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EE22BTECH11059

47.2023 Let (X, Y) have joint probability mass function

$$p(x, y) = \begin{cases} \frac{c}{2^{x+y+2}} & if\, x = 0, 1, 2, \dots y = 0, 1, 2, \dots; x \neq y \\ 0 & otherwise \end{cases} \tag{1}$$

Then which of the following is true?

1) $c = \frac{1}{2}$
2) $c = \frac{1}{4}$
3) $c > 1$
4) X and Y are independent

Solution: For p(x, y) to be joint probability mass function

$$\sum_{y=-\infty}^{\infty} \sum_{x=-\infty}^{\infty} p(x, y) = 1 \, | x \neq y \tag{2}$$

$$\sum_{y=0}^{\infty} \sum_{x=0}^{\infty} \frac{c}{2^{x+y+2}} - \sum_{x=y} \frac{c}{2^{x+y+2}} = 1 \tag{3}$$

$$\sum_{y=0}^{\infty} \frac{c}{2^{y+2}} \sum_{x=0}^{\infty} 2^{-x} - \frac{c}{4} \sum_{x=0}^{\infty} \frac{1}{4^x} = 1 \tag{4}$$

$$\sum_{y=0}^{\infty} \frac{2c}{2^{y+2}} - \frac{c}{3} = 1 \tag{5}$$

$$\frac{2c}{4} \sum_{y=0}^{\infty} 2^{-y} - \frac{c}{3} = 1 \tag{6}$$

$$c - \frac{c}{3} = 1 \tag{7}$$

$$c = \frac{3}{2} \tag{8}$$

1) Marginal probability mass function of X

$$p_X(x) = \sum_{y=0}^{\infty} p(x, y) \tag{9}$$

$$= \sum_{y=0}^{\infty} \frac{3}{2^{x+y+3}} \tag{10}$$

$$= \frac{3}{2^{x+3}} \sum_{y=0}^{\infty} 2^{-y} \tag{11}$$

$$= \frac{3}{2^{x+2}} \tag{12}$$

2) Marginal probability mass function of Y

$$p_Y(y) = \sum_{x=0}^{\infty} p(x, y) \tag{13}$$

$$= \sum_{x=0}^{\infty} \frac{3}{2^{x+y+3}} \tag{14}$$

$$= \frac{3}{2^{y+3}} \sum_{x=0}^{\infty} 2^{-x} \tag{15}$$

$$= \frac{3}{2^{y+2}} \tag{16}$$

For X and Y to be independent,

$$p(x, y) = p_X(x) \, p_Y(y) \tag{17}$$

$$p_X(x) \, p_Y(y) = \frac{3}{2^{x+2}} \frac{3}{2^{y+2}} \tag{18}$$

$$= \frac{9}{2^{x+y+4}} \tag{19}$$

$$p(x, y) \neq p_X(x) \, p_Y(y) \tag{20}$$

Option (4) is incorrect.
$\therefore$ Only option (3) is correct.
Joint cumulative distribution function F(x, y)

$$= P(X \leq x, Y \leq y) \tag{21}$$

$$= \sum_i \left(\sum_j p(i, j) \right) | x \neq y \tag{22}$$

1) x<y

$$F(x,y) = \sum_{i=0}^{x} \sum_{j=1}^{i} p(i,j) \tag{23}$$

$$= \sum_{i=0}^{x} \sum_{j=1}^{i} \frac{3}{2^{i+j+3}} \tag{24}$$

$$= \frac{3}{8} \sum_{i=0}^{x} \frac{1}{2^i} \sum_{j=1}^{i} 2^{-j} \tag{25}$$

$$= \frac{3}{8} \sum_{i=0}^{x} \frac{1}{2^i} \frac{\left(1 - \frac{1}{2^i}\right)}{\frac{1}{2}} \tag{26}$$

$$= \frac{3}{4} \left(\sum_{i=0}^{x} \frac{1}{2^i} - \frac{1}{4^i} \right) \tag{27}$$

$$= \frac{3}{4} \left(\left(\frac{1 - \frac{1}{2^{x+1}}}{\frac{1}{2}} \right) - \left(\frac{1 - \frac{1}{4^{x+1}}}{\frac{3}{4}} \right) \right) \tag{28}$$

$$= \frac{3}{2} - \frac{3}{2^{x+2}} - 1 + \frac{1}{2^{2x+2}} \tag{29}$$

$$= \frac{1}{2} + \frac{1}{2^{2x+2}} - \frac{3}{2^{x+2}} \tag{30}$$

2) x>y

$$F(x,y) = \sum_{j=0}^{y} \sum_{i=1}^{j} p(i,j) \tag{31}$$

$$= \sum_{j=0}^{y} \sum_{i=1}^{j} \frac{3}{2^{i+j+3}} \tag{32}$$

$$= \frac{3}{8} \sum_{j=0}^{y} \frac{1}{2^j} \sum_{i=1}^{j} 2^{-i} \tag{33}$$

$$= \frac{3}{8} \sum_{j=0}^{y} \frac{1}{2^j} \frac{\left(1 - \frac{1}{2^j}\right)}{\frac{1}{2}} \tag{34}$$

$$= \frac{3}{4} \left(\sum_{j=0}^{y} \frac{1}{2^j} - \frac{1}{4^j} \right) \tag{35}$$

$$= \frac{3}{4} \left(\left(\frac{1 - \frac{1}{2^{y+1}}}{\frac{1}{2}} \right) - \left(\frac{1 - \frac{1}{4^{y+1}}}{\frac{3}{4}} \right) \right) \tag{36}$$

$$= \frac{3}{2} - \frac{3}{2^{y+2}} - 1 + \frac{1}{2^{2y+2}} \tag{37}$$

$$= \frac{1}{2} + \frac{1}{2^{2y+2}} - \frac{3}{2^{y+2}} \tag{38}$$

Simulations:

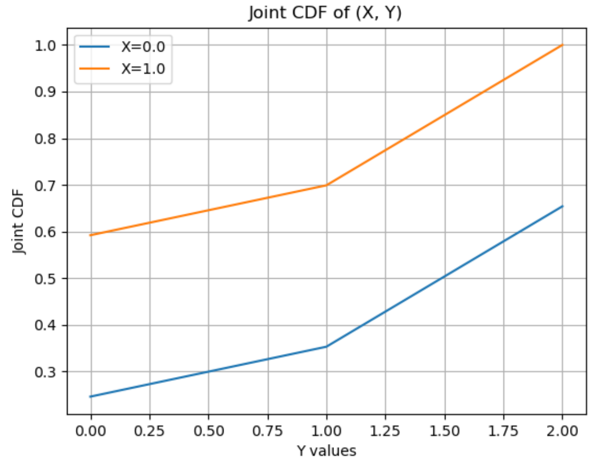


Fig. 2. Joint CDF (for some values of X)

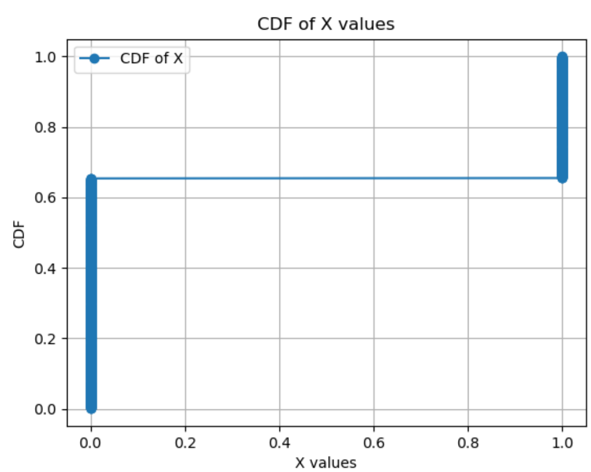
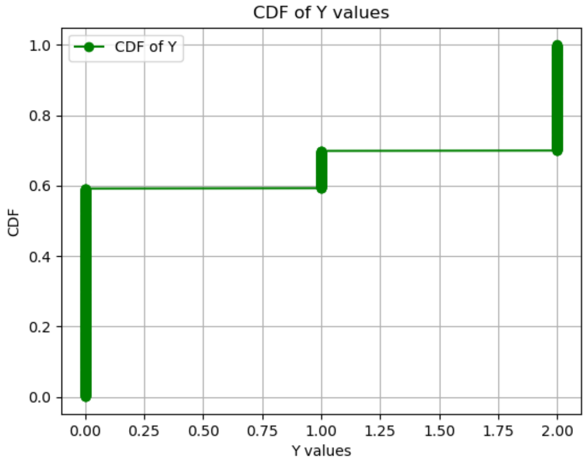


Fig. 2. CDF of X



CDF of Y values
CDF of Y
1.0
0.8
0.6
0.4
0.2
0.0
CDF
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
Y values

Fig. 2. CDF of Y