Lecture 2 Practice Session
15-424/15-624/15-824 Logical Foundations of Cyber-Physical Systems

1 Agenda

- Lecture video/slides Q&A (15-20 mins)
- ODEs review (30 mins)
- FOL review (30 mins)

You are encouraged to try the problems marked "(Try this at home)" yourself to obtain a better understanding of the course material. A few of the questions go beyond required course material and are marked with an additional *.

2 Q&A

3 ODEs review

This course does not require advanced ODE/IVP solving techniques. However, some basic intuition with ODEs and their solutions is useful.

Exercise 1:
What is an Initial Value Problem (IVP)?
Answer: It comes in two parts:

1. A system of (first-order) *ordinary differential equations* (ODEs):
$$y'(t) = f(t, y)$$
Note: $y = (y_1, \ldots, y_n)$ is a n-dimensional *system* of ODEs.
Domain: $f : D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$.
2. Initial values $(t_0, y_0) \in D$
$$y(t_0) = y_0$$

Exercise 2:
What is required for $Y(t)$ to be a solution of this IVP?
Answer:

1. First, Y should assign values in the domain to the state variables y at each point in time. So $Y : I \to \mathbb{R}^n$ is a function from time interval I to $\mathbb{R}^n$. Moreover, it should be in the domain of the ODE at all times $t \in I$:
$$(t, Y(t)) \in D$$

2. Second, it should obey the ODE. Thus, $Y'(t)$ must exist and satisfy:

$$Y'(t) = f(t, Y(t))$$

3. Finally, it should obey the initial values:

$$Y(t_0) = y_0$$

Exercise 3:
Textbook 2.6. How to check that a claimed IVP "solution" is really a solution?
Answer: By plugging in the solution and checking that it obeys all of the above!

1. $x' = x, x(0) = x_0$. Solution: $X(t) = x_0 e^t$. Solution defined for all time t.

2. $x' = x^2, x(0) = x_0$. Solution: $X(t) = \frac{x_0}{1-tx_0}$

   Exercise 4:
   For what values of t is the solution $X(t)$ defined for a given x_0?

   Answer: For $x_0 > 0$, solution is defined for $(-\infty, \frac{1}{x_0})$. For $x_0 < 0$, solution is defined for $(\frac{1}{x_0}, \infty)$ For $x_0 = 0$, solution is defined for $(-\infty, \infty)$

3. $x' = \sqrt{x}, x(0) = x_0$. Solution $X(t) = \frac{t^2}{4} \pm t\sqrt{x_0} + x_0$.

   Exercise 5:
   For what values of t is the solution $X(t)$ defined for a given x_0?

   Answer: Defined for all time for $x_0 \geq 0$. **The IVP itself is undefined otherwise, because $\sqrt{\cdot}$ does not make sense for a negative number.**

   Exercise 6:
   Do you notice anything special about $x_0 = 0$?

   Answer: From solution above: $X(t) = \frac{t^2}{4}$ is a solution, but plugging in $X(t) = 0$ gives us yet another solution!

   Exercise 7:
   (Try this at home *). Construct infinitely many solutions to the IVP in case $x_0 = 0$.

   Fortunately, the ODEs we will need in this course have unique solutions.

4. $x' = y, y' = x, x(0) = 0, y(0) = 1$. Solution: $X(t) = \sin(t), Y(t) = \cos(t)$.

   Oops! We dropped a $-$ sign for the $y' = -x$ equation. If we knew the solution, it is easy to find the error.

Exercise 8:
How do we know that the ODEs we write down make sense?

Answer: This is a very important question, which you should keep in mind for the course. Make sure that the ODEs you write down make sense (physical or otherwise) for what you are modeling!

For this particular (planar) ODE, we can get away with sketching the direction field.

Exercise 9:
(Try this at home) Sketch the direction fields for $x' = y, y' = x$ and $x' = y, y' = -x$.

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(* Left  *) StreamPlot[{y, x}, {x, -2, 2}, {y, -2, 2}]
(* Right *) StreamPlot[{y, -x}, {x, -2, 2}, {y, -2, 2}]
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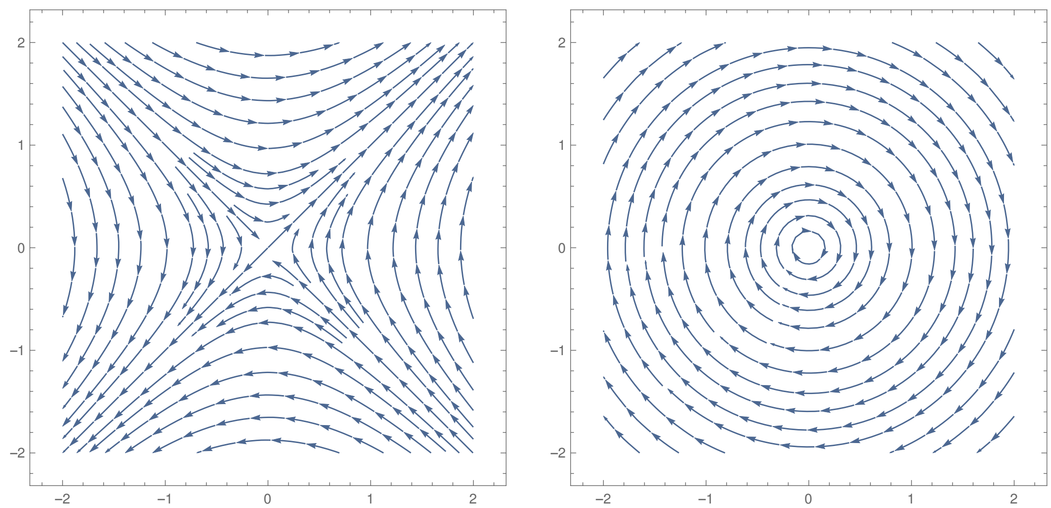

Later in the course, we will learn advanced techniques for analyzing ODEs. With those techniques, you can also check that certain expected properties of your ODE hold.

Exercise 10:
Textbook 2.3.
Answer: Draw a picture and model the situation with an ODE.

1. A (simple) ODE for the car while it is driving forwards:

$$x' = v, v' = a$$

2. For simplicity, suppose car was initially at $x_0 = 0, v_0 \geq 0$. Solve the IVP.

$$V(t) = v_0 + at$$
$$X(t) = x_0 + v_0 t + a\frac{t^2}{2} = v_0 t + a\frac{t^2}{2}$$

3. When will the car come to a stop, and how far would it have traveled by then ($x(t)$)?
4. How fast will the speeding car crash into the deer ($v(t)$, for a slightly different problem)?
5. (Try this at home) How could we account for a driver delay of 2 seconds?

Exercise 11:
(Try this at home) Can you solve this problem directly with material you know from physics?

Exercise 12:
(Try this at home *) From physics, you might be familiar with thinking of acceleration a as the second (time) derivative of position x, i.e., $x'' = a$. How is this (second-order) ODE related to the car ODE used above? For this course, we will only use *systems* of first-order differential equations.

Exercise 13:
(Try this at home *) Another second-order ODE describing the motion of a pendulum is:

$$\theta'' = -\frac{g}{l}\sin(\theta) - \theta'$$

Can you represent this as a system of first-order ODEs? Hint: introduce a variable $\omega = \theta'$.

4 FOL review

In the last problem, the car was really decelerating instead of braking, because its velocity v continues to decrease even after the car has reached $v = 0$! To specify that the ODE should stop evolving when $v = 0$, this course uses **differential equations with evolution domains**, where domain Q restricts where the ODEs are allowed to evolve:

$$\{x' = f(x) \,\&\, Q\}$$

Notice that the RHS, $f(x)$ can now only depend on x but not t explicitly. This is also called an **autonomous** system of ODEs.

Exercise 14:
How might we say that the ODE holds only while the velocity is still nonnegative?
Answer: Using constraint $v \geq 0$

Exercise 15:
How might we augment this system if we needed to mention t in the RHS?
Answer: Adding $t' = 1$ with appropriate initial value.

The preceding section introduces the *mathematical* definition of ODEs/IVP. We will instead give a *logical* definition of the semantics of ODE $\{x' = f(x) \,\&\, Q\}$ that interfaces well with the semantics of discrete programs. For that purpose, we first need to understand term and formula semantics.

Exercise 16:
Textbook 2.1
Answer:

1. Given $\omega(x) = 7$, what is the value of $4 + x \cdot 2$ in ω?

$$\begin{aligned}\omega[\![4 + x \cdot 2]\!] &= \omega[\![4]\!] + \omega[\![x \cdot 2]\!] \\ &= 4 + \omega[\![x]\!] \cdot \omega[\![2]\!] = 4 + \omega(x) \cdot 2 \\ &= 4 + 7 \cdot 2 = 18\end{aligned}$$

2. Given $\nu(x) = -4$, what is the value of $4 + x \cdot 2$ in ν?

$$\begin{aligned}\nu[\![4 + x \cdot 2]\!] &= \nu[\![4]\!] + \nu[\![x \cdot 2]\!] \\ &= 4 + \nu[\![x]\!] \cdot \nu[\![2]\!] = 4 + \nu(x) \cdot 2 \\ &= 4 + (-4) \cdot 2 = -4\end{aligned}$$

3. What is the value of term $4 + x \cdot 2 + x \cdot x \cdot x$ in the same state ν?

$$\begin{aligned}\nu[\![4 + x \cdot 2 + x \cdot x \cdot x]\!] &= \nu[\![4 + x \cdot 2]\!] + \nu[\![x \cdot x \cdot x]\!] \\ &= -4 + \nu[\![x \cdot x \cdot x]\!] = -4 + \nu[\![x]\!] \cdot \nu[\![x]\!] \cdot \nu[\![x]\!] \\ &= -4 + (-4) \cdot (-4) \cdot (-4) = -68\end{aligned}$$

4. How does its value change if also $\nu(y) = 7$? (It does not).

5. (Try this at home) Finish the rest of this question.

Exercise 17:
Textbook 2.2
Answer:

1. Define $-e \stackrel{\text{def}}{\equiv} -1 \cdot e$

2. General divisions $\frac{e}{e}$ are **not** directly definable in the syntax. (Why?) Divisions by (non-zero) constants are fine though, because $\frac{x}{2} = \frac{1}{2} \cdot x$.

   In practice, we will use divisions anyway because they are so useful for modeling. But we need to be extra careful about ill-definedness!

   For example, what is $\omega[\![\frac{1}{0}]\!]$? Is it some real number? If so, which one?

3. General exponentiation $e^{\tilde{e}}$ is **not** directly definable. (Why?) Recall earlier example: $x^{\frac{1}{2}}$ is ill-behaved when $x < 0$!

   Again in practice, some fractional powers like $\frac{1}{2}$ are useful, but we need to be extra careful about ill-definedness.

Exercise 18:
Calculate and explain whether each of the following formulas is true in a state with $\omega(x) = 1$.

1. $x \geq 5 \land x \leq 4$.

   Answer:

$$\begin{aligned} \omega \models x \geq 5 \land x \leq 4 &\text{ iff } \omega \models x \geq 5 \text{ and } \omega \models x \leq 4 \\ &\text{ iff } \omega[\![x]\!] \geq \omega[\![5]\!] \text{ and } \omega[\![x]\!] \leq \omega[\![4]\!] \\ &\text{ iff } 1 \geq 5 \text{ and } 1 \leq 4 \\ &\text{ iff } false \end{aligned}$$

2. $x > 1 \to x = 42$.

   Answer:

$$\begin{aligned} \omega \models x > 1 \to x = 42 &\text{ iff } \omega \not\models x > 1 \text{ or } \omega \models x = 42 \\ &\text{ iff } \text{not}(\omega[\![x]\!] > \omega[\![1]\!]) \text{ or } \omega[\![x]\!] = \omega[\![42]\!] \\ &\text{ iff } \text{not}(1 > 1) \text{ or } 1 = 42 \\ &\text{ iff } true \end{aligned}$$

   If this result surprised you, review the meaning of implications carefully.

3. $\forall x\, (x > 1 \to x = 42)$.

   Answer:

$$\omega \models \forall x\, (x > 1 \to x = 42) \text{ iff } \omega_x^d \models x > 1 \to x = 42 \text{ for all } d \in \mathbb{R}$$

   Try $d = 2$:

$$\begin{aligned} \omega_x^2 \models x > 1 \to x = 42 &\text{ iff } \omega_x^2 \not\models x > 1 \text{ or } \omega_x^2 \models x = 42 \\ &\text{ iff } \text{not}(\omega_x^2[\![x]\!] > \omega_x^2[\![1]\!]) \text{ or } \omega_x^2[\![x]\!] = \omega_x^2[\![42]\!] \\ &\text{ iff } \text{not}(2 > 1) \text{ or } 2 = 42 \\ &\text{ iff } false \end{aligned}$$

   Thus,the formula $\forall x\, (x > 1 \to x = 42)$ is false in state ω. **If this result surprised you, review the meaning of the universal quantifier carefully.**

4. $\exists x\,(x > 1 \to x = 42)$.

$$\omega \models \exists x\,(x > 1 \to x = 42) \text{ iff } \omega_x^d \models x > 1 \to x = 42 \text{ for some } d \in \mathbb{R}$$

(Try this at home) Which choices of d would satisfy $\omega_x^d \models x > 1 \to x = 42$?

Hint: It is not just $d = 42$.

Exercise 19:
Which of the above formulas are: (un)satisfiable? Which of them are valid?
Answer: (In order) Unsat, Sat (not valid), Unsat, Valid (also Sat).

Exercise 20:
(Try this at home) For terms, we used a minimal grammar of operators consisting of $+, \cdot$ and treated other operators like $-$ as definable. Can you identify a minimal set of logical operators for formulas and define the others out of them? Which of your definitions rely on logical equivalences? Which of your them rely on properties of the real numbers?

Exercise 21:
(Try this at home) Textbook 2.9

Exercise 22:
Define the semantics of ODE $\{x' = f(x) \,\&\, Q\}$.
Answer: See the lecture slides (Slide 17)