Design and Analysis of Algorithms

Heap Sort

Lecture 9-10

Instructor: Dr. G P Gupta

Heapsort

- Combines the better attributes of merge sort and insertion sort.
  - » Like merge sort, but unlike insertion sort, running time is $O(n \lg n)$.
  - » Like insertion sort, but unlike merge sort, ***sorts in place.***
- Introduces an algorithm design technique
  - » Create data structure (*heap*) to manage information during the execution of an algorithm.
- The *heap* has other applications beside sorting.
  - » Priority Queues

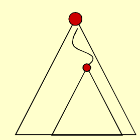
Binary heap

- A binary tree where the value of a parent is greater than or equal to the value of it's children

- Additional restriction: all levels of the tree are **complete** except the last

Max heap vs. Min heap

Heap Property (Max and Min)



- **Max-Heap**
  - » For every node excluding the root,

    $A[parent[i]] \geq A[i]$
- **Largest element is stored at the root.**
- In any subtree, no values are larger than the value stored at subtree root.
- **Min-Heap**
  - » For every node excluding the root,

    $A[parent[i]] \leq A[i]$
- **Smallest element is stored at the root.**
- In any subtree, no values are smaller than the value stored at subtree root

Binary Heap

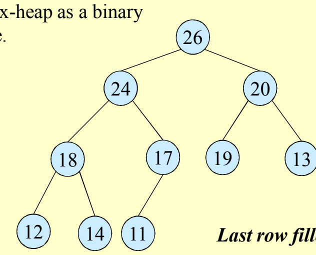
- **Array viewed as a nearly complete binary tree.**
  - » Physically – linear array.
  - » Logically – binary tree, filled on all levels (except lowest.)
- **Map from array elements to tree nodes and vice versa**
  - » Root – $A[1]$
  - » Left[i] – $A[2i]$
  - » Right[i] – $A[2i+1]$
  - » Parent[i] – $A[\lfloor i/2 \rfloor]$
- length[A] – number of elements in array A.
- heap-size[A] – number of elements in heap stored in A.
  - » heap-size[A] $\leq$ length[A]

Heaps – Example

26	24	20	18	17	19	13	12	14	11
1	2	3	4	5	6	7	8	9	10

Max-heap as an array.

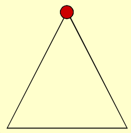
Max-heap as a binary tree.



Last row filled from left to right.

Height

- ***Height of a node in a tree*:** the number of edges on the longest simple downward path from the node to a leaf.
- ***Height of a tree*:** the height of the root.
- Height of a heap: $\lfloor \lg n \rfloor$
  - » Basic operations on a heap run in $O(\lg n)$ time



Binary heap - operations

Maximum(S) - return the largest element in the set

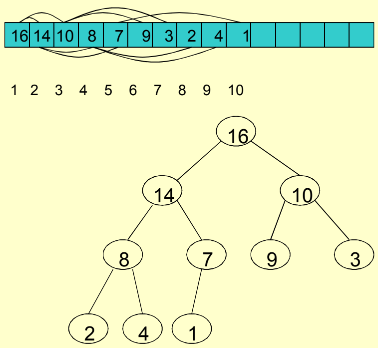
ExtractMax(S) – Return and remove the largest element in the set

Insert(S, val) – insert val into the set

IncreaseElement(S, x, val) – increase the value of element x to val

BuildHeap(A) – build a heap from an array of elements

Binary heap representations



Heapify

Assume left and right children are heaps, turn current set into a valid heap

```
HEAPIFY(A, i)
l ← LEFT(i)
r ← RIGHT(i)
largest ← i
if l ≤ heap-size[A] and A[l] > A[i]
        largest ← l
if r ≤ heap-size[A] and A[r] > A[largest]
        largest ← r
if largest ≠ i
        swap A[i] and A[largest]
        HEAPIFY(A, largest)
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find out which is largest: current, left of right

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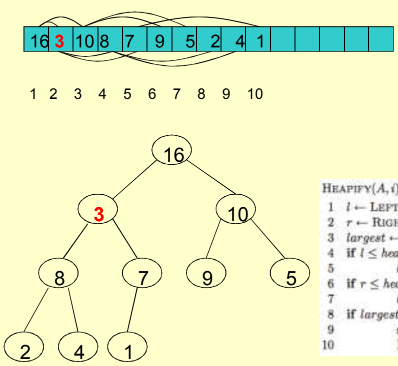
if a child is larger, swap and recurse

Heapify

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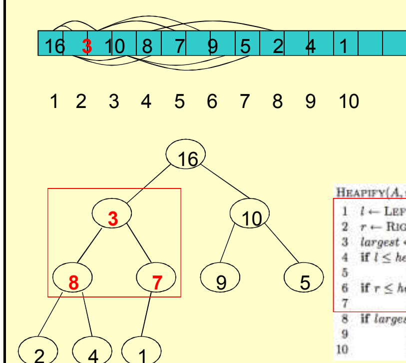


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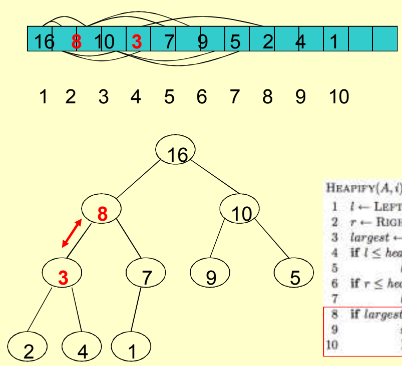


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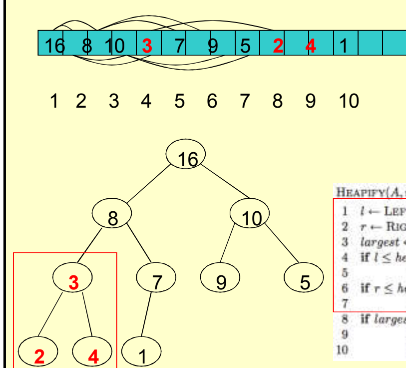


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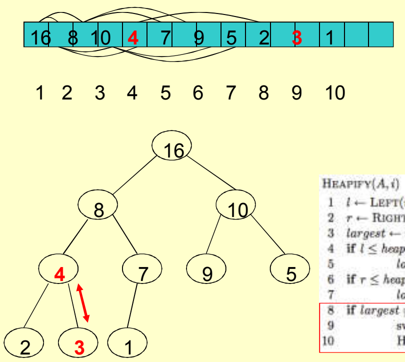


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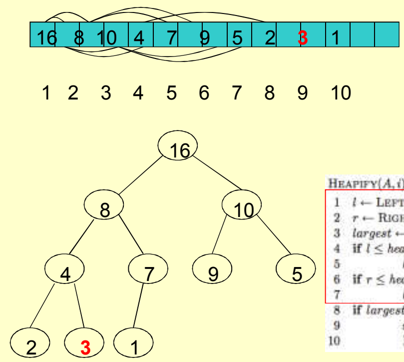


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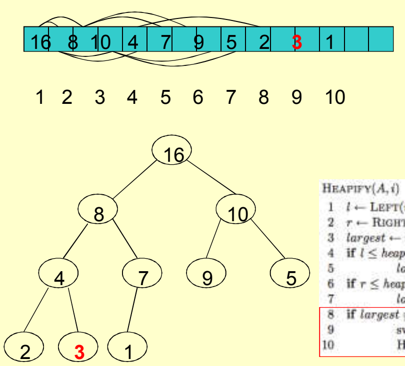
Heapify

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Correctness of Heapify

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Correctness of Heapify

Base case:

- Heap with a single element
- Trivially a heap

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```

Correctness of Heapify

Both children are valid heaps

Three cases:

Case 1: A[i] (current node) is the largest

```
  if largest ≠ i
        swap A[i] and A[largest]
        HEAPIFY(A, largest)
```

- parent is greater than both children
- both children are heaps
- current node is a valid heap

Correctness of Heapify

Case 2: left child is the largest

```
if largest ≠ i
      swap A[i] and A[largest]
      HEAPIFY(A, largest)
```

- When Heapify returns:
  - Left child is a valid heap
  - Right child is unchanged and therefore a valid heap
  - Current node is larger than both children since we selected the largest node of current, left and right
  - current node is a valid heap

Case 3: right child is largest

- similar to above

Running time of Heapify

What is the cost of each individual call to Heapify (not counting recursive calls)?

» $\Theta(1)$

How many calls are made to Heapify?

» O(height of the tree)

What is the height of the tree?

» Complete binary tree, except for the last level

$$2^h \leq n$$

$$h \leq \log_2 n$$

O(log n)

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 l ← LEFT(i)
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 largest ← i
 if l ≤ heap-size[A] and A[l] > A[i]
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Binary heap - operations

Maximum(S) - return the largest element in the set

ExtractMax(S) – Return and remove the largest element in the set

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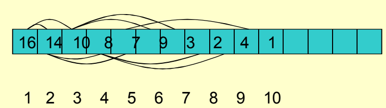
Maximum

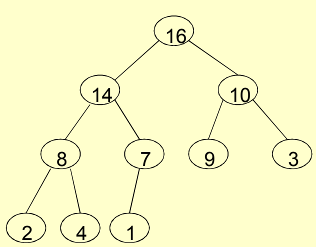
Return the largest element from the set

Return A[1]



ExtractMax

Return and remove the largest element in the set



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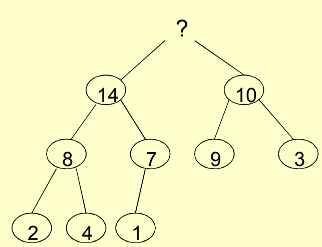

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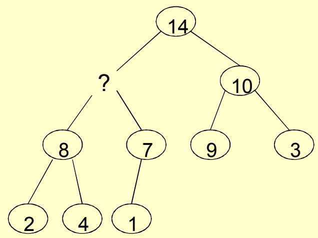

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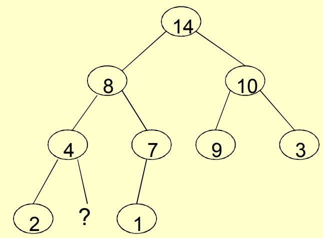

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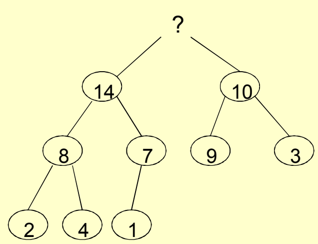

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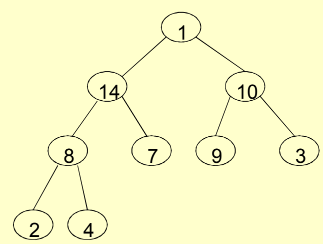

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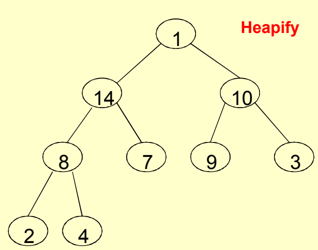

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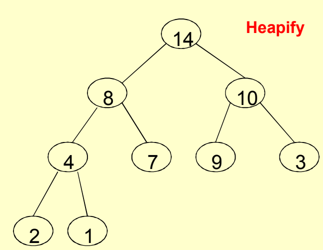

ExtractMax

Return and remove the largest element in the set

```
HEAP-EXTRACT-MAX(A)
if A.heap-size < 1
    error "heap underflow"
max = A[1]
A[1] = A[A.heap-size]
A.heap-size = A.heap-size - 1
MAX-HEAPIFY(A, 1)
return max
```

ExtractMax running time

Constant amount of work plus one call to Heapify – O(log n)

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if A.heap-size < 1
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max = A[1]
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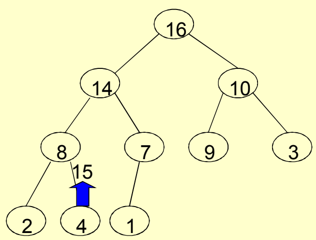
IncreaseElement

Increase the value of element *x* to *val*



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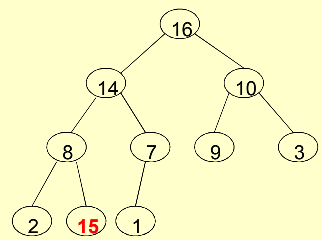

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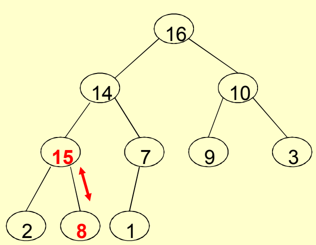

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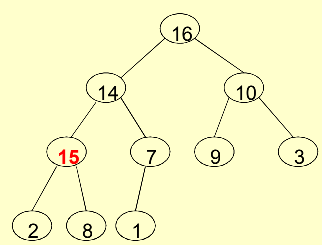

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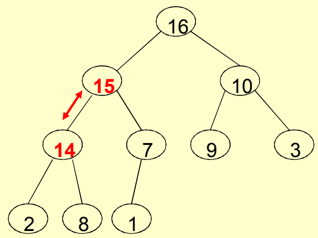

IncreaseElement

Increase the value of element *x* to *val*

```
INCREASE-ELEMENT(A, i, val)
if val < A[i]
        error
A[i] ← val
while i > 1 and A[PARENT(i)] < A[i]
        swap A[i] and A[PARENT(i)]
        i ← PARENT(i)
```

Correctness of IncreaseElement

Why is it ok to swap values with parent?

```
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if val < A[i]
        error
A[i] ← val
while i > 1 and A[PARENT(i)] < A[i]
        swap A[i] and A[PARENT(i)]
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Correctness of IncreaseElement

Stop when heap property is satisfied

```
INCREASE-ELEMENT(A, i, val)
if val < A[i]
        error
A[i] ← val
while i > 1 and A[PARENT(i)] < A[i]
        swap A[i] and A[PARENT(i)]
        i ← PARENT(i)
```

Running time of IncreaseElement

Follows a path from a node to the root

Worst case O(height of the tree)

O(log n)

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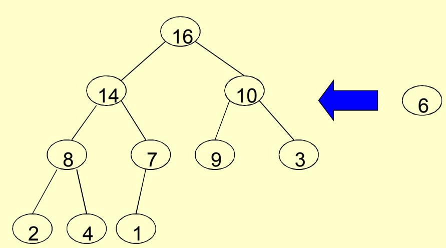
Insert

Insert *val* into the set



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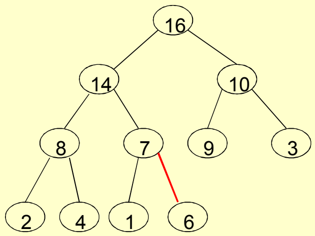

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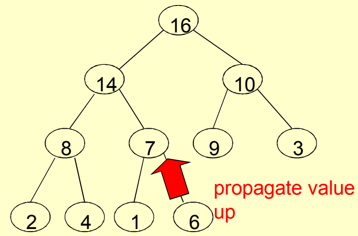

Insert

```
INSERT(A, val)
heap-size[A] ← heap-size[A] + 1
A[heap-size[A]] ← −∞
INCREASE-ELEMENT(A, heap-size[A], val)
```

Running time of Insert

Constant amount of work plus one call to IncreaseElement – O(log n)

```
INSERT(A, val)
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INCREASE-ELEMENT(A, heap-size[A], val)
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Building a heap

Can we build a heap using the functions we have so far?

- Maximum(S)
- ExtractMax(S)
- Insert(S, val)|
- IncreaseElement(S, x, val)

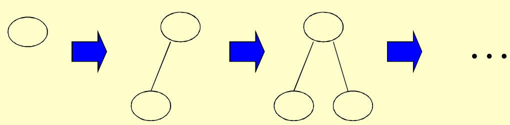
Building a heap

```
BUILD-HEAP1(A)
copy A to B
heap-size[A] ← 0
for i ← 1 to length[B]
        INSERT(A, B[i])
```

Running time of BuildHeap1

n calls to Insert – O(n log n)

Can we do better?



Building a heap: take 2

```
BUILD-HEAP2(A)
1  heap-size[A] ← (length)[A]
2  for i ← ⌊(length)[A]/2⌋ to 1
3        HEAPIFY(A, i)
```

Start with n/2 "simple" heaps

call Heapify on element n/2-1, n/2-2, n/2-3 …

all children have smaller indices

building from the bottom up, makes sure that all the children are heaps

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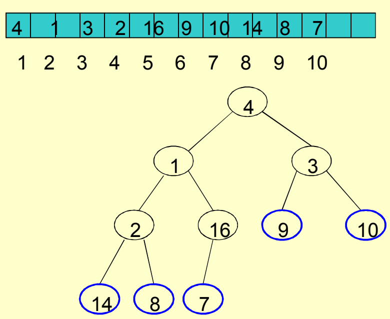


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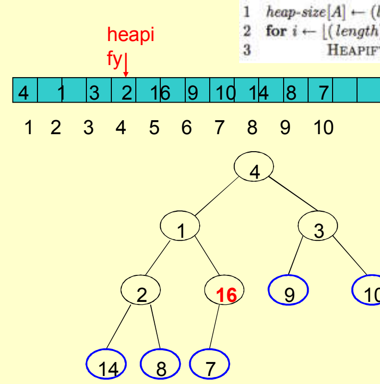


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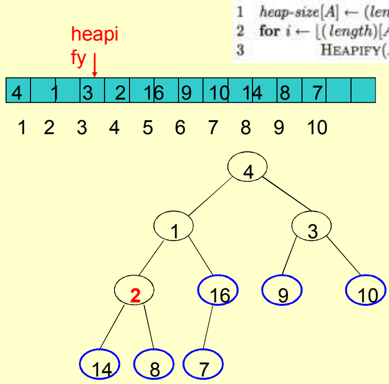
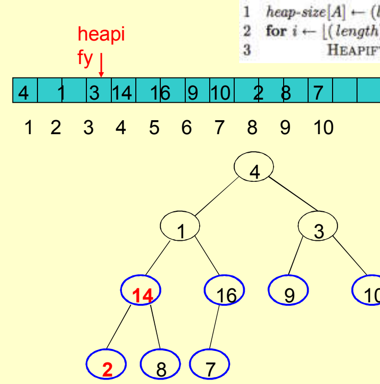
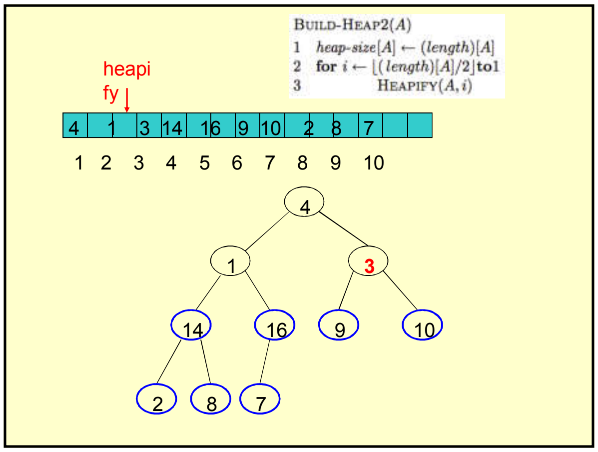
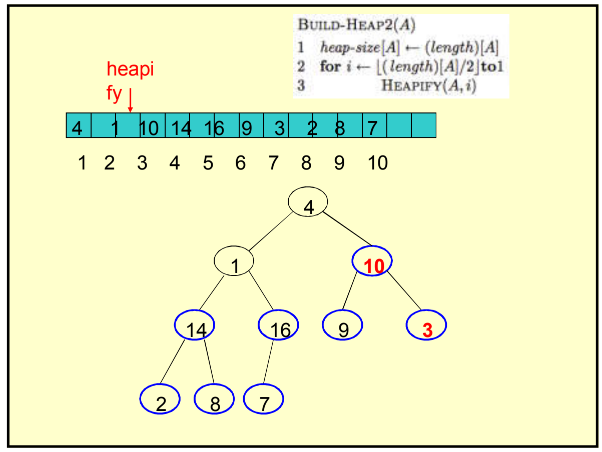



```
BUILD-HEAP2(A)
1  heap-size[A] ← (length)[A]
2  for i ← ⌊(length)[A]/2⌋ to 1
3        HEAPIFY(A, i)
```




BUILD-HEAP2(A)
1 heap-size[A] ← (length)[A]
2 for i ← ⌊(length)[A]/2⌋to1
3 HEAPIFY(A, i)
heapi
fy
4 1 3 14 16 9 10 2 8 7
1 2 3 4 5 6 7 8 9 10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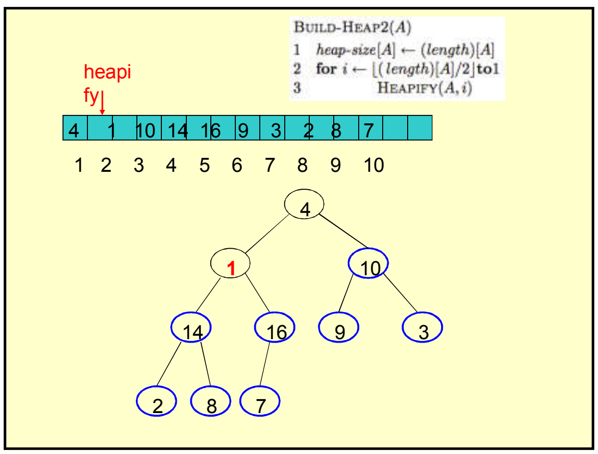
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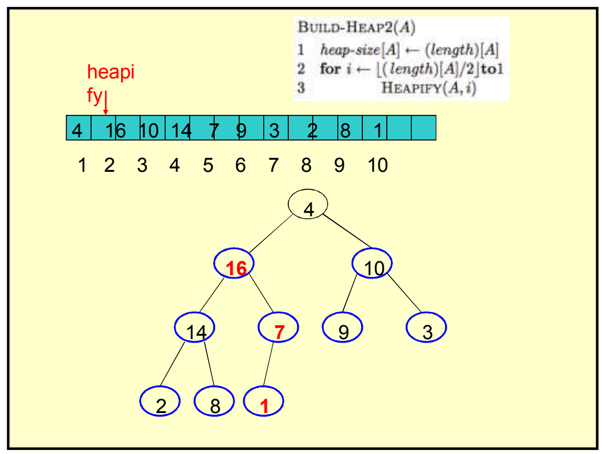
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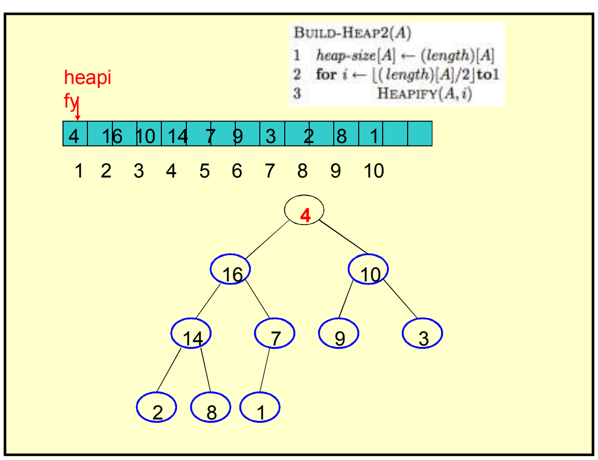
fy
4 16 10 14 7 9 3 2 8 1
1 2 3 4 5 6 7 8 9 10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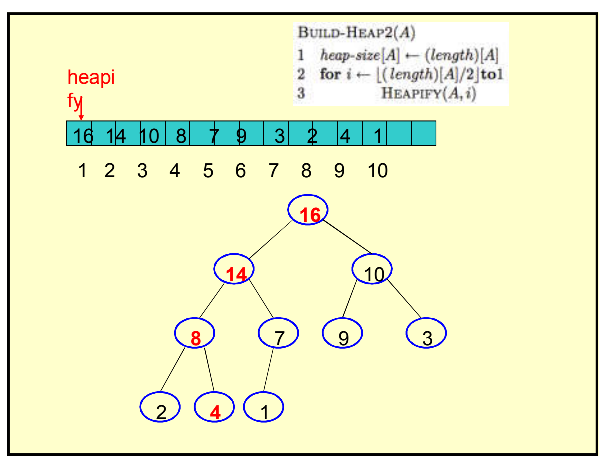
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heapi
fy
16 14 10 8 7 9 3 2 4 1
1 2 3 4 5 6 7 8 9 10

Correctness of BuildHeap2

Invariant:

```
BUILD-HEAP2(A)
heap-size[A] ← (length)[A]
for i ← ⌊(length)[A]/2⌋ to 1
        HEAPIFY(A, i)
```

Correctness of BuildHeap2

```
BUILD-HEAP2(A)
heap-size[A] ← (length)[A]
for i ← ⌊(length)[A]/2⌋ to 1
        HEAPIFY(A, i)
```

Invariant: elements A[i+1…n] are all heaps

Base case: i = floor(n/2). All elements i+1, i+2, …, n are "simple" heaps

Inductive case: We know i+1, i+2, .., n are all heaps, therefore the call to Heapify(A,i) generates a heap at node i

Termination?

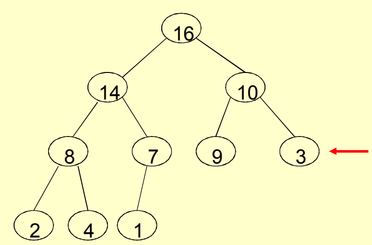
Running time of BuildHeap2

n/2 calls to Heapify – O(n log n)

Can we get a tighter bound?

```
BUILD-HEAP2(A)
heap-size[A] ← (length)[A]
for i ← ⌊(length)[A]/2⌋ to 1
        HEAPIFY(A, i)
```

Running time of BuildHeap2



all nodes at the same level will have the same cost

How many nodes are at level *d*? 2^d

Running time of BuildHeap2

$$T(n) = \sum_{d=0}^{\log n} 2^d O(d)$$

?

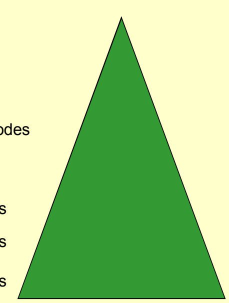
Nodes at height *h*

h < ceil(n/2^{h+1}) nodes

h=2 < ceil(n/8) nodes

h=1 < ceil(n/4) nodes

h=0 < ceil(n/2) nodes



Running time of BuildHeap2

$$T(n) = \sum_{h=0}^{\log n} \left\lceil \frac{n}{2^{h+1}} \right\rceil O(h)$$

$$= O\left(n \sum_{h=0}^{\log n} \left\lceil \frac{1}{2^{h+1}} \right\rceil h \right)$$

$$= O\left(n \sum_{h=0}^{\log n} \frac{h}{2^h} \right)$$

$$= O\left(n \sum_{h=0}^{\infty} \frac{h}{2^h} \right)$$

$$= O(n)$$

$$\boxed{\sum_{h=0}^{\infty} \frac{h}{2^h} = \frac{1/2}{(1-1/2)^2} = 2}$$

BuildHeap1 vs. BuildHeap2

```
BUILD-HEAP1(A)
copy A to B
heap-size[A] ← 0
for i ← 1 to length[B]
        INSERT(A, B[i])
```

```
BUILD-HEAP2(A)
heap-size[A] ← (length)[A]
for i ← ⌊(length)[A]/2⌋ to 1
        HEAPIFY(A, i)
```

Runtime

- O(n) vs. O(n log n)

Memory

- Both O(n)
- BuildHeap1 requires an additional array, i.e. 2n memory

Complexity/Ease of implementation

Heap uses

Could we use a heap to sort?

Heap uses

Heapsort

- Build a heap
- Call ExtractMax for all the elements
- O(n log n) running time

Priority queues

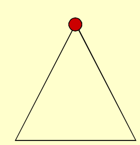
- scheduling tasks: jobs, processes, network traffic
- A* search algorithm

Heaps in Sorting

- **Use max-heaps for sorting.**
- **The array representation of max-heap is not sorted.**
- **Steps in sorting**
  - » Convert the given array of size n to a max-heap (*BuildMaxHeap*)
  - » Swap the first and last elements of the array.
    - Now, the largest element is in the last position – where it belongs.
    - That leaves $n - 1$ elements to be placed in their appropriate locations.
    - However, the array of first $n - 1$ elements is no longer a max-heap.
    - Float the element at the root down one of its subtrees so that the array remains a max-heap (MaxHeapify)
    - Repeat step 2 until the array is sorted.

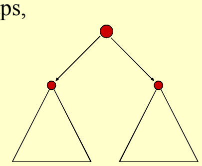
Heap Characteristics

- *Height* $= \lfloor \lg n \rfloor$
- No. of *leaves* $= \lceil n/2 \rceil$
- No. of nodes of height h $\leq \lceil n/2^{h+1} \rceil$



Maintaining the heap property

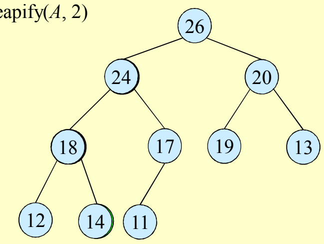
- Suppose two subtrees are max-heaps, but the root violates the max-heap property.



- Fix the offending node by exchanging the value at the node with the larger of the values at its children.
  - » May lead to the subtree at the child not being a heap.
- Recursively fix the children until all of them satisfy the max-heap property.

MaxHeapify – Example

MaxHeapify(A, 2)



Procedure MaxHeapify

```
MaxHeapify(A, i)
l ← left(i)
r ← right(i)
if l ≤ heap-size[A] and A[l] > A[i]
   then largest ← l
   else largest ← i
if r ≤ heap-size[A] and A[r] > A[largest]
   then largest ← r
if largest≠ i
   then exchange A[i] ↔ A[largest]
        MaxHeapify(A, largest)
```

Assumption:
Left(i) and Right(i) are max-heaps.

Running Time for MaxHeapify

```
MaxHeapify(A, i)
l ← left(i)
r ← right(i)
if l ≤ heap-size[A] and A[l] > A[i]
   then largest ← l
   else largest ← i
if r ≤ heap-size[A] and A[r] > A[largest]
   then largest ← r
if largest≠ i
   then exchange A[i] ↔ A[largest]
        MaxHeapify(A, largest)
```

Time to fix node i and its children = $\Theta(1)$

PLUS

Time to fix the subtree rooted at one of i's children = T(size of subree at *largest*)

Running Time for MaxHeapify(A, n)

- $T(n) = T(largest) + \Theta(1)$
- $largest \leq 2n/3$ (worst case occurs when the last row of tree is exactly half full)
- $T(n) \leq T(2n/3) + \Theta(1) \Rightarrow T(n) = O(\lg n)$
- Alternately, MaxHeapify takes $O(h)$ where h is the height of the node where MaxHeapify is applied

Building a heap

- Use *MaxHeapify* to convert an array A into a max-heap.
- How?
- Call MaxHeapify on each element in a bottom-up manner.

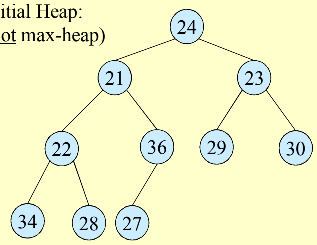
```
BuildMaxHeap(A)
heap-size[A] ← length[A]
for i ← ⌊length[A]/2⌋ downto 1
    do MaxHeapify(A, i)
```

BuildMaxHeap – Example

Input Array:

24	21	23	22	36	29	30	34	28	27

Initial Heap:
(<u>not</u> max-heap)



BuildMaxHeap – Example

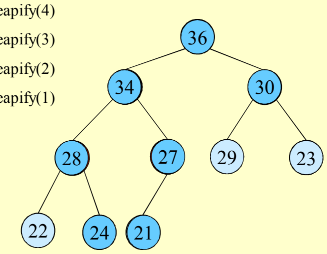
MaxHeapify(⌊10/2⌋ = 5)

MaxHeapify(4)

MaxHeapify(3)

MaxHeapify(2)

MaxHeapify(1)



Correctness of *BuildMaxHeap*

- <u>**Loop Invariant:**</u> At the start of each iteration of the **for** loop, each node $i+1$, $i+2$, …, n is the root of a max-heap.
- <u>**Initialization:**</u>
  - » Before first iteration $i = \lfloor n/2 \rfloor$
  - » Nodes $\lfloor n/2 \rfloor+1$, $\lfloor n/2 \rfloor+2$, …, n are leaves and hence roots of max-heaps.
- <u>**Maintenance:**</u>
  - » By LI, subtrees at children of node i are max heaps.
  - » Hence, MaxHeapify(i) renders node i a max heap root (while preserving the max heap root property of higher-numbered nodes).
  - » Decrementing i reestablishes the loop invariant for the next iteration.

Running Time of *BuildMaxHeap*

- Loose upper bound:
  - » Cost of a *MaxHeapify* call × No. of calls to *MaxHeapify*
  - » $O(\lg n) \times O(n) = O(n \lg n)$
- Tighter bound:
  - » Cost of a call to *MaxHeapify* at a node depends on the height, h, of the node – $O(h)$.
  - » Height of most nodes smaller than n.
  - » Height of nodes h ranges from 0 to $\lfloor \lg n \rfloor$.
  - » No. of nodes of height h is $\lceil n/2^{h+1} \rceil$

Running Time of *BuildMaxHeap*

Tighter Bound for $T(BuildMaxHeap)$

$T(BuildMaxHeap)$

$$\sum_{h=0}^{\lfloor \lg n \rfloor} \left\lceil \frac{n}{2^{h+1}} \right\rceil O(h) = O\left(n \sum_{h=0}^{\lfloor \lg n \rfloor} \frac{h}{2^h} \right)$$

$$\sum_{h=0}^{\lfloor \lg n \rfloor} \frac{h}{2^h} \le \sum_{h=0}^{\infty} \frac{h}{2^h} = \frac{1/2}{(1-1/2)^2} = 2 \qquad , x = 1/2 \text{ in (A.8)}$$

$$O\left(n \sum_{h=0}^{\lfloor \lg n \rfloor} \frac{h}{2^h} \right) = O\left(n \sum_{h=0}^{\infty} \frac{h}{2^h} \right) = O(n)$$

Can build a heap from an unordered array in linear time

Heapsort

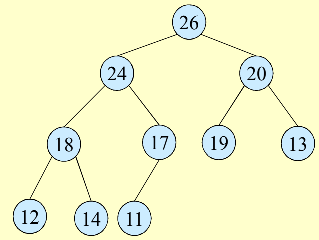
- Sort by maintaining the as yet unsorted elements as a max-heap.
- Start by building a max-heap on all elements in A.
  - » Maximum element is in the root, $A[1]$.
- Move the maximum element to its correct final position.
  - » Exchange $A[1]$ with $A[n]$.
- Discard $A[n]$ – it is now sorted.
  - » Decrement heap-size[A].
- Restore the max-heap property on $A[1..n–1]$.
  - » Call *MaxHeapify*(A, 1).
- Repeat until heap-size[A] is reduced to 2.

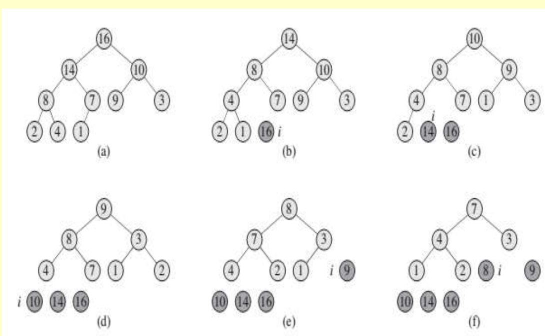
Heapsort(A)

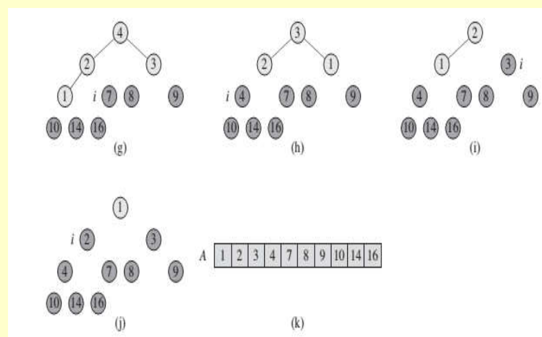
HeapSort(A)
1. Build-Max-Heap(A)
2. **for** $i \leftarrow length[A]$ **downto** 2
3. **do** exchange $A[1] \leftrightarrow A[i]$
4. $heap\text{-}size[A] \leftarrow heap\text{-}size[A] - 1$
5. $MaxHeapify(A, 1)$

Heapsort – Example

26	24	20	18	17	19	13	12	14	11
1	2	3	4	5	6	7	8	9	10







Algorithm Analysis

HeapSort(A)
1. Build-Max-Heap(A)
2. **for** $i \leftarrow length[A]$ **downto** 2
3. **do** exchange $A[1] \leftrightarrow A[i]$
4. $heap\text{-}size[A] \leftarrow heap\text{-}size[A] - 1$
5. $MaxHeapify(A, 1)$

- In-place
- Not Stable
- Build-Max-Heap takes $O(n)$ and each of the n-1 calls to Max-Heapify takes time $O(\lg n)$.
- Therefore, $T(n) = O(n \lg n)$

Heap Procedures for Sorting

- MaxHeapify $O(\lg n)$
- BuildMaxHeap $O(n)$
- HeapSort $O(n \lg n)$

Priority Queue

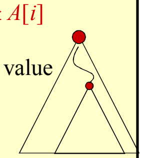
- Popular & important application of heaps.
- Max and min priority queues.
- Maintains a *dynamic* set *S* of elements.
- Each set element has a *key* – an associated value.
- Goal is to support insertion and extraction efficiently.
- Applications:
  - » Ready list of processes in operating systems by their priorities – the list is highly dynamic
  - » In event-driven simulators to maintain the list of events to be simulated in order of their time of occurrence.

Basic Operations

- Operations on a max-priority queue:
  - » Insert(S, x) - inserts the element x into the set S
    - $S \leftarrow S \cup \{x\}$.
  - » Maximum(S) - returns the element of S with the largest key.
  - » Extract-Max(S) - removes and returns the element of S with the largest key.
  - » Increase-Key(S, x, k) – increases the value of element x's key to the new value k.
- Min-priority queue supports Insert, Minimum, Extract-Min, and Decrease-Key.
- Heap gives a good compromise between fast insertion but slow extraction and vice versa.

Heap Property (Max and Min)

- Max-Heap
  - » For every node excluding the root, value is at most that of its parent: $A[parent[i]] \geq A[i]$
- Largest element is stored at the root.
- In any subtree, no values are larger than the value stored at subtree root.



- Min-Heap
  - » For every node excluding the root, value is at least that of its parent: $A[parent[i]] \leq A[i]$
- Smallest element is stored at the root.
- In any subtree, no values are smaller than the value stored at subtree root

Heap-Extract-Max(A)

Implements the Extract-Max operation.

```
Heap-Extract-Max(A)
if heap-size[A] < 1
   then error "heap underflow"
max ← A[1]
A[1] ← A[heap-size[A]]
heap-size[A] ← heap-size[A] - 1
MaxHeapify(A, 1)
return max
```

Running time : Dominated by the running time of MaxHeapify
$= O(\lg n)$

Heap-Insert(A, key)

```
Heap-Insert(A, key)
heap-size[A] ← heap-size[A] + 1
   i ← heap-size[A]
while i > 1 and A[Parent(i)] < key
   do A[i] ← A[Parent(i)]
        i ← Parent(i)
A[i] ← key
```

Running time is $O(\lg n)$

The path traced from the new leaf to the root has length $O(\lg n)$

Heap-Increase-Key(A, i, key)

```
Heap-Increase-Key(A, i, key)
If key < A[i]
   then error "new key is smaller than the current key"
A[i] ← key
while i > 1 and A[Parent[i]] < A[i]
      do exchange A[i] ↔ A[Parent[i]]
           i ← Parent[i]
```

```
Heap-Insert(A, key)
heap-size[A] ← heap-size[A] + 1
A[heap-size[A]] ← –∞
Heap-Increase-Key(A, heap-size[A], key)
```

Examples

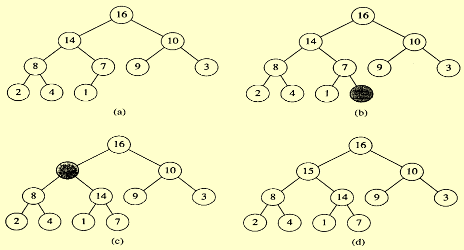


Figure 7.5 The operation of HEAP-INSERT. **(a)** The heap of Figure 7.4(a) before we insert a node with key 15. **(b)** A new leaf is added to the tree. **(c)** Values on the path from the new leaf to the root are copied down until a place for the key 15 is found. **(d)** The key 15 is inserted.