Why digital ?

→ Noise immunity is better than ~~at~~ analog.
→ error detection and correction can be performed
→ repeaters can be used as it is possible to separate out noise and signal.
→ Improves the quality of signal
→ ~~ad~~ we can add new features
→ reduce the cost of conventional voice services.

But at the cost of higher BW requirement for increased bit rates.

1. INTRODUCTION

1.1 Digital Communication sources, transmitters, transmission channels and receivers.

Sources

It can be defined as the origin of any information that needs to transmitted or coveyed to any particular destination. Based on the nature of electrical signal produced at the origin, the source can be classified as analog or digital source.

Analog sources produces electrical signals that are continuous f^n of time for example, output of microphone, TV, camera etc.

Digital sources produce a sequence of symbols at fixed interval of time. for example, A, f, @, #, etc.

Now since we are more concerned with digital communications, we will proceed with discrete nature of sources.

i. Discrete communication sources.

In case of digital communication, the information or communication source produces a message signal that is not continuously varying with time. The output of such discrete sources such as a teletype or the numerical output of a computer consists of a sequence of discretes symbols or letters. Also, an ~~at~~ analog information can be converted into a discrete form by sampling and quantizing. Such discrete information sources can be characterized by following parameters,

a. source alphabets (letter, digits, special characters.) or symbols.
b. symbol rate (rate in which information source generates source alphabet).
unit → symbols / sec
c. source alphabet probabilities.
d. probabilistic dependence of symbol in a sequence.
e. entropy (H) and
f. information rate (R).

ii) Source encoder :

It converts ~~sequency~~ sequence of symbols at its input into binary sequence of 1's and 0's, by assigning codeward to each individual source alphabets or symbols.

A codeword is a combination of 1's and 0's and thus can have bit length equal to any power of 2. So, the number of bits used decide the number of symbols represented by a codeword.

So if we use 8 bits, we can have $2^8 = 256$ distinct codeward representing 256 distinct symbols.

Source encoder must have following important parameters,

a. Block size : It is the maximum number of distinct codeword that can be represented by a source encoder.
Block size of 4 bit source encoder is $2^4 = 16$.
b. codeword length: It is the number of bits used to represent each symbol.
c. Average data rate (ADR) : It is the rate at which a source encoder produces an output.

So, if the symbol rate is 10 symbols per second and the length of codeword is 8, then

ADR = 10 symbols / sec × 8 bits / symbol
= 80 bits / sec

A, B → Source Encoder → 11001, 10110
1, 2 → 01010, 10101

This process of converting the source alphabets into their unique codeword is known as source coding.

Basically there are two methods in source coding,

1. Fixed length coding (FLC)
→ It is used to convert the symbols with equal probability into binary streams.

2. Variable length coding (VLC)
→ It is used to covert the symbols with different probabilities into binary stream.

eg. Shannon-Fano & Huffmand coding.

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iii) Channel encoder :

It is known that a communication channel adds noise and interference to the signal being transmitted. So, if the output of the source encoder is directly fed to the channel, errors will be introduced in the binary sequence received at the receiver end. To minimize these errors, channel encoding is done. The channel encoder adds one or more bits as error control bits (redundant bits). Thus a channel encoder is used to enhance the reliability and efficiency of digital signal transmission by adding extra bits to the actual message sequence.

eg. Block coding, cyclic coding etc.

iv) Channel modulator :

It is intended to convert the bit streams from channel encoder to electrical waveform for suitable transmission over the communication channel. The proper design of the channel modulator minimizes the effect of noise. It increases the matching of signal characteristics (frequency, power

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and bandwidth) with channel characteristics. It provides multiple data communication over the same physical channel i.e. multiplexing.

v. Channel :

It is a medium through which the encoded signal traverses. For wired communication, the channel can be a cable or optical fiber whereas for wireless communication, free space is the available channel. These channels have ~~limited~~ parameters that limit the speed or the rate of data transmission.

Now, the channel has finite frequency bandwidth. The signal power attenuates as it travels along the channel and noise too added at the channel. All this results in distorted signal at the receiver end.

Thus the capacity of a channel can be defined in terms of the given bandwidth (channel) and the required level of SNR or ~~BR~~ bit error rate (BER).

So, the Shannon channel capacity theorem equates,

$$C = B \log_2 (1 + SNR) \text{ bits/sec}$$

where C = channel capacity or maximum speed of data transmission

B = channel bandwidth

SNR = signal to noise ratio.

vi) Channel demodulator :

A channel demodulator converts the received electrical signals into bit streams.

vii) Channel decoder :

A channel decoder recovers the information bearing bit stream from the channel demodulator output with minimum error and maximum efficiency.

viii) Source decoder :

It converts the binary output of the channel decoder into the sequence of symbols with minimum errors and maximum efficiency.

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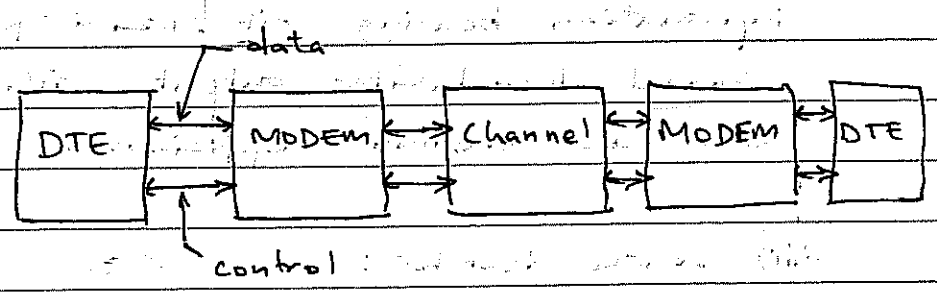
ix) Destination :

It is the desired location where any message needs to be passed to. It is the final stage used to represent the output of soure decoder conveying the original message through the sequence of symbols.

With digital nature of source and its communication form, the whole digital communication form can be taken as a Data Terminal Equipment (DTE) which comprises of computer, data-logger etc, connected through MODEM (modulator - demodulator) unit.



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So, the block diagram of a digital communication source can be viewed as,

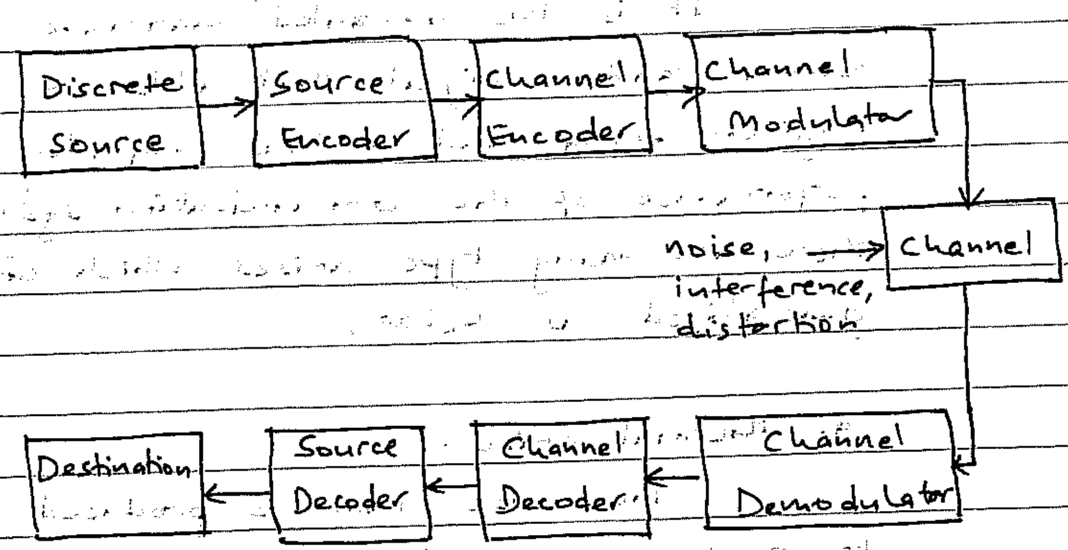

fig. Block diagram of a DCS.

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1.2 Noise, interference and distortion

i. Noise:

It is the unwanted manmade or natural random signal that adds to the original signal and ~~d~~ thus degrades the performance of the communication system. There are many type noises which can be generalized as below,

a. Thermal noise.

The noise that is produced due to random movement of free electrons within the conducting portion of electrical ~~or~~ circuitry of the system. The average thermal noise power can be estimated using,

$$P = KTB \text{ watts}$$

where,

P = noise power

K = Boltzmann's constant (1.38×10^{-23} J/°K

T = Temperature of conductors in °K

B = Bandwidth of noise spectrum, Hz.

b. Shot noise:

In semiconductors, the shot noise is described as the variation in number of electrons crossing the potential barrier. In case of vacuum tubes, it is the anode current noise resulting from the random ~~for~~ fluctuation in electron emission from the cathode.

c. Partition noise:

Partition noise ~~are~~ arises due to the random fluctuations occuring when current is divided into two or more paths.

d. Flicker or low frequency noise:

For frequencies lower than a few KHz, a noise appear in the devices whose spectral density increases as the frequency is decreased. Such noise is known as ~~ff~~ flicker noise or low frequency noise. It is also known as $1/f$ noise.

e. Generation- recombination noise :

This type of noise arises due to the random ionization of impurities produced in the semiconductor device.

f. Transit-time or HF noise :

For very high frequency signals, the signal period is low which may lead to the carriers diffuse back to the source before crossing the junction barrier and thus produces noise. The psdf of this kind of noise increases with frequency.

g. white noise :

It is a random signal that has equal intensities at different frequencies giving it a constant power spectral density. It is analogous to white light.

ii) Interference :

It is referred to as the contamination of received signal by other extraneous, usually manmade, signals similar to the desired signal. Manmade interferences are usually the signals from various broadcasting and communication systems.

Intersymbol interference (ISI) is a special case of interference in digital transmission system when the interference is occurred within the system itself. ISI is thus the effect of all other transmitted bits on the decoding of any ~~particu~~ particular received bit.

iii) Distortion :

Distortion is any unwanted change in the waveform at the output of any device as compared to the input waveform. We can have linear or non-linear distortion due to linear ~~and~~ or non-linear characteristics of semiconductor devices respectively.

So, the block diagram of a DCS can be summed up as,

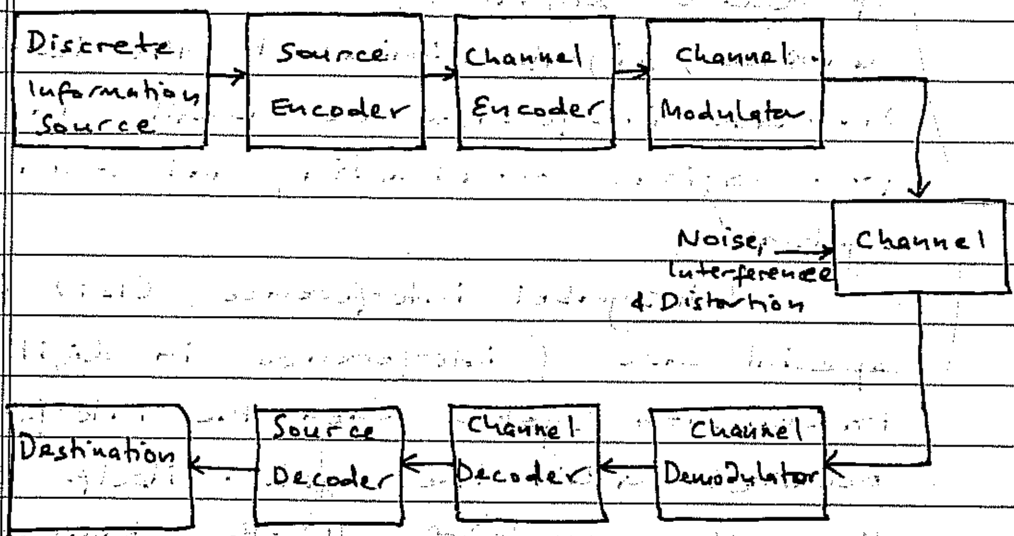
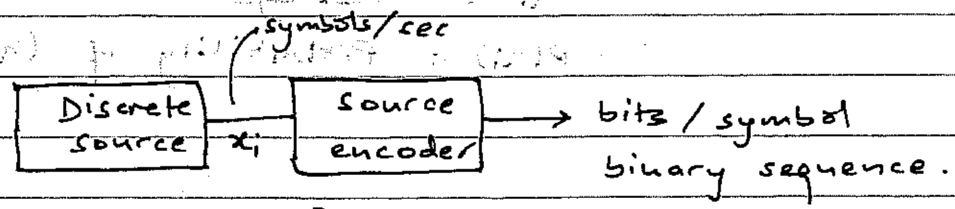


fig. Block diagram of a digital communication system.

1.3. Source coding, coding efficiency, Shannon-fano and Huffman codes, coding of continuous time signals (A/D conversion)

i. Source coding :

It is a type of coding which converts the output of discrete source symbols into a sequence of binary bits (i.e. codeword).



$X = \{x_1, x_2, .. x_m\}$

The objective of source coding is to minimize the average bit rate required for the representation of source by reducing the redundancy of information source.

The codewords generated by source coding can be characterized by

i) Codeword length (n) : It is the number of bits required to represent a symbol.

ii) Average codeword length (L).

It is the average number of bits required to represent the symbols in a message vector.

$$L = \sum_{i=1}^{m} n_i \cdot P(x_i)$$

where,

n_i = number of bits required for individual symbols

x_i = message vector

$P(x_i)$ = Probability of (x_i).

iii) Code efficiency

$$\eta = \frac{H(X)}{L}$$

where $H(X)$ = entropy

L = average codeword length

iv) Code redundancy

$$\gamma = 1 - \eta$$

The codewords can be classified into two types.

a. Fixed length coding (FLC):

It assigns same number of bits to represent different symbols.

i.e. for 26 alphabetic letters, we need 5 bits i.e. $2^5 = 32$. such that, A to Z, all are represented by 5 bit codewords.

A - 00001
B - 00010
Z - 11110

Just an example.

b. ~~Vary~~ Variable length coding (VLC):

It assigns varying number of bits to represent different symbols.

A - 00
B - 01
C - 010
D - 110
E - 1101

Just an example.

There are two methods generally used for variable length coding, namely,

a. Shannon-Fano coding
b. Huffman coding

a. Shannon-Fano coding

Procedure.

i) Arrange the source symbols in the order of decreasing probability.

ii) Separate the set into two parts such that they are as equiprobable as possible.
Assign '0' to upper set
Assign '1' to lower set.

iii) Continue separating the upper and lower sets into two sets each of equiprobable value until no further partitioning is possible.
Assign '0' to each upper set and '1' to each lower set.

(iv) Take the codewords for each symbol.

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Given, probabilities for 6 symbols are 0.12, 0.08, 0.05, 0.25, 0.20 and 0.30. Find the codewords for each symbol.

First step is to arrange the symbols in their decreasing order, i.e.

x_i	$P(x_i)$	Step 1	Step 2	Step 3	Step 4		Codeword
x_1	0.30	0	0			2	00
x_2	0.25	0	1			1	01
x_3	0.20	1	0			2	10
x_4	0.12	1	1	0		3	110
x_5	0.08	1	1	1	0	4	1110
x_6	0.05	1	1	1	1		1111

The first partition results in x_1 & x_2 in upper half with sum of probabilities = 0.55 so, put '0' in for x_1 & x_2. and put '1' in for x_3, x_4, x_5 & x_6.

Now for x_1 & x_2, upper probability is set '0' and lower '1'.

And for x_3, x_4, x_5 & x_6, dividing them into equiprobable halves leaves x_3 on upper half and x_4, x_5 & x_6 on lower half.

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Continuing the process above we get only two probabilities at the bottom.
Hence we get the codewards,

$x_1 = 00$ $n_1 = 2$
$x_2 = 01$ $n_2 = 2$
$x_3 = 10$ $n_3 = 2$
$x_4 = 110$ $n_4 = 3$
$x_5 = 1110$ $n_5 = 4$
$x_6 = 1111$ $n_6 = 4$

Now, the average codeword length,

$$L = \sum_{i=1}^{m} n_i \, P(x_i)$$

$$= \sum_{i=1}^{6} n_i \, P(x_i)$$

$$= 2\times 0.30 + 2\times 0.25 + 2\times 0.20 + 3\times 0.12 + 4\times 0.08 + 4\times 0.05$$

$\therefore L = 2.38$ bits / symbol.

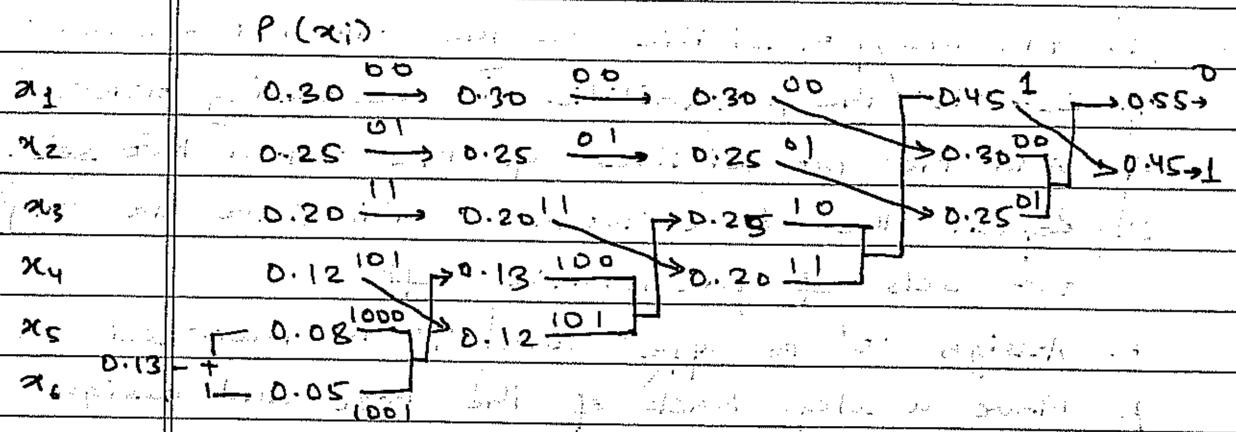
\# In Shannon-Fano coding, ~~ambi~~ ambiguity (confusion) may arise in the choice of approximating the ~~equiprobt~~ equiprobable sets.

b. Huffman-coding

Procedure

1. Arrange the symbols in decreasing probability.
2. Add the probabilities of the bottom two sets.
3. Re-order the probabilities in decreasing order.
4. Add the probabilities of the bottom two sets.
5. Repeat the above process until there are only two sets of probabilities left.
6. Assign '0' to upper and '1' to lower set.
7. Move a step back of the tree and assign '0' and '1' to the respective branches.
8. The for the 'added' probabilites put '0' and '1' to the upper and lower branch respectively.
9. Keep moving back repeating process '8' till all the sets gain an individual codeword.

Working out on the same set of example as for shannon-fano, we have probabilities of 6 symbols as 0.12, 0.08, 0.05, 0.25, 0.20 and 0.30.



So,

$x_1 = 00$ $\qquad n_1 = 2$

$x_2 = 01$ $\qquad n_2 = 2$

$x_3 = 11$ $\qquad n_3 = 2$

$x_4 = 101$ $\qquad n_4 = 3$

$x_5 = 1000$ $\qquad n_5 = 4$

$x_6 = 1001$ $\qquad n_6 = 4$

$$L = \sum_{i=1}^{6} n_i \, P(x_i)$$

$$= 2\times0.30 + 2\times0.25 + 2\times0.20 + 3\times0.12 + 4\times0.08 + 4\times0.05$$

$= \cancel{2.38}$ 2.38 bits / symbol

c. coding of continuous time signals (A/D conversion)

Analog to digital conversion is a process in which a continuously variable (analog) is changed without altering its essential content into a digital signal.

The analog signal consists of varying voltage levels eg. sinewaves, human speech waveforms etc. These almost infinite number of voltage levels are converted to defined number of levels with the help of an analog to digital converter.

The number of levels now assigned are always a power of two i.e. 2, 4, 8, 16 etc.

Examples

1. Construct Shannon-fano, Huffman and fixed length codes for following symbols, with probabilities,

0.25, 0.20, 0.15, 0.12, 0.08, 0.07, 0.07, 0.06

for, shannon-fano coding,

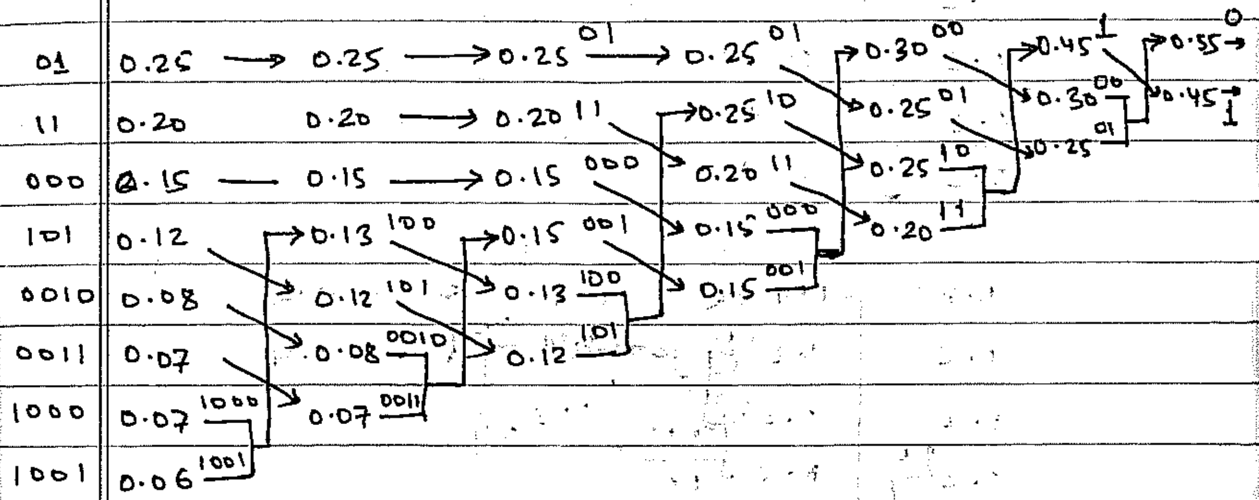
						Codeword
x_1	0.25	0	0			00
x_2	0.20	0	1			01
x_3	0.15	1	0	0		100
x_4	0.12	1	0	1		101
x_5	0.08	1	1	0	0	1100
x_6	0.07	1	1	0	1	1101
x_7	0.07	1	1	1	0	1110
x_8	0.06	1	1	1	1	1111

for fixed length codes, we have eight different symbols, thus the number of bits required = lo antilog$_2$ 8 = 3.

∴ codewords can be,

000	100
001	101
010	110
011	111

Huffman coding



(#) Construct Huffman, shannon-fano and fixed length codes where,

$(P_1, P_2, P_3, P_4, P_5, P_6) = (0.3, 0.2, 0.2, 0.1, 0.1, 0.1)$

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2. Sampling theory.

2.1 Sampling:

It is a process performed to conver analog signals to digital format. A continuou time signal is converted to discrete time signal by measuring the signal at periodic instants of time.

A sufficient number of samples of the analog signal must be taken such that the original signal is represented in its samples completely. And these samples ~~sou~~ should lead to the reconstruction of the original signal completely.

Thus, a sampling theorem stated by Nyquist - Kotelnikov is,

i) A band-limited signal of finite energy, which has no frequency-component higher than f_m Hz, is completely described by its sample values at uniform intervals less than or equal to $1/2f_m$ seconds apart.

i.e. $T_s \leq \frac{1}{2f_m}$.

ii) A bandlimited signal of finite energy which has no frequency component higher than f_m Hz, may be completely recovered from ~~its~~ the knowledge of its samples taken at the rate of $2f_m$ samples per second.

i.e. $f_s = 2f_m$.

The first part of the theorem is implemented to the transmitter section whereas the second part indicates the receiver section.

The sampling theorem thus can be summarized as,

'A continuous time signal may be completely represented in its samples and recovered back if the sampling frequency is

$$f_s \geq 2f_m$$

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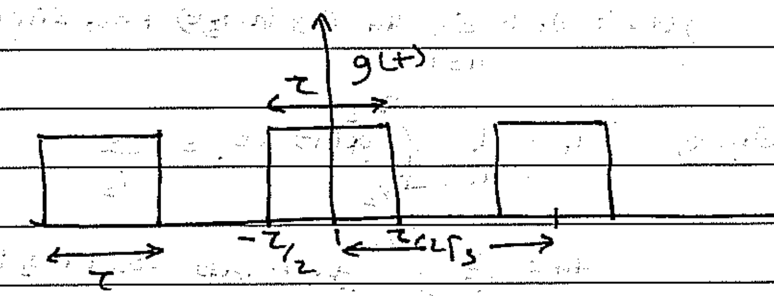
If the signal $x(t)$ to be sampled is band-limited then the sampled signal can be represented as

$$x_s(t) = x(t) \cdot g(t)$$

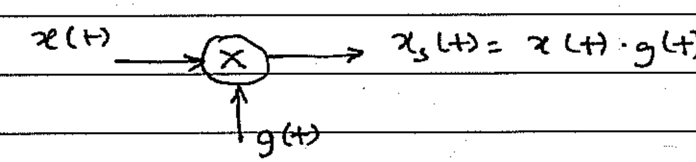
where $g(t)$ is sampling function shown as below

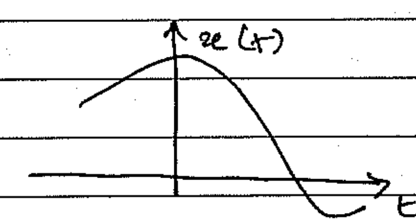


where, τ = duration of sampling pulse

T_s : sampling period.

The sampler can be implemented using the following arrangement,





$x(t)$ with max frequency f_m.

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Now, to prove the two sampling theorem stated above, let us find the spectrum of $x_s(t)$. For that the function $g(t)$ can be expressed in terms of Fourier series as,

$$g(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_s t) + b_n \sin(n\omega_s t)$$

where, $a_0 = \frac{1}{T_s}\int_{-\tau/2}^{\tau/2} g(t)\,dt = \frac{\tau}{T_s}$

$$a_n = \frac{2}{T_s}\int_{-\tau/2}^{\tau/2} g(t)\cos(n\omega_s t)\,dt$$

$$= 2f_s\cdot\tau \ \text{Sinc}[nf_s\cdot\tau] \quad [g(t) \text{ with amplitude } 1]$$

For even signal,

$b_n = 0$

So,

$$g(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_s t)$$

so we have $x_s(t)$ as,

$$x_s(t) = x(t)\cdot g(t)$$
$$= a_0\cdot x(t) + a_1 x(t)\cdot\cos(\omega_s t)$$
$$+ a_2 x(t)\cdot\cos(2\omega_s t)$$
$$+ a_3 x(t)\cdot\cos(3\omega_s t)$$
$$+ a_n x(t)\cdot\cos(n\omega_s t) + \dots$$

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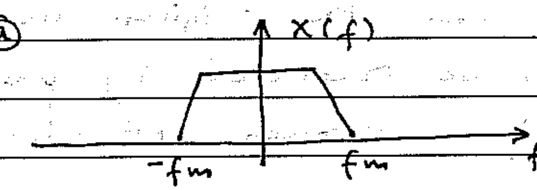
Now, the Fourier transform of $x_s(t)$ is,

$$X_s(f) = a_0 X(f) + a_1[X(f-f_s) + X(f+f_s)]$$
$$+ a_2[X(f-2f_s) + X(f+2f_s)]$$
$$+ a_3[X(f-3f_s) + X(f+2f_s)]$$
$$+ \dots$$

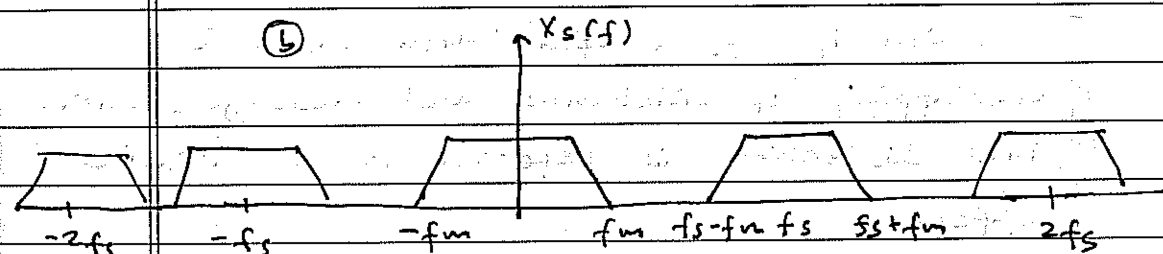
or $X_s(f) =$

The above series can be graphically represented

(a)



(b)



The figure (a) represent the FT of the original signal $x(t)$ and figure (b) is the spectrum of the signal at the output of the sampler. The figure (b) indicates that it contains the spectrum of the original message signal.

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From the figure it is clear that for distortionless recovery of original message signal from the spectrum of the sampled signal following condition should be met,

$$f_s - f_m \geq f_m$$

$$\text{or } f_s \geq 2f_m$$

In this case the original message spectra can be recovered by passing the sampled signal through LPF with bandwidth equal to $\pm f_m$.

Now if $f_s < 2f_m$ then there is overlapping of sidebands and message spectra. This distortion is referred to as aliasing effect.

Therefore the minimum sampling rate for distortionless recovery of message spectrum is known as Nyquist's sampling rate.

i.e.,

$$f_{sN} = 2f_m$$

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And the Nyquist interval is ~~known~~ derived as,

$$T_{S,N} = \frac{1}{2f_m}$$

where f_m = Max frequency of message signal

An analog signal is expressed by the equation $x(t) = 3\cos 50\pi t + 10\sin 300\pi t - \cos 100\pi t$. Calculate the Nyquist rate for this signal.

Here,

$$x(t) = 3\cos 50\pi t + 10\sin 300\pi t - \cos 100\pi t \quad \text{-(i)}$$

let three frequencies present be f_1, f_2 & f_3

then,

$$x(t) = 3\cos 2\cdot\pi\cdot \overset{f_1}{\cancel{25}}t + 10\sin 2\pi f_2 t - \cos 2\pi f_3 t \quad \text{(ii)}$$

Comparing (i) & (ii)

$$2\pi f_1 t = 50\pi t \Rightarrow f_1 = 25\,Hz$$

$$2\pi f_2 t = 300\pi t \Rightarrow f_2 = 150\,Hz$$

$$2\pi f_3 t = 100\pi t \Rightarrow f_3 = 50\,Hz$$

Therefore, the maximum frequency present in $x(t)$ is $f_2 = 150\,Hz$.

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Now Nyquist rate is given as,

$f_s = 2f_m$

Here $f_m = f_2 = 150\ Hz$.

$\therefore f_s = 2f_2 = 2 \times 150 = 300\ Hz$.

(#) Find the Nyquist rate and Nyquist interval for the signal

$x(t) = \frac{1}{2\pi} \cos(4000\pi t) \cdot \cos(1000\pi t)$

Here,

$x(t) = \frac{1}{2\pi} \cos(4000\pi t) \cdot \cos(1000\pi t)$

$= \frac{1}{4\pi} \cdot 2\cos(4000\pi t) \cdot \cos(1000\pi t)$

$= \frac{1}{4\pi} \left[\cos(4000\pi t + 1000 t) + \cos(4000\pi t - 1000\pi t) \right]$

$\left[\because 2\cos A \cdot \cos B = \cos(A+B) + \cos(A-B) \right]$

or $x(t) = \frac{1}{4\pi} \left[\cos 5000\pi t + \cos 3000\pi t \right]$

Taking two frequencies as f_1 & f_2, we get,

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$x(t) = \frac{1}{4\pi} \left[\cos 2\pi f_1 t + \cos 2\pi f_2 t \right]$

Comparing two eq^n we get,

$2\pi f_1 t = 5000\pi t$

or $f_1 = 2500\ Hz$

$2\pi f_2 t = 3000\pi t$

$f_2 = 1500\ Hz$

$\therefore f_m = f_1 = 2500\ Hz$,

And $f_s = 2f_m$ = Nyquist rate

$= 2 \times 2500\ Hz$

$= 5000\ Hz$

$= 5\ kHz$

and Nyquist interval,

$T_s = \frac{1}{2f_m} = \frac{1}{2 \times 2500} = \frac{1}{5000}$

$\therefore T_s = 0.2 \times 10^{-3}$ seconds

$= 0.2$ msec.

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(#) A continuous time signal is given below:

$x(t) = 8 \cos 200\pi t$

Determine,

i) Minimum sampling rate

ii) If sampling frequency is 400 Hz. What is the discrete-time signal $x(n)$ or $x(nT_s)$ obtained after sampling?

iii) If the sampling frequency $f_s = 150$ Hz. What is the discrete-time signal $x(n)$ or $x(nT_s)$ obtained after sampling?

iv) What is the frequency $0 < f < f_s/2$ of the signal input that yields samples identical to those obtained in part (iii).

Here,

(i) $x(t) = 8 \cos 200\pi t$

or $2\pi f t = 200\pi t$

$\Rightarrow f = 100$ Hz

$\therefore f_s = 2f = 200$ Hz.

(ii) Given $f_s = 400$ Hz.

The frequency of discrete time signal,

$F = \frac{f}{f_s} = \frac{100}{400} = \frac{1}{4}$

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$\therefore$ The discrete time signal will be given as,

$x(n) = 8 \cos 2\pi F \cdot n$

$= 8 \cos 2\pi \cdot \frac{1}{4} n$

$= 8 \cos \frac{\pi n}{2}$

iii) Here, $f_s = 150$ Hz.

So, $F = \frac{100}{150} = \frac{2}{3}$

Therefore,

$x(n) = 8 \cos 2\pi F . n$

$= 8 \cos 2\pi . \frac{2}{3} . n$

$= 8 \cos \frac{4\pi}{3} n = 8 \cos \left(2\pi - \frac{2\pi}{3}\right) n$

$\therefore x(n) = 8 \cos \frac{2\pi n}{3} \Rightarrow F = \frac{1}{3}$

(iv) For sampling rate of $f_s = 150$ Hz

$F = \frac{f}{f_s}$ or $f = F \times f_s = \frac{1}{3} \times 150 = 50$ Hz

Then the sinusoidal signal will be,

$y(t) = 8 \cos 2\pi f t = 8 \cos 2\pi \times 50 \times t$

$= 8 \cos 100\pi t$

Reconstruction of sampled signal.

The process of reconstructing a continuous time signal $x(t)$ from its samples is known as interpolation.

Now, we had,

$$x_s(t) = x(t) \cdot g(t)$$

where $g(t)$ is a rectangular pulse train, periodic at T_s such that the fourier series representation of $g(t)$ is,

$$g(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_s t) + b_n \sin(n\omega_s t)$$

where, $a_0 = \frac{1}{T_s} \int_{-\tau/2}^{\tau/2} g(t)\, dt$

and $a_n = \frac{2}{T_s} \int_{-\tau/2}^{\tau/2} g(t) \cdot \cos(n\omega_s t)\, dt$

$b_n = \frac{2}{T_s} \int_{-\tau/2}^{\tau/2} g(t) \sin(n\omega_s t)\, dt$.

Now, if $g(t) = \delta(t)$ i.e. delta f^n,

then,

$$x_s(t) = x(t) \cdot \delta(t)$$

And Fourier series expansion of $\delta(t)$, is

$$\delta(t) = \frac{1}{T_s} + \sum_{n=1}^{\infty} \frac{2}{T_s} \cos(n\omega_s t)$$

$$= \frac{1}{T_s} \left[1 + 2\cos\omega_s t + 2\cos 2\omega_s t + 2\cos 3\omega_s t \ldots \right.$$

$\therefore$ $x_s(t) = x(t) \cdot \delta(t)$

$$= \frac{1}{T_s} \left[x(t) + 2x(t)\cos\omega_s t + 2x(t)\cos 2\omega_s t + 2x(t)\cos 3\omega_s t + \ldots \right]$$

So, from the above equation, it is clear that the sampled signal contains a component $\frac{1}{T_s} \times x(t)$.

Such that,

$$X_s(f) = \frac{1}{T_s} X(f) + \frac{1}{T_s}\left[X(f-f_s) + X(f+f_s)\right] + \ldots$$

So, in order to reconstruct $X(f)$ from $X_s(f)$, we need to pass $X_s(f)$ through an ideal low pass filter with cut-off frequency $\pm f_m$ Hz.

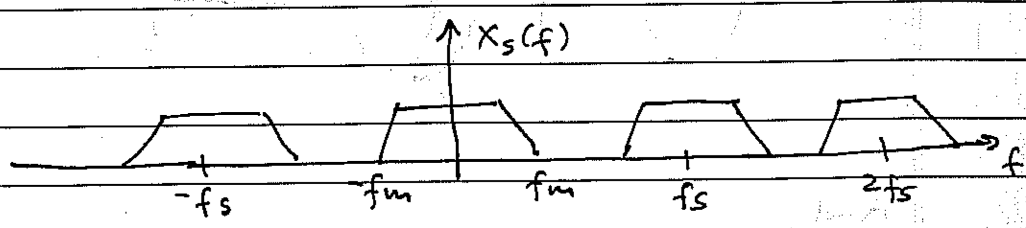
i.e. $H(f) = T_s \times \mathrm{rect}\left(\frac{f}{2f_m}\right)$

such that,

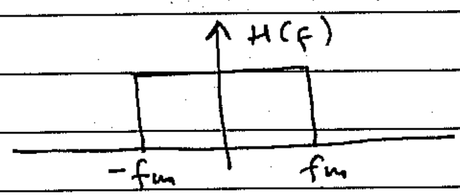
$$X_s(f) \times H(f) = \left[\frac{1}{T_s} X(f) + \frac{1}{T_s}\{X(f-f_s)+\dots\}\right] \cdot \times T_s \, \text{rect}\left(\frac{f}{2f_m}\right)$$

$$= X(f) \quad \text{for } f \leq |f_m|$$

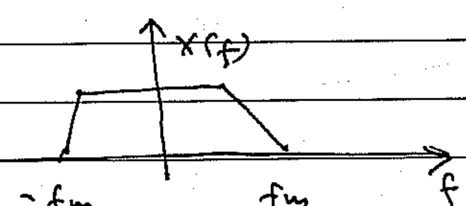
i.e.



$\times$



$=$



Now, the impulse response of the filter with transfer function $H(f)$ is,

$$h(t) = F^{-1}[H(f)]$$

$$= F^{-1}\left[T_s \, \text{rect}\left(\frac{f}{2f_m}\right)\right]$$

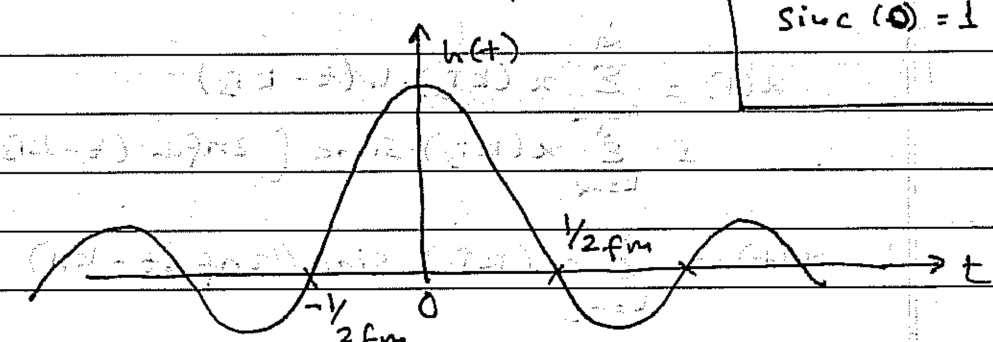
$$= 2f_m \cdot T_s \, \text{Sinc}(2f_m t)$$

Now, if the sampling is done with $f_s = 2f_m$ then, $\frac{1}{T_s} = 2f_m$ or $T_s = \frac{1}{2f_m}$

i.e. $2f_m \cdot T_s = 1$

So, $h(t) = \text{Sinc}(2f_m t)$

$\text{Sinc}(0) = 1$.



Now, $x_s(t) \otimes h(t) = x(t)$

or $x(t) = \Sigma\, x(\tau) \cdot h(t-\tau)$

i.e. when the sampled signal $x_s(t)$ is applied at the input of a filter with $h(t) = \text{sinc}(2\pi f_m t)$, the output will be $x(t)$.

Each sample in $x_s(t)$, being an impulse, produces a sinc pulse of height equal to the strength of the sample.

Addition of these sinc pulses produced by all the samples result in $x(t)$.

Say for the k^{th} sample of $x_s(t)$ is the impulse $x(kT_s)\,\delta(t-kT_s)$ and the filter output of this impulse will be $x(kT_s)\cdot h(t-kT_s)$ and

$$x(t) = \sum_{k=-\infty}^{\infty} x(kT_s)\cdot h(t-kT_s)$$

$$= \sum_{k=-\infty}^{\infty} x(kT_s)\,\text{sinc}\left[2\pi f_m (t-kT_s)\right]$$

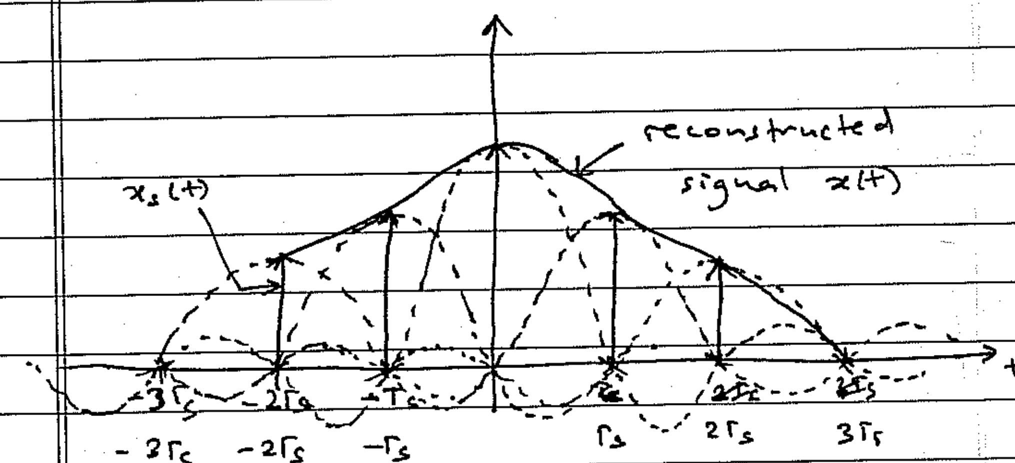
$$\therefore\ x(t) = \sum_{k=-\infty}^{\infty} x(kT_s)\,\text{sinc}(2\pi f_m t - k\pi)$$

The equation above states that $x(t)$ is the weighted sum of all the sample values.

Say for $k = -2$ to 2,

$$x(t) = x[-2T_s]\cdot\text{sinc}\left[2\pi f_m(t+2T_s)\right]$$
$$+\ x[-T_s]\ \text{sinc}\left[2\pi f_m(t+T_s)\right]$$
$$+\ x[0]\ \text{sinc}\left[2\pi f_m t\right]$$
$$+\ x[T_s]\ \text{sinc}\left[2\pi f_m(t-T_s)\right]$$
$$+\ x[2T_s]\ \text{sinc}\left[2\pi f_m(t-2T_s)\right]$$

So, each sampled value is weighted by time shifted sinc function.



$T_s = \frac{1}{2f_m}$

Sampling processes - Types.

1. Ideal sampling / instantaneous sampling

In this type of sampling, the sampling function is a train of impulses. The train of impulses may be represented as,

$$\delta_{T_s}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

let $x(t)$ be the signal to be sampled. So, the sampled signal $x_s(t)$ is expressed as,

$$x_s(t) = x(t) \cdot \delta_{T_s}(t)$$

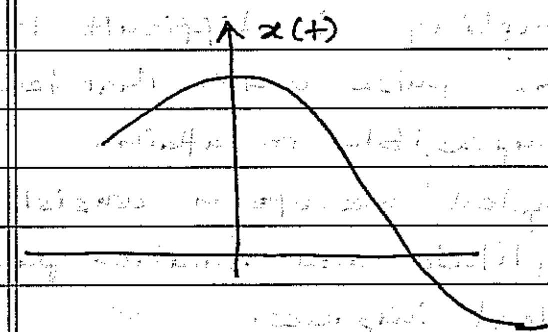
$$= x(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

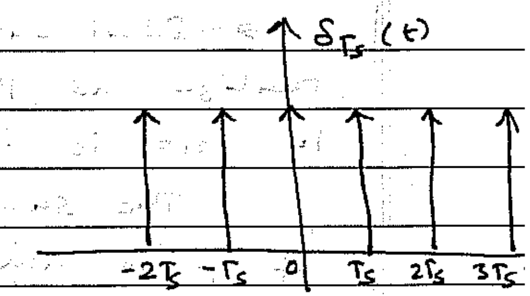
$$\therefore \quad x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\, \delta(t - nT_s)$$

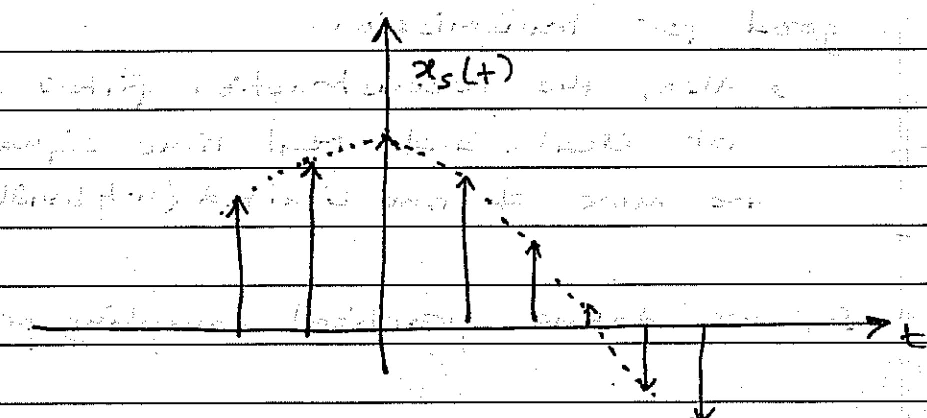
And the F.T. of $x_s(t)$ is,

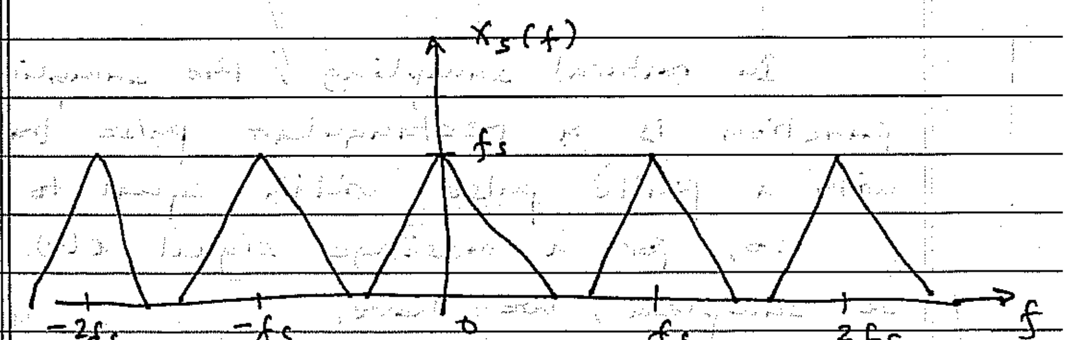
$$X_s(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s)$$

Graphically,









⊕ Practical considerations

→ Ideal sampling is difficult to realize as the pulse width that tends to zero is impossible to attain.

The sampled waveform consists of finite amplitude and duration pulses rather than ideal impulses.

→ Even the power content of the impulses is very small so it is not good for transmission.

→ Also, the reconstruction filters are not ideal. and real time signals are more of time limited (not bandlimited).

So, we define practical sampling processes,

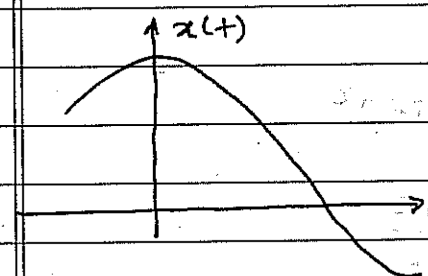
1. Natural sampling process.

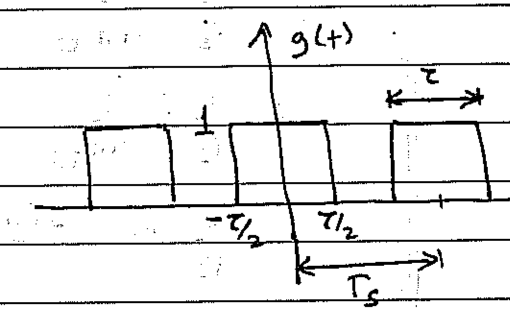
In natural sampling, the sampling function is a rectangular pulse train with a finite pulse width equal to 'τ'.

So, for a message signal $x(t)$ to be sampled, we have,

$$x_s(t) = x(t) \cdot g(t)$$

where, $g(t)$ is a rectangular pulse train.





Now,

$g(t)$ can be represented in term of fourier series as,

$$g(t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi f_s n t}$$

where, $C_n = \frac{1}{T_s} \int_{-T_s/2}^{T_s/2} g(t) e^{-j2\pi f_s n t} dt$

$$= \frac{1}{T_s} \int_{-\tau/2}^{\tau/2} g(t) e^{-j2\pi f_s n t} dt$$

$$= \frac{1}{T_s} \int_{-\tau/2}^{\tau/2} e^{-j2\pi f_s n t} dt \quad [\because g(t) = 1]$$

$$= \frac{1}{T_s} \cdot \frac{1}{-j2\pi f_s \cdot n} e^{-j2\pi f_s n t} \Big|_{-\tau/2}^{\tau/2}$$

or $C_n = \frac{1}{T_s} \cdot \frac{1}{-j2\pi f_s n} \left[e^{-j2\pi f_s n \cdot \tau/2} - e^{+j2\pi f_s n \tau/2} \right]$

$$\text{or}\ C_n = \frac{1}{T_s}\cdot\frac{1}{\pi f_s n}\left[\frac{e^{j2\pi f_s n \cdot \tau/2} - e^{-j2\pi f_s n \tau/2}}{2j}\right]$$

$$= \frac{1}{T_s}\cdot\frac{1}{n\pi f_s}\ \sin 2\pi f_s n\cdot \tau/2$$

$$= \frac{1}{T_s}\cdot\frac{1}{n\pi f_s}\ \sin \pi f_s n \tau$$

$$= \frac{\tau}{T_s}\cdot\frac{\sin \pi f_s n \tau}{n\pi f_s \tau}$$

$$= \frac{\tau}{T_s}\ \mathrm{sinc}(n f_s \tau)$$

Therefore,

$$g(t) = \sum_{n=-\infty}^{\infty} \frac{\tau}{T_s}\ \mathrm{sinc}(n f_s \tau)\, e^{j2\pi n f_s t}$$

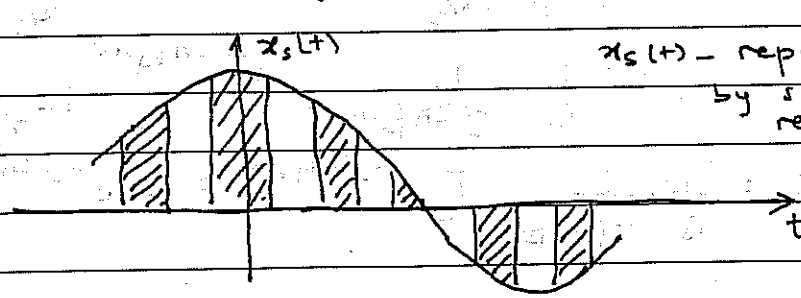
And,

$$x_s(t) = x(t)\cdot g(t)$$

$$= \frac{\tau}{T_s}\sum_{n=-\infty}^{\infty} \mathrm{sinc}(n f_s \tau)\cdot x(t)\cdot e^{j2\pi n f_s t}$$

So, the sampled signal is,

$$x_s(t) = \frac{\tau}{T_s}\sum_{n=-\infty}^{\infty} \mathrm{sinc}(n f_s \tau)\cdot x(t)\ e^{j2\pi n f_s t}$$

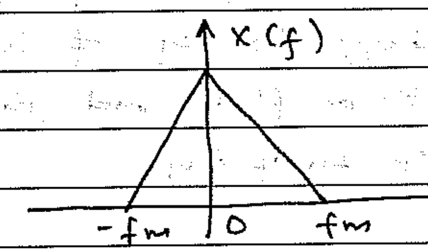


$x_s(t)$ – represented by shaded region.

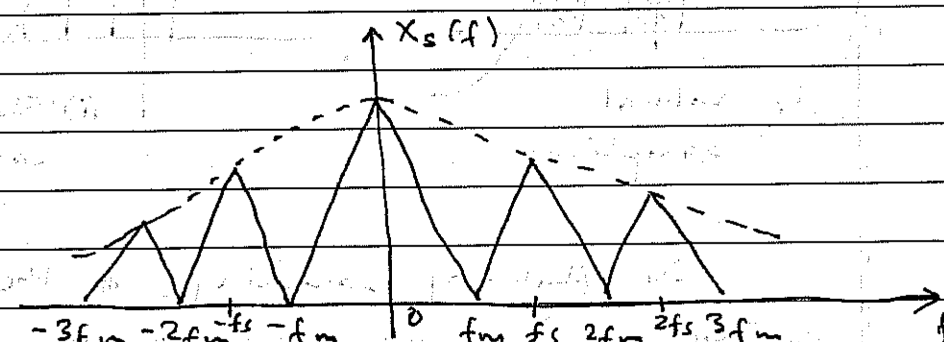
Taking the Fourier transform of $x_s(t)$, we get,

$$X_s(f) = \frac{\tau}{T_s}\sum_{n=-\infty}^{\infty} \mathrm{sinc}(n f_s \tau)\cdot X(f - n f_s)$$

For $x(t)$ with maximum frequency component f_m, we have spectrum of $x(t)$ as,



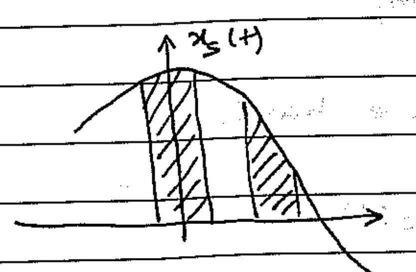
and for $X_s(f)$, we have,



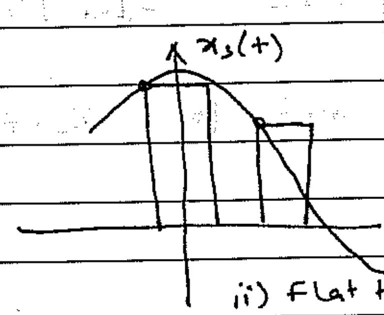
2. Flat top sampling

In case of natural sampling process we can notice that for any individual sampling instances, the amplitude follows the amplitude of the input signal which is in analog form.

Thus, a better or more practical sampling process is flat top-sampling. In such sampling, the top of the samples remain constant or flat and thus it is called flat top sampling.



(i) natural sampling



ii) Flat top sampling

In flat-top sampling, ~~to~~ the top of the samples are equal to the instantaneous value of the bandlimited or baseband signal $x(t)$ at the start of sampling.

The duration or the width of each sample is τ and sampling rate is equal to $f_s = \frac{1}{T_s}$.

Now, the flat top pulse of $x_s(t)$ is mathematically equivalent to the convolution of instantaneous sample and a pulse $h(t)$. i.e. the width of $x_s(t)$ is determined by the width of $h(t)$.

So, for a baseband signal $x(t)$ to be sampled, we first need to multiply the signal $x(t)$ with a train of impulses such that,

$$s(t) = x(t) \cdot \delta_{T_s}(t)$$
$$= x(t) \sum \delta(t - nT_s)$$
$$= \sum_{n=-\infty}^{\infty} x(nT_s)\, \delta(t - nT_s)$$

And to derive, $x_s(t)$ i.e. sampled signal, convolution of $s(t)$ and $h(t)$ is performed where, $h(t)$ is the constant pulse width function.

i.e.

$$x_s(t) = s(t) \otimes h(t)$$
$$= \int_{-\infty}^{\infty} s(\tau) \cdot h(t - \tau)\, d\tau$$

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or $x_s(t) = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x(nT_s)\, \delta(\tau - nT_s)\, h(t-\tau)\, d\tau$

$= \sum_{n=-\infty}^{\infty} x(nT_s) \int_{-\infty}^{\infty} \delta(\tau - nT_s)\, h(t-\tau)\, d\tau$

We have,

$\int_{-\infty}^{\infty} f(t) \cdot \delta(t - t_0)\, dt = f(t_0)$

$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\, h(t - nT_s)$

$\because \int_{-\infty}^{\infty} \delta(\tau - nT_s)\, h(t-\tau)\, d\tau = h(t - nT_s)$

So, the sampled signal due to flat-top sampling is,

$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\, h(t - nT_s)$

Also, $x_s(t) = s(t) \otimes h(t)$

$\therefore X_s(f) = S(f) \cdot H(f)$

$= f_s \sum_{n=-\infty}^{\infty} X(f - nf_s) \cdot H(f)$

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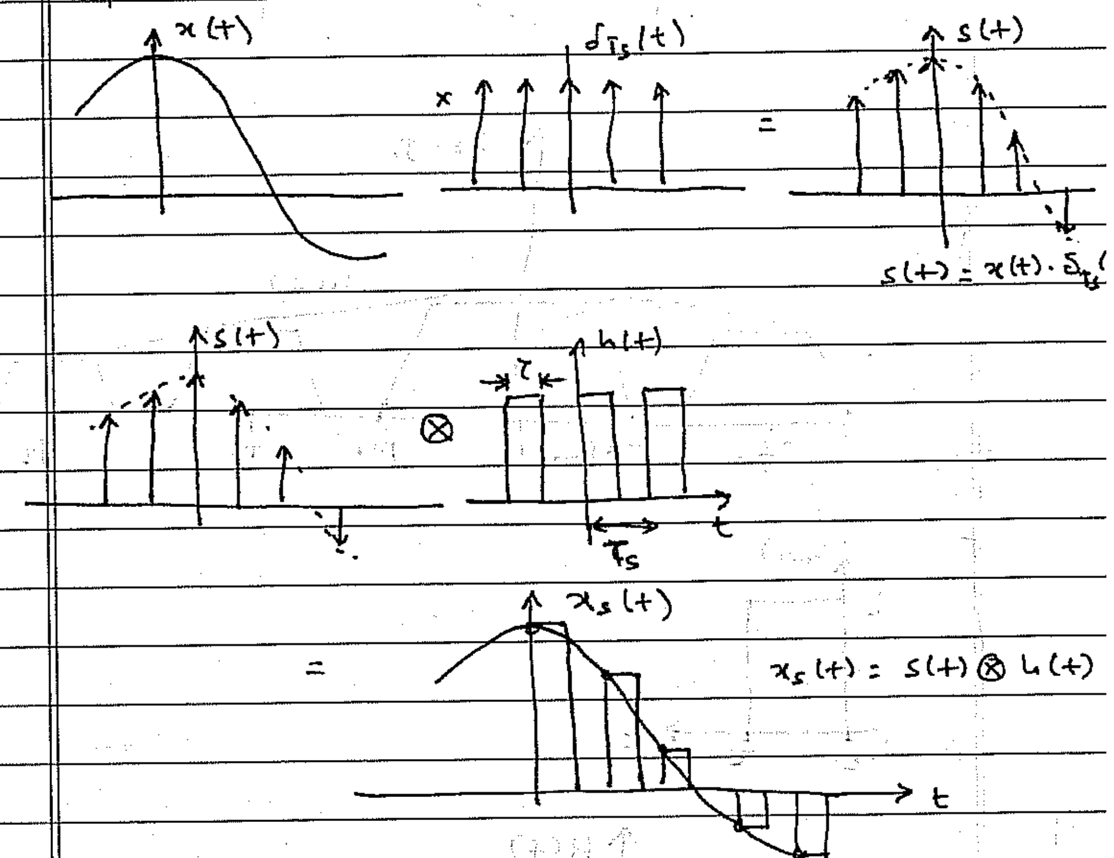
And, $H(f) = \tau\, \mathrm{sinc}(f \cdot \tau)\, e^{-j\pi f \tau}$

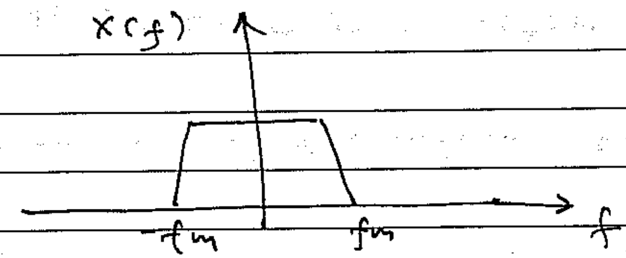
~~Visually,~~ Graphically, it can be ~~sup~~ summed up as,

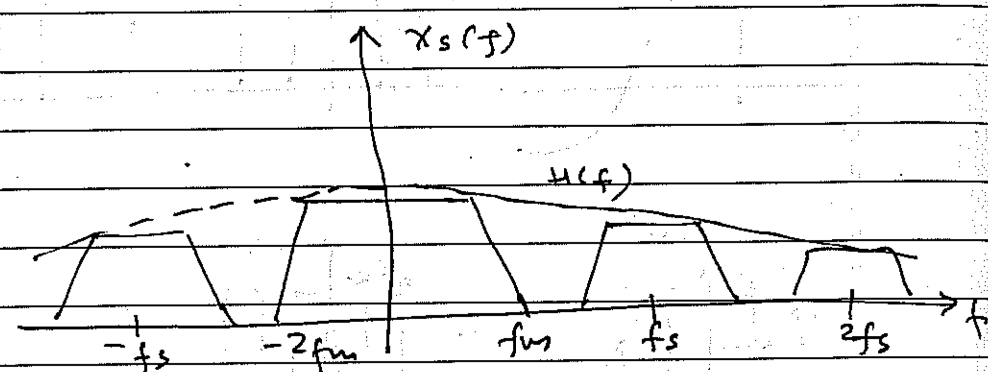


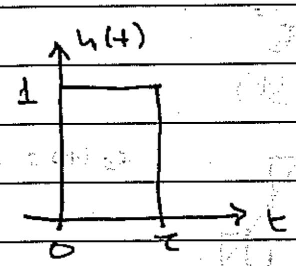
So, we have,

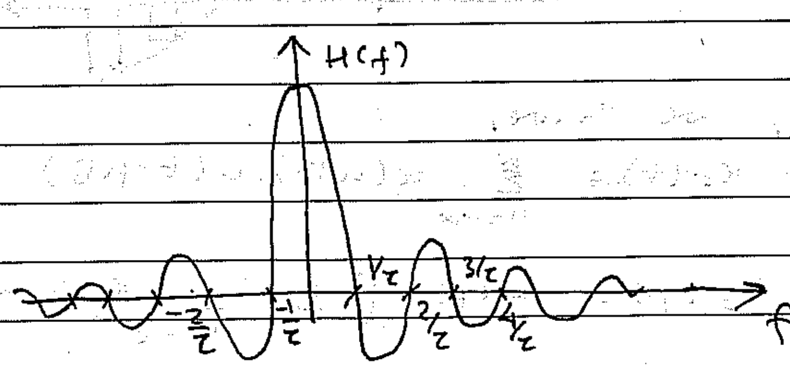
$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \cdot h(t - nT_s)$ &

$X_s(f) = H(f) \cdot \left[f_s \cdot \sum_{n=-\infty}^{\infty} X(f - nf_s) \right]$









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Now, since $H(f)$ is a sinc function, the primary effect of flat-topped sampling is the attenuation of higher frequency components of the message signals.

This attenuation of higher frequency components is known as aperture effect. This effect can be neutralized by using an equalizing filter with a transfer function

$$H_{eq}(f) = 1/H(f)$$

And, also, if we take the width (τ) of the sampling function $h(t)$ very very less than the sampling time T_s, i.e. $\tau << T_s$ then the amplitude of $H(f)$ remains more or less constant over message frequency band and thus the aperture effect can be neglected.

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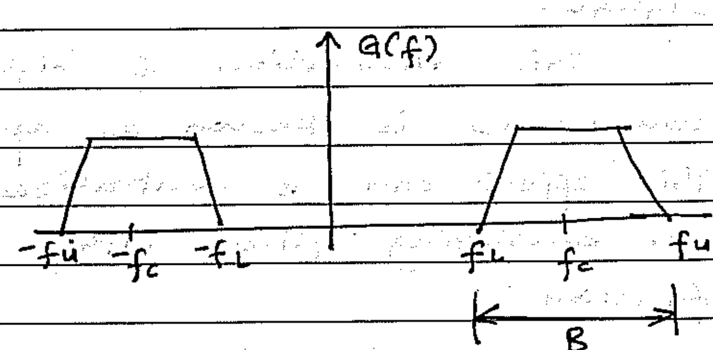
Sampling of bandpass signals :

Consider a band-pass signal $g(t)$ with the spectrum $G(f)$ as,



Here, B = bandwidth

f_u = upper cutoff frequency

f_L = lower limit.

Now, the signal $g(t)$ can be represented by instantaneous values, $g(nT_s)$ if the sampling rate

$$f_s = \frac{2f_u}{m}$$

where, 'm' is an integer defined by

$$\frac{f_u}{B} - 1 < m \leq \frac{f_u}{B}.$$

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So, if a bandpass signal is sampled at $f_s = 2f_u/m$, then the original message can be recovered without distortion.

This theorem is also known as sub-sampling theory.

⊕ Consider a signal $g(t)$ having upper cutoff frequency f_u = 100KHz & the lower cutoff frequency f_L = 80KHz. find the sampling frequency.

Here, f_u = 100 KHz

f_L = 80 KHz.

$\therefore B = f_u - f_L = 100 - 80 = 20$ KHz.

Now,

$$m = \frac{f_u}{B} = \frac{100}{20} = 5 = m$$

and $$f_s = \frac{2f_u}{m} = \frac{2 \times 100}{5} = 40 \text{ KHz}.$$

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for $f_u = 120\ KHz$

$f_L = 70\ KHz$

find f_s.

$B = 120 - 70 = 50\ KHz$.

$\frac{f_u}{B} = \frac{120}{50} = 2.4$

$\frac{f_u}{B} - 1 = 1.4$.

$1.4 < m \leq 2.4$

$\therefore\ m = 2$

Now,

$f_s = \frac{2 \cdot f_u}{m} = \frac{2 \times 120}{2} = \frac{240}{2}$

$= 120\ KHz$.

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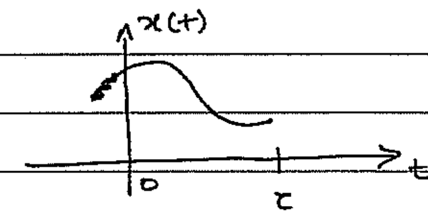
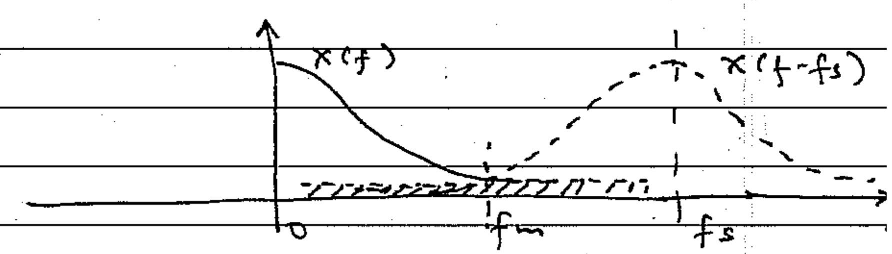
Till now, we have been studying that the signals are bandlimited but in real cases, the signals are time limited i.e. any signal starts at some point and ends at some other point.

Now, for such time limited signal, the spectrum is theoretically unlimited and thus there will always be aliasing effect.





Thus we see that the spectrum of $X(f)$ and $X(f-f_s)$ overlap resulting in aliasing.

To overcome this effect pre aliasing filter is used such that the time limited signal are restricted to bandlimited signal with highest frequency component f_m. Then after band limited sampling can be performed.