13.2 FINDING LIMITS ALGEBRAICALLY

■ Limit Laws ■ Applying the Limit Laws ■ Finding Limits Using Algebra and the Limit Laws ■ Using Left- and Right-Hand Limits

In Section 13.1 we used calculators and graphs to guess the values of limits, but we saw that such methods don't always lead to the correct answer. In this section we use algebraic methods to find limits exactly.

■ Limit Laws

We use the following properties of limits, called the *Limit Laws*, to calculate limits.

LIMIT LAWS

Suppose that c is a constant and that the following limits exist:

$$\lim_{x \to a} f(x) \qquad \text{and} \qquad \lim_{x \to a} g(x)$$

Then

1. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$ — Limit of a Sum
2. $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$ — Limit of a Difference
3. $\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$ — Limit of a Constant Multiple
4. $\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$ — Limit of a Product
5. $\displaystyle\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \quad \text{if } \lim_{x \to a} g(x) \neq 0$ — Limit of a Quotient

These five laws can be stated verbally as follows:

Limit of a Sum
1. The limit of a sum is the sum of the limits.

Limit of a Difference
2. The limit of a difference is the difference of the limits.

Limit of a Constant Multiple
3. The limit of a constant times a function is the constant times the limit of the function.

Limit of a Product
4. The limit of a product is the product of the limits.

Limit of a Quotient
5. The limit of a quotient is the quotient of the limits (provided that the limit of the denominator is not 0).

It's easy to believe that these properties are true. For instance, if $f(x)$ is close to L and $g(x)$ is close to M, it is reasonable to conclude that $f(x) + g(x)$ is close to $L + M$. This gives us an intuitive basis for believing that Law 1 is true.

If we use Law 4 (Limit of a Product) repeatedly with $g(x) = f(x)$, we obtain the following Law 6 for the limit of a power. A similar law holds for roots.

LIMIT LAWS

6. $\lim_{x\to a} [f(x)]^n = [\lim_{x\to a} f(x)]^n$ where n is a positive integer — Limit of a Power

7. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ where n is a positive integer — Limit of a Root

[If n is even, we assume that $\lim_{x\to a} f(x) > 0$.]

In words, these laws say the following:

Limit of a Power
6. The limit of a power is the power of the limit.

Limit of a Root
7. The limit of a root is the root of the limit.

EXAMPLE 1 ■ Using the Limit Laws

Use the Limit Laws and the graphs of f and g in Figure 1 to evaluate the following limits if they exist.

(a) $\lim_{x\to -2} [f(x) + 5g(x)]$ **(b)** $\lim_{x\to 1} [f(x)g(x)]$

(c) $\lim_{x\to 2} \dfrac{f(x)}{g(x)}$ **(d)** $\lim_{x\to 1} [f(x)]^3$

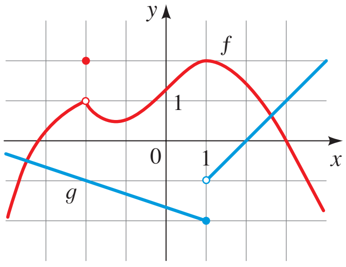


FIGURE 1

SOLUTION

(a) From the graphs of f and g we see that

$$\lim_{x\to -2} f(x) = 1 \qquad \text{and} \qquad \lim_{x\to -2} g(x) = -1$$

Therefore we have

$$\begin{aligned}\lim_{x\to -2}[f(x)+5g(x)] &= \lim_{x\to -2} f(x) + \lim_{x\to -2}[5g(x)] && \text{Limit of a Sum}\\ &= \lim_{x\to -2} f(x) + 5\lim_{x\to -2} g(x) && \text{Limit of a Constant Multiple}\\ &= 1 + 5(-1) = -4\end{aligned}$$

(b) We see that $\lim_{x\to 1} f(x) = 2$. But $\lim_{x\to 1} g(x)$ does not exist because the left- and right-hand limits are different:

$$\lim_{x\to 1^-} g(x) = -2 \qquad \lim_{x\to 1^+} g(x) = -1$$

So we can't use Law 4 (Limit of a Product). The given limit does not exist, since the left-hand limit is not equal to the right-hand limit.

(c) The graphs show that

$$\lim_{x\to 2} f(x) \approx 1.4 \qquad \text{and} \qquad \lim_{x\to 2} g(x) = 0$$

Because the limit of the denominator is 0, we can't use Law 5 (Limit of a Quotient). The given limit does not exist because the denominator approaches 0 while the numerator approaches a nonzero number.

(d) Since $\lim_{x\to 1} f(x) = 2$, we use Law 6 to get

$$\begin{aligned}\lim_{x\to 1}[f(x)]^3 &= [\lim_{x\to 1} f(x)]^3 && \text{Limit of a Power}\\ &= 2^3 = 8\end{aligned}$$

Now Try Exercise 3 ■

■ Applying the Limit Laws

In applying the Limit Laws, we need to use four special limits.

SOME SPECIAL LIMITS

1. $\lim_{x\to a} c = c$
2. $\lim_{x\to a} x = a$
3. $\lim_{x\to a} x^n = a^n$ where n is a positive integer
4. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer and $a > 0$

Special Limits 1 and 2 are intuitively obvious—looking at the graphs of $y = c$ and $y = x$ will convince you of their validity. Limits 3 and 4 are special cases of Limit Laws 6 and 7 (Limits of a Power and of a Root).

EXAMPLE 2 ■ Using the Limit Laws

Evaluate the following limits, and justify each step.

(a) $\lim_{x\to 5}(2x^2 - 3x + 4)$ **(b)** $\lim_{x\to -2}\dfrac{x^3 + 2x^2 - 1}{5 - 3x}$

SOLUTION

(a)

$$\begin{aligned}\lim_{x\to5}(2x^2-3x+4) &= \lim_{x\to5}(2x^2) - \lim_{x\to5}(3x) + \lim_{x\to5}4 && \text{Limits of a Difference and Sum}\\ &= 2\lim_{x\to5}x^2 - 3\lim_{x\to5}x + \lim_{x\to5}4 && \text{Limit of a Constant Multiple}\\ &= 2(5^2) - 3(5) + 4 && \text{Special Limits 3, 2, and 1}\\ &= 39\end{aligned}$$

(b) We start by using Law 5, but its use is fully justified only at the final stage when we see that the limits of the numerator and denominator exist and the limit of the denominator is not 0.

$$\begin{aligned}\lim_{x\to-2}\frac{x^3+2x^2-1}{5-3x} &= \frac{\lim_{x\to-2}(x^3+2x^2-1)}{\lim_{x\to-2}(5-3x)} && \text{Limit of a Quotient}\\ &= \frac{\lim_{x\to-2}x^3 + 2\lim_{x\to-2}x^2 - \lim_{x\to-2}1}{\lim_{x\to-2}5 - 3\lim_{x\to-2}x} && \text{Limits of Sums, Differences, and Constant Multiples}\\ &= \frac{(-2)^3+2(-2)^2-1}{5-3(-2)} && \text{Special Limits 3, 2, and 1}\\ &= -\frac{1}{11}\end{aligned}$$

Now Try Exercises 9 and 11 ■

If we let $f(x) = 2x^2 - 3x + 4$, then $f(5) = 39$. In Example 2(a) we found that $\lim_{x\to5} f(x) = 39$. In other words, we would have gotten the correct answer by substituting 5 for x. Similarly, direct substitution provides the correct answer in part (b). The functions in Example 2 are a polynomial and a rational function, respectively, and similar use of the Limit Laws proves that direct substitution always works for such functions. We state this fact as follows.

LIMITS BY DIRECT SUBSTITUTION

If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x\to a} f(x) = f(a)$$

Functions with this direct substitution property are called **continuous at** a. You will learn more about continuous functions when you study calculus.

EXAMPLE 3 ■ Finding Limits by Direct Substitution

Evaluate the following limits.

(a) $\lim_{x\to3}(2x^3 - 10x - 8)$

(b) $\lim_{x\to-1}\dfrac{x^2+5x}{x^4+2}$

SOLUTION

(a) The function $f(x) = 2x^3 - 10x - 12$ is a polynomial, so we can find the limit by direct substitution:

$$\lim_{x\to 3}(2x^3 - 10x - 12) = 2(3)^3 - 10(3) - 8 = 16$$

(b) The function $f(x) = (x^2 + 5x)/(x^4 + 2)$ is a rational function, and $x = -1$ is in its domain (because the denominator is not zero for $x = -1$). Thus we can find the limit by direct substitution:

$$\lim_{x\to -1}\frac{x^2 + 5x}{x^4 + 2} = \frac{(-1)^2 + 5(-1)}{(-1)^4 + 2} = -\frac{4}{3}$$

Now Try Exercise 13 ■

■ Finding Limits Using Algebra and the Limit Laws

As we saw in Example 3, evaluating limits by direct substitution is easy. But not all limits can be evaluated this way. In fact, most of the situations in which limits are useful require us to work harder to evaluate the limit. The next three examples illustrate how we can use algebra to find limits.

EXAMPLE 4 ■ Finding a Limit by Canceling a Common Factor

Find $\displaystyle\lim_{x\to 1}\frac{x - 1}{x^2 - 1}$.

SOLUTION Let $f(x) = (x - 1)/(x^2 - 1)$. We can't find the limit by substituting $x = 1$ because $f(1)$ isn't defined. Nor can we apply Law 5 (Limit of a Quotient) because the limit of the denominator is 0. Instead, we need to do some preliminary algebra. We factor the denominator as a difference of squares:

$$\frac{x - 1}{x^2 - 1} = \frac{x - 1}{(x - 1)(x + 1)}$$

The numerator and denominator have a common factor of $x - 1$. When we take the limit as x approaches 1, we have $x \neq 1$, and so $x - 1 \neq 0$. Therefore we can cancel the common factor and compute the limit as follows.

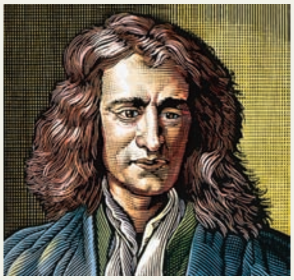
$$\begin{aligned} \lim_{x\to 1}\frac{x - 1}{x^2 - 1} &= \lim_{x\to 1}\frac{x - 1}{(x - 1)(x + 1)} && \text{Factor} \\ &= \lim_{x\to 1}\frac{1}{x + 1} && \text{Cancel} \\ &= \frac{1}{1 + 1} = \frac{1}{2} && \text{Let } x \to 1 \end{aligned}$$

This calculation confirms algebraically the answer we got numerically and graphically in Example 1 in Section 13.1.

Now Try Exercise 19 ■

EXAMPLE 5 ■ Finding a Limit by Simplifying

Evaluate $\displaystyle\lim_{h\to 0}\frac{(3 + h)^2 - 9}{h}$.



B. Sanerson/Science Source

SIR ISAAC NEWTON (1642–1727) is universally regarded as one of the giants of physics and mathematics. He is well known for discovering the laws of motion and gravity and for inventing calculus, but he also proved the Binomial Theorem and the laws of optics, and he developed methods for solving polynomial equations to any desired accuracy. He was born on Christmas Day, a few months after the death of his father. After an unhappy childhood, he entered Cambridge University, where he learned mathematics by studying the writings of Euclid and Descartes.

During the plague years of 1665 and 1666, when the university was closed, Newton thought and wrote about ideas that, once published, instantly revolutionized the sciences. Imbued with a pathological fear of criticism, he published these writings only after many years of encouragement from Edmund Halley (who discovered the now-famous comet) and other colleagues.

Newton's works brought him enormous fame and prestige. Even poets were moved to praise; Alexander Pope wrote:

> Nature and Nature's Laws
> lay hid in Night.
> God said, "Let Newton be"
> and all was Light.

Newton was far more modest about his accomplishments. He said, "I seem to have been only like a boy playing on the seashore . . . while the great ocean of truth lay all undiscovered before me." Newton was knighted by Queen Anne in 1705 and was buried with great honor in Westminster Abbey.

SOLUTION We can't use direct substitution to evaluate this limit, because the limit of the denominator is 0. So we first simplify the limit algebraically.

$$\begin{aligned}
\lim_{h\to 0}\frac{(3+h)^2-9}{h} &= \lim_{h\to 0}\frac{(9+6h+h^2)-9}{h} && \text{Expand}\\
&= \lim_{h\to 0}\frac{6h+h^2}{h} && \text{Simplify}\\
&= \lim_{h\to 0}(6+h) && \text{Cancel } h\\
&= 6 && \text{Let } h\to 0
\end{aligned}$$

Now Try Exercise 25 ■

EXAMPLE 6 ■ Finding a Limit by Rationalizing

Find $\displaystyle\lim_{t\to 0}\frac{\sqrt{t^2+9}-3}{t^2}$.

SOLUTION We can't apply Law 5 (Limit of a Quotient) immediately, since the limit of the denominator is 0. Here the preliminary algebra consists of rationalizing the numerator:

$$\begin{aligned}
\lim_{t\to 0}\frac{\sqrt{t^2+9}-3}{t^2} &= \lim_{t\to 0}\frac{\sqrt{t^2+9}-3}{t^2}\cdot\frac{\sqrt{t^2+9}+3}{\sqrt{t^2+9}+3} && \text{Rationalize numerator}\\
&= \lim_{t\to 0}\frac{(t^2+9)-9}{t^2\left(\sqrt{t^2+9}+3\right)} = \lim_{t\to 0}\frac{t^2}{t^2\left(\sqrt{t^2+9}+3\right)}\\
&= \lim_{t\to 0}\frac{1}{\sqrt{t^2+9}+3} = \frac{1}{\sqrt{\lim_{t\to 0}(t^2+9)}+3} = \frac{1}{3+3} = \frac{1}{6}
\end{aligned}$$

This calculation confirms the guess that we made in Example 2 in Section 13.1.

Now Try Exercise 27 ■

■ Using Left- and Right-Hand Limits

Some limits are best calculated by first finding the left- and right-hand limits. The following theorem is a reminder of what we discovered in Section 13.1. It says that *a two-sided limit exists if and only if both of the one-sided limits exist and are equal.*

$$\lim_{x\to a} f(x) = L \quad\text{if and only if}\quad \lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$$

When computing one-sided limits, we use the fact that the Limit Laws also hold for one-sided limits.

EXAMPLE 7 ■ Comparing Right and Left Limits

Show that $\displaystyle\lim_{x\to 0}|x| = 0$.

SOLUTION Recall that

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

The result of Example 7 looks plausible from Figure 2.

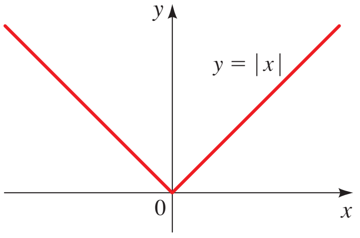


FIGURE 2

Since $|x| = x$ for $x > 0$, we have

$$\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0$$

For $x < 0$ we have $|x| = -x$, so

$$\lim_{x \to 0^-} |x| = \lim_{x \to 0^-} (-x) = 0$$

Therefore

$$\lim_{x \to 0} |x| = 0$$

Now Try Exercise 37 ■

EXAMPLE 8 ■ Comparing Right and Left Limits

Prove that $\displaystyle\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

SOLUTION Since $|x| = x$ for $x > 0$ and $|x| = -x$ for $x < 0$, we have

$$\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = \lim_{x \to 0^+} 1 = 1$$

$$\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = \lim_{x \to 0^-} (-1) = -1$$

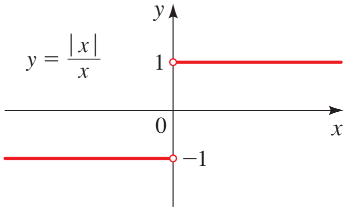


FIGURE 3

Since the right-hand and left-hand limits exist and are different, it follows that $\lim_{x \to 0} |x|/x$ does not exist. The graph of the function $f(x) = |x|/x$ is shown in Figure 3 and supports the limits that we found.

Now Try Exercise 39 ■

EXAMPLE 9 ■ The Limit of a Piecewise Defined Function

Let

$$f(x) = \begin{cases} \sqrt{x - 4} & \text{if } x > 4 \\ 8 - 2x & \text{if } x < 4 \end{cases}$$

Determine whether $\displaystyle\lim_{x \to 4} f(x)$ exists.

SOLUTION Since $f(x) = \sqrt{x - 4}$ for $x > 4$, we have

$$\lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} \sqrt{x - 4} = \sqrt{4 - 4} = 0$$

Since $f(x) = 8 - 2x$ for $x < 4$, we have

$$\lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} (8 - 2x) = 8 - 2 \cdot 4 = 0$$

The right- and left-hand limits are equal. Thus the limit exists, and

$$\lim_{x \to 4} f(x) = 0$$

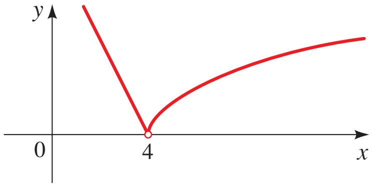


FIGURE 4

The graph of f is shown in Figure 4.

Now Try Exercise 43 ■