Today's Content :

Wednesday → {Recursion}

[Slightly Easier]

Revision: Recursion

- : Addition & Multiplication Rule
- : Permutation Basics
- : Combination Basics & Properties
- : $({}^{N}C_{R}) \% P$ → way 1, → way 2
- : Permutation with Repetition {If Time permits}

Given 3 T/F questions, every question have to answered (T/F)

How many ways can we answer all questions?

8, 3!, 6, $^3C_2$

F	F	F
F	F	T
F	T	F
F	T	T
T	F	F
T	F	T
T	T	F
T	T	T

→ 8 ways:

$\underset{\{T\ F\}}{\underline{2}} \times \underset{\{T\ F\}}{\underline{2}} \times \underset{\{T,\ F\}}{\underline{2}}$ → 8 ways

Ex1: Given 10 girls & 7 Boys, How many different pairs?

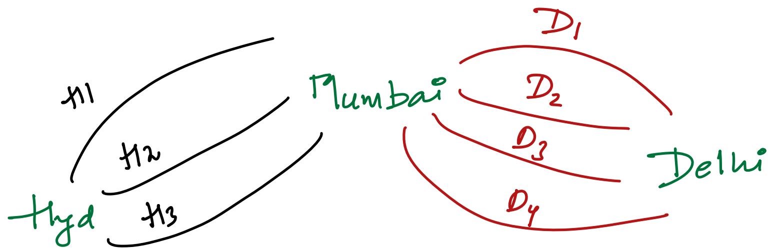
Girl & Boy

Boys:	girls
B_1	G_1
B_2	G_2
B_3	G_3
B_4	⋮
B_5	
B_6	G_{10}
B_7	

// girl & Boy

$\underline{10} \times \underline{7}$ → 70 ways pairs?

Ex3:

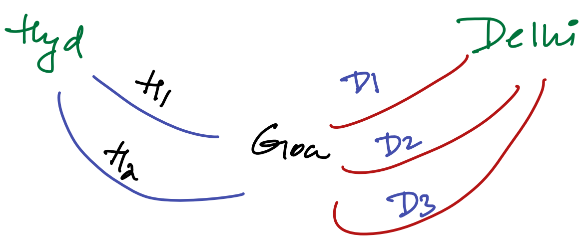


ways to reach Hyd to Delhi via Mumbai?

Hyd → Mumbai && Mumbai → Delhi : 12 ways

4 ways　　　3 ways

Ex4:

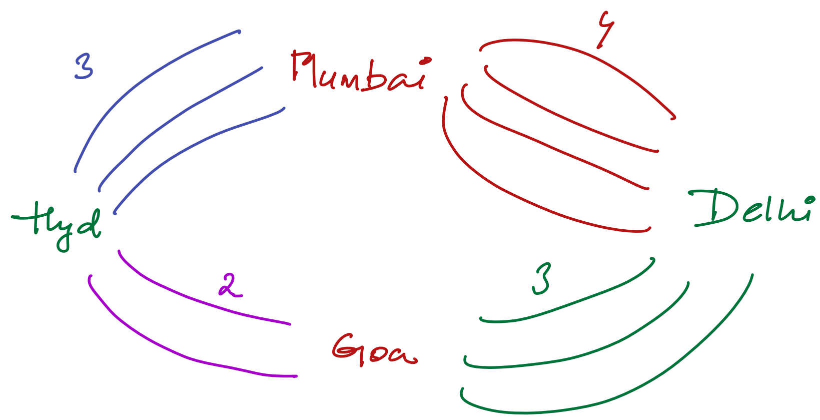


ways to reach Hy to Delhi via Goa?

Hy → Goa && Goa → Delhi

2ways　　　3ways → 6ways

Qns:



ways to reach from Hyd to Delhi ?

Hyd → Delhi via Mumbai **OR** Hyd → Delhi via Goa

12 ↓ + 6 → 18

// Say we need to buy a valentines gift?

(pen & book) or (flowers & Chocolates) or (ring)

↓ ↓ ↓ ↓ ↓

3 5 7 3 3

#ways: 15 + 21 + 3 = 39 ways

AND	OR
*	+

Permutations : arrangements of objects

In general : $(i, j) \neq (j, i)$: order matters

$(i, j) = (j, i)$

Q) Given 3 distinct characters ?

How many ways we can arrange them ?

s = "a c d"

a c d
a d c
c a d
c d a
d a c
d c a

→ 6 arrangements :

$\underline{3} \times \underline{2} \times \underline{1} \Rightarrow 3! = 6$

a < c – d, d – c

c < a – d, d – a

d < a – c, c – a

Q2) How many ways we can arrange 4 distinct char?

$\underline{4} \times \underline{3} \times \underline{2} \times \underline{1} \Rightarrow 4 \times 3 \times 2 \times 1 \Rightarrow 4!$

a $\begin{Bmatrix} b \\ c \\ d \end{Bmatrix}$

b - $\begin{Bmatrix} a \\ c \\ d \end{Bmatrix}$

c - $\begin{Bmatrix} a \\ b \\ d \end{Bmatrix}$

d - $\begin{Bmatrix} a \\ b \\ c \end{Bmatrix}$

Q3) How many ways to arrange n distinct characters?

$\underline{N} \times \underline{N-1} \times \underline{N-2} \times \underline{\ .\ } \ \underline{\ .\ } \ . \ \underline{\ \ } \ \underline{\ .\ } \ 1 = N!$

How many ways to arrange N distinct characters $\Rightarrow$ N!

Q3) : Given 5 distinct characters, In how many can we arrange them in 2 place? $\Rightarrow {}^5P_2 \Rightarrow \frac{5!}{3!} \Rightarrow \frac{120}{6} \Rightarrow 20$

{a b c d e}

$\left.\begin{array}{ll} a & \{b\ c\ d\ e\} \\ b & \{a\ c\ d\ e\} \\ c & \{a\ b\ d\ e\} \\ d & \{a\ b\ c\ e\} \\ e & \{a\ b\ c\ d\} \end{array}\right\}$ # 20 ways

Q) N distinct characters, need to arrange 3 characters?

$\underline{N} * \underline{N-1} * \underline{N-2} \Rightarrow (N)(N-1)(N-2)$

Q) N distinct characters, need to arrange 4 characters?

$\underline{N} * \underline{N-1} * \underline{N-2} * \underline{N-3} = (N)(N-1)(N-2)(N-3)$

Q) N distinct characters, need to arrange r characters?

$\underline{N} \quad \underline{N-1} \quad \underline{N-2} \quad \underline{N-3} \quad \ldots \quad \underline{N-r+1}$

r-1 are already fixed

N-(r-1) are left out

N-r+1

$$\#\text{ways} = \frac{N * (N-1) * (N-2) * \ldots (N-r+1) * (N-r) * (N-r-1) * (N-r-2) \ldots 1}{(N-r) * (N-r-1) * (N-r-2) \ldots \ldots !}$$

$$= \boxed{\frac{N!}{(N-r)!} = {}^{N}P_{r}}$$

$$\Rightarrow {}^{5}P_{2} = \frac{5!}{3!} \Rightarrow \frac{5 * 4 * \cancel{3!}}{\cancel{3!}} \Rightarrow 20 :$$

Combinations: → No: of ways to Select → (i, j) == (j, i)

Q) Given 4 cricketers, count ways of Selecting 3 players?

$P_1 \quad P_2 \quad P_3 \quad P_4$ } → arrangement won't matter

$P_1 \quad P_2 \quad P_3$

$P_1 \quad P_2 \quad P_4$

$P_1 \quad P_3 \quad P_4$

$P_2 \quad P_3 \quad P_4$

Q) No: of ways to arrange the players in 3 slots:

$P_1\ P_2\ P_3$	$P_1\ P_2\ P_4$	$P_1\ P_3\ P_4$	$P_2\ P_3\ P_4$
$P_1\ P_3\ P_2$	$P_1\ P_4\ P_2$	$P_1\ P_4\ P_3$	$P_2\ P_4\ P_3$
$P_2\ P_1\ P_3$	$P_2\ P_1\ P_4$	$P_3\ P_1\ P_4$	$P_3\ P_2\ P_4$
$P_2\ P_3\ P_1$	$P_2\ P_4\ P_1$	$P_3\ P_4\ P_1$	$P_3\ P_4\ P_2$
$P_3\ P_1\ P_2$	$P_4\ P_1\ P_2$	$P_4\ P_1\ P_3$	$P_4\ P_2\ P_3$
$P_3\ P_2\ P_1$	$P_4\ P_2\ P_1$	$P_4\ P_3\ P_1$	$P_4\ P_3\ P_2$

arrang

Sel $\{P_1\ P_2\ P_3\}$ $\{P_1\ P_2\ P_4\}$ $\{P_1\ P_3\ P_4\}$ $\{P_2\ P_3\ P_4\}$

// 6 arrang — 1 Select
// 24 arra — n

$n * 6 = 24$

$n = \frac{24}{6}$ $\quad n = 24/3!$

no: of Sel = 4

$n = 4$

// given N distinct how many ways we can select r time?

// given N distinct, Arrange r times → ^NP_r → $\frac{N!}{(N-r)!}$

// arrange r items → $r!$

arrange		selection
$r!$	×	1
$\frac{N!}{(N-r)!}$		n

$n \times r! = \frac{N!}{(N-r)!}$

$$\boxed{n = \frac{N!}{(N-r)!\,r!}}$$

// given N distinct select R items = $\frac{N!}{(N-r)!\,r!}$ → $\frac{^NP_r}{r!}$

$$^NC_r = \frac{N!}{(N-r)!\,r!}$$

$\{0! = 1\}$

// $^NC_0$ → $\frac{N!}{(N-0)!\,0!}$ → $\frac{N!}{(N!)}$ → 1

How many to select 0 items from N items = 1 }

1 way

Don't pick any item

Subsets:

$\{a \quad b \quad c\}$

$^3C_0 \rightarrow \{\ \}$

$^3C_1 \rightarrow \{a\}\ \{b\}\ \{c\}$

$^3C_2 \rightarrow \{a\ b\}\ \{a\ c\}\ \{b\ c\}$

$^3C_3 \rightarrow \{a\ b\ c\}$

$^3C_0 + {^3C_1} + {^3C_2} + {^3C_3} \Rightarrow 2^3$

$\underline{a} \quad \underline{b} \quad \underline{c}$

$\{2 \times 2 \times 2\} \Rightarrow 8$

$$\boxed{^NC_0 + {^NC_1} + {^NC_2} + \dots {^NC_N} = 2^N}$$

4 distinct:

$^4C_0 = 1$

$+$

$^4C_1 = 4$

$+$

$^4C_2 = 6$

$+$

$^4C_3 = 4$

$+$

$^4C_4 = 1$

$\rightarrow 16 \Rightarrow 2^4$

10:30 break

Properties: $\underbrace{N \text{ distinct Select } R \text{ items}}$

N^{th}

Pick : ${}^{N-1}C_{R-1}$

OR $\quad +$

Don't Pick : ${}^{N-1}C_{R}$

$$\boxed{{}^{N}C_{R} = {}^{N-1}C_{R-1} + {}^{N-1}C_{R}}$$ → property 1

$R! = R * (R-1)!$

$(N-R)! = (N-R) * (N-R-1)!$

$$= \frac{(N-1)!}{(N-R)!\,(R-1)!} + \frac{(N-1)!}{(N-R-1)!\,R!}$$

$$= \frac{(N-1)!}{(N-R) * (N-R-1)! * (R-1)!} + \frac{(N-1)!}{(N-R-1)! * (R * (R-1)!)}$$

$$= \frac{(N-1)!}{(N-R-1)! * (R-1)!} \left\{ \frac{1}{(N-R)} + \frac{1}{R} \right\}$$

$$= \frac{(N-1)!}{(N-R-1)! * (R-1)!} \left\{ \frac{(R) + (N-R)}{(N-R)(R)} \right\}$$

$$= \frac{(N-1)!}{(N-R-1)! * (R-1)!} \left\{ \frac{N}{(N-R)(R)} \right\}$$

$$= \frac{N!}{(N-R)!\, R!}$$

$$= \boxed{{}^{N}C_{R} = {}^{N-1}C_{R-1} + {}^{N-1}C_{R}}$$ property

// 5 Boys → B_1 B_2 B_3 B_4 B_5 $\boxed{{}^5C_2 = {}^5C_3}$

B_1 B_2 : B_3 B_4 B_5

B_1 B_3 : B_2 B_4 B_5

B_1 B_4 : _ _ _

B_1 B_5 : _ _ _

B_2 B_3 : _ _ _

B_2 B_4 : _ _ _

B_2 B_5 : _ _ _

B_3 B_4 : _ _ _

B_3 B_5 : _ _ _

B_4 B_5 : _ _ _

// $\boxed{{}^{N}C_{R} = {}^{N}C_{N-R}}$ → 2nd proper

$${}^5C_2 \Rightarrow \frac{5!}{(5-2)!\,2!} \Rightarrow \frac{5!}{3!\,2!}$$

$${}^5C_3 \Rightarrow \frac{5!}{(5-3)!\,3!} \Rightarrow \frac{5!}{2!\,3!}$$

//

Q18: given N, R, p calculate $({}^{N}C_{R})\%p$

// → p is prime, N & R < p

Constraints
$1 <= (N, R) <= 10^5$
$R < N < p$
p is prime

$${}^{N}C_{R}\%p \rightarrow \left(\frac{N!}{(N-R)! * R!}\right)\%p$$

// Idea: $$a^{-1}\%p \xrightarrow[\text{if p is prime}]{\text{if } gcd(a,p)==1} a^{p-2}\%p$$ } Fermats

$$\boxed{({}^{N}C_{R})\%p \rightarrow (N!)\%p * \{((N-R)!)^{-1} * (R!)^{-1}\}\%p}$$

$$= (N!)\%p * \{\underbrace{((N-R)!)^{-1}\%p}_{(a^{-1})\%p} * \underbrace{((R!)^{-1})\%p}_{(a^{-1})\%p}\}\%p$$

✓ ✓ ✓

TC: $\underbrace{(N!\%p)}_{O(N)} + \underbrace{((N-R)!\%p)^{p-2}}_{O(N-R)+O(\log p)} + \underbrace{((R!)\%p)^{p-2}}_{O(R)+O(\log P)}$

TC: $2N + 2\log P \Rightarrow O(N+\log P)$ SC: O(1)

$\left((N-R)!\right)^{-1} \% p$ → $(a^{-1}) \% p$

$a = (N-R)!$ p is prime →

gcd(a, p) = 1 only than we can apply ✓

// gcd ((N-R)!, p) = 1 ? // $(a^n) \% p \rightarrow ((a \% p)^n) \% p$

// given N < p

// (N-r) < p → p is prime

$(N-r)! = 1 * 2 * 3 * 4 * \ldots (N-r-1) \cdot (N-r) < p$

gcd ((N-r)!, p) = 1 → (Inverse model ✓)

$(a^{-1}) \% p \xrightarrow[\text{p is pri}]{\gcd(a,p)=1} (a^{p-2}) \% p$

// a = (N-R)!

// p is prime

→ $\left((N-R)!\right)^{p-2} \% p$

→ $\left(\underbrace{(N-R)! \% p}_{n}\right)^{p-2} \% p$

→ $(n)^{p-2} \% p$ → use your recursion pow fun to solve

$$\left((R!)^{-1}\right) \% p \quad \left\{ \begin{array}{l} a = R! \quad p \text{ is prime.} \\ \gcd(R!, p) = 1 \end{array} \right. \longrightarrow \left(a^{p-2}\right) \% p$$

$$\left(a^{-1}\right) \% p$$

$$\rightarrow \left(\left(R!\right)^{p-2}\right) \% p = \left(\underbrace{(R!) \% p}_{x \,\to\, \checkmark}\right)^{p-2} \% p$$

$$= \underbrace{\left(x^{p-2}\right) \% p}$$

→ use your recursive pow fun to solve

Q) given N, R, p calculate $({}^{N}C_R)\%p$ p is not prime

Applying inverse modulo is not possible

$${}^{N}C_R = {}^{N-1}C_{R-1} + {}^{N-1}C_R$$

$${}^{N}C_R \% p = ({}^{N-1}C_{R-1} + {}^{N-1}C_R)\% p$$

→ Solve the value?

```
comb(N, R, P)   it will return (NCR) % p
{
    if (N < R) { return 0 }
    if (R == 0) { return 1 }
    a = comb(N-1, R-1, p)
    b = comb(N-1, R, p)
    return (a + b) % p
}
```

// p can be anything

TC: ${}^{N}C_R$ → $O({}^{N}C_R)$

${}^{N}C_R$

$${}^{N}C_R = {}^{N-1}C_R + {}^{N-1}C_{R-1}$$

$${}^{1}C_0 = {}^{0}C_0 + {}^{0}C_{-1} \rightarrow$$

N < R : If no: of items to choose are more than available we cannot choose

// Space min:

N = 4, R = 3 → ${}^4C_3 \to {}^3C_2 \to {}^2C_1 \to {}^1C_0$

4th row → 3rd row → 2nd row

// fill top → bottom

$${}^NC_R = {}^{N-1}C_R + {}^{N-1}C_{R-1}$$

N \ R	0	1	2	3
0	1	0	0	0
1	1	1	0	0
2	1	2	1	0
3	1	3	3	1
4	1	4	6	4

$${}^1C_1 \to {}^0C_1 + {}^0C_0$$

$${}^1C_2 \to {}^0C_2 + {}^0C_1$$

$${}^2C_2 \to {}^1C_2 + {}^1C_1$$

$${}^3C_2 \to {}^2C_2 + {}^2C_1$$

$${}^3C_3 \to {}^2C_3 + {}^2C_2$$

$${}^4C_1 \to {}^3C_1 + {}^3C_0$$

// Can we simply crea [][] & get value?

// In genera N, R

V[N+1][R+1] → TC: $O(N^* R)$

SC: $O(N^* R)$

for (j = 0; j <= R; j++){ V[0][j] = 0 }

for (i = 0; i <= N; i++){ V[i][0] = 1 }

dynamic Programming ?

→ Memoization ?

→ When in DP

```
for(i = 1; i <= N; i = i+1){
    for(j = 1; j <= R; j = j+1){
        V[i][j] = (V[i-1][j] + V[i-1][j-1]) % p
    }
}
```

$\underbrace{V[i][j]}_{{}^{i}C_{j}} = (\underbrace{V[i-1][j]}_{{}^{i-1}C_{j}} + \underbrace{V[i-1][j-1]}_{{}^{i-1}C_{j-1}}) \% p$

// return V[N][R]

$\longrightarrow$ (TLE) $\rightarrow$ (TA) $\rightarrow$

Elem = $\boxed{33}$ * 100 * 21 * 66

Hashset →

prime fns

$$\left\{ \begin{matrix} 3 & 2 \\ 11 & 5 \\ 7 & \end{matrix} \right\}$$

ans = 5