Solving Recurrences

Substitution Method

- comprises of 2 steps:
  1. guess the form of the solution
  2. use mathematical induction to find the constants and show that the solution works
- we substitute the guessed solution for the function when applying the inductive hypothesis to smaller values

Example

$$T(n) = 2T(\lfloor \frac{n}{2} \rfloor) + n$$
$$T(1) = 1$$

- *guess* that the solution is $T(n) = O(n \lg n)$[1]
- induction requires us to show that $T(n) \leq cn \lg n$ for an appropriate choice of the constant $c > 0$
- assume that this bound holds for all positive $m < n$, in particular for $m = \lfloor n/2 \rfloor$
- this yields $T(\lfloor n/2 \rfloor) \leq c\lfloor n/2 \rfloor \lg (\lfloor n/2 \rfloor)$[2]
- substitute the above [2] back into the original

$$\begin{aligned} T(n) &\leq 2(c\lfloor n/2 \rfloor \lg (\lfloor n/2 \rfloor)) + n \\ &\leq cn \lg (n/2) + n \\ &= cn \lg n - cn \lg 2 + n \\ &= cn \lg n - cn + n \\ &\leq cn \lg n && \text{if } c \geq 1 \end{aligned}$$

- induction also requires us to show the solution holds for the boundary conditions
  - recall asymptotic notation requires us to prove for "sufficiently large n" or $n \geq n_0$ where we get to choose what n_0 is
  - the base case $T(1) = 1 \not\leq c(1) \lg (1) = 0$ goes against out hypothesis
  - notice that for any $n > 3$, our relation does not depend on $T(1)$
  - this leaves us with $n = 2$ and $n = 3$ that we must prove works with our hypothesis

$$\begin{aligned} T(2) &= 2T(1) + 2 = 4 \\ &\leq c(2) \lg (2) = 2c \\ T(3) &= 2T(1) + 3 = 5 \\ &\leq c(3) \lg (3) \approx 4.75c \end{aligned}$$

  - lastly, we complete the proof $T(n) \leq cn \lg n$ by choosing $c \geq 2$

Subtracting lower-order from the guess

$$T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1$$

- *guess* that the solution is $T(n) = O(n)$
- we must show $T(n) \leq cn$ [1] for some choice of c

- assuming the boundary holds for all $m < n$, in particular $m = \lfloor n/2 \rfloor$ and $m = \lceil n/2 \rceil$
- this yields $T(\lfloor n/2 \rfloor) \le c\lfloor n/2 \rfloor$ and $T(\lceil n/2 \rceil) \le c\lceil n/2 \rceil$ [2]
- substituting the above [2] back into the original

$$\begin{aligned} T(n) &\le c\lfloor n/2 \rfloor + c\lceil n/2 \rceil + 1 \\ &= cn + 1 \end{aligned}$$

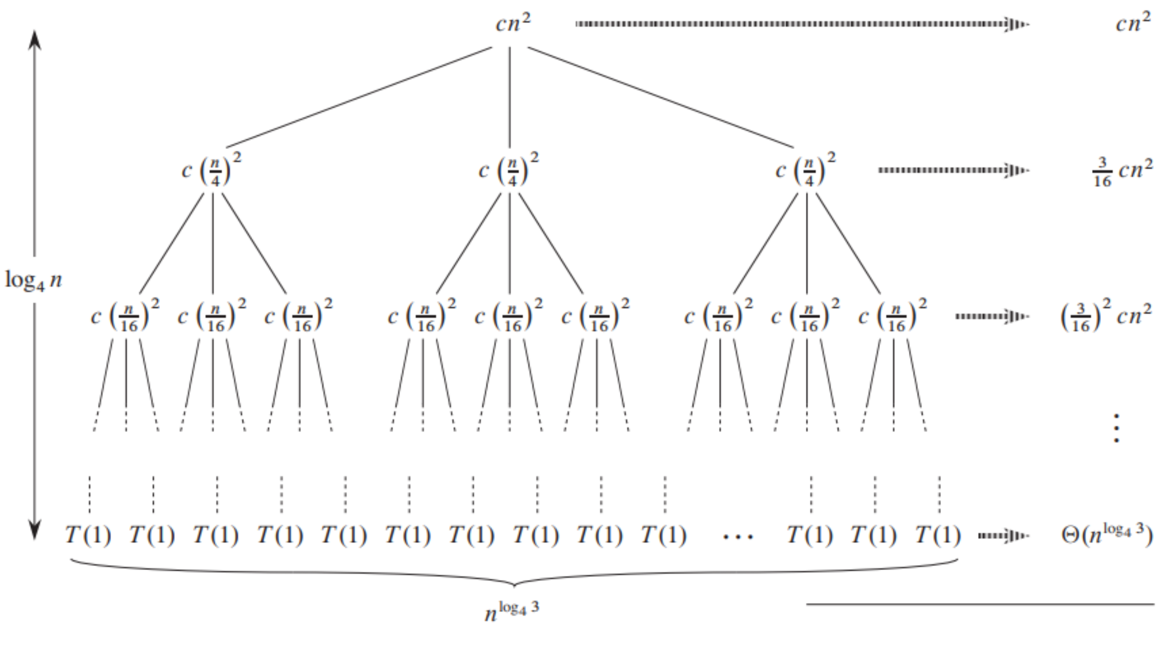
- however, this doesn't imply $T(n) \le cn$ for any c
- we can solve this by *strengthening* our hypothesis to $T(n) \le cn - d$ [3]
- again, assuming the boundary holds for all $m < n$, in particular $m = \lfloor n/2 \rfloor$ and $m = \lceil n/2 \rceil$
- this yields $T(\lfloor n/2 \rfloor) \le c\lfloor n/2 \rfloor - d$ and $T(\lceil n/2 \rceil) \le c\lceil n/2 \rceil - d$ [4]
- substituting the above [4] back into the original

$$\begin{aligned} T(n) &\le (c\lfloor n/2 \rfloor - d) + (c\lceil n/2 \rceil - d) + 1 \\ &= cn - 2d + 1 \\ &\le cn - d && \text{if } d \ge 1 \end{aligned}$$

Recursion Tree

Example 1

Solve the upper bound of $T(n) = 3T(\frac{n}{4}) + cn^2$.



(d)

- how far from the root do we reach 1?
  - the subproblem for a node at depth i is $\frac{n}{4^i}$
  - the subproblem size is 1 when $n(\frac{1}{4})^i = 1$ or when $i = \log_4 n$
  - thus, the tree has $\log_4 n + 1$ levels (at depths $0, 1, 2, \ldots, \log_4 n$)
- what is the cost at each level of the tree excluding the final level?

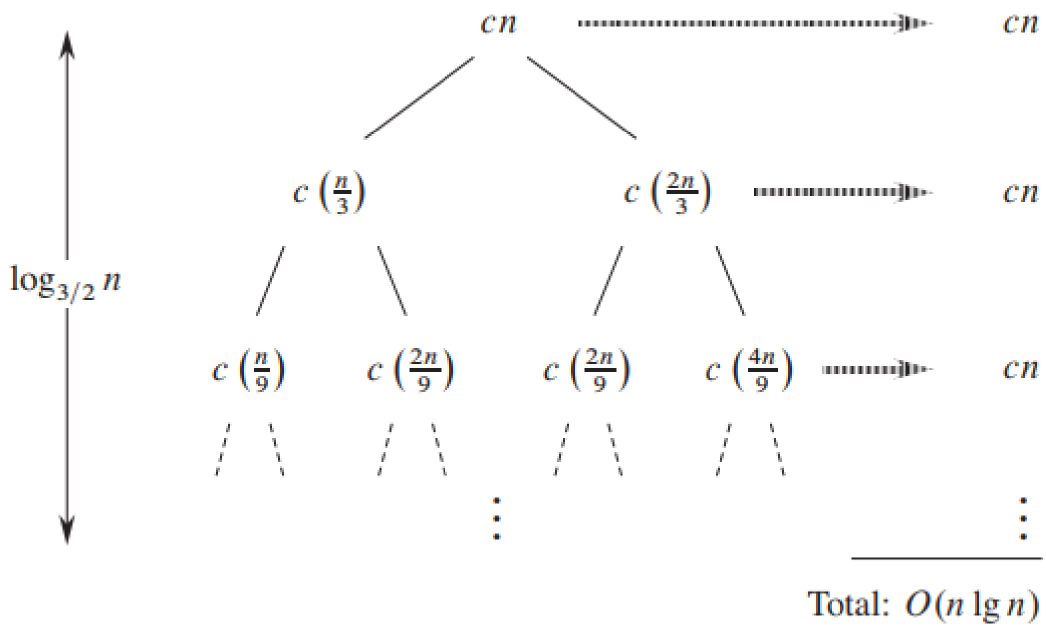
- each level has 3 times more nodes than the one before so the number of nodes at depth i is 3^i
- the subproblem size reduces by a factor of 4 for each level we go down, each node at depth i for $i = 0, 1, 2, \dots, \log_4 n - 1$ (i.e. all the levels but the last) has a cost of $c(\frac{n}{4^i})^2$ such that the total cost at depth i is $3^i c(\frac{n}{4^i})^2 = (\frac{3}{16})^i cn^2$

- what is the cost at the final level of the tree?
  - the final level at depth $\log_4 n$ has $3^{\log_4 n} = n^{\log_4 3}$ nodes
  - each node has a cost $T(1)$ which we can assume is a constant
  - thus, the final level of the tree has a cost of $\Theta(n^{\log_4 3})$
- what is the sum of all the costs?

$$\begin{aligned} T(n) &= cn^2 + \frac{3}{16}cn^2 + (\frac{3}{16})^2 cn^2 + \dots + (\frac{3}{16})^{\log_4 n - 1} cn^2 + \Theta(n^{\log_4 3}) \\ &= k * cn^2 + \Theta(n^{\log_4 3}) \\ &= O(n^2) \end{aligned}$$

Example 2

Solve the upper bound for $T(n) = T(\frac{n}{3}) + T(\frac{2n}{3}) + cn$.



- how far from the root do we reach 1?
  - notice that the tree can't be balanced as some branch paths will reach further to get to 1
  - the longer branch path will be that of $T(\frac{2n}{3})$
  - using this longer branch path, the subproblem size is 1 when $(\frac{2}{3})^i n = 1$ or when $i = \log_{\frac{3}{2}} n$

$$\begin{aligned} i &= \log_{\frac{3}{2}} n \\ &= \frac{\lg n}{\lg \frac{3}{2}} \\ &= c \lg n \\ &= O(\lg n) \end{aligned}$$

- what is the cost at each level of the tree?

- notice at every level, it is cn
- what is the sum of all costs?
  - if we pretend the tree is balanced (which is fine for our upper bound since we'll be overestimating)
  - there are $O(\lg n)$ levels
  - each level is cn
  - thus, $T(n) \leq O(\lg n) * cn = O(n \lg n)$

Master Theorem

Applicable for recurrence relations in the form of:

$$T(n) = aT(\frac{n}{b}) + f(n)$$

> Note that master theorem only solves *some* case.

Cases

Case 1

- if $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$,
- then $T(n) = \Theta(n^{\log_b a})$.

Case 2

- if $f(n) = \Theta(n^{\log_b a})$,
- then $T(n) = \Theta(n^{\log_b a} \lg n)$

Case 3

- if $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$ and
- if $af(\frac{n}{b}) \leq cf(n)$ for some constant $c > 1$ and sufficiently large n,
- then $T(n) = \Theta(f(n))$

Example 1

- solve the asymptotic bound of $T(n) = T(\frac{2n}{3}) + 1$
- $a = 1, b = \frac{3}{2}, f(n) = 1$
- $\log_b a = \log_{\frac{3}{2}} 1 = 0$
- we can easily show the tight bound by $f(n) = 1 = \Theta(1) = \Theta(n^0) = \Theta(n^{\log_b a})$
- this is case 2 such that $T(n) = \Theta(n^{\log_b a} \lg n) = \Theta(\lg n)$

Example 2

- solve the asymptotic bound of $T(n) = 3T(\frac{n}{4}) + n \lg n$
- $a = 3, b = 4, f(n) = n \lg n$
- $\log_b a = \log_4 3 = 0.8$
- we can show the lower bound by $f(n) = n \lg n = \Omega(n^{0.8+0.2}) = \Omega(n^{\log_b a + \epsilon})$

    - where $\epsilon = 0.2$
- this is case 3 such that $T(n) = \Theta(f(n)) = \Theta(n \lg n)$

Example 3

- solve the asymptotic bound of $T(n) = 2T(\frac{n}{2}) + n^2$
- $a = 2, b = 2, f(n) = n^2$
- $\log_b a = \log_2 2 = 1$
- we can show the lower bound by $f(n) = n^2 = \Omega(n^{1+1}) = \Omega(n^{\log_b a + \epsilon})$
    - where $\epsilon = 1$
- this is case 3 such that $T(n) = \Theta(f(n)) = \Theta(n^2)$

Extended Form of Master Theorem

$$T(n) = aT(\frac{n}{b}) + f(n)$$

Extended Form Cases

Case 1

- if $af(\frac{n}{b}) = cf(n)$ is true for some constant $c < 1$,
- then $T(n) = \Theta(f(n))$

Case 2

- if $af(\frac{n}{b}) = cf(n)$ is true for some constant $c > 1$,
- then $T(n) = \Theta(n^{\log_b a})$

Case 3

- if $af(\frac{n}{b}) = f(n)$ is true,
- then $T(n) = \Theta(f(n) \log_b n)$

Methodology

1. list values of $a, b, f(n)$
2. plug a, b in to evaluate $af(\frac{n}{b})$
3. set $af(\frac{n}{b}) = cf(n)$ and solve for c
4. match the value of c to the above cases

Example

- solve the asymptotic bound of $T(n) = 3T(\frac{n}{2}) + n$
- $a = 3, b = 2, f(n) = n$

$$af(\frac{n}{b}) = cf(n)$$
$$\frac{3}{2}n = cn$$
$$c = \frac{3}{2}$$
$$c > 1$$

- this is case 2 such that $T(n) = \Theta(n^{\log_b a}) = \Theta(n^{\lg 3})$