(2) Find radius of curvature at $ay^2 = x^3$

(3) $a y^2 = a^3 - x^3$ at $(a,0)$ on the curve.

(2) $ay^2 = x^3$

$2ayy' = 3x^2$

$y' = \frac{3x^2}{2ay}$

$y'' = \frac{\cancel{2}ay(6x) - 3x^2(\cancel{2}ay')}{(\cancel{2}ay)^2}$

$= \frac{6\cancel{a}xy - 3x^2\cancel{a}y'}{2\cancel{a}^2y^2}$

$= \frac{6xy - 3x^2y'}{2ay^2}$

$s = \frac{\left[1+\left(\frac{3x^2}{2ay}\right)^2\right]^{3/2}}{\frac{6xy - 3x^2y'}{2ay^2}}$

$= \left(\frac{4a^2y^2 + 9x^4}{4a^2y^2}\right)^{3/2} \frac{2ay^2}{6xy - 3x^2y'}$

$= \frac{(4a^2y^2 + 9x^4)^{3/2}}{(2ay)^3} \frac{2ay^2}{6xy - 3x^2\left(\frac{3x^2}{2ay}\right)}$

$= \frac{(4a^2y^2 + 9x^4)^{3/2}}{(2ay)^3} \frac{2ay^2}{12axy^2 - 9x^4} \times 2ay$

$$\frac{(4a^2y^2+9x^4)^{3/2}}{2a(12axy^2-9x^4)}$$

$a^2y^2 = x^3$

$2a^2yy_1 = 3x^2$

$y_1 = \dfrac{3x^2}{2ay}$

$y_2 = \dfrac{2ay(6x) - \frac{3x^2}{2ay}(2ay')}{(2ay)^2}$

$= \dfrac{6xy - 3x^2y'}{2ay^2}$

$= \dfrac{6xy - 3x^2\left(\frac{3x^2}{2ay}\right)}{2ay^2}$

$\dfrac{12axy^2 - 9x^4}{4a^2y^3}$

$\rho = \left[1+\left(\dfrac{3x^2}{2ay}\right)^2\right]^{3/2}$

---

(3) $a^2y^2 = a^3 - x^3$ $(a,0)$

$a^2(2yy') = -3x^2$

$y' = \dfrac{-3x^2}{2a^2y}$

$= -3a$

---

(2) $ay^2 = x^3$

$\log a + 2\log y = 3\log x$

$\dfrac{2}{y}y' = \dfrac{3}{x}$

$y' = \dfrac{3y}{2x}$

$y'' = \dfrac{2x\cdot 3y' - 3y\cdot 2}{(2x)^2}$

$= \dfrac{6xy' - 6y}{4x^2}$

$\rho = \dfrac{\left[1+\left(\frac{3y}{2x}\right)^2\right]^{3/2}}{\frac{6xy'-6y}{4x^2}}$

$= \left[\dfrac{4x^2+9y^2}{4x^2}\right]^{3/2} \dfrac{4x^2}{6xy'-6y}$

$= \dfrac{(4x^2+9y^2)^{3/2}}{\left(\cancel{6}^{3}x\left(\frac{3y}{\cancel{2x}}\right) - 6y\right)2x}$

$= \dfrac{(4x^2+9y^2)^{3/2}}{3xy\,x}$

* Find radius of curvature for the curve whose intrinsic equation is $s = a\log\tan(\pi/4 + \psi/2)$

$\rightarrow \quad s = a\log\tan(\pi/4 + \psi/2)$

$$\frac{ds}{d\psi} = \frac{a}{\tan(\pi/4+\psi/2)}\left(\sec^2(\pi/4+\psi/2)\right)\frac{1}{2}$$

$$= \frac{a}{2}\,\frac{1}{\sin(\ )\cos(\ )}.$$

$$= \frac{a}{\sin 2(\pi/4+\psi/2)} = \frac{a}{\sin\left(\frac{\pi}{2}+\psi\right)} = \frac{a}{\cos\psi}$$

$$\frac{ds}{d\psi} = a\sec\psi = \rho$$

* Show that the radius of curvature for the catenary of uniform $y = a\log\sec(x/a)$ is $a\sec x/a$

$$\rightarrow \quad \frac{ds}{d\psi} = \frac{1}{a}\,\frac{a}{\sec x/a}\sec(x/a)\tan x/a$$

$$y_1 = \tan(x/a).$$

$$y_2 = \sec^2 x/a$$

$$\rho = \frac{(1+\tan^2 x/a)^{3/2}}{\sec^2(x/a)\frac{1}{a}} = \frac{a(\sec^2 x/a)^{3/2}}{\sec^2 x/a}$$

$$= a\sec(x/a)$$

* Show that for the catenary $y = c\cosh(x/c)$ the radius of curvature $= y^2/c$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$\frac{dy}{dx} = c\sinh(x/c)\frac{1}{c} = \sinh\left(\frac{x}{c}\right).$$

$$y_2 = \cosh\left(\frac{x}{c}\right)\frac{1}{c}$$

$$\rho = \frac{(1+\sinh^2 x/c)^{3/2}}{\cosh(x/c)\frac{1}{c}} = \frac{c(\cosh^2 x/c)^{3/2}}{\cos(x/c)}$$

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$= c\cosh^2(x/c)$

$= \frac{y^2}{c}$

* Find the radius of curvature for the curve $y^2 = \frac{4a^2(2a-x)}{x}$, where the curve meets the x-axis

→ $\frac{4a^2(2a-x)}{x} = 0$ (since curve touches x axis $y=0$)

$4a^2(2a-x) = 0$

$x = 2a$ $(2a,0)$ is the pt. on the curve at which we have to find ρ.

the given equation can be put in the form

$y^2 = \frac{8a^3}{x} - 4a^2$

$2yy_1 = -\frac{\cancel{8}^{4}a^3}{x^2}$

$y_1 = \frac{-4a^3}{x^2 y}$

At $(2a,0)$ $y_1 = \infty$ & hence we have to consider $\frac{dx}{dy} = x_1$.

$x_1 = \frac{-x^2 y}{4a^3} = 0$

$x_2 = -\frac{1}{4a^3}(2x x_1 + x^2)$

At $(2a,0)$

$x_2 = -\frac{1}{\cancel{4}a^3}(\cancel{4}a^2) = -\frac{1}{a}$

$\rho = \frac{(1+x_1^2)^{3/2}}{x_2} = -a$

$|\rho| = a$

* Find the radius of curvature for the curve $x^2 y = a(x^2+y^2)$ at the pt. $(-2a, 2a)$

$\Rightarrow$ $x^2 y' + 2xy = a(2x + 2yy')$

$= 2ax + 2ayy'$

$x_1 = \dfrac{x(x^2-y^2)}{-2y^3}$

$y' = \dfrac{2ax - 2xy}{x^2 - 2ay}$

At $(-2a, 2a)$

$y' = \dfrac{2a(-2a) - 2(-2a)(2a)}{(-2a)^2 - 2a(2a)} = \dfrac{-4a^2 + 8a^2}{4a^2 - 4a^2}$

$\underline{\underline{x_1 = 0}}$

$x_1 = \dfrac{x^2 - 2ay}{2ax - 2xy} = x^2 - 2ay$

$x_2 = \dfrac{2ax - 2xy(2xx_1 - 2a) - (x^2-2ay)(2ax_1 - 2x - 2yx_1)}{(2ax - 2xy)^2}$

$=$

$\dfrac{-2(2a)^3(\cancel{3\times4a^2} - 4a^2) - (-8a^3 - (-2a)4a^2)(-6\times4a^2)}{(2 \cdot 8a^3)^2}$

$-8a^3 - 8a^3$ 24

$-16a^3$

$64a^5$

2
24×16

144
$24+$

384

64

320

$\dfrac{\cancel{160}\,320a^5}{144a^6}$

$\cancel{80}$
$\cancel{40}$
20

$\cancel{72}$

$\cancel{36}\ \cancel{18}\ 9$

$\dfrac{20}{9a}$//

$$x^2 y = a(x^2+y^2)$$

Method 1

$$x^2y' + 2xy = 2ax + 2ayy'$$

$$x^2y' - 2ayy' = 2ax - 2xy$$

$$y' = \frac{2ax - 2xy}{x^2 - 2ay}$$

$$2xx'y + x^2 = 2xx'a + 2$$

$$2xx'y - 2xx'a = 2ya - x^2$$

$$x' = \frac{2ay - x^2}{2xy - 2xa}$$

Method 2

$$2\log x + \log y = \log a + \log(x^2+y^2)$$

$$\frac{2}{x} + \frac{1}{y}y' = \frac{1}{x^2+y^2}(2x+2yy')$$

$$= \frac{2x}{x^2+y^2} + \frac{2yy'}{x^2+y^2}$$

$$-\frac{2}{x} - \frac{2x}{x^2+y^2} = y'\left(\frac{-1}{y} + \frac{2y}{x^2+y^2}\right)$$

$$= y'\left(\frac{-x^2 - y^2 + 2y^2}{x^2y + y^3}\right)$$

$$y' = \left(\frac{2}{x} - \frac{2x}{x^2+y^2}\right)\left(\frac{x^2y+y^3}{\cancel{x^2+3y^2}\ y^2 - x^2}\right)$$

$$= \left(\frac{\cancel{2x^2} + 2y^2 - \cancel{2x^2}}{x^3+xy^2}\right)\left(\frac{x^2y+y^3}{\cancel{x^2+3y^2}\ y^2 - x^2}\right)$$

$$= \left(\frac{2y^2}{x^3+xy^2}\right)\left(\frac{x^2y+y^3}{\cancel{x^2+3y^2}\ y^2 - x^2}\right)$$

$$= \frac{2y^3}{x\cancel{(x^2+y^2)}} \frac{\cancel{x^2+y^2}}{y^2-x^2}$$

$$= \frac{-2y^3}{x(x^2-y^2)} \qquad x' = \frac{x(x^2-y^2)}{-2y^3}$$

1, 2

$\frac{16}{3}$ $-\frac{16}{3}$ $\cancel{-2x}$ $-\frac{2}{1}(8)$ -4

$$x'' = \frac{-2y^3\left[x'(x^2-y^2)+x(2xx'-2y)\right]-x(x^2-y^2)(-6y^2)}{(-2y^3)^2}$$

$$= \frac{-2y^3(-2xy)+6xy^2(x^2-y^2)}{4y^6}$$

$$= \frac{2(2a)^3(2)(-2a)(2a) + 6(-2a)(2a)^2}{4(2a)^{\cancel{6}\,2}}$$

$$= \frac{-\cancel{8}a}{\cancel{4}\cdot 4a^2} = -\frac{1}{2a}$$

$$|s| = \frac{(1+x_1^2)^{3/2}}{x_2} = \underline{\underline{2a}}$$

① Find radius of curvature for the curve $\sqrt{x}+\sqrt{y}=4$ at the pt. where it cuts the line passing through origin makes an angle $45°$ with x axis.

→ Equ^n of line is $x=y$ & we ~~found~~ find the pt. of intersection of line with curve $\sqrt{x}+\sqrt{y}=4$

$\sqrt{x}+\sqrt{x}=4$ $\quad 2\sqrt{x}=4$ $\quad \sqrt{x}=2$ $\quad \underline{\underline{x=4}}$

pt is $(4,4)$.

$$\sqrt{x}+\sqrt{y}=4$$

$$\frac{1}{\cancel{2}\sqrt{x}}+\frac{1}{\cancel{2}\sqrt{y}}y'=0$$

$$\frac{1}{\sqrt{x}}=-\frac{y'}{\sqrt{y}}$$

$$y'=-\sqrt{\frac{y}{x}}$$

At $(4,4)$ $\underline{\underline{y_1=-1}}$

$$y''=\frac{\sqrt{x}\left(\frac{-1}{2\sqrt{y}}y'\right)+\sqrt{y}\left(\frac{1}{2\sqrt{x}}\right)}{x}$$

$$=\frac{\sqrt{x}\left(\frac{+1}{2\sqrt{y}}\right)+\frac{1}{2}\sqrt{\frac{y}{x}}}{x}$$

$$=\frac{\cancel{2}\left(\frac{1}{\cancel{2}\cdot 2}\right)+\frac{1}{2}}{4}$$

$$=\frac{1}{4}//.$$

$$\rho=\frac{(1+1)^{3/2}}{\frac{1}{4}}$$

$$=2^{3/2}\cdot 2^{2}$$

$$=2^{7/2}=\underline{\underline{8\sqrt{2}}}$$

② If ρ be the radius of curvature at any pt. on the parabola $y=4ax$, show that ρ^2 varies as $(SP)^3$ where S is focus of parabola.

→ Consider $y^2=4ax$

$$2yy'=4a$$

$$y'=\frac{\cancel{2}a}{y}$$

$$y''=-\frac{2a}{y^2}y'$$

$$=-\frac{4a^2}{y^3}$$

$$\rho = \frac{\left[1+\left(\frac{2a}{y}\right)^2\right]^{3/2}}{-\frac{4a^2}{y^3}} = \frac{-y^3}{4a^2}\left[\frac{(y^2+4a^2)}{y^3}\right]^{3/2}$$

$$= -\frac{1}{4a^2}\left[\cancel{y}\,(4ax+4a^2)^{3/2}\right]$$

$$= \frac{(4a)^{3/2}\,(x+a)^{3/2}}{-4a^2}$$

By squaring both sides

$$\rho^2 = \frac{(4a)^3(x+a)^3}{16a^4} = \frac{64a^3(x+a)^3}{16a^4}$$

$$= \frac{4}{a}(x+a)^3 \quad — (1)$$

Focus of parabola is $S=(a,0)$ & we have (x,y).

$$SP = \sqrt{(x-a)^2+(y-0)^2}$$

$$= \sqrt{x^2+a^2-2xa+y^2}$$

$$= \sqrt{x^2+a^2-2ax+4ax}$$

$$= x+a \quad — (2)$$

Using (2) in (1)

$$\rho^2 = \frac{4}{a}(SP)^3$$

Thus $\rho^2 \propto (SP)^3$

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An expression for radius of curvature for polar curve

$$\boxed{\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}}$$

① S.T for equiangular spiral $r = ae^{\theta\cot\alpha}$ where a & α are constants ρ/r is const.

$$r = ae^{\theta\cot\alpha}$$

$$\log r = \log a + \theta\cot\alpha$$

$$\frac{1}{r}\frac{dr}{d\theta} = \cot\alpha$$

$$r_1 = r\cot\alpha$$

$$r_2 = r_1\cot\alpha$$

$$\rho = \frac{[r^2 + (r\cot\alpha)^2]^{3/2}}{r^2 + 2(r\cot\alpha)^2 - (r)(r_1\cot\alpha)}$$

$$= \frac{[r^2 + r^2\cot^2\alpha]^{3/2}}{r^2 + 2r^2\cot^2\alpha - r(r\cot\alpha)(\cot\alpha)}$$

$$= \frac{r^3(1+\cot^2\alpha)^{3/2}}{r^2 + 2r^2\cot^2\alpha - r^2\cot^2\alpha} = \frac{r^3(\operatorname{cosec}^2\alpha)^{3/2}}{r^2 + r^2\cot^2\alpha}$$

$$= \frac{r^3\operatorname{cosec}^3\alpha}{r^2\operatorname{cosec}^2\alpha}$$

$$= \cancel{\frac{1}{r\sin\alpha}} \quad r\operatorname{cosec}\alpha$$

$$\frac{\rho}{r} = \operatorname{cosec}\alpha$$

$$= \text{const.}$$

(2) S.T the radius of curvature of the curve $r^n = a^n \cos n\theta$ varies inversly as r^{n-1}.

$\rightarrow$ $n \log r = n \log a + \log \cos n\theta$

$\frac{\cancel{n}}{r} r_1 = \frac{\cancel{n}}{\cos n\theta}(-\sin n\theta)$

$r_1 = -r \tan n\theta$

$r_2 = -r\sec^2 n\theta\,(n) + \tan n\theta(-r')$

$= -rn\sec^2 n\theta + r\tan^2 n\theta$

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}$$

$$= \frac{(r^2 + r^2\tan^2 n\theta)^{3/2}}{r^2 + 2r^2\tan^2 n\theta - r^2\tan^2 n\theta + r^2 n\sec^2 n\theta}$$

$$= \frac{r^3\sec^3 n\theta}{r^2 + r^2\tan^2 n\theta + r^2 n\sec^2 n\theta}$$

$$= \frac{r\sec^3 n\theta}{\sec^2 n\theta + n\sec^2 n\theta} = \frac{r\sec n\theta}{1+n}$$

$\sec n\theta = \frac{a^n}{r^n}$ $\qquad \rho = \frac{r}{1+n}\left(\frac{a^n}{r^n}\right)$

$= \frac{a^n}{1+n} \cdot \frac{1}{r^{n-1}}$

$$\boxed{\rho \propto \frac{1}{r^{n-1}}}$$

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(8) S.T for the curve $r(1-\cos\theta) = 2a$ ρ^2 varies as r^3.

$\cos 2\theta <$ 1...

$$r(1-\cos\theta) = 2a$$

$$\log r + \log(1-\cos\theta) = \log 2a$$

$$\frac{1}{r} r' + \frac{(\sin\theta)}{1-\cos\theta} = 0$$

$$\frac{r'}{r} = -\frac{\sin\theta}{1-\cos\theta} = -\frac{\cancel{2}\sin\theta/2 \cos\theta/2}{\cancel{2}\sin^2\theta/2}$$

$$= -\cot\theta/2$$

$$r' = -r\cot\theta/2$$

$$r'' = +\frac{r}{2}\operatorname{cosec}^2\theta/2 + \cot\theta/2(-r_1)$$

$$= \frac{r\operatorname{cosec}^2\theta/2}{2} + r\cot^2\theta/2$$

$$S = \frac{(r^2 + r^2\cot^2\theta/2)^{3/2}}{r^2 + 2r^2\cot^2\theta/2 - \frac{r^2\operatorname{cosec}^2\theta/2}{2} - r^2\cot^2\theta/2}$$

$$= \frac{r^3\operatorname{cosec}^3\theta/2}{r^2 + r^2\cot^2\theta/2 - \frac{r^2\operatorname{cosec}^2\theta/2}{2}}$$

$$= \cancel{\frac{2r^3\operatorname{cosec}^3\theta/2}{2r^2 - r^2\operatorname{cosec}^2\theta/2 + r^2\cot^2\theta/2}}$$

$$= \frac{2r^3\operatorname{cosec}^3\theta/2}{2r^2\operatorname{cosec}^2\theta/2 - r^2\operatorname{cosec}^2\theta/2} = \frac{2r^3\operatorname{cosec}^3\theta/2}{r^2\operatorname{cosec}^2\theta/2}$$

$$= 2r\operatorname{cosec}\theta/2$$

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$r(1-\cos\theta) = 2a$

$\cancel{2} r \sin^2 \theta/2 = \cancel{2} a$

$\operatorname{cosec}^2 \theta/2 = \dfrac{r}{a}$

$\rho^2 = 4r^2 \operatorname{cosec}^2 \theta/2$

$= 4r^2 \left(\dfrac{r}{a}\right)$

$\rho^2 = \left(\dfrac{4}{a}\right) r^3$

$\underline{\underline{\rho^2 \propto r^3}}$

④ ~~Find~~ ~~radius~~ $r = a\sin n\theta$ at the pole. $(\theta = 0)$

$r_1 = na\cos n\theta \qquad r_2 = -an^2 \sin n\theta$

At the pole $\theta = 0$

when $\theta = 0$, $r = 0$, $r_1 = an$, $r_2 = 0$

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - r r_2}$$

$$= \frac{(an)^3}{2(an)^2} = \underline{\underline{\frac{an}{2}}}$$

⑤ S.T at the pt. where the curve $r = a\theta$ intersect the curve $r = \dfrac{a}{\theta}$, their curvatures are in the ratio $3:1$

→ $r = a\theta$

$r_1 = a$

$r_2 = 0$

$r = a\theta \qquad r = \dfrac{a}{\theta}$

$a\theta = \dfrac{\cancel{a}}{\theta} \qquad \boxed{\theta = \pm 1}$

At $\theta = 1$

$r = a$, $r_1 = a$, $r_2 = 0$

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - r r_2} = \frac{(a^2 + a^2)^{3/2}}{a^2 + 2a^2} = \frac{2^{3/2} a^3}{3a^2}$$

$= \dfrac{2\sqrt{2}}{3} a$ — ①

$r = \dfrac{a}{\theta} \qquad r_1 = -\dfrac{a}{\theta^2}$

$r_2 = \dfrac{2a}{\theta^3}$

At $\theta = 1$

$r = a$, $r_1 = -a$, $r_2 = 2a$

$$\rho = \frac{(a^2 + a^2)^{3/2}}{a^2 + 2a^2 - 2a^2}$$

$$= \frac{2\sqrt{2}\, a^3}{a^2}$$

$= 2\sqrt{2}\, a$ — ②

① ÷ ②

$\dfrac{2\sqrt{2}}{3} \times 2\sqrt{2} = \underline{1:3}$

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Imp ⑥ (a) Show that for the ~~pole~~ curve $r = a(1+\cos\theta)$ ρ^2/r is a const.

Imp (b) If ρ_1 & ρ_2 be the radii of curvatures at the extremities of the polar chord of the cardioide S.T $\rho_1^2 + \rho_2^2 = 16a^2/9$.

(a) →

$$r = a(1+\cos\theta)$$

$$\log r = \log a + \log(1+\cos\theta)$$

$$\frac{1}{r} r' = \frac{1}{1+\cos\theta}(-\sin\theta)$$

$$= \frac{-2\sin\theta/2\cos\theta/2}{2\cos^2\theta/2}$$

$$r' = -r\tan\theta/2$$

$$r'' = -\frac{r}{2}\sec^2\theta/2 - r'\tan\theta/2$$

$$= r\tan^2\theta/2 - \frac{r}{2}\sec^2\theta/2$$

$$\rho = \frac{(r^2 + r^2\tan^2\theta/2)^{3/2}}{r^2 + 2r^2\tan\theta/2 - r^2\tan^2\theta/2 + \frac{r}{2}\sec^2\theta/2}$$

$$= \frac{r^3\sec^3\theta/2}{r^2\sec^2\theta/2 + \frac{r}{2}\sec^2\theta/2} = \frac{2}{3} r\sec\theta/2$$

$$r = 2a\cos^2\theta/2$$

$$\sec^2\theta/2 = \frac{2a}{r}$$

$$\rho^2 = \frac{4}{9} r^2\sec^2\theta/2$$

$$= \frac{4}{9} r^2 \cdot \frac{2a}{r}$$

$$= \frac{8}{9} ar$$

$$\frac{\rho^2}{r} = \frac{8a}{9}$$

$$= \text{const}$$

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