12.13.3.12

12.13.3.12 If X is the number of tails in three tosses of coin, determine the standard deviation of X.
Solution: Let number of tails obtained be defined by random variable

$$X = 0, 1, 2, 3 \tag{1}$$
$$n = \text{Number of trails} \tag{2}$$
$$p = \text{Probability of getting tails in a toss} \tag{3}$$
$$= \frac{1}{2} \tag{4}$$
$$q = \text{Probability of not getting tails} \tag{5}$$
$$= (1 - p) \tag{6}$$
$$= \frac{1}{2} \tag{7}$$
$$\sigma^2_{X^2} = E\left(X - E\left(X\right)\right)^2 \tag{8}$$
$$= E\left(X^2 - 2XE\left(X\right) + E\left(X\right)^2\right) \tag{9}$$
$$= E\left(X^2\right) - 2E\left(X\right).E\left(X\right) + E\left(X\right)^2 \tag{10}$$
$$= E\left(X^2\right) - E\left(X\right)^2 \tag{11}$$
$$E[X^n] = \frac{d^n M\left(z^{-1}\right)}{dz^n} \tag{12}$$
$$M\left(z^{-1}\right) = \sum_{-\infty}^{\infty} P_X\left(k\right) z^k \tag{13}$$
$$E[X] = \frac{d\left(q + pz\right)^n}{dz}|_{z=1} \tag{14}$$
$$= np\left(q + pz\right)^{n-1}|_{z=1} \tag{15}$$
$$= np\left(p + q\right)^{n-1} \tag{16}$$
$$= np \tag{17}$$
$$E[X^2] = \frac{d^2 M\left(z^{-1}\right)}{dz^2}|_{z=1} \tag{18}$$
$$\frac{d\left(M_X\left(z^{-1}\right)\right)}{dz} = \sum_{-\infty}^{\infty} kP_X\left(k\right) z^{k-1} \tag{19}$$
$$\frac{d\left(q + pz\right)^n}{dz} = \sum_{-\infty}^{\infty} kP_X\left(k\right) z^{k-1} \tag{20}$$
$$np\left(q + pz\right)^{n-1} = \sum_{-\infty}^{\infty} kP_X\left(k\right) z^{k-1} \tag{21}$$
$$\tag{22}$$

Multiplying z on both sides, we get

$$znp(q+pz)^{n-1} = \sum_{-\infty}^{\infty} kP_X(k)z^k \tag{23}$$

Differentiating both sides,

$$zn(n-1)p^2(q+pz)^{n-2} + np(q+pz)^{n-1} = \sum_{-\infty}^{\infty} k^2 P_X(k)z^{k-1} \tag{24}$$

$$n^2p^2 - np^2 + np = \sum_{-\infty}^{\infty} k^2 P_X(k) \tag{25}$$

$$E[X^2] = n^2p^2 - np^2 + np \tag{26}$$

$$\sigma_{X^2}^2 = n^2p^2 - np^2 + np - (np)^2 \tag{27}$$

$$\sigma_{X^2}^2 = np - np^2 \tag{28}$$

$$= np(1-p) \tag{29}$$

$$= npq \tag{30}$$

$$\tag{31}$$

Standard deviation

$$= \sigma_{X^2} \tag{32}$$

$$= \sqrt{npq} \tag{33}$$

$$= \sqrt{\frac{3}{4}} \tag{34}$$