Simple puzzles- For beginners, and for the mathematically challenged.

1. Given an n over m chocolate bar, what is the minimum number of times you have to break the bar, in order to get n*m pieces of 1 over 1 (you can only break a bar in a straight line, and only one bar can be broken at a time).

2. Prove that every prime number other than 2, can be expressed as a difference of two squares.

3. The plane is divided into states by straight infinite lines. Show that a tour, that starts and ends in the same state, must involve an even number of border crossings (a border cannot be crossed in the intersection of two or more border lines).

4. A physicist that lives in a two dimensional universe, wants to see if his universe is a plane (topologically) or the face of a torus. Offer him an easy way to distinguish between the two, using only a rope.

5. The plane is divided into areas by n straight infinite lines. Show that the maximal number of areas that can be obtained in that manner is given by $(n^2+n+2)/2$. How does this figure change, when we substitute the plane with a sphere and the lines with circles on that sphere?
   (Hint available on page 8)

6. Show that $2^{(2^n)-1}$ divides $(2^n)!$

7. An Israeli armored brigade consists of two armored battalions and one infantry battalion, while an infantry brigade consists of two infantry battalions and one armored battalion. An armored battalion consists of two armored squadrons and one infantry squadron, and an infantry battalion consists of two infantry squadrons and one that is armored.
   It keeps on this way through squadrons, platoons, squads, until finally, an armored squad consists of two armory soldiers and one infantry soldier, and an infantry squad ... you know already.
   How many soldiers should be retrained, in order to transform an infantry brigade into an armored brigade?

8. How many squares can be drawn on a checkers board, given that these squares should consist of whole black-white squares (the ones that are already painted on the board)?

9. You have 9 brass rings, three of which are gold. can you find the three gold rings using a balance scale three times?
   Thanks to Sean Butler for this puzzle.

10. You have two sandglasses, one which measures 7 minutes and one which measures 11 minutes. How can one use them to measure 15 minutes?

These puzzles were collected by Guy Kindler (http://www.math.tau.ac.il/~puzne/) of Tel-Aviv University.

Easy puzzles- No previous knowledge required.

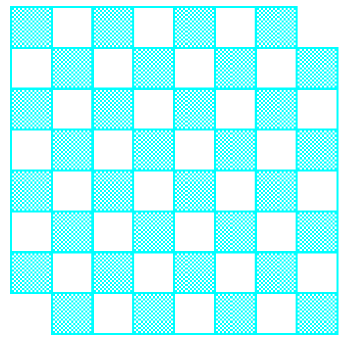
1. The plane is divided into areas by straight infinite lines. Show that these areas can be colored using only two colors, so that any two states that share a border line, have different colors.

2. Given a group of children, show that they can be divided into two subgroups, such that at least half of the friends of any child, are in the opposite subgroup (friendship is considered to be mutual).
   (Hint available on page 8)

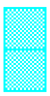
3. You have an empty room, and a group of people waiting outside the room. At each step, you may either get one person into the room, or get one out. Can you make subsequent steps, so that every possible combination of people is achieved exactly once?

4. Is there a sphere in R^3, where exactly one point on it has only rational coordinates?

5. Suppose you have a chess board like the following, that lacks two of its corners:

   And domino pieces, each with size equal to two squares in the chess board:

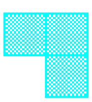
   Can the board be covered with domino pieces, so that each square on the board is covered by exactly one domino piece (pieces should not exceed the border of the board)?

6. Show how to cut a cube into six tetrahedra, all with the same volume. Remember that a tetrahedron is a pyramid with a triangle base, and that its volume is given by hb/3, where h is its height, and b is the area of its base.

7. Can you put six boxes (that are, say, made of cement) in space, so there would be a point in space, from which you will not be able to see any of the vertices of any of the boxes?
   You may assume whatever you want about the boxes' size.

8. Are there positive irrational real numbers a and b for which a^b is rational?
   (Hint available on page 8)

9. Show that a board of 2^n over 2^n squares, that lacks its topmost rightmost corner, can be covered by pieces of the following form:

   (rotations are permitted)
   Each square on the board should be covered by exactly one piece, and pieces should not exceed the border of the board.
   Thanks to Kolby Kappes for this puzzle.

10. Consider the following game of two players:

    There's a square table, a pile of identical coins standing near it, and one coin is placed on the table with its center just above the center of the table.
    Each player in his turn must take a coin from the pile, and put it on the table. In placing the coin, the player should follow these rules:

    1. None of the other coins must be moved.
    2. The coin should not exceed the border of the table.
    3. The coin should not be placed partly or entirely above another coin.

    A player LOSES the game when he has no room to place a coin on his turn.
    Assuming the players are infinitely intelligent, which of the players will win the game (the one who plays the first move or the one that plays second)?

11. Trianglia is an island where no road has a dead end, and all the crossroads are "Y" shaped. The young prince of Trianglia mounts his horse, and is about to go on a quest to explore the land of Trianglia. He gets to the road by his palace, when the mother queen comes out and shouts: "But Charles, how will you find your way back?". "Don't worry Elizabeth", the prince replies, "I will turn right in every second crossroad to which I arrive, and left otherwise. Thus I shall surely return to the palace sooner or later." Is the prince right?
    Thanks to Paul Earwicker for this puzzle.

12. Show how a torus can be intersected with a plane so that the edges of the cut area will form two dissecting circles.
    Thanks to Paul Earwicker for this puzzle. The two last puzzles are from the book "Selected problems and theorems in elementary mathematics" by D.O. Shktyarsky, N.N. Chentsov and I.M. Yaglom Mir (1976)

13. You have two threads. Each of them will burn for exactly one hour if one of its sides is set on fire. How can one measure 45 minutes exactly, using only these threads and a lighter (no regularity can be assumed on the physical structure of the threads, and they are not promised to be identical)?

Mathematical puzzles- First year undergraduate knowledge might be needed.

1. Show a 3D closed convex object, such that the set of its extremal points is not closed.

2. Show that in any non countable subset of the line, there is at least one point in the set, that is an accumulation point of it.

3. Show that in any non-countable subset S of the line, there is a non-countable subset of S that contains only accumulation points of S.

4. Let C denote the continuum cardinality. Prove that the set of functions on the real line, that have the Darbaux property, has cardinality 2^C.

5. Prove that any closed subset of the real line, is the zero set of an infinitely derivable function (maybe you would like to prove it for a continuous function first).

6. Show a one to one function from the real line onto the unit square in the plane.

7. Show that there is no continuous one to one function from the real line onto the unit square in the plane.

8. Show that in R^n, the unit cube can be divided into n! simplexes all with the same volume.
   (Hint available on page 8)

9. The distance between Cairo and Damascus is 1000 miles. Your mission is to move a 10,000 kilograms load of grass from Cairo to Damascus using your camel, but you have two problems:
   1. The camel won't budge unless you let it to continuously chew grass - it consumes 1 kilogram of grass per mile.
   2. The camel's maximum load is 1000 kilograms.

   Can you manage to get ANY of the grass to Damascus?
   What is the maximum amount of grass that you can get there?
   Thanks to Carlos Ibarra for this puzzle.
   (Hint available on page 8)

10. Does 7 appear as the leftmost digit in the decimal representation of any power of 2 (For example, 5 is the leftmost digit of 512)? What percent of the powers of 2 have 1 as their leftmost digit?

11. Prove that any 4 regular graph on 7 vertices is non-planar.
    Thanks to N Arvindkumar for this puzzle.

12. You have a set of 11 integers with the following peculiar property. One can choose any one of them, and divide the remaining ten to two groups of five, such that the sum of first five equals the sum of the second. Show that all the 11 integers are equal.

13. Solve mathematical puzzle #12, where the numbers are not necessarily integers.

Hard puzzles- Heavier use of advanced undergraduate mathematics might be needed.

1. Construct an open set O, that contains all the rational numbers in the segment [0,1], such that m(O) < 0.5, where m is the Lebesgue measure.

2. Show a partition of the real line, into a set of measure zero, and a set of category 1.

3. Let f be an analytic function, with an essential singularity. Show that the set of points in the complex plane, that are the image under f, of infinitely many points, is dense.

4. Let f be an analytic function on the complex plane, that is also one to one. Show that f has the form "ax+b".

5. Show that every polytope in R^n, has at least one extremal point.

6. Show that every polytope in R^n that is not a single point, has at least two extremal points.

7. Consider a Euclidean space with infinite dimension. How many balls with radius 1, can you fit into another ball of radius 2?

8. How about when the other ball's radius is 3?

9. Show that every compact subset of a metric space is of no more than continuum cardinality.

10. Let K be a compact subset of the plane. can K be sliced using two straight lines, into four pieces that have the same Lebesgue measure?

11. Does there exist a measurable space, such that the set of measurable subsets is infinite, yet countable?

12. Show that for every Lebesgue measurable subset A of the line with positive measure, there is a sequence $X_1 \ldots X_{1996}$ that satisfies:
    1. X_i is in A for all i.
    2. For some positive scalar r, $X_i - X_{i-1} = r$ (for all i).

13. Find an (efficient) algorithm, which given a set of vectors in R^n spanning a lattice, finds a minimal (in size) set of vectors that span the same lattice.

14. Let A be a finite set of real numbers. Show that there exists a non-zero integer n, such that each member of n. A has an integer within distance < 0.0001 from it.

These puzzles were collected by Guy Kindler (http://www.math.tau.ac.il/~puzne/) of Tel-Aviv University.

Impossible puzzles- *Enter at your own risk.*

1. Divide the set of positive real numbers into three subsets, such that the equation "a + b = 5c" is not solvable in any of the subsets, i.e. the equation does not hold, for any three numbers a, b, c, that come from the same subset. Can this be done for the equation "a + b = 3c", maybe by using more subsets?
   (Hint available on page 8)

2. Can positive real numbers be divided into a finite number of subsets, such that the equation "a + b = 1.5c" is not solvable in any of the subsets?

3. Show that bits (zeros and ones) can be arranged in a directed circle, so that for every binary string of length n, there exists exactly one subsequence of the circle that is equal to that string.
   (Hint available on page 8)

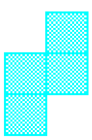
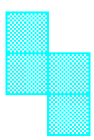
4. There are 120 members in the Israeli parliament, and 40 committees. A committee can be assembled only when the number of committee members that are now present in the parliament hall, is odd (so that votes would be decisive). Show a way to assign the parliament members to committees, so that in all times, except for when the hall is completely empty, there would be at least one committee that can be assembled, or show that it is impossible.
   (Hint available on page 8)

5. Show that every convex closed bounded set in R^n, either has at least n+1 extremal points, or is contained in an affine subspace of R^n, with dimension n-1.

6. Construct a set S of continuum cardinality, containing subsets of the natural numbers, such that the disjunction of any two different members of S would be finite.

7. Let m be a natural number, and let $n := 2^m$. Show an orthogonal basis of R^n, that consists of vectors with coefficients 1 and -1 only.
   (Hint available on page 8)

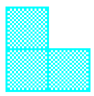
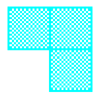
8. Let n be a natural number bigger than 5, and suppose you have an n over n+1 board, and an unlimited supply of pieces of the following form:

    and  (rotation is permitted)

   Can the board be covered with these pieces, so that each square on the board is covered by exactly one piece (pieces should not exceed the border of the board)?How about when you are allowed not to cover any of the corners of the board?
   (Hint available on page 8)

9. a) Show a continuous function from the real line onto the unit square in the plane.
   b) Show a connected compact subset of the plain that is not the range of any continuous function from the real line.

10. Suppose we have a coin that, when tossed, has probability P to obtain tails, and probability 1-P to obtain heads. Suppose we toss it again and again, until an equal number of heads and tails is obtained, and then the process ends. Compute the probability that this process would end eventually.
    Thanks to Yoad Lustig for this puzzle.
    (Hint available on page 8)

11. You have an n by n board, that you want to cover completely, using the following shapes:
     and  (rotation is NOT permitted).
    There may be overlapping between the shapes, but when two shapes overlap, the squares that are overlapped are canceled, and so you have to cover them again. For which n can the board be completely covered?
    Thanks to Tamer Kahveci for this puzzle.

12. Show the existence of a continuous function f from the real line onto the unit square in the plane, such that for every point p on the square, the set {t | f(t)=p} has continuum cardinality.
    (Hint available on page 8)

13. Let F be a finite field and c, d positive integers. Suppose for each x in F^{c+d} we have a real number v(x), such that for each c-dimensional **affine** subspace of F^{c+d}, the sum of v(x) over all the points in the subspace yields zero. Prove that v(x) equals 0 for all x in F^{c+d}.

14. Show two explicit irrational numbers a, b, such that a^b is rational.

Hints-

Simple Puzzles:

5. Use Induction

Easy Puzzles:

2. Assign the children into groups arbitrarily, and then try to gradually improve the assignment.

8. The natural log of 2 is irrational.

Mathematical Puzzles:

8. 1) Try constructing a solution using induction.
   2) Not all the simplexes should be isometric.

9. Even the longest journey starts with one little step.

Impossible Puzzles:

1. MAYBE it would be easier to do this with four subsets first.

3. The answer lies where Hamilton and Euler meet.

4. Use linear algebra.

7. Use recursion.

8. You may first think of the problem when you have this additional type of piece:

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10. 1) Compute recursively the probability that the process would end after 2n tosses.
    2) Use a generating function to find the elaborate equation for the latter probability.

12. Solve riddle number 9 first.