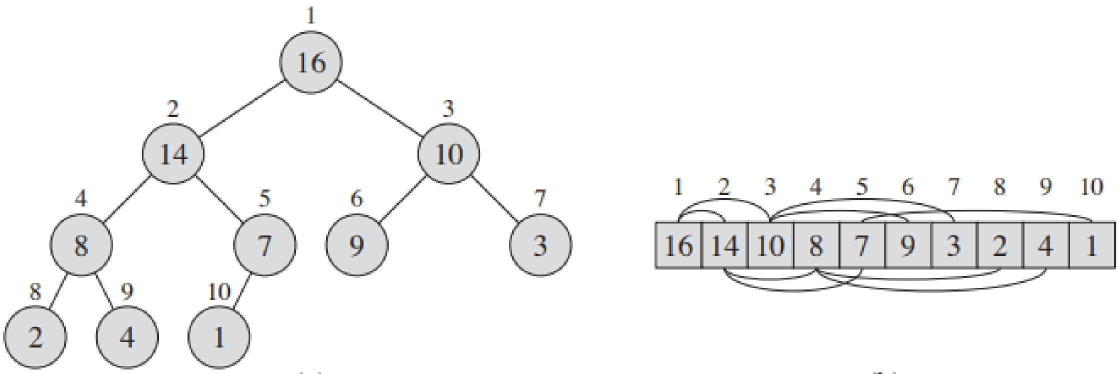
Heap Sort

Heaps 🧺

Binary Heap



(a) (b)

- a binary heap is an almost complete binary tree that is stored in an array
  - we will use a variable `size` to determine which parts of the array constitue the heap
- given index `i`, the following can be used to calculate a node's parent and left / right children

```
const parent = (i: number) => Math.floor(i / 2) + 1;
const left = (i: number) => 2 * i + 1;
const right = (i: number) => 2 * i + 2;
```

Max Heap Property

For any node i (excluding the root), $A[\text{parent}(i)] \geq A[i]$.

Code

`maxHeapify`

```
function maxHeapify(A: number[], i: number, size: number) {
    const l = left(i);
    const r = right(i);
    let largest: number;
    // find the largest tree
    if (l < size && A[l] > A[i]) largest = l;
    else largest = i;
    if (r < size && A[r] > A[largest]) largest = r;
    if (largest != i) {
        // swap
        swap(A, i, largest);
        // recurse
        maxHeapify(A, largest, size);
    }
}
```

- `maxHeapify` assumes that the trees at `left(i)` and `right(i)` are already max heaps, but the property is violiated at `i`
- it finds the largest of the two children and swaps with the parent `i`. Then it recurses down the child branch it swapped with
- it terminates when `i` is already the largest

`buildMaxHeap`

```
function buildMaxHeap(A: number[]) {
    for (let i = parent(A.length - 1); i >= 0; i--) {
        maxHeapify(A, i, A.length);
    }
    return A.length;
}
```

- takes an array `A` and builds a max heap out of it
- it returns the `size` of the heap
    - which is just `A.length` since a max heap was built using the entire array

`heapSort`

```
function heapSort(A: number[]) {
    let size = buildMaxHeap(A);
    for (let i = A.length - 1; i > 0; i--) {
        swap(A, 0, i);
        size--;
        maxHeapify(A, 0, size);
    }
}
```

- starting from the lastmost element, we swap it with the root of the tree
    - where the root was previously the largest element the tree as a result of the max heap property
- next, we decrement the size of the heap as the element last swapped with is the next largest element
- finally we call `maxHeapify` to correct for the swapped element in the root
    - notice that the root's left and right children are *still* max heaps as we excluded the largest element from our heap using `size--`

Correctness

Runtime Analysis

Max Heapify

- children subtrees have a size of at most $\frac{2n}{3}$ such that we can describe the recurrence relation as

$$T(n) \leq T(\frac{2n}{3}) + \Theta(1)$$

- we can solve this using the master thereom

$$a = 1, b = \frac{3}{2}, f(n) = \Theta(1)$$
$$\log_a a = \log_{\frac{3}{2}} 1 = 0$$
$$f(n) = \Theta(1) = \Theta(n^0) = \Theta(n^{\log_b a})$$

- this is case 2 such that $T(n) = \Theta(n^0 \lg n) = \Theta(\lg n)$
- alternatively, you can characterize the runtime on the height of a binary tree which would be $\lg n$

Build Max Heap

- each call to `maxHeapify` is $O(\lg n)$
- n of these calls are made such that the upper bound is $O(n \lg n)$
- *we can show a tight bound but I don't feel like it* 🙃

Heap Sort

- if we put the previous stuff together we get

$$\begin{aligned} T(n) &= O(n \lg n) + O(n \lg n) + O(1) \\ &= O(n \lg n) \end{aligned}$$

- the nice thing about heap sort is that we can sort in place meaning only a constant number of array elements are stored outside the input array
  - notice in merge sort we do not have this feature