CS 170

Lecture 1. Introduction Dec. 12. 2020.

Arithmetic algorithm

1. Addition ($X+Y$, $x, y \le 10^n$)

(1) add 1 each time, y times totally.

Add 1 run time: $\le n$, eg. $999+1=1000$

Add y times: $\le 10^n$

Total $T(n) \le 10^n \cdot n$

(2)

$$\begin{array}{r} a\,b\,c\,d \\ +\ e\,f\,d\,h \\ \hline \end{array}$$

$T(n) = n$, linear time

$\boxed{\text{Question 1}}$ Is there an alg. better than n?

No. Each digits in x and y matters.

$\boxed{\text{Question 2}}$ What if use based 1000 or n?

Still the same. Makes the hard-coded more complex.

2. Multiplication ($X \cdot Y$, $x, y \le 10^n$)

(1) add x, y times

add x: n

$T(n) \le n \cdot y \le n \cdot 10^n$

(2)

$$\begin{array}{r} a\,b\,c\,d \\ \times\ e\,f\,g\,h \\ \hline \end{array}$$

$T(n) = n^2$, square time

(3) Divide & Conquer

X_h X_l $X = X_h \cdot 10^{n/2} + X_l$

$Y_h \quad Y_l \qquad Y = Y_h \cdot 10^{n/2} + Y_l$

Result $= \underline{X_h Y_h} \, 10^n + (\underline{X_h Y_l} + \underline{X_l Y_h}) \, 10^{n/2} + \underline{X_l Y_l}$

$T(n) \leq 4 \cdot T(\frac{n}{2}) + \underbrace{cn}_{\text{addition}}$

$$T(n) \leq \begin{cases} 4 \cdot T(\frac{n}{2}) + cn & n > 1 \\ 1 & n = 1 \end{cases}$$

$T(n) = 4 \cdot c\frac{n}{2} + c \cdot n + T(\frac{n}{4}) \cdot 4^2$

$= \ldots\ldots$

just look at $\underline{c \cdot n}$:

$c \cdot n$

$c\frac{n}{2} \quad c\frac{n}{2} \quad c\frac{n}{2} \quad c\frac{n}{2}$

Total

cn

$2 \cdot c \cdot n$

$\frac{n}{4} \cdot 16 \cdot c = 4c \cdot n$

$T(n) = c \cdot n + 2 \cdot c \cdot n + 2^2 \cdot c \cdot n + \cdots + 2^k \cdot c \cdot n$

$= c \cdot n \, (1 + 2 + 2^2 + \cdots + 2^k)$

Here, k = layer of tree, $k = \log_2 n$

$S = 1 + 2 + 2^2 + \cdots + 2^k$

$= \frac{1 - 2^{k+1}}{1 - 2} = 2^{k+1} - 1$

$= 2n - 1$

Karatsuba Algorithm

$T(n) = (2n-1) \, c \cdot n = O(n^2)$

Optimization: $A = X_h Y_h$

$B = X_l Y_l$

$D = (X_h + X_l)(Y_h + Y_l)$

each $c \cdot n$ has three children

Run time $T(n) \leq \begin{cases} 3 \cdot T(\frac{n}{2}) + c \cdot n & n > 1 \\ 1 & n = 1 \end{cases}$

$c \cdot n$

$c\frac{n}{2} \quad c\frac{n}{2} \quad c\frac{n}{2}$

$c\frac{n}{4} \quad c\frac{n}{4} \quad c\frac{n}{4}$

$c \cdot n$

$3 \cdot \frac{n}{2} \cdot c = (\frac{3}{2}) cn$

$3^2 \frac{n}{4} \cdot c = (\frac{3}{2})^2 cn$

$T(n) \leq c \cdot n \, (1 + \frac{3}{2} + (\frac{3}{2})^2 + \cdots + (\frac{3}{2})^k)$

$= c \cdot n \cdot \frac{1 - (\frac{3}{2})^{k+1}}{1 - \frac{3}{2}}$

$$\leq 2Cn \cdot \left(\frac{3}{2}\right)^{[illegible]} = 2Cn \cdot \left(\frac{3}{2}\right)^{\log_2 n + 1}$$

$$= 3Cn \left(\frac{3}{2}\right)^{\log_2 n} = 3C\, 3^{\log_2 n}$$

$$\cancel{= 3C \cdot} \log = 3C \cdot \left(2^{\log_2 3}\right)^{\log_2 n}$$

$$= \underline{3C \cdot n^{\log_2 3}} \leq 3C \cdot n^2$$

$$= O(n^{1.585\cdots})$$