6. RANDOM SIGNALS and NOISE IN COMMUNICATION

Random variables and processes.

Variables can be classified on the basis of uncertainty about an outcome at any instant of time, which can be,

i) Deterministic variables

ii) Random variables.

i) Deterministic variable :

→ If the outcome of an experiment can be exactly predicted or counted then it is said to be deterministic. i.e. all data are known beforehand. For a given input, the output is always same.

example: If you know the initial deposit in a bank account and you know the interest rate, you can determine the amount in the account after a year.

ii) Random variable : (Denoted by capital letter)

→ If the outcome of an experiment can only be predicted with some probabilities then the outcome of that experiment is referred to as a random variable. For an experiment there is a set of possible outcomes known as sample space, denoted by 'S'.

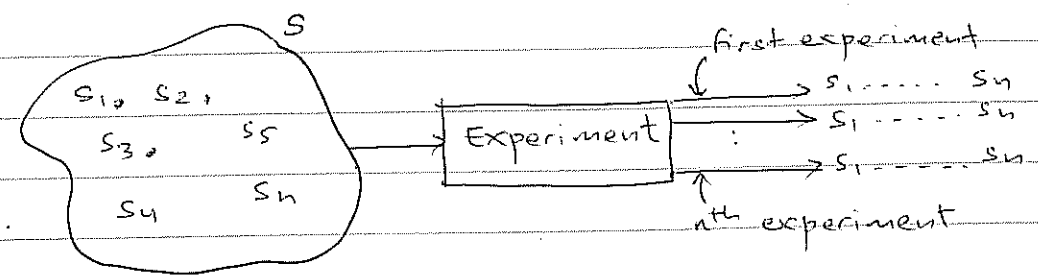
For example, an experiment can be throwing of a die. The possible outcomes for this particular experiment,

$S = \{1, 2, 3, 4, 5, 6\}$

So, the experiment can take any of the possible outcome at a particular instant.

Thus, a random variable can be defined as a function which can take on any value from the sample space and its range is some set of real numbers. You know the likelihood of something to happen but you don't know when.

Discrete and Continuous Random variable.



In the figure above, the sample space S contains the real numbers $s_1, s_2 \ldots s_n$.

Now, if the ~~nth~~ numbers s_1 to s_n are countable then the random variable is known as discrete random variable.

eg. Random variable associated with the outcomes of a coin-toss, card draws etc.

And if the numbers s_1 to s_n are uncountable then they are known as continuous random variable.

eg. measuring the time taken for a task to be done.

Random process : $x(t)$

In many real life systems, observations are made over a period of time and they are influenced by random effects, not just at a single instant but throughout the entire interval of time.

A random process in such cases assigns a time function to every outcome of ~~an~~ a random experiment.

Cumulative Distribution function [CDF].

→ CDF of a random variable 'X' can be defined as the probability of that a random variable 'X' takes a value less than or equal to a 'x'.

So, for any event $X \leq x$, the probability $P(X \leq x)$ is the cumulative distribution function, denoted by $F_X(x)$

$\therefore F_X(x) = P(X \leq x)$

→ It can also be called distribution f^n or probability distribution f^n.

Properties of $F_X(x)$

1. $0 \leq F_X(x) \leq 1$
2. $F_X(x_1) \leq F_X(x_2)$ if $x_1 < x_2$

   i.e. $F_X(x)$ is non decreasing.
3. $P(a < X \leq b) = F_X(b) - F_X(a)$
4. $F_X(-\infty) = 0$ and

   $F_X(\infty) = 1$.

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(4) Probability density function

The derivative of cumulative distribution function, $F_X(x)$, is known as probability density function expressed as,

$$PDF = f_X(x) = \frac{d}{dx} F_X(x)$$

Properties of PDF.

i) PDF is always non-zero for all values of x.
i.e. $f_X(x) \geq 0$ for all values of x.

ii) The area under the PDF curve is always equal to unity.
i.e. $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

iii) $F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$

iv) Probability of the event $\{x_1 < X \leq x_2\}$ is given by the area under PDF curve in range $x_1 < X \leq x_2$.
i.e.
$$P(x_1 < X \leq x_2) = \int_{x_1}^{x_2} f_X(x)\,dx$$

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R.V → Random Variable

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Some examples of important random variables are,

i) Uniform Random variable;
Its pdf is given by;

$$f_X(x) = \begin{cases} 1/b-a, & a < x < b \\ 0 & \text{otherwise} \end{cases}$$

This is a model for continuous R.V. whose range is known but nothing else is known about the likelihood of various values that the RV can assume.
eg. When ~~the~~ ~~po~~ phase of a sinusoid is random it is usually modeled as a uniform random variable between 0 & 2π.

Now,

$$F_X(x) = \begin{cases} 0 & \text{if } x < a \\ \frac{x-a}{b-a} & \text{if } a < x < b \\ 0 & \text{if } x > b \end{cases}$$

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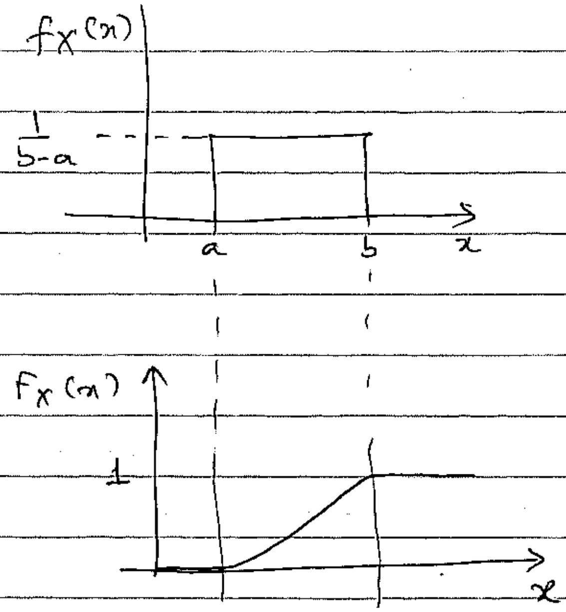

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~~Stastiti~~ Statistical and time averaged moments.

Mean or Average (m_x)

$$m_x = E[X]$$

where, E[ ] represents expectation operator

→ It is expressed by the summation of the values of random variable 'X' weighted by their probabilities.

For, discrete RV,

$$m_x = E[X] = \bar{X} = \sum_{i=1}^{n} x_i \cdot P(x_i)$$

For continuous RV

$$m_x = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx$$

where, $f_X(x) \to$ pdf.

Moments and variance.

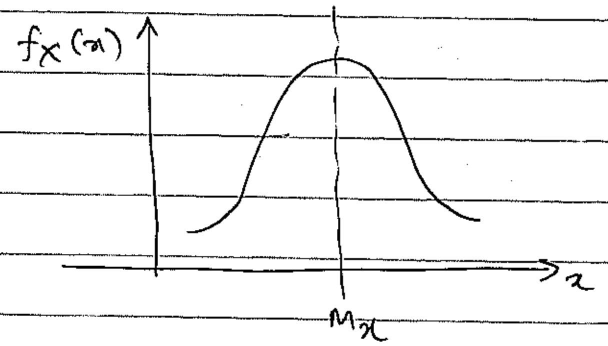
The n^{th} moment of a pdf $f_X(x)$ is defined as

$$E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x)\, dx$$

Now,
for the first moment i.e. $n=1$, above eq^n becomes,

$$E[X] = m_x = \int_{-\infty}^{\infty} x\, f_X(x)\, dx.$$

$\therefore$ first moment of a random variable 'X' will be same as its mean value.



for $n=2$,

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx$$

which gives the mean square value of 'X'.

Now, the central moments are the moments of difference between 'X' and its mean 'm_x'. The n^{th} central moment is given by,

$$E[(X-m_x)^n] = \int_{-\infty}^{\infty} (x-m_x)^n f_X(x)\, dx$$

The first central moment is always zero, whereas the second central moment is called the variance of random variable 'X' given by,

$$\text{Variance}[X] = \sigma_x^2 = E[(X-m_x)^2] = \int_{-\infty}^{\infty} (x-m_x)^2 f_X(x)\, dx$$

$$\text{or } \sigma_x^2 = E[X^2] - m_x^2 = \int_{-\infty}^{\infty} (x-m_x)^2 f_X(x)\, dx$$

The variance σ_x^2 is the measure of dispersion of pdf of X.

Square root of σ_x^2 ie σ_x is standard deviation.

Correlation function.

The joint moment of first order of two RV X & Y is called correlation.

i.e.

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{XY}(x,y)\, dx\, dy$$

and the joint central moment of X & Y is called covariance of X & Y and is given by,

$$cov[XY] = E\{[X - E(X)][Y - E(Y)]\}$$

$$= \cancel{E[(x-E(x)}$$

$$= E[(x-m_x)(y-m_y)]$$

$$= E[X \cdot Y] - m_x \cdot m_y$$

or $E[X \cdot Y] = cov[XY] + m_x \cdot m_y$

Now $\rho_{xy} = \dfrac{Cov[XY]}{\sigma_x \cdot \sigma_y}$ = correlation coefficient

where,

σ_x = standard deviation of X

σ_y = standard deviation of Y.

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Now,

i. X & Y are uncorrelated if and only if $cov[XY]$ = zero.

ii) X & Y are orthogonal if and only if $E[X \cdot Y] = 0$

iii) If both X & Y have zero mean i.e. $m_x = m_y = 0$ and are orthognal random variable i.e $E[XY] = 0$ then they are uncorrelated.

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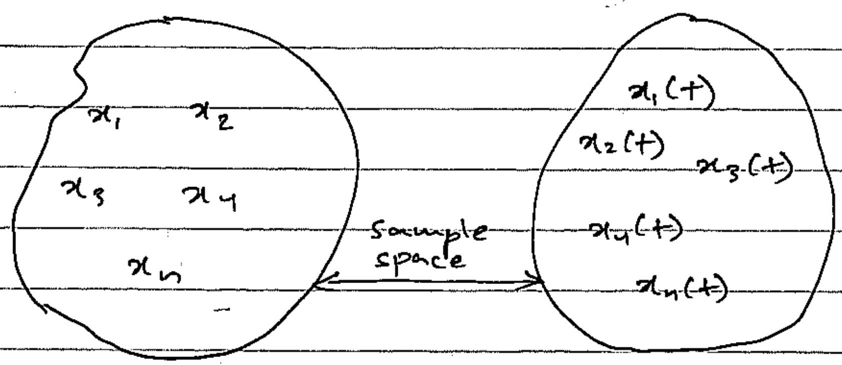
Random Process

→ In communication system, both the source signal as well as the noise are functions of time. So, the random variable associated with such signals must also have some relation with time.

So, in random process every random variable is assigned a time function to every outcome of any random experiment.

i.e. At any instance t, the value of the signal is random variable ~~p but~~ and such process or signal is called random process.



$x_i \to$ independent of time

$x_i(t) \to$ function of time.

A random process can be described using its statistical average (ensemble averages). The most common are means and autocorrelation f^n.

so, we have,

$$M_x(t) = E[X(t)] \text{ - mean value}$$

& $$R_{xx}(t) = E[X(t_1)X(t_2)] \text{ - autocorrelation.}$$

Stationary Process :

A random process is said to be stationary in strict sense if its statistics are not affected by any shift in time origin,

i.e.

$$X(t) = X(t+\Delta) \quad \text{where } \Delta = \text{any time shift.}$$

or

$$f_{X(t)}(x) = f_{X(t+\Delta)}(x)$$

which states that any time translation of a realization of random process results in another realization of random process having same statistical properties.

Statistical properties refer to all the moments.

In practice, these statistical properties are confined to mean, covariance & autocorrelation, as dealing with higher order moments are complex.

So, in practical use, strict sense reduces to wide sense where any random process is called stationary in wide sense if its mean and autocorrelation function donot vary with a shift in time origin.

i.e.

$$E[X(t)] = E[X(t+\tau)] = \text{constant and}$$

$$R_{XX}(t_1 t_2) = R_{XX}\{t+t_1, t+t_2\}$$
$$= R_{XX}\{|t_2 - t_1|\}$$
$$= R_{XX}(\tau)$$

or $R_{XX}(\tau) \triangleq E[X(t)X(t+\tau)]$

Naturally all strict sense stationary process (SSSP) is wide sense stationary process (WSSP) but vice versa is not true always.

AC → auto correlation

Properties of AC function of a WSSP.

i) $R_{XX}(\tau) = R_{XX}(-\tau)$

→ symmetry condition.

ii) $R_{XX}(0) = E[X(t-0)X(t)] = E[X^2(t)]$

→ mean square value

iii) $|R_{XX}(\tau)| \leq R_X(0)$

→ maximum value at zero time lag.

~~Time averaging~~

RP → random process

Time averaging

→ average quantity of a single system over a certain time.

Ensemble averaging

→ ~~many~~ average quantity of many identical system at a certain time.

Ergodic process.

→ In general time average and ensemble average are not equal for most of RP. But there exists a random process for which the time average and ~~ensem~~ ensemble average are equal. Such RP are called ergodic random process or ergodic processes.

A RP is said to be ergodic in general form if its statistical average mean value m_x and AC, $R_{xx}(\tau)$ are equal to time average value $\langle m_x \rangle$ and $\langle R_{xx}(\tau) \rangle$.

i.e.

$$E[\langle m_x \rangle] = m_x$$

$$E[\langle R_{xx}(\tau)] = E[\langle x(t)x(t-\tau)\rangle] = R_{xx}(\tau)$$

where,

time averages

$$\langle m_x \rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\,dt = \langle x(t) \rangle$$

&

$$\langle R_{xx}(\tau) \rangle = \langle x(t)x(t-\tau) \rangle$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\,x(t-\tau)\,dt$$

The autocorrelation (AC) and power spectral density function (psdf) of an ergodic process.

For a RP, the fourier transform of its AC function is called psdf.

i.e.

$$S_x(f) = \int_{-\infty}^{\infty} R_{xx}(\tau)\, e^{-j2\pi\tau}\, d\tau$$

provided

$$\int_{-\infty}^{\infty} R_{xx}(\tau)\, d\tau < \infty$$

In fact $R_{xx}(\tau)$ is the measure of average power dissipated in 1Ω resistance,

i.e. $E[x^2(t)] = R_{xx}(0) = \int_{-\infty}^{\infty} S_x(f)\,df$.

$\tau = 0$

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Now, if the process is ergodic then,

$$\langle S_x(f)\rangle = S_x(f)$$

where,

$$\langle S_x(f)\rangle = \lim_{T\to\infty} \frac{1}{T}\left[\int_{-T/2}^{T/2} x(t)\, e^{-j2\pi f t}\, dt\right]^2$$

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Noise in communication

1) White noise (WN)

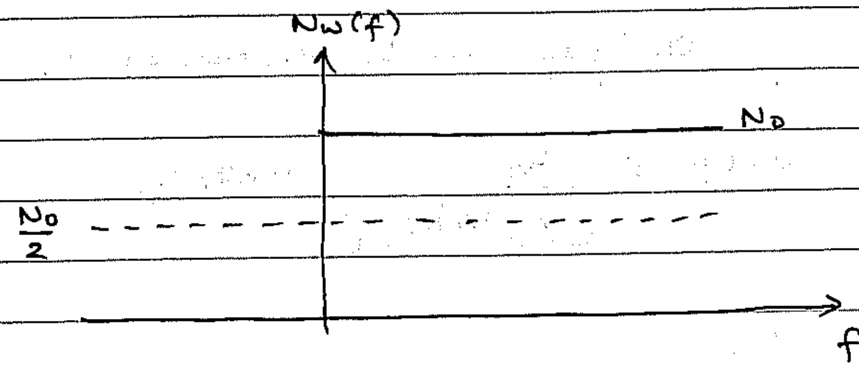
→ It is the ideal case of description of noise in communication. It can be taken as any random signal having equal intensity at different frequencies. It is analogous to white light which is the mixture of many different lights.

A WN has a flat spectral density $N_w(f)$ over $-\infty < f < \infty$ and has a zero mean value.

i.e., the psdf can be represented as,

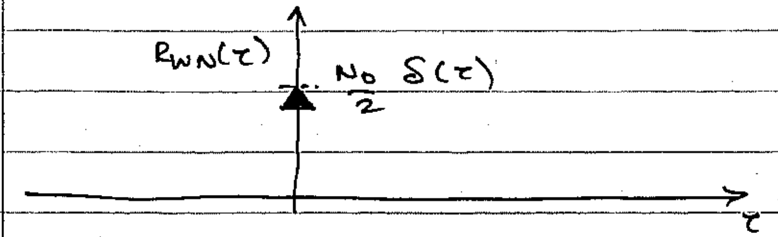
$$N_w(f) = N_0/2 \quad \text{for } -\infty < f < \infty$$

$$= N_0 \quad \text{for } 0 \le f < \infty \text{ or } -\infty < f < 0$$



The AC function of the white noise is therefore a delta function.

i.e. $R_{WN}(\tau) = \frac{N_0}{2}\delta(\tau)$



As $R_{WN}(\tau) = 0$ for $\tau \neq 0$, two different samples of WN, no matter how close they are in time shift $(\tau \to 0)$, are uncorrelated.

#2) Thermal noise.

The noise produced by the random movement of electrons in any conducting material due to thermal energy is termed as thermal noise.

Its pdf can be expressed as,

$$N_T(f) = \frac{hf}{2\left(e^{hf/KT} - 1\right)} \quad \text{watt/Hz}$$

where,

h = Planck's constant

K = Boltzman's constant

T = Temperature in Kelvin

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④ Passage of wide-sense stationary random signals through a LTI

Let a wide sense stationary random process (WSSP), $x(t)$, applied to the input of a LTI system having impulse response $h(t)$ & transfer function $H(f)$ respectively.

The output of the LTI system $y(t)$ is the convolution of input signal and the impulse response of the system,

i.e.

$$y(t) = \int_{-\infty}^{\infty} x(\alpha)\, h(t-\alpha)\, d\alpha$$

$$= \int_{-\infty}^{\infty} x(t-\alpha)\, h(\alpha)\, d\alpha$$

The mean value of $y(t)$ is thus equal to,

$$E[y(t)] = E\left[\int_{-\infty}^{\infty} x(\alpha) h(t-\alpha)\, d\alpha\right]$$

$$= \int_{-\infty}^{\infty} E[x(\alpha)]\, h(t-\alpha)\, d\alpha$$

$$= m_x \int_{-\infty}^{\infty} h(t-\alpha)\, d\alpha$$

$$= m_x H(0)$$

$\because$ $H(f) = \int_{-\infty}^{\infty} h(\alpha) e^{-j2\pi f\alpha} d\alpha$, so for $f=0$

$$H(0) = \int_{-\infty}^{\infty} h(\alpha)\, d\alpha$$

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As $E[y(t)] = m_x H(0)$, it is obvious that the mean value of the output signal is also independent of the shift in time.

Now, to determine the AC f^n of $y(t)$, let us determine the cross correlation between $x(t)$ & $y(t)$.

The cross correlation bet^n $x(t)$ & $y(t)$ can be expressed as,

$$R_{xy}(\tau) = E[x(t_1)\, y(t_2)] = E\left[x(t_1) \int_{-\infty}^{\infty} x(s)\, h(t_2-s)\, ds\right]$$

$$= \int_{-\infty}^{\infty} E[x(t_1)\, x(s)]\, h(t_2-s)\, ds = \int R_{xx}(t_1-s)\, h(t_2-s)\, ds$$

$$\cancel{= \int_{-\infty}^{\infty} R_{xx}(\tau)\, h(t_2-s)\, ds}$$

let $u = s - t_2$, then,
$s = u + t_2$

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} R_{xx}(t_1 - t_2 - u)\, h(-u)\, du$$

$$= \int_{-\infty}^{\infty} R_{xx}(\tau - u)\, h(-u)\, du \qquad [\tau = t_1 - t_2]$$

$$= R_{xx}(\tau) \circledast h(-\tau)$$

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Now,

the AC function of the output signal $y(t)$ is,

$$R_{YY}(t_1, t_2) = E[y(t_1) \cdot y(t_2)]$$

$$= E\left[y(t_2) \int_{-\infty}^{\infty} x(s) h(t_1 - s) ds \right]$$

$$= \int_{-\infty}^{\infty} E[x(s) \cdot y(t_2)] h(t_1 - s) ds$$

$$= \int_{-\infty}^{\infty} R_{XY}(s - t_2) h(t_1 - s) ds$$

Again,

let $u = s - t_2$, $s = u + t_2$,

$$R_{YY}(t_1, t_2) = \int_{-\infty}^{\infty} R_{XY}(u + t_2 - t_2) h(t_1 - t_2 - u) ds$$

$$= \int_{-\infty}^{\infty} R_{XY}(u) h(\tau - u) du$$

$$= R_{XY} \otimes h(\tau)$$

substituting ~~the~~ ~~val~~ for R_{XY} in above eqn,

$$R_{YY}(t_1, t_2) = R_{XX}(\tau) \otimes h(-\tau) \otimes h(\tau)$$

or $R_{YY}(\tau) = R_{XX}(\tau) \otimes h(\tau) \otimes h(-\tau)$

It is apparent from the above eqn that the AF for the output of the LTI

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system is not dependent on the shift in time.

Thus with mean & AF both independent of any shift in time, we can conclude that the ~~ot~~ output of LTI to a WSSP signal is also a WSSP.

$\therefore$ Now, we have psdf as the FT of AF.

$\therefore$ $S(f) = FT[R(\tau)]$

such that

$h(\tau) \xleftrightarrow{FT} H(f)$ where, $H^*(f)$ is complex

$h(-\tau) \xleftrightarrow{FT} H^*(f)$ conjugate of $H(f)$

and $R_{XX}(\tau) \xleftrightarrow{FT} S_X(f)$

$$\therefore S_Y(f) = FT[R_{XX}(\tau) \otimes h(\tau) \otimes h(-\tau)]$$

$$= S_X(f) \cdot H(f) \cdot H^*(f)$$

$$\therefore S_Y(f) = S_X(f) \cdot |H(f)|^2$$

which is the psds or simply pdf ~~of~~ the output signal.

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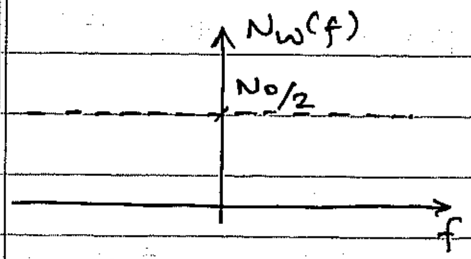
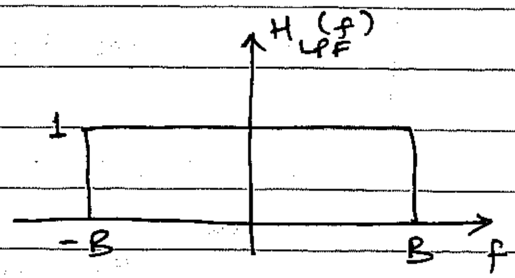
White noise - WN

(3) Ideal low-pass filtering of white noise.

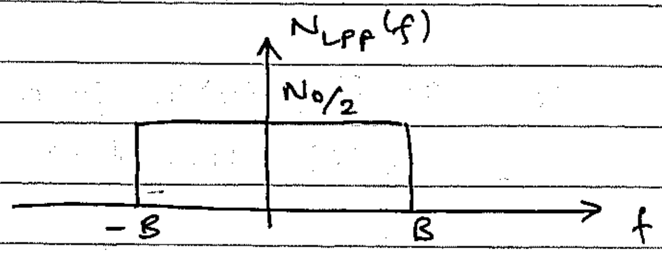
Let $N_W(f)$ be the frequency spectrum of a white noise with pdf $N_0/2$. The WN is applied to the input of an ideal LPF with bandwidth B.

Such that,





When passed through ideal LPF, the pdf of noise at the output of filter is represented as,



i.e.

$$N_{LPF}(f) = \begin{cases} N_0/2 & \text{for } -B < f < B \\ 0 & \text{elsewhere} \end{cases}$$

The autocorrelation function is derived as,

$$R_{N_{LPF}}(f) = \int_{-B}^{B} \frac{N_0}{2} e^{j2\pi f\tau} df$$

$$= \frac{N_0}{2} \cdot \frac{1}{j2\pi\tau} \cdot \left[e^{j2\pi f\tau} \right]_{-B}^{B}$$

$$= \frac{N_0}{2} \cdot \frac{1}{j2\pi\tau} \left[e^{j2\pi B\tau} - e^{-j2\pi B\tau} \right]$$

$$= \frac{N_0}{2} \cdot \frac{1}{\pi\tau} \cdot \left[\frac{e^{j2\pi B\tau} - e^{-j2\pi B\tau}}{2j} \right]$$

$$= \frac{N_0}{2} \cdot \frac{1}{\pi\tau} \cdot \sin 2\pi B\tau$$

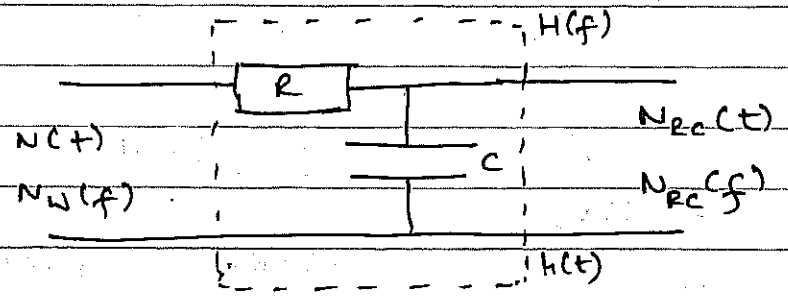
$$= N_0 \cdot B \cdot \frac{1}{2\pi B\tau} \cdot \sin 2\pi B\tau$$

$$R_{N_{LPF}}(f) = N_0 B \cdot \text{Sinc}(2B\tau)$$

where, $\frac{\sin \pi \cdot 2B\tau}{\pi \cdot 2B\tau} = \text{Sinc}(2B\tau)$.

White noise through RC filtering

Let a white noise $N(t)$ with spectrum $N_W(f)$ pass through a RC LPF as shown below,



The frequency response of RC filter can be expressed as,

$$H(f) = \frac{1}{1+j\omega RC} = \frac{1}{1+j\,2\pi f RC}$$

$$\Rightarrow |H(f)|^2 = \left|\frac{1}{1+j2\pi f RC}\right|^2 = \frac{1}{1+(2\pi f RC)^2}$$

Now we have,

$$N_{RC}(f) = N_W(f)\cdot|H(f)|^2$$

$$= \frac{N_0}{2}\cdot\frac{1}{1+(2\pi f\cdot RC)^2} \qquad [\because N_W(f) = N_0/2]$$

i.e. $$N_{RC}(f) = \frac{N_0}{2}\cdot\frac{1}{1+(2\pi f_c RC)^2}$$

Now, autocorrelation f^n, at output

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} N_{RC}(f)\, e^{j2\pi f\tau}\, df$$

$$= \int_{-\infty}^{\infty} \frac{N_0}{2}\cdot\frac{1}{1+(2\pi f RC)^2}\, e^{j2\pi f\tau}\, df$$

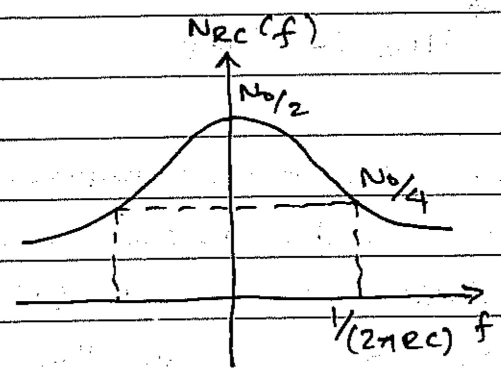
$$= \frac{N_0}{4\pi}\int_{-\infty}^{\infty} \frac{1}{1+(\omega RC)^2}\, e^{j\omega\tau}\, d\omega$$

$\Rightarrow$

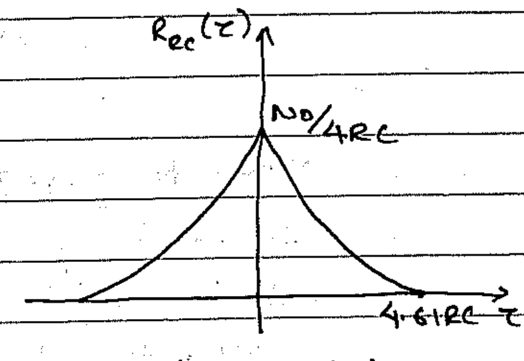
$$R_{RC}(\tau) = \frac{N_0}{2\pi}\int_{-\infty}^{\infty} \frac{\cos\omega\tau}{1+(\omega RC)^2}\, d\omega$$

$$R_{RC}(\tau) = \frac{N_0}{2\pi}\times\frac{\pi}{2RC}\cdot e^{-|\tau|/RC}$$

Such that the spectrum and the AC f^n of a white noise when passed through an RC LPF will have the following patterns.



Freq. spectrum



Autocorrelation

$$\boxed{P = \int_{-\alpha}^{\alpha} S(f)\, |H(f)|^2\, df}$$

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Noise equivalent bandwidth.

We can conclude from the derivations made earlier that the noise power at the output of an ideal LPF is finite and proportional to the bandwidth (B) of the filter, i.e.

$$P_{o/p}(\text{Ideal LPF}) = \frac{N_0}{2} \cdot 2B = N_0 B.$$

Also, the noise power at the output of an RC filter is theoretically infinite and is defined only by the transfer f^n of the RC filter,

i.e.

$$P_{o/p}(RC) = \frac{N_0}{2} \int_{-\alpha}^{\alpha} |H_{RC}(f)|^2\, df$$

$$= \frac{N_0}{2} \cdot 2 \int_{0}^{\alpha} |H_{RC}(f)|^2\, df$$

$$= N_0 \int_{0}^{\alpha} |H_{RC}(f)|^2\, df.$$

Now, to generalize the power expression for all kinds of LPF, we ~~de~~ define a parameter called noise equivalent bandwidth (B_N).

With the help of B_N we can then calculate average noise power.

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As for ideal LPF,

$$P_{o/p}(\text{Ideal}) = N_0 B$$

which can be written as,

$$P_{o/p}(\text{ideal}) = N_0 \cdot B_N \cdot |H(0)|^2$$

where, B_N = equivalent B.W.

$H(0)$ = transfer f^n of ideal LPF.

Also,

$$P_{o/p}(RC) = N_0 \int_{0}^{\alpha} |H(f)|^2\, df.$$

Now, with the equivalent bandwidth B_N applied we have,

$$P_{o/p}(\text{ideal}) = P_{o/p}(RC)$$

i.e.

$$N_0 \cdot B_N \cdot H^2(0) = N_0 \int_{0}^{\alpha} |H(f)|^2\, df$$

or

$$B_N = \frac{\int_{0}^{\alpha} |H(f)|^2\, df}{|H^2(0)|}$$

Similarly, for BPF,

$$B_N = \frac{\int_{0}^{\alpha} |H(f)|^2\, df}{|H^2(f_c)|}$$

And in general, B_N can be expressed as,

$$B_N = \frac{1}{g_a} \int_{0}^{\alpha} |H(f)|^2\, df$$ where g_a = max value of $|H(0)|^2$ or $|H(f_c)|^2$.

Now, $P_{o/p} = N_0 B_N \cdot g_a$

Flair

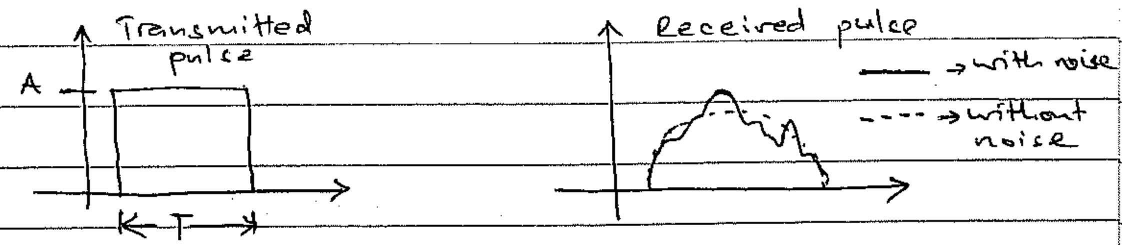
Optimum detection of a pulse in additive white noise.

In digital communication, combination of 1's & 0's are used to transmit any message.

The amplitude and duration of any pulse determines ~~wether~~ whether a signal for message contains 1's or 0's. So, when detecting the received signals, the detection of exact replica of the transmitted pulse is not important as the shape of the pulse is known most of the times. So, the only required function of the receiver is to decide the presence or absence of the pulses in the mix of received signal and noice.

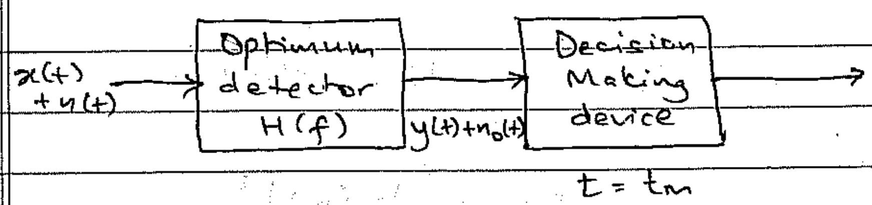
So, at the receiver end a detector is implemented such that it examines the entire duration of pulses and decides whether any pulse is present or absent in presence of any noise.

For such purpose optimum detector are vastly used as the device has least probability of error in making the decision in favour of 1 or 0.



Now the basic idea in determining the best received signal, the received signal is first passed through a filter that suppresses the noise and if any pulse is present then gives a sharp peak to the signal at decision making instance.

This results in maximum signal to noise ratio at decision making instance.



Now, the objective of optimum detector is to make $\left.\dfrac{y^2(t)}{n_o^2(t)}\right|_{max}$ at $t = t_m$.

i.e. $$\max\{SNR\} = \frac{y^2(t)}{n_o^2(t)} \quad \text{for } t = t_m.$$

$$= \frac{y^2(t_m)}{n_o^2(t_m)}$$

Now, for the optimum detector,

$$y(t) = F^{-1}[X(\omega)\cdot H(\omega)]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)\cdot X(\omega)\cdot e^{j\omega t}\, d\omega$$

and for $t = t_m$,

$$y(t_m) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)\, X(\omega)\, e^{j\omega t_m}\, d\omega$$

Also, average noise power,

$$n_o^2(t_m) = \frac{1}{2\pi}\int_{-\infty}^{\infty} N(\omega)\cdot|H(\omega)|^2\, d\omega$$

wher $N(\omega)$ = spectral density of white noise

$$\therefore\ n_o^2(t_m) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{N_o}{2}\cdot|H(\omega)|^2\, d\omega$$

$$= \frac{N_o}{4\pi}\int_{-\infty}^{\infty}|H(\omega)|^2\, d\omega$$

$$\therefore (SNR)_{t_m} = \frac{y^2(t_m)}{n_o^2(t_m)}$$

$$= \frac{\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)\cdot X(\omega)\, e^{j\omega t_m}\, d\omega\right)^2}{\frac{N_o}{4\pi}\int_{-\infty}^{\infty}|H(\omega)|^2\, d\omega}$$

or, $$SNR_{t_m} = \frac{1}{N_o\pi}\cdot\frac{\left(\int_{-\infty}^{\infty} H(\omega)\cdot X(\omega)\, e^{j\omega t_m}\, d\omega\right)^2}{\int_{-\infty}^{\infty}|H(\omega)|^2\, d\omega}$$

Now,

to maximize the SNR we apply schwartz's inequality principle which states,

$$\left|\int_{-\infty}^{\infty} f_1(x)\cdot f_2(x)\, dx\right|^2 \le \int_{-\infty}^{\infty}|f_1(x)|^2 dx\cdot\int_{-\infty}^{\infty}|f_2(x)|^2 dx$$

or $$\frac{\left|\int_{-\infty}^{\infty} f_1(x)\, f_2(x)\, dx\right|^2}{\int_{-\infty}^{\infty}|f_1(x)|^2 dx} \le \int_{-\infty}^{\infty}|f_2(x)|^2 dx$$

which holds true only if,

$$f_1(x) = k\, f_2^*(x)$$

where, k = arbitrary constant

$f_2^*(x)$ = complex conjugate of $f_2(x)$.

Now substituting $f_1(x)$ by $H(\omega)$ & $f_2(x)$ by $X(\omega)\, e^{j\omega t_m}$

$$\frac{\left[\int_{-\infty}^{\infty} H(\omega)\cdot X(\omega)\, e^{j\omega t_m}\, d\omega\right]^2}{\int_{-\infty}^{\infty}|H(\omega)|^2 d\omega} \le \int_{-\infty}^{\infty}|X(\omega)\, e^{j\omega t_m}|^2 d\omega$$

or $$\frac{1}{N_o\pi}\cdot\frac{\left[\int_{-\infty}^{\infty} H(\omega)\cdot X(\omega)\, e^{j\omega t_m}\, d\omega\right]^2}{\int_{-\infty}^{\infty}|H(\omega)|^2 d\omega} \le \frac{1}{N_o\pi}\int_{-\infty}^{\infty}|X(\omega)|^2 d\omega$$

$$\because\ |e^{j\omega t_m}|^2 = 1$$

$$\therefore \; SNR_{t_m} \leq \frac{1}{N_0 \pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

And the SNR is maximum ~~when~~,

$$SNR_{max} \leq \frac{1}{N_0 \pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega,$$

when,

$$H(\omega) = K \cdot \{X(\omega) e^{j\omega t_m}\}^*$$
$$= K \cdot X^*(\omega) \cdot e^{-j\omega t_m}$$
$$= K \cdot X(-\omega) \, e^{-j\omega t_m}$$

Now, the impulse response of the optimum detector corresponding to the transfer function $H(\omega)$ will be,

$$h(t) = F^{-1}[H(\omega)]$$
$$= F^{-1}\left[K \cdot X(-\omega) e^{-j\omega t_m}\right]$$

$$\therefore h(t) = K \cdot x(t_m - t)$$

assuming $K = 1$, we have

$$h(t) = x(t_m - t)$$

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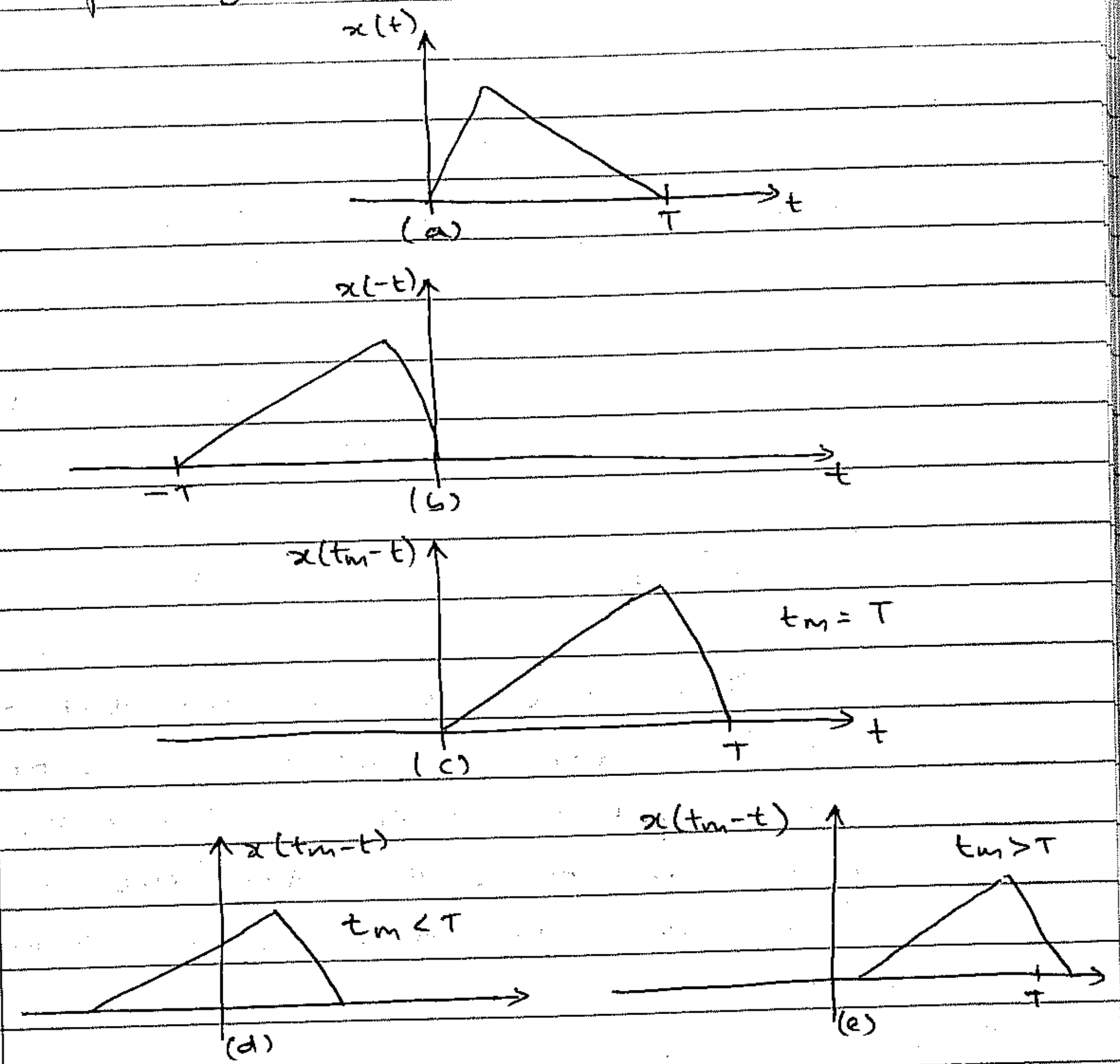
Graphically,



The impulse response, as can be seen above is the mirror image of incoming signal shifted by t_m. Thus it is also known as matched filter.

In practise we use $t_m \geq T$.

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$$\boxed{\frac{1}{N_0\pi}\int_{-\infty}^{\infty}|X(\omega)|^2 d\omega = \frac{2}{N_0}\int_{-\infty}^{\infty}|X(f)|^2 df}$$

Also, at the time $t = t_m$, the maximum output signal to noise ratio is given by

$$\max\left(\frac{S}{N}\right)_{\max} = \frac{2}{N_0}\int_{-\infty}^{\infty}|X(f)|^2 df$$

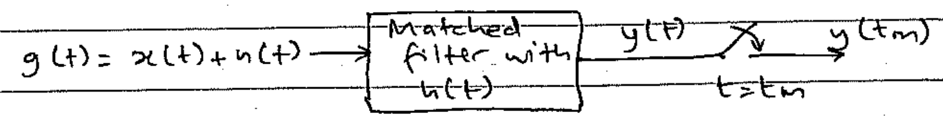
And for input signal $x(t)$, its energy is given by,

$$E = \int_{-\infty}^{\infty} x^2(t)\,dt = \int_{-\infty}^{\infty}|X(f)|^2 df \quad \left[\text{Parseval's theorem}\right]$$

$$\therefore \left(\frac{S}{N}\right)_{t_m} = \frac{2}{N_0}\cdot E = \frac{E}{N_0/2}$$

Thus it can be deduced that the maximum output signal to noise ratio depends on the input signal energy 'E' and the power spectral density of the noise '$N_0/2$' and not on the particular shape of the waveform that is used.

3 ~~Reali~~ Realization of Matched filter.



The output $y(t)$ for the above arrangement can be expressed as,

$$y(t) = \int_{-\infty}^{\infty} g(s)\cdot h(t-s)\,ds$$

Now,

$$h(t) = x(t_m - t)$$

$$h(t-s) = x(t_m - t + s) = x(s + t_m - t)$$

So,

when detection is done at time $t = t_m$,

$$h(t_m - s) = x(s + t_m - t_m) = x(s)$$

$$\therefore y(t_m) = \int_{-\infty}^{\infty} g(s)\,x(s)\,ds.$$

The above equation states that the arrangement shown above is a time correlator and is a synchronous detector.

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Matched filter for rectangular pulse.

Let us consider a signal $x(t)$ with a rectangular pulse of amplitude A and duration T such that it has a unit area

i.e. $A \cdot T = 1$

and

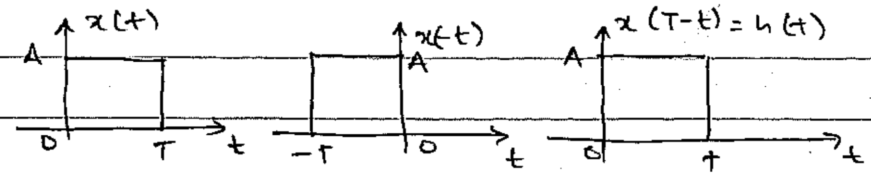
$$x(t) = A \quad \text{for} \quad 0 \le t \le T$$
$$= 0 \quad \text{elsewhere},$$

And,

$$X(f) = \text{sinc}(fT)\, e^{-j\pi f \cdot T}$$

Now,

$$h(t) = x(T-t) = A \quad \text{for} \quad 0 \le t \le T.$$



$$\therefore \; H(f) = \text{sinc}(f \cdot T)\, e^{-j\pi f \cdot T}$$

Now,

$$Y(f) = X(f) \cdot H(f)$$
$$= \text{sinc}^2(fT) \cdot e^{-j2\pi f \cdot T}$$

where,

$\text{sinc}^2(f \cdot T)$ is the energy spectrum density of a rectangular pulse having unit area.

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Also we have $\left(\frac{S}{N}\right)_{max} = E/_{N_0/2}$

where,

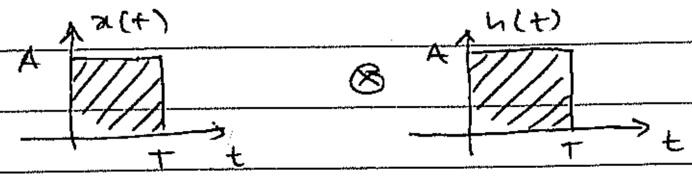
$$E = \int_{-\infty}^{\infty} x^2(t)\, dt = \int_0^T A^2\, dt = A^2 T$$

$$\therefore \left(S/N\right)_{max} = A^2T/_{N_0/2} = \frac{2A^2T}{N_0}.$$

Now,

$$Y(f) = X(f) \cdot H(f)$$
$$\Rightarrow y(t) = x(t) \otimes h(t).$$
$$= \int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau)\, d\tau.$$

The output wave form will thus be,



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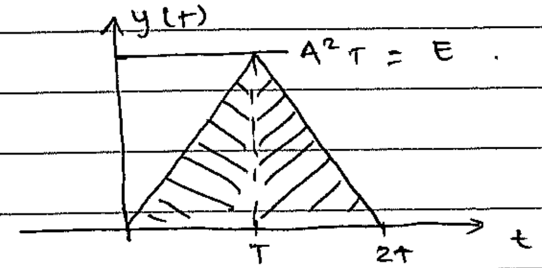

where, $y = A^2 t \quad \text{for} \quad 0 \le t \le T$
$= \cancel{A^2(2t-T)} \quad \text{for} \quad T \le t \le 2T$
$= A^2(2T-t)$
$= 0 \quad \text{elsewhere}$

Flair

It can be seen that the matched filter gives the maximum output SNR only when the entire signal component has entered the filter i.e. if the signal duration is 'T' then the decision can be made only when time elapses by '2T'.

This may be impractical when 'T' is large.

Therefore, for a matched filter to be practically realizable the following criteria should be met,

1. The maximum SNR should be achieved at some instance less than 'T'.

Ideal LPF and RC filters as matched filters.

for a $x(t) = A \quad 0 \le t \le T$,

We have for ideal LPF,

$$SNR_{LPF} = \frac{(2A/\pi)^2 \cdot Si^2(\pi B T)}{B \cdot N_0} \qquad \left[Si(x) = \int_0^x \frac{\sin \tau}{\tau} d\tau \right]$$

And for RC filter,

$$SNR_{RC} = \frac{1.628 A^2 T}{N_0}$$

And for rectangular pulse,

$$SNR_{MF} = \frac{2A^2T}{N_0}.$$

7hrs.

4. Baseband Data Communication Systems

4.1 Introduction to information theory, measure of information, entropy, symbol rates and data (bit) rates.