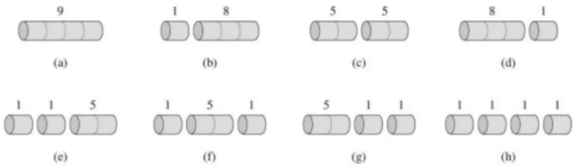
Rod Cutting

Design

- we have rod of length $n = 4$
- rods sell for different prices depending on their length. For example

length i	1	2	3	4
price p_i	1	5	8	9

- what is the optimal way we can cut up our rod to get the most revenue? Here are some example ways to cut our rod



- the number of potential cuts for a rod of length n is $2^{n-1} = 2^{4-1} = 8$
- this is exponential so testing every solution is not feasible
- assume the optimal solution cuts the rod into k pieces where $1 \leq k \leq n$
  - the optimal decomposition is $n = i_1 + i_2 + \ldots + i_k$
  - the corresponding optimal revenue is $r_n = p_1 + p_2 \ldots + p_k$

length i	1	2	3	4	5	6	7	8	9	10
price p_i	1	5	8	9	10	17	17	20	24	30

- find the optimal revenue r for each subproblem
  - $r_1 = \max(1) = 1$
  - $r_2 = \max(2, 5) = 5$
    - where you can cut the rod into $[1, 1]$ for price $1 + 1 = 2$ or
    - $[2]$ for 5
  - $r_3 = \max(3, 6, 6, 8) = 8$
    - $[1, 1, 1]$ for $1 + 1 + 1 = 3$
    - $[1, 2]$ for $1 + 5 = 6$
    - $[2, 1]$ for $5 + 1 = 6$
    - $[3]$ for 8
  - notice we can actually reuse previous overlapping solutions
  - $r_4 = \max(9, 9, 19, 8) = 10$ where can reuse previous solutions by
    - $0 + p_4 = 9$
    - $r_1 + p_3 = 1 + 8 = 9$
    - $r_2 + p_2 = 5 + 5 = 10$
    - $r_3 + p_1 = 8 + 1 = 9$
  - $r_5 = \max(10, 13, 13, 11) = 13$
    - $0 + p_5 = 10$

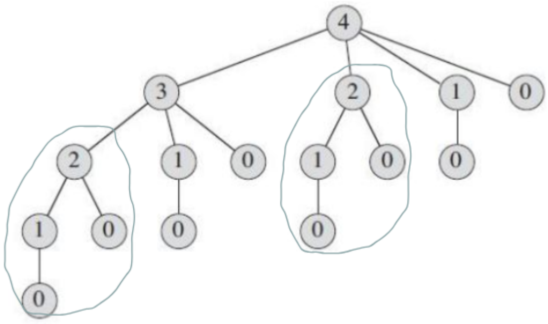
- $0 + p_5 = 10$
- $r_1 + p_4 = 1 + 9 = 10$
- $r_2 + p_3 = 5 + 8 = 13$
- $r_3 + p_2 = 8 + 5 = 13$
- $r_4 + p_1 = 10 + 1 = 11$

Without Memoization

Code

```
CUT-ROD(p, n)
if n == 0
    return 0
q = -∞
for i = 1 to n
    q = max(q, p[i] + CUT-ROD(p, n - i))
return q
```

Runtime Analysis



- our recursion tree is made by taking initial cuts (the nodes) then recursing in `cutRod` to find the next maximum
- notice the redundancies in the tree

$$T(n) = 1 + \sum_{j=0}^{n-1} T(j)$$
$$T(0) = 1$$
$$T(1) = 1 + 1 = 2$$
$$T(2) = 1 + T(0) + T(1) = 4$$
$$T(3) = 1 + T(0) + T(1) + T(2) = 8$$
$$T(n) = 2^n$$

- note that you can use an inductive prove to show this
- we can use **dynamic programming** which uses additional memory to save previous computations
- there are 2 equivalent ways to reduce the repeated computation:
    - **top down**
    - **bottom up**

Top Down

We write the procedure recursively in a natural manner, but modified to save results of subproblems.

Code

```
MEMOIZED-CUT-ROD(p, n)
let r[0 . . n] be a new array
for i = 0 to n
    r[i] = -∞
return MEMOIZED-CUT-ROD-AUX(p, n, r)
```

```
MEMOIZED-CUT-ROD-AUX(p, n, r)
if r[n] ≥ 0
    return r[n]
if n == 0
    q = 0
else q = -∞
    for i = 1 to n
        q = max(q, p[i] + MEMOIZED-CUT-ROD-AUX(p, n - i, r))
r[n] = q
return q
```

Bottom Up

When solving a particular subproblem, we have already solved all of the smaller subproblems its solutions depends on and have those solutions saved.

Code

```
BOTTOM-UP-CUT-ROD(p, n)
let r[0 . . n] be a new array
r[0] = 0
for j = 1 to n
    q = -∞
    for i = 1 to j
        q = max(q, p[i] + r[j - i])
    r[j] = q
return r[n]
```