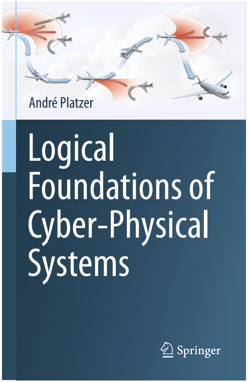
14: Hybrid Systems & Games

Logical Foundations of Cyber-Physical Systems



André Platzer

Carnegie Mellon University
Computer Science Department

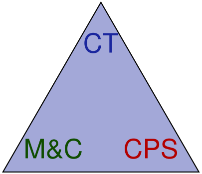
Outline

Outline

1. Learning Objectives
2. Motivation
3. A Gradual Introduction to Hybrid Games
   - Choices & Nondeterminism
   - Control & Dual Control
   - Demon's Derived Controls
4. Differential Game Logic
   - Syntax of Hybrid Games
   - Syntax of Differential Game Logic Formulas
   - Examples
   - Push-around Cart
   - Robot Dance
   - Example: Robot Soccer
5. An Informal Operational Game Tree Semantics
6. Summary

Learning Objectives
Hybrid Systems & Games

fundamental principles of computational thinking
logical extensions
PL modularity principles
compositional extensions
differential game logic
best/worst-case analysis
models of alternating computation



adversarial dynamics
conflicting actions
multi-agent systems
angelic/demonic choice

multi-agent state change
CPS semantics
reflections on choices

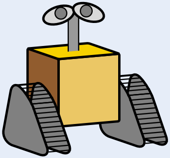
Outline

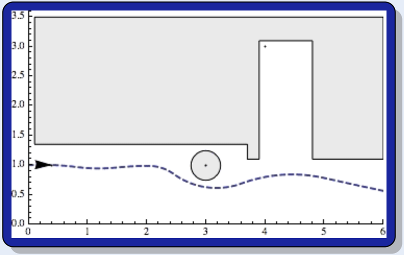
CPS Analysis: Robot Control

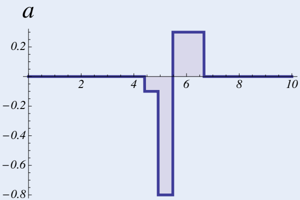
Challenge (Hybrid Systems)

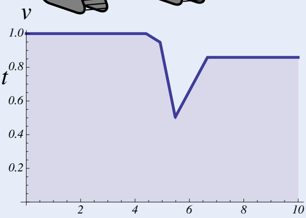
Fixed rule describing state evolution with both

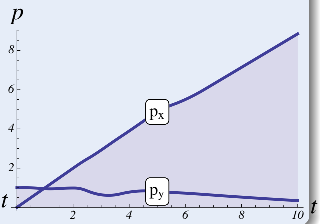
- Discrete dynamics (control decisions)
- Continuous dynamics (differential equations)







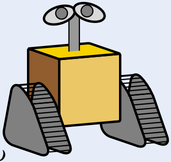


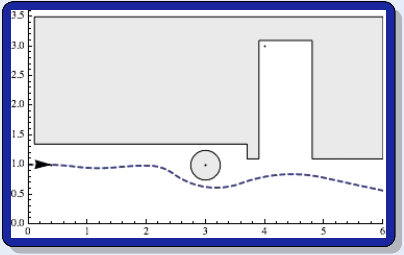


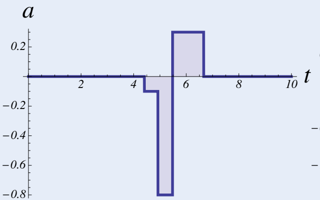
Challenge (Hybrid Systems)

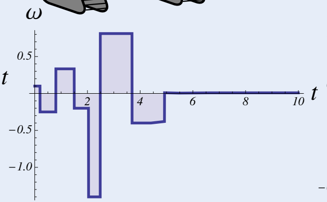
Fixed rule describing state evolution with both

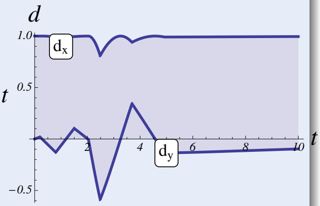
- Discrete dynamics (control decisions)
- Continuous dynamics (differential equations)

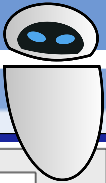








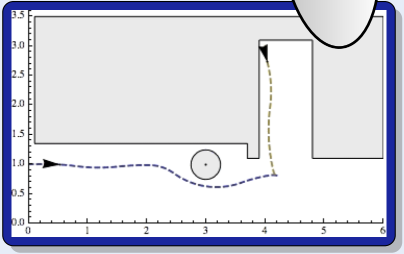


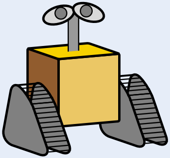


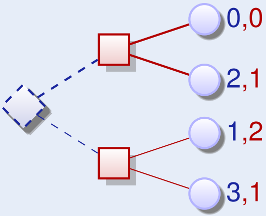
Challenge (Games)

Game rules describing play evolution with both

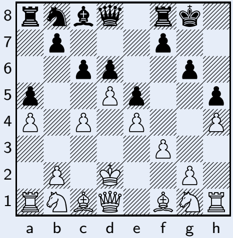
- Angelic choices (player ◇ Angel)
- Demonic choices (player □ Demon)

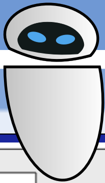






◇\□	Tr	Pl
Trash	1,2	0,0
Plant	0,0	2,1

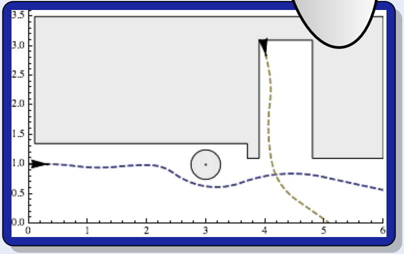


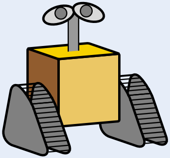


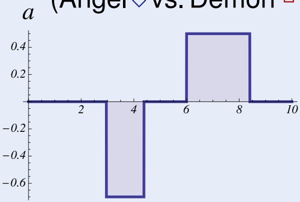
Challenge (Hybrid Games)

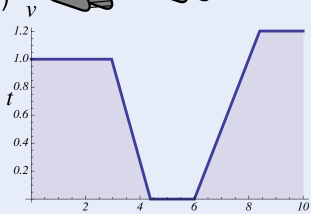
Game rules describing play evolution with

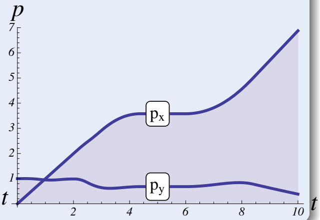
- Discrete dynamics (control decisions)
- Continuous dynamics (differential equations)
- Adversarial dynamics (Angel ◇ vs. Demon ◻)









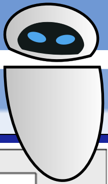


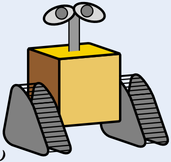
CPS Analysis: Robot Control

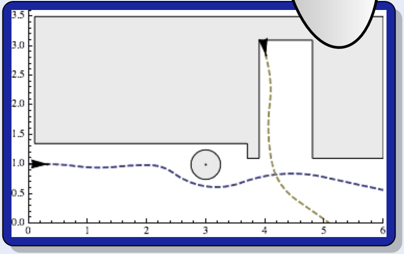
Challenge (Hybrid Games)

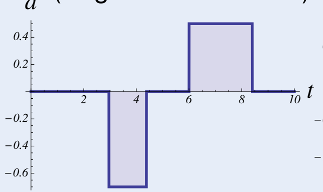
Game rules describing play evolution with

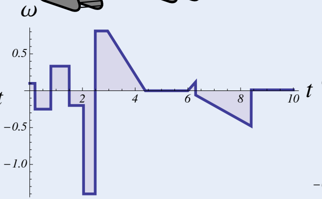
- Discrete dynamics (control decisions)
- Continuous dynamics (differential equations)
- Adversarial dynamics (Angel $\diamond$ vs. Demon $\square$)

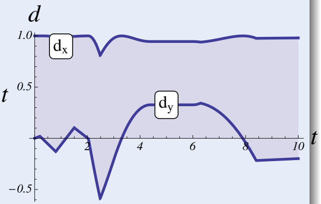










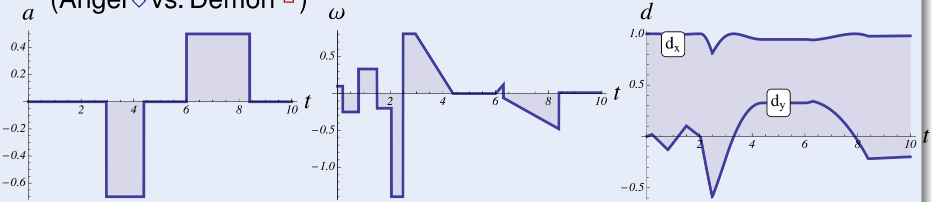


CPS Analysis: RoboCup Soccer

Challenge (Hybrid Games)

Game rules describing play evolution with

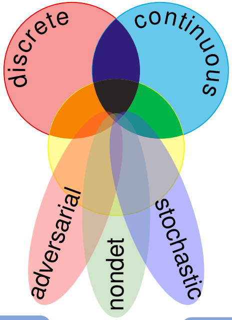
- Discrete dynamics (control decisions)
- Continuous dynamics (differential equations)
- Adversarial dynamics (Angel $\diamond$ vs. Demon $\square$)



CPSs are Multi-Dynamical Systems

CPS Dynamics

CPS are characterized by multiple facets of dynamical systems.



CPS Compositions

CPS combines multiple simple dynamical effects.

Descriptive simplification

Tame Parts

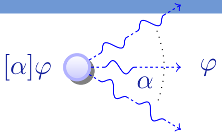
Exploiting compositionality tames CPS complexity.

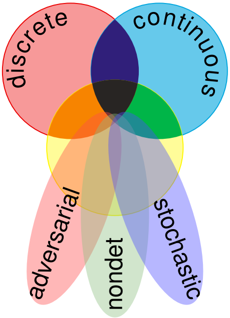
Analytic simplification

Dynamic Logics for Dynamical Systems

differential dynamic logic

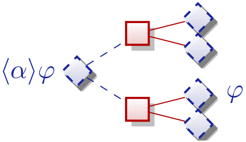
$\mathsf{dL} = \mathsf{DL} + \mathsf{HP}$





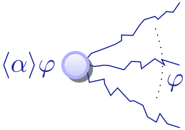
differential game logic

$\mathsf{dGL} = \mathsf{GL} + \mathsf{HG}$



stochastic differential DL

$\mathsf{SdL} = \mathsf{DL} + \mathsf{SHP}$



quantified differential DL

$\mathsf{QdL} = \mathsf{FOL} + \mathsf{DL} + \mathsf{QHP}$

Outline

Differential Dynamic Logic dL: Syntax

Definition (Hybrid program α)

$$x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha ; \beta \mid \alpha^*$$

Definition (dL Formula P)

$$e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, P \mid [\alpha]P \mid \langle \alpha \rangle P$$

Differential Dynamic Logic dL: Syntax

Discrete Assign | Test Condition | Differential Equation | Nondet. Choice | Seq. Compose | Nondet. Repeat

Definition (Hybrid program α)

$$x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha ; \beta \mid \alpha^*$$

Definition (dL Formula P)

$$e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, P \mid [\alpha] P \mid \langle \alpha \rangle P$$

All Reals | Some Reals | All Runs | Some Runs

Nondet. Choice

Definition (Hybrid program α)

$$x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^*$$

Definition (dL Formula P)

$$e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, P \mid [\alpha]P \mid \langle\alpha\rangle P$$

Nondeterminism during HP runs

Differential Equation

Nondet. Choice

Nondet. Repeat

Definition (Hybrid program α)

$$x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^*$$

Definition (dL Formula P)

$$e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, P \mid [\alpha]P \mid \langle\alpha\rangle P$$

Nondeterminism during HP runs

Differential Equation

Nondet. Choice

Nondet. Repeat

Definition (Hybrid program α)

$$x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^*$$

Definition (dL Formula P)

$$e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, P \mid [\alpha]P \mid \langle\alpha\rangle P$$

All Choices

Some Choice

All choices resolved in one way

Differential Equation

Nondet. Choice

Nondet. Repeat

Definition (Hybrid program α)

$$x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha ; \beta \mid \alpha^*$$

Definition (dL Formula P)

$$e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, P \mid [\alpha] P \mid \langle \alpha \rangle P$$

Modality decides the mode: help/hurt

All Choices

Some Choice

All choices resolved in one way

Differential Equation

Nondet. Choice

Nondet. Repeat

Definition (Hybrid program α)

$$x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^*$$

Definition (dL Formula P)

$$e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, P \mid [\alpha]P \mid \langle\alpha\rangle P$$

Modality decides the mode: help/hurt

All Choices

Some Choice

$[\alpha_1]\langle\alpha_2\rangle[\alpha_3]\langle\alpha_4\rangle P$ only fixed interaction depth

Control & Dual Control Operators

◇ Angel Ops

$\cup$	choice
$*$	repeat
$x' = f(x)$	evolve
$?Q$	challenge

Let Angel be one player

Control & Dual Control Operators

◇ Angel Ops

$\cup$	choice
$*$	repeat
$x' = f(x)$	evolve
$?Q$	challenge

□ Demon Ops

$\cap$	choice
$\times$	repeat
$x' = f(x)^{\mathrm{d}}$	evolve
$?Q^{\mathrm{d}}$	challenge

Let Angel be one player

Let Demon be another player

Control & Dual Control Operators

◇ Angel Ops

$\cup$	choice
$*$	repeat
$x' = f(x)$	evolve
$?Q$	challenge

d

d

□ Demon Ops

$\cap$	choice
$\times$	repeat
$x' = f(x)^{d}$	evolve
$?Q^{d}$	challenge

Duality operator ^{d} passes control between players

Game Operators

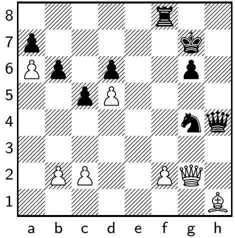
◇ Angel Ops	
$\cup$	choice
$*$	repeat
$x' = f(x)$	evolve
$?Q$	challenge

d

d

□ Demon Ops	
$\cap$	choice
$\times$	repeat
$x' = f(x)^d$	evolve
$?Q^d$	challenge

Duality operator ^{d} passes control between players



Game Operators

◇ Angel Ops

$\cup$	choice
$*$	repeat
$x' = f(x)$	evolve
$?Q$	challenge

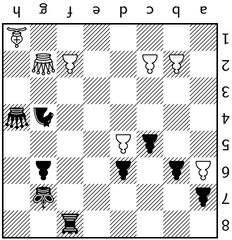
d

d

□ Demon Ops

$\cap$	choice
$\times$	repeat
$x' = f(x)^{d}$	evolve
$?Q^{d}$	challenge

Duality operator ^{d} passes control between players



◇ Angel Ops

$\cup$	choice
$*$	repeat
$x' = f(x)$	evolve
$?Q$	challenge

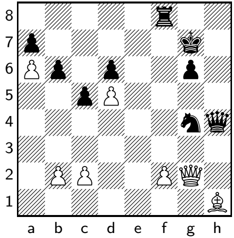
d

d

□ Demon Ops

$\cap$	choice
$\times$	repeat
$x' = f(x)^d$	evolve
$?Q^d$	challenge

Duality operator ^{d} passes control between players



Definable Game Operators

$$\text{Angel Ops} \underset{d}{\overset{d}{\rightleftarrows}} \text{Demon Ops}$$

◇ Angel Ops

$\cup$	choice
$*$	repeat
$x' = f(x)$	evolve
$?Q$	challenge

□ Demon Ops

$\cap$	choice
$\times$	repeat
$x' = f(x)^d$	evolve
$?Q^d$	challenge

$$
\begin{aligned}
\text{if}(Q)\,\alpha\,\text{else}\,\beta &\equiv \\
\text{while}(Q)\,\alpha &\equiv \\
\alpha \cap \beta &\equiv \\
\alpha^{\times} &\equiv \\
(x' = f(x)\,\&\,Q)^d &\quad x' = f(x)\,\&\,Q \\
(x := e)^d &\quad x := e \\
?Q^d &\quad ?Q
\end{aligned}
$$

Definable Game Operators

◇ Angel Ops

$\cup$	choice
$*$	repeat
$x' = f(x)$	evolve
$?Q$	challenge

$\xrightarrow{\;d\;}$ (Angel Ops to Demon Ops)

$\xleftarrow{\;d\;}$ (Demon Ops to Angel Ops)

□ Demon Ops

$\cap$	choice
$\times$	repeat
$x' = f(x)^d$	evolve
$?Q^d$	challenge

$$
\begin{aligned}
\text{if}(Q)\,\alpha\,\text{else}\,\beta &\equiv (?Q;\alpha)\cup(?\neg Q;\beta)\\
\text{while}(Q)\,\alpha &\equiv \\
\alpha\cap\beta &\equiv \\
\alpha^\times &\equiv \\
(x' = f(x)\,\&\,Q)^d &\quad x' = f(x)\,\&\,Q\\
(x := e)^d &\quad x := e\\
?Q^d &\quad ?Q
\end{aligned}
$$

$$\text{d}: \text{Angel Ops} \rightleftarrows \text{Demon Ops}$$

◇ Angel Ops

$\cup$	choice
$*$	repeat
$x' = f(x)$	evolve
$?Q$	challenge

□ Demon Ops

$\cap$	choice
$\times$	repeat
$x' = f(x)^{\text{d}}$	evolve
$?Q^{\text{d}}$	challenge

$$\text{if}(Q)\,\alpha\,\text{else}\,\beta \equiv (?Q;\alpha) \cup (?\neg Q;\beta)$$

$$\text{while}(Q)\,\alpha \equiv (?Q;\alpha)^*;?\neg Q$$

$$\alpha \cap \beta \equiv$$

$$\alpha^\times \equiv$$

$$(x' = f(x)\,\&\,Q)^{\text{d}} \qquad x' = f(x)\,\&\,Q$$

$$(x := e)^{\text{d}} \qquad x := e$$

$$?Q^{\text{d}} \qquad ?Q$$

◇ Angel Ops

$\cup$	choice
$*$	repeat
$x' = f(x)$	evolve
$?Q$	challenge

d

d

□ Demon Ops

$\cap$	choice
$\times$	repeat
$x' = f(x)^d$	evolve
$?Q^d$	challenge

$$\mathrm{if}(Q)\,\alpha\,\mathrm{else}\,\beta \equiv (?Q;\alpha)\cup(?\neg Q;\beta)$$

$$\mathrm{while}(Q)\,\alpha \equiv (?Q;\alpha)^*;?\neg Q$$

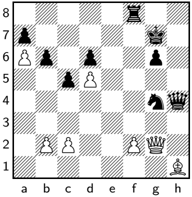
$$\alpha\cap\beta \equiv$$

$$\alpha^\times \equiv$$

$$(x' = f(x)\,\&\,Q)^d \qquad x' = f(x)\,\&\,Q$$

$$(x := e)^d \qquad x := e$$

$$?Q^d \qquad ?Q$$



Definable Game Operators

◇ Angel Ops

$\cup$	choice
$*$	repeat
$x' = f(x)$	evolve
$?Q$	challenge

d

d

□ Demon Ops

$\cap$	choice
$\times$	repeat
$x' = f(x)^d$	evolve
$?Q^d$	challenge

$$\text{if}(Q)\,\alpha\,\text{else}\,\beta \equiv (?Q;\alpha) \cup (?\neg Q;\beta)$$

$$\text{while}(Q)\,\alpha \equiv (?Q;\alpha)^*;?\neg Q$$

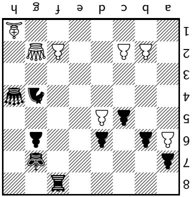
$$\alpha \cap \beta \equiv (\alpha^d \cup \beta^d)^d$$

$$\alpha^\times \equiv$$

$$(x' = f(x)\,\&\,Q)^d \qquad x' = f(x)\,\&\,Q$$

$$(x := e)^d \qquad x := e$$

$$?Q^d \qquad ?Q$$



Definable Game Operators

◇ Angel Ops	
$\cup$	choice
$*$	repeat
$x' = f(x)$	evolve
$?Q$	challenge

$$\text{Angel Ops} \underset{\mathrm{d}}{\overset{\mathrm{d}}{\rightleftarrows}} \text{Demon Ops}$$

□ Demon Ops	
$\cap$	choice
$\times$	repeat
$x' = f(x)^{\mathrm{d}}$	evolve
$?Q^{\mathrm{d}}$	challenge

$$\mathrm{if}(Q)\,\alpha\,\mathrm{else}\,\beta \equiv (?Q;\alpha) \cup (?\neg Q;\beta)$$

$$\mathrm{while}(Q)\,\alpha \equiv (?Q;\alpha)^*;?\neg Q$$

$$\alpha \cap \beta \equiv (\alpha^{\mathrm{d}} \cup \beta^{\mathrm{d}})^{\mathrm{d}}$$

$$\alpha^{\times} \equiv ((\alpha^{\mathrm{d}})^*)^{\mathrm{d}}$$

$$\begin{array}{rl} (x' = f(x)\,\&\,Q)^{\mathrm{d}} & x' = f(x)\,\&\,Q \\ (x := e)^{\mathrm{d}} & x := e \\ ?Q^{\mathrm{d}} & ?Q \end{array}$$

Definable Game Operators

◇ Angel Ops

$\cup$	choice
$*$	repeat
$x' = f(x)$	evolve
$?Q$	challenge

d

d

□ Demon Ops

$\cap$	choice
$\times$	repeat
$x' = f(x)^{d}$	evolve
$?Q^{d}$	challenge

$$\mathrm{if}(Q)\,\alpha\,\mathrm{else}\,\beta \equiv (?Q;\alpha) \cup (?\neg Q;\beta)$$

$$\mathrm{while}(Q)\,\alpha \equiv (?Q;\alpha)^{*};?\neg Q$$

$$\alpha \cap \beta \equiv (\alpha^{d} \cup \beta^{d})^{d}$$

$$\alpha^{\times} \equiv ((\alpha^{d})^{*})^{d}$$

$$(x' = f(x)\,\&\,Q)^{d} \not\equiv x' = f(x)\,\&\,Q$$

$$(x := e)^{d} \qquad x := e$$

$$?Q^{d} \qquad ?Q$$

Definable Game Operators

◇ Angel Ops

$\cup$	choice
$*$	repeat
$x' = f(x)$	evolve
$?Q$	challenge

d

d

□ Demon Ops

$\cap$	choice
$\times$	repeat
$x' = f(x)^{\mathrm{d}}$	evolve
$?Q^{\mathrm{d}}$	challenge

$$\mathrm{if}(Q)\,\alpha\,\mathrm{else}\,\beta \equiv (?Q;\alpha) \cup (?\neg Q;\beta)$$

$$\mathrm{while}(Q)\,\alpha \equiv (?Q;\alpha)^*;?\neg Q$$

$$\alpha \cap \beta \equiv (\alpha^{\mathrm{d}} \cup \beta^{\mathrm{d}})^{\mathrm{d}}$$

$$\alpha^{\times} \equiv ((\alpha^{\mathrm{d}})^*)^{\mathrm{d}}$$

$$(x' = f(x)\,\&\,Q)^{\mathrm{d}} \not\equiv x' = f(x)\,\&\,Q$$

$$(x := e)^{\mathrm{d}} \equiv x := e$$

$$?Q^{\mathrm{d}} \qquad ?Q$$

Definable Game Operators

◇ Angel Ops	
$\cup$	choice
$*$	repeat
$x' = f(x)$	evolve
$?Q$	challenge

$\xrightarrow{d}$ / $\xleftarrow{d}$

◻ Demon Ops	
$\cap$	choice
$\times$	repeat
$x' = f(x)^d$	evolve
$?Q^d$	challenge

$$\text{if}(Q)\,\alpha\,\text{else}\,\beta \equiv (?Q;\alpha) \cup (?\neg Q;\beta)$$

$$\text{while}(Q)\,\alpha \equiv (?Q;\alpha)^*;?\neg Q$$

$$\alpha \cap \beta \equiv (\alpha^d \cup \beta^d)^d$$

$$\alpha^\times \equiv ((\alpha^d)^*)^d$$

$$(x' = f(x)\,\&\,Q)^d \not\equiv x' = f(x)\,\&\,Q$$

$$(x := e)^d \equiv x := e$$

$$?Q^d \not\equiv ?Q$$

Outline

Definition (Hybrid game α)

$$\alpha, \beta \ ::= \ x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^* \mid \alpha^d$$

Hybrid Games: Syntax

Discrete Assign | Test Game | Differential Equation | Choice Game | Seq. Game | Repeat Game

Definition (Hybrid game α)

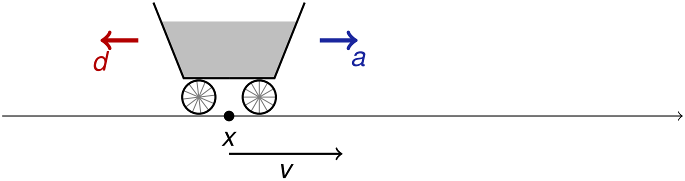
$$\alpha, \beta ::= x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha ; \beta \mid \alpha^* \mid \alpha^d$$

Discrete Assign | Test Game | Differential Equation | Choice Game | Seq. Game | Repeat Game | Dual Game

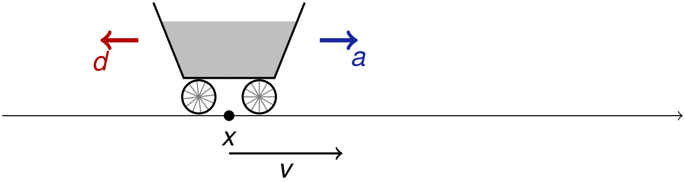
Definition (Hybrid game α)

$$\alpha, \beta \ ::= \ x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha ; \beta \mid \alpha^* \mid \alpha^d$$

Example: Push-around Cart

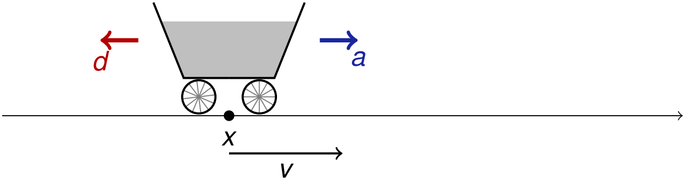


Example: Push-around Cart



$$\Big((a := 1 \cup a := -1); (d := 1 \cup d := -1)^{\mathrm{d}}; \{x' = v, v' = a + d\}\Big)^*$$

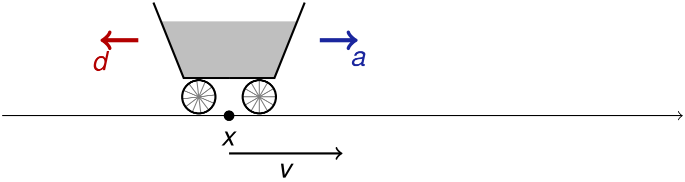
Example: Push-around Cart



$$\big((a:=1 \cup a:=-1);(d:=1 \cup d:=-1)^{d};\{x'=v,v'=a+d\}\big)^{*}$$

$$\big((d:=1 \cup d:=-1)^{d};(a:=1 \cup a:=-1);\{x'=v,v'=a+d\}\big)^{*}$$

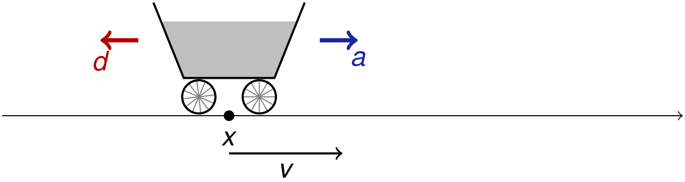
Example: Push-around Cart



$$\left((a := 1 \cup a := -1); (d := 1 \cap d := -1); \{x' = v, v' = a + d\}\right)^*$$

$$\left((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\}\right)^*$$

Example: Push-around Cart

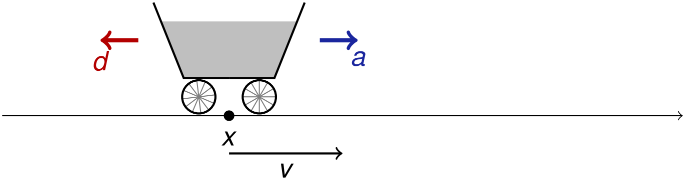


$$\big((a:=1 \cup a:=-1);(d:=1 \cap d:=-1);\{x'=v,v'=a+d\}\big)^*$$

$$\big((d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v,v'=a+d\}\big)^*$$

HP $\big((d:=1 \cup d:=-1);(a:=1 \cup a:=-1);\{x'=v,v'=a+d\}\big)^*$

Example: Push-around Cart



$$\big((a:=1 \cup a:=-1);(d:=1 \cap d:=-1);\{x'=v,v'=a+d\}\big)^*$$

$$\big((d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v,v'=a+d\}\big)^*$$

HP $\big((d:=1 \cup d:=-1);(a:=1 \cup a:=-1);\{x'=v,v'=a+d\}\big)^*$

Hybrid systems can't say that a is Angel's choice and d is Demon's

Differential Game Logic: Syntax

Definition (Hybrid game α)

$$\alpha, \beta ::= x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha ; \beta \mid \alpha^* \mid \alpha^d$$

Definition (Hybrid game α)

$$\alpha, \beta ::= x := e \mid ?Q \mid x' = f(x) \& Q \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^* \mid \alpha^d$$

Definition (dGL Formula P)

$$P, Q ::= e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, P \mid \langle\alpha\rangle P \mid [\alpha]P$$

Differential Game Logic: Syntax

Discrete Assign
Test Game
Differential Equation
Choice Game
Seq. Game
Repeat Game

Definition (Hybrid game α)

$$\alpha, \beta \ ::= \ x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^* \mid \alpha^d$$

Definition (dGL Formula P)

$$P, Q \ ::= \ e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, P \mid \langle\alpha\rangle P \mid [\alpha] P$$

All Reals
Some Reals

Differential Game Logic: Syntax

Discrete Assign | Test Game | Differential Equation | Choice Game | Seq. Game | Repeat Game | Dual Game

Definition (Hybrid game α)

$$\alpha, \beta \ ::= \ x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha ; \beta \mid \alpha^* \mid \alpha^d$$

Definition (dGL Formula P)

$$P, Q \ ::= \ e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x \, P \mid \exists x \, P \mid \langle \alpha \rangle P \mid [\alpha] P$$

All Reals | Some Reals

Differential Game Logic: Syntax

Discrete Assign | Test Game | Differential Equation | Choice Game | Seq. Game | Repeat Game | Dual Game

Definition (Hybrid game α)

$$\alpha, \beta \ ::= \ x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha ; \beta \mid \alpha^* \mid \alpha^d$$

Definition (dGL Formula P)

$$P, Q \ ::= \ e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, P \mid \langle \alpha \rangle P \mid [\alpha] P$$

All Reals | Some Reals | Angel Wins

Differential Game Logic: Syntax

Discrete Assign | Test Game | Differential Equation | Choice Game | Seq. Game | Repeat Game | Dual Game

Definition (Hybrid game α)

$$\alpha, \beta ::= x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha ; \beta \mid \alpha^* \mid \alpha^d$$

Definition (dGL Formula P)

$$P, Q ::= e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, P \mid \langle \alpha \rangle P \mid [\alpha] P$$

All Reals | Some Reals | Angel Wins | Demon Wins

Simple Examples

$$\langle (x := x+1; (x' = 1)^{d} \cup x := x-1)^{*} \rangle\, (0 \le x < 1)$$

$$\langle (x := x+1; (x' = 1)^{d} \cup (x := x-1 \cap x := x-2))^{*} \rangle (0 \le x < 1)$$

Simple Examples

$$\vDash \langle (x := x+1; (x' = 1)^{d} \cup x := x-1)^{*} \rangle (0 \leq x < 1)$$

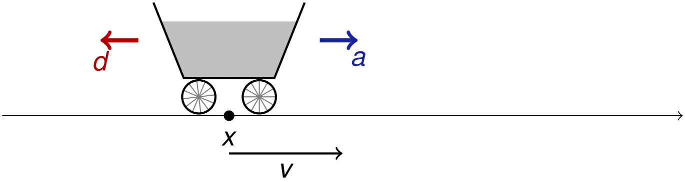
$$\langle (x := x+1; (x' = 1)^{d} \cup (x := x-1 \cap x := x-2))^{*} \rangle (0 \leq x < 1)$$

Simple Examples

$$\vDash \langle (x := x+1; (x' = 1)^{d} \cup x := x-1)^{*} \rangle (0 \leq x < 1)$$

$$\nvDash \langle (x := x+1; (x' = 1)^{d} \cup (x := x-1 \cap x := x-2))^{*} \rangle (0 \leq x < 1)$$

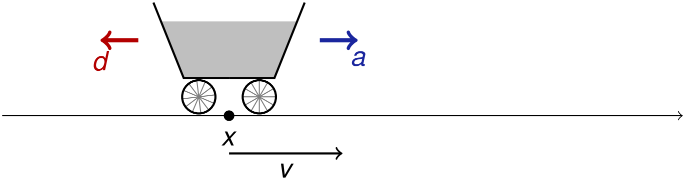
Example: Push-around Cart



$$v \geq 1 \rightarrow$$

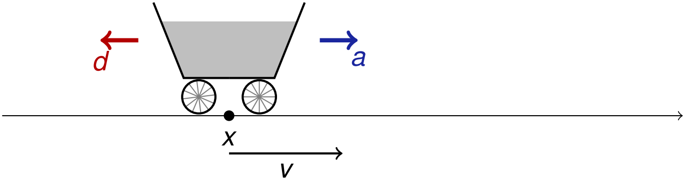
$$\Big[\big((d := 1 \cup d := -1)^{d}; (a := 1 \cup a := -1); \{x' = v, v' = a + d\}\big)^{*}\Big] v \geq 0$$

Example: Push-around Cart



$$\vDash v \geq 1 \rightarrow$$
$$\Big[\big((d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v, v'=a+d\}\big)^*\Big]\, v \geq 0$$

Example: Push-around Cart



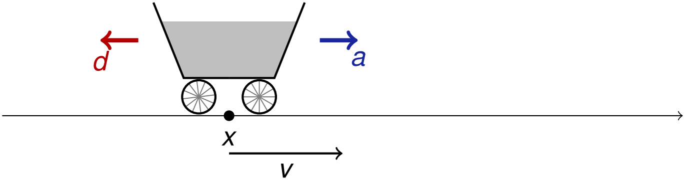
$\vDash v \geq 1 \rightarrow$ 　　 *d* before *a* can compensate

$$\left[\left((d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v, v'=a+d\}\right)^*\right] v \geq 0$$

$x \geq 0 \land v \geq 0 \rightarrow$

$$\left[\left((d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v, v'=a+d\}\right)^*\right] x \geq 0$$

Example: Push-around Cart



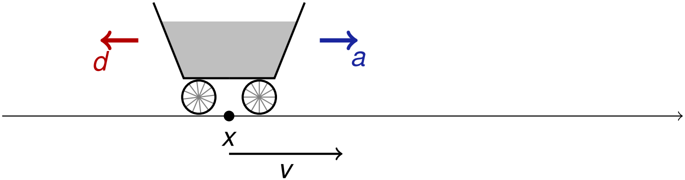
$\vDash v \geq 1 \rightarrow$ $\quad$ d before a can compensate

$$\left[\left((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\}\right)^*\right] v \geq 0$$

$\not\vDash x \geq 0 \land v \geq 0 \rightarrow$ $\quad$ d before a can compensate

$$\left[\left((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\}\right)^*\right] x \geq 0$$

Example: Push-around Cart

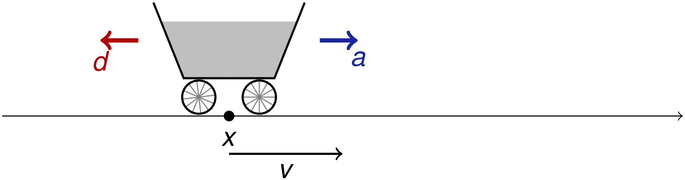


$\vDash v \geq 1 \rightarrow$ 　　　　 *d* before *a* can compensate

$$[((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\})^*] v \geq 0$$

$x \geq 0 \quad \rightarrow$

$$\langle ((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\})^* \rangle x \geq 0$$

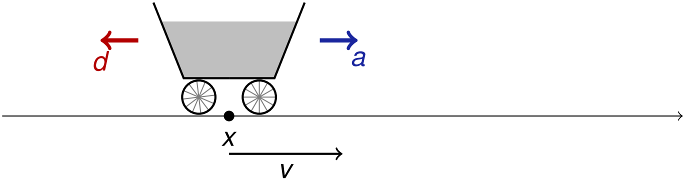


$\vDash v \geq 1 \rightarrow$ $\quad$ d before a can compensate

$$[((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\})^*]\, v \geq 0$$

$\vDash x \geq 0 \quad \rightarrow$ $\quad$ boring by skip

$$\langle((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\})^*\rangle\, x \geq 0$$

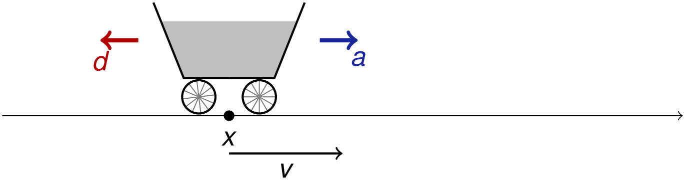


$\vDash v \geq 1 \rightarrow$ $\quad$ *d* before *a* can compensate

$$[((d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v, v'=a+d\})^*]\, v \geq 0$$

$$\langle((d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v, v'=a+d\})^*\rangle\, x \geq 0$$

Example: Push-around Cart



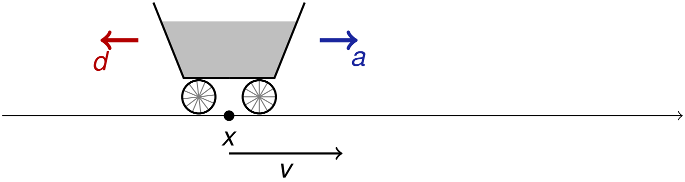
$\vDash v \geq 1 \rightarrow$ $\qquad$ d before a can compensate

$$[((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\})^*] v \geq 0$$

$\nvDash$ $\qquad$ counterstrategy $d := -1$

$$\langle((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\})^*\rangle x \geq 0$$

Example: Push-around Cart



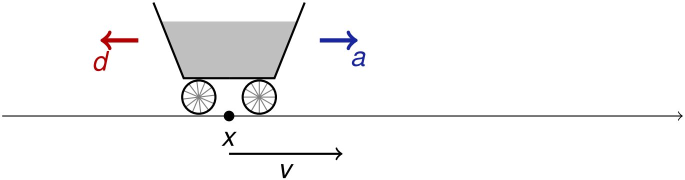
$\vDash v \geq 1 \rightarrow$ d before a can compensate

$$[\big((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\}\big)^*]\, v \geq 0$$

$\not\vDash$ counterstrategy $d := -1$

$$\langle\big((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\}\big)^*\rangle\, x \geq 0$$

$$\langle\big((d := 1 \cap d := -1); (a := 2 \cup a := -2); \{x' = v, v' = a + d\}\big)^*\rangle\, x \geq 0$$



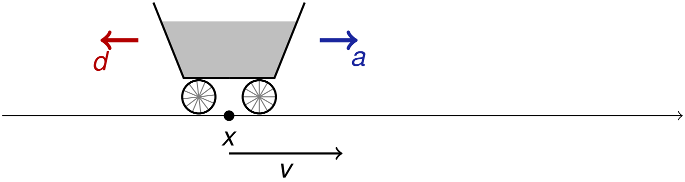
$\vDash v \geq 1 \rightarrow$ $\quad$ *d* before *a* can compensate

$$[((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\})^*]\, v \geq 0$$

$\nvDash$ $\quad$ counterstrategy $d := -1$

$$\langle((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\})^*\rangle\, x \geq 0$$

$$\vDash \langle((d := 1 \cap d := -1); (a := 2 \cup a := -2); \{x' = v, v' = a + d\})^*\rangle\, x \geq 0$$



$\vDash v \geq 1 \rightarrow$ d before a can compensate

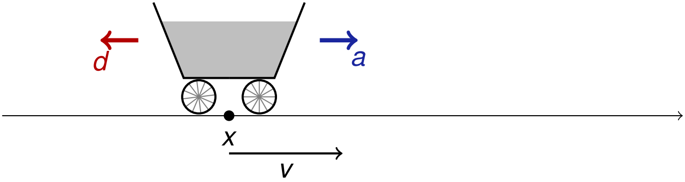
$$[\big((d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v,v'=a+d\}\big)^*]\,v \geq 0$$

$\not\vDash$ counterstrategy $d:=-1$

$$\langle\big((d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v,v'=a+d\}\big)^*\rangle\, x \geq 0$$

$$\vDash \langle\big((d:=1 \cap d:=-1);\ (a:=2 \cup a:=-2);\ \{x'=v,v'=a+d\}\big)^*\rangle\, x \geq 0$$

$$\langle\big((d:=2 \cap d:=-2);\ (a:=2 \cup a:=-2);$$
$$t:=0;\ \{x'=v,v'=a+d,t'=1\ \&\ t \leq 1\}\big)^*\rangle\, x^2 \geq 100$$



$\vDash v \geq 1 \rightarrow$ $\qquad$ *d* before *a* can compensate

$$\big[\big((d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v,v'=a+d\}\big)^*\big]\, v \geq 0$$

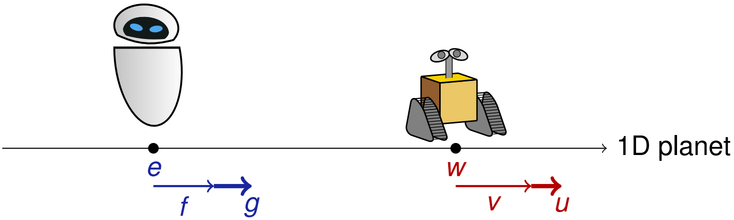
$\not\vDash$ $\qquad$ counterstrategy $d:=-1$

$$\big\langle\big((d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v,v'=a+d\}\big)^*\big\rangle\, x \geq 0$$

$$\vDash \big\langle\big((d:=1 \cap d:=-1);\,(a:=2 \cup a:=-2);\,\{x'=v,v'=a+d\}\big)^*\big\rangle\, x \geq 0$$

$$\vDash \big\langle\big((d:=2 \cap d:=-2);\,(a:=2 \cup a:=-2);$$ $\qquad$ $a:=d$ then $a:=2\,\mathrm{sign}\,v$

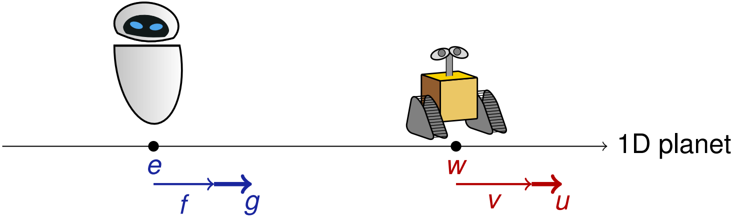
$$t:=0;\,\{x'=v,v'=a+d,t'=1\,\&\,t \leq 1\}\big)^*\big\rangle\, x^2 \geq 100$$



$$
\begin{aligned}
&(w-e)^2 \le 1 \land v = f \rightarrow \\
&\langle ((u := 1 \cap u := -1); \\
&\quad (g := 1 \cup g := -1); \\
&\quad t := 0; \\
&\quad \{w' = v, v' = u, e' = f, f' = g, t' = 1 \,\&\, t \le 1\}^{d} \\
&)^{\times} \rangle \, (w-e)^2 \le 1
\end{aligned}
$$

EVE at e plays Angel's part controlling g

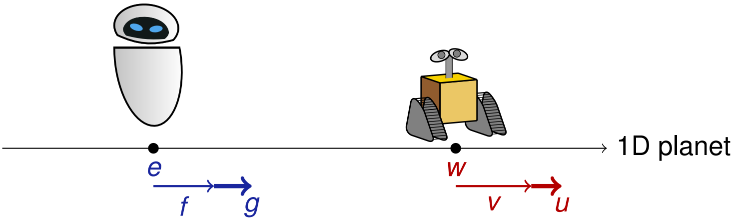
WALL·E at w plays Demon's part controlling u



$$
\begin{aligned}
&(w-e)^2 \le 1 \land v = f \rightarrow \\
&\langle \big((u := 1 \cap u := -1); \\
&\quad (g := 1 \cup g := -1); \\
&\quad t := 0; \\
&\quad \{w' = v, v' = u, e' = f, f' = g, t' = 1 \,\&\, t \le 1\}^d \\
&\big)^\times \rangle \, (w-e)^2 \le 1
\end{aligned}
$$

EVE at e plays Angel's part controlling g

WALL·E at w plays Demon's part controlling u and world time

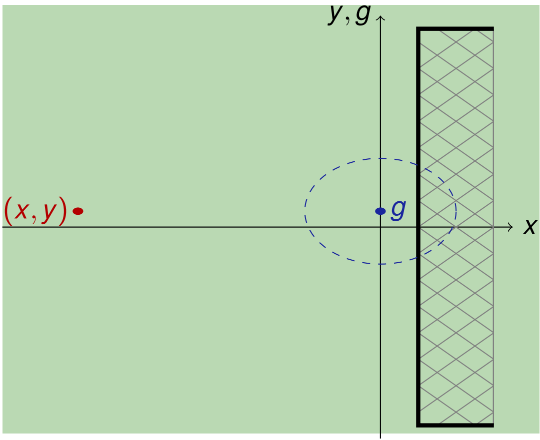


$$(w-e)^2 \le 1 \land v = f \rightarrow$$
$$[((u := 1 \cap u := -1);$$
$$(g := 1 \cup g := -1);$$
$$t := 0;$$
$$\{w' = v, v' = u, e' = f, f' = g, t' = 1 \,\&\, t \le 1\}$$
$$)^{\times}]\ (w-e)^2 > 1$$

WALL·E at w plays Demon's part controlling u and world time

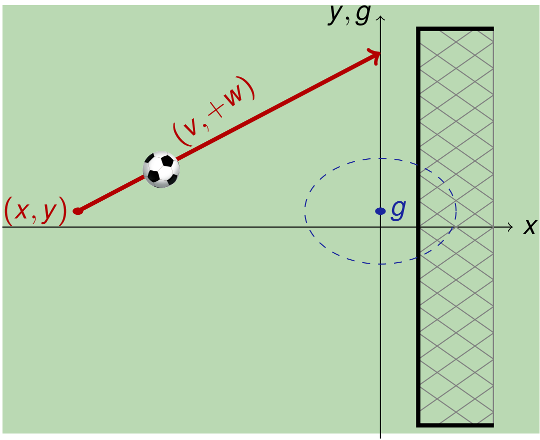
EVE at e plays Angel's part controlling g

Example: Goalie in Robot Soccer



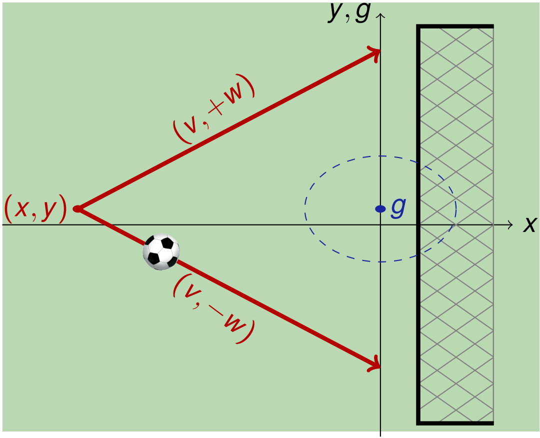
$$x < 0 \land v > 0 \land y = g \rightarrow \Big\langle (w := +w \cap w := -w); \Big((u := +u \cup u := -u); \{x' = v, y' = w, g' = u\} \Big)^* \Big\rangle \, x^2 + (y-g)^2 \leq 1$$

Example: Goalie in Robot Soccer



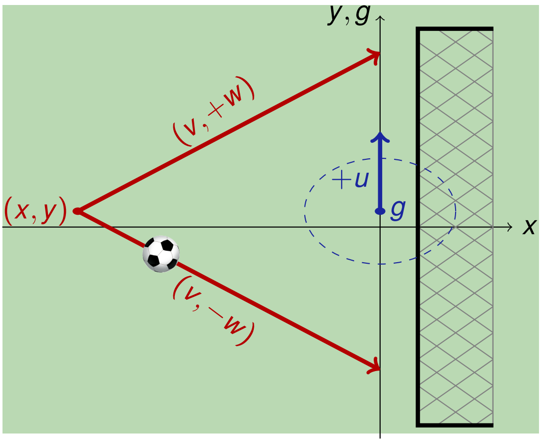
$$x < 0 \land v > 0 \land y = g \rightarrow$$
$$\big\langle (w := +w \cap w := -w);$$
$$\big((u := +u \cup u := -u); \{x' = v, y' = w, g' = u\}\big)^* \big\rangle\, x^2 + (y-g)^2 \le 1$$

Example: Goalie in Robot Soccer



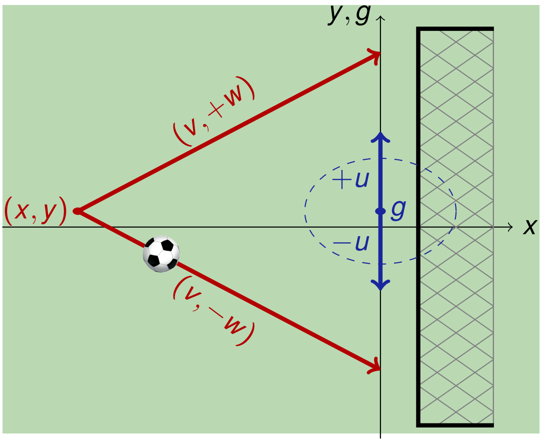
$$x<0 \land v>0 \land y=g \rightarrow$$
$$\langle (w:=+w \cap w:=-w);$$
$$\big((u:=+u \cup u:=-u);\{x'=v, y'=w, g'=u\}\big)^*\rangle\, x^2+(y-g)^2 \le 1$$

Example: Goalie in Robot Soccer



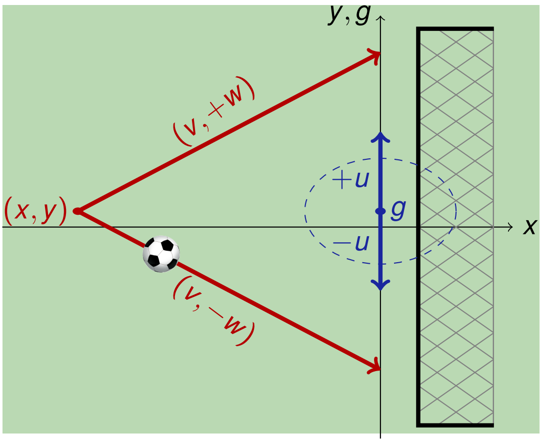
$$x < 0 \land v > 0 \land y = g \rightarrow$$
$$\langle (w := +w \cap w := -w);$$
$$\big((u := +u \cup u := -u); \{x' = v, y' = w, g' = u\}\big)^* \rangle\, x^2 + (y-g)^2 \leq 1$$

Example: Goalie in Robot Soccer



$$x < 0 \land v > 0 \land y = g \rightarrow$$
$$\Big\langle (w := +w \cap w := -w);$$
$$\big((u := +u \cup u := -u); \{x' = v, y' = w, g' = u\}\big)^* \Big\rangle\, x^2 + (y - g)^2 \leq 1$$

Example: Goalie in Robot Soccer



Goalie's Secret

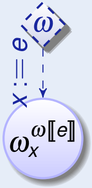
$$\left(\frac{x}{v}\right)^2 (u-w)^2 \le 1 \land$$

$$x < 0 \land v > 0 \land y = g \rightarrow$$
$$\langle (w := +w \cap w := -w);$$
$$\big((u := +u \cup u := -u); \{x' = v, y' = w, g' = u\}\big)^* \rangle\, x^2 + (y-g)^2 \le 1$$

Outline

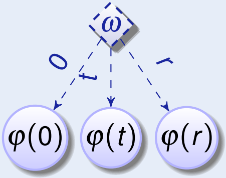
Definition (Hybrid game α: operational semantics)

$x := e$

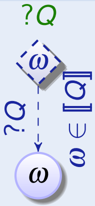


Definition (Hybrid game α: operational semantics)

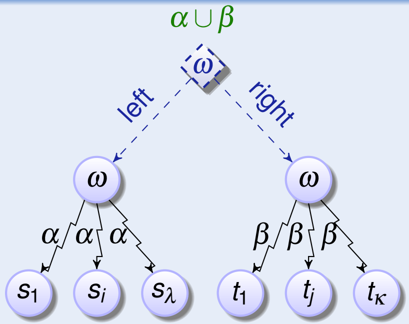
$x' = f(x) \,\&\, Q$



Definition (Hybrid game α: operational semantics)

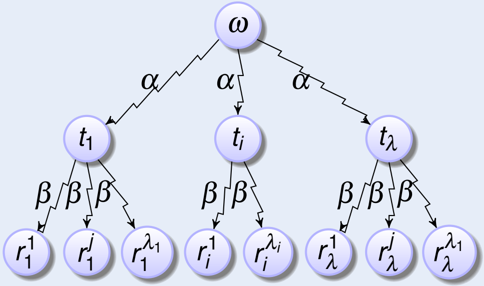


Definition (Hybrid game α: operational semantics)

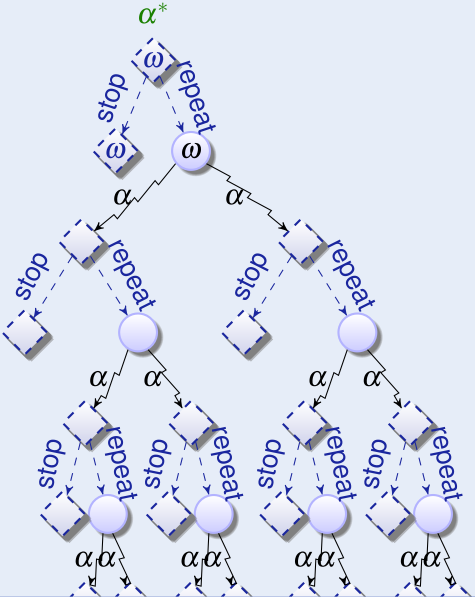


Definition (Hybrid game α: operational semantics)

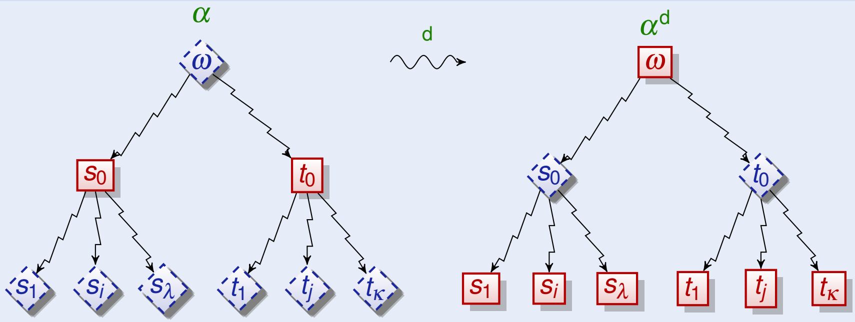
$\alpha;\beta$



Definition (Hybrid game α: operational semantics)

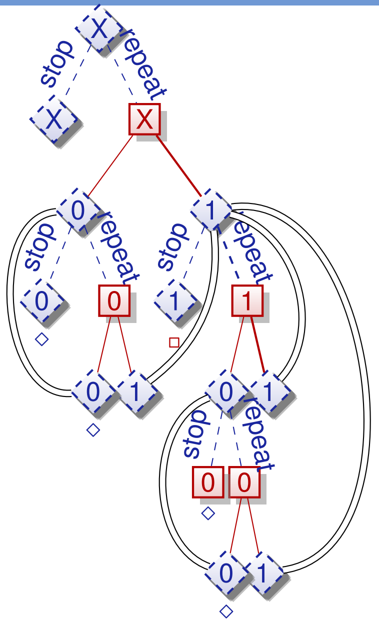


Definition (Hybrid game α: operational semantics)



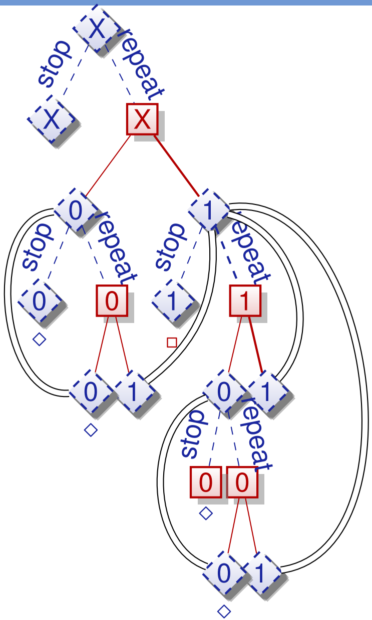
Filibusters

$$\langle (x := 0 \cap x := 1)^* \rangle x = 0$$



$\langle (x := 0 \cap x := 1)^* \rangle x = 0$

$\overset{\text{wfd}}{\rightsquigarrow}$ false unless $x = 0$

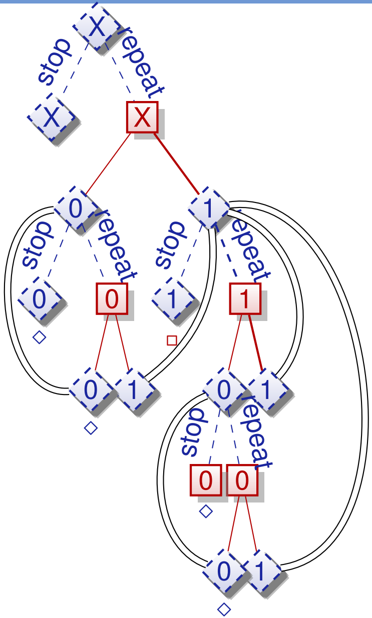


$$\langle (x' = 1^{d}; x := 0)^{*} \rangle x = 0$$

$$\langle (x := 0; x' = 1^{d})^{*} \rangle x = 0$$

$$\langle (x := 0 \cap x := 1)^{*} \rangle x = 0$$

$\overset{\text{wfd}}{\rightsquigarrow}$ false unless $x = 0$



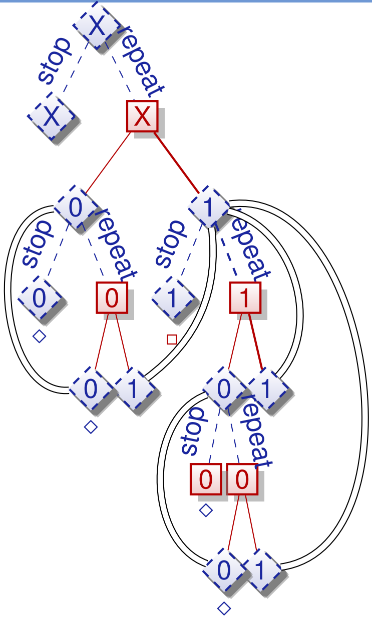
$\overset{<\infty}{\rightsquigarrow}$ true

$$\langle (x' = 1^{\mathrm{d}}; x := 0)^* \rangle x = 0$$

$$\langle (x := 0; x' = 1^{\mathrm{d}})^* \rangle x = 0$$

$$\langle (x := 0 \cap x := 1)^* \rangle x = 0$$

$\overset{\text{wfd}}{\rightsquigarrow}$ false unless $x = 0$



$\overset{<\infty}{\rightsquigarrow}$ true

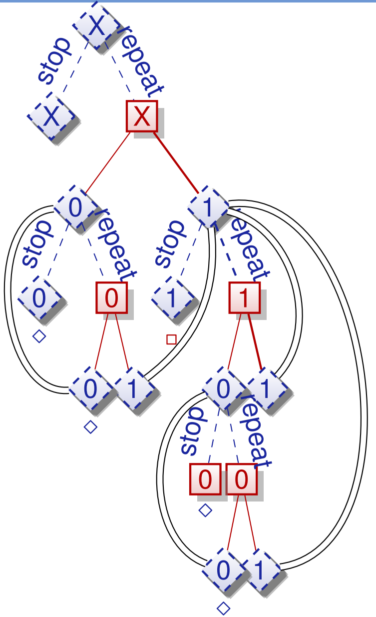
$\langle(x' = 1^{\mathrm{d}}; x := 0)^*\rangle x = 0$

$\langle(x := 0; x' = 1^{\mathrm{d}})^*\rangle x = 0$

$\langle(x := 0 \cap x := 1)^*\rangle x = 0$

$\overset{\mathrm{wfd}}{\rightsquigarrow}$ false unless $x = 0$

Well-defined games
can't be postponed forever



Outline

Differential Game Logic: Syntax

Discrete Assign | Test Game | Differential Equation | Choice Game | Seq. Game | Repeat Game | Dual Game

Definition (Hybrid game α)

$$\alpha, \beta \ ::= \ x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha ; \beta \mid \alpha^* \mid \alpha^d$$

Definition (dGL Formula P)

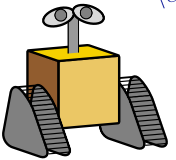
$$P, Q \ ::= \ e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x \, P \mid \exists x \, P \mid \langle \alpha \rangle P \mid [\alpha] P$$

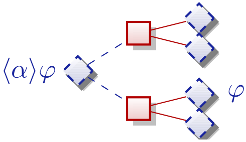
All Reals | Some Reals | Angel Wins | Demon Wins

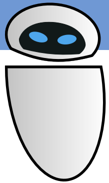
Summary

differential game logic

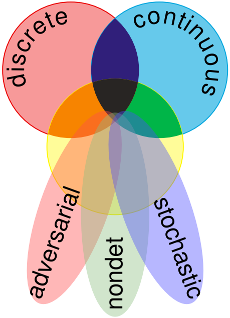
$\text{dGL} = \text{GL} + \text{HG} = \text{dL} + {}^{d}$







- Differential game logic
- Logic for hybrid games
- Compositional PL + logic
- Discrete + continuous + adversarial
- Operational semantics (informally)

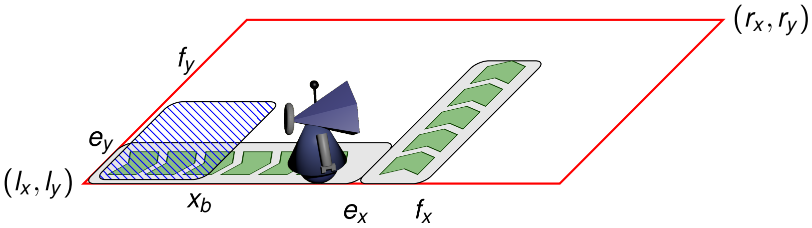


Next chapter

1. Formal semantics

Outline

Example: Robot Factory Decentralized Automation



Model

- (x, y) robot coordinates
- (v_x, v_y) velocities
- conveyor belts may instantaneously increase robot's velocity by (c_x, c_y)

Primary objectives of the robot

- Leave within time ε
- Never leave outer

Challenges

- Distributed, physical environment
- Possibly conflicting secondary objectives

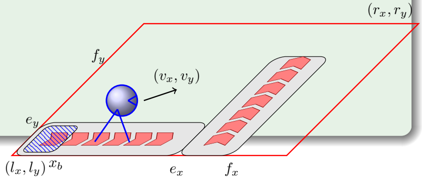
Robot Factory Automation (*RF*)

Example (Robot-Demon vs. Angel-Factory Environment)

$$\Big(\big(?true \cup (?(x < e_x \wedge y < e_y \wedge \mathrm{eff}_1 = 1);\ v_x := v_x + c_x;\ \mathrm{eff}_1 := 0) \qquad \text{// belt}$$

$$\cup (?(e_x \leq x \wedge y \leq f_y \wedge \mathrm{eff}_2 = 1);\ v_y := v_y + c_y;\ \mathrm{eff}_2 := 0)\big);$$

$$\Big)^*$$



Robot Factory Automation (*RF*)

Example (Robot-Demon vs. Angel-Factory Environment)

$$\Big(\big(?\mathit{true} \cup (?(x < e_x \land y < e_y \land \mathrm{eff}_1 = 1);\ v_x := v_x + c_x;\ \mathrm{eff}_1 := 0) \qquad \text{// belt}$$

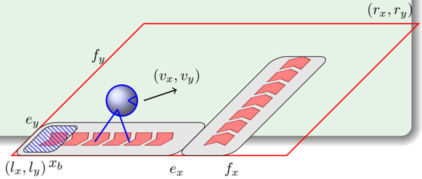
$$\cup (?(e_x \le x \land y \le f_y \land \mathrm{eff}_2 = 1);\ v_y := v_y + c_y;\ \mathrm{eff}_2 := 0) \big);$$

$$(\, a_x := *;\ ?(-A \le a_x \le A);$$

$$a_y := *;\ ?(-A \le a_y \le A); \qquad \text{// “independent” robot acceleration}$$

$$t_s := 0 \,)^{\mathrm{d}};$$

$$\Big)^*$$



Example (Robot-Demon vs. Angel-Factory Environment)

$$\Big(\big(?true \cup (?(x < e_x \wedge y < e_y \wedge \mathrm{eff}_1 = 1);\ v_x := v_x + c_x;\ \mathrm{eff}_1 := 0)$$ // belt

$$\cup (?(e_x \le x \wedge y \le f_y \wedge \mathrm{eff}_2 = 1);\ v_y := v_y + c_y;\ \mathrm{eff}_2 := 0)\big);$$

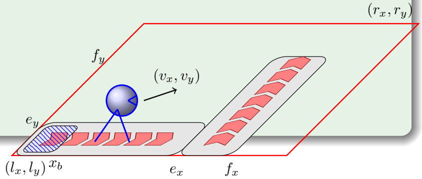
$$(a_x := *;\ ?(-A \le a_x \le A);$$

$$a_y := *;\ ?(-A \le a_y \le A);$$ // "independent" robot acceleration

$$t_s := 0)^d;$$

$$(x' = v_x, y' = v_y, v_x' = a_x, v_y' = a_y, t' = 1, t_s' = 1\ \&\ t_s \le \varepsilon);$$

$$\Big)^*$$



Example (Robot-Demon vs. Angel-Factory Environment)

$$\Big(\big(?true \cup (?(x < e_x \wedge y < e_y \wedge \mathrm{eff}_1 = 1);\ v_x := v_x + c_x;\ \mathrm{eff}_1 := 0) \qquad \text{// belt}$$

$$\cup (?(e_x \le x \wedge y \le f_y \wedge \mathrm{eff}_2 = 1);\ v_y := v_y + c_y;\ \mathrm{eff}_2 := 0)\big);$$

$$(a_x := *;\ ?(-A \le a_x \le A);$$

$$a_y := *;\ ?(-A \le a_y \le A); \qquad \text{// “independent” robot acceleration}$$

$$t_s := 0\ )^d;$$

$$\big((x' = v_x, y' = v_y, v_x' = a_x, v_y' = a_y, t' = 1, t_s' = 1\ \&\ t_s \le \varepsilon\ );$$

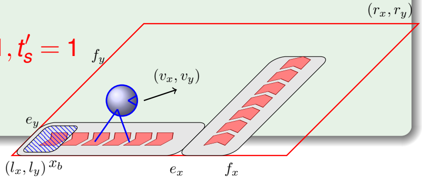
$$\cap (?(a_x v_x \le 0 \wedge a_y v_y \le 0)^d; \qquad \text{// brake}$$

$$\text{if } v_x = 0 \text{ then } a_x := 0 \text{ fi}; \qquad \text{// per direction: no time lock}$$

$$\text{if } v_y = 0 \text{ then } a_y := 0 \text{ fi};$$

$$(x' = v_x, y' = v_y, v_x' = a_x, v_y' = a_y, t' = 1, t_s' = 1$$

$$\&\ t_s \le \varepsilon \wedge a_x v_x \le 0 \wedge a_y v_y \le 0))\big)\Big)^*$$



Robot Factory Automation (*RF*)

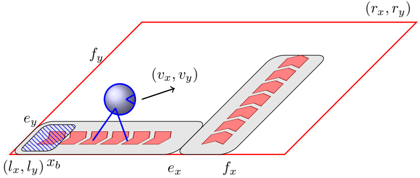
Proposition (Robot stays in □)

$$\models (x = y = 0 \land v_x = v_y = 0 \land \text{Controllability Assumptions})$$
$$\rightarrow [RF](x \in [l_x, r_x] \land y \in [l_y, r_y])$$

Proposition (Stays in □ and leaves ▨ on time)

$RF|_x$: RF projected to the x-axis

$$\models (x = 0 \land v_x = 0 \land \text{Controllability Assumptions})$$
$$\rightarrow [RF|_x](x \in [l_x, r_x] \land (t \geq \varepsilon \rightarrow x \geq x_b))$$



André Platzer.
Logical Foundations of Cyber-Physical Systems.
Springer, Switzerland, 2018.
URL: http://www.springer.com/978-3-319-63587-3,
doi:10.1007/978-3-319-63588-0.

André Platzer.
Differential game logic.
ACM Trans. Comput. Log., 17(1):1:1–1:51, 2015.
doi:10.1145/2817824.

André Platzer.
Logics of dynamical systems.
In LICS [12], pages 13–24.
doi:10.1109/LICS.2012.13.

André Platzer.
Logic & proofs for cyber-physical systems.
In Nicola Olivetti and Ashish Tiwari, editors, *IJCAR*, volume 9706 of
LNCS, pages 15–21, Berlin, 2016. Springer.
doi:10.1007/978-3-319-40229-1_3.

André Platzer.
Differential dynamic logic for hybrid systems.
J. Autom. Reas., 41(2):143–189, 2008.
doi:10.1007/s10817-008-9103-8.

André Platzer.
A complete uniform substitution calculus for differential dynamic logic.
J. Autom. Reas., 59(2):219–265, 2017.
doi:10.1007/s10817-016-9385-1.

André Platzer.
Differential hybrid games.
ACM Trans. Comput. Log., 18(3):19:1–19:44, 2017.
doi:10.1145/3091123.

André Platzer.
The complete proof theory of hybrid systems.
In LICS [12], pages 541–550.
doi:10.1109/LICS.2012.64.

André Platzer.

A complete axiomatization of quantified differential dynamic logic for distributed hybrid systems.
Log. Meth. Comput. Sci., 8(4:17):1–44, 2012.
Special issue for selected papers from CSL'10.
doi:10.2168/LMCS-8(4:17)2012.

André Platzer.
Stochastic differential dynamic logic for stochastic hybrid programs.
In Nikolaj Bjørner and Viorica Sofronie-Stokkermans, editors, *CADE*, volume 6803 of *LNCS*, pages 446–460, Berlin, 2011. Springer.
doi:10.1007/978-3-642-22438-6_34.

Jan-David Quesel and André Platzer.
Playing hybrid games with KeYmaera.
In Bernhard Gramlich, Dale Miller, and Ulrike Sattler, editors, *IJCAR*, volume 7364 of *LNCS*, pages 439–453, Berlin, 2012. Springer.
doi:10.1007/978-3-642-31365-3_34.

Logic in Computer Science (LICS), 2012 27th Annual IEEE Symposium on, Los Alamitos, 2012. IEEE.