Schramm Loewner Evolution: A Brief Introduction

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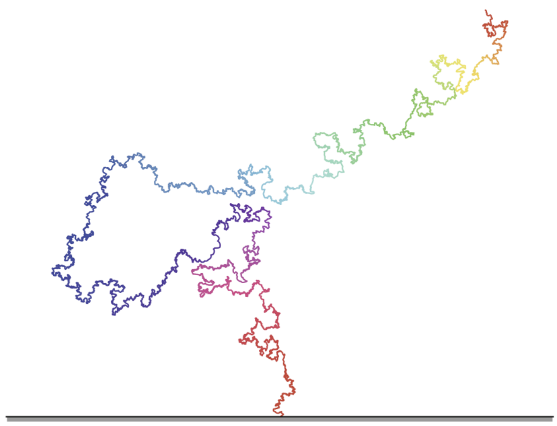

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Figure 1: Oded Schramm

Motivation and Objectives

This work is concerned with the study of random planar curves using tools from probability theory and complex analysis. The main object we aim to understand is the Schramm–Loewner Evolution (SLE), a family of random curves that arise as scaling limits of discrete models from statistical physics.

The starting point of SLE is a classical result due to Loewner, who showed that any growing family of simply connected domains in the complex plane can be encoded by a real-valued driving function through a differential equation. In 2000, Oded Schramm had the idea of replacing this deterministic function with a Brownian motion. This led to a new class of random curves with conformal invariance and Markovian properties — features expected in the scaling limits of critical models such as percolation and the Ising model.

The goal of this report is to prepare the ground for a rigorous understanding of SLE by presenting the main probabilistic and analytic tools involved. The first part recalls basic notions from stochastic calculus: Brownian motion and martingales, which will play a central role in the probabilistic formulation. The second part introduces key notions from conformal geometry: simply connected domains, conformal maps, Martin boundaries, and compact subsets of the upper half-plane. Finally, these ideas are brought together in the last sections to define Loewner chains and introduce their stochastic version.

Throughout, we aim to give a clear and structured path from classical concepts to the modern theory of conformally invariant random curves.

1 Probabilistic Background

This first section is devoted to recalling the probabilistic tools that will later be used to define and analyze random conformal evolutions. We focus in particular on Brownian motion and martingales, which will appear naturally in the construction of Loewner chains. Throughout this paper, we work on a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathcal{F}$ is the $\mathbb{P}$-completion[1] of a sigma-algebra for Ω.

We begin with Brownian motion, the fundamental stochastic process underlying the random dynamics we aim to study.

1.1 Brownian Motion

Definition 1.1 (Standard Brownian Motion)**.** A stochastic process $(B_t)_{t\geq 0}$, with values in $\mathbb{R}$, is called a *standard Brownian motion* if it satisfies the following properties:

- $B_0 = 0$ almost surely;
- (B_t) has independent and stationary increments, i.e. $B_t - B_s$ is independent of B_s, and $B_t - B_s \overset{\mathcal{L}}{=} B_{t-s}$ $\forall 0 \leq s \leq t$;
- The paths $t \mapsto B_t$ are almost surely continuous.

Remark 1.2. From now on, we fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ where $(\mathcal{F}_t)_{t\geq 0}$ is the natural filtration generated by the Brownian motion:

$$\mathcal{F}_t := \sigma(B_s : 0 \leq s \leq t).$$

Remark 1.3. A planar Brownian motion (or two-dimensional Brownian motion) is a stochastic process $(B_t)_{t\geq 0}$ with values in $\mathbb{R}^2$, where

$$(B_t)_{t\geq 0} = (B_t^{(1)}, B_t^{(2)})_{t\geq 0}$$

[1]The $\mathbb{P}$-completion of a sigma-algebra $\mathcal{F}_0$ on Ω is the smallest sigma-algebra $\mathcal{F} \supseteq \mathcal{F}_0$ such that every subset of a $\mathcal{F}_0$-measurable set of $\mathbb{P}$-measure zero is included in $\mathcal{F}$

and $(B_t^{(1)})_{t\geq 0}$, $(B_t^{(2)})_{t\geq 0}$ are two independent standard Brownian motions in $\mathbb{R}$..

Definition 1.4 (Complex Brownian Motion)**.** If $B^{(1)}$, $B^{(2)}$ are independent real Brownian motions, then $B_t^{\mathbb{C}} := B_t^{(1)} + iB_t^{(2)}$ is a complex Brownian motion.

Figure 2: 2D Brownian Motion

Proposition 1.5 (Rotational Invariance)**.** *Let $B = (B_t^{(1)}, B_t^{(2)})_{t\geq 0}$ be a planar Brownian motion, and let $O \in O_2(\mathbb{R})$ be an orthogonal matrix. Define $\widetilde{B}_t := OB_t$. Then $(\widetilde{B}_t)_{t\geq 0}$ is also a planar Brownian motion.*

Proof. We verify the defining properties:

Initial condition: Since $B_0 = (0, 0)$ almost surely, we have

$$\widetilde{B}_0 = OB_0 = (0, 0) \quad \text{almost surely.}$$

Independent and stationary increments: For $0 \leq s \leq t$, we have

$$\widetilde{B}_t - \widetilde{B}_s = O(B_t - B_s).$$

Since $B_t - B_s$ is independent of $\mathcal{F}_s$ and O is deterministic and linear, the random variable $\widetilde{B}_t - \widetilde{B}_s$ remains independent of $\mathcal{F}_s$.

Moreover, the law of $\widetilde{B}_t - \widetilde{B}_s$ only depends on $t - s$. Indeed, $B_t - B_s$ is a centered Gaussian vector with covariance matrix $(t - s)I_2$, and applying O, we get

$$\mathrm{Cov}(\widetilde{B}_t - \widetilde{B}_s) = O\,\mathrm{Cov}(B_t - B_s)O^\top = (t - s)OI_2O^\top = (t - s)I_2,$$

since O is orthogonal ($OO^\top = I_2$).

Thus, $\widetilde{B}_t - \widetilde{B}_s$ is Gaussian, centered, with covariance matrix $(t - s)I_2$, independently of the past. Therefore, $(\widetilde{B}_t)_{t\geq 0}$ has independent and stationary increments.

Continuity of paths: Finally, since B has continuous trajectories and O is linear and continuous, the process $\widetilde{B}$ also has continuous trajectories.

Thus, $(\widetilde{B}_t)_{t\geq 0}$ is a planar Brownian motion. □

Remark 1.6. The rotational invariance of planar Brownian motion can equivalently be expressed in complex notation. Writing $B_t = B_t^{(1)} + iB_t^{(2)}$, and letting $\theta \in [0; 2\pi[$, we have:

$$e^{i\theta}B_t \overset{\mathcal{L}}{=} B_t,$$

where $\stackrel{\mathcal{L}}{=}$ denotes equality in distribution. That is, multiplication by a complex number of unit modulus corresponds to a planar rotation and preserves the law of Brownian motion.

The following result plays an important role in the study of extremal behavior of Brownian motion and will be used to describe the distribution of its supremum.

Theorem 1.1 (Reflection principle for Brownian motion)**.** *Let $(B_t)_{t\geq 0}$ be a standard real Brownian motion. For any $a > 0$ and $t > 0$,*

$$\mathbb{P}\left(\sup_{0\leq s\leq t} B_s \geq a\right) = 2\,\mathbb{P}(B_t \geq a).$$

In particular,

$$\sup_{0\leq s\leq t} B_s \stackrel{\mathcal{L}}{=} |B_t|.$$

Remark 1.7. For a proof of the reflection principle, see [6, Section 3.6].

The reflection principle will be instrumental when describing how Brownian trajectories interact with boundaries, a crucial aspect in the study of hitting distributions.

We conclude this subsection with a fundamental characterization of Brownian motion among continuous processes with stationary and independent increments. It will later serve as a key tool when identifying the driving function of SLEs.[2]

Theorem 1.2 (Lévy–Khinchin characterization of Brownian motion)**.** *Let $(M_t)_{t\geq 0}$ be a continuous real-valued stochastic process. Then the following properties are equivalent:*

1. $M_t = \sqrt{\kappa}\, B_t$ for some $\kappa \geq 0$ and a standard Brownian motion $(B_t)_{t\geq 0}$;

2. (M_t) has stationary and independent increments, continuous trajectories, and satisfies $M_0 = 0$;

3. (M_t) is invariant in distribution under Brownian scaling: for all $\lambda > 0$,

$$(M_t)_{t\geq 0} \stackrel{\mathcal{L}}{=} (\lambda M_{\lambda^{-2}t})_{t\geq 0}\,, \quad \forall \lambda > 0.$$

Proof. This characterization can be viewed as a consequence of the Lévy–Khintchine representation: the only continuous Lévy processes are Brownian motions with constant drift, and the scaling invariance forces the drift to vanish. For more details, see [7]. □

1.2 Martingales

To control the randomness of Brownian motion in a structured way, we rely on martingales and their powerful optional stopping properties. Intuitively, they model "fair" random processes whose future evolution, on average, is equal to their current state.

Definition 1.8 (Martingale and Local Martingale)**.** An adapted process $M = (M_t)_{t\geq 0}$ is a *martingale* if:

$$\mathbb{E}[|M_t|] < \infty \quad \text{and} \quad \mathbb{E}[M_t \mid \mathcal{F}_s] = M_s, \quad \text{for all } 0 \leq s \leq t.$$

It is a *local martingale* if there exists an increasing sequence of stopping times $(T_n) \uparrow \infty$ such that each stopped process $M^{T_n} := (M_{t\wedge T_n})_{t\geq 0}$ is a martingale.

Local martingales behave like martingales up to random finite horizons and form a more general class, especially useful when integrability is lacking. A fundamental tool involving martingales is the optional sampling theorem, which ensures consistency when stopping at random times.

[2]See Section 3, for the definition of these concepts

Proposition 1.9 (Optional Sampling Theorem). *Let $(M_t)_{t\geq 0}$ be a continuous local martingale. Let $S \leq T$ be stopping times such that there exists a constant $C > 0$ with $S, T \leq C$ a.s. Then:*

$$\mathbb{E}[M_T \mid \mathcal{F}_S] = M_S \quad \textit{a.s.}, \quad \textit{and in particular} \quad \mathbb{E}[M_S] = \mathbb{E}[M_T].$$

Having set the probabilistic foundations, we now turn to the complex-analytic structures that will encode the geometry of growing random sets.

2 SLE's Key Theoretical Foundations

Our goal in this section is to first formalize the notion of evolving compact subsets of the upper half-plane $\mathbb{H}$, also known as $\mathbb{H}$-*hulls*. These will serve as the geometric objects encoded by the solutions to Loewner's differential equation. To this end, we begin by recalling classical results from complex analysis on conformal mappings of simply connected domains.

2.1 Conformal Mappings: Definitions and Examples

A domain $D \subset \mathbb{C}$ is a non-empty open connected set. It is said to be *simply connected* if every closed loop in D can be continuously contracted to a point within D, or equivalently, if $\mathbb{C} \cup \{\infty\} \setminus D$ is connected.

We now present a few classical examples of simply connected domains, which will serve as canonical models throughout the theory. For each one, we verify the connectedness of the complement in the Riemann sphere.

Example 2.1 (Unit Disc). The open unit disc

$$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$$

is simply connected. Its complement in $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is the closed set $\{|z| \geq 1\} \cup \{\infty\}$, which is connected.

Example 2.2 (Upper Half-Plane). The upper half-plane

$$\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$$

is also simply connected. Its complement is the real line $\mathbb{R}$, which is closed, together with the lower half-plane and the point at infinity. This union forms a connected subset of $\widehat{\mathbb{C}}$.

Example 2.3 (Infinite Horizontal Strip). The infinite strip

$$S := \{z \in \mathbb{C} : 0 < \mathrm{Im}(z) < 1\}$$

is another simply connected domain. Its complement in the Riemann sphere is the union of the two unbounded half-planes $\{\mathrm{Im}(z) \leq 0\}$ and $\{\mathrm{Im}(z) \geq 1\}$, plus the point at infinity. This set is connected.

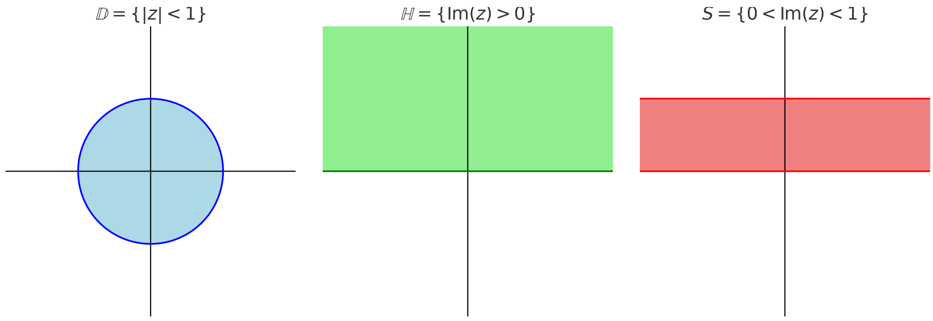

Figure 3: Examples of simply connected domains: open unit disk $\mathbb{D}$, upper half-plane $\mathbb{H}$, and infinite strip S.

We introduce several fundamental concepts from complex analysis that will be used in the study of conformal mappings and the Loewner evolution.

Definition 2.4 (Holomorphic function)**.** Let $f : D \subset \mathbb{C} \to \mathbb{C}$ be a complex-valued function over D some open non-empty subset of the complex plane. We say that f is *holomorphic* at a point $z_0 \in D$ if the complex derivative

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. If this holds for all $z_0 \in D$, then f is said to be holomorphic on D.

We now introduce conformal maps, which are fundamental to the constructions studied in this work.

Definition 2.5 (Conformal Map and Conformal Isomorphism)**.** Let $f : D \to \mathbb{C}$ be a holomorphic function on a domain $D \subset \mathbb{C}$.

We say that f is *conformal at* $z_0 \in D$ if $f'(z_0) \neq 0$. If f is conformal at every point of D, we say that f is *conformal on* D.

Moreover, a *conformal isomorphism* between two domains $D, D' \subset \mathbb{C}$ is a bijective conformal map $\varphi : D \to D'$ whose inverse $\varphi^{-1} : D' \to D$ is also holomorphic.

Remark 2.6. Conformal maps preserve angles and local shapes.

These notions allow us to rigorously describe conformal equivalences between domains.

Theorem 2.1 (Riemann Mapping Theorem)**.** *Let $D \subsetneq \mathbb{C}$ be a proper simply connected domain. Then there exists a conformal isomorphism $\phi : D \to \mathbb{D}$, where $\mathbb{D}$ is the open unit disc.*

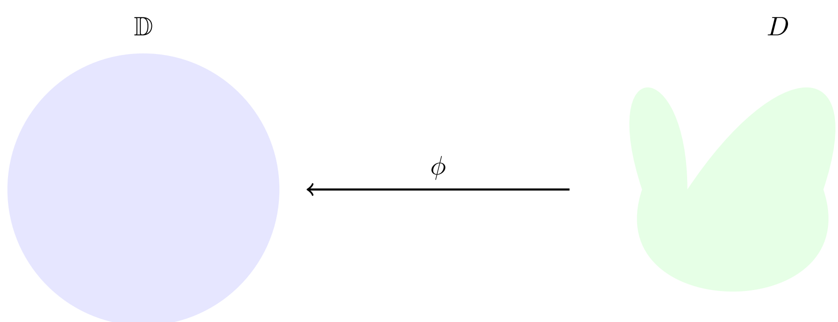

Figure 4: A simply connected domain D and a conformal isomorphism $\phi : D \to \mathbb{D}$.

This powerful result allows us to identify any proper simply connected domain $D \subsetneq \mathbb{C}$ with the unit disc $\mathbb{D}$ up to a conformal map. In particular, any two such domains D and D' are conformally equivalent: if $\varphi : D \to \mathbb{D}$ and $\psi : D' \to \mathbb{D}$ are conformal isomorphisms, then the composition

$$\psi^{-1} \circ \varphi : D \to D'$$

defines a conformal isomorphism between D and D'.

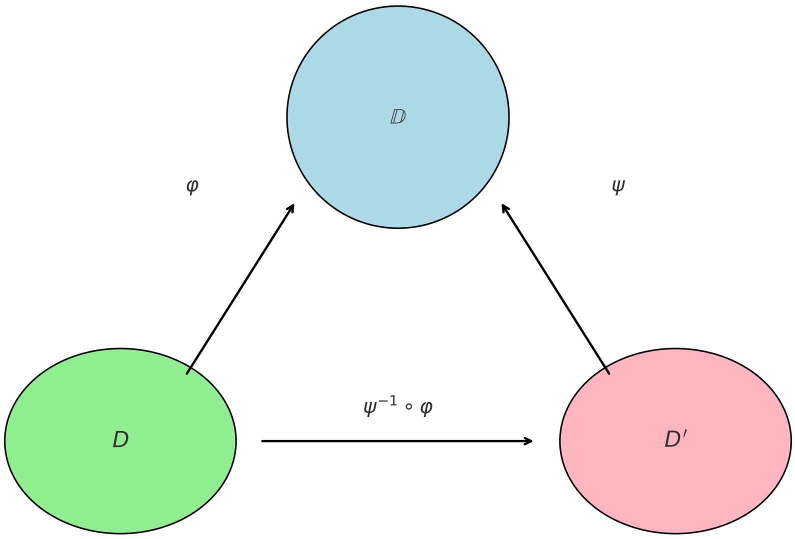

Figure 5: Conformal equivalence between two domains via the unit disc

This provides a canonical method to relate two simply connected domains by mapping each one conformally onto the unit disc and composing the resulting maps.

Let us give another property that will be useful in the next sections.

Property 2.1.1. *Let D be a proper simply connected domain and let $w \in D$. Then there exists a unique conformal isomorphism $\psi_w : D \to \mathbb{D}$ such that $\psi_w(w) = 0$ and $\arg \psi'_w(w) = 0$.*

Proof. We will not prove uniqueness nor that $\arg \psi'_w(w) = 0$. By the Riemann mapping theorem, there exists a conformal isomorphism $\phi_0 : D \to \mathbb{D}$. Set $v = \phi_0(w)$ and $\theta = -\arg \phi'_0(w)$, and define

$$\psi_w := \Phi_{\theta,v} \circ \phi_0,$$

where, for $\theta \in [0, 2\pi[$ and $w \in \mathbb{D}$, the automorphism $\Phi_{\theta,w} : \mathbb{D} \to \mathbb{D}$ is defined by

$$\Phi_{\theta,w}(z) = e^{i\theta} \cdot \frac{z - w}{1 - \bar{w}z}.$$

Then $\psi_w : D \to \mathbb{D}$ is a conformal isomorphism satisfying $\psi_w(w) = 0$. $\square$

Remark 2.7. Note that in Corollary 1.3 of [4], it is explained that φ can be extended to a homeomorphism of $\overline{\mathbb{D}}$.

2.2 Martin Boundary

The Martin boundary offers a refined way to understand the behavior of conformal maps near the boundary of a simply connected domain. Unlike the usual topological boundary, it captures finer information about the way sequences approach the boundary.

Let $D \subset \mathbb{C}$ be a proper simply connected domain, and let $\phi : D \to \mathbb{D}$ be a conformal isomorphism. Using ϕ, we can define a new metric d_ϕ on D by

$$d_\phi(z, z') := |\phi(z) - \phi(z')|.$$

This metric reflects the conformal geometry of D through its image in $\mathbb{D}$. We may then consider the metric completion $\widehat{D}$ of D with respect to d_ϕ. The metric completion $\widehat{D}$ of D with respect to d_ϕ is the space obtained by formally adding all the limit points of Cauchy sequences in D (under the metric d_ϕ). In other words, it is the smallest complete metric space containing D as a dense subset.

This construction allows us to distinguish different "modes of approach" to the boundary of D, depending on how sequences behave under the conformal mapping ϕ.

Definition 2.8 (Martin Boundary)**.** The *Martin boundary* of D, denoted δD, is defined as

$$\delta D := \widehat{D} \setminus D,$$

where $\widehat{D}$ is the completion of D under the metric d_ϕ.

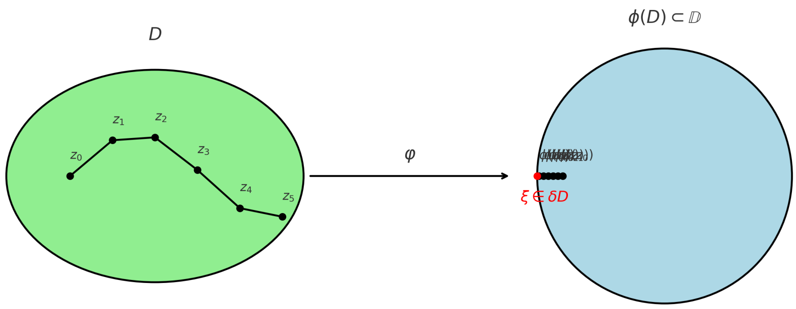

Figure 6: A Cauchy sequence $(z_n) \subset D$ whose image under the conformal map $\varphi : D \to \mathbb{D}$ converges to a point $\xi \in \partial\mathbb{D}$. This limit point ξ defines a Martin boundary point of D.

Remark 2.9. Although the metric d_ϕ is defined using a conformal isomorphism $\phi : D \to \mathbb{D}$, the resulting Martin boundary δD is independent of the choice of ϕ. In this sense, the Martin boundary is a canonical construction associated to the domain D. See [4, Section 1.3] for further details.

Remark 2.10. In the case $D = \mathbb{H}$, the upper half-plane, $\delta\mathbb{H}$ corresponds to $\mathbb{R} \cup \{\infty\}$.

2.3 Compact $\mathbb{H}$-Hulls and Mapping-Out Functions

We now formalize the class of compact subsets of the upper half-plane that will play a key role in Schramm–Loewner Evolution (SLE). These sets will be progressively "removed" from $\mathbb{H}$ via conformal maps to encode geometric growth.

Definition 2.11 ($\mathbb{H}$-hull)**.** A compact set $K \subset \overline{\mathbb{H}}$ is called an $\mathbb{H}$-*hull* if the complement $\mathbb{H} \setminus K$ is simply connected.

Such sets include slits, finite unions of compact curves, or even more general closed sets, provided their removal does not disconnect the upper half-plane.

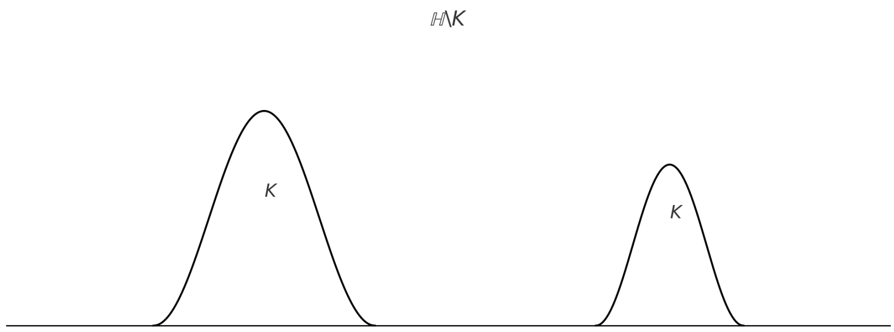

Figure 7: Example of a compact $\mathbb{H}$-hull

Proposition 2.12 (Existence and Uniqueness of the Mapping-Out Function)**.** *Let $K \subset \overline{\mathbb{H}}$ be a compact $\mathbb{H}$-hull. Then there exists a unique conformal map*

$$g_K : \mathbb{H} \setminus K \longrightarrow \mathbb{H}$$

such that

$$g_K(z) - z \longrightarrow 0 \quad \textit{as} \quad |z| \longrightarrow \infty,$$

and moreover, there exists a constant $a_K \in \mathbb{R}$ such that

$$g_K(z) = z + \frac{a_K}{z} + o\left(\frac{1}{z}\right) \quad \textit{as} \quad |z| \longrightarrow \infty.$$

The map g_K is called the mapping-out function *of K, and a_K is its* half-plane capacity*.*

The function g_K is often called the *mapping-out function* of K, as it conformally maps away the "defect" K, sending $\mathbb{H} \setminus K$ back to the standard upper half-plane $\mathbb{H}$. It plays a central role in Loewner theory.

Property *Let K be a compact $\mathbb{H}$-hull. Let $r > 0$ and $x \in \mathbb{R}$. Set*

$$rK = \{rz : z \in K\}, \quad K + x = \{z + x : z \in K\}.$$

Then rK and $K + x$ are compact $\mathbb{H}$-hulls and we have

$$g_{rK}(z) = r g_K(z/r), \quad g_{K+x}(z) = g_K(z - x) + x.$$

Proposition 2.13 (Composition of $\mathbb{H}$-hulls)**.** *Let K_0 and K_1 be compact $\mathbb{H}$-hulls, and define*

$$K := K_0 \cup g_{K_0}^{-1}(K_1).$$

Then K is a compact $\mathbb{H}$-hull containing K_0, and we have

$$g_K = g_{K_1} \circ g_{K_0}, \quad \textit{and} \quad a_K = a_{K_0} + a_{K_1},$$

where a_K denotes the half-plane capacity of the hull K.

Remark 2.14. This result shows that any compact $\mathbb{H}$-hull containing a fixed K_0 can be obtained by composing its mapping-out function with that of a hull K_1 in the complement domain.

We now turn to several quantitative estimates that will be crucial later. These describe how the mapping-out function g_K behaves analytically, especially near infinity and near the boundary. They provide refined control over how the geometry of the hull K is encoded in the asymptotics of g_K.

From now on, we will denote

$$H := \mathbb{H} \setminus K$$

for K a compact $\mathbb{H}$-hull.

Differentiability control. There exists an absolute constant $C < \infty$ such that for any $r > 0$, any $\xi \in \mathbb{R}$, and any compact $\mathbb{H}$-hull $K \subset r\overline{\mathbb{D}} + \xi$, we have:

$$\left| g_K(z) - z - \frac{a_K}{z - \xi} \right| \leq \frac{Cra_K}{|z - \xi|^2}, \quad \text{for all } |z - \xi| \geq 2r.$$

Boundary behavior. Since g_K extends to a homeomorphism from δH onto $\mathbb{R} \cup \{\infty\}$, its boundary values can be interpreted probabilistically via harmonic measure[3]. For any measurable set $S \subset \delta H$, one has:

$$\lim_{y \to \infty,\, x/y \to 0} \pi y \, \mathbb{P}_{x+iy}(\widehat{B}_{T(H)} \in S) = \mathrm{Leb}(g_K(S)),$$

where $\widehat{B}_{T(H)}$ denotes the limit point of Brownian motion in the Martin boundary of H, and Leb denotes Lebesgue measure on $\mathbb{R} \cup \{\infty\}$.

Before constructing Schramm–Loewner Evolutions (SLE), it is essential to introduce a fundamental property of Brownian motion in the complex plane: its conformal invariance.

2.4 Conformal Invariance

Theorem 2.2. *Let D and D' be domains and let $\varphi : D \to D'$ be a conformal isomorvarphism. Fix $z \in D$ and set $z' = \varphi(z)$. Let $(B_t)_{t \geq 0}$ and $(B'_t)_{t \geq 0}$ be complex Brownian motions starting from z and z' respectively. Set*

$$T = \inf\{t \geq 0 : B_t \notin D\}, \quad T' = \inf\{t \geq 0 : B'_t \notin D'\}.$$

Set

$$\widetilde{T} = \int_0^T |\varphi'(B_t)|^2 dt$$

and define for $t < \widetilde{T}$:

$$\tau(t) = \inf\{s \geq 0 : \int_0^s |\varphi'(B_r)|^2 dr = t\}, \quad \widetilde{B}_t = \varphi(B_{\tau(t)}).$$

Then $(\widetilde{T}, (\widetilde{B}_t)_{t<\widetilde{T}})$ and $(T', (B'_t)_{t<T'})$ have the same distribution, and $\varphi(B_{\tau(t)})$ defines a Brownian Motion with another time parametrization.

This fundamental result ensures that Brownian motion remains invariant under conformal mappings. This property will also be crucial for introducing and manipulating the harmonic measure.

Property 2.4.1. *Let D be a proper simply connected domain. Fix $z \in D$ and let $(B_t)_{t \geq 0}$ be a complex Brownian motion starting from z. Set*

$$T(D) = \inf\{t \geq 0 : B_t \notin D\}.$$

[3] See Section 3.1 for the construction and definition of harmonic measure via Brownian motion and the Martin boundary.

Then $\mathbb{P}_z(T(D) < \infty) = 1$ i.e almost surely, brownian motion leaves any proper domain he starts from.

Proof. Let $(B_t)_{t\geq 0}$ be a complex Brownian motion starting from any point $z \in \mathbb{D}$. We want to show that the hitting time

$$T := \inf\{t \geq 0 : B_t \notin \mathbb{D}\}$$

is almost surely finite, i.e., $\mathbb{P}(T < \infty) = 1$.

Take any integer $N \geq 1$. We have the following chain of inequalities:

$$\begin{aligned}
\mathbb{P}(T < \infty) &\geq \mathbb{P}\left(\exists n \in \mathbb{N}, \sup_{0\leq s\leq n} |B_s|^2 \geq 2\right) \\
&\geq \mathbb{P}\left(\exists n \in \mathbb{N}, \sup_{0\leq s\leq n} |B_s^{(1)}| \geq 1 \text{ and } \sup_{0\leq s\leq n} |B_s^{(2)}| \geq 1\right) \\
&\geq \mathbb{P}\left(\sup_{0\leq s\leq N} |B_s^{(1)}| \geq 1 \text{ and } \sup_{0\leq s\leq N} |B_s^{(2)}| \geq 1\right)
\end{aligned}$$

where $B^{(1)}$ and $B^{(2)}$ are the real and imaginary parts of B, which are independent standard Brownian motions.

By independence:

$$\mathbb{P}\left(\sup_{0\leq s\leq N} |B_s^{(1)}| \geq 1 \text{ and } \sup_{0\leq s\leq N} |B_s^{(2)}| \geq 1\right) \geq \left(\mathbb{P}\left(\sup_{0\leq s\leq N} B_s \geq 1\right)\right)^2.$$

Using the reflexion principle of the Brownian motion 1.1

$$\mathbb{P}\left(\sup_{0\leq s\leq N} B_s \geq 1\right) = \mathbb{P}\left(|B_N| \geq 1\right) = \mathbb{P}\left(|B_1| \geq \frac{1}{\sqrt{N}}\right).$$

Therefore,

$$\mathbb{P}(T < \infty) \geq \left(\mathbb{P}\left(|B_1| \geq \frac{1}{\sqrt{N}}\right)\right)^2 \underset{N\to\infty}{\longrightarrow} 1$$

So that :

$$\mathbb{P}(T < \infty) = 1.$$

Now, to generalize this for any domain D, we know that there exists a conformal map $\varphi : \mathbb{D} \to D$ and we conclude by conformal invariance of brownian motion. □

2.5 Harmonic measure

Let D be a proper domain of $\mathbb{C}$. Given a point $z_0 \in D$, and a Brownian motion B starting from z_0, the random variable B_{T_D} is an element of ∂D almost surely. By definition, the **harmonic measure** in D viewed from z_0 is the law of B_{T_D}. We call $h_D(z, \cdot)$ the *harmonic measure for D starting from z* . First, we give an example of such a measure in the most common setting, then by conformal invariance we will extend our knowledge.

Property 2.5.1. *The* harmonic measure *in the unit disc $\mathbb{D}$, viewed from 0, is the uniform distribution on the unit circle $\partial\mathbb{D}$ that is : $h_{\mathbb{D}}(0, I) = \frac{1}{2\pi}\mathbf{1}_I$, $I \subset \partial\mathbb{D}$*

Proof. We recall that Brownian motion1.5 is rotationally invariant: that is, if B is a Brownian motion, then $e^{i\theta}B_t$ is also a Brownian motion. Consequently, if μ denotes the harmonic measure in $\mathbb{D}$ viewed from 0, then μ is also rotationally invariant: that is,

$$\mu(I) = \mu(e^{i\theta}I)$$

for any circular arc $I \subset \partial\mathbb{D}$.

If we define $F(\theta) = \mu(I_\theta)$, where I_θ is the circular arc between 1 and $e^{i\theta}$ for $\theta \in (0, 2\pi)$, then

$$F(\theta + \theta') = F(\theta) + F(\theta')$$

whenever $0 \leq \theta + \theta' \leq 2\pi$.

From this and the right continuity of F, it follows that F is linear, and the result follows. $\square$

We now present a brief development to compute harmonic measures explicitly in three distinct settings, which will help illustrate the role of conformal invariance in concrete examples.

2.5.1 Examples of Harmonic Measure

Let (B_t) be a brownian motion starting from z. By conformal invariance, as $t \uparrow T(D)$, B_t converges in $\bar{D}$ to a limit $\hat{B}_T \in \delta D$. Denote by $h_D(z, \cdot)$ the distribution of $\hat{B}_T$ on δD. Let us take $\varphi : D \to \mathbb{D}$ a conformal isomorphism. By conformal invariance of Brownian motion, for $s_1, s_2 \in \delta D$ and $\theta_1, \theta_2 \in [0, 2\pi)$ with $\theta_1 \leq \theta_2$ and $\varphi(s_1) = e^{i\theta_1}$ and $\varphi(s_2) = e^{i\theta_2}$, we have

$$h_D(z, [s_1, s_2]) = \mathbb{P}_z(\hat{B}_T \in [s_1, s_2]) = \mathbb{P}_0(B_T^{(\mathbb{D})} \in [e^{i\theta_1}, e^{i\theta_2}]) = \frac{\theta_2 - \theta_1}{2\pi}.$$

We often take an interval $I \subseteq \mathbb{R}$ and a parametrization $s : I \to \delta D$ of the Martin boundary. We may then be able to find a density function $h_D(z, \cdot)$ on I such that

$$\int_I h_D(z, t)\, dt = h_D(z, s(I)).$$

There exists $t_1, t_2 \in I$ such that $s(t_1) \leq s(t) \leq s(t_2)\,, t \in I$. Thus, we should get:

$$\int_{t_1}^{t_2} h_D(z, t)\, dt = \frac{\theta(t_2) - \theta(t_1)}{2\pi} = \frac{1}{2\pi} \int_{t_1}^{t_2} \theta'(t)\, dt.$$

It follows that

$$h_D(z, t) = \frac{\theta'(t) dt}{2\pi}$$

where dt represents the lebesgue measure over I. Here we would like to find the law of the hitting time of the frontier of the unit disk with a Brownian motion that does not starts anymore from 0, but from another point $z \in \mathbb{D}$. In that case, we want to send the Brownian motion from z to 0 while staying in $\mathbb{D}$, from 2.1.1, there exists a unique conformal automorphism ψ_z such that :

$$\psi_z(s) = \frac{s - z}{1 - \bar{z} \cdot s}, \forall s \in \mathbb{D}$$

Now let us take the following parametrization : $s(t) = e^{it} \in \delta\mathbb{D}$ for all $t \in [0; 2\pi[$. If we find a map θ such that for any t :

$$\psi_z(s(t)) = e^{i\theta(t)}$$

we will be able to express the harmonic measure of the brownian motion starting from z. Thus[4], we differentiate in t and obtain :

$$\theta'(t) = \frac{1 - \bar{z}^2}{|e^{it} - z|^2}$$

so that we have $\forall 0 \leq t < 2\pi$:

$$h_{\mathbb{D}}(z, t) = \frac{1}{2\pi} \cdot \frac{1 - (\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2)}{(\cos t - \mathrm{Re}(z))^2 + (\sin t - \mathrm{Im}(z))^2}$$

[4]From 2.7, one can legally extend ψ over the frontier of the disk.

Therefore, for $t_1 < t_2 \ \in [0; 2\pi[$

$$\mathbb{P}_z(\hat{B}_T \in [e^{it_1}, e^{it_2}]) = \frac{1 - (\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2)}{2\pi} \int_{t_1}^{t_2} \frac{1}{(\cos t - \mathrm{Re}(z))^2 + (\sin t - \mathrm{Im}(z))^2}\, dt$$

Notice that if $z = 0$, we recover the uniform law over $\partial\mathbb{D}$.

Concluding this part with harmonic measure, we would like to consider a Brownian motion starting from a point of modulus larger than 1. Our goal is again to determine the law of the hitting point $\widehat{B}_T$ on the unit circle $\partial\mathbb{D}$, where $T := \inf\{t \geq 0 : |B_t| = 1\}$.

To analyze this, we use again the conformal mapping technique. Define the inversion map:

$$\varphi(x) = \frac{1}{x}, \quad x \in \mathbb{C} \setminus \{0\}.$$

This map sends the exterior of the unit disk onto the open unit disk, and the unit circle is invariant under φ. Then define:

$$z_0 := \varphi(z) = \frac{1}{z} \in \mathbb{D}, \text{ since } |z| > 1.$$

Invock again the automorphism $\psi_{z_0} : \mathbb{D} \to \mathbb{D}$ from 2.1.1 which sends z_0 to 0 and satisfies $\arg \psi'_{z_0}(z_0) = 0$. That is,

$$\psi_{z_0}(x) = \frac{x - z_0}{1 - \overline{z_0} x},$$

We now examine the harmonic measure via this parametrization of the boundary, searching for: $s(t) := e^{-it} \in \delta(\mathbb{C} \setminus \overline{\mathbb{D}})$, and define:

$$\theta \text{ such that } \psi(e^{-it}) = e^{i\theta(t)} \forall t \in [0; 2\pi[.$$

Again, we differentiate in t

$$\psi(e^{-it})e^{-i\theta(t)} = 1 \quad \text{thus} \quad \partial_t \left(\psi(e^{-it})e^{-i\theta(t)}\right) = 0 \;\text{ whence }\; \theta'(t) = -\frac{\psi'(e^{-it})e^{-it}}{\psi(e^{-it})}$$

And we already know that :

$$\psi'(x) = \frac{1 - |z_0|^2}{(1 - \overline{z_0} x)^2}$$

So :

$$\theta'(t) = -(1 - |z_0|^2) \cdot \frac{e^{-it}}{(1 - \overline{z_0} e^{-it})^2} \cdot \frac{1 - \overline{z_0} e^{-it}}{e^{-it} - z_0} = -\frac{1 - |z_0|^2}{|1 - z_0 e^{it}|^2}.$$

Finally, the harmonic measure is:

$$h_{\mathbb{C}\setminus\overline{\mathbb{D}}}(z_0, t) = \frac{|z_0|^2 - 1}{2\pi |1 - \overline{z_0} e^{-it}|^2} \ , t \in [0; 2\pi[$$

2.6 Half-plane capacity and estimates :

Let us define another geometric parameter, the so-called "radius" :

$$\mathrm{rad}(K) = \inf\{r \geq 0 : K \subseteq r\overline{\mathbb{D}} + x \text{ for some } x \in \mathbb{R}\}.$$

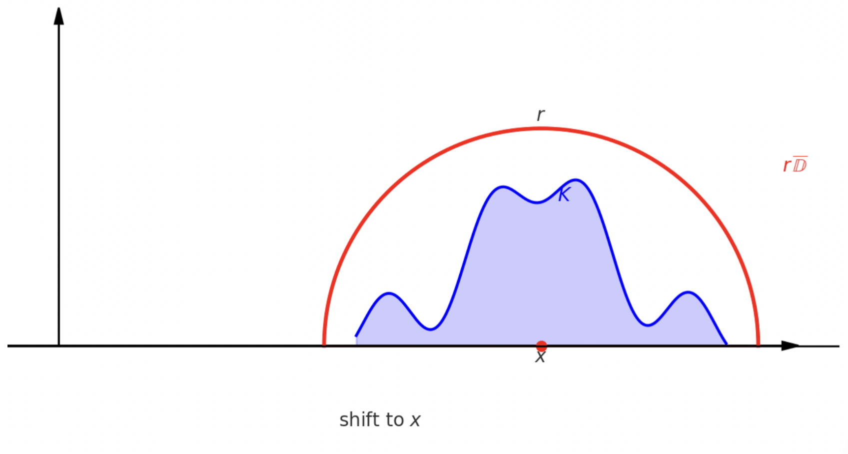

Figure 8: An r-disk enclosing a compact $\mathbb{H}$-hull

Hence, from this illustration, $\text{rad}(K)$ is just the infinimum taken over all such $r \geq 0$ covering the figure above.

Definition 3.5. *The half-plane capacity of the compact $\mathbb{H}$-hull K is the quantity*

$$\text{hcap}(K) = \lim_{z \to \infty} z(g_K(z) - z) = a_K.$$

As examples, let us determine the hcap of these two compact $\mathbb{H}$-hulls :

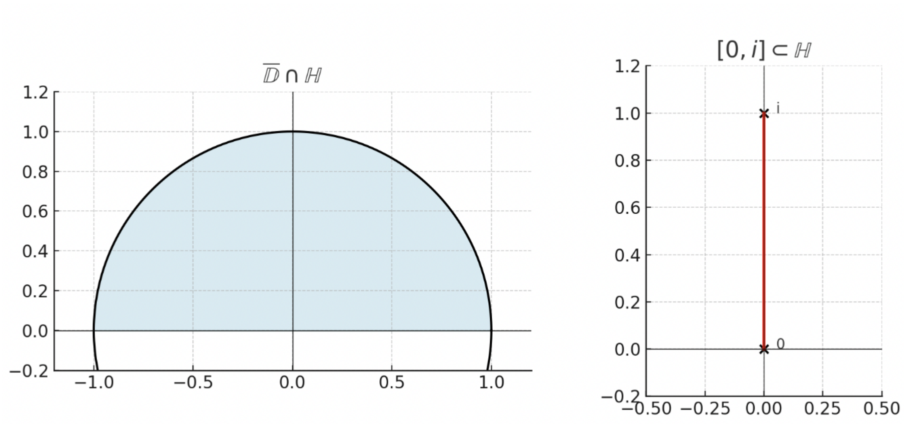

Figure 9:

The mapping-out function has a simple form for the half-disc $\overline{\mathbb{D}} \cap \mathbb{H}$ and for the slit $]0, i] = \{iy : y \in]0, 1]\}$ being :

$$g_{\overline{\mathbb{D}} \cap \mathbb{H}}(z) = z + \frac{1}{z}, \qquad g_{(0,i]}(z) = \sqrt{z^2 + 1} = z + \frac{1}{2z} + \mathcal{O}(|z|^{-2}).$$

Hence :

$$\text{hcap}(\overline{\mathbb{D}} \cap \mathbb{H}) = 1, \qquad \text{hcap}([0, i]) = \frac{1}{2}.$$

For a Brownian motion $B = (B_t)_{t \geq 0}$ starting from some $z \in H$, let

$$T = T(H) = \inf\{t > 0 : B_t \notin H\}$$

the first hitting time of $K \cup \mathbb{R}$.

Property 2.6.1. *For any K, compact $\mathbb{H}$-hull, we have*

$$hcap(K) = \lim_{y\to\infty} y\mathbb{E}_{iy}[Im(B_T)].$$

In particular $hcap(K) \geq 0$.

Proof. Fix $z \in H$ and consider a complex Brownian motion $(B_t)_{t\geq 0}$ starting from z. For $t < T(H)$, $g_K(B_t)$ is a time-changed Brownian motion in the upper-half plane $\mathbb{H}$, starting from $g_K(z)$, and so converges a.s. to a random variable, denoted $g_K(\hat{B}_T) \in \mathbb{R}$, as $t \uparrow T$ - $\hat{B}_T \in \delta H$, the Martin boundary. Recall also that $g_K(z) - z$ is a bounded holomorphic function on H :

$$|g_K(z) - z| \sim |\frac{a_K}{z}| \quad \text{as } |z| \to \infty.$$

If we set $M_t = g_K(B_t) - B_t$ for $t < T$, then $(M_t)_{t<T}$ is a bounded martingale which converges to $g_K(\hat{B}_T) - B_T$ as $t \to T$. Hence, by optional stopping,

$$g_K(z) - z = \mathbb{E}_z(g_K(\hat{B}_T) - B_T).$$

Recall also $z(g_K(z) - z) \to \text{hcap}(K)$ as $z \to \infty$. Take $z = iy$, hence we have

$$\text{hcap}(K) = \lim_{y\to\infty} iy\mathbb{E}_{iy}[g_K(\hat{B}_T) - B_T].$$

On the other hand, hcap(K) is known to be real so we can take the real part on both sides. Noting that $g_K(\hat{B}_T) \in \mathbb{R}$ by definition, we deduce that

$$\lim_{y\to\infty} y\mathbb{E}_{iy}[\text{Im}(B_T)] = \text{hcap}(K)$$

as desired. □

Property 2.6.2. *Let K and K' be compact $\mathbb{H}$-hulls with $K \subseteq K'$. Set $\widetilde{K} = g_K(K' \setminus K)$. Then*

$$hcap(K) \leq hcap(K) + hcap(\widetilde{K}) = hcap(K').$$

Moreover, for any $r > 0$ and $x \in \mathbb{R}$, we have

$$hcap(rK) = r^2\, hcap(K), \qquad hcap(K + x) = hcap(K).$$

Proof. We start from this :

$$g_K(z) = z + \frac{a_K}{z} + O(|z|^{-2}), \quad |z| \to \infty.$$

Now consider the scaled hull $rK = \{rz : z \in K\}$, and let g_{rK} be its mapping-out function. We stated in last chapter that :

$$g_{rK}(z) = rg_K(z/r).$$

Plugging the asymptotic expansion of g_K into this expression, we get

$$g_{rK}(z) = r\left(\frac{z}{r} + \frac{a_K}{z/r} + O(|z/r|^{-2})\right) = z + \frac{r^2 a_K}{z} + O(|z|^{-2}).$$

It holds immediatly that :

$$\text{hcap}(rK) = a_{rK} = r^2 a_K = r^2\text{hcap}(K).$$

□

We deduce that, for $K \subseteq r\mathbb{D}$, we have $\text{hcap}(K) \le \text{hcap}(r\overline{\mathbb{D}} \cap \mathbb{H}) = r^2$. Hence, for all compact $\mathbb{H}$-hulls K,

$$\text{hcap}(K) \le \text{rad}(K)^2.$$

From [4], one can also establish that :

$$\text{hcap}(K) = \frac{2}{\pi} \int_0^\pi \mathbb{E}_{e^{i\theta}} \left(\text{Im}(B_{T(H)})\right) \sin\theta \, d\theta,$$

which shows that $\text{hcap}(K) > 0$ for all non-empty compact $\mathbb{H}$-hulls. The proof will not be given here as it is not relevant for our work but is available on the quoted paper.

3 Schramm-Loewner Theory.

This final section brings together the tools developed so far to present the core ideas of Loewner Theory and its stochastic extension.

We begin in the deterministic setting, where we explain the two fundamental directions of construction: from a given driving function to a family of growing hulls, and conversely, from a suitable family of hulls back to a driving function.

Building on this dual perspective, we then follow Schramm's key insight: to introduce randomness by choosing the driving function to be a Brownian motion, thereby defining the Schramm–Loewner Evolution (SLE), which connects probabilistic dynamics with conformal geometry.

3.1 From Compact $\mathbb{H}$-hulls to *Loewner Transform*

Let $(K_t)_{t\ge0}$ be a family of compact $\mathbb{H}$-hulls.

– The family is said to be *increasing* if $K_s \subsetneq K_t$ whenever $s < t$.

– Define the right-limit hull at time t by

$$K_{t+} := \bigcap_{s>t} K_s.$$

– For $s < t$, define :

$$K_{s,t} := g_{K_s}(K_t \setminus K_s).$$

– The family $(K_t)_{t\ge0}$ is said to satisfy the *local growth property* if

$$\text{rad}(K_{t,t+h}) \to 0 \quad \text{as } h \downarrow 0 \text{ uniformly on compacts in } t.$$

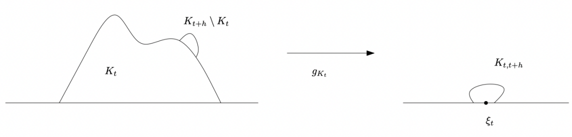

Figure 5: The local growth property and the Loewner transform.

Figure 10: Caption

Property 3.1.1. *Let $(K_t)_{t\ge0}$ be an increasing family of compact $\mathbb{H}$-hulls having the local growth property. Then $K_{t+} = K_t$ for all t. Moreover, the map $t \mapsto hcap(K_t)$ is continuous and strictly increasing on $[0, \infty[$, and :*

$$\forall\, t \ge 0 : \bigcap_{h>0} \overline{K_{t,t+h}} \;=: \xi_t$$

$\xi : t \mapsto \xi_t$ *defines the Loewner process.*

To prove it, we just give a useful continuity estimate :

Let K be a compact $\mathbb{H}$-hull. Then

$$|g_K(z) - z| \leq 3\,\mathrm{rad}(K), \forall z \in \mathbb{H}.$$

Proof. Set $K_{t,t+h} = g_{K_t}(K_{t+h} \setminus K_t)$. For all $t \geq 0$ and $h > 0$, we shown before that :

$$\mathrm{hcap}(K_{t+h}) = \mathrm{hcap}(K_t) + \mathrm{hcap}(K_{t,t+h}).$$

And one has that :

$$\mathrm{hcap}(K_{t,t+h}) \leq \mathrm{rad}(K_{t,t+h})^2$$

Hence :

$$\mathrm{hcap}(K_{t+h}) - \mathrm{hcap}(K_t) \leq \mathrm{rad}(K_{t,t+h})^2 \underset{h\downarrow 0}{\longrightarrow} 0 \quad \text{by local growth property} \tag{1}$$

Moreover : $\mathrm{hcap}(K_{t,t+}) = 0$, so $K_{t,t+} = \emptyset$ i.e $K_{t+} = K_t$ since g_{K_t} is conformal. Conversely, for any $h > 0 : K_{t,t+h} \neq \emptyset$ so $\mathrm{hcap}(K_{t,t+h}) > 0$. Hence : $t \mapsto \mathrm{hcap}(K_t)$ is continuous and increasing from $\mathbb{R}_+^*$ towards itself : it is an homeomorphism. Now, for any $t, h > 0$, the $\overline{K_{t,t+h}}$ are a family of non-empty non-decreasing compacts sets. Therefore, their intersection is a singleton, denote it ξ_t for any $t > 0$. For $t \geq 0$ and $h > 0$, choose $z \in K_{t+2h} \setminus K_{t+h}$ and set $w = g_{K_t}(z)$ and $w' = g_{K_{t+h}}(z)$. Then $w \in K_{t,t+2h}$ and $w' \in K_{t+h,t+2h}$, with $w' = g_{K_{t,t+h}}(w) = (g_{K_{t,t+h}} \circ g_{K_t})(z)$ according to Proposition 2.13. Whence

$$|\xi_t - w| \leq |\xi_t| + |w| \leq 2\,\mathrm{rad}(K_{t,t+2h}), \quad |\xi_{t+h} - w'| \leq 2\,\mathrm{rad}(K_{t+h,t+2h}), \quad |w - w'| \leq 3\,\mathrm{rad}(K_{t,t+h}),$$

Indeed, since $\xi_t \in \overline{K_{t,t+2h}}$, and $w \in K_{t,t+2h}$, by the very definition of $\mathrm{rad}(K_{t,t+2h})$, we do have that

$$\xi_t, w \in \mathrm{rad}(K_{t,t+2h})\overline{\mathbb{D}} \implies \xi_t, w \in \overline{B}_{|.|}(0; \mathrm{rad}(K_{t,t+2h}))$$

We use the same argument for ξ_{t+h}, and the continuity estimate stated just before for the last inequality. Finally :

$$|\xi_{t+h} - \xi_t| \leq 2\,\mathrm{rad}(K_{t+h,t+2h}) + 3\,\mathrm{rad}(K_{t,t+h}) + 2\,\mathrm{rad}(K_{t,t+2h}) \to 0$$

as $h \to 0$, uniformly on compacts in t, by local growth property. □

$(\xi_t)_{t\geq 0}$ is the *Loewner transform* of $(K_t)_{t\geq 0}$. We shall see that the family of compact $\mathbb{H}$-hulls $(K_t)_{t\geq 0}$ can be reconstructed from its Loewner transform.

We say that $(K'_t)_{t\in]0,T[}$ is **parametrized by half-plane capacity** if for any $t \geq 0$: $\mathrm{hcap}(K_t) = 2t$.

We now come to Loewner's crucial observation: the local growth property implies that the mapping-out functions satisfy a differential equation.

Proposition 3.1. *Let $(K_t)_{t\geq 0}$ be an increasing family of compact $\mathbb{H}$-hulls, satisfying the local growth property and parametrized by half-plane capacity, and let $(\xi_t)_{t\geq 0}$ be its Loewner transform. Set $g_t = g_{K_t}$ and $\zeta(z) = \inf\{t \geq 0 : z \in K_t\}$. Then, for all $z \in \mathbb{H}$, the function $(g_t(z) : t \in [0, \zeta(z)[)$ is differentiable, and satisfies Loewner's differential equation*

$$\dot{g}_t(z) = \frac{2}{g_t(z) - \xi_t}.$$

Moreover, if $\zeta(z) < \infty$, then $g_t(z) - \xi_t \to 0$ as $t \to \zeta(z)$.

Proof. Let $0 \leq s < t < \zeta(z)$ and set $z_t = g_t(z)$. Note that $\mathrm{hcap}(K_s) + \mathrm{hcap}(K_{s,t}) = \mathrm{hcap}(K_t)$, so $\mathrm{hcap}(K_{s,t}) = 2(t - s)$. Also, $g_{K_{s,t}}(z_s) = z_t$ and $K_{s,t} \subseteq \xi_s + 2\mathrm{rad}(K_{s,t})\overline{\mathbb{D}}$. We apply a continuity estimate

:

$$|z_t - z_s| \leq 3\,\mathrm{rad}(K_{s,t}),$$

By local growth property, we see that $(z_t : t \in [0, \zeta(z)))$ is continuous. Now, provided $|z_s - \xi_s| \geq 4\,\mathrm{rad}(K_{s,t})$, from the differentiability estimate in Section 2.3:

$$\left| z_t - z_s - \frac{2(t-s)}{z_s - \xi_s} \right| \leq \frac{C \cdot 2\mathrm{rad}(K_{s,t})\mathrm{hcap}(K_{s,t})}{|z_s - \xi_s|^2} = \frac{4C \cdot \mathrm{rad}(K_{s,t})(t-s)}{|z_s - \xi_s|^2}$$

Then $t \mapsto |z_t - \xi_t|$ is positive and continuous on $[0, \zeta(z)[$, and so is locally uniformly positive. Again, local growth property show that $(z_t : t \in [0, \zeta(z)[)$ is differentiable with $\dot{z}_t = 2/(z_t - \xi_t)$. Moreover, if $\zeta(z) < \infty$, then for $s < \zeta(z) < t$ we have $z \in K_t \setminus K_s$, so $z_s \in K_{s,t}$, so $|z_s - \xi_s| \leq 2\,\mathrm{rad}(K_{s,t})$, and so by the local growth property $|z_s - \xi_s| \to 0$ as $s \to \zeta(z)$. □

3.1.1 Loewner's example

Let $(\gamma_t)_{t \geq 0}$ a family of curves and the hulls $(K_t)_{t\geq 0}$ given by $K_t = \gamma((0,t])$ with Loewner transform $(\xi_t)_{t\geq 0}$.

This section serves as a guide to understanding how the geometry of a curve $(\gamma_t)_{t\geq 0}$ is encoded through the Loewner transform $(\xi_t)_{t\geq 0}$ associated with the hulls $(K_t)_{t\geq 0}$, where $K_t = \gamma((0,t])$. This discussion helps clarify how the choice of driving function influences the geometry of the generated curve.

Let$\alpha \in (0, \pi/2)$ and take $\gamma(t) = r(t)e^{i\alpha}$, where $r(t)$ is chosen so that $\mathrm{hcap}(K_t) = 2t$. Note that the scaling map $z \mapsto \lambda z$ takes $H_t := \mathbb{H} \setminus K_t$ to $H_{\lambda^2 t}$, so the mapping-out functions $g_t = g_{K_t}$ satisfy $g_{\lambda^2 t}(z) = \lambda g_t(z/\lambda)$. Differentiating both sides with respect to t, we get:

$$\partial_t(g_{\lambda^2 t})(z) = \lambda^2 \cdot \partial_t g_{\lambda^2 t}(z) = \lambda^2 \cdot \frac{2}{g_{\lambda^2 t}(z) - \xi_{\lambda^2 t}},$$

while

$$\partial_t \lambda g_t\left(\frac{z}{\lambda}\right) = \lambda \cdot \frac{2}{g_t(z/\lambda) - \xi_t} = \frac{2\lambda^2}{g_{\lambda^2 t}(z) - \lambda\xi_t}.$$

Identifying both expressions yields:

$$\frac{2\lambda^2}{g_{\lambda^2 t}(z) - \lambda\xi_t} = \lambda^2 \cdot \frac{2}{g_{\lambda^2 t}(z) - \xi_{\lambda^2 t}},$$

which implies that

$$\xi_{\lambda^2 t} = \lambda \xi_t.$$

So $\xi_t = c_\alpha \sqrt{t}$, where $c_\alpha = \xi_1$. The value of c_α is known, but we will prove that $c_\alpha > 0$. To see this, fix τ so that $\mathrm{rad}(K_\tau) = 1$ and from J.Norris paper, some property about the mapping out function allows us to take $\varepsilon > 0$ such that there exists $b > 1$ with $g_\tau(b) \leq b + \varepsilon$ and $g_\tau(-b) \geq -b - \varepsilon$. Write δ^- for the interval of δH_τ from $-b$ to γ_τ and δ^+ for the interval of δH_τ from γ_τ to b. Then, for $y > 1$

$$\mathbb{P}_{iy}(\hat{B}_{T(H_\tau)} \in \delta^-) \geq \mathbb{P}_{iy}(\hat{B}_{T(H_\tau)} \in \delta^+).$$

Indeed, this is pretty intuitive since we have an acute angle $\alpha \in]0; 2\pi[$ from γ_τ to b, and the brownian motion starts from above at some iy. The following picture gives us a geometric idea :

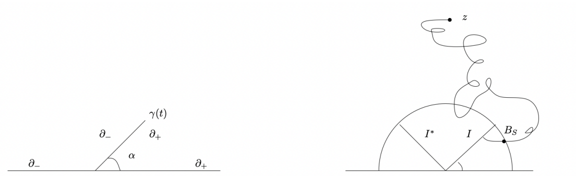

Figure 6: a. ∂_- and ∂_+. b. If $\arg(B_S) \leqslant \alpha$ then $B_T \in \partial_+$.

Figure 11: Caption

Another idea would be the following, from the results about harmonic measure, one could use conformal invariance of brownian motion and try to parametrize $\delta H_\tau = \delta^+ \cup \delta^-$. In that case, one should have some conformal isomorphism sending $\mathbb{D} \cap \mathbb{H}$ over H_τ such that

$$\begin{aligned}
\mathbb{P}_{iy}(\hat{B}_{T(H_\tau)} \in \delta^-) &= \mathbb{P}_0(B_{T(\mathbb{D})} \in U_\alpha) && (2)\\
&\geq \mathbb{P}_0(B_{T(\mathbb{D})} \in [0; 2\pi[\setminus U_\alpha) && (3)\\
&= \mathbb{P}_{iy}(\hat{B}_{T(H_\tau)} \in \delta^-) && (4)
\end{aligned}$$

where $U_\alpha \subset]0; 2\pi[$ depending on α. Now multiply by πy and let $y \to \infty$, by a result stated before, we deduce that

$$g_\tau(\gamma_\tau) - g_\tau(-b) \geq g_\tau(b) - g_\tau(\gamma_t).$$

Now $g_\tau(\gamma_\tau) = \xi_\tau = c_\alpha\sqrt{\tau}$, so $2c_\alpha\sqrt{\tau} = 2\xi_\tau \geq g_\tau(b) + g_\tau(-b) \geq 2\varepsilon$. Since $\varepsilon > 0$ was arbitrary, this implies that $c_\alpha \geq 0$. But we cannot have $c_\alpha = 0$, since this corresponds to the case $\alpha = \pi/2$. In fact, c_α is decreasing in α with $c_\alpha \to \infty$ as $\alpha \to 0$. Note the infinite initial velocity required for the Loewner transform needed to achieve a “turn to the right” with greater angle of turn for greater c_α. For a “turn to the left”, we take $\xi_t = -c_\alpha\sqrt{t}$.

3.2 From the Loewner transform to the Compact $\mathbb{H}$-hulls..

In the previous sections, we focused on the forward direction of Loewner theory: given a driving function $(\xi_t)_{t\geq 0}$, we generate a family of growing hulls $(K_t)_{t\geq 0}$ by solving Loewner's differential equation. We now study the inverse direction, which establishes that, under appropriate conditions, any continuous real-valued function can serve as a driving term to define such a family.

This section formalizes the so-called inverse Loewner transform. Starting from a function $t \mapsto \xi_t$, we solve the Loewner equation to construct conformal maps and associated hulls. This construction provides a foundation for identifying the scaling limits of discrete random curves with SLE, since it shows how geometry emerges from a stochastic process.

Proposition 3.2. *For all $z \in \mathbb{C} \setminus \{\xi_0\}$, there is a unique $\zeta(z) \in]0, \infty[\cup \{\infty\}$ and a unique continuous map $t \mapsto g_t(z) : [0, \zeta(z)[\to \mathbb{C}$ such that, for all $t \in [0, \zeta(z)[$, we have $g_t(z) \neq \xi_t$ and*

$$g_t(z) = z + \int_0^t \frac{2}{g_s(z) - \xi_s}\, ds$$

and such that $|g_t(z) - \xi_t| \to 0$ as $t \to \zeta(z)$ whenever $\zeta(z) < \infty$. Set $\zeta(\xi_0) = 0$ and define $C_t = \{z \in \mathbb{C} : \zeta(z) > t\}$. Then, for all $t \geq 0$, C_t is open, and $g_t : C_t \to \mathbb{C}$ is holomorphic.

The process $(g_t(z) : t \in [0, \zeta(z)))$ is the *maximal solution starting from* z, and $\zeta(z)$ is its *lifetime*. Define

$$K_t = \{z \in \mathbb{H} : \zeta(z) \leq t\}, \quad H_t = \{z \in \mathbb{H} : \zeta(z) > t\} = \mathbb{H} \setminus K_t.$$

The family of maps $(g_t)_{t\geq 0}$ is then called the *Loewner flow* (in $\mathbb{H}$) *with driving function* $(\xi_t)_{t\geq 0}$.

Proposition 3.3. *The family of sets $(K_t)_{t\geq 0}$ is an increasing family of compact $\mathbb{H}$-hulls having the local growth property. Moreover hcap$(K_t) = 2t$ and $g_{K_t} = g_t$ for all t. Moreover the driving function $(\xi_t)_{t\geq 0}$ is the Loewner transform of $(K_t)_{t\geq 0}$.*

The proof will not be given for the sake of conciseness but can be found in [4]

3.2.1 The Loewner Kufarev Theorem

The following theorem unifies the previous two parts by canonically associating a family of compact hulls to a given process, now within a stochastic framework that we proceed to describe. Write $\mathcal{K}$ for the set of all compact $\mathbb{H}$-hulls. Fix a metric d_∞ of uniform convergence on compacts for $C(\mathbb{H}, \mathbb{H})$, in the sense that

$$d_\infty(f, h) := \sup_{z\in\mathbb{H}} |f(z) - h(z)| \;\; \forall f, h \in C(\mathbb{H}, \mathbb{H})$$

We make $\mathcal{K}$ into a metric space using this metric :

$$d_{\mathcal{K}}(K_1, K_2) = d_\infty(g_{K_1}^{-1}, g_{K_2}^{-1}).$$

Write $\mathcal{L}$ for the set of increasing families of compact $\mathbb{H}$-hulls $(K_t)_{t\geq 0}$ having the local growth property and such that hcap$(K_t) = 2t$ for all t. Then $\mathcal{L} \subset C([0, \infty[, \mathcal{K})$. We fix on $C([0, \infty[, \mathcal{K})$ a metric of uniform convergence on compact time intervals.

Theorem 3.1. *There exists a bi-adapted homeomorphism $L : C([0, \infty[, \mathbb{R}) \to \mathcal{L}$ given by*

$$L((\xi_t)_{t\geq 0}) = (K_t)_{t\geq 0}, \quad K_t = \{z \in \mathbb{H} : \zeta(z) \leq t\}$$

where $\zeta(z)$ is the lifetime of the maximal solution to Loewner's differential equation

$$\dot{z}_t = \frac{2}{z_t - \xi_t}$$

starting from z.

Moreover $(\xi_t)_{t\geq 0}$ is then the Loewner transform *of $(K_t)_{t\geq 0}$, given by*

$$\{\xi_t\} = \bigcap_{s>t} \overline{K}_{t,s}, \quad K_{t,s} = g_{K_t}(K_s \setminus K_t)$$

where g_{K_t} is the mapping-out function for K_t.

We call L the Loewner map. *The proof that L and its inverse are continuous and adapted will not be given.*

However, from 3.3, it can be easily proved that L is surjective since we can associate to any $(K_t) \in \mathcal{L}$, its Loewner Transform.

The Loewner–Kufarev theorem formalizes the link between a continuous driving function and the growth of compact sets in the upper half-plane. This result will serve as the basis for introducing randomness into the picture.

Rather than fixing a deterministic function, we now equip the space of hulls with a probabilistic structure that allows us to define Loewner evolutions driven by random processes. This is the framework in which Schramm's construction of SLE takes place

3.2.2 About the Probabilistic Structure

We equip the space $\mathcal{K}$ of compact $\mathbb{H}$-hulls with the topology induced by the uniform convergence on compact subsets of the associated inverse Loewner maps. Since $C(\mathbb{H}, \mathbb{H})$, the space of continuous functions from $\mathbb{H}$ to itself endowed with the topology of uniform convergence on compact subsets, is Polish (separable by density of rational points and complete by standard arguments), it follows that $\mathcal{K}$, being a closed subset of $C(\mathbb{H}, \mathbb{H})$, is itself Polish.

As a consequence, the space of continuous trajectories

$$\Omega^{\mathcal{K}} = C([0, \infty), \mathcal{K})$$

equipped with the topology of uniform convergence on compact time intervals, is also Polish (see Ethier & Kurtz, *Markov Processes*, 1986, Proposition 7.5). We endow $\Omega^{\mathcal{K}}$ with its Borel σ-algebra $\mathcal{F}^{\mathcal{K}} = \mathcal{B}(\Omega^{\mathcal{K}})$.

The driving process $(\xi_t)_{t\geq 0}$ is modeled as a complex Brownian motion,

$$\xi_t = B_t^{(1)} + iB_t^{(2)},$$

where $B^{(1)}$ and $B^{(2)}$ are independent standard real Brownian motions. The canonical space associated with ξ is

$$\Omega^{\xi} = C([0, \infty), \mathbb{C})$$

equipped with the topology of uniform convergence on compact subsets.

We define the probability measure $\mathbb{P}^{\xi}$ on Ω^{ξ} as the product of two standard Wiener measures.

We then define the product probability space:

$$(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}) = (\Omega^{\xi} \times \Omega^{\mathcal{K}},\ \mathcal{F}^{\xi} \otimes \mathcal{F}^{\mathcal{K}},\ \mathbb{P}^{\xi} \otimes \mathbb{P}^{\mathcal{K}}),$$

where $\mathbb{P}^{\mathcal{K}}$ may be, for instance, the pushforward of $\mathbb{P}^{\xi}$ under the (continuous) Loewner transform

$$L : \Omega^{\xi} \to \Omega^{\mathcal{K}}, \quad \xi \mapsto (K_t^{\xi})_{t\geq 0}.$$

We equip $\tilde{\Omega}$ with the canonical filtration

$$\mathcal{F}_t = \sigma\left(\xi_s, K_s : 0 \leq s \leq t\right),$$

which contains all the information available up to time t about both the driving function and the associated hulls.

Under this structure, the driving process (ξ_t) is naturally adapted to $(\mathcal{F}_t)$, and, by continuity and bijectivity of the Loewner transform, the hull process (K_t) is adapted as well.

3.2.3 From deterministic Loewner chains to SLE

We now continue within the probabilistic framework introduced above.
A random Loewner chain is a stochastic process $(K_t)_{t\geq 0}$ with values in the space of compact $\mathbb{H}$-hulls $\mathcal{K}$, defined on the product space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ introduced in Section 4.3.2, and adapted to the canonical filtration $(\mathcal{F}_t)_{t\geq 0}$. The goal is to characterize those random evolutions that are generated by Brownian motion as the driving function. More precisely, given $\kappa > 0$, we define SLE_κ as the Loewner chain $(K_t)_{t\geq 0}$ obtained by solving the Loewner equation with driving function $\xi_t = \sqrt{\kappa} B_t$, where $(B_t)_{t\geq 0}$ is a standard Brownian motion.

As we will see, such evolutions are distinguished by two properties: scale invariance, and the domain Markov property. We define these below.

Scale invariance. Let $(K_t)_{t\geq 0} \in \mathcal{L}$ be a random Loewner chain. For any $\lambda > 0$, define the rescaled family

$$K_t^{\lambda} := \lambda K_{t/\lambda^2}.$$

We say that $(K_t)_{t\geq 0}$ is *scale invariant* if $(K_t^{\lambda})_{t\geq 0}$ has the same distribution as $(K_t)_{t\geq 0}$ for all $\lambda > 0$.

Domain Markov property. Let $(K_t)_{t\geq 0} \in \mathcal{L}$ be a random Loewner chain, and let $s \geq 0$. Define the shifted hulls

$$K_t^{(s)} := g_{K_s}(K_{s+t} \setminus K_s) - \xi_s,$$

where g_{K_s} is the mapping-out function and ξ_s the driving function at time s. Then $(K_t)_{t\geq 0}$ is said to satisfy the *domain Markov property* if $(K_t^{(s)})_{t\geq 0}$ has the same distribution as $(K_t)_{t\geq 0}$, and is independent of $\mathcal{F}_s$.

This expresses the memoryless nature of the evolution: given the domain at time s, the future dynamics are independent of the past and evolve as a fresh copy of the process in the remaining domain.

We are now ready to state the central result of this theory, due to Rohde and Schramm, which shows that these two natural properties characterize the law of SLE_κ uniquely among random Loewner chains.

Theorem 3.2. *Let $(K_t)_{t\geq 0}$ be a random Loewner chain in $\mathcal{L}$. Then $(K_t)_{t\geq 0}$ is distributed as* SLE_κ *if and only if it is scale invariant and satisfies the domain Markov property.*

This result confirms that the SLE is the only natural candidate for a conformally invariant scaling limit of planar models with local Markovian growth.

Proof. Let $(\xi_t)_{t\geq 0}$ the Loewner transform of $(K_t)_{t\geq 0}$ and remember that $(\xi_t)_{t\geq 0}$ is continuous. For $\lambda > 0$ and $s \in [0, \infty[$, define

$$\xi_t^\lambda = \lambda \xi_{\lambda^{-2}t} \quad \text{and} \quad \xi_t^{(s)} = \xi_{s+t} - \xi_s.$$

Then $(K_t^\lambda)_{t\geq 0}$ has Loewner transform $(\xi_t^\lambda)_{t\geq 0}$ and $(K_t^{(s)})_{t\geq 0}$ has Loewner transform $(\xi_t^{(s)})_{t\geq 0}$. Hence, $(K_t)_{t\geq 0}$ has the domain Markov property if and only if $(\xi_t)_{t\geq 0}$ has stationary independent increments. Also, $(K_t)_{t\geq 0}$ is scale invariant if and only if the law of $(\xi_t)_{t\geq 0}$ is invariant under Brownian scaling. By the Lévy–Khinchin Theorem (See Theorem 1.2)As usual, our default assumption is that Brownian motion starts at 0, $(\xi_t)_{t\geq 0}$ has both these properties if and only if it is a Brownian motion of some diffusivity $\kappa \in [0, \infty)$, that is to say, if and only if $(K_t)_{t\geq 0}$ is an SLE.

$\square$

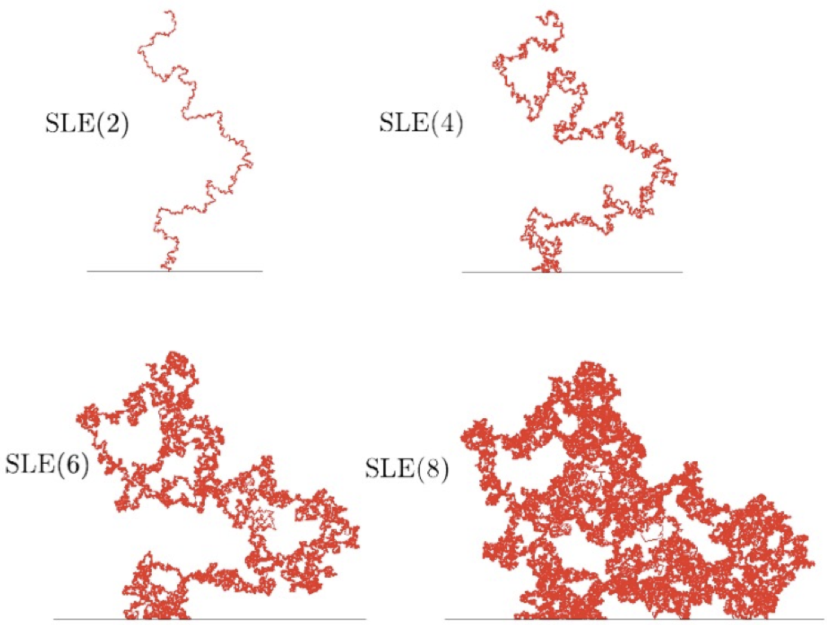

Figure 12: Simulation of SLE_κ for $\kappa = 2, 4, 6, 8$. Simulations produced by Tom Kennedy and Hao Wu.

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