Shortest Paths

- our input its
  - a directed graph $G = (V, E)$
  - a weight function $w : E \to \mathbb{R}$
- **weight of a path** $p =< v_0, v_1, \ldots, v_k >$ is the sum of its edge weights
- **shortest path** from u to v is any path p such that $w(p) = \delta(u, v)$

$$\delta(u,v) = \begin{cases} \min\{w(p) : u \overset{p}{\rightsquigarrow} v|\} & \text{if a path } u \rightsquigarrow v \text{ exists} \\ \infty & \text{otherwise} \end{cases}$$

Variants

- **single-source**: find shortest path from a given *source* vertex s to every vertex $v \in V$
- **single-destination**: find shortest path to a given destination
- **single-pair**: find shortest path u to v
  - there is no way known to solve that's better in the worst case than single-source
- **all-pairs**: find shortest path from u to v for all $u, v \in V$

"Gotchyas"

Negative-weight Edges

- they are okay so long as no negative-weight cycles are reachable form the source
  - if we have a negative-weight cycle, just keep going around it and we get $w(s, v) = -\infty$ for all v on the cycle
  - some algorithms work only if there are *no* negative-weight edges in the graph

Can a path contain a cycle?

- a path can't contain a negative cycle because you can always loop it again to decrease the path length
- a path can't contain a positive cycle because you can remove it to decrease the path length
- a path also can't a zero-weight cycle
- thus paths do not have cycles

Optimal substructure

- **lemma**: any sub-path of a shortest path is also a shortest path
- **proof**: using "cut and paste"
  - suppose p is a shortest path from u to v where $\delta(p) = w(p_{ux}) + w(p_{xy}) + w(p_{yv})$
  - suppose there is a shorter path p'_{xy} such that $w(p'_{xy}) < w(p_{xy})$
  - thus we can get a $\delta(p') = w(p_{ux}) + w(p'_{xy}) + w(p_{yv}) < p$ which contradicts p being a shortest path

Single Source Algorithm: Bellman Ford

- can have negative weighted edges (but no cycles)

Variable Conventions

- $d[v]$ is a **shortest-path estimate** from the source s to some v
  - initially $d[v] = \infty$
  - always maintain $d[v] \geq \delta(s, v)$
- $\pi[v]$ is the predecessor of v on a shortest path from s
  - if there's no predecessor then $\pi[v] = \mathrm{NIL}$ (this is also our initialization)
  - π induces a **shortest-path tree**

Initialization

All the shortest-paths algorithms start with INIT-SINGLE-SOURCE.

INIT-SINGLE-SOURCE*(V, s)*

for each $v \in V$

  $d[v] \leftarrow \infty$

  $\pi[v] \leftarrow \mathrm{NIL}$

$d[s] \leftarrow 0$

Relax

Can we improve the shortest-path estimate (best seen so far) for v by going through u and taking (u, v)?

RELAX*(u, v,w)*

if $d[u] + w(u, v) <= d[v]$

then $d[v] \leftarrow d[u] + w(u, v)$

  $\pi[v] \leftarrow u$

Code

BELLMAN-FORD*(V, E,w, s)*

INIT-SINGLE-SOURCE*(V, s)*

for $i \leftarrow 1$ to $|V|-1$ **// for each vertex in V**
  for each edge $(u, v) \in E$ **// all edges, in any order**
    RELAX*(u,v,w)*

for each edge $(u, v) \in E$
  if $d[v] > d[u] + w(u, v)$
    then return FALSE

return TRUE

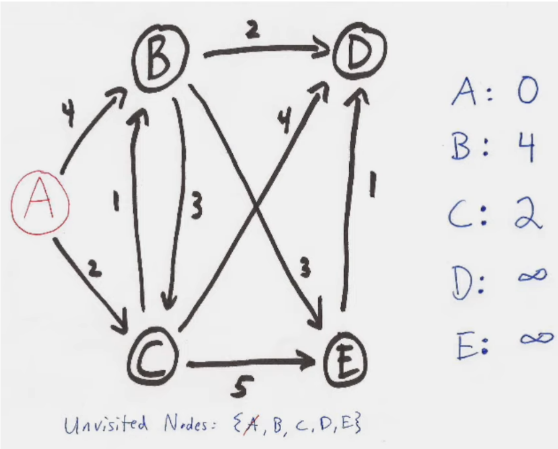
The first **for** loop relaxes all edges $|V|-1$ times.
$O(VE+E)=O(VE)$
$= O(V^3)$

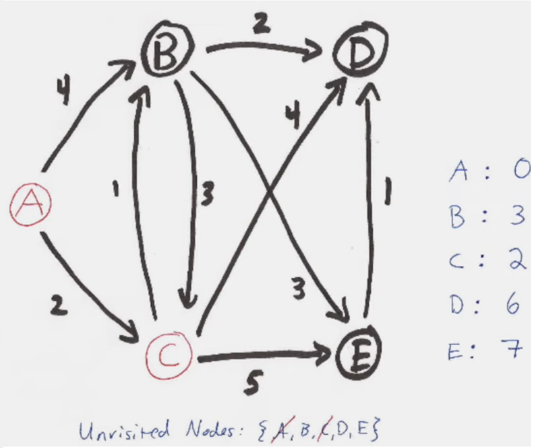
```
def bellman_ford(G, start):
```

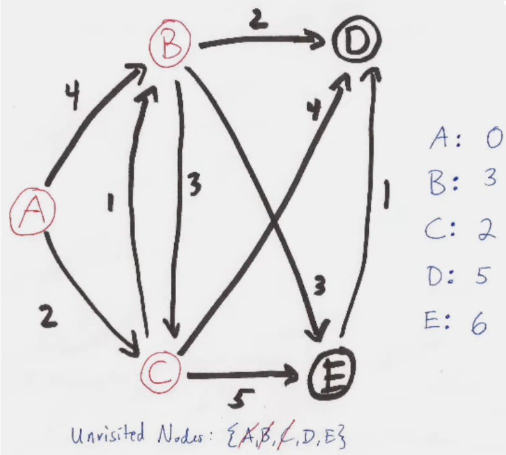
```
    shortest_paths = {}
    for node in G:
        shortest_paths[node] = infinity
    shortest_paths[start] = 0
    size = len(G)
    for _ in range(size - 1):
        for node in G:
            for edge in G[node]:
                cost = edge[0]
                to_node = edge[1]
                if shortest_paths[node] + cost < shortest_paths[to_node]:
                    shortest_paths[to_node] = shortest_paths[node] + cost
    # iterate once more and check for negative cycle
    for node in G:
        for edge in G[node]:
            cost = edge[0]
            to_node = edge[1]
            if shortest_paths[node] + cost < shortest_paths[to_node]:
                return 'INVALID - negative cycle detected'
    return shortest_paths
```

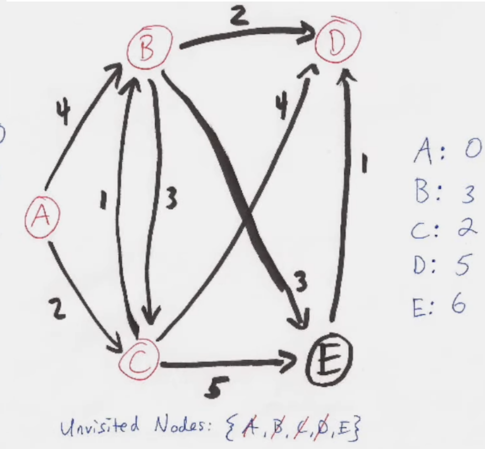
Single Source Algorithm: Dijkstra's Algorithm

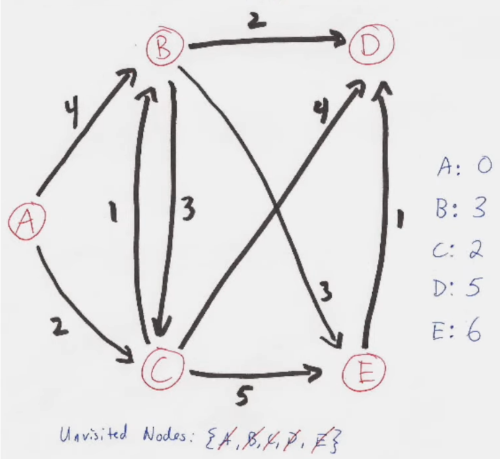
- no negative-weight edges
- is basically a weighted version of BFS
- instead of a FIFO queue, it used a priority queue using $d[v]$
- has 2 sets of vertices
    - S for vertices whose final shortest-path weights are determined
    - Q is a priority queue











- to pick the next vertex, pick the one that hasn't been chosen with the smallest $d[v]$
- if we implement the priority queue with a binary heap
  - $O(E \lg V)$
- proving greedy choice

Greedy Choice – pick the vertex with the smallest shortest path estimate (not including the vertices we are done with)
Assume we have a solution: we know the shortest path from s to every other vertex. “S” is the set of edges in the solution. If S does not contain the greedy choice at the last step, we can remove the non-greedy last edge added to S and add the greedy choice to S and get just as good a solution.