Lie Groups and Lie Algebra

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1 Introduction

What do a circle and the real number line under the addition operation have in common? Well, they are both members of what we call a Lie group. A rigorous understanding of Lie Groups requires a solid grasp of differential geometry and manifold theory. However, rudimentary understanding is possible through knowledge of the material in a one semester abstract algebra course or even just linear algebra as I discovered recently. Lie Groups are groups which have a differentiable manifold structure over the field they are associated with. A differentiable manifold being a manifold that is locally similar to a vector space such that differential calculus can be applied on it. The applicability of differential calculus might lead you think that of Lie Groups being continuous and that is indeed the case. Lie Groups are commonly used to study continuous symmetry as is the case with a circle. This paper will focus on the definition of a Lie Group, how they can be used to employ Lie Algebra, and some properties Lie Groups and Lie Algebras exhibit exhibits. They are used in analysis, combinatorics, number theory, and a wide variety of mathematical topics. In the paper, I will be proving the homomorphism of Lie Groups.

2 Groups

To begin with, we need to define what a group is. A group is a set together with a law of composition that fulfills some axioms. Before we proceed, the law of composition is a function that changes an ordered pair from the set back to an element of the set. An example is addition on the set of real numbers, $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Addition takes two numbers from $\mathbb{R} \times \mathbb{R}$ say (2,3) and returns an element of $\mathbb{R}$ in this case 5. Going back to the definition of groups, the axioms a set G and a corresponding law of composition denotes as $\circ$ must fulfill to be a group are as follows[8].

- The law of composition on the set is associative.

$$a \circ (b \circ c) = (a \circ b) \circ c$$

- An identity element e exists such that e $\in$ G and for every element a $\in$ G

$$a \circ e = e \circ a = a$$

- For every element a $\in$ G, there exists an inverse element a^{-1} $\in$ G such that

$$a \circ a^{-1} = a^{-1} \circ a = e$$

Example 1. *The set of real numbers and addition operation($\mathbb{R}$, +) is a group.*

3 Manifolds

The next object we need to define is a manifold. A topological m-manifold, in the most simple sense, is a smooth entity that can be broken down into simpler components that can be parameterized. An example you might be familiar with is a curve in a Cartesian plane as can be seen in Fig. 1. The curve is a one dimensional manifold parameterized by a certain quantity say time. The formal definition of a topological m-manifold is as follows.

Definition 1. *Let m be a non-negative integer. A topological $\boldsymbol{m}$-manifold is a Hausdorff* *and second-countable topological space M with the following property: For each $a \in M$, there is an open neighborhood of a homeomorphic to an open subset of $\mathbb{R}^m$. [7]*

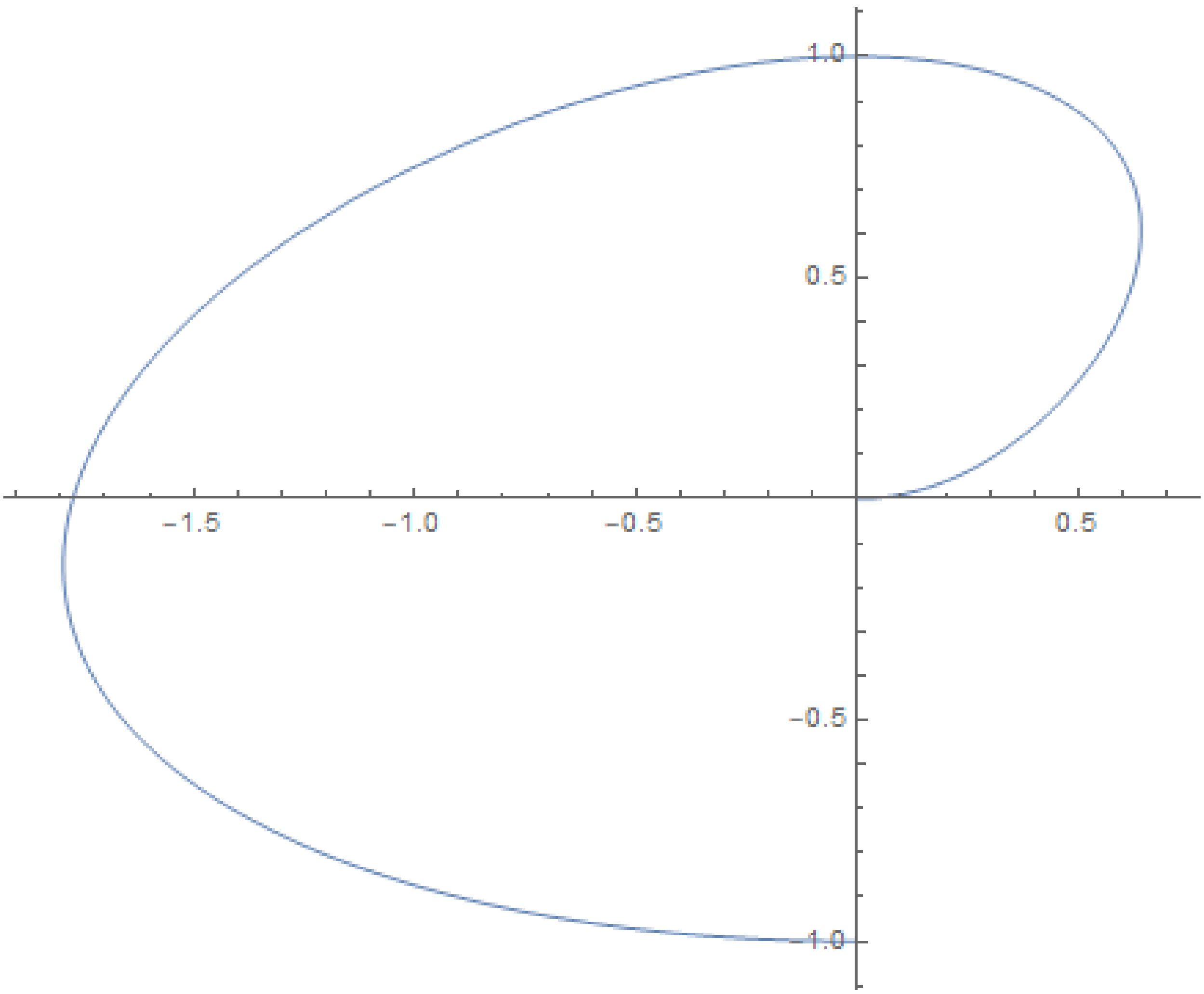

Figure 1: A parameterized curve in the Cartesian plane. The function is f(t) = ($\sqrt{t}$cos(t), sin(t)).The parameter of the curve is t which ranges from 0 to $\frac{3\pi}{2}$.

Knowledge of two categories of topological manifolds is required to understand what exactly Lie Groups are. These two categories are analytic and smooth differentiable manifolds. Analytic manifolds are just differentiable manifolds with analytic transition maps so let us start with the definition and some properties of differentiable manifolds.

3.1 Smooth Manifolds

We need to give a quick definition to what an atlas is before we capable of defining what a smooth manifold is. An atlas denoted as $\mathfrak{U}$ is a collection of coordinate charts with a smooth transition function on a manifold. A smooth manifold is then a topological manifold together with a maximal atlas[6]. This maximal atlas is unique and atlas contains every other atlas in a locally Euclidean space.

4 Lie Groups

We are now finally ready to define Lie Groups. A Lie Group is both a manifold and a group. A property that defines Lie Groups is that the group operations on Lie Groups are smooth. Thus, a more formal definition of Lie Groups is as follows.

Definition 2. *A Lie group is a group that is also a smooth manifold such that the group operations of multiplication and inversion are infinitely differentiable.*

The dimension of the Lie Group in consideration is the dimension of its manifold aspect [7]. A few elements of the Lie Group you might be familiar with are the space $\mathbb{R}^n$ along with the operation of vector addition, a unit circle on the complex plane and the general linear group over the field $\mathbb{R}$. A unit circle in the complex plane is displayed in Fig. 2. The specially linear group $SL(n, \mathbb{R})$ which a subgroup of the general linear group $GL(n, \mathbb{R})$ and also a Lie Group can be written as follows.

Example 2. $SL(n, \mathbb{R}) = \{M \in \mathcal{M}(n, \mathbb{R}) : det(\mathcal{M} = 1)\}$,

where $\mathcal{M}(n, \mathbb{R})$ is the set of all n by n matrices over the field $\mathbb{R}$ [2]. Another property Lie Groups have is that they are locally similar around a point. This is due to the fact that left translation by an element of the Lie Group is a diffeomorphism of the group back to itself. Left translation is a bijective map such that for $x \in G$, $L_a : G \to G$ given by $x \mapsto a$ For

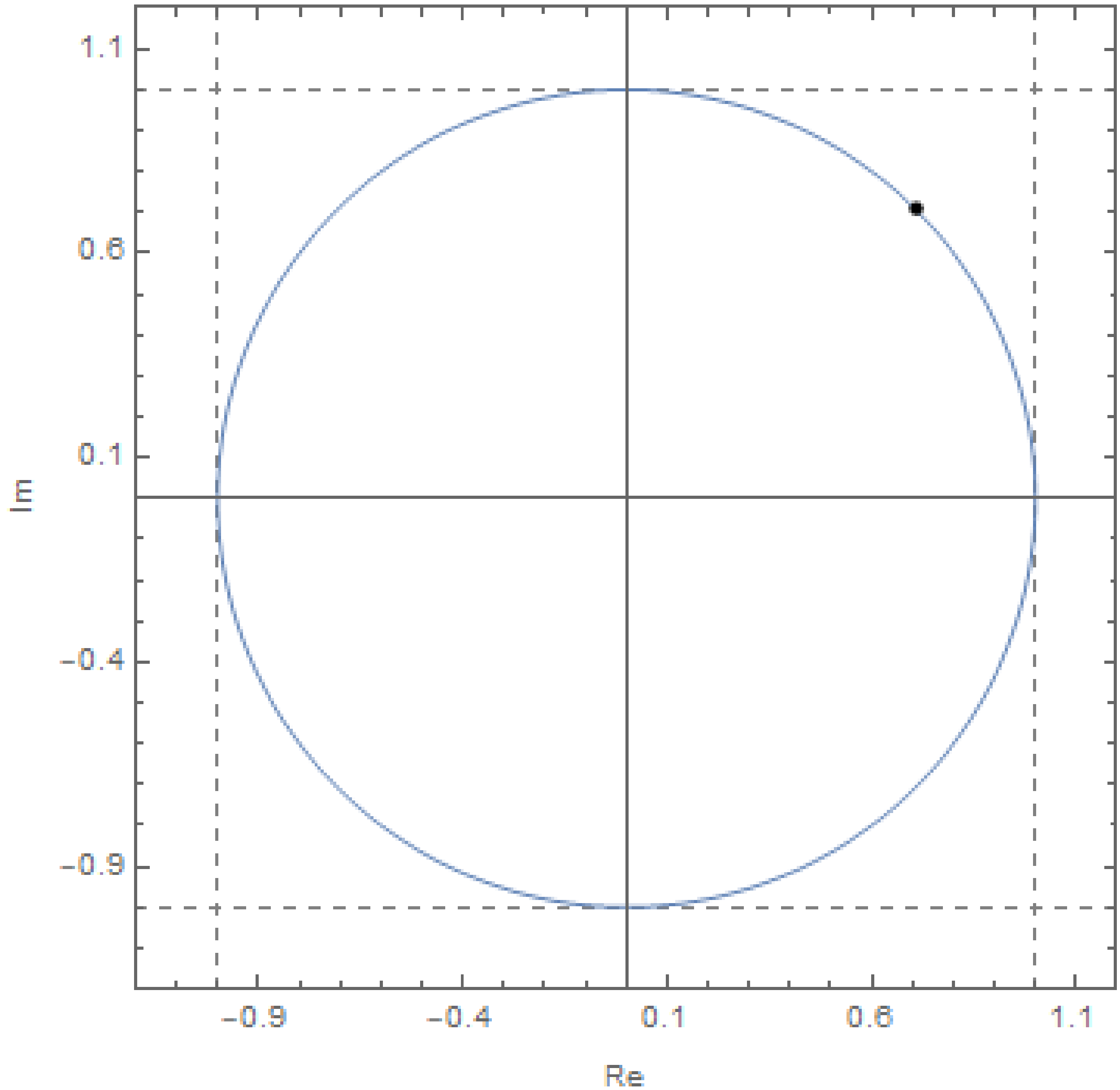

Figure 2: A unit circle in the complex plane. The y-axis is the imaginary axis while the x-axis is the real axis.

context, a diffeomorphism is an isomorphism of smooth manifolds. A diffeomorphism can be formally expressed as follows

$$L : G \to G : L(x) = gx,$$

where L is left translation map and g,x are elements of the group. Left translation is also sometimes called left multiplication. Let L be left multiplication defined by

$$L : G \to G : L(x) = gx,$$

then for $X \in G$, $(L)_*$ X denotes a left multiplication acting on X by L.

Lie Groups have a wide variety of application and connectivity with and to multiple branches of mathematics ranging from algebraic topology, analysis to group theory. They are also applicable in physics. One area of physics that uses Lie groups a lot is the study of symmetries in particle spins. The special unitary group SU is specifically used to study symmetries in particle physics. Poincare spheres are used in unitary presentations in quantum field theory[4].

5 Lie Group Homomorphism and Lie Algebra

A group homomorphism is a type of map between two groups that is similar to an isomorphism but also different since in contrast to an isomorphic map, a homomorphic map is not bijective. In fact, all linear maps are homomorphisms but not all homomorphisms are linear maps. A homomorphism between two groups (G, +) and $(H, \circ)$ can be expressed as

$$f(h + g) = f(h) \circ f(g) \tag{1}$$

where h,g $\in$ G. As is the case with other groups, it is possible to construct a map between two Lie Groups. Given the condition that this map is smooth and a group homomorphism, it is called a Lie Group homomorphism[6]. If a Lie Group homomorphism happens to also be a linear map,Thus referring back to Eq. 1, it should be clear that a group homomorphism of the identity of the group maps the identity back to itself. In order to use Lie Group homomorphism in a proof, I will introduce the concept of Lie algebra.

The local similarity of elements of a Lie Group around a point due to the diffeomorphism from left translation gives a unique property to a tangent space at the identity of the Lie group. The addition of a Lie bracket [,] to this tangent space T_eG makes it a Lie Algebra[6]. A property to note is that the differential of a Lie Group homomorphism becomes a Lie Algebra homomorphism[6].

Theorem 1. *Let G and H be Lie Groups with Lie Algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively. Suppose L: $G \rightarrow H$ is a Lie Group homomorphism. Then the differential $D(F)_e : \mathfrak{g} \rightarrow \mathfrak{h}$ is a Lie Algebra homomorphism[3].*

Proof. We will be using a direct proof. $D(F)_e$ is by definition a linear homomorphism from $\mathfrak{g}$ to $\mathfrak{h}$. So in order to show $D(F)_e$ is a Lie Algebra homomorphism, we need to show that $D(F)_e[v, w] = [D(F)_e v, D(F)_e w]$ for all $v, w \in \mathfrak{g}$. By definition, $D(F)_e[v, w] = D(F)_e[L^v, L^w]_e$.

Lemma 1. *Let F: $G \rightarrow H$ be a Lie Group homomorphism. Then $\forall X \in$ Lie(G), $\exists\ Y \in$ Lie(G), $Y = (F)_* X$.*

Proof for Lemma1. We will use a direct proof. The map F is a Lie Group homomorphism thus

$$F \circ L_g = L_{F(g)} \circ F.$$

Thus, for all g and h,

$$D(F)_{L_g(h)} \circ D\left(L_g\right)_h = D\left(L_{F(g)}\right)_{F(h)} \circ D(F)_h.$$

Given X = L^v for some $v \in T_eG$, and let let $Y = L^{D(F)_e v}$. We then have

$$Y_{F(g)} = D\left(L_{F(g)}\right)_e D(F)_e v = D(F)_g D\left(L_g\right)_e v = D(F)_g X_g$$

for all g, so $Y = (F)_* X$. T □

Lemma. 1 enables us to write $D(F)_e\left[L^v, L^w\right]_e$ as $\left((F)_*\left[L^v, L^w\right]\right)_e$. Since, $(F)_* L^v$ is a left invariant field taking $D(F)_e$ at the identity, $(F)_* L^v = L^{D(F)_e v}$, and similarly for w.Thus

$$D(F)_e\left[L^v, L^w\right]_e = \left[(F)_* L^v, (F)_* L^w\right]_e = \left[L^{D(F)_e v}, L^{D(F)_w}\right]_e.$$

This implies the result by definition [3]. Therefore, if L: G →H is a Lie Group homomorphism. Then the differential $D(F)_e : \mathfrak{g} \to \mathfrak{h}$ is a Lie Algebra homomorphism. Therefore, if L: G →H is a Lie Group homomorphism, then the differential $D(F)_e : \mathfrak{g} \to \mathfrak{h}$ is a Lie Algebra homomorphism. $\square$

6 Conclusion

Lie Groups are a fascinating group that are both a group and manifold. They have the property that the manifold they are based on is infinitely differentiable at every point. They help in the study of symmetries as they are locally similar around a point. A Lie algebra is a Lie group together with a Lie bracket. Lie algebras allow us to apply algebra to a Lie group thereby allowing us to understand the structure of the manifold more thoroughly. Lie groups are applicable in many areas of mathematics and physics. In fact, it could be said that Lie groups and algebras lie around the center of every math principle.

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