Chapter 03
Spectral Analysis

Fourier analysis

- **Fourier analysis** is a method of defining periodic waveforms in terms of **trigonometric function**s.
- The method gets its name from a French mathematician and physicist named **Jean Baptiste Joseph, Baron de Fourier**, who lived during the 18th and 19th centuries.
- Fourier analysis is used in **electronics, acoustics,** and **communications**.

- Many waveforms consist of energy at a fundamental frequency and also at harmonic frequencies. The relative proportions of energy in the fundamental and the harmonics determines the shape of the wave.
- The wave function (usually amplitude, frequency, or phase versus time) can be expressed as of a sum of sine and cosine functions called a Fourier Series, uniquely defined by constants known as Fourier coefficients.
- If these coefficients are represented by a, a_1, a_2, a_3, ..., a_n, ... and b_1, b_2, b_3, ..., b_n, ..., then the Fourier series $F(x)$, where x is an independent variable (usually time), has the following form:

$$F(x) = a/2 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \dots + a_n \cos nx + b_n \sin nx + \dots$$

- In Fourier analysis, the objective is to calculate coefficients a, a_1, a_2, a_3, ..., a_n and b_1, b_2, b_3, ..., b_n up to the largest possible value of n. The greater the value of n, the more accurate is the Fourier-series representation of the waveform.

Fourier series and Fourier transform

- To know the **amplitude** and **phase** of each frequency component contained in the signal, **Fourier analysis** is performed.
- If signal is **power signal**, then **fourier series** is performed.
- If signal is **energy signal**, then **fourier transform** is performed.
- Fourier transform is the process of converting **time domain** signal to **frequency domain.**

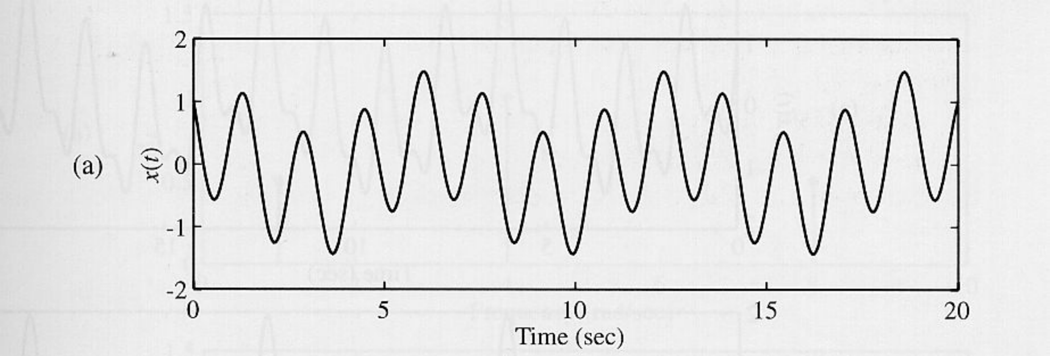
Example: Sum of Sinusoids

- Consider the CT signal given by

$$x(t) = A_1 \cos(t) + A_2 \cos(4t + \pi/3) + A_3 \cos(8t + \pi/2),$$

- The signal has only **three frequency components** at 1,4, and 8 *rad/sec,* amplitudes A_1, A_2, A_3 and phases $0, \pi/3, \pi/2$
- The shape of the signal *x*(*t*) depends on the relative magnitudes of the frequency components, specified in terms of the amplitudes A_1, A_2, A_3

$$\begin{cases} A_1 = 0.5 \\ A_2 = 1 \\ A_3 = 0 \end{cases}$$



Fourier Series

- As a matter of fact, sine waves and cosine waves are the basic building functions for any periodic signal.
- Fourier series is a tool used to analyze any periodic signal.
- We obtain the following information about the signal:
  - ❖ How many frequency component are present in the signal?
  - ❖ Their amplitudes
  - ❖ Their relative phase difference between these frequency components.

Types:

1. Trigonometric
2. Exponential
3. Polar

Drichlet's condition for Fourier Series

To be described by the **Fourier Series** the waveform *f(t)* must satisfy the following mathematical properties:

1. *f(t)* is a ***single-value function*** except at possibly a finite number of points.

2. The integral $\int_{t_0}^{t_0+T} |f(t)|\, dt < \infty$ for any t_0.

3. *f(t)* has a finite number of *discontinuities* within the period *T*.
4. *f(t)* has a finite number of ***maxima*** and ***minima*** within the period *T*.

In practice, *f(t)* = *v(t)* or *i(t)* so the above 4 conditions are always satisfied.

Trigonometric Form

- It is simply a linear combination of sines and cosines at multiples of its fundamental frequency, $f_0 = 1/T$.

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(2\pi f_0 n t) + \sum_{n=1}^{\infty} b_n \sin(2\pi f_0 n t)$$

- a_0 counts for any dc offset in $x(t)$.
- a_0, a_n and b_n are called the trigonometric Fourier Series Coefficients.
- The n^{th} harmonic frequency is nf_0.

Trigonometric Form

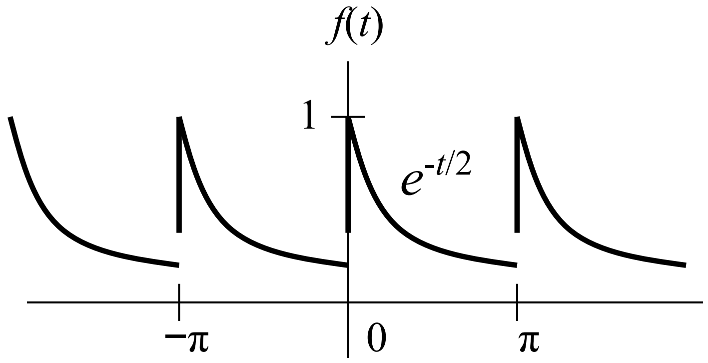
To find a_{0}, a_n, b_n :

$$a_0 = \frac{1}{T_0} \int_{T_0} x(t)dt$$

$$a_n = \frac{2}{T_0} \int_{T_0} x(t)\cos(2\pi n f_0 t)dt$$

$$b_n = \frac{2}{T_0} \int_{T_0} x(t)\sin(2\pi n f_0 t)dt$$

Example



- Fundamental period

  $T_0 = \pi$

- Fundamental frequency

  $f_0 = 1/T_0 = 1/\pi$ Hz

  $\omega_0 = 2\pi/T_0 = 2$ rad/s

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(2nt) + b_n \sin(2nt)$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} e^{-\frac{t}{2}} dt = -\frac{2}{\pi}\left(e^{-\frac{\pi}{2}} - 1\right) \approx 0.504$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} e^{-\frac{t}{2}} \cos(2nt)\, dt = 0.504\left(\frac{2}{1+16n^2}\right)$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} e^{-\frac{t}{2}} \sin(2nt)\, dt = 0.504\left(\frac{8n}{1+16n^2}\right)$$

a_n and b_n decrease in amplitude as $n \to \infty$.

$$f(t) = 0.504\left[1 + \sum_{n=1}^{\infty} \frac{2}{1+16n^2}\left(\cos(2nt) + 4n\sin(2nt)\right)\right]$$

Polar Form

- Using single sinusoid,

$$x(t) = \underbrace{C_0}_{\text{dc component}} + \sum_{n=1}^{\infty} \underbrace{C_n \cos(2\pi n f_0 t + \theta_n)}_{\text{nth harmonic}}$$

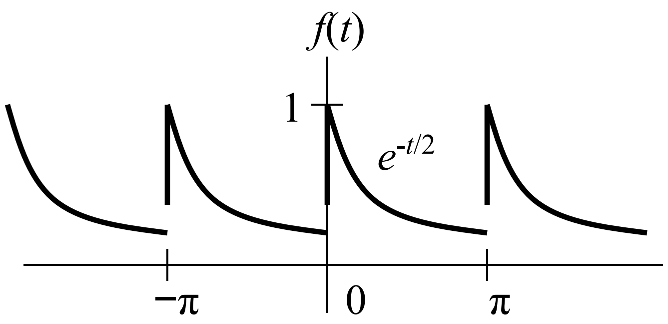
Where, $C_0 = a_0$ and C_n, and θ_n are related to the trigonometric coefficients a_n and b_n as:

$$C_n = \sqrt{a_n^{\,2} + b_n^{\,2}}$$

and

$$\theta_n = -\tan^{-1}\left(\frac{b_n}{a_n}\right)$$

Compact Trigonometric



- Fundamental period

  $T_0 = \pi$

- Fundamental frequency

  $f_0 = 1/T_0 = 1/\pi$ Hz

  $\omega_0 = 2\pi/T_0 = 2$ rad/s

$$f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(2nt - \theta_n)$$

$$a_0 \approx 0.504$$

$$a_n = 0.504\left(\frac{2}{1+16n^2}\right)$$

$$b_n = 0.504\left(\frac{8n}{1+16n^2}\right)$$

$$C_0 = a_o = 0.504$$

$$C_n = \sqrt{a_n^2 + b_n^2} = 0.504\left(\frac{2}{\sqrt{1+16n^2}}\right)$$

$$\theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right) = -\tan^{-1} 4n$$

$$f(t) = 0.504 + 0.504\sum_{n=1}^{\infty} \frac{2}{\sqrt{1+16n^2}} \cos\left(2nt - \tan^{-1} 4n\right)$$

Exponential Fourier Series

COMPLEX EXPONENTIALS

$$e^{jn\omega_0 t} = \cos n\omega_0 t + j\sin n\omega_0 t$$

$$e^{-jn\omega_0 t} = \cos n\omega_0 t - j\sin n\omega_0 t$$

$$\cos n\omega_0 t = \frac{1}{2}\left(e^{jn\omega_0 t} + e^{-jn\omega_0 t}\right)$$

$$\sin n\omega_0 t = \frac{1}{2j}\left(e^{jn\omega_0 t} - e^{-jn\omega_0 t}\right) = -\frac{j}{2}\left(e^{jn\omega_0 t} - e^{-jn\omega_0 t}\right)$$

Exponential Fourier Series

A periodic signal x(t) is expressed in the exponential Fourier Series form as:

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

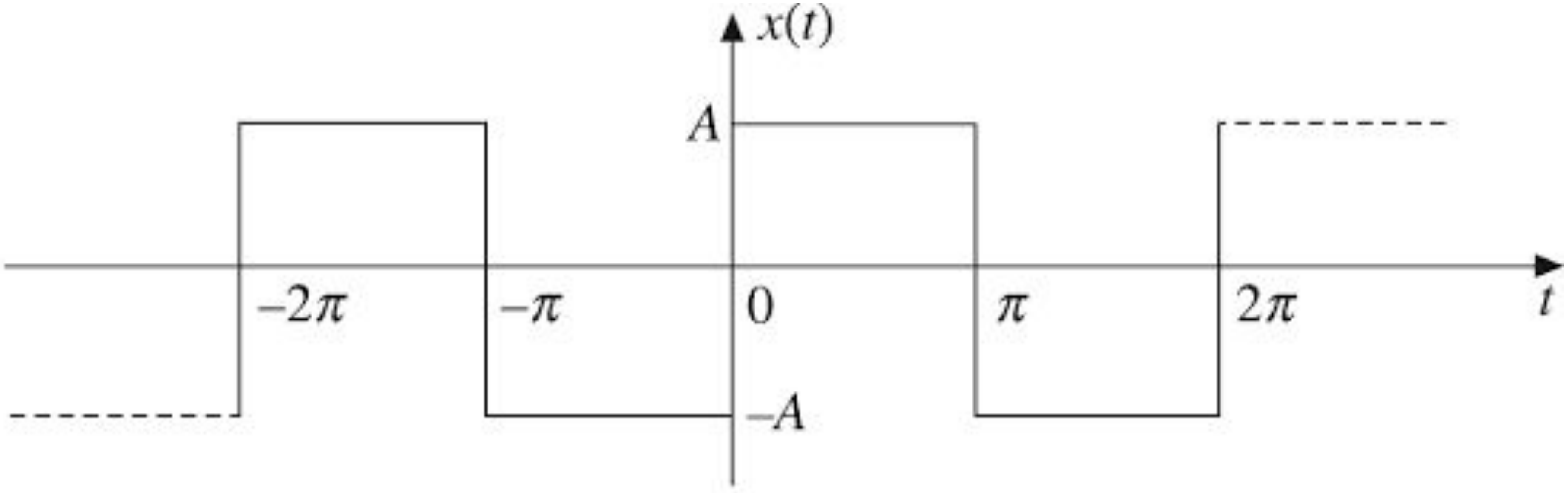
Where, C_n is the coefficient of exponential Fourier series and can be calculated as:

$$C_n = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) e^{-jn\omega_0 t} dt$$

$$\boxed{C_0 = A_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) dt}$$

Example:

Obtain the exponential Fourier series for the waveform shown in Figure. Also draw the frequency spectrum.



Figure

Solution:

The periodic waveform shown in Figure 3 with a period T = 2π can be expressed as:

$$x(t) = \begin{cases} A, & 0 \leq t \leq \pi \\ -A, & \pi \leq t \leq 2\pi \end{cases}$$

Let $t_0 = 0, t_0 + T = 2\pi$

And fundamental frequency: $\omega_0 = \dfrac{2\pi}{T} = \dfrac{2\pi}{2\pi} = 1$

Solution:

Exponential Fourier Series:

$$C_0 = \frac{1}{T}\int_0^T x(t)dt = \frac{1}{2\pi}\left(\int_0^{\pi} A dt + \int_{\pi}^{2\pi} -A dt\right) = \frac{A}{2\pi}\left[(t)_0^{\pi} - (t)_{\pi}^{2\pi}\right] = 0$$

$$C_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}dt$$

$$= \frac{1}{2\pi}\left(\int_0^{\pi} A(t)e^{-jnt}dt + \int_{\pi}^{2\pi} A(t)e^{-jnt}dt\right) = \frac{A}{2\pi}\left[\left(\frac{e^{-jnt}}{-jn}\right)_0^{\pi} - \left(\frac{e^{-jnt}}{-jn}\right)_{\pi}^{2\pi}\right]$$

Cont.…

$$= -\frac{A}{jn2\pi}\left[\left(e^{-jn\pi} - e^{0}\right)_{0}^{\pi} - \left(e^{-j2\pi n} - e^{-jn\pi}\right)_{\pi}^{2\pi}\right]$$

$$= -\frac{A}{jn2\pi}\left[(-1)^{n} - 1\right] - \left[1 - (-1)^{n}\right] = -j\frac{2A}{n\pi}$$

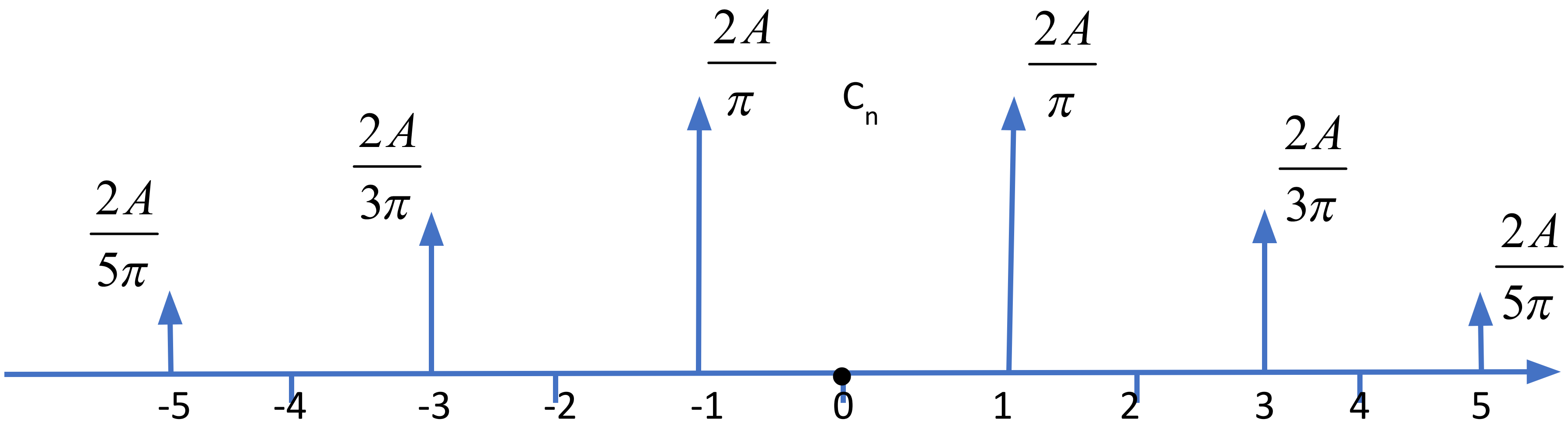
$$C_n = \begin{cases} -j\dfrac{2A}{n\pi}, & for\_odd\_n \\ 0, & for\_even\_n \end{cases}$$

$$x(t) = C_0 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} C_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} -j\frac{2A}{n\pi}e^{jnt}, for\_odd\_n$$

$$\boxed{C_0 = 0, C_1 = C_{-1} = \frac{2A}{\pi}, C_3 = C_{-3} = \frac{2A}{3\pi}, C_5 = C_{-5} = \frac{2A}{5\pi}}$$

Cont....

$$\boxed{C_0 = 0, C_1 = C_{-1} = \frac{2A}{\pi}, C_3 = C_{-3} = \frac{2A}{3\pi}, C_5 = C_{-5} = \frac{2A}{5\pi}}$$



Fourier Transform

- We have seen that periodic signals can be represented with the Fourier series
- Can aperiodic signals be analyzed in terms of frequency components?
- Yes, and the Fourier transform provides the tool for this analysis
- The major difference w.r.t. the line spectra of periodic signals is that the spectra of aperiodic signals are defined for all real values of the frequency variable ω not just for a discrete set of values

The Fourier Transform in the General Case

- Given a signal *x*(*t*), its ***Fourier transform*** $X(\omega)$ is defined as:

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

- A signal *x*(*t*) is said to have a *Fourier transform in the ordinary sense* if the above integral converges
- Given a signal *x*(*t*) with Fourier transform $X(\omega)$, *x*(*t*) can be recomputed from $X(\omega)$ by applying the **inverse Fourier transform** given by:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$$x(t) \leftrightarrow X(\omega)$$

Drichlet condition for fourier transform

- The function x(t) should be single valued in any finite time interval T.
- The signal x(t) is absolutely integral i. e. $\int_{-\infty}^{\infty} x(t)dt < \infty$
- The number of maxima and minima in any finite length of real time is finite.
- The discontinuities in any finite length of real time is finite.

Properties of the Fourier Transform

$$x(t) \leftrightarrow X(\omega) \qquad y(t) \leftrightarrow Y(\omega)$$

- Linearity:

$$\alpha x(t) + \beta y(t) \leftrightarrow \alpha X(\omega) + \beta Y(\omega)$$

- Left or Right Shift in Time:

$$x(t - t_0) \leftrightarrow X(\omega) e^{-j\omega t_0}$$

- Time Scaling:

$$x(at) \leftrightarrow \frac{1}{a} X\left(\frac{\omega}{a}\right)$$

Properties of the Fourier Transform

- *Time Reversal:* $x(-t) \leftrightarrow X(-\omega)$

- *Multiplication by a Power of t:* $t^n x(t) \leftrightarrow (j)^n \dfrac{d^n}{d\omega^n} X(\omega)$

- *Multiplication by a Complex Exponential:* $x(t)e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0)$

- *Multiplication by a Sinusoid (Modulation):*

$$x(t)\sin(\omega_0 t) \leftrightarrow \frac{j}{2}\left[X(\omega + \omega_0) - X(\omega - \omega_0)\right]$$

$$x(t)\cos(\omega_0 t) \leftrightarrow \frac{1}{2}\left[X(\omega + \omega_0) + X(\omega - \omega_0)\right]$$

Properties of the Fourier Transform

- *Differentiation in the Time Domain:* $\frac{d^n}{dt^n}x(t) \leftrightarrow (j\omega)^n X(\omega)$

- *Convolution in the Time Domain:* $x(t) * y(t) \leftrightarrow X(\omega)Y(\omega)$

- *Multiplication in the Time Domain:* $x(t)y(t) \leftrightarrow X(\omega) * Y(\omega)$

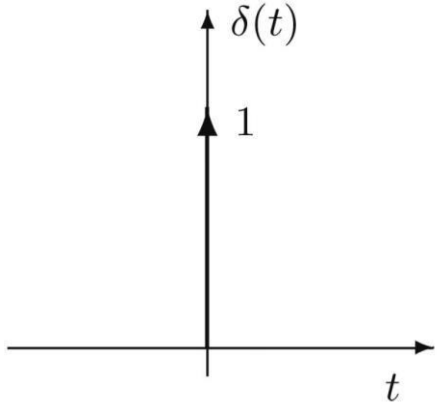
TABLE 4.1 PROPERTIES OF THE FOURIER TRANSFORM

Property	Transform Pair/Property
Linearity	$ax(t) + bv(t) \leftrightarrow aX(\omega) + bV(\omega)$
Right or left shift in time	$x(t - c) \leftrightarrow X(\omega)e^{-j\omega c}$
Time scaling	$x(at) \leftrightarrow \frac{1}{a}X\left(\frac{\omega}{a}\right) \quad a > 0$
Time reversal	$x(-t) \leftrightarrow X(-\omega) = \overline{X(\omega)}$
Multiplication by a power of t	$t^n x(t) \leftrightarrow j^n \frac{d^n}{d\omega^n} X(\omega) \quad n = 1, 2, \ldots$
Multiplication by a complex exponential	$x(t)e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0) \quad \omega_0$ real
Multiplication by sin $\omega_0 t$	$x(t)\sin\omega_0 t \leftrightarrow \frac{j}{2}[X(\omega + \omega_0) - X(\omega - \omega_0)]$
Multiplication by cos $\omega_0 t$	$x(t)\cos\omega_0 t \leftrightarrow \frac{1}{2}[X(\omega + \omega_0) + X(\omega - \omega_0)]$
Differentiation in the time domain	$\frac{d^n}{dt^n}x(t) \leftrightarrow (j\omega)^n X(\omega) \quad n = 1, 2, \ldots$
Integration	$\int_{-\infty}^{t} x(\lambda)\, d\lambda \leftrightarrow \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)$
Convolution in the time domain	$x(t) * v(t) \leftrightarrow X(\omega)V(\omega)$
Multiplication in the time domain	$x(t)v(t) \leftrightarrow \frac{1}{2\pi}X(\omega) * V(\omega)$
Parseval's theorem	$\int_{-\infty}^{\infty} x(t)v(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{X(\omega)}V(\omega)\, d\omega$
Special case of Parseval's theorem	$\int_{-\infty}^{\infty} x^2(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2\, d\omega$
Duality	$X(t) \leftrightarrow 2\pi x(-\omega)$

Example: Impulse Function δ(t)

$$F(\omega) = \mathfrak{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = e^{-j\omega t}\Big|_{t=0} = e^{j0} = 1$$

$$\boxed{\begin{array}{l} \delta(t) \Leftrightarrow 1 \\ 1 \Leftrightarrow 2\pi\delta(\omega) \end{array}}$$



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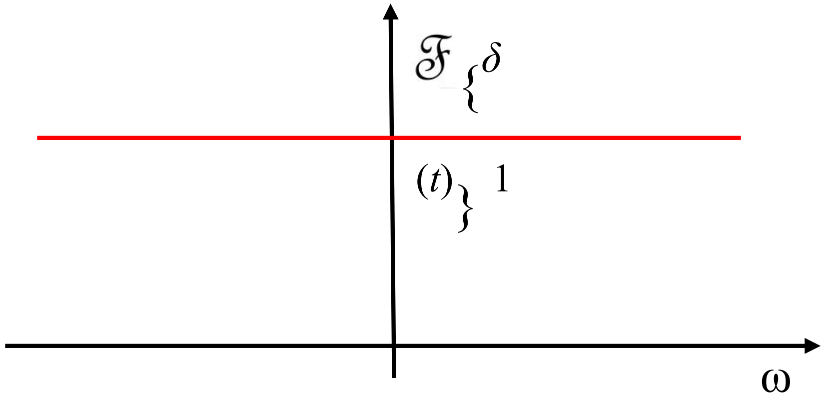


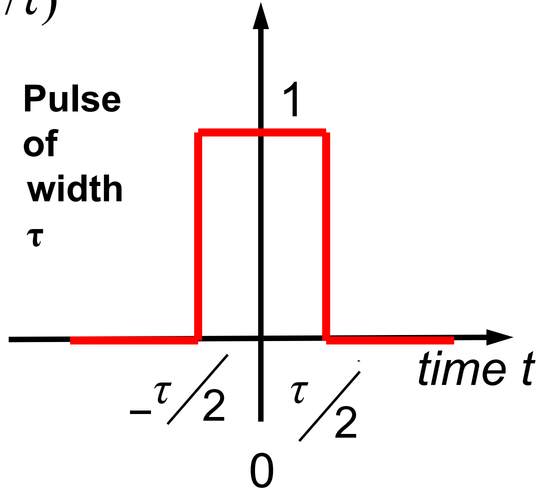
$$\delta(t) = \begin{cases} 1 \text{ if } t = 0 \\ 0 \text{ if } t \neq 0 \end{cases}$$

Delta function has unity area.

Example: Fourier Transform of Single Rectangular Pulse

$$f(t) = \mathrm{rect}(t) = \Pi(t/\tau) = \begin{cases} 1 & \text{for } -\frac{\tau}{2} \le t \le \frac{\tau}{2} \\ 0 & \text{for all } |t| > \frac{\tau}{2} \end{cases}$$

$f(t) = \mathrm{rect}(t) = \Pi(t/\tau)$



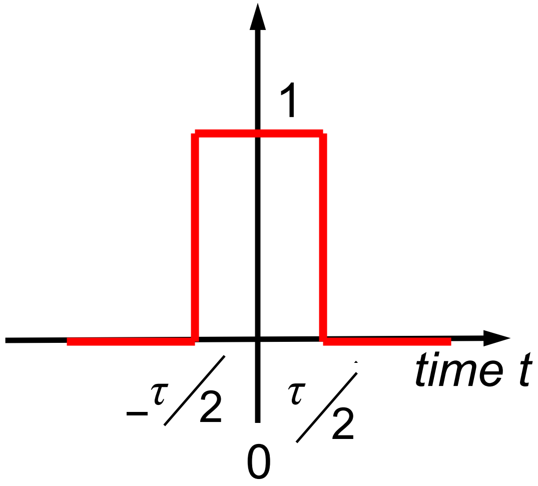
Remember $\omega = 2\pi f$

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt = \int_{-\tau/2}^{\tau/2} e^{-j\omega t}\, dt$$

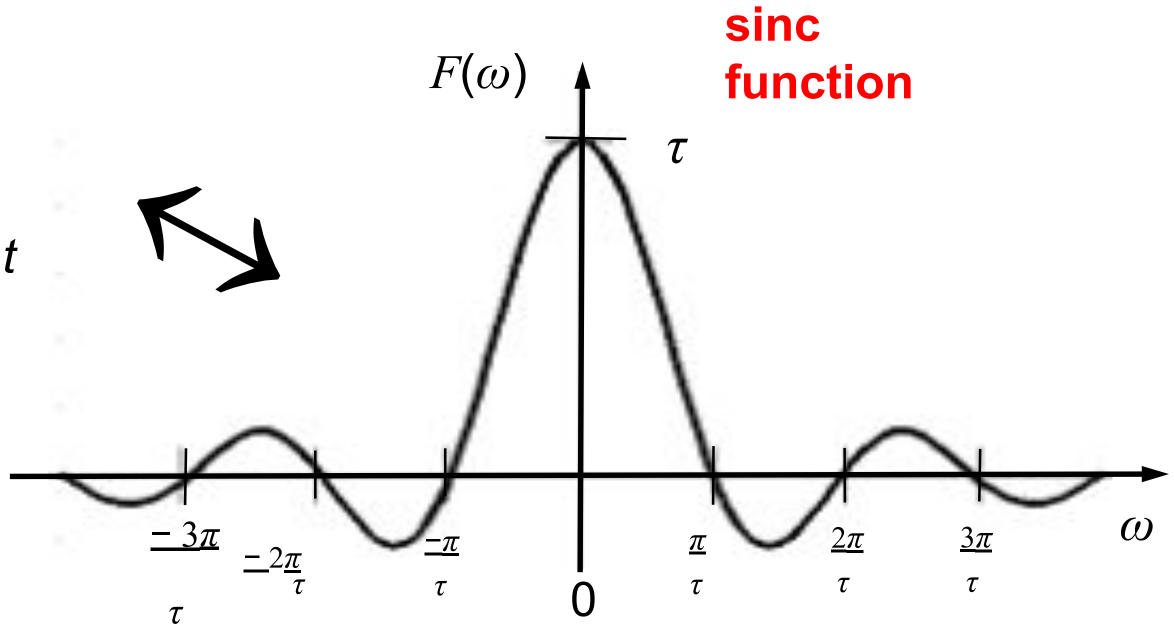
$$= \left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-\tau/2}^{\tau/2} = \frac{e^{-j\omega\tau/2} - e^{j\omega\tau/2}}{-j\omega}$$

$$= \frac{-j2\sin(\omega\tau/2)}{-j\omega} = \tau \cdot \frac{\sin(\omega\tau/2)}{(\omega\tau/2)}$$

$$F(\omega) = \tau \cdot \frac{\sin(\omega\tau/2)}{(\omega\tau/2)} = \tau \cdot \mathrm{sinc}(\pi f \tau)$$



Note the pulse is time centered



Fourier Transform of Complex Exponentials

$$\mathfrak{F}^{-1}[\delta(f-f_c)] = \int_{-\infty}^{\infty} \delta(f-f_C)e^{-j2\pi f t}df$$

Evaluate for $f = f_c$

$$\mathfrak{F}^{-1}[\delta(f-f_c)] = \int_{f=f_c} e^{-j2\pi f_c t}df = e^{-j2\pi f_c t}$$

$$\therefore \boxed{\delta(f-f_c) \Leftrightarrow e^{-j2\pi f_c t}} \quad and$$

$$\mathfrak{F}^{-1}[\delta(f+f_c)] = \int_{-\infty}^{\infty} \delta(f+f_C)e^{-j2\pi f t}df$$

Evaluate for $f = -f_c$

$$\mathfrak{F}^{-1}[\delta(f+f_c)] = \int_{f=-f_c} e^{j2\pi f_c t}df = e^{j2\pi f_c t}$$

$$\therefore \boxed{\delta(f+f_c) \Leftrightarrow e^{j2\pi f_c t}}$$

Fourier Transform of Sinusoidal Functions

Taking $\delta(f - f_c) \Leftrightarrow e^{-j2\pi f_c t}$ and $\delta(f + f_c) \Leftrightarrow e^{j2\pi f_c t}$

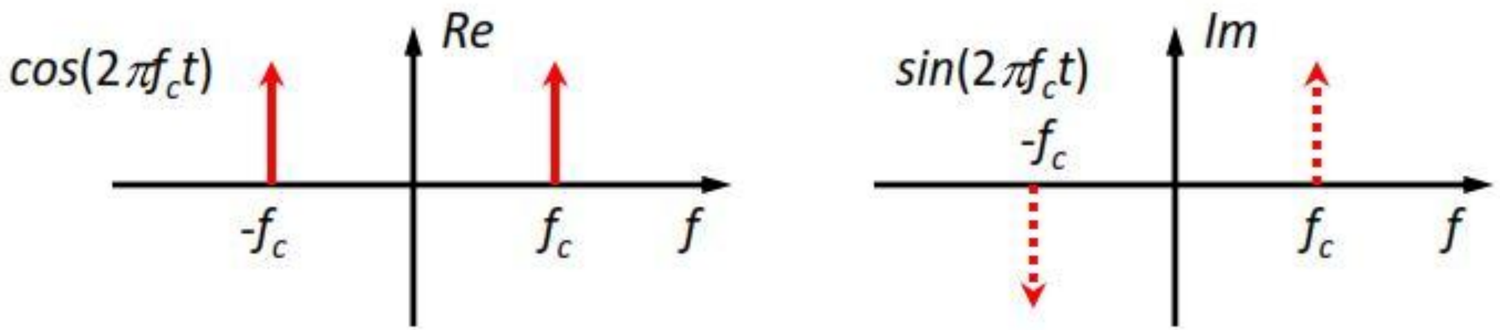
We use these results to find FT of $\cos(2\pi ft)$ and $\sin(2\pi ft)$

Using the identities for $\cos(2\pi ft)$ and $\sin(2\pi ft)$,

* $\cos(2\pi ft) = \frac{1}{2}\left[e^{j2\pi f_c t} + e^{-j2\pi f_c t}\right]$ & $\sin(2\pi ft) = \frac{1}{2j}\left[e^{j2\pi f_c t} - e^{-j2\pi f_c t}\right]$

Therefore,

$$\boxed{\begin{aligned} \cos(2\pi ft) &\Leftrightarrow \tfrac{1}{2}[\delta(f + f_c) + \delta(f - f_c)], \\ \sin(2\pi ft) &\Leftrightarrow \tfrac{1}{2j}[\delta(f + f_c) - \delta(f - f_c)] \end{aligned}}$$ and



Power and Energy of signal

Energy signal

- Energy is finite and non zero.
- Non periodic signal are energy signal.
- These signals are time limited.
- The power of energy signal is zero.
- Example: A single rectangular pulse.

Power signal

- The normalized average power is finite and non zero.
- Periodic (practical) signal are power signals.
- These signals can exist infinite time.
- Energy of power signal is infinite.
- Example: A train of rectangular pulse.

Parseval's Theorem for Energy signals:

- Parseval's theorem refers to that information is not lost in Fourier transform.
- It states that the total energy of the signal x(t) is equal to the sum of energies of the individual spectral components in the frequency domain.
- We can find the energy of signal without knowing its time domain according to Parseval's theorem.
- Suppose that x(t) and X(ω) are the fields in the time and frequency domain respectively, where X(ω) is obtained by the Fourier transform of x(t). According to Parseval's theorem, the following equation holds:

$$E = \int_{-\infty}^{\infty} x^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

Proof:

- Let there be a signal x(t) with its Fourier transform X($\boldsymbol{\omega}$) . The energy E of signal x(t) is expressed as:

$$E = \int_{-\infty}^{\infty} x^2(t) dt = \int_{-\infty}^{\infty} x(t) * x(t) dt$$

Inverse Fourier transform x(t) can be defined as:

$$x(t) = F^{-1}[X(\omega)] = \frac{1}{2\Pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

Replacing one x(t)

$$E = \int_{-\infty}^{\infty} x(t) \left[\frac{1}{2\Pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega \right] dt$$

- Interchanging the order of integration, we get

$$E = \frac{1}{2\Pi}\int_{-\infty}^{\infty} X(\omega) \int_{-\infty}^{\infty} x(t)e^{j\omega t}\, dt\, d\omega$$

We know that,

X(ω) $= \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$ so, X($-\omega$) $= \int_{-\infty}^{\infty} x(t)\, e^{j\omega t}\, dt$

Therefore,

$$E = \frac{1}{2\Pi}\int_{-\infty}^{\infty} X(\omega)X(-\omega)\, d\omega$$

For a real signal x(t), the Fourier transform X(ω) and X(-ω) are complex conjugates.

Hence

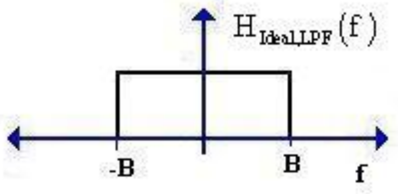
$$E = \int_{-\infty}^{\infty} x^2(t)dt = \frac{1}{2\Pi}\int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega = \int_{-\infty}^{\infty} |X(\omega)|^2\, df$$

Energy Spectral Density:

- The squared amplitude spectrum $|X(f)|^2$ is called as the Energy spectral density or energy density spectrum. Defined as:

$$\mathbf{ESD = \psi(f) = |X(f)|^2}$$

- Let us apply x(t) to an ideal low pass filter
- The response or output of a system is expressed as:



$$Y(\omega)=X(\omega)H(\omega)$$

Where, $Y(\omega)$ and $X(\omega)$ are Fourier transform of y(t) and x(t)

- Using Parseval's theorem

$$E_0 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |Y(\omega)|^2 \, d\omega$$

- $$E_0 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(\omega).X(\omega)|^2\, d\omega$$
- Here H(ω)=0 for all the frequencies except for the narrow band - B_m to B_m for which it is unity.

Therefore,

$$E_0 = \frac{1}{2\pi}\int_{-B_m}^{B_m} |1.X(\omega)|^2\, d\omega = \frac{1}{2\pi}\int_{-B_m}^{B_m} |X(\omega)|^2\, d\omega$$

- Again we may assume that FT X(ω) is constant with frequency for a narrow band - B_m to B_m .
- The energy of the signal over transmission band $\Delta B = 2B_m$ will be

$$E_0 = \frac{1}{2\pi}|X(\omega)|^2 \int_{-B_m}^{B_m} 1\, d\omega = \frac{1}{2\pi}|X(\omega)|^2\,(2B_m)$$

Substituting $2B_m = \Delta\omega, we\ get$

$$E_0 = \frac{1}{2\pi}|X(\omega)|^2\,(\Delta\omega) = |X(\omega)|^2\,(\Delta f)$$

$$\frac{E_0}{\Delta f} = |X(\omega)|^2$$

- Where, $|X(\omega)|^2$ represents energy per unit bandwidth and is known as ESD or energy density spectrum, denoted by $\mathbf{\Psi}(\omega)$.

$$\mathbf{\Psi}(\omega) = |X(\omega)|^2$$

$$E = \frac{1}{2\Pi}\int_{-\infty}^{\infty}\mathbf{\Psi}(\omega)\, d\omega$$

Parseval's Theorem for Power Signals:

- This theorem relates average power of a periodic signal to its fourier series coefficients
- It states the total average power of a periodic signal x(t) is equal to the sum of average powers of individual fourier coefficients (C_n).
- The average power of a signal x(t) is defined as:

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)\, dt \qquad P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2\, dt$$

- If the signal can be represented as sum of fourier series components and power is defined as:

$$\mathrm{P} = \sum_{n=-\infty}^{\infty} \left| c_n \right|^2$$

Where C_n is the amplitude of n^{th} harmonic component of fourier series.

Proof:

Parseval's theorem relates the average power P of a periodic signal to its Fourier coefficients.

$$P = \frac{1}{T_o}\int_{T_o} |v(t)|^2\,dt = \frac{1}{T_o}\int_{T_o} v(t)v^*(t)\,dt$$

$$\boxed{v^*(t) = \left[\sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_0 t}\right]^* = \sum_{n=-\infty}^{\infty} c_n^* e^{-j2\pi n f_0 t}}$$

$$P = \frac{1}{T_o}\int_{T_o} v(t)\left[\sum_{n=-\infty}^{\infty} c_n^* e^{-j2\pi n f_0 t}\right]dt$$

$$= \sum_{n=-\infty}^{\infty}\left[\frac{1}{T_0}\int_{T_0} v(t)e^{-j2\pi n f_0 t}\,dt\right]c_n^*$$

$$P = \sum_{n=-\infty}^{\infty} c_n c_n^* = \sum_{n=-\infty}^{\infty} |c_n|^2$$

Power Spectral Density Function:

- In applying frequency-domain techniques to the analysis of random signals the natural approach is to Fourier transform the signals.

- Unfortunately the Fourier transform of a stochastic process does not, strictly speaking, exist because it has infinite signal energy.

- But the Fourier transform of a truncated version of a stochastic process does exist.

For a CT stochastic process let

$$\mathrm{X}_T(t) = \begin{cases} \mathrm{X}(t) & , \ |t| \le T/2 \\ 0 & , \ |t| > T/2 \end{cases} = \mathrm{X}(t)\operatorname{rect}(t/T)$$

The Fourier transform is

$$\mathcal{F}\left(\mathrm{X}_T(t)\right) = \int_{-\infty}^{\infty} \mathrm{X}_T(t) e^{-j2\pi f t} dt \quad , \quad T < \infty$$

Using Parseval's theorem,

$$\int_{-T/2}^{T/2} \left|\mathrm{X}_T(t)\right|^2 dt = \int_{-\infty}^{\infty} \left|\mathcal{F}\left(\mathrm{X}_T(t)\right)\right|^2 df$$

Dividing through by T,

$$\frac{1}{T}\int_{-T/2}^{T/2} \mathrm{X}_T^2(t) dt = \frac{1}{T}\int_{-\infty}^{\infty} \left|\mathcal{F}\left(\mathrm{X}_T(t)\right)\right|^2 df$$

$$\frac{1}{T}\int_{-T/2}^{T/2} \mathrm{X}_T^2(t)\,dt = \frac{1}{T}\int_{-\infty}^{\infty} \left|\mathcal{F}\left(\mathrm{X}_T(t)\right)\right|^2 df \leftarrow \boxed{\text{Average signal power over time, } T}$$

If we let T approach infinity, the left side becomes the average power over all time. On the right side, the Fourier transform is not defined in that limit. But it can be shown that even though the Fourier transform does not exist, *its expected value does*. Then

$$\mathrm{E}\left(\frac{1}{T}\int_{-T/2}^{T/2} \mathrm{X}_T^2(t)\,dt\right) = \mathrm{E}\left(\frac{1}{T}\int_{-\infty}^{\infty} \left|\mathcal{F}\left(\mathrm{X}_T(t)\right)\right|^2 df\right)$$

Taking the limit as T approaches infinity,

$$\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\mathrm{E}\left(\mathrm{X}^2\right)dt=\lim_{T\to\infty}\frac{1}{T}\int_{-\infty}^{\infty}\mathrm{E}\left[\left|\mathcal{F}\left(\mathrm{X}_T\left(t\right)\right)\right|^2\right]df$$

$$\left\langle\mathrm{E}\left(X^2\right)\right\rangle=\int_{-\infty}^{\infty}\lim_{T\to\infty}\mathrm{E}\left[\frac{\left|\mathcal{F}\left(\mathrm{X}_T\left(t\right)\right)\right|^2}{T}\right]df$$

The integrand on the right side is identified as **power spectral density (PSD)**.

$$\mathrm{G}_X\left(f\right)=\lim_{T\to\infty}\mathrm{E}\left(\frac{\left|\mathcal{F}\left(\mathrm{X}_T\left(t\right)\right)\right|^2}{T}\right)$$

$$\int_{-\infty}^{\infty} \mathrm{G}_X(f)\,df = \text{mean} - \text{squared value of } \{\mathrm{X}(t)\}$$

$$\int_{-\infty}^{\infty} \mathrm{G}_X(f)\,df = \text{average power of } \{\mathrm{X}(t)\}$$

PSD is a description of the variation of a signal's power versus frequency.

PSD can be (and often is) conceived as single-sided, in which all the power is accounted for in positive frequency space.

- The expression for power spectral density may be derived by assuming the power signal as a limiting case of an energy signal.
- Let us consider a power signal x(t) which is extended to infinity.
- The signal $x_\tau(t)$ may be expressed as

$$x_\tau(t) = \begin{cases} x(t) & |t| < \tau/2 \\ 0 & \text{elsewhere} \end{cases}$$

Now the terminated signal is of finite duration τ, therefore it is an energy signal.

- Let the energy of this signal is denoted by E_τ

$$E_\tau = \int_{-\infty}^{\infty} |x_\tau(t)|^2\, dt = \int_{-\infty}^{\infty} |X_\tau(\omega)|^2\, df$$

Where $X_\tau(\omega)$ if fourier transfer of $x_\tau(t)$

- It may be observed that x(t) over the interval (- τ/2, τ/2) will be same as $x_\tau(t)$ over the interval (-∞, ∞)

$$\int_{-\infty}^{\infty} |x_\tau(t)|^2\, dt = \int_{-\tau/2}^{\tau/2} |x(t)|^2\, dt$$

So,

$$\frac{1}{\tau}\int_{-\infty}^{\infty} |X_\tau(\omega)|^2\, df = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} |x(t)|^2\, dt$$

$$\lim_{\tau\to\infty} \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} |x(t)|^2\, dt = \lim_{\tau\to\infty} \frac{1}{\tau} \int_{-\infty}^{\infty} |X_\tau(\omega)|^2\, df$$

And,

$$P=\int_{-\infty}^{\infty} \lim_{\tau\to\infty} \frac{|X_\tau(\omega)|^2}{\tau}\, df$$

In the limit $\tau \to \infty$, the ratio $\frac{|X_\tau(\omega)|^2}{\tau}$ may be approach a finite value and is represented by $S(\omega)$ such that $S(\omega)=\lim_{\tau\to\infty} \frac{|X_\tau(\omega)|^2}{\tau}$ known as PSDF.

Auto correlation function and Psdf:

- The correlation function of two signal x1(t) and x2(t) is defined as,

$$R_{1,2}(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-\tau/2}^{\tau/2} x1(t).\, x(t-\tau)d\tau$$

Where, τ is delayed parameter.

- It is measure of similarity between the signal x1(t) and time delay version of another signal x2(t). This correlation is called cross-correlation.
- Special case of cross-correlation is auto correlation in which both signal are same. i.e. x1(t) = x2(t) = x(t)

$$R(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-\tau/2}^{\tau/2} x(t).\, x(t-\tau)d\tau$$

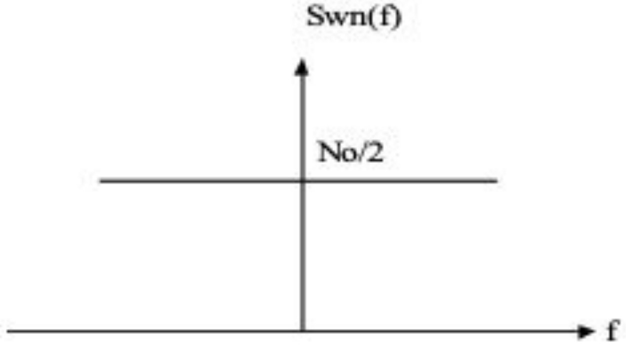
Properties of Autocorrelation function:

1. The autocorrelation function and psdf form a Fourier transform pair.
2. If $R(\tau)$ and $R^{*}(-\tau)$ are the complex conjugates of each other then they must be equal.
3. The autocorrelation function at origin is equal to the average power of a signal.
4. The autocorrelation function has maximum value at the origin.

Psdf of white noise

- Noise signals are random signals so, any two samples of noise signals are uncorrelated.
- The special case of noise signal is white noise whose psdf is independent of frequency (i.e, constant over entire frequency range).
- The signal contains equal power within a fixed bandwidth at any center frequency.

$S_{wn}(f) = N_o/2.\,, \quad -\infty < f < \infty$



The AC function of white noise,

$R_{WN}(\tau) = N_o/2 \;.\; \delta(\tau)$

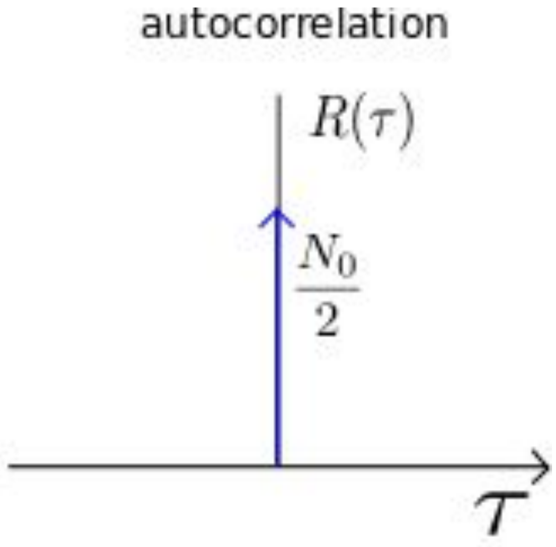
i.e $R_{WN}(\tau) = 0$ for all $\tau \neq 0$

Relation between AC function and PSDF:

$S_x(f) = \int_{-\infty}^{\infty} R_x(\tau)\, e^{-j2\pi f\tau}\, d\tau$

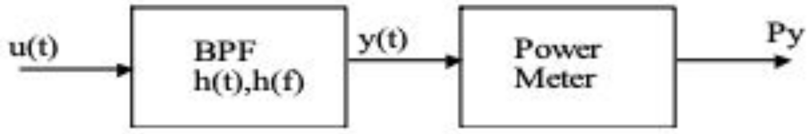
And

$R_x(\tau) = \int_{-\infty}^{\infty} s_x(f)\, e^{-j2\pi f\tau}\, df$



Interpretation of power spectral density

- Suppose a power signal u(t) is applied to a band pass filter followed by a power meter as shown in figure below.



- Assume a filter has narrow bandwidth Δ f centered on some frequency fc.
  Then the average power of the signal u(t) be

$$P = \int_{-\infty}^{\infty} s_u(w)\, df = 1/2\pi \int_{-\infty}^{\infty} S_\lambda(w)\, dw$$

As the PSD is an even function of w the power contain in the positive side of frequency spectrum will be :

$$P = 2.\ 1/2\pi \int^{\infty}_{-\infty} S_u(w)dw$$
$$= 1/\pi \int^{\infty}_{o} s_u(w)\ dw$$

The o/p power of the BPF will be ,

$$P_y = \bar{y}^2(t) = \lim \to T \quad \infty\ 1/T \int^{T/2}_{-T/2} u^2(t)\ h^2(t)dt.$$
$$= \int^{\infty}_{-\infty} \lim T \to \infty\ 1/T\ |u(w)|^2\ |H(w)|^2\ dw$$
$$\approx\ 2S_u(w)\ \Delta f$$

$$S_u(w) = P_y/2\Delta f$$

What is SPECTRUM ANALYZER

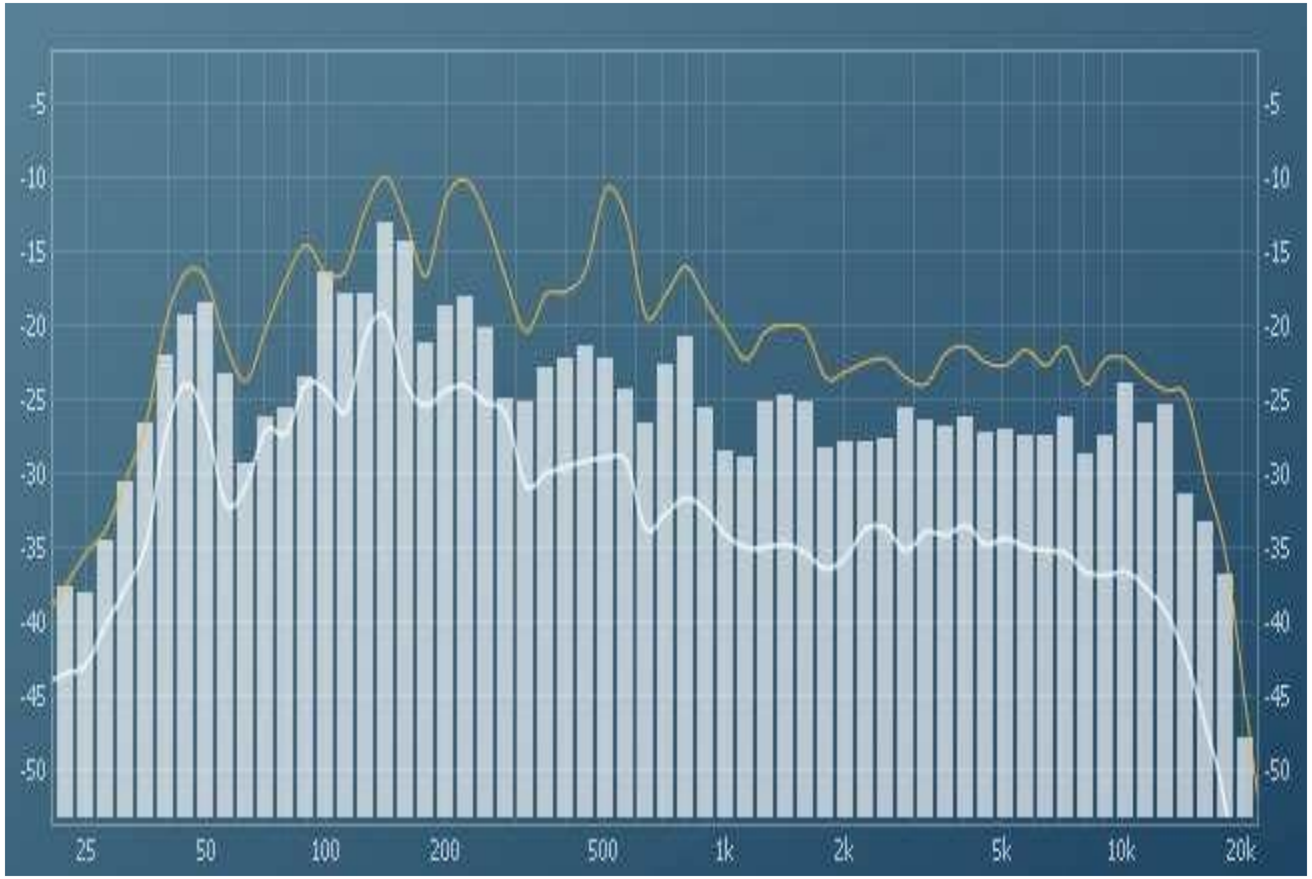
- The problems associated with non-real-time analysis in the frequency domain can be eliminated by using a spectrum analyzer. A spectrum analyzer is a real-time analyzer, which means that it simultaneously displays the amplitude of all the signals in the frequency range of the analyzer.

- Spectrum analyzers, like wave analyzers, provide information about the voltage or energy of a signal as a function of frequency. Spectrum analyzers provide a graphical display on a CRT.

SPECTRUM ANALYZER

- A spectrum analyzer measures the magnitude of an input signal versus frequency within the full frequency range of the instrument.

- The input signal that a spectrum analyzer measures is electrical, however, spectral compositions of other signals, such as acoustic pressure waves and optical light waves, can be considered through the use of an appropriate transducer.



SPECTRUM ANALYZER

- By analyzing the spectra of electrical signals, dominant frequency, power, distortion, harmonics, bandwidth, and other spectral components of a signal can be observed that are not easily detectable in time domain waveforms.

- The display of a spectrum analyzer has frequency on the horizontal axis and the amplitude displayed on the vertical axis.

Analog spectrum analyzer

- A spectrum analyzer is an instrument is used to visualize and measure PSDF of power type signal.
- Measurement of PSDF is equivalent to passing the signal x(t) through a narrow band filter whose centered frequency can be sweep from near DC to infinity and observing or measuring output of the filter at each and every tuned frequency.
- As we know if x(t) is passed through a NB(narrow band) filter tuned at w_o the average power of the output signal y(t) would be:

$$P_y = \lim_{T \to \infty} 1/T \int_{-T/2}^{T/2} y^2(t)dt \approx 2k^2 s_x(t)\, \Delta f$$

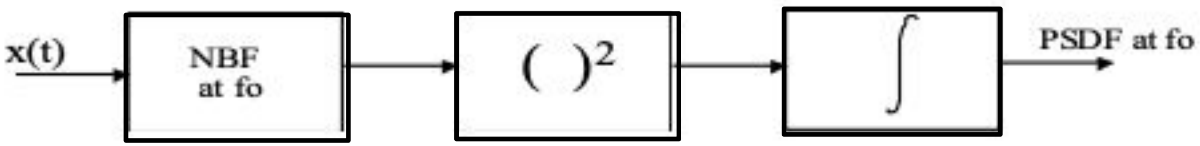
Where, k = the gain/ attenuation of filter within Δf .

Now , Let us consider $k^2 = 1/2\Delta f$ then,

$$P_y \approx s_x(f) \ldots\ldots\ldots\ldots(i)$$

which is the PSDF of x(t).

The above (i) can be obtained by passing x(t) through the NB filter centered at f_0, squaring the o/p of the filter and integrating it.



Now if we want to measure PSDF at frequency range we need weep the centered frequency of NF form zero to ∞ .

The accuracy of PSDF measurement increases with decrease in the bandwidth of narrow band filter.

In practice it is very difficult to weep the centre frequency of narrow band filter as they are constructed using quartz crystals.

Therefore another method in which narrow band filter is tuned to a precise frequency and the x(t) is frequency shifted in such a way that the frequency range of frequency shifted x(t) sequentially fall within the pass band of the filter, is employed to measure PSDF.

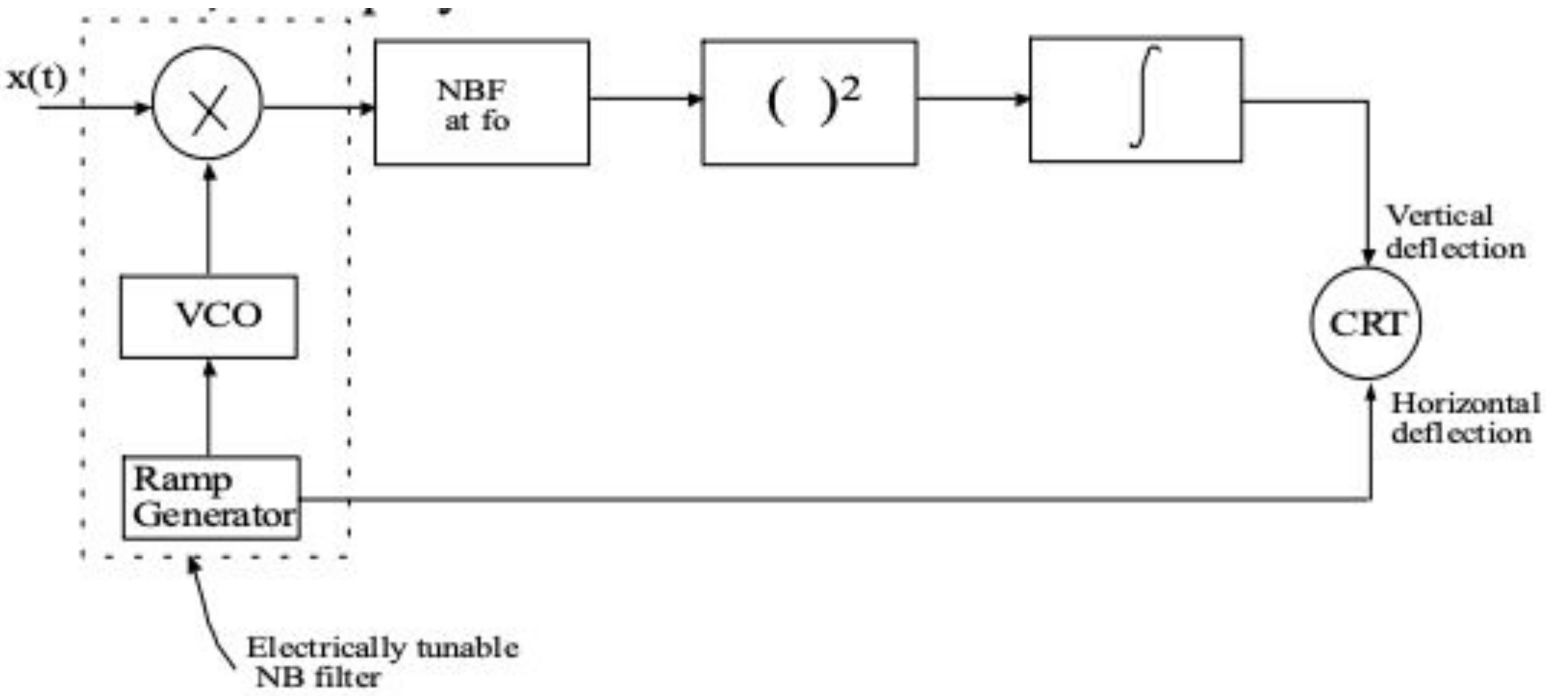


Figure : Analog spectrum analyzer

Assignment

- PSDF of harmonic signal
- Prove that in a LTI system when a power signal x(t) is applied the PSD of its output is equal to the PSD of its input multiplied by the squared amplitude response of the system.

Thank You