Efficient Local Search for Nonlinear Real Arithmetic

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Outline

1. Problem - Nonlinear Real Arithmetic
   - Search Space of SMT(NRA)
   - Current Existing Methods
2. Incremental Computation of Variable Scores
   - Scoring Boundary for Arithmetic Variable
   - Incremental Computation
3. Temporary Relaxation of Equality (Non-Strick) Constraints
   - Difficulty in Local Search
   - Relaxation Method
4. Experiment

Outline

Syntax of SMT(NRA)

polynomial: $p ::= x \mid c \mid p + p \mid p - p \mid p \times p$
atoms: $a ::= b \mid p = 0 \mid p > 0 \mid p < 0$
formula: $f ::= a \mid \neg f \mid f \wedge f \mid f \vee f$

SMT: Determine whether the formula is satisfied by some assignment (local search focuses), or prove unsat

Example:
$x^2 + y^2 \leq 1 \wedge x + y < 1 \wedge x + z > 0$
assignment with $\{x \to 0, y \to 0, z \to 1\}$ satisfies all clauses.

Fragment of Local Search (1)

Input : A set of clauses F
Output: An assignment of variables that satisfy F, or failure

Initialize assignment to variables;
while $\top$ **do**
  if *all clauses satisfied* **then**
    return *success with assignment;*
  end
  if *time or step limit reached* **then**
    return *failure;*
  end
  Critical move procedure.
end

Algorithm 1: Basic Fragment of Local Search[a]

[a] Shaowei Cai, Bohan Li, and Xindi Zhang. "Local Search for SMT on Linear Integer Arithmetic". In: *Computer Aided Verification - 34th International Conference, CAV 2022, Haifa*

Fragment of Local Search (2)

$var, new\_value, score \leftarrow$ best move according to make-break score;
if *score* $>$ *0* **then**
  Move var to new_value;
end
else
  Update clause weight;
end

repeat
  $cls \leftarrow$ random unsatisfied clause;
  $var, new\_value, score \leftarrow$ critical move making cls satisfied;
  if *score* $\neq -\infty$ **then**
    move var to new_value;
  end
until *3 times*;
if *no move performed* **then**
  Move some variables in unsatisfied clauses;
end

Local Search for SAT and SMT

LS \ Problem	SAT	SMT
Operation (Move)	Flip	Critical Move
Score Definition	Weighted unsat clauses	
Score Computation	Cached score	No Caching, time costly

What LS for SAT brings us:

Maintain scoring information after each iteration.

Difficulty:

Predetermine critical move shift value.

Our Solution:

Introduce Scoring Boundaries.

Outline

Make-break Intervals

make-break intervals[1]
Combination of (in)feasible intervals of arithmetic varaible x with respect to all clauses.

Example

Current assignment: $\{x \mapsto 1, y \mapsto 1, z \mapsto 1\}$
Calculate infeasible set for each clause.

- $x^2 + y^2 \leq 1$ (unsat): $(-\infty, 0) \cup (0, \infty)$.
- $x + y < 1$ (unsat): $[0, \infty)$.
- $x + z > 0$ (sat): $(-\infty, -1]$.

Combined information: x: $(-\infty, -1] \mapsto 0$, $(-1, 0) \mapsto 1$, $[0, 0] \mapsto 1$, $(0, \infty) \mapsto 0$.

[1] Bohan Li and Shaowei Cai. "Local Search For SMT On Linear and Multilinear Real Arithmetic". In: *CoRR* abs/2303.06676 (2023). accepted for FMCAD.. DOI: 10.48550/arXiv.2303.06676. arXiv: 2303.06676. URL: https://doi.org/10.48550/arXiv.2303.06676.

Traditional Computation

Input : unsat clauses F
Output: Best critical move (variable, value)
foreach *variable* v *in unsat clauses* **do**
  foreach *unsat clause* c *with* v **do**
    Compute interval-score info of v in c.
  end
  Combine interval-score information.
  Update best var-value move.
end
return *best critical move*

Repeated computation:

- variable's (in)feasible set
- clause's sat staus

Boundary

Definition. A quadruple $\langle val, is\_open, is\_make, cid \rangle$, where val is a real number, is_open and is_make are boolean values, and cid is a clause identifier.

Meaning

- val : make-break value.
- is_open : active or not at val point.
- is_make : make or break, increase or decrease score.
- cid : causing clause.

Sorting First ordered by val, then by is_open $(\bot < \top)$.

Boundary

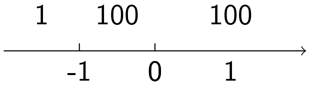
Current assignment: $\{x \mapsto 1, y \mapsto 1, z \mapsto 1\}$

- $x^2 + y^2 \leq 1$: starting score 0, boundary set $\{(0, \bot, \top, 1), (0, \top, \bot, 1)\}$, indicating no change for large negative values, *make* at boundary $[0, \cdots$, followed by *break* at boundary $(0, \cdots$.
- $x + y < 1$: starting score 3, boundary set $\{(0, \bot, \bot, 2)\}$, indicating *make* at large negative values, and *break* at boundary $[0, \ldots$.
- $x + z > 0$: starting score -2, boundary set $\{(-1, \top, \top, 3)\}$, indicating *break* at large negative values, and *make* at boundary $(-1, \ldots$.

sorted boundary set:
$\{(-1, \top, \top, 3), (0, \bot, \top, 1), (0, \bot, \bot, 2), (0, \top, \bot, 1)\}$

Boundary Example

boundary set:
$\{(-1, \top, \top, 3), (0, \bot, \top, 1), (0, \bot, \bot, 2), (0, \top, \bot, 1)\}$



Starting score: Score when x moves to -∞.
Maintain and Change: We maintain the boundary info for all arithmetic varaibles, unless the neighbour does a critical move.

Algorithm for computing boundary

Input : Variable v that is modified
Output: Make-break score for all variables

```
S ← {} ;            // set of updated variables
for clause cls that contains v do
    for variable v' appearing in cls do
        add v' to S;
        recompute starting score and boundary of v'
          with respect to cls;
    end
end
for variable v' in S do
    recompute best critical move and score in terms
      of boundary information;
end
```

Outline

Previous Algorithm and Difficulty

Number complexation in Local Search

When a variable chooses a complex value, the iteration is much slower, but sometimes we have to ...

Reference[2] ignores equalities constraints due to its accurate value complexation.

We introduce Relaxation into strick equality constraints, resulting in temporary interval candidate (rather than a point).

[2] Bohan Li and Shaowei Cai. "Local Search For SMT On Linear and Multilinear Real Arithmetic". In: *CoRR* abs/2303.06676 (2023). accepted for FMCAD.. DOI: 10.48550/arXiv.2303.06676. arXiv: 2303.06676. URL: https://doi.org/10.48550/arXiv.2303.06676.

Algebraic Numbers Situation

Definition (Complexity of values)

We define a preorder $\prec_c$ on algebraic numbers as follows. $x \prec_c y$ if x is rational and y is irrational, or if both x and y are rational numbers, and the denominator of x is less than that of y. We write $x \sim_c y$ if neither $x \prec_c y$ nor $y \prec_c x$.

Algebraic (irrational) numbers have the largest complexity.

Relaxation

Example

Given assignment $\{x \mapsto 1, y \mapsto 1\}$

$$z^2 = x^2 + y^3 \qquad\qquad z^3 \geq 5x^2+y \vee z^3 \leq 3x+3y$$

Both situations force z to an irrational number.

Relaxation

- If the constraint is of the form $p = 0$, it is relaxed into the pair of inequalities $p < \epsilon_p$ and $p > -\epsilon_p$.
- If the constraint is of the form $p \geq 0$, it is relaxed into $p > -\epsilon_p$. Likewise, if the constraint is of the form $p \leq 0$, it is relaxed into $p < \epsilon_p$.

Local Search with Relaxation

Input : A set of clauses F
Output: An assignment of variables that satisfy F, or failure

```
Initialize assignment to variables;
while ⊤ do
    if all clauses satisfied then
        success ← find exact solution;
        if success then
            return success with assignment;
        end
        else
            Restore relaxed constraints to original form;
            success ← find exact solution by limited local search;
            if success then
                return success with assignment;
            end
        end
    end
    if time or step limit reached then
        return failure;
    end
    Proceed traditional local search (slack).
end
```

Outline

Overall Result

Category	#inst	Z3	cvc5	Yices	Ours	Unique
20161105-Sturm-MBO	120	0	0	0	**88**	88
20161105-Sturm-MGC	2	**2**	0	0	0	0
20170501-Heizmann	60	3	1	0	**8**	6
20180501-Economics-Mulligan	93	**93**	89	91	90	0
2019-ezsmt	61	**54**	51	52	19	0
20200911-Pine	237	**235**	201	**235**	224	0
20211101-Geogebra	112	**109**	91	99	101	0
20220314-Uncu	74	73	66	**74**	70	0
LassoRanker	351	155	**304**	122	272	13
UltimateAtomizer	48	**41**	34	39	27	2
hycomp	492	**311**	216	227	304	11
kissing	42	**33**	17	10	**33**	1
meti-tarski	4391	**4391**	4345	4369	4351	0
zankl	133	70	61	58	**100**	27
Total	6216	5570	5476	5376	**5687**	148

Comparison

References I

[CLZ22] Shaowei Cai, Bohan Li, and Xindi Zhang. "Local Search for SMT on Linear Integer Arithmetic". In: *Computer Aided Verification - 34th International Conference, CAV 2022, Haifa, Israel, August 7-10, 2022, Proceedings, Part II*. Ed. by Sharon Shoham and Yakir Vizel. Vol. 13372. Lecture Notes in Computer Science. Springer, 2022, pp. 227–248. DOI: 10.1007/978-3-031-13188-2_12. URL: https://doi.org/10.1007/978-3-031-13188-2_12.

References II

[LC23] Bohan Li and Shaowei Cai. “Local Search For SMT On Linear and Multilinear Real Arithmetic”. In: *CoRR* abs/2303.06676 (2023). accepted for FMCAD. DOI: 10.48550/arXiv.2303.06676. arXiv: 2303.06676. URL: https://doi.org/10.48550/arXiv.2303.06676.