Banach Spaces:
Complete Normed Spaces

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"In Riemann, Hilbert or in Banach space Let superscripts and subscripts go their ways. Our symptotes no longer out of phase, We shall encounter, counting, face to face."

Stanislaw Lem

1 Introduction

The concept of functions have been around for quite a while now. Functions are basically just maps that take a point or a collection of points and transfers them into some other place. For a long time, functions were used in conjunction with values that occurred in reality be it astronomy, probability of events, or any other phenomenon[5]. It was Dirichlet who gave the first formal definition of an arbitrary function on some closed interval in 1837 [5].

The idea of mathematical "spaces" developed even later than that of functions. In 1844, Arthur Cayley and Hermann Grassmann independently published papers that drew the picture of dimensional spaces of n dimensions. Vector spaces were first formally introduced formally by Giuseppe Peano in 1888. Peano was also the first mathematician to introduce vector spaces with an inner product. Finally, in 1851, Bernhard Riemann formally introduced the idea of a function space. Riemann also later constructed the idea of function spaces with infinite dimensions. One famous function space is the Banach spaces.

Function spaces have a wide range of applications. They appear in many areas of mathematics including but not limited to functional analysis, set theory, topology, and algebraic topology. However, function spaces are not limited to mathematics, they are employed in other fields like physics. In physics the Hilbert space, another function space, is used for quantum mechanics formalism. Banach spaces are generalization of Hilbert spaces and this might be an indication that they can also be employed in the formalism of quantum mechanics. Although not as commonly seen, Banach spaces can also be used for formalism in quantum mechanics [3]

Before we begin to define what a normed space is we need to give a few key definitions. Consider a set S and a field $\mathbb{F}$. A field is set containing a minimum of two elements 0 and 1, with addition and multiplication satisfying the properties listed in the vector space properties 6. Some fields you might be familiar with are $\mathbb{R}$ and $\mathbb{C}$. A visual representation of $\mathbb{C}$ is presented in Fig. 1.

Let $x, y \in S$ and $z \in \mathbb{F}$. Then, a metric $d(,)$ is a function that maps an ordered pair (x, y) to some $u \in \mathbb{F}$ and follows the following properties [6]

1. $d(x,y) \geq 0$;
2. $d(x,y) = 0$ if and only if $x = y$;
3. $d(x,y) = d(y,x)$;
4. $d(x,z) \leq d(x,y) + d(y,z)$.

The set S with the metric d denoted as (S,d) is called a metric space.

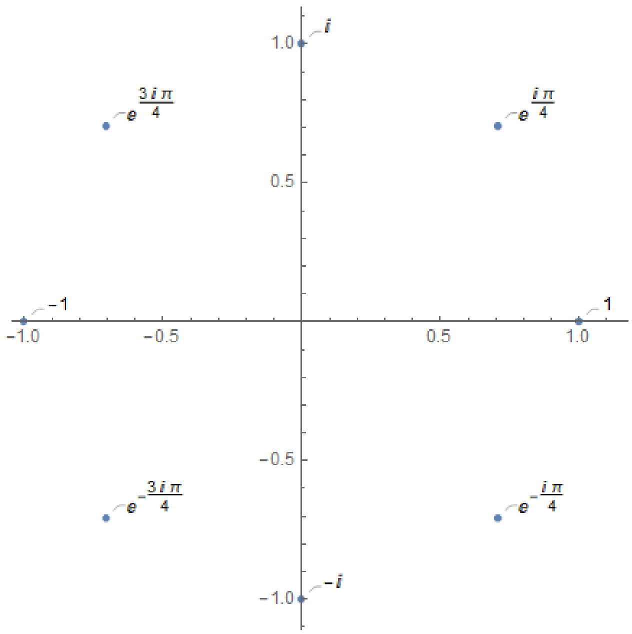

Figure 1: A visual representation of $\mathbb{C}$

2 Vector Spaces

Another key term is a vector space. Vector spaces are sets of functions with an algebraic structure and metric. A vector space V is a set X along with addition and multiplication on the set over a field $\mathbb{F}$. [9]. The properties a set needs to possess to be considered a vector space are as follows

1. additive identity
   - There exists $e \in X$ such that for any $a \in X$, $a + e = e$
2. additive inverse
   - For all $a \in X$, there exists $c \in X$ such that $a + c = 0$;
3. commutativity
   - for all $a, b \in X$. $a + b = b + a$;
4. associativity
   - for all $a, b, c \in X$, $a + (b + c) = (a + b) + c$ and $(uv)a = u(vc)$; for $u, v \in \mathbb{F}$;
5. multiplicative inverse
   - There exists $i \in \mathbb{F}$ for all $a \in X$ such that $ia = a$.
6. distributive
   - For all $a, b \in X$ and $u, v \in \mathbb{F}$, $u(a + b) = ua + ub$ and $a(u + v) = au + av$.

An instance of a vector space is presented in Ex. 1.

Example 1. $\mathbb{R}^2$ *is a vector space of* $\mathbb{R}$ *over* $\mathbb{R}$.

Vector spaces often contain a subspace. A subspace of a vector space V is a subset of the underlying set of the vector space that is also a subset. There are three sufficient conditions in order for a subset of V to be a subspace. One of these conditions is that the subset contains the additive identity of V, in the case of the V being a vector space over $\mathbb{R}$ or $\mathbb{C}$ the additive identity would be the number zero. Second the conditions is that the subset is closed under addition, meaning that the sum of any two elements of the subset is also an element of the subspace. Finally, the subspace is closed under scalar multiplication. For clarification on this last condition, let $x \in S$ where S is a subset of the vector space over $\mathbb{R}$, V. Then if $\lambda \in \mathbb{R}$, then $\lambda x \in U$[9].

3 Normed Spaces

In order to define the norm, we need to first define what an inner product is.

Definition 1. *The inner product* $\langle , \rangle$ *is a function that maps some ordered pair* (a, b)*, where* $a, b \in V$*, to some* $c \in \mathbb{F}$.

The inner product has the following properties [9]

1. $\langle a, b \rangle \geq 0$ for any $a, b \in V$;
2. $\langle a, b \rangle = 0$ if and only if either one of a or b is 0.
3. $\langle a + b, c \rangle = \langle a, c \rangle + \langle b, c \rangle$ for all $a, b, c \in \mathbb{F}$;
4. $\langle \lambda a, b \rangle = \lambda \langle a, b \rangle$ for all $\lambda \in \mathbb{F}$ and $a, b \in V$
5. $\langle a, b \rangle = \overline{\langle b, a \rangle}$ for all $a, b \in V$.

Suppose we have a vector space V over some field F. Then the norm of some $x \in V$ is

$$\|x\| = \sqrt{\langle x, x \rangle}.$$

The norm $\|.\|$ has the following properties [4]. For some $x \in V$ and $\lambda \in \mathbb{F}$,

1. $\|x\| \geq 0$;
2. $\|\lambda x\| = \lambda \|x\|$;
3. $\|x + y\| \leq \|x\| + \|y\|$.

The most important of the metrics used on vector spaces are normed. A vector space v along with the norm $\|.\|$ forms a normed vector space $(V, \|.\|)$. The metric on the normed vector space is defined as $d(x, y) = \|x - y\|$. An example of normed vector space is presented in Ex. 2 [2].

Example 2. *Suppose the norm* $\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p}$ *acting on a vector space* V*. Then the lp space is the set of all sequences* $x_n \in V$ *such that* $\|x\|_p$ *is finite.*

4 Banach Spaces

We are now finally ready to define what a Banach space. Banach spaces are complete normed spaces or more formally presented,

Definition 2. *A Banach space is a normed vector space for which every Cauchy sequence of the underlying set of the metric space converges.*

For the purpose of this paper, we are considering vector spaces over $\mathbb{R}$. It happens to be the case that normed spaces are also metric spaces. This can be verified by noticing that the properties of normed spaces fulfill the metric space properities. We define the completeness of a normed space by looking at its completness as a metric space [2]. We say a metric space (X,d) is complete if every Cauchy sequence $x_n \in X$ converges to some $x \in X$. A Cauchy sequence being a sequence (x_n) where for every $\epsilon > 0$ there exists a positive integer N such that if $n,m \geq N$, then $|x_n - x_m| < \epsilon$.

Let us consider a subspace of a Banach space. An interesting result is as follows.

Theorem 1. *A subspace of a Banach space is a Banach space if and only if it is closed.*

In order to prove this theorem, let us first define what the term "closed" means.

Definition 3. *A set is closed if it equal to its closure. In other words, a set is closed if it contains all its limit points.*

We shall prove Thm. 1 as follows.

Proof. Let the Banach space B be defined as follows

$$B = (V, \|.\|).$$

In order to prove the forward direction, suppose $X \subset V$ and let $(X, \|.\|)$ be a Banach space. Then, since $X \subset V$, if a Cauchy sequence $x_n \in X$ then $x_n \in V$. Then by the definition of Banach spaces, x_n converges to some value $v \in V$. Let $\epsilon > 0$ be given. Then there exists $N \in \mathbb{N}$ such that if $n \geq N$.

$$|x_n - x| < \epsilon.$$

In order to show $x \in X$, consider the completeness of V. Since V is complete and $X \subset V$, there exists some $x \in V$ that is also an element of X such that $x_n \to x$. A set is closed if any Cauchy sequence in the set converges to some value in the set[10]. Thus, X is closed .

For the reverse direction, suppose $X \subset V$ and let X be closed. Since X is closed, if any sequence $x_n \in X$ converges, it converges to some $x \in X$. Then by the same logic as the forward direction, if a Cauchy sequence is an element of X then it is an element of V and thus converges. Then since all Cauchy sequences in X converge, it forms a complete normed space(Banach Space). □

Although Banach spaces are often abstract and of higher dimension that we can possible comprehend, there are some visualizations that can aid when studying them. A visualization of a quasi-Banach space is presented in Fig. 2 [8].

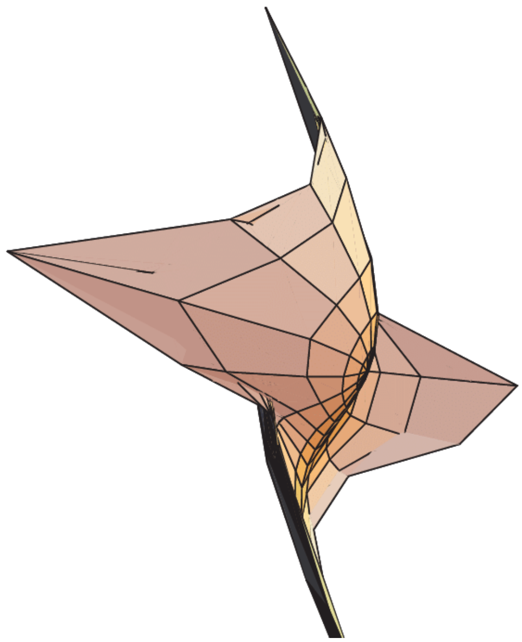

Figure 2: A visual representation of quasi-Banach space

5 Conclusion

Functions have a range of real world applications and they seem to appear in nature. Although considering these functions helps us understand and apply our understanding, it is also important to understand the structure of abstract functions. Spaces allow us to analyze some structure of a quantity we are considering. The structure the space has depends on what type of space we are considering. For example vector spaces posses vector properties of under vector multiplication and addition.

Vector spaces under an inner product space and more specifically, a norm provide a metric to measure the relation of any two points in the space. We call these normed spaces.Vector spaces, characterized by their algebraic structure and metric, provide a framework for studying linear relationships and transformations. Normed spaces, defined by norms that quantify the size of vectors provide a method to analyze the convergence and continuity of the set of spaces in our space. Normed spaces have a wide range of applications in both mathematics and physics. A Banach space is a complete normed space. In mathematics, the Banach space can applied in order to analyze differentiation and integration of hyderdimensional spaces and in my physics, they are used to formalize abstract spaces.

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