A Polynomial Approach to the van Everdingen-Hurst Dimensionless Variables for Water Encroachment

M.A. Klins,* SPE, Pennsylvania State U.
A.J. Bouchard, SPE, Pennsylvania State U.
C.L. Cable,** SPE, Pennsylvania State U.

SPE 15433

Summary. Traditional water-influx calculations rely on accurate values of the van Everdingen and Hurst dimensionless variables p_D and q_D. We have presented six sets of simple polynomials that provide a fast, simple method to determine p_D, p_D', and q_D for finite or infinite radial aquifers. The results yield values as accurate as the original tables and are up to 15 times more efficient.

Introduction

Classic reservoir engineering reserve estimates and simulation studies for water-driven reservoirs have relied on the traditional van-Everdingen-Hurst approach or Carter-Tracy modification to estimate water encroachment. The predicted volume of water influx into a reservoir is a function of one of two dimensionless variables, p_D or q_D, depending on which encroachment technique is used. To estimate values of p_D or q_D, table lookup and interpolation between time entries may be needed, and for finite aquifers, an additional interpolation between aquifer sizes may be needed. The van Everdingen-Hurst or Carter-Tracy table lookup and interpolation approach has several drawbacks: (1) storage requirements for computer applications are large, (2) the application is cumbersome and time-consuming, (3) there are questions regarding accuracy because of interpolation, (4) the tables are limited to finite aquifer/reservoir size ratios of less than 10 : 1, and (5) the Carter-Tracy approach requires derivatives of the p_D tabular values.

This paper presents four sets of simple polynomials that are easy to implement to obtain accurate values of p_D or q_D for either the finite or infinite radial aquifer case. Derivatives of the p_D polynomials have also been prepared. The average absolute errors between polynomially determined values of p_D for finite and infinite aquifers and the numerically correct solution are less than 0.03 and 0.02%, respectively. Similarly, average absolute errors between finite and infinite q_D estimates and their numerically correct counterparts are less than 0.10 and 0.05%, respectively.

The magnitude of these errors is on the same order as that exhibited by the original van Everdingen and Hurst tables. In addition, because aquifer size ratio and dimensionless time are implicit, no interpolation is needed. These analytic expressions also extend application to aquifer/reservoir ratios of 25 : 1. Last, the polynomial approach to estimate water encroachment is well suited for computer-based reservoir engineering studies, especially reservoir simulation, because a wider range of reservoir properties can be examined and no time-consuming matrix-search techniques are used.

Classic Water Encroachment

Estimating water influx into a producing reservoir holds inherent uncertainty. Aquifer characteristics including pressure, permeability, thickness, areal extent, and shape are usually all unknown and make aquifer fitting/prediction a high-risk trial-and-error procedure. Many authors, starting with Schilthuis in 1936,[1] have presented models for estimating water influx.[1-11] These models have application for different flow geometries (bottom, edge, linear, radial, etc.) and flow regimes, including steady-, modified steady-, pseudosteady-, and unsteady-state.

*Now at Chevron USA Inc.
**Now at Standard Alaska Production Co.

The most generally accepted solution to the water-encroachment problem was developed by van Everdingen and Hurst in 1949.[3] It provides a rigorously correct method for estimating water encroachment under all flow regimes practically encountered in water-influx calculations. In addition to their radial development, the van Everdingen-Hurst solutions can be equally applied to linear aquifers.

van Everdingen and Hurst describe the aquifer as a hollow, right cylinder with flow perpendicular into the center axis line. The inside surface of the cylinder represents the reservoir/aquifer interface, while the outer surface is the aquifer's boundary. Flow is radial, isothermal, single-phase, unsteady-state with the pressure distribution in the aquifer at any time being mathematically developed from solutions to the radial diffusivity equation:

$$\frac{\partial^2 p}{\partial r^2}+\frac{1}{r}\frac{\partial p}{\partial r}=\frac{\partial p}{\partial t}. \quad (1)$$

Two solutions were developed with Laplace transforms: the constant-terminal-rate case and the constant-terminal-pressure case. For the constant-terminal-rate case, flow rate across one of the two cylinder boundaries is given as a constant for a given time period, and one solves for the pressure drop throughout the reservoir as a function of time. In the constant-terminal-pressure case, pressure at one of the boundaries is fixed as a constant over a period of time, and one solves for flow rate.

It is this latter case, van Everdingen and Hurst's constant-terminal-pressure solution, that has found great utility for the water-influx problem. From production history, pressures at the reservoir/aquifer boundary are assumed to be known. If some average pressure is specified at the interface over a given time, flow rate and hence water influx into the reservoir can be estimated. If pressure continues to drop at the oil/water contact (OWC) over time, a number of constant-pressure steps can replace this declining pressure and superposition can be used for the solution to Eq. 1.

In short, water influx into a reservoir can be calculated for radial aquifers by use of superposition and the van Everdingen-Hurst constant-terminal-pressure solution to the diffusivity equation such that

$$W_e(t_{D_j})=B\sum_{k=0}^{j-1}(\Delta p_k)q_D(t_{D_j}-t_{D_k}), \quad (2)$$

where

$$B=1.119\phi h c_{wr} r_o^2 f, \quad (3)$$

$$\Delta p_k=\frac{p_{k-1}-p_{k+1}}{2}, \quad (4)$$

and

$$t_{D_j}=\frac{6.33kt_j}{\phi\mu_w c_{wr} r_o^2} \qquad (5)$$

are functions of time, reservoir geometry, and fluid properties. Dimensionless flow rate, q_D, is usually determined from table lookup with reduced time and dimensionless aquifer size (r_e/r_o).

Hurst[5] and later Carter and Tracy[6] used the van Everdingen-Hurst constant-terminal-rate solution to develop an alternative approach to calculate water influx that eliminated superposition. They proposed that water influx be estimated additively by

$$W_e(t_{D_j})=W_e(t_{D_{j-1}})+\left[\frac{B(\Delta p)-W_e(t_{D_{j-1}})p_D'{}_j}{p_{D_j}-t_{D_{j-1}}p_D'{}_j}\right](t_{D_j}-t_{D_{j-1}}), \qquad (6)$$

such that

$$p=p(0)-p(t_{D_j}). \qquad (7)$$

Dimensionless time, t_D, and the proportionality constant, B, are previously defined in Eqs. 3 and 5. Like q_D, the van Everdingen and Hurst reduced pressure function, p_D, is usually determined by table lookup and is listed as a function of reduced time and reduced aquifer size.

Estimating water influx requires multiple values of q_D for the van Everdingen-Hurst constant-pressure solution or p_D for the Carter-Tracy constant-terminal-rate solution. Two approaches have been developed to accomplish this: (1) table lookup and the accompanying interpolation, and (2) graphic analysis for hand calculations. Both approaches exhibit speed, storage, or accuracy shortcomings.

The focus of this paper, then, is to augment the above approaches to estimate p_D and q_D with a set of simple, yet accurate, polynomials. Several authors have presented equations for limited portions of either the p_D or q_D tables.[8,12-15] A complete and accurate set to replace the van Everdingen and Hurst tables adequately for both the constant-terminal-pressure and constant-terminal-rate, radial-flow cases (finite and infinite aquifers) is presented in the appendices.

Constant-Terminal-Rate Case, p_D

Finite Aquifers. van Everdingen and Hurst's dimensionless pressure is given by

$$p(t_D)=\frac{2}{(r_D^2-1)}\left(\frac{1}{4}+t_D\right)-\frac{(3r_D^4-4r_D^4\log_e r_D-2r^2{}_D-1)}{4(r_D^2-1)^2}$$
$$+2\sum_{n=1}^{\infty}\frac{e^{-\beta_n^2 t_D}J_1^2(\beta_n r_D)}{\beta_n^2[J_1^2(\beta_n r_D)-J_1^2(\beta_n)]}, \qquad (8)$$

where t_D is given in Eq. 5, r_D is defined as the ratio of aquifer radius to reservoir radius, and J_1 is a Bessel function of the first kind of order 1 (see Appendix A).

$\beta_1, \beta_2 \ldots \beta_n$ are defined as the roots of

$$[J_1(\beta_n r_D)Y_1(\beta_n)-J_1(\beta_n)Y_1(\beta_n r_D)]=0, \qquad (9)$$

where J_1 and Y_1 are Bessell functions of the first and second kind of order 1.

To use Eq. 8 to determine p_D in a straightforward, algebraic fashion, the number of terms in the infinite summation was limited to two and β_1 and β_2 were evaluated as continuous functions of r_D rather than through table lookup.

Values of p_D were calculated with up to 10 terms of the infinite-summation series. Results were then compared with values of p_D calculated with only two terms of the series. A $<0.1\%$ difference in answers resulted. This conclusion—that only two terms of the expansion are necessary to give the accuracy needed—was the same as that reached by van Everdingen and Hurst.

By using Eq. 9 and varying r_D from 2 to 25, 24 values of β_1 and β_2 were calculated. These values were fit with nonlinear regression techniques and found to have the form

$$\beta=b_0+b_1[\operatorname{csch}(r_D)]+b_2(r_D)^{b_3}+b_4(r_D)^{b_5}. \qquad (10)$$

Error analysis between the β roots predicted by the regression, Eq. 10, and the actual values determined by use of Eq. 9 shows that β_1 has an average difference of -0.0002%, an average absolute difference of 0.039%, and a maximum difference of 0.0863%. β_2 has an average difference of -0.0001%, an average absolute difference of 0.047%, and a maximum difference of 0.0995%.

The actual coefficients and calculation procedure for calculating β_1, β_2, and p_D for finite aquifers is given in Appendix B.

Infinite Aquifers. For infinite aquifers, the value of p_D as a function of dimensionless time is defined by van Everdingen and Hurst as

$$p_D=\frac{4}{\pi^2}\int_0^{\infty}\frac{(1-e^{-u^2 t_D})du}{u^3[J_1^2(u)+Y_1^2(u)]}. \qquad (11)$$

An analytical solution of this integral is not available, and numerical methods are difficult to use near the origin because of the asymptotic nature of the function. For evaluation, the integral was broken into two parts such that Eq. 11 becomes

$$p_D=\frac{4}{\pi^2}\int_0^{\delta}\frac{(1-e^{-u^2 t_D})du}{u^3[J_1^2(u)+Y_1^2(u)]}+\frac{4}{\pi^2}\int_{\delta}^{\infty}\frac{(1-e^{-u^2 t_D})du}{u^3[J_1^2(u)+Y_1^2(u)]}. \qquad (12)$$

If δ is chosen sufficiently small as to be the minimum of 0.001 or the square root of $(0.001/t_D)$, the following three substitutions can be made into the first integral of Eq. 12:

$J_1(u)=0.5$,

$Y_1(u)=2/(\pi u)$,

and

$(1-e^{-u^2 t_D})=u^2 t_D$.

This first integral in Eq. 12 can then be solved analytically such that

$$p_D=\frac{2t_D}{\pi}\left[\tan^{-1}\left(\frac{\delta^2\pi}{4}\right)\right]+\frac{4}{\pi^2}\int_{\delta}^{\infty}\frac{(1-e^{-u^2 t_D})du}{u^3[J_1^2(u)+Y_1^2(u)]}. \qquad (13)$$

With Simpson's rule, p_D can then be evaluated from Eq. 13. For a given value of t_D, more than 25,000 segments were used to evaluate p_D. The order of error in the integration was less than 1×10^{-8}. More than 500 values of p_D were calculated with this integration technique and curve fit by nonlinear regression analysis. The resultant polynomials are given in Appendix C.

Finite Aquifers Acting Infinitely. It is apparent that all aquifers act as if they are infinite for small values of dimensionless time. At later times, boundary effects are felt and finite aquifer behavior deviates accordingly. The purpose of this section, then, is to estimate for a given aquifer-to-reservoir ratio the dimensionless time at which boundary effects are felt. Once this crossover value of t_D is determined, the user can decide whether the finite or infinite set of polynomials is appropriate to calculate p_D because the finite polynomials do not yield accurate answers of p_D for values of t_D less than this crossover point.

By examining the intersection points of infinite and finite p_D curve fits and using regression analysis, the value of t_D at which boundary effects are exhibited can be estimated from

$$t_{cross}=b_0(r_D-1)+b_1(r_D-1)^{b_2}, \quad (14)$$

where
b_0 = 0.0980958,
b_1 = 0.100683, and
b_2 = 2.03863.

For values of $t_D<t_{cross}$, the aquifer is infinite-acting and one should use the infinite-aquifer approach in Appendix C to calculate p_D. If $t_D \geq t_{cross}$, use Appendix B for finite aquifers.

Derivatives of Dimensionless Pressure, p_D'. The Carter-Tracy method of estimating water influx requires values of the p_D derivative for implementation. Appendix D has been prepared with the information in Appendices B and C for evaluation. The decision to use either the finite- or infinite-acting case for finite aquifers still follows the crossover logic previously presented.

Constant-Terminal-Pressure Case, q_D

van Everdingen and Hurst's traditional approach to estimate water encroachment requires the determination of multiple values of q_D for use in Eq. 2. A parallel discussion similar to the development of the constant-terminal-rate case, p_D, for Carter-Tracy applications follows.

Finite Aquifers. van Everdingen and Hurst's dimensionless flow rate, sometimes described as reduced flow rate or flow rate influence function, is given by

$$q_D=\frac{r_D^2-1}{2}-2\sum_{n=1}^{\infty}\frac{e^{-\alpha_n^2 t_D}J_1^2(\alpha_n r_D)}{\alpha_n^2[J_0^2(\alpha_n)-J_1^2(\alpha_n r_D)]}, \quad (15)$$

where t_D is defined in Eq. 5, r_D is the ratio of aquifer radius to reservoir radius, and J_0 and J_1 are Bessel functions of the first kind of orders 0 and 1, respectively (see Appendix A).

$\alpha_1, \alpha_2 \ldots \alpha_n$ are defined as the roots of

$$[J_1(\alpha_n r_D)Y_0(\alpha_n)-Y_1(\alpha_n r_D)J_0(\alpha_n)]=0, \quad (16)$$

where J_0 and J_1 are Bessell functions of the first kind of orders 0 and 1, and Y_0 and Y_1 are Bessel functions of the second kind of orders 0 and 1.

To use Eq. 15 to determine q_D, the number of terms in the infinite summation was again limited to two, and α_1 and α_2 were evaluated as continuous functions of r_D rather than table lookup being used.

Values of q_D were calculated with up to 10 terms of the infinite-summation series in Eq. 15. Results were then compared with those values of q_D calculated by only two terms of the series. As with p_D, they yielded a <0.1% difference.

By use of Eq. 16 and with r_D varied from 2 to 25, 24 values of α_1 and α_2 were calculated. These values were fit with nonlinear-regression techniques and found to have the form

$$\alpha=b_0+b_1[\mathrm{csch}(r_D)]+b_2(r_D)^{b_3}+b_4(r_D)^{b_5}. \quad (17)$$

Error analysis between alpha values estimated by the regression-fit polynomials and the actual roots of Eq. 16 are such that α_1 values show an average difference of −0.0007%, an average absolute difference of 0.073%, and a maximum difference of 0.1566%. α_2 has an average error of −0.0003%, an average absolute difference of 0.048%, and a maximum difference of 0.0981%.

The actual coefficients and calculation procedure for α_1, α_2, and q_D in finite aquifers are presented in Appendix E.

Infinite Aquifers. For infinite aquifers, the value of q_D as a function of dimensionless time is defined by the integral

$$q_D=\frac{4}{\pi^2}\int_0^{\infty}\frac{(1-e^{-u^2 t_D})\mathrm{d}u}{u^3[J_0^2(u)+Y_0^2(u)]}. \quad (18)$$

An analytical solution of this integral has not been found and numerical methods are again difficult to use near the origin because of the asymptotic nature of the function. For evaluation purposes, a combination numerical/analytical approach was used. With the integral broken into two parts, Eq. 18 now becomes

$$q_D=\frac{4}{\pi^2}\int_0^{\delta}\frac{(1-e^{-u^2 t_D})\mathrm{d}u}{u^3[J_0^2(u)+Y_0^2(u)]}+\frac{4}{\pi^2}\int_{\delta}^{\infty}\frac{(1-e^{-u^2 t_D})\mathrm{d}u}{u^3[J_0^2(u)+Y_0^2(u)]}. \quad (19)$$

If δ is chosen sufficiently small as to be the minimum of 0.001 or the square root of $(0.001/t_D)$, the following three substitutions can be made into the first integral of Eq. 19:

$$J_0(u)=1.0,$$

$$Y_0(u)=\frac{2}{\pi}(\log_e u-0.115931508),$$

and

$$(1-e^{-u^2 t_D})=u^2 t_D.$$

This first portion of the integral in Eq. 19 can then be solved analytically such that

$$q_D=\frac{2t_D}{\pi}\left(\tan^{-1}\left\{\frac{2}{\pi}[\log_e(\delta)-0.115931508]\right\}+\frac{\pi}{2}\right)+\frac{4}{\pi^2}\int_{\delta}^{\infty}\frac{(1-e^{-u^2 t_D})\mathrm{d}u}{u^3[J_0^2(u)+Y_0^2(u)]}. \quad (20)$$

With Simpson's rule to approximate the second integral numerically, q_D can be evaluated. For a given value of t_D, more than 25,000 segments were used to evaluate q_D in Eq. 20. The order of error in the integration was $<1\times10^{-8}$.

More than 500 values of q_D were calculated with this integration technique. Comparison with the values of q_D presented by van Everdingen and Hurst showed a maximum table error of 4% for small values of dimensionless time. For values of $t_D>1$, there is virtually no difference between the values of q_D presented by van Everdingen and Hurst and those calculated with Eq. 20. The values of q_D from Eq. 20 were curve fit by nonlinear-regression techniques with the resultant polynomials given in Appendix F.

Finite Aquifers Acting Infinitely. All aquifers act as if they are infinite for small values of dimensionless time. At later times, boundary effects are felt and finite aquifer behavior deviates accordingly. The purpose of this section, then, is to determine, for a given aquifer-to-reservoir ratio, the value of t_D at which boundary effects are felt. It is this crossover value of t_D that determines which set of polynomials should be used to estimate q_D because the finite polynomials in Appendix E do not yield accurate answers of q_D for values of t_D less than this crossover point.

Therefore, the value of t_D at which boundary effects are exhibited is given by

$$t_{cross}=b_0+b_1(r_D)+b_2(r_D)^{b_4}+b_3[\log_e(r_D)]^{b_5}, \quad (21)$$

where
b_0 = −1.767,
b_1 = −0.606,

$b_2 = 0.12368,$
$b_3 = 3.02,$
$b_4 = 2.25,$ and
$b_5 = 0.50.$

One would expect the values of t_{cross} from Eqs. 14 and 21 to be the same. The curve fits for finite and infinite cases of q_D and p_D, however, intersect at slightly different values of dimensionless time. These equations, Eqs. 14 and 21, predict the points at which these two distinct sets intersect. While the intersection points for q_D and p_D are different, they introduce virtually no error into the dimensionless rate/pressure calculations.

For values of $t_D < t_{cross}$ calculated by Eq. 21, use the Appendix F infinite-aquifer approach to estimate q_D. If $t_D \geq t_{cross}$, use Appendix E for finite aquifers.

Sample Calculations

Because of the large number of coefficients presented in this paper, examples have been assembled for all cases (finite and infinite) in Appendices G and H. The reader is asked to refer to these samples as an aid before preparing his/her own applications.

Summary

The van Everdingen-Hurst approach and Carter-Tracy modification provide rigorous solutions to the radial-diffusivity equation. Application of these solutions relies on accurate values of either the dimensionless pressure function, p_D, or the dimensionless rate influence function, q_D. Values of p_D and q_D are generally derived from tables presented in the original work of van Everdingen and Hurst.

Table lookup is cumbersome, time consuming, and limited to r_e/r_o values <10 for finite aquifers (unless supplemental values have been calculated by the user), and usually requires large storage with single/double interpolation. In addition, if the Carter-Tracy water-influx technique is being applied, values of the derivative of p_D are needed.

This paper, then, has presented six sets of polynomials for the following dimensionless pressure/rate cases: p_D, finite aquifers; p_D, infinite aquifers; p_D', finite aquifers; p_D', infinite aquifers; q_D, finite aquifers; and q_D, infinite aquifers.

These simple equations provide values of p_D and q_D as accurate as the original van Everdingen and Hurst tables, use up to 15 times less computational time than traditional table lookup, and because r_D and t_D are implicit in the calculations, require no interpolation. For water-influx procedures, these equations represent a tractable replacement to tabular listings of the van Everdingen and Hurst dimensionless functions.

Nomenclature

b_m = mth regression coefficient
B = aquifer constant, bbl/psi [m^3/kPa]
c_r = rock compressibility, psi^{-1} [kPa^{-1}]
c_w = water compressibility, psi^{-1} [kPa^{-1}]
c_{wr} = effective compressibility of water and rock in aquifer, $c_w + c_r$, psi^{-1} [kPa^{-1}]
f = fraction of perimeter of circle that original oil/water boundary constitutes
F = argument of Bessel function
h = aquifer thickness, ft [m]
J_0 = Bessel function of first kind of order 0
J_1 = Bessel function of first kind of order 1
k = effective aquifer permeability to water, darcies
p = pressure at OWC, psi [kPa]
Δp = pressure drop at OWC, psi [kPa]
p_D = dimensionless pressure
p_D' = first derivative of dimensionless pressure
q_D = dimensionless flow rate
r = radius, ft [m]
r_D = dimensionless radius, r_e/r_o
r_e = radius to perimeter of aquifer, ft [m]
r_o = radius to perimeter of reservoir, ft [m]
t = time, days
t_D = dimensionless time
t_j = cumulative elapsed time at end of jth interval, days
u = argument of integration
W_e = cumulative water influx, bbl [m^3]
x = argument of Bessel function
Y_0 = Bessel function of second kind of order 0
Y_1 = Bessel function of second kind of order 1
α_n = nth root of Eq. 16
β_n = nth root of Eq. 9
δ = limit of integration
ϵ = error
θ = argument of Bessel function, degrees [rad]
μ_w = water viscosity, cp [Pa·s]
ϕ = aquifer porosity, fraction

Subscripts

j = value at a point in time
k = value at a point in time
cross = dimensionless time at which finite- and infinite-acting solutions are equivalent

References

1. Schilthuis, R.J.: "Active Oil and Reservoir Energy," *Trans.*, AIME (1936) **118**, 33–52.
2. Hurst, W.: "Water Influx Into a Reservoir and Its Application to the Equation of Volumetric Balance," *Trans.*, AIME (1943) **151**, 57–72.
3. van Everdingen, A.F. and Hurst, W.: "The Application of the Laplace Transformation to Flow Problems in Reservoirs," *Trans.*, AIME (1949) **186**, 305–24.
4. Pirson, S.J.: *Oil Reservoir Engineering*, McGraw-Hill Book Co., New York City (1959) 608.
5. Hurst, W.: "Simplification of the Material Balance Formulas by the Laplace Transformation," *Trans.*, AIME (1958) **213**, 292–308.
6. Carter, R.D. and Tracy, G.W.: "An Improved Method for Calculating Water Influx," *JPT* (Dec. 1960) 58–60; *Trans.*, AIME, **219**.
7. Fetkovich, M.J.: "A Simplified Approach to Water Influx Calculations—Finite Aquifer Systems," *JPT* (July 1971) 814–28.
8. Dake, L.P.: *Fundamentals of Reservoir Engineering*, Elsevier Science Publishers, New York City (1978) 303.
9. Allard, D.R. and Chen, S.M.: "Calculation of Water Influx for Bottom-Water Drive Reservoirs," paper SPE 13170 presented at the 1984 SPE Annual Technical Conference and Exhibition, Houston, Sept. 16–19.
10. Leung, W.F.: "A Fast Convolution Method for Implementing Single-Porosity Finite/Infinite Aquifer Models for Water-Influx Calculations," *SPERE* (Sept. 1986) 490–510.
11. Vogt, J.P. and Wang, B.: "Accurate Formulas for Calculating the Water Influx Superposition Integral," paper SPE 17066 presented at the 1987 SPE Eastern Regional Meeting, Pittsburgh, Oct. 21–23.
12. Matthews, C.S. and Russell, D.G.: *Pressure Buildup and Flow Tests in Wells*, Monograph Series, SPE, Richardson, TX (1967) **1**, 130–32.
13. Edwardson, M.J. *et al.*: "Calculation of Formation Temperature Disturbances Caused by Mud Circulation," *JPT* (April 1962) 416–25; *Trans.*, AIME, **225**.
14. Fanchi, J.R.: "Analytical Representation of the van Everdingen-Hurst Influence Functions for Reservoir Simulation," *SPEJ* (June 1985) 405–06.
15. *Reservoir Engineering Manual*, Scientific Software Corp., Denver, prepared for the U.S. Geological Survey (14-08-0001-13926) (July 1954).
16. Abramowitz, M. and Stegun, I.A.: *Handbook of Mathematical Functions*, Dover Publications, New York City (1968) 369–70.

Appendix A—Applied Mathematical Functions

A number of functions including the Bessel and hyperbolic cosecant[16] may not be supported by the computational facilities of the user. They are included here for completeness.

Bessel Functions of the First Kind of Order 0, J_0.

$0 \leq x < 3.0$.

$$J_0(x) \cong b_0 + b_1(x/3)^2 + b_2(x/3)^4 + b_3(x/3)^6 + b_4(x/3)^8 + b_5(x/3)^{10} + b_6(x/3)^{12}, \quad \text{(A-1)}$$

where

b_0 = 1.000,
b_1 = −2.2499997,
b_2 = 1.2656208,
b_3 = −0.3163866,
b_4 = 0.0444479,
b_5 = −0.0039444, and
b_6 = 0.0002100.

3.00≤x<∞.

$$J_0(x) \cong (x)^{-1/2}(F_0)[\cos(\theta_0)]. \quad \text{(A-2)}$$

$$F_0 = b_0 + b_1(3/x) + b_2(3/x)^2 + b_3(3/x)^3 + b_4(3/x)^4 + b_5(3/x)^5 + b_6(3/x)^6, \quad \text{(A-3)}$$

where

b_0 = 0.79788456,
b_1 = −0.00000077,
b_2 = −0.00552740,
b_3 = −0.00009512,
b_4 = 0.00137237,
b_5 = −0.00072805, and
b_6 = 0.00014476.

$$\theta_0 = x + b_0 + b_1(3/x) + b_2(3/x)^2 + b_3(3/x)^3 + b_4(3/x)^4 + b_5(3/x)^5 + b_6(3/x)^6, \quad \text{(A-4)}$$

where

b_0 = −0.78539816,
b_1 = −0.04166397,
b_2 = −0.00003954,
b_3 = 0.00262573,
b_4 = −0.00054125,
b_5 = −0.00029333, and
b_6 = 0.00013558.

Bessel Functions of the First Kind of Order 1, J_1.

0≤x<3.00.

$$(x)^{-1}J_1(x) \cong b_0 + b_1(x/3)^2 + b_2(x/3)^4 + b_3(x/3)^6 + b_4(x/3)^8 + b_5(x/3)^{10} + b_6(x/3)^{12}, \quad \text{(A-5)}$$

where

b_0 = 0.5000,
b_1 = −0.56249985,
b_2 = 0.21093573,
b_3 = −0.03954289,
b_4 = 0.00443319,
b_5 = −0.00031761, and
b_6 = 0.00001109.

3.000≤x<∞.

$$J_1(x) \cong (x)^{-1/2}(F_1)[\cos(\theta_1)]. \quad \text{(A-6)}$$

$$F_1 = b_0 + b_1(3/x) + b_2(3/x)^2 + b_3(3/x)^3 + b_4(3/x)^4 + b_5(3/x)^5 + b_6(3/x)^6, \quad \text{(A-7)}$$

where

b_0 = 0.79788456,
b_1 = 0.00000156,
b_2 = 0.01659667,
b_3 = 0.00017105,
b_4 = −0.00249511,
b_5 = 0.00113653, and
b_6 = −0.00020033.

$$\theta_1 = x + b_0 + b_1(3/x) + b_2(3/x)^2 + b_3(3/x)^3 + b_4(3/x)^4 + b_5(3/x)^5 + b_6(3/x)^6, \quad \text{(A-8)}$$

where

b_0 = −2.35619449,
b_1 = 0.12499612,
b_2 = 0.00005650,
b_3 = −0.00637879,
b_4 = 0.00074348,
b_5 = 0.00079824, and
b_6 = −0.00029166.

Hyperbolic Cosecant, csch(x).

$$\mathrm{csch}(x) = \frac{2}{e^x - e^{-x}}. \quad \text{(A-9)}$$

Appendix B—Constant-Terminal-Rate Case: p_D for Finite Aquifers

1. Enter t_D and r_D.
2. If $t_D < t_{cross}$ (see Eq. 14), the aquifer is acting infinitely; use Appendix C.
3. Estimate β_1 and β_2 from

$$\beta_1 = b_0 + b_1[\mathrm{csch}(r_D)] + b_2(r_D)^{b_3} + b_4(r_D)^{b_5},$$

where

b_0 = −0.00870415,
b_1 = −1.08984,
b_2 = 12.4458,
b_3 = −2.8446,
b_4 = 3.4234, and
b_5 = −0.949162; and

$$\beta_2 = b_0 + b_1[\mathrm{csch}(r_D)] + b_2(r_D)^{b_3} + b_4(r_D)^{b_5},$$

where

b_0 = −0.0191642,
b_1 = −2.47644,
b_2 = 25.3343,
b_3 = −2.73054,
b_4 = 6.13184, and
b_5 = −0.939529.

4. Calculate p_D such that

$$p_D = \frac{2}{(r_D^2 - 1)}\left(\frac{1}{4} + t_D\right) - \frac{(3r_D^4 - 4r_D^4 \log_e r_D - 2r_D^2 - 1)}{4(r_D^2 - 1)^2} + \frac{2e^{-\beta_1^2 t_D} J_1^2(\beta_1 r_D)}{\beta_1^2 [J_1^2(\beta_1 r_D) - J_1^2(\beta_1)]} + \frac{2e^{-\beta_2^2 t_D} J_1^2(\beta_2 r_D)}{\beta_2^2 [J_1^2(\beta_2 r_D) - J_1^2(\beta_2)]} + \epsilon.$$

The error analysis is the difference between p_D values with the five terms in Eq. 8 and the two-term polynomial approach presented here. The average error is −0.0100%, average absolute error is 0.0203%, and maximum absolute error is 0.0764%.

Appendix C—Constant-Terminal-Rate Case: p_D for Infinite Aquifers

1. $t_D \le 0.01$.

$$p_D = \frac{2}{\pi}\sqrt{t_D}.$$

2. $0.01 \le t_D < 500$.

$$p_D = \frac{b_0(t_D)^{b_6} + b_1(t_D) + b_2(t_D)^{b_7}}{b_3 + b_4(t_D)^{b_6} + b_5(t_D) + (t_D)^{b_7}} + \epsilon,$$

where
b_0 = 107.5868,
b_1 = 37.60613,
b_2 = 7.038188,
b_3 = 95.13748,
b_4 = 77.0034,
b_5 = 16.63856,
b_6 = 0.5003552, and
b_7 = 1.338479.

The error analysis is the difference between p_D values with Eq. 13 and the polynomials presented here. The average error is 0.0044%, average absolute error is 0.0051%, and maximum absolute error is 0.0167%.

3. $500 \le t_D$.

$$p_D = \frac{1}{2}[\log_e(t_D)]\left(1 + \frac{1}{2t_D}\right) + 0.40454\left(1 + \frac{1}{2t_D}\right) + \epsilon.$$

The error analysis is the difference between p_D values with Eq. 13 and the polynomials presented here. The average error is 0.0193%, average absolute error is 0.0193%, and maximum absolute error is 0.0238%.

Appendix D—Constant-Terminal-Rate Case: p_D' for Finite and Infinite Aquifers

Infinite Aquifers.

1. $t_D \le 0.01$.

$$p_D' = 1/\sqrt{\pi t_D}.$$

2. $0.01 \le t_D < 500$.

$$p_D' = \frac{b_0 + b_1(t_D)^{b_6} + b_2(t_D)^{b_7} + b_3(t_D)^{b_8} + b_4(t_D)^{b_9} + b_5(t_D)^{b_{10}}}{[b_{11} + b_{12}(t_D)^{b_7} + b_{13}(t_D) + t_D{}^{b_9}]^2},$$

where

b_0 = 3577.752441,
b_1 = 5121.404179,
b_2 = 552.462473,
b_3 = 364.062209,
b_4 = 26.908805,
b_5 = 896.239475,
b_6 = −0.499645,
b_7 = 0.5003552,
b_8 = 0.838834,
b_9 = 1.338479,
b_{10} = 0.338479,
b_{11} = 95.13748,
b_{12} = 77.0034, and
b_{13} = 16.63856.

3. $500 \le t_D$.

$$p_D' = \frac{1}{2t_D}\left[1 - \frac{\log_e(t_D)}{2t_D} + \frac{0.09546}{t_D}\right].$$

Finite Aquifers

1. $t_{cross} \le t_D$.

$$p_D' = \frac{2}{r_D^2 - 1} - \frac{2e^{-\beta_1^2 t_D} J_1^2(\beta_1 r_D)}{J_1^2(\beta_1 r_D) - J_1^2(\beta_1)} - \frac{2e^{-\beta_2^2 t_D} J_1^2(\beta_2 r_D)}{J_1^2(\beta_2 r_D) - J_1^2(\beta_2)}.$$

Appendix E—Constant-Terminal-Pressure Case: q_D for Finite Aquifers

1. Enter t_D and r_D.
2. If $t_D < t_{cross}$ (see Eq. 21), the aquifer is acting infinitely; use Appendix F.
3. Estimate α_1 and α_2 from

$$\alpha_1 = b_0 + b_1[\text{csch}(r_D)] + b_2(r_D)^{b_3} + b_4(r_D)^{b_5},$$

where
b_0 = −0.00222107,
b_1 = −0.627638,
b_2 = 6.277915,
b_3 = −2.734405,
b_4 = 1.2708, and
b_5 = −1.100417; and

$$\alpha_2 = b_0 + b_1[\text{csch}(r_D)] + b_2(r_D)^{b_3} + b_4(r_D)^{b_5},$$

where
b_0 = −0.00796608,
b_1 = −1.85408,
b_2 = 18.71169,
b_3 = −2.758326,
b_4 = 4.829162, and
b_5 = −1.009021.

4. Calculate q_D such that

$$q_D = \frac{r_D^2 - 1}{2} - \frac{2e^{-\alpha_1^2 t_D} J_1^2(\alpha_1 r_D)}{\alpha_1^2[J_0^2(\alpha_1) - J_1^2(\alpha_1 r_D)]} - \frac{2e^{-\alpha_2^2 t_D} J_1^2(\alpha_2 r_D)}{\alpha_2^2[J_0^2(\alpha_2) - J_1^2(\alpha_2 r_D)]} + \epsilon.$$

The error analysis is the difference between q_D values with the five terms in Eq. 15 and the two-term polynomial approach presented here. The average error is 0.0029%, average absolute error is 0.0915%, and maximum absolute error is 0.2957%.

Appendix F—Constant-Terminal-Pressure Case: q_D for Infinite Aquifers

1. $t_D \le 0.01$.

$$q_D = (2/\sqrt{\pi})(\sqrt{t_D})$$

2. $0.01 \le t_D < 200$.

$$q_D = \frac{b_0(t_D)^{b_7} + b_1(t_D) + b_2(t_D)^{b_8} + b_3(t_D)^{b_9}}{b_4(t_D)^{b_7} + b_5(t_D) + b_6},$$

where
b_0 = 1.129552,
b_1 = 1.160436,
b_2 = 0.2642821,

$b_3 = 0.01131791,$
$b_4 = 0.5900113,$
$b_5 = 0.04589742,$
$b_6 = 1.00,$
$b_7 = 0.5002034,$
$b_8 = 1.500,$ and
$b_9 = 1.979139.$

The error analysis is the difference between q_D values with Eq. 20 and the polynomials presented here. The average error is 0.0021%, average absolute error is 0.0022%, and maximum absolute error is 0.0297%.

3. $200 \leq t_D \leq 2.0 \times 10^{12}$

$$q_D = 10^{\{b_0 + b_1 \log_e(t_D) + b_2 [\log_e(t_D)]^{b_3}\}} + \epsilon,$$

where
$b_0 = 4.39890,$
$b_1 = 0.43693,$
$b_2 = -4.16078,$ and
$b_3 = 0.090.$

The error analysis is the difference between q_D values with Eq. 20 and the polynomials presented here. The average error is 0.0002%, average absolute error is 0.0431%, and maximum absolute error is 0.2294%.

Appendix G—Constant-Terminal-Rate Case, p_D and p_D'

Finite Case Using Equations From Appendices A, B, and D.
1. Given $t_D = 20.0$ and $r_D = 10$.
2. Determine $t_{cross} = 9.7606$; because $t_D > t_{cross}$, the aquifer is acting finitely.

$\beta_1 = 0.3939,$
$J_1(\beta_1) = 0.1931,$
$J_1(\beta_2) = 0.3422,$
$p_D = 1.9690,$
$p_D' = 0.0247,$
$\beta_2 = 0.7325,$
$J_1(\beta_1 r_D) = -0.0424,$ and
$J_1(\beta_2 r_D) = 0.0895.$

Infinite Case Using Equations From Appendices C and D.
1. $t_D < 500$.
Given $t_D = 20.0$, determine $p_D = 1.9589$ and $p_D' = 0.0228$.
2. $t_D > 500$.
Given $t_D = 1000.0$, determine $p_D = 3.8606$ and $p_D' = 0.0005$.

Appendix H—Constant-Terminal-Pressure Case, q_D

Finite Case Using Equations From Appendices A and E.
1. Given $t_D = 20.0$ and $r_D = 10$.
2. Determine $t_{cross} = 18.7494$; because $t_D > t_{cross}$, the aquifer is acting finitely.

$\alpha_1 = 0.1101,$
$J_0(\alpha_1) = 0.9970,$
$J_0(\alpha_2) = 0.9391,$
$q_D = 12.2640,$
$\alpha_2 = 0.4975,$
$J_1(\alpha_1 r_D) = 0.4713,$ and
$J_1(\alpha_2 r_D) = -0.3247.$

Infinite Case Using Equations From Appendices A and F.
1. $t_D < 200$.
Given $t_D = 20.0$, determine $q_D = 12.3195$.
2. $t_D \geq 200$.
Given $t_D = 1000.0$, determine $q_D = 292.3163$.

SPERE

Original SPE manuscript received for review Oct. 5, 1986. Paper accepted for publication May 26, 1987. Revised manuscript received July 10, 1987. Paper (SPE 15433) first presented at the 1986 SPE Annual Technical Conference and Exhibition held in New Orleans, Oct. 5–8.