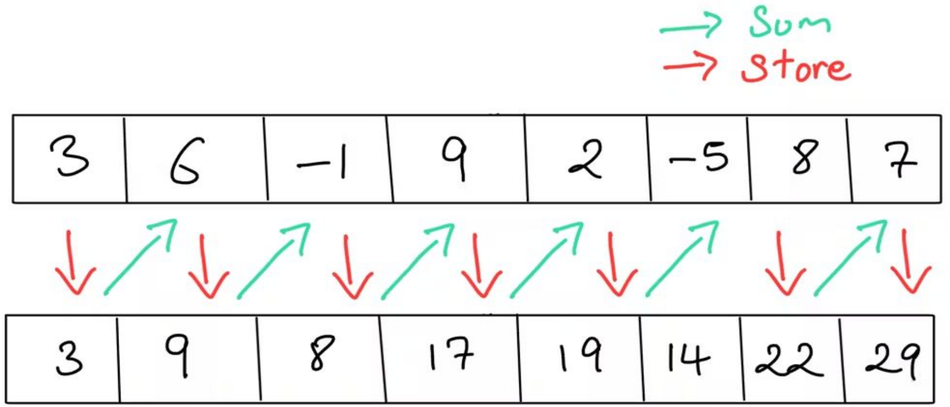
Prefix Sum



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Lecture Flow

- Pre-requisites
- Motivation
- Definition
- Variants
- Common Pitfalls
- Complexity Analysis
- Applications of prefix sum
- Practice Questions

Pre-requisites

1. Arrays and Matrices
2. Familiarity with Arithmetic Operations
3. Understanding Pointers or Indices

Motivation

Given an integer array `nums`, handle multiple queries of the following type:

1. Calculate the sum of the elements of `nums` between indices `left` and `right` inclusive where `left <= right`.

Implement the `NumArray` class:

- `NumArray(int[] nums)` Initializes the object with the integer array `nums`.
- `int sumRange(int left, int right)` Returns the sum of the elements of `nums` between indices `left` and `right` inclusive (i.e. `nums[left] + nums[left + 1] + ... + nums[right]`).

Example:

Input: ["NumArray", "sumRange", "sumRange", "sumRange"]

[[[-2, 0, 3, -5, 2, -1]], [0, 2], [2, 5], [0, 5]]

Output: [null, 1, -1, -3]

Prefix Sum

- Prefix sum, also known as cumulative sum, is a technique used to efficiently answer range sum queries on an array of numbers. It involves precomputing the sum of elements from the start of the array up to each index and storing the results in a separate array.

Variants

There are several variants of Prefix Sum:

1. Running Sum
2. 1D Prefix Sum
3. Inclusive Prefix Sum
4. Exclusive Prefix Sum
5. Range Updates
6. 2D Prefix Sum

Running Sum

- The summation of a sequence of numbers which is updated each time a new number is added to the sequence
  - done by adding the value of the new number to the previous running total.

- It allows the total to be stated at any point in time **without** having to sum the entire sequence each time
- It can save having to record the sequence itself, if the particular numbers are **not** individually important.

Running Sum

Example: consider the sequence <5, 8, 3, 2>

Total = 5 + 8 + 3 + 2

Now insert 6 to the sequence <5, 8, 3, 2, 6>

Total = 5 + 8 + 3 + 2 + 6

But if we regarded running_sum = 18 at first

now running_sum = 18 + 6 = 24

- we would not even need to know the sequence at all

Pair Programming

Problem Link

1D Prefix Sum

- The 1-dimensional prefix sum algorithm is a simple and efficient technique for calculating the cumulative sum of a sequence of numbers.
- The basic idea is to iterate over the sequence and compute the sum of all previous elements for each element in the sequence.
- The resulting array is called the prefix sum array.

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1D Prefix Sum

- Here's an example to illustrate the 1-dimensional prefix sum algorithm:

```
numbers    = [3,  1,  7,  0,  4,   1,   6]
prefix_sum = [3, 4, 11, 11, 15, 16, 22]
```

Pair Programming

Problem Link

Problem Pattern

Given an array of integers nums, calculate the pivot index of this array.

The pivot index is the index where the sum of all the numbers strictly to the left of the index is equal to the sum of all the numbers strictly to the index's right.

If the index is on the left edge of the array, then the left sum is 0 because there are no elements to the left. This also applies to the right edge of the array.

Return the leftmost pivot index. If no such index exists, return -1.

Input: nums = [1,7,3,6,5,6]

Output: 3

Explanation: The pivot index is 3.

Left sum = nums[0] + nums[1] + nums[2] = 1 + 7 + 3 = 11

Right sum = nums[4] + nums[5] = 5 + 6 = 11

Inclusive Prefix Sum

- Sums up all the elements **before** and **including** the current element

Given an array of n elements

$A = [a_0, a_1, a_2, a_3, \ldots, a_{n-1}]$

The inclusive prefix sum will be:

$prefix\_sum = [a_0,\ a_0 + a_1,\ a_0 + a_1 + a_2,\ \ldots,\ a_0 + \ldots + a_{n-2} + a_{n-1}]$

Inclusive Prefix Sum

Example:

A = [3, 1, 7, 0, 4, 1, 6, 3]

prefix_sum = [3, 4, 11, 11, 15, 16, 22, 25]

Exclusive Prefix Sum

- Sums up all the elements **before** and **not including** the current element

Given an array of n elements

$$A = [a_0,\ a_1,\ a_2,\ a_3, \ldots,\ a_{n-1}]$$

The exclusive prefix sum will be:

$$prefix\_sum = [i,\ i + a_0,\ a_0 + a_1,\ a_0 + a_1 + a_2,\ \ldots,\ a_0 + \ldots + a_{n-2} + a_{n-1}]$$

Where i is the identity / zero element

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Exclusive Prefix Sum

Example:

A = [3, 1, 7, 0, 4, 1, 6, 3]

prefix_sum = [0, 3, 4, 11, 11, 15, 16, 22, 25]

1D range updates using prefix sum

Motivating problem: Array Manipulation

Problem Summary:

- We are given an array and multiple queries asking us to add a constant value to all elements in the range l to r

1D range updates using prefix sum

- What if we were asked to add a constant value to the suffix of the array starting at index l, for multiple queries?

Example: given nums = [1, 3, 5, 7] and queries = [[1, 2], [2, 1]]

where queries[i] = [l, k]   for: 0 < i < len(queries),

l = index(0 - indexed),

k = constant value

1D range updates using prefix sum

- Instead of updating the whole suffix for each query, we can only increment the first index of the suffix for each query,
  - prefix_sum = [0,0,0,0]

    i.e. prefix_sum[l] += k.

    For i = 0: prefix_sum = [0, 2, 0, 0]

    For i = 1: prefix_sum = [0, 2, 1, 0]

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1D range updates using prefix sum

- Then after all queries, we take the prefix sum of the whole array.

prefix_sum = [0, 2, 3, 3]

- Add the prefix sum back to the original array

nums = [1, 5, 8, 10]

1D range updates using prefix sum

Now, what about updating the range l to r ?

- We can increment the suffix starting at index l by k and decrement the suffix starting at index r + 1 by k, for each query

  i.e nums[l] += k and

  nums[r + 1] -= k

  then take prefix sum of the array as before.

Pair Programming

Problem Link

2D Prefix Sum

- The 2D prefix sum is a technique used to efficiently calculate the sum of a sub-rectangle in a 2D array.
- The basic idea is to precompute the sum of all elements in the rectangle with upper left corner (0, 0) and lower right corner (i, j), where i and j are the row and column indices, respectively.

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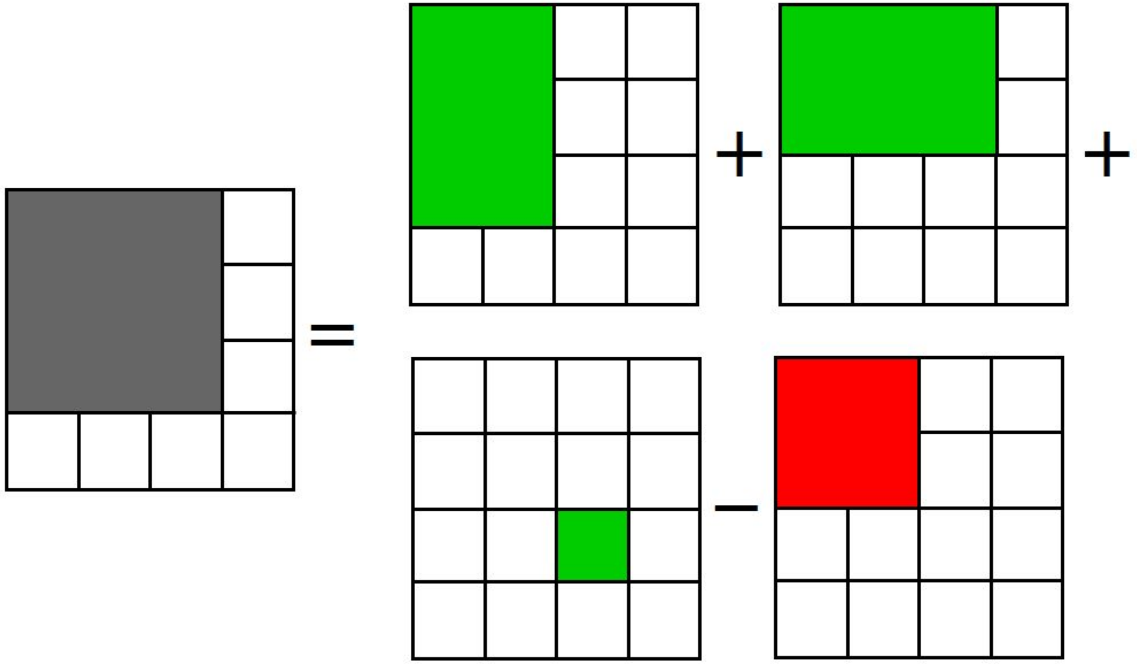
2D Prefix Sum

- This can be done using the 2D prefix sum array, which stores the cumulative sum of elements from (0, 0) to (i, j) for all i and j.

Formula: prefix[i][j] = prefix[i–1][j] + prefix[i][j–1] – prefix[i–1][j–1] + arr[i][j]

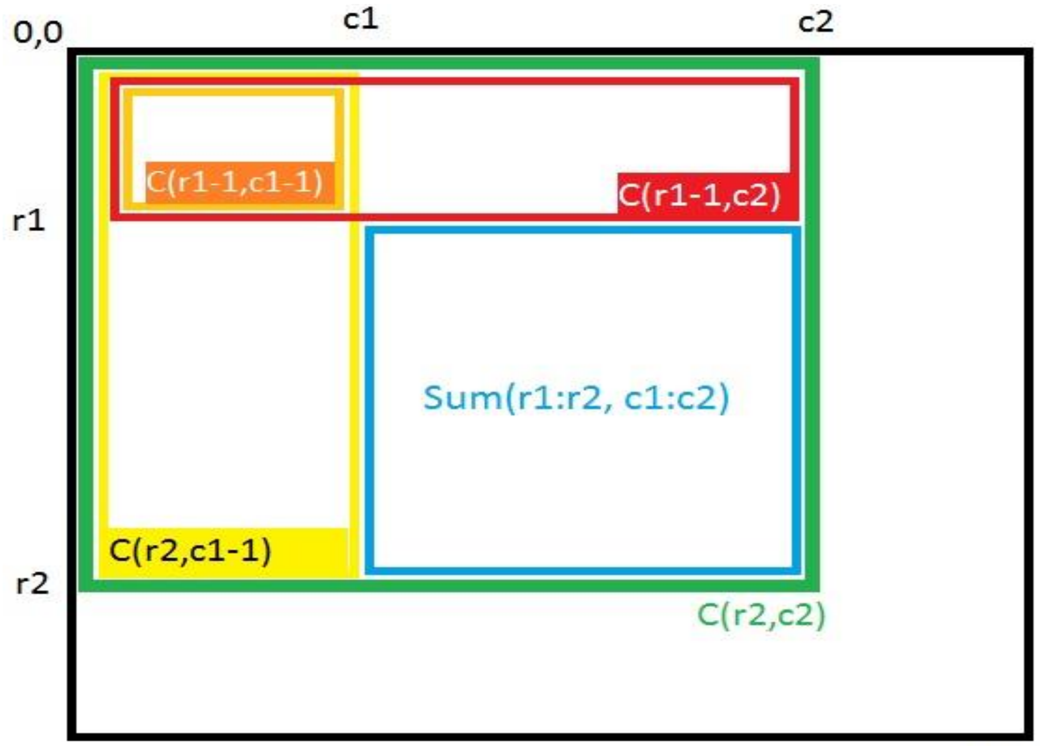
Constructing a 2D Prefix Sum Array

prefix[i][j] = prefix[i−1][j] + prefix[i][j−1] − prefix[i−1][j−1] + arr[i][j]



Accessing from a 2D Prefix Sum Array

submatrix_sum = prefix[r2][c2] – prefix[r2][c1 - 1] - prefix[r1 - 1][c2] + prefix[r1 - 1][c1 - 1]



Problem Pattern

Given a 2D matrix `matrix`, handle multiple queries of the following type:

- Calculate the sum of the elements of `matrix` inside the rectangle defined by its upper left corner `(row1, col1)` and lower right corner `(row2, col2)`.

Implement the `NumMatrix` class:

- `NumMatrix(int[][] matrix)` Initializes the object with the integer matrix `matrix`.
- `int sumRegion(int row1, int col1, int row2, int col2)` Returns the sum of the elements of `matrix` inside the rectangle defined by its upper left corner `(row1, col1)` and lower right corner `(row2, col2)`.

You must design an algorithm where `sumRegion` works on ***O(1)*** time complexity.

Pair Programming

[Problem Link]()

Common Pitfalls

There are several pitfalls on prefix sum:

1. Off-by-one Errors
2. Inconsistent Indexing
3. Forgetting to Initialize the Prefix Sum
4. Misunderstanding the Problem Requirements
5. Not Considering Edge Cases

Off-by-one Errors

These errors occur when we incorrectly compute the indices or bounds of the array, resulting in incorrect or unexpected results.

For example: suppose we have an array A of size n and we want to compute the sum of elements in the range [l, r]. If we use the prefix sum technique, we can compute the sum of this range as sum[r] - sum[l - 1], where sum[i] represents the cumulative sum of the first i elements of A.

Off-by-one Errors

original_arr = [1, 2, 3, 4, 5]

prefix_sum = [1, 3, 6, 10, 15]

Get the sum from index [0, 2] ?

Inconsistent Indexing

This can happen when we are computing the prefix sum and using different indexing schemes for the array and the prefix sum.

For example: suppose we have an array A of size n and we want to compute the prefix sum array S, where S[i] represents the sum of the first i elements of A. If we use a **0-based** indexing scheme for A but mistakenly use a **1-based** indexing scheme for S, we may end up with an inconsistent prefix sum that doesn't match the original array.

Forgetting to Initialize the Prefix Sum

This can result in an incorrect or unexpected results when computing range sums or updates.

For example: suppose we have an array A of size n and we want to compute the prefix sum array S. If we forget to initialize S[0] to A[0], we may end up with incorrect prefix sums for the rest of the array.

Misunderstanding the Problem Requirements

- Misunderstanding the Input Requirements:

  Don’t assume that the input array must be sorted or that it must have certain properties

- Misunderstanding the Output Requirements:

  Don’t assume that the output array must be sorted, or that it must have a specific size or format.

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Misunderstanding the Problem Requirements

- Misunderstanding the Algorithm Requirements:

  The optimal algorithm for prefix sum problems is to use a single pass over the input array to compute the prefix sums in linear time.

- Misunderstanding the Problem Constraints:

  It's essential to understand the problem constraints to ensure the solution is valid.

Not Considering Edge Cases

Edge cases to look out for while working with prefix sum questions:

- Empty array
- Single element array
- Array with negative elements
- Array with all negative elements
- Subarrays with first or last index of the input array
- Subarray of size 1
- Subarray with zero-sum
- Large input sizes

Complexity Analysis

- 1D Prefix Sum
  - Preprocessing
    - Time complexity: ?
    - Space complexity: ?
  - Query
    - Time complexity: ?
    - Space complexity: ?

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Complexity Analysis

- 1D Prefix Sum
  - Preprocessing
    - Time complexity: O(n)
    - Space complexity: O(n)
  - Query
    - Time complexity: O(1)
    - Space complexity: O(1)

Complexity Analysis

- 2D Prefix Sum
  - Preprocessing
    - Time complexity: ?
    - Space complexity: ?
  - Query
    - Time complexity: ?
    - Space complexity: ?

Complexity Analysis

- 2D Prefix Sum
  - Preprocessing
    - Time complexity: O(n*m)
    - Space complexity: O(n*m)
  - Query
    - Time complexity: O(1)
    - Space complexity: O(1)

Complexity Analysis

- 1D range updates for adding a constant to the range
  - Preprocessing
    - Time complexity: ?
    - Space complexity: ?
  - Query
    - Time complexity: ?
    - Space complexity: ?

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Complexity Analysis

- 1D range updates for adding a constant to the range
  - Preprocessing
    - Time complexity: O(n)
    - Space complexity: O(n)
  - Query
    - Time complexity: O(1)
    - Space complexity: O(1)

Applications of Prefix Sum

1. Computing subarray sums
2. Range updates and queries
3. Computing running averages
4. Counting occurrences
5. Dynamic programming

Practice Questions

- Array Manipulation
- Range Sum Query 2D - Immutable - LeetCode
- Subarray Sum Equals K - LeetCode
- Maximum Subarray - LeetCode
- Karen and Coffee
- Product of Array Except Self - LeetCode

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Quote of the Day

“By failing to prepare, you are preparing to fail”

Benjamin Franklin

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