Name - Yogesh Agarwala
EE19B130

Assignment - 2

(1) Solution :

$4n\log n + 2n \longrightarrow O(n\log n)$

$3n + 100\log n \longrightarrow O(n)$

$n^2 + 10n \longrightarrow O(n^2)$

$2^{10} \longrightarrow O(1)$

$4n \longrightarrow O(n)$

$n^3 \longrightarrow O(n^3)$

$2^n \longrightarrow O(2^n)$

$n\log n \longrightarrow O(n\log n)$

$2^{\log n} = n^{\log 2}$ →
- Case 1: base of log is 2 $= O(n)$
- Case 2: base of log > 2 $= O(n^{\log 2})$

Case 1: base of log is 2

Now, $O(1) < O(n) < O(n\log n) < O(n^2) < O(n^3) < O(2^n)$

$\boxed{2^{10}} < \boxed{2^{\log n} = 3n + 100\log n = 4n} < \boxed{n\log n = 4n\log n + 2n} < \boxed{n^2 + 10n} < \boxed{n^3} < \boxed{2^n}$

though
$2^{\log n} < 3n + 100\log n < 4n$
as n approaches ∞
but their asymptotic growth rate is same.

Case 2: base of log > 2

Now, $O(1) < O(n^{\log 2}) < O(n) < O(n\log n) < O(n^2) < O(n^3) < O(2^n)$

$\boxed{2^{10}} < \boxed{2^{\log n}} < \boxed{3n + 100\log n = 4n} < \boxed{n\log n = 4n\log n + 2n} < \boxed{n^2 + 10n} < \boxed{n^3} < \boxed{2^n}$

2

- **Method 1** : Formal defination

$f(n) = O(g(n))$

if there exist constants $c, n_0 > 0$

Such that

$0 \le f(n) \le c\,g(n)$ for all $n \geq n_0$

$f(n) = \Omega(g(n))$

if there exist constants $c, n_0 > 0$

Such that

$0 \le c\,g(n) \le f(n)$ for all $n \geq n_0$

$f = \Theta(g(n))$

if there exist constants c_1, c_2 & $n_0 > 0$

Such that

$0 \le c_1 g(n) \le f(n) \le c_2 g(n)$ for all $n \geq n_0$

- **Method 2** : Limit

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \begin{cases} 0 & , f(n) = O(g(n)) \\ \infty & , f(n) = \Omega(g(n)) \\ \text{otherwise}, \rightarrow & \begin{cases} f = O(g(n)) \\ \quad = \Omega(g(n)) \\ \quad = \Theta(g(n)) \end{cases} \end{cases}$$

(a) $f(n) = n - 100$ $\qquad$ $g(n) = n - 200$

Method 1 :

$f(n) \leq c\, g(n)$

$n - 100 \leq c\,(n - 200)$

Let $c = 2$

$\& \ n_0 = 300$

$\Rightarrow$ for all $n > n_0$

$f(n) \leq c\, g(n)$

$f(n) = O(g(n))$

$f(n) \geq c\, g(n)$

$n - 100 \geq c\,(n - 200)$

Let $c = 1$

$n_0 = 1$

$\Rightarrow$ for all $n > n_0$

$f(n) \geq c\, g(n)$

$f(n) = \Omega(g(n))$

$f(n) = \Theta(g(n))$

So

for $f(n) = n - 100$, $g(n) = n - 200$ $\longrightarrow$

$$\boxed{\begin{aligned} f(n) &= O(g(n)) \\ f(n) &= \Omega(g(n)) \\ f(n) &= \Theta(g(n)) \end{aligned}}$$

method 2 :

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \lim_{n \to \infty} \frac{n - 100}{n - 200} = 1$$

$$\therefore \boxed{\begin{aligned} f(n) &= O(g(n)) \\ f(n) &= \Omega(g(n)) \\ f(n) &= \Theta(g(n)) \end{aligned}}$$

(b) $f(n) = 100n + \log n \qquad g(n) = n + (\log n)^2$

method 1:

$f(n) \le c\,g(n)$

$100n + \log n \le c(n + (\log n)^2)$

Let
$c = 100$
$n_0 = 2$

$\Rightarrow$ for all $n \ge n_0$
$f(n) \le c\,g(n)$

$\downarrow$

$f(n) = O(g(n))$

$f(n) \ge c\,g(n)$

$100n + \log n \ge c(n + \log^2 n)$

Let $c = 1$
$n_0 = 1$

Satisfies
$100n + \log n \ge c(n + \log^2 n)$
for all $n \ge n_0$

$\downarrow$

$f(n) = \Omega(g(n))$

$\downarrow$

$f(n) = \Theta(g(n))$

So for

$$\left.\begin{array}{l} f(n) = 100n + \log n \\ g(n) = n + (\log n)^2 \end{array}\right\} \rightarrow \boxed{\begin{array}{l} f(n) = O(g(n)) \\ f(n) = \Omega(g(n)) \\ f(n) = \Theta(g(n)) \end{array}}$$

method 2:

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{100n + \log n}{n + \log^2 n} = \frac{100 + \log n/n}{1 + \log^2 n/n} =$$

$$= 100 \longrightarrow \boxed{\begin{array}{l} f(n) = O(g(n)) \\ f(n) = \Omega(g(n)) \\ f(n) = \Theta(g(n)) \end{array}}$$

(c) $f(n) = \log(2n)$
$g(n) = \log(3n)$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{\log(2n)}{\log(3n)}$$

$$= \lim_{n\to\infty} \frac{\left(\frac{1}{2n}\right)(2)}{\left(\frac{1}{3n}\right)(3)}$$

$$= 1$$

$$\therefore \boxed{\begin{aligned} f(n) &= O(g(n)) \\ f(n) &= \Omega(g(n)) \\ f(n)) &= \Theta(g(n)) \end{aligned}}$$

Alternate method:

$f(n) = \log(2n) = \log 2 + \log n$
$g(n) = \log(3n) = \log 3 + \log n$

So both $f(n)$ and $g(n)$ have same asymptotic growth rate of $O(\log n)$

$$\therefore \boxed{\begin{aligned} & f(n) = \Theta(g(n)) \\ & \text{which also means} \\ & f(n) = O(g(n)) \\ \& \ & f(n) = \Omega(g(n)) \end{aligned}}$$

(d) $f(n) = n^{1.01}$

$g(n) = n \log^2 n$

$$\lim_{n\to\infty} \frac{n^{1.01}}{n\log^2 n} = \frac{n^{0.01}}{\log^2 n}$$

$$= \frac{0.01 \; n^{-0.99}}{2\log n \left(\frac{1}{n}\right)}$$ (using L hopital rule)

$$= \frac{0.01}{2} \frac{n^{0.01}}{\log n}$$

$$= \frac{0.01^2}{2} \frac{n^{-0.99}}{1/n}$$

$$= \frac{0.01^2}{2} n^{0.01}$$

$$= \infty$$

$$\therefore \lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$$

$$\Rightarrow \boxed{f(n) = \Omega(g(n))}$$

③ Describe an efficient algorithm for finding the ten largest elements in a sequence of size n. What is the running time of your algorithm.

Pseudocode :

Part 1 : Sorting the array using mergesort.

```
def merge (A,B) :

    c = [ ]
    i = 0
    j = 0
    n = len(A) + len(B)
    for k in range (0,n) :
        if A[i] < B[j] :
            c[k] = A[i]
            i++
        else :
            c[k] = B[j]
            j++

    return c

def mergesort (arr) :

    n = len(arr)
    if (n == 1) :
        return arr

    else : mid = n//2
        A = mergesort (arr[:mid])
        B = mergesort (arr [mid:])

        merged array = merge (A, B)
        return mergedarray.
```

Part 2: now from the sorted array, insert the 10 largest element into a new array

```
def tenlargest (sortedarray):

    n = len( sortedarray)
    arr = [ ] // output array

    for i in range (n-1, n-11, -1):

        arr[n-i-1] = sortedarray [i]

    return arr
```

Part 3: main function :

```
inputarray = [ 2, 3, 1, 7, 8, 23, 11, 9, 1, 3, 5, 12]
sorted array = mergesort (input array)
final array = ten largest (sorted array)
print (final array)
```

Output:

[23, 12, 11, 9, 8, 7, 5, 3, 3, 2]

Time complexity:

$$\text{Timecomplexity} = \text{mergesort} + \text{tenlargest}$$

$$= O(n\log n) + O(1)$$

$$= O(n\log n)$$

(4) Use the divide and conquer integer multiplication algorithm to multiply the two binary integers 10011011 and 10111010

Logic / algorithm:

Divide each of the two binary number in two halves

$$x = 2^{n/2} x_L + x_R$$

$$y = 2^{n/2} y_L + y_R$$

So

$$xy = (2^{n/2} x_L + x_R)(2^{n/2} y_L + y_R)$$

$$= 2^n x_L y_L + 2^{n/2}(x_L y_R + x_R y_L) + x_R y_R$$

$$= 2^n y_L x_L + 2^{n/2}\left[(x_L + x_R)(y_L + y_R) - x_L y_L - x_R y_R\right] + x_R y_R$$

So we have three subproblem

$$a = x_L y_L$$

$$b = x_R y_R$$

$$c = (x_L + x_R)(y_L + y_R)$$

$$xy = 2^n a + 2^{n/2}(c - a - b) + b$$

then compute
a, b, c recursively

Example:

in our question: we need to multiply 10011011 & 10111010

$$x = 10011011 = 2^4(1001) + 1011$$
$$y = 10111010 = 2^4(1011) + 1010$$

$$xy = 2^8(x_L y_L) + 2^4\left[(x_L + x_R)(y_L + y_R) - x_L y_L - x_R y_R\right] + x_R y_R$$

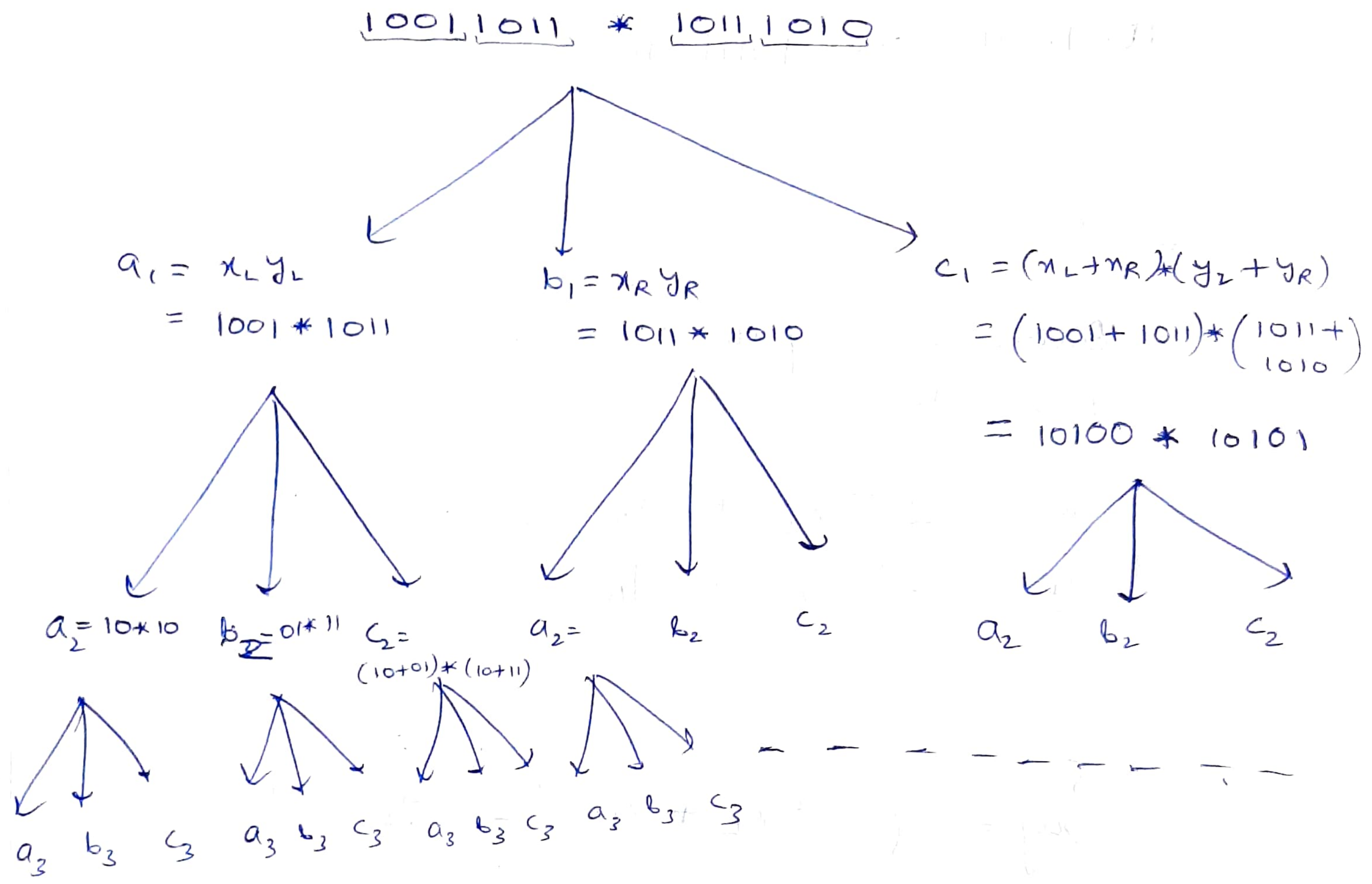
$$a = x_L y_L = 1001 \times 1011$$
$$b = x_R y_R = 1011 \times 1010$$
$$c = (x_L + x_R)(y_L + y_R) = (1001 + 1011) \times (1011 + 1010)$$
$$= 10100 \times 10101$$

Now a, b, c will be computed recursively

$$\Rightarrow \quad a = 1100011$$
$$b = 1101110$$
$$c = 0110100100$$

$$xy = 2^8(a) + 2^4(c - a - b) + b$$
$$\therefore xy = 2^8(1100011) + 2^4(11010011) + 1101110$$
$$= (1110000100111110)_2$$
$$= (28830)_2$$

Logic visualisation tree:



Time complexity:

$$T(n) = 3T(n/2) + O(n)$$

using master theorem we get

$$T(n) = O(n^{\log_2 3})$$

$$= O(n^{1.59})$$

Implementation in Python → P.T.O

Q4. Use the divide and conquer integer multiplication algorithm to multiply the two binary integers 10011011 and 10111010.

In [8]:
```
# function that multiplies two bit strings X and Y and returns
# the product in decimal format

from math import floor, ceil

def karatsuba(x, y):

    """converting int to strings, for easy access to digits"""
    sx = str(x)
    sy = str(y)
    n = max(len(sx), len(sy))

    """base case of recursion"""
    if len(sx) == 1 and len(sy) == 1:
        return x*y


    else:
        """split the digit sequences about the middle"""
        m = ceil(n/2)
        a = int(x // (10**m))
        b = int(x % (10**m))
        c = int(y // (10**m))
        d = int(y % (10**m))


        """recursively calculate the 3 products"""
        ac = karatsuba(int(a), int(c))
        bd = karatsuba(int(b), int(d))
        adbc = karatsuba(int(a)+int(b), int(c)+int(d)) - ac - bd

        """this little trick, writing n as 2*m takes care of both
        even and odd n"""
        return (2**(2*m))*ac + (2**m)*adbc + bd
```

In [11]:
```
# Python program to convert decimal to binary

def decimalToBinary(n):
    return bin(n).replace("0b", "")
```

In [10]:
```
# program to take inputs from the user and then print the result

x = int(input("Enter x: "))
y = int(input("Enter y: "))

product = karatsuba(x,y)
product_in_binary = decimalToBinary(product)

print ("x*y in binary = ",product_in_binary)
print ("x*y in decimal = ",product)
```

```
Enter x: 10011011
Enter y: 10111010
x*y in binary =  111000010011110
x*y in decimal =  28830
```

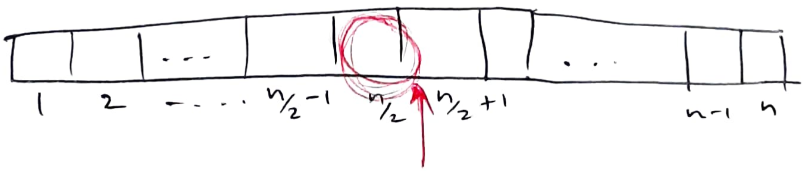
(5) Algorithm:

```
maxelement (arr, i, j):
    n = (i+j)/2
    if arr[n-1] <= arr[n] >= arr[n+1]:
        return n
    elif arr[n-1] > arr[n]:
        return (maxelement (arr, i, n-1))
    elif arr[n] < arr[n+1]
        return (maxelement (arr, n+1, j))

# example unimodal array.
arr = [1, 2, 4, 7, 3, 0]

print (maxelement (arr, 0, len(arr)-1))
```

Explanation:



If $arr[n/2] < arr[n/2 - 1]$ then only look at left half i.e $1 \dots n/2 - 1$ for finding max element.

elif $arr[n/2] < arr[n/2 + 1]$ then only look at right half i.e $n/2 + 1 \dots n$ for finding max element

elif $arr[n/2 - 1] \leq arr[n/2] \geq arr[n/2 + 1]$ then $n/2$ is the max element position

Time complexity of above algorithm:

$$T(n) = T(n/2) + c$$

$$T(n) = \left(T(n/4) + c\right) + c$$

$$T(n) = \left(T(n/8) + c\right) + c + c$$

$$T(n) = T\left(\frac{n}{2^k}\right) + ck$$

where

$k = \log_2 n$

$$\therefore T(n) = T\left(\frac{n}{2^{\log_2 n}}\right) + c\log_2 n$$

$$= T\left(\frac{n}{n}\right) + c\log_2 n$$

$$= T(1) + c\log_2 n$$

$$T(n) \to T(n/2) \to T(n/4) \to \cdots \to T(1)$$

$\log_2 n$ time

$$\therefore T(n) = \Theta(\log n)$$

⑥ Pseudocode:

Part 1: Sorting the array using mergesort

```
def merge (A, B) :
        c = [ ] //outputarray
        i = 0
        j = 0
        n = len (A) + len (B)
        for k in range(0, n) :
                if A[i] < B[j] :
                        c [k] = A [i]
                        i++
                else :
                        c [k] = B [j]
                        j++
        return c

def mergesort (arr):
        n = len (arr)
        if (n==1) :
                return arr
        else :
                mid = n // 2
                A = mergesort (arr[:mid])
                B = mergesort (arr [mid:])

                mergedarray = merge (A, B)
                return mergedarray.
```

Part 2: Removing duplicates from the sorted array.

```
def removeduplicates (sortedarray):
        arr = [] // output array with no duplicate
        i=0
        j=0
        n = len(sortedarray)
        while (i< n) :
                //copying value to output array from sorted array.
                arr [j] = sorted array [i]
                while ( sortedarray[i] = = sortedarray[i+1])
                        i = i+1
                        //when duplicates is found then
                          don't copy to output array.
                i = i+1
                j = j+1

        return arr// this is the final array with
                                          no duplicates
```

Part 3: Main function (taking input & printing result)

```
inputarray = [2,3,1, 3, 6, 2,1,3]
sortedarray = mergesort (inputarray)
final_array = removeduplicates ( sorted array)
```

Concept used:

To remove all duplicates in $O(n \log n)$, we first sort the array using mergesort & then remove duplicates by traversing the sorted array

Time complexity = mergesort + linear traversal of array

$= O(n \log n) + O(n)$

$= O(n \log n)$