13.5 AREAS

■ The Area Problem ■ Definition of Area

We have seen that limits are needed to compute the slope of a tangent line or an instantaneous rate of change. Here we will see that they are also needed to find the area of a region with a curved boundary. The problem of finding such areas has consequences far beyond simply finding area. (See the *Focus on Modeling* on page 944.)

■ The Area Problem

One of the central problems in calculus is the *area problem*: Find the area of the region S that lies under the curve $y = f(x)$ from a to b. This means that S, illustrated in Figure 1, is bounded by the graph of a function f (where $f(x) \geq 0$), the vertical lines $x = a$ and $x = b$, and the x-axis.

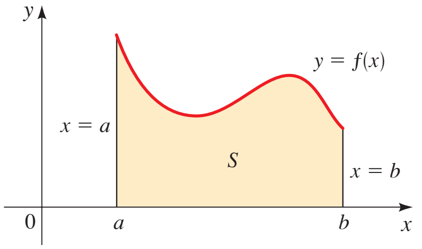


FIGURE 1

In trying to solve the area problem, we have to ask ourselves: What is the meaning of the word *area*? This question is easy to answer for regions with straight sides. For a rectangle, the area is defined as the product of the length and the width. The area of a triangle is half the base times the height. The area of a polygon is found by dividing it into triangles (as in Figure 2) and adding the areas of the triangles.

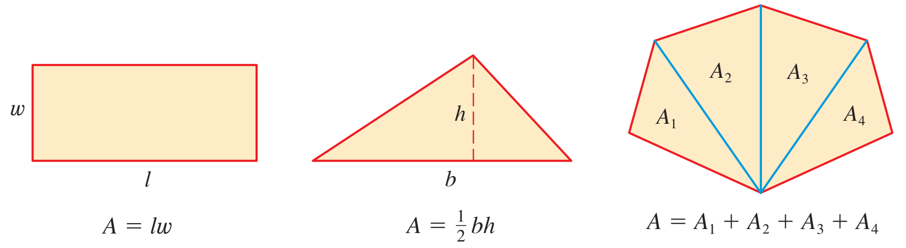


FIGURE 2

However, it is not so easy to find the area of a region with curved sides. We all have an intuitive idea of what the area of a region is. But part of the area problem is to make this intuitive idea precise by giving an exact definition of area.

Recall that in defining a tangent, we first approximated the slope of the tangent line by slopes of secant lines, and then we took the limit of these approximations. We pursue a similar idea for areas. We first approximate the region S by rectangles, and then we take the limit of the areas of these rectangles as we increase the number of rectangles. The following example illustrates the procedure.

EXAMPLE 1 ■ Estimating an Area Using Rectangles

Use rectangles to estimate the area under the parabola $y = x^2$ from 0 to 1 (the parabolic region S illustrated in Figure 3).

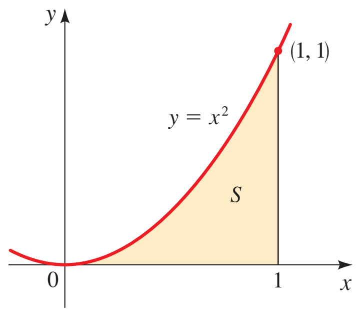


FIGURE 3

SOLUTION We first notice that the area of S must be somewhere between 0 and 1 because S is contained in a square with side length 1, but we can certainly do better than that. Suppose we divide S into four strips S_1, S_2, S_3, and S_4 by drawing the vertical lines $x = \frac{1}{4}$, $x = \frac{1}{2}$, and $x = \frac{3}{4}$ as in Figure 4(a). We can approximate each strip by a rectangle whose base is the same as the strip and whose height is the same as the right edge of the strip (see Figure 4(b)). In other words, the heights of these rectangles are the values of the function $f(x) = x^2$ at the right endpoints of the subintervals $\left[0, \frac{1}{4}\right]$, $\left[\frac{1}{4}, \frac{1}{2}\right]$, $\left[\frac{1}{2}, \frac{3}{4}\right]$, and $\left[\frac{3}{4}, 1\right]$.

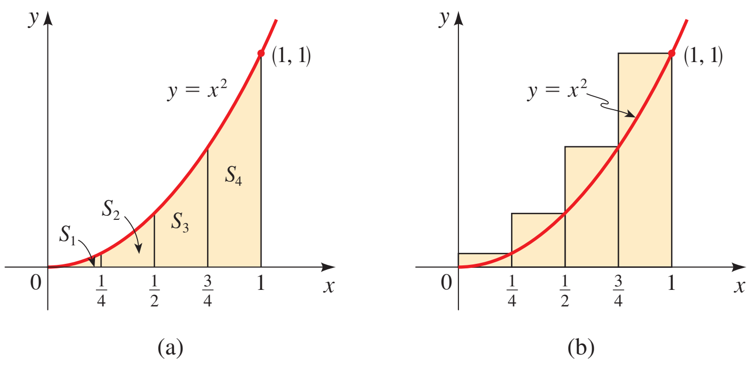


FIGURE 4

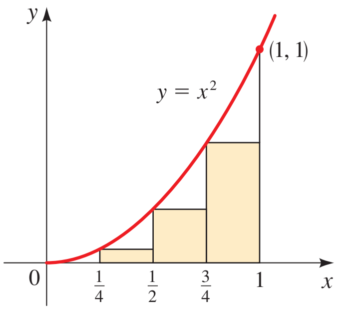


FIGURE 5

Each rectangle has width $\frac{1}{4}$, and the heights are $\left(\frac{1}{4}\right)^2$, $\left(\frac{1}{2}\right)^2$, $\left(\frac{3}{4}\right)^2$, and 1^2. If we let R_4 be the sum of the areas of these approximating rectangles, we get

$$R_4 = \frac{1}{4}\cdot\left(\frac{1}{4}\right)^2 + \frac{1}{4}\cdot\left(\frac{1}{2}\right)^2 + \frac{1}{4}\cdot\left(\frac{3}{4}\right)^2 + \frac{1}{4}\cdot 1^2 = \frac{15}{32} = 0.46875$$

From Figure 4(b) we see that the area A of S is less than R_4, so

$$A < 0.46875$$

Instead of using the rectangles in Figure 4(b), we could use the smaller rectangles in Figure 5 whose heights are the values of f at the left endpoints of the subintervals. (The leftmost rectangle has collapsed because its height is 0.) The sum of the areas of these approximating rectangles is

$$L_4 = \frac{1}{4}\cdot 0^2 + \frac{1}{4}\cdot\left(\frac{1}{4}\right)^2 + \frac{1}{4}\cdot\left(\frac{1}{2}\right)^2 + \frac{1}{4}\cdot\left(\frac{3}{4}\right)^2 = \frac{7}{32} = 0.21875$$

We see that the area of S is larger than L_4, so we have lower and upper estimates for A:

$$0.21875 < A < 0.46875$$

We can repeat this procedure with a larger number of strips. Figure 6 shows what happens when we divide the region S into eight strips of equal width. By computing the sum of the areas of the smaller rectangles (L_8) and the sum of the areas of the larger rectangles (R_8), we obtain better lower and upper estimates for A:

$$0.2734375 < A < 0.3984375$$

So one possible answer to the question is to say that the true area of S lies somewhere between 0.2734375 and 0.3984375.

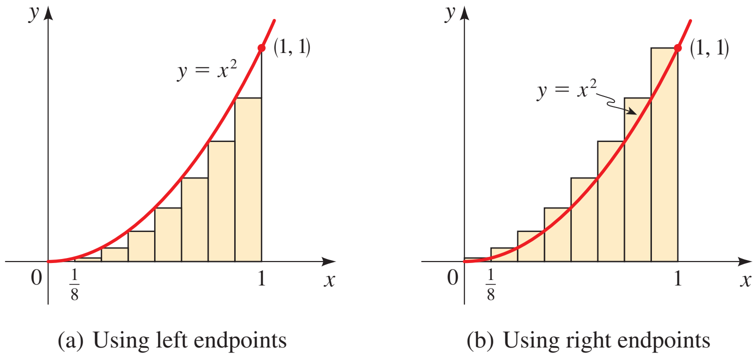


FIGURE 6 Approximating S with eight rectangles

n	L_n	R_n
10	0.2850000	0.3850000
20	0.3087500	0.3587500
30	0.3168519	0.3501852
50	0.3234000	0.3434000
100	0.3283500	0.3383500
1000	0.3328335	0.3338335

We could obtain better estimates by increasing the number of strips. The table in the margin shows the results of similar calculations (with a computer) using n rectangles whose heights are found with left endpoints (L_n) or right endpoints (R_n). In particular, we see by using 50 strips that the area lies between 0.3234 and 0.3434. With 1000 strips we narrow it down even more: A lies between 0.3328335 and 0.3338335. A good estimate is obtained by averaging these numbers: $A \approx 0.3333335$.

Now Try Exercise 3 ■

From the values in the table it looks as if R_n is approaching $\frac{1}{3}$ as n increases. We confirm this in the next example.

EXAMPLE 2 ■ The Limit of Approximating Sums

For the region S in Example 1, show that the sum of the areas of the upper approximating rectangles approaches $\frac{1}{3}$, that is,

$$\lim_{n\to\infty} R_n = \frac{1}{3}$$

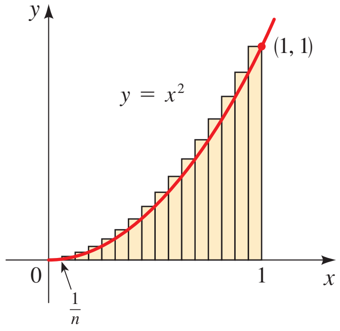


FIGURE 7

This formula was discussed in Section 12.5.

SOLUTION Let R_n be the sum of the areas of the n rectangles shown in Figure 7. Each rectangle has width $1/n$, and the heights are the values of the function $f(x) = x^2$ at the points $1/n, 2/n, 3/n, \ldots, n/n$. That is, the heights are $(1/n)^2, (2/n)^2, (3/n)^2, \ldots, (n/n)^2$. Thus

$$\begin{aligned} R_n &= \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \frac{1}{n}\left(\frac{3}{n}\right)^2 + \cdots + \frac{1}{n}\left(\frac{n}{n}\right)^2 \\ &= \frac{1}{n} \cdot \frac{1}{n^2}(1^2 + 2^2 + 3^2 + \cdots + n^2) \\ &= \frac{1}{n^3}(1^2 + 2^2 + 3^2 + \cdots + n^2) \end{aligned}$$

Here we need the formula for the sum of the squares of the first n positive integers:

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Putting the preceding formula into our expression for R_n, we get

$$R_n = \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6n^2}$$

Thus we have

$$\begin{aligned} \lim_{n\to\infty} R_n &= \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2} \\ &= \lim_{n\to\infty} \frac{1}{6}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) \\ &= \lim_{n\to\infty} \frac{1}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) \\ &= \tfrac{1}{6} \cdot 1 \cdot 2 = \tfrac{1}{3} \end{aligned}$$

Now Try Exercise 13 ■

It can be shown that the lower approximating sums also approach $\frac{1}{3}$, that is,

$$\lim_{n\to\infty} L_n = \tfrac{1}{3}$$

From Figures 8 and 9 it appears that as n increases, both R_n and L_n become better and better approximations to the area of S. Therefore we *define* the area A to be the limit of the sums of the areas of the approximating rectangles, that is,

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n = \tfrac{1}{3}$$

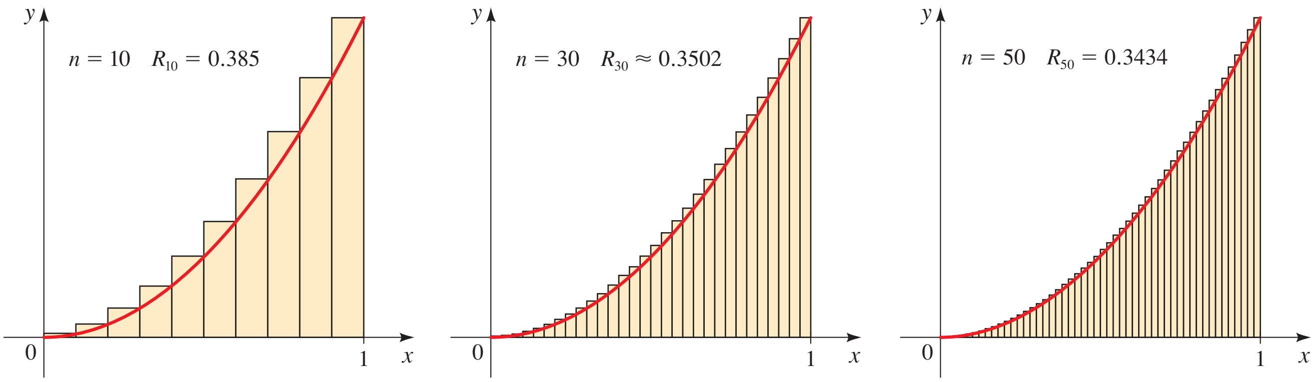


FIGURE 8

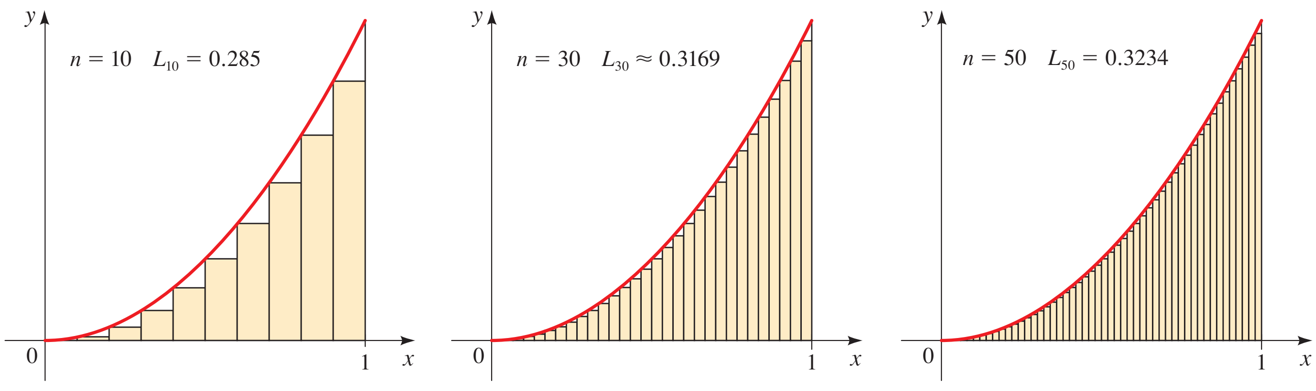


FIGURE 9

Definition of Area

Let's apply the idea of Examples 1 and 2 to the more general region S of Figure 1. We start by subdividing S into n strips $S_1, S_2, \ldots, S_n$ of equal width as in Figure 10.

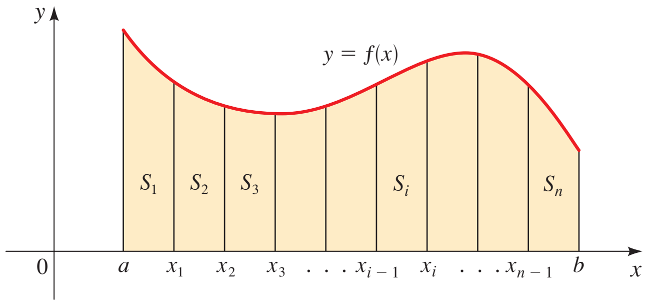


FIGURE 10

The width of the interval $[a, b]$ is $b - a$, so the width of each of the n strips is

$$\Delta x = \frac{b - a}{n}$$

These strips divide the interval $[a, b]$ into n subintervals

$$[x_0, x_1], \quad [x_1, x_2], \quad [x_2, x_3], \quad \ldots, \quad [x_{n-1}, x_n]$$

where $x_0 = a$ and $x_n = b$. The right endpoints of the subintervals are

$$x_1 = a + \Delta x, \quad x_2 = a + 2\,\Delta x, \quad x_3 = a + 3\,\Delta x, \quad \ldots, \quad x_k = a + k\,\Delta x, \quad \ldots$$

Let's approximate the kth strip S_k by a rectangle with width Δx and height $f(x_k)$, which is the value of f at the right endpoint (see Figure 11). Then the area of the kth rectangle is $f(x_k)\,\Delta x$. What we think of intuitively as the area of S is approximated by the sum of the areas of these rectangles, which is

$$R_n = f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x$$

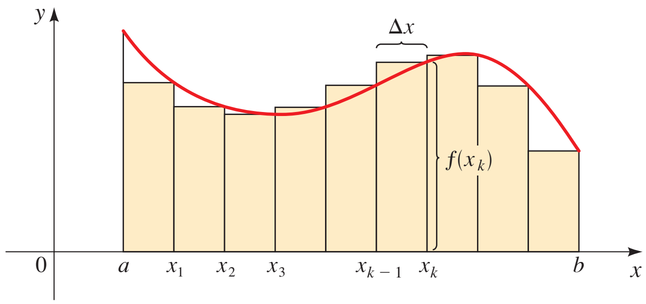


FIGURE 11

Figure 12 shows this approximation for $n = 2, 4, 8$, and 12.

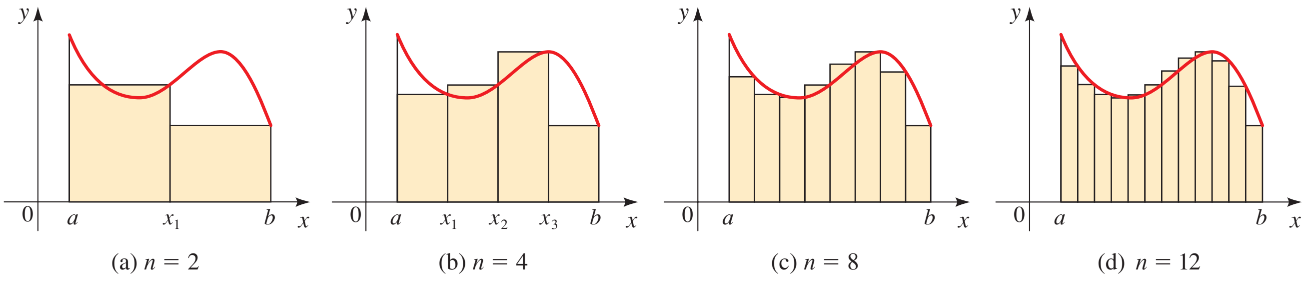


FIGURE 12

Notice that this approximation appears to become better and better as the number of strips increases, that is, as $n \to \infty$. Therefore we define the area A of the region S in the following way.

DEFINITION OF AREA

The **area** A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty}[f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x]$$

Using sigma notation, we write this as follows:

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\,\Delta x$$

In using this formula for area, remember that Δx is the width of an approximating rectangle, x_k is the right endpoint of the kth rectangle, and $f(x_k)$ is its height. So

Width: $\qquad \Delta x = \dfrac{b-a}{n}$

Right endpoint: $\qquad x_k = a + k\,\Delta x$

Height: $\qquad f(x_k) = f(a + k\,\Delta x)$

When working with sums, we will need the following properties from Section 12.1:

$$\sum_{k=1}^{n}(a_k \pm b_k) = \sum_{k=1}^{n} a_k \pm \sum_{k=1}^{n} b_k \qquad \sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$$

We will also need the following formulas for the sums of the powers of the first n natural numbers from Section 12.5.

$$\sum_{k=1}^{n} c = nc \qquad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

EXAMPLE 3 ■ Finding the Area Under a Curve

Find the area of the region that lies under the parabola $y = x^2$, where $0 \le x \le 5$.

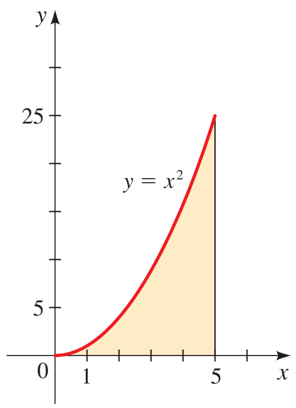


FIGURE 13

SOLUTION The region is graphed in Figure 13. To find the area, we first find the dimensions of the approximating rectangles at the nth stage.

$$\text{Width:} \qquad \Delta x = \frac{b-a}{n} = \frac{5-0}{n} = \frac{5}{n}$$

$$\text{Right endpoint:} \qquad x_k = a + k\,\Delta x = 0 + k\left(\frac{5}{n}\right) = \frac{5k}{n}$$

$$\text{Height:} \qquad f(x_k) = f\left(\frac{5k}{n}\right) = \left(\frac{5k}{n}\right)^2 = \frac{25k^2}{n^2}$$

Now we substitute these values into the definition of area.

$$\begin{aligned}
A &= \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\,\Delta x && \text{Definition of area}\\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{25k^2}{n^2}\cdot\frac{5}{n} && f(x_k) = \frac{25k^2}{n^2},\ \Delta x = \frac{5}{n}\\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{125k^2}{n^3} && \text{Simplify}\\
&= \lim_{n\to\infty} \frac{125}{n^3}\sum_{k=1}^{n} k^2 && \text{Factor } \frac{125}{n^3}\\
&= \lim_{n\to\infty} \frac{125}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} && \text{Sum of Squares Formula}\\
&= \lim_{n\to\infty} \frac{125(2n^2+3n+1)}{6n^2} && \text{Cancel } n\text{, and expand the numerator}\\
&= \lim_{n\to\infty} \frac{125}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) && \text{Divide the numerator and denominator by } n^2\\
&= \frac{125}{6}(2+0+0) = \frac{125}{3} && \text{Let } n \to \infty
\end{aligned}$$

We can also calculate the limit by writing

$$\frac{125}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} = \frac{125}{6}\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)$$

as in Example 2.

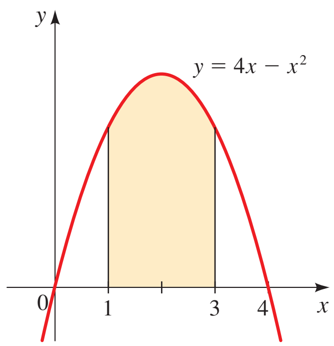
Thus the area of the region is $\frac{125}{3} \approx 41.7$.

Now Try Exercise 15 ■

EXAMPLE 4 ■ Finding the Area Under a Curve

Find the area of the region that lies under the parabola $y = 4x - x^2$, where $1 \le x \le 3$.

The figure below shows the region whose area is computed in Example 4.



SOLUTION We start by finding the dimensions of the approximating rectangles at the nth stage.

$$\text{Width:} \qquad \Delta x = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n}$$

$$\text{Right endpoint:} \qquad x_k = a + k\,\Delta x = 1 + k\left(\frac{2}{n}\right) = 1 + \frac{2k}{n}$$

$$\begin{aligned}
\text{Height:} \qquad f(x_k) &= f\left(1 + \frac{2k}{n}\right) = 4\left(1 + \frac{2k}{n}\right) - \left(1 + \frac{2k}{n}\right)^2\\
&= 4 + \frac{8k}{n} - 1 - \frac{4k}{n} - \frac{4k^2}{n^2}\\
&= 3 + \frac{4k}{n} - \frac{4k^2}{n^2}
\end{aligned}$$

Thus according to the definition of area, we get

$$
\begin{aligned}
A &= \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\,\Delta x = \lim_{n\to\infty} \sum_{k=1}^{n} \left(3 + \frac{4k}{n} - \frac{4k^2}{n^2}\right)\left(\frac{2}{n}\right) \\
&= \lim_{n\to\infty} \left(\sum_{k=1}^{n} 3 + \frac{4}{n}\sum_{k=1}^{n} k - \frac{4}{n^2}\sum_{k=1}^{n} k^2 \right)\left(\frac{2}{n}\right) \\
&= \lim_{n\to\infty} \left(\frac{2}{n}\sum_{k=1}^{n} 3 + \frac{8}{n^2}\sum_{k=1}^{n} k - \frac{8}{n^3}\sum_{k=1}^{n} k^2 \right) \\
&= \lim_{n\to\infty} \left(\frac{2}{n}(3n) + \frac{8}{n^2}\left[\frac{n(n+1)}{2}\right] - \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] \right) \\
&= \lim_{n\to\infty} \left(6 + 4\cdot\frac{n}{n}\cdot\frac{n+1}{n} - \frac{4}{3}\cdot\frac{n}{n}\cdot\frac{n+1}{n}\cdot\frac{2n+1}{n} \right) \\
&= \lim_{n\to\infty} \left[6 + 4\left(1 + \frac{1}{n}\right) - \frac{4}{3}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) \right] \\
&= 6 + 4\cdot 1 - \frac{4}{3}\cdot 1\cdot 2 = \frac{22}{3}
\end{aligned}
$$

Now Try Exercise 17 ■