{H^2(0)}\,df}$$

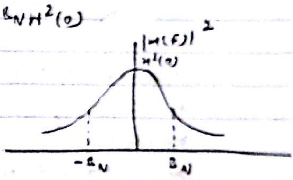


fig:→ Noise equivalent bandwidth for LPF.

In similar way we can also find the noise equivalent bandwidth for a band pass filter given as

$$\boxed{B_N = \frac{\int_0^{\infty}|H(f)|^2\,df}{|H(f_c)|^2}}$$

where $|H(f_c)|$ is the amplitude response at center frequency 'f_c' of the filter.

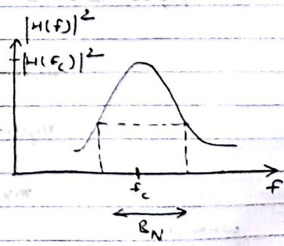
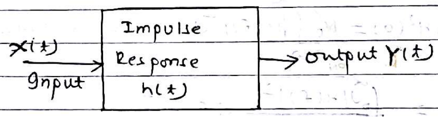


fig:→ Noise equivalent bandwidth for band pass filter.

* Passage of wide-sense Random Signals through a LTI

Let a random process $x(t)$ be applied to the input of a Linear Time Invariant (LTI) filter having an impulse response $h(t)$. If the input random process $x(t)$ produces random process $Y(t)$ at the filter output.



Assuming $x(t)$ is a wide-sense stationary process (WSSP) we determine the mean and auto-Correlation function of the output random process $Y(t)$ in terms of input random process $x(t)$.

Mean of output Random process

$$m_Y(t) = E[Y(t)] = E\left[\int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau\right]$$

$$m_Y(t) = \int_{-\infty}^{\infty} h(\tau)\, E[x(t-\tau)]\, d\tau$$

$$m_Y(t) = \int_{-\infty}^{\infty} h(\tau)\, m_x(t-\tau)\, d\tau$$

$$m_Y(t) = \int_{-\infty}^{\infty} h(\tau)\, m_x(t-\tau)\, d\tau \quad -(1)$$

when $x(t)$ is WSS, the mean $m_x(t)$ is a constant m_x.

$$\therefore\ m_Y = m_x \int_{-\infty}^{\infty} h(\tau)\, d\tau$$

$$\boxed{m_Y = m_x H(0)} \quad -(2)$$

where $H(0)$ is the zero frequency response of the system.

Eq^n (2) clearly shows that the mean of output random process of a stable LTI system is equal to the mean of the input random process multiplied by the zero frequency response of the system.

Now calculating the autocorrelation function of the output random process $Y(t)$. If t and u denote two values of time at which output process is observed.

$$R_Y(t,u) = E[Y(t)\, Y(u)]$$

By using the Convolution integral we get

$$R_Y(t,u) = E\left[\int_{-\infty}^{\infty} h(\tau_1)\, x(t-\tau_1)\, d\tau_1 \int_{-\infty}^{\infty} h(\tau_2)\, x(u-\tau_2)\, d\tau_2\right]$$

$$R_Y(t,u) = \int_{-\infty}^{\infty} h(\tau_1)\, d\tau_1 \int_{-\infty}^{\infty} h(\tau_2)\, d\tau_2\ E[(x(t-\tau_1)\, x(u-\tau_2)]$$

$$R_Y(t,u) = \int_{-\infty}^{\infty} h(\tau_1)\, d\tau_1 \int_{-\infty}^{\infty} h(\tau_2)\, d\tau_2\ R_x[t-\tau_1-(u-\tau_2)]$$

$$R_Y(t,u) = \int_{-\infty}^{\infty} h(\tau_1)\, d\tau_1 \int_{-\infty}^{\infty} h(\tau_2)\, d\tau_2\ R_x[t-\tau_1-u+\tau_2] \quad -(3)$$

Note:→ $\because R_x(t_j - t_k) = E[x(t_j)\, x(t_k)]$

when the input is WSSP, the autocorrelation function of $x(t)$ is the functional difference between the observation time $(t-\tau_1)$ and $(u-\tau_2)$. put $\tau = t-u$

$$\boxed{R_Y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)\, h(\tau_2)\, R_x(\tau-\tau_1+\tau_2)\, d\tau_1\, d\tau_2} \quad -(4)$$

From above expression ① & ④ we conclude that if the input to a stable LTI filter is WSSP, then the output of the filter is also a WSSP.

put $\tau = 0$, in eqn ④

$$\boxed{R_Y(0) = E[Y^2(t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_X(\tau_2-\tau_1)\,d\tau_1 d\tau_2}$$

* Thermal Noise :→

Thermal noise arises or introduced by thermally induced motion of electrons in conducting media. In a conductor their is large number of free electrons and equally large number of ions bound by strong molecular forces. If the thermally induced electrons collides with vibrating ions randomly. Random electric current are generated in the form of Thermal Noise.

Therefore power spectral Density (PSD) of thermal noise produced is

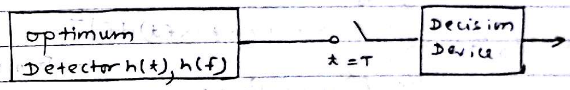
$$\boxed{S_{TN}(f) = \frac{2hf}{\exp(hf/_{KT})-1} = \frac{2hf}{e^{hf/KT}-1}}$$

$T \Rightarrow$ Absolute temperature in degree kelvin.

$h \Rightarrow$ plancks constant

$K \Rightarrow$ Boltzman constant

* optimum detection of pulse in additive white noise / Matched Filter



let $s(t) \rightleftharpoons s(f)$ and $s_o(t) \rightleftharpoons s_o(f)$

Let $s(t)$ be the received signal. The output due to signal component $s(t)$ is

$$s_o(t) = IFT[s_o(f)]$$

$$s_o(t) = \int_{-\infty}^{\infty} H(f)s(f)\exp(j2\pi ft)\,dt \quad — ①$$

PSDF of output noise $S_{N_o}(t)$ due to the application of white noise $w(t)$ at the input.

$$S_{N_o}(f) = \frac{N_o}{2}|H(f)|^2 \quad — ②$$

The average noise power is

$$P_N = \int_{-\infty}^{\infty} S_{N_o}(f)\,df \quad — ③$$

$$P_N = \int_{-\infty}^{\infty} \frac{N_o}{2}|H(f)|^2 df \quad [\text{From eq}^n ②]$$

$$P_N = \frac{N_o}{2}\int_{-\infty}^{\infty} |H(f)|^2 df \quad — ④$$

The objective of the optimum detector is to make $(SNR)_o$ output signal or received to Noise Ratio maximum at sampling time $t = T$.

$\therefore (SNR)_0 = \frac{|s_0(t)|^2}{P_N}$

$$(SNR)_0 = \frac{\left|\int_{-\infty}^{\infty} H(f)\,S(f)\exp(j2\pi ft)\,df\right|^2}{\frac{N_0}{2}\int_{-\infty}^{\infty}|H(f)|^2\,df}$$ From eqn (1) & (4).

To maximize the $(SNR)_0$ we can apply schwarz's inequality principle which states

$$\left|\int_{-\infty}^{\infty} H(f)\,S(f)\exp(j2\pi ft)\,df\right|^2 \le \int_{-\infty}^{\infty}|H(f)|^2 df \cdot \int_{-\infty}^{\infty}|S(f)|^2 df \quad -(5)$$

Equation (5) is satisfied if and only if the first function H(f) is complex conjugate of second function $S(f)\exp(j2\pi ft)$

Now,

$$(SNR)_{opt} = \frac{\cancel{\int_{-\infty}^{\infty}|H(f)|^2 df}\int_{-\infty}^{\infty}|S(f)|^2 df}{\frac{N_0}{2}\cancel{\int_{-\infty}^{\infty}|H(f)|^2 df}}$$

$$(SNR)_{opt} = \frac{2}{N_0}\int_{-\infty}^{\infty}|S(f)|^2 df$$

$$\boxed{(SNR)_{opt} = \frac{2E}{N_0}} \quad -(6)$$

This condition is fulfilled when $\boxed{H_{opt}(f) = S^*(f)\exp(-j2\pi fT)}$ except for the exponential factor $\exp(-j2\pi fT)$ representing a constant time delay T, the transfer function of optimum filter (detector) is same as complex conjugate of the spectrum of the input filter. such a filter is known as matched filter.

Now $h_{opt}(t) = IFT[H_{opt}(f)]$

$$h_{opt}(t) = \int_{-\infty}^{\infty} S^*(f)\exp(-j2\pi fT)\exp(j2\pi ft)\,df$$

$$h_{opt}(t) = \int_{-\infty}^{\infty} S^*(f)\exp[-j2\pi f(T-t)]\,df$$

For real value signal $s(t)$, $S^*(f) = S(-f)$

$$h_{opt}(t) = \int_{-\infty}^{\infty} S(-f)\exp[-j2\pi f(T-t)]\,d$$

$$\boxed{\therefore h_{opt}(t) = s(T-t)}$$

Therefore the impulse response of the matched filter is time reversed and delayed version of the input signal $s(t)$.

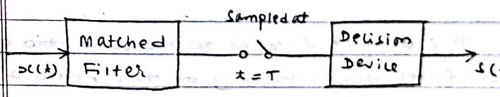


fig:→ Matched Filter Receiver.

For a matched filter operating in real time to be physically realizable, it must be casual given as

$h_{opt}(t) = 0 \quad$ for $t < 0$

$h_{opt}(t) = s(T-t) \quad$ for $t \ge 0$

* properties of Matched Filter

*** property 1** The spectrum of the output signal of a matched filter with the matched signal as input is except for time delay factor is proportional to the energy spectral density of the input signal given as

$$S_o(f) = H_{opt}(f)\, S(f)$$

$$S_o(f) = S^*(f)\, S(f) \exp(-j2\pi fT)$$

$$S_o(f) = |S(f)|^2 \exp(-j2\pi fT)$$

*** property 2**

The output signal of a matched filter is proportional to the shifted version of the autocorrelation function of the input signal to which the filter is matched.

$$\boxed{S_o(t) = R_S(t-T)}$$

*** property 3**

The output signal to noise ratio of a matched filter depends only on the ratio of the signal energy to the power spectral density of white noise at the filter input.

$$\boxed{(SNR)_{o/opt} = \frac{2E}{N_0}}$$

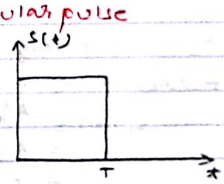
*** property 4 :→**

The matched filtering operation is seperated into two matching condition namely spectral phase matching that produces the desired output at time T, and spectral amplitude matching that maximizes the output signal to noise ratio at time $t = T$. given as

$$\boxed{|H(f)| = |S(f)|}$$

* Matched Filter for Rectangular pulse

Consider a rectangular pulse $s(t)$ of duration T and amplitude A given as



$$s(t) = A \quad 0 \le t \le T \quad -①$$
$$\quad\;\, 0 \quad \text{otherwise}$$

fig :→ Rectangular pulse input

For ease we assume pulse $s(t)$ has unit area.
i.e $AT = 1$. Then fourier transform of $s(t)$ is

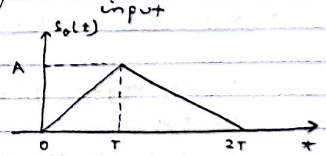


fig :→ Matched filter output

Assuming $AT = 1$.

$$S(f) = \text{sinc}(fT)\exp(-j\pi fT) \quad -②$$

The impulse response of matched filter to the rectangular pulse $s(t)$ is also a rectangular pulse as

$$h_{opt}(t) = A \quad 0 \le t \le T \quad -③$$
$$\qquad = 0 \quad \text{otherwise}$$

The transfer function of matched filter is

$$H_{opt}(f) = \text{sinc}(fT)\exp(-j\pi fT) \quad -④$$

The Fourier transform of the matched filter output is

$$S_0(f) = H_{opt}(f) \cdot S(f)$$

$$= \text{sinc}(ft)\exp(-j\pi ft)\ \text{sinc}(ft)\exp(-j\pi ft)$$

$$\therefore \boxed{S_0(f) = \text{sinc}^2(ft)\exp(-j2\pi ft)} \quad -(5)$$

$\text{sinc}^2(ft)$ is the energy spectral density of the rectangular pulse. which satisfies property 1 of matched filter.

Again convolving $s(t)$ and $h_{opt}(t)$ eqn (1) & (3), the matched filter output is

$$s_0(t) = \begin{cases} At/T & 0 < t \le T \\ A(2 - t/T) & T \le t < 2T \\ 0 & \text{otherwise} \end{cases} \quad -(6)$$

* Matched Filter for Rectangular pulse

For convenience the rectangular pulse $s(t)$ of amplitude A and duration T as A T = 1. we examine two different low pass structures for approximating the matched filter. They are

1. Ideal low pass filter with variable bandwidth

The transfer function of the matched filter

$$H_{opt}(f) = \text{sinc}(fT)\exp(-j\pi fT) \quad -(1)$$

The average Signal to Noise ratio of the ideal low pass filter is power

$$(SNR)'_0 = \frac{(2A/\pi)^2 \sin^2(\pi BT)}{BN_0} \quad -(2)$$

The output signal to Noise ratio of matched filter for rectangular pulse is

$$(SNR)_0 = \frac{2A^2T}{N_0} \quad -(3)$$

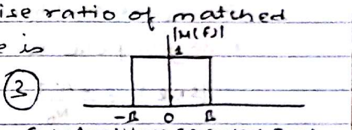


fig:→ Amplitude Response of Ideal LPF approximating matched filter.

Therefore

$$\frac{(SNR)'_0}{(SNR)_0} = \frac{(2A/\pi)^2 \sin^2(\pi BT)}{B\cancel{N_0}} \times \frac{\cancel{N_0}}{2A^2T}$$

$$\boxed{\frac{(SNR)'_0}{(SNR)_0} = \frac{2\sin^2(\pi BT)}{\pi^2 BT}} \quad -(4)$$

2. RC Low pass filter of variable bandwidth

RC Low pass Filter (LPF) provides best approximation to the matched filter for a rectangular pulse $s(t)$ of amplitude A and duration T. If $s_0'(t)$ is the RC LPF output. The response $s_0'(t)$ of RC LPF reaches peak value at time $t = T$

$$s_0'(T) = A\left[1 - \exp\left(-\frac{T}{RC}\right)\right] \quad -(1)$$

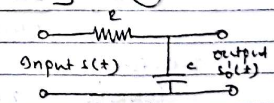


fig:→ RC LPF

As, RC is time constant of filter. 3-dB bandwidth 'B' of the filter is

$$B = \frac{1}{2\pi RC} \quad -(2)$$

putting this value in eqn (1) we get

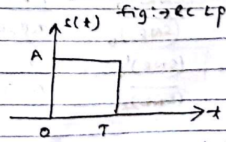


fig:→ Rectangular pulse Input

$$s_0'(T) = A[1-\exp(-2\pi BT)] \quad -(3)$$

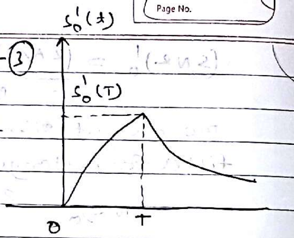


fig:→ Response of the filter.

SNR of the matched filter of rectangular pulse is

$$(SNR)_0 = \frac{2A^2T}{N_0} \quad -(4)$$

To calculate the SNR of RC low pass filter. The transfer function of LPF

$$|H(f)| = \frac{1}{1+j2\pi fRC} = \frac{1}{1+jf/B}$$

Average Noise power at the LPF output is

$$\boxed{N_0' = \int_{-\infty}^{\infty} \frac{N_0}{2}|H(f)|^2 df}$$

$$N_0' = \frac{\pi N_0 B}{2}$$

$$N_0' = \frac{\pi N_0 B}{2} \quad -(5)$$

SNR of RC LPF on solving eq^n (3) & (5) — For power calculation square the eq^n (3)

$$(SNR)_0' = \frac{2A^2}{\pi N_0 B}\left[1-\exp(-2\pi BT)\right]^2$$

Therefore

$$\frac{(SNR)_0'}{(SNR)_0} = \frac{2A^2}{\pi N_0 B}\left[1-\exp(-2\pi BT)\right]^2 \times \frac{N_0}{2A^2T}$$

$$\boxed{\frac{(SNR)_0'}{(SNR)_0} = \frac{1}{\pi BT}\left[1-\exp(-2\pi BT)\right]^2} \quad -(6)$$

* Impulse Response of Matched Filter

The transfer function of optimum filter is

$$H(f) = K\frac{X^*(f)}{S(f)}e^{-j2\pi fT}$$

$$H(f) = K\frac{X^*(f)}{S(f)}e^{-j2\pi fT}$$

$$S(f) = \frac{N_0}{2}$$

$$\therefore H(f) = \frac{2K}{N_0}X^*(f)e^{-j2\pi fT}$$

Complex Conjugate $X^*(f) = X(-f)$

$$H(f) = \frac{2K}{N_0}X(-f)e^{-j2\pi fT}$$

Taking Inverse Fourier transform (IFT) we get

$$\boxed{h(t) = \frac{2K}{N_0}x(T-t)}$$

* Realization of matched Filter (Time Correlators)

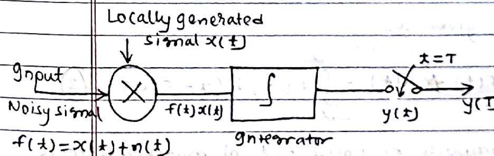


fig:→ Block Diagram of Correlator

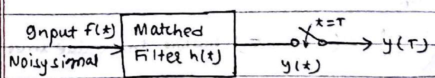


fig:→ Block Diagram of Matched Filter Receiver

The Correlator is a coherent reception which correlates the received signal $f(t)$ with a stored replica of the known signal $x(t)$.

Above figure (a) shows the block diagram of the correlator where received signal $f(t)$ is introduced to product modulator with locally generated signal $x(t)$.

It is then passed through integrator and is sampled by sampler for $t = T$.

The output of integrator is

$$y(t) = \int_0^T f(t)\, x(t)\, dt \quad \text{—(1)}$$

At $t = T$, output of correlator is

$$y(T) = \int_0^T f(t)\, x(t)\, dt \quad \text{—(2)}$$

For matched filter

fig (b) the output is given as

$$y(t) = f(t) \overset{\text{convolution}}{*} h(t) = \int_{-\infty}^{\infty} f(\tau)\, h(t-\tau)\, d\tau \quad \text{—(3)}$$

As we know, the impulse response $h(t)$ of matched filter is

$$h(t) = \frac{2K}{N_0}\, x(T-t) \quad \text{—(4)}$$

$\therefore$ $h(t-\tau) = \frac{2K}{N_0}\, x(T-t+\tau)$ putting this value in eqn (3)

we get, $y(t) = \int_{-\infty}^{\infty} f(\tau)\, \frac{2K}{N_0}\, x(T-t+\tau)\, d\tau$

when passed through the sampler at $t = T$, the output of matched filter received is

$$y(T) = \int_{-\infty}^{\infty} f(\tau)\, \frac{2K}{N_0}\, x(T-T+\tau)\, d\tau$$

$$\boxed{y(T) = \frac{2K}{N_0} \int_{-\infty}^{\infty} f(\tau)\, x(\tau)\, d\tau}$$

For convenience

$$\boxed{y(T) = \frac{2K}{N_0} \int_{-\infty}^{\infty} f(t)\, x(t)\, dt}$$

* Gaussian Distribution or process

Gaussian random variable is the most widely used random variable in the statistical analysis of communication system. Gaussian distribution or process gives the analytical result of the system. It is a mathematical justification of study of physical phenomena for random variable and random events.

For a gaussian random variable 'x' of mean m_x and variance σ_x^2 has the probability density function (PDF)

$$\boxed{f_x(x) = \frac{1}{\sqrt{2\pi}\,\sigma_x} \exp\left[-\frac{1}{2\sigma_x^2}(x-m_x)^2\right]}$$

For second form of integral for $f_x(x) \geq 0$

$$\boxed{\int_{-\infty}^{\infty} f_x(dx) = \frac{1}{\sqrt{2\pi}\,\sigma_x} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2\sigma_x^2}(x-m_x)^2\right] dx}$$

Gaussian process can also be expressed as, If $n(t)$ is assumed to be zero mean Gaussian process with variance N_0, the noise $n_0(t)$ will also be Gaussian process with variance N_0

$$\boxed{P_e = \frac{1}{2}\int_{|x|>A}^{\infty} \frac{1}{\sqrt{2\pi N_0}} \exp\left(-\frac{x^2}{2N_0}\right) dx}$$

$$\boxed{P_e = \int_{A}^{\infty} \frac{1}{\sqrt{2\pi N_0}} \exp\left(-\frac{x^2}{2N_0}\right) dx}$$

6.6 performance limitation of base bond Communication due to noise

Noise has been limiting factor for every communication system whether it is digital or analog. Interception of noise decreases the performance of the system. practically, we can't get completely noise free system but we can take necessary steps or measures to reduce or mitigitae the effect due to noise.

performance of digital Communication System is evaluated in terms of probability. That means decoder receives or decodes the receive bit rightly in absence of noise effect or receives wrongly in presence of noise. This presence of noise and introduced error is measured in terms of probablity and named as error probabilities.

Flair

* Error probabilities in binary and M-ary baseband data Communication

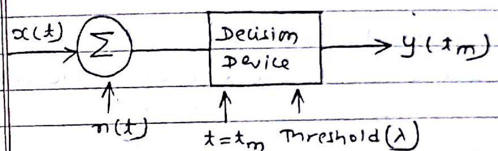


fig:→ Base band data Communication system

For baseband data communication (PAM, PPM, PDM, PCM) System the error probability is calculated as

output

$$\boxed{y(t_m) = A_m + n_0(t_m)} \quad \text{—(1)}$$

where,

$A_m = +A$ If $b_m = 1$ bit 1 transmitted.

$A_m = -A$ If $b_m = 0$ bit 0 transmitted.

In absence of Noise, Decision device compares $y(t_m)$ against the threshold (λ) level.

i.e $y(t_m) > 0$ bit 1 was transmitted.

$y(t_m) < 0$ bit 0 was transmitted. —(2)

For presence of Noise error will occur and decision is made in favour of bit '1' when actually bit '0' was transmitted and favour of bit '0' when actually bit '1' was transmitted.

In other words the condition for occurence of the error can be stated as

$y(t_m) < 0$ when $A_m = +A$ bit 1 transmitted

$y(t_m) > 0$ when $A_m = -A$ bit 0 transmitted —(3)

Flair

Scanned by CamScanner

The total probability of error is sum of each error probabilities given as

$$P_e = p[y(t_m)>0/b_m=0] \times p(b_m=0) + p[y(t_m)<0/b_m=1] \times p(b_m=1)$$

As bit sequence 1 and 0 are equi-probable and independent. $\left[p(b_m=0) = p(b_m=1) = \frac{1}{2}\right]$

$$P_e = \frac{1}{2} p[y(t_m)>0/b_m=0] + \frac{1}{2} p[y(t_m)<0/b_m=1]$$

In terms of Noise

$$P_e = \frac{1}{2}\left[p[n_0(t_m)>A] + p[n_0(t_m)<-A]\right]$$

$$\boxed{P_e = \frac{1}{2} p\left[|n_0(t_m)|>A\right]} \quad — (4)$$

Probability density function (pdf) of error probability for $n(t)$ assumed zero mean Gaussian process with variance 'N_0' is given as

$$P_e = \frac{1}{2}\int_{|x|>A}^{\infty} \frac{1}{\sqrt{2\pi N_0}} \exp\left(-\frac{x^2}{2N_0}\right) dx$$

$$P_e = \int_{A}^{\infty} \frac{1}{\sqrt{2\pi N_0}} \exp\left(-\frac{x^2}{2N_0}\right) dx$$

put $z^2 = \frac{x^2}{2N_0} \Rightarrow x = \sqrt{2N_0}\, z \Rightarrow \frac{dx}{dz} = \sqrt{2N_0} \Rightarrow dx = \sqrt{2N_0}\, dz$

Chainging limit :-> $(x = A)$ lower limit

$$\boxed{\frac{A}{\sqrt{2N_0}} = z}$$

Flair

$$\therefore P_e = \int_{A/\sqrt{2N_0}}^{\infty} \frac{1}{\sqrt{2\pi N_0}} \exp(-z^2)\sqrt{2N_0}\, dz$$

$$P_e = \int_{A/\sqrt{2N_0}}^{\infty} \frac{1}{\sqrt{\pi}} \exp(-z^2)\, dz$$

will be given as Complimentary error function

$$\mathrm{erfc}(u) = \int_{u}^{\infty} \frac{1}{\sqrt{\pi}} \exp(-z^2)\, dz$$

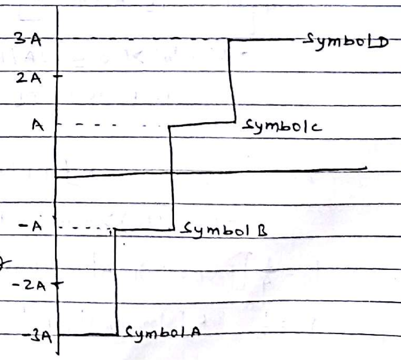
Therefore error probability can be calculated using error function erf(u) described as

$$\boxed{\mathrm{erf}(u) = \int_{0}^{u} \frac{1}{\sqrt{\pi}} \exp(-z^2)\, dz}$$

$$\boxed{P_e = \frac{1}{2}\mathrm{erfc}\left(\sqrt{\frac{A^2 T_b}{4N_0}}\right)}$$

* Error probability for M-ary system

For M-ary data communication system the error probability (Pe) can be calculated for different value of M or level. For generalizing taking M=4, shown in figure shows four



Flair

different level and represented with four different symbol.

For $M=4$, the decoding algorithm is as

$y(t_m) > 2A$ Symbol D

$0 < y(t_m) \le 2A$ Symbol C

$-2A < y(t_m) \le 0$ Symbol B

$y(t_m) \le -2A$ Symbol A

Probability of error is

$$P_e = P(\text{error}/D\text{sent}) \times P(D\text{sent}) + P(\text{error}/A\text{sent}) \times P(A\text{sent}) + \ldots$$

For equiprobable

$$P(D\text{sent}) = P(A\text{sent}) = P(C\text{sent}) = P(A\text{sent}) = \frac{1}{4}$$

$$P_e = \frac{1}{4}\Big[P[y(t_m) \le 2A / A_m = 3A] + P[y(t_m) > 2A / A_m = -3A] + P[y(t_m) > 0 \text{ or } \le -2A / A_m = -A] + P[y(t_m) > 2A \text{ or} \le 0 / A_m = A]\Big] \quad -(1)$$

Eq^n (1) expressed in terms of noise as

$$P_e = \frac{1}{4}\Big[P[n_0(t_m) < -A] + P[n_0(t_m) > A \text{ or } n_0(t_m) < -A] + P[n_0(t_m) > A \text{ or } n_0(t_m) < -A] + P[n_0(t_m) > A]\Big]$$

$$P_e = \frac{1}{4}\Big[P[|n_0(t_m)| > A] + P[|n_0(t_m)| > A] + P[|n_0(t_m)| > A]\Big]$$

$$P_e = \frac{1}{4}\, 3P[|n_0(t_m)| > A]$$

$$\boxed{P_e = \frac{3}{4} P[|n_0(t_m)| > A]} \quad -(2)$$

Again probability density function (pdf) of error probability for $n(t)$ assumed zero mean Gaussian process with variance 'N_0' is given as

$$P_e = \frac{3}{4} \times \frac{1}{2} \int_{|x|>A}^{\infty} \frac{1}{\sqrt{2\pi N_0}} \exp\left(-\frac{x^2}{2N_0}\right) dx$$

$$P_e = \frac{3}{4} \int_{A}^{\infty} \frac{1}{\sqrt{2\pi N_0}} \exp\left(-\frac{x^2}{2N_0}\right) dx$$

put $z^2 = \frac{x^2}{2N_0} \Rightarrow x = \sqrt{2N_0}\, z \Rightarrow \frac{dx}{dz} = \sqrt{2N_0} \Rightarrow dx = \sqrt{2N_0}\, dz$

Changing Limit :> ($x = A$ For lower limit)

$$\frac{A}{\sqrt{2N_0}} = z$$

$$P_e = \frac{3}{4} \int_{A/\sqrt{2N_0}}^{\infty} \frac{1}{\sqrt{2\pi N_0}} \exp(-z^2) \sqrt{2N_0}\, dz$$

$$\boxed{P_e = \frac{3}{4} \int_{A/\sqrt{2N_0}}^{\infty} \frac{1}{\sqrt{\pi}} \exp(-z^2)\, dz = \frac{m-1}{m} \int_{A/\sqrt{2N_0}}^{\infty} \frac{1}{\sqrt{\pi}} \exp(-z^2)\, dz}$$

will be given as Complementary error function

$$\text{erfc}(u) = \frac{3}{4} \int_{u}^{\infty} \frac{1}{\sqrt{\pi}} \exp(-z^2)\, dz$$

Therefore error probability can be calculated using error function erf(u) described as

$$erf(u) = \frac{3}{4}\int_0^u \frac{1}{\sqrt{\pi}} \exp(-z^2)\, dz$$

$$\boxed{erf(u) = \frac{m-1}{m} \int_0^u \frac{1}{\sqrt{\pi}} \exp(-z^2)\, dz}$$

which shows their is high error probability in M-ary system then binary system.