

3. Bayesian Inference

Network Data Analysis - NDA'22
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Sorbonne-LIP6



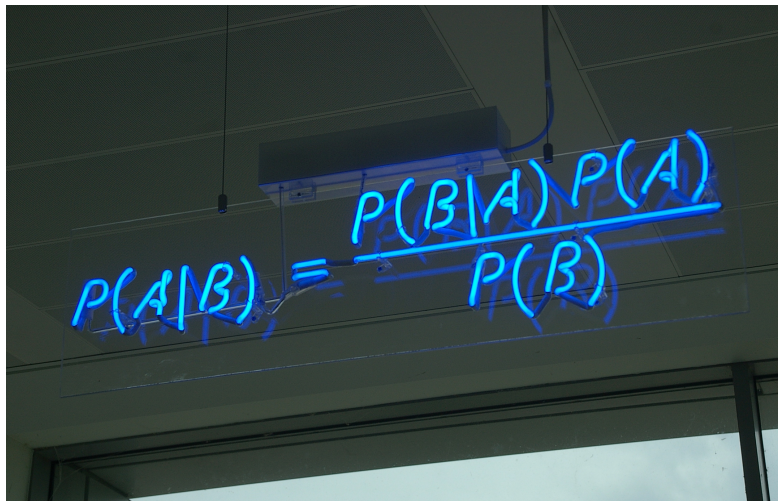
Octobre 03, 2022

Bibliography

- B.1 Christopher M. Bishop, "Pattern Recognition and Machine Learning", Springer 2006.
 - B.2 H. Pishro-Nik, "Introduction to probability, statistics, and random processes", available at <https://www.probabilitycourse.com>, Kappa Research LLC, 2014.
- 👉 Chapter 8.3, 8.4

Bayesian Art

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A photograph of a blue neon sign mounted on a ceiling, displaying the Bayesian formula:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Heads or Tails?

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Suppose we toss a coin three times: (H, H, H)



What can we say about the probability to get heads (H) in the next toss?

Probability of Heads

We remind the frequentist estimation (Sample Mean):

$$\hat{\theta} = \bar{X} = \frac{1 + 1 + 1}{3} = 1$$

☞ The estimated probability for heads (H) is 1, thus we expect surely to get heads next time we throw the coin.

Is this a good estimate?

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Is this a good estimate?

This is the best we can do, given the information we have.

Limited experience

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In the "Heads or Tails" game, we can repeat the experiment several times, until we get a good "frequentist" estimate of the chance to fall Heads (H).

If the coin is fair, the unknown parameter will obviously be $1/2$. The sample mean will "eventually" converge to this value because of zero bias.

But, there are also other events that cannot be repeated many times:

Will the Arctic ice cap have disappeared by the end of the century?

Revise Uncertainty

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☞ By obtaining fresh data, we can revise every year our opinion on the rate of ice loss, given some previous idea that we had.

Thomas Bayes (1701-1761)

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- ▶ Theologist, scientist, mathematician.
- ▶ **Inverse Probability** "Essay towards solving a problem in the doctrine of chances" (1764)
- ▶ The name "Bayes Theorem" was given by Poincaré.

Pierre-Simon Laplace (1749-1827)

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- ▶ "Best mathematician in France" at that time.
- ▶ "Théorie Analytique des Probabilités" (1812)

Bayes rule

Back to our estimation problem. Suppose that we observe data $\mathcal{D} = \{x_1, \dots, x_n\}$, and we want to estimate θ .

In the Heads-Tails example, the estimate was the probability of Heads.

Bayes rule, assumes a **prior distribution** $f_{\Theta}(\theta)$ over the value of θ .

$$f_{\Theta|\mathcal{D}}(\theta|\mathcal{D}) = \frac{P(\mathcal{D}|\theta) \cdot f_{\Theta}(\theta)}{P(\mathcal{D})}$$

The **posterior density** $f_{\Theta|\mathcal{D}}(\theta|\mathcal{D})$ can be used to infer Θ .

Bayes rule assumes that **the unknown is a random variable Θ rather than fixed and deterministic.**

Prior and Posterior distributions

- ▶ $P(\mathcal{D}|\theta)$ is just the **likelihood function** ! How probable is the observed data given the parameter θ and the distribution.
- ▶ $P(\mathcal{D})$ is the overall probability to observe the data

$$P(\mathcal{D}) = \int P(\mathcal{D}|\theta) f_{\Theta}(\theta) d\theta.$$

Note: It is a normalisation constant.

Bayes theorem in simple words

$$\text{posterior} \propto \text{likelihood} \times \text{prior}$$

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Bayes theorem in simple words

$$\text{posterior} \propto \text{likelihood} \times \text{prior}$$

👉 The prior distribution summarises our initial **uncertainty** over the parameter value θ , and the posterior, how this uncertainty is updated after the data is taken into account.

Application: Wireless Communications

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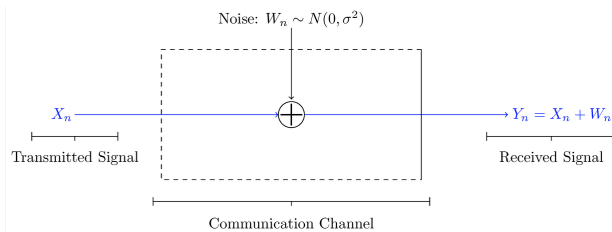


Figure: Source H. Pishro-Nik (B.2)

Application: Wireless Communications

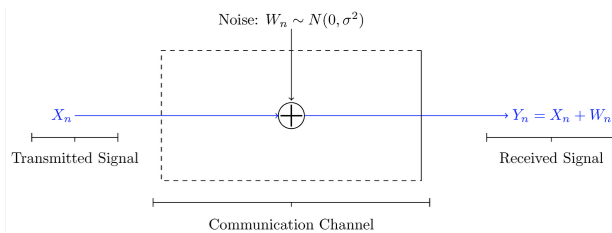


Figure: Source H. Pishro-Nik (B.2)

👉 We want to estimate X_n based on the received Y_n , and assuming we know the prior distribution. Then, the posterior (pdf) is

$$f_{X_n|Y_n}(x|y) = \frac{f_{Y_n|X_n}(y|x) \cdot f_X(x)}{f_Y(y)}.$$

Application: Spam filter

Given that a certain word W appears in an email, is it **Spam** or **Ham**?

The Software applies Bayes' theorem (**PMF**):

$$P(S|W) = \frac{P(W|S) \cdot P(S)}{P(W|S) \cdot P(S) + P(W|H) \cdot P(H)}$$

$Pr(S W)$	probability that a message is a spam, given it contains "W"
$Pr(S)$	overall probability that any message is spam
$Pr(W S)$	probability that the word "W" appears in spam messages
$Pr(H)$	overall probability that any given message is not spam
$Pr(W H)$	probability that the word "W" appears in "ham" messages.

Example: Inference

☞ 3 coins in my pocket

1. Biased 3:1 in favour of Tails
2. Fair coin
3. Biased 3:1 in favour of Heads

I randomly pick one coin, flip it and get Heads (H). What is the probability that I have chosen coin No.3?

INPUT

$X = 1$ means Heads, $X = 0$ means Tails, θ is the mean.

Prior: $P(\theta = 0.25) = P(\theta = 0.5) = P(\theta = 0.75) = \frac{1}{3}$.

Example: Inference cont'd

		Prior	Likelihood	Posterior	Posterior Norm.
Coin	θ	$P(\theta)$	$P(X = 1 \theta)$	$P(X = 1 \theta)P(\theta)$	$\frac{P(X=1 \theta)P(\theta)}{P(X=1)}$
No.1	1/4	1/3	1/4	1/12	1/6
No.2	1/2	1/3	1/2	1/6	1/3
No.3	3/4	1/3	3/4	1/4	1/2

where, the normalising constant is $P(X = 1) = 1/12 + 1/6 + 1/4 = 1/2$.

👉 **Answer:** I have chosen No.3 with probability 50%, No.2 with probability 33.3% and No.1 with probability 16.7%.

Coin No.3 is both the ML estimate as well as the "MAP" estimate.

Example: Inference (variation)

☞ 3 coins in my pocket

1. Fair coin (x2)
2. Biased 3:2 in favour of Heads (x1)

I randomly pick one coin, flip it and get Heads (H). What is the probability that I have chosen coin No.2?

INPUT

$X = 1$ means Heads, $X = 0$ means Tails, θ is the mean.

Prior: $P(\theta = 0.25) = 2/3$ and $P(\theta = 0.75) = 1/3$.

Inference (variation) cont'd

Coin	θ	Prior $P(\theta)$	Likelihood $P(X = 1 \theta)$	Posterior $P(X = 1 \theta)P(\theta)$	Posterior Norm. $\frac{P(X=1 \theta)P(\theta)}{P(X=1)}$
No.1 (x2)	0.500	2/3	1/2	1/3	5/8
No.2 (x1)	0.600	1/3	3/5	1/5	3/8

where, the normalising constant is $P(X = 1) = 1/3 + 1/5 = 8/15$.

👉 **Answer:** I have chosen No.1 with probability 62.5%, and No.2 with probability 37.5%.

Coin No.1 is the "MAP" estimate.

👉 But, coin No.2 has higher Likelihood than coin No.1. The maximum likelihood estimator (ML) is coin No.2! (why?)

MAP Estimator

Maximum Likelihood (ML) estimator

$$\theta_{ML} = \arg \max_{\theta} P(\mathcal{D} \mid \theta)$$

Maximum A Posteriori (MAP) estimator

$$\begin{aligned}\theta_{MAP} &= \arg \max_{\theta} P(\theta \mid \mathcal{D}) \\ &= \arg \max_{\theta} P(\mathcal{D} \mid \theta) \cdot P_{\Theta}(\theta)\end{aligned}$$

Note: When the prior is a uniform distribution, then $P_{\Theta}(\theta)$ is a constant and $\theta_{ML} = \theta_{MAP}$.

👉 The MAP is a summary statistic of the posterior distribution, which corresponds to the **mode** (arg max).

MMSE

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We saw that the **MAP** corresponds to the estimator that maximizes the posterior distribution.

Are there other possibilities?

The **posterior mean**

$$\hat{\theta} = \mathbb{E}[\theta \mid \mathcal{D}].$$

is called the **Minimum Mean Squared Error Estimate (MMSE)**.

👉 It is the best estimate, in terms of the mean squared error.

👉 It is an **unbiased** estimator!

Minimise the MSE

Let a general estimate for θ , given data \mathcal{D} be a function of the data

$$\hat{\Theta} := g(\mathcal{D}).$$

The mean squared error (MSE) is given by

$$\mathbb{E} \left[(\theta - g(\mathcal{D}))^2 \mid \mathcal{D} \right].$$

By developing this we get

$$\mathbb{E} [\theta^2 - 2\theta g(\mathcal{D}) + g(\mathcal{D})^2 \mid \mathcal{D}] = \mathbb{E} [\theta^2] - 2g(\mathcal{D})\mathbb{E} [\theta \mid \mathcal{D}] + g(\mathcal{D})^2.$$

To minimize, we differentiate over $g(\mathcal{D})$ and set to 0

$$-2\mathbb{E} [\theta \mid \mathcal{D}] + 2g(\mathcal{D}) = 0.$$

Normal distribution

Consider a single real-valued variable x that follows a Gaussian distribution

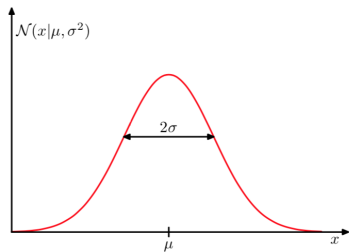
$$\mathcal{N}(x|\mu, \sigma^2) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp \left\{ -\frac{(x - \mu)^2}{2\sigma^2} \right\},$$

- ▶ with **mean** μ ,
- ▶ **variance** σ^2 ,
- ▶ **standard deviation** σ (derived as $\sqrt{\text{Var}(X)}$),
- ▶ (Sometimes we use **precision** $\beta = 1/\sigma^2$ instead of variance).

Gaussian PDF: properties

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- ▶ Positive: $\mathcal{N}(x|\mu, \sigma^2) > 0$,
- ▶ Valid probability density: $\int_{-\infty}^{+\infty} \mathcal{N}(x|\mu, \sigma^2) dx = 1$
- ▶ Mean: $\mathbb{E}[X] = \int_{-\infty}^{\infty} \mathcal{N}(x|\mu, \sigma^2) x dx = \mu$,
- ▶ Second moment: $\mathbb{E}[X^2] = \mu^2 + \sigma^2$,
- ▶ Variance: $\mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \mu^2 + \sigma^2 - \mu^2 = \sigma^2$.



Source: Bishop (B.2)

Gaussian inference

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- Data

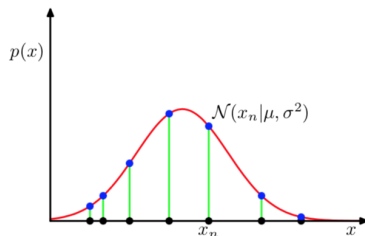
$$\mathcal{D} = \{x_1, \dots, x_N\}$$

- Data i.i.d. from Gaussian PDF

$$\mathcal{N}(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

- Unknown parameters

$$\theta = \{\mu, \sigma^2\}$$



Source: Bishop (B.2)

Gaussian ML

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► Likelihood

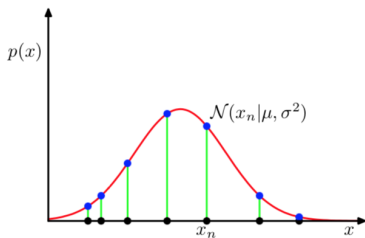
$$P(\mathcal{D}|\theta) = \prod_{n=1}^N \mathcal{N}(x_n|\mu, \sigma^2)$$

► Maximum likelihood

$$\theta_{ML} = \arg \max_{\theta} P(\mathcal{D}|\theta)$$

► equivalent problem

$$\theta_{ML} = \arg \max_{\theta} \log P(\mathcal{D}|\theta)$$



Source: Bishop (B.2)

Gaussian ML solution

$$(\mu_{ML}, \sigma_{ML}) = \arg \max_{\mu, \sigma} \left\{ -\frac{1}{2\sigma^2} \sum_{n=1}^N (x_n - \mu)^2 - \frac{N}{2} \log \sigma^2 - \frac{N}{2} \log(2\pi) \right\}$$

- Maximise first over μ

$$\frac{\partial \log P(\mathcal{D}|\theta)}{\partial \mu} = 0 \Rightarrow \mu_{ML} = \frac{1}{N} \sum_{n=1}^N x_n =: \bar{X}_n$$

- Then, maximise over σ^2

$$\frac{\partial \log P(\mathcal{D}|\theta)}{\partial \sigma^2} = 0 \Rightarrow \sigma_{ML}^2 = \frac{1}{N} \sum_{n=1}^N (x_n - \mu_{ML})^2.$$

Problems related to ML solution

The available dataset, could be a result of i.i.d Gaussian realisations, but the available values contain uncertainty. Let us observe the extreme case for $N = 1$

- ▶ ML estimate of μ

$$\mu_{ML} = x_1$$

- ▶ and ML estimate of σ^2

$$\sigma_{ML}^2 = (x_1 - \mu_{ML})^2 = (x_1 - x_1)^2 = 0.$$

How about the Bayesian approach?

Gaussian posterior

- Assume for simplicity known $\{\sigma^2\}$ variance.

Unknown parameter $\theta = \{\mu\}$.

- Likelihood

$$P(\mathcal{D}|\mu, \sigma^2) = \prod_{n=1}^N \mathcal{N}(x_n|\mu, \sigma^2)$$

- Combine likelihood with a **Gaussian prior** over μ

$$P(\mu) = \mathcal{N}(\mu | m_0, s_0^2)$$

- The **posterior** is proportional to

$$P(\mu | \mathcal{D}, \sigma^2) \propto P(\mathcal{D}|\mu, \sigma^2) \cdot P(\mu)$$

Bayesian update

$$\begin{aligned}
 P(\mu \mid \mathcal{D}, \sigma^2) &\propto P(\mathcal{D} \mid \mu, \sigma^2) \cdot P(\mu) \\
 &= \prod_{i=1}^N \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x_i - \mu)^2}{2\sigma^2}\right) \cdot \frac{1}{\sqrt{2\pi s_0^2}} \exp\left(-\frac{(\mu - m_0)^2}{2s_0^2}\right) \\
 &= \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \frac{1}{\sqrt{2\pi s_0^2}} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^N (x_i - \mu)^2 - \frac{1}{2s_0^2} (\mu - m_0)^2\right) \\
 &= C_1 \cdot \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^N (x_i^2 + \mu^2 - 2\mu x_i) - \frac{1}{2s_0^2} (\mu^2 + m_0^2 - 2\mu m_0)\right) \\
 &= C_2 \cdot \exp\left(-\frac{1}{2\hat{\sigma}_{\mathcal{N}}^2} \left[\mu^2 - 2\mu \hat{\sigma}_{\mathcal{N}}^2 \left(\frac{N\mu_{ML}}{\sigma^2} + \frac{m_0}{s_0^2} \right) + C_3 \right] \right).
 \end{aligned}$$

C_2 and C_3 are such that $\frac{P(\mathcal{D} \mid \mu, \sigma^2) \cdot P(\mu)}{Z}$ is a probability density function.

For the Gaussian pdf the max coincides with the mean due to symmetry!

Gaussian MAP for (μ, σ^2)

The posterior distribution $P(\mu \mid \mathcal{D}, \sigma^2) \sim \mathcal{N}(\mu \mid \hat{\mu}_N, \hat{\sigma}_N^2)$:

- ▶ $\frac{1}{\hat{\sigma}_N^2} = \frac{1}{s_0^2} + \frac{N}{\sigma^2} \Rightarrow \hat{\sigma}_N^2 = \frac{\sigma^2 s_0^2}{Ns_0^2 + \sigma^2}$ (**post-variance**)
- ▶ $\hat{\mu}_N = \frac{\sigma^2}{Ns_0^2 + \sigma^2} m_0 + \frac{Ns_0^2}{Ns_0^2 + \sigma^2} \mu_{ML}$. (**post-mean**)

where $\hat{\mu}_N, \hat{\sigma}_N^2$ are the Bayesian MAP estimates, $\mu_{ML} = \frac{1}{N} \sum_{i=1}^N x_i$.

Limiting cases

	$N = 0$	$N \rightarrow \infty$
$\hat{\sigma}_N^2$	s_0^2	0
$\hat{\mu}_N$	m_0	μ_{ML}

Posterior for the mean

- Posterior of the Gaussian mean for increasing data size N
(The variance reduces with N .)

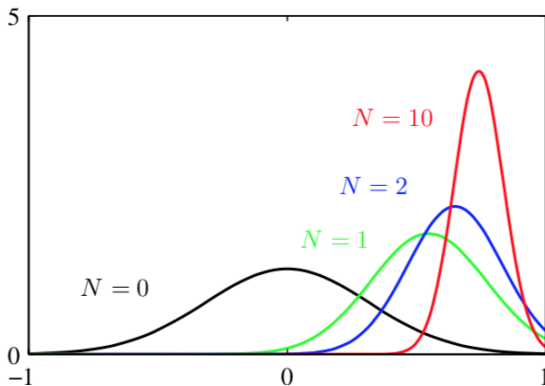


Figure: Bishop (B.2), p.99

Conjugate Priors

☞ In the above case:

Observe already that the posterior distribution has the same shape (Gaussian) as the prior!

$P(\theta)$ is a **conjugate prior** for a particular likelihood $P(\mathcal{D} \mid \theta)$ if the posterior is of the same functional form as the prior.

*For all members of the **exponential family** it is possible to construct a conjugate prior

$$P(\mathbf{x} \mid \theta) = h(\mathbf{x})g(\theta) \exp(\theta^T u(\mathbf{x})).$$

Conjugate Priors for Gaussian variance*

► Likelihood

$$P(\mathcal{D}|\mu, \sigma^2) = \prod_{n=1}^N \mathcal{N}(x_n|\mu, \sigma^2)$$
$$\stackrel{\beta := 1/\sigma^2}{=} \left(\frac{\beta}{2\pi}\right)^{N/2} \exp\left\{-\frac{\beta}{2} \sum_{i=1}^N (x_i - \mu)^2\right\}$$

► For **known mean**, the suitable prior is:

$$P(\beta) = \text{Gam}(\beta \mid a, b) = \frac{1}{\Gamma(a)} b^a \beta^{a-1} \exp(-b\beta).$$

Note: $\Gamma(x) := \int_0^\infty u^{x-1} e^{-u} du$. Also $\Gamma(x+1) = x\Gamma(x)$. It holds: $\Gamma(1) = 1$ and hence $\Gamma(x+1) = x!$ when x is integer.

Properties: $\mathbb{E}[\lambda] = \frac{a}{b}$, and $\text{Var}[\lambda] = \frac{a}{b^2}$.

Conjugate Priors for Gaussian (general)* A. Giovanidis 2022

Likelihood reformulated

$$\begin{aligned}
 P(\mathcal{D}|\mu, \sigma^2) &\stackrel{\beta:=1/\sigma^2}{=} \left(\frac{\beta}{2\pi}\right)^{N/2} \exp\left\{-\frac{\beta}{2} \sum_{i=1}^N (x_i - \mu)^2\right\} \\
 &\propto \left[\beta^{1/2} \exp\left(-\frac{\beta\mu^2}{2}\right)\right]^N \exp\left\{\beta\mu \sum_{i=1}^N x_i - \frac{\beta}{2} \sum_{i=1}^N x_i^2\right\}
 \end{aligned}$$

☞ For **unknown mean and variance** the conjugate prior is

$$P(\mu, \beta) = \mathcal{N}(\mu \mid \mu_0, (\beta)^{-1}) \cdot \text{Gam}(\beta \mid a, b).$$

Normal-Gamma distribution (coupling between μ and β)

Bernoulli inference

Consider a number N of Bernoulli realisations with parameter θ

$$P(x = 1 \mid \theta) = \theta.$$

The probability distribution for Bernoulli is given by

$$\text{Bernoulli}(x \mid \theta) = \theta^x (1 - \theta)^{1-x},$$

which has variance $\text{Var}[x] = \theta(1 - \theta)$.

The **Likelihood function**, given i.i.d. observations from $P(x = 1 \mid \theta)$

$$P(\mathcal{D} \mid \theta) = \prod_{i=1}^N \theta^{x_i} (1 - \theta)^{1-x_i}.$$

Bernoulli ML

From the **Maximum Likelihood** estimate, we get:

$$\begin{aligned}\theta_{ML} &= \arg \max_{\mu} \log P(\mathcal{D} \mid \theta) \\ &= \arg \max_{\theta} \sum_{i=1}^N \log P(x_i \mid \theta) \\ &= \arg \max_{\theta} \sum_{i=1}^N \{x_i \log(\theta) + (1 - x_i) \log(1 - \theta)\}\end{aligned}$$

$$\frac{d}{d\theta} \log P(\mathcal{D} \mid \theta) = 0 \quad \Rightarrow \quad \sum_{i=1}^N \frac{x_i}{\theta} = \sum_{i=1}^N \frac{1 - x_i}{1 - \theta}$$

$$\theta_{ML} = \frac{1}{N} \sum_{i=1}^N x_i.$$

Binomial Heads

Equivalently, we see that

$$\theta_{ML} = \frac{m(N)}{N},$$

where $m(N)$ is the number of Heads (H) in a Heads-Tails experiment of size N .

The number $m(N)$ follows the [Binomial distribution](#)

$$m(N) \sim \text{Binomial}(m \mid N, \theta) = \binom{N}{m} \theta^m (1 - \theta)^{N-m},$$

where $\binom{N}{m} = \frac{N!}{(N-m)!m!}$ is the number of choosing m objects out of N identical ones.

Binomial Distribution

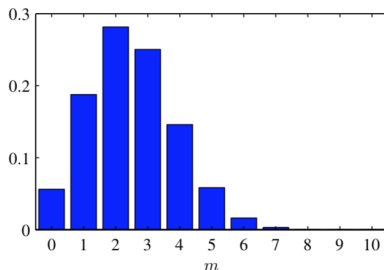


Figure: Bishop (B.2), p.70

- ▶ $\mathbb{E}[m] := \sum_{m=0}^N m \text{Binomial}(m | N, \theta) = N\theta.$
- ▶ $\text{Var}[m] := \sum_{m=0}^N (m - \mathbb{E}[m])^2 \text{Binomial}(m | N, \theta) = N\theta(1 - \theta).$

Bernoulli ML issues

☞ The ML estimator for the Bernoulli is based strongly on the available data, and tends to **severely overfit** the estimated value for small data-sets.

Remember the Heads-Tails example $\{H, H, H\}$.

$$\theta_{ML} = \frac{1}{3} \sum_{i=1}^3 x_i = \frac{1+1+1}{3} = 1.$$

Prediction: From the above the coin should always (a.s.) give Heads !

Bayesian approach

- We will use the Bayesian approach and will propose a **conjugate prior** that keeps the same shape when multiplied by the Likelihood function.
- ▶ We saw that the Likelihood function is

$$P(\mathcal{D} \mid \theta) = \prod_{i=1}^N \theta^{x_i} (1 - \theta)^{1-x_i} = \theta^m (1 - \theta)^\ell,$$

where m is the count of (H), ℓ is the count of (T) and $\ell = N - m$.

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where m is the count of (H), ℓ is the count of (T) and $\ell = N - m$.

- ▶ The **Beta function** has the conjugate property

$$\text{Beta}(\theta \mid a, b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \theta^{a-1} (1-\theta)^{b-1}.$$

Beta Moments

The mean and variance of the Beta distribution are given by

$$\begin{aligned}\mathbb{E}[\theta] &= \frac{a}{a+b}, \\ \text{Var}[\theta] &= \frac{ab}{(a+b)^2(a+b+1)}.\end{aligned}$$

Beta plots

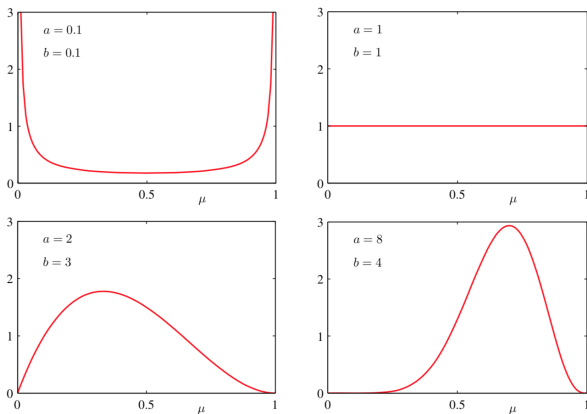


Figure 2.2 Plots of the beta distribution $\text{Beta}(\mu|a, b)$ given by (2.13) as a function of μ for various values of the hyperparameters a and b .

Bernoulli posterior

- Suppose $\text{Beta}(\theta|a, b)$ is the **prior distribution** for θ and multiply with the binomial likelihood function.

👉 Posterior distribution $\text{Beta}(\theta|a + m, b + \ell)$:

$$P(\theta \mid m, \ell, a, b) \propto \theta^{m+a-1}(1 - \theta)^{\ell+b-1}.$$

Bernoulli posterior cont'd

Taking into account the normalisation, we have:

$$P(\theta \mid m, \ell, a, b) = \frac{\Gamma(m+a+\ell+b)}{\Gamma(m+a)\Gamma(\ell+b)} \theta^{m+a-1} (1-\theta)^{\ell+b-1}.$$

□ Observing m Heads in data, adds m to a . Similarly, observing ℓ Tails in data adds ℓ to b . → These **hyperparameters** can be seen as the **effective number of observations of $x = 1$ and $x = 0$** .

□ The posterior probability distribution has an **updated mean**

$$P(x = 1 \mid \mathcal{D}) = \frac{m+a}{m+a+\ell+b}$$

When $m, \ell \rightarrow \infty$: $P(x = 1 \mid \mathcal{D}) \approx \frac{m}{m+\ell} = \frac{m}{N} = \theta_{ML}$.

Sequential Learning

- ▶ Very often we do not have the whole dataset \mathcal{D} available.
- ▶ Data arrives sequentially, and we need to **update** our estimates using the new info.
- ▶ e.g. $\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_t, \dots$
- ▶ In the simplest case, each data-set consists of 1 single new data (measurement)

Question: How do the ML and MAP (Bayesian) estimators update sequentially?

Sequential ML

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Consider we have observed data from $\mathcal{D}' = \{x_1, \dots, x_{N-1}\}$, estimate $\theta_{ML}^{(N-1)}$, and then observe x_N and update the estimate.

► Gaussian, Bernoulli:

$$\begin{aligned}\theta_{ML}^{(N)} &= \frac{1}{N} \sum_{i=1}^N x_i \\ &= \dots \\ &= \theta_{ML}^{(N-1)} + \frac{1}{N} \left(x_N - \theta_{ML}^{(N-1)} \right)\end{aligned}$$

Robbins-Monro algorithm

In the more general case, we can use following sequential algorithm:

$$\theta^{(N)} = \theta^{(N-1)} - a_{N-1} \frac{\partial}{\partial \theta^{(N-1)}} \left[-\log P(x_N | \theta^{(N-1)}) \right]$$

where

- ▶ $\lim_{N \rightarrow \infty} a_N = 0$,
- ▶ $\sum_{N=1}^{\infty} a_N = \infty$,
- ▶ $\sum_{N=1}^{\infty} a_N^2 < \infty$

Note that in the case of Gaussian $-\log P(x|\mu, \sigma^2) = \frac{1}{2\sigma^2} (x - \mu)^2$,

$$\mu^{(N)} = \mu^{(N-1)} + a_{N-1} \frac{1}{\sigma^2} (x_N - \mu^{(N-1)}) .$$

What is a_{N-1} for the Gaussian ML?

Sequential MAP

- ▶ We consider again a data set \mathcal{D}' of $N - 1$ data points, and observation x_N .
- ▶ Posterior distribution:

$$\begin{aligned} P(\theta \mid \{\mathcal{D}', x_N\}) &\propto \prod_{i=1}^N P(x_i \mid \theta) \cdot P(\theta) \\ &= \left[\prod_{i=1}^{N-1} P(x_i \mid \theta) \cdot P(\theta) \right] P(x_N \mid \theta) \\ &= P(x_N \mid \theta) \cdot P(\theta \mid \mathcal{D}'). \end{aligned}$$

The **posterior distribution** after $N - 1$ observations, becomes the **new prior**!

Exercise 1: RADAR

A radar scans a surface for dangerous targets every time unit [hour].

- ▶ The detection mechanism of the radar can detect a real target in 99% of all cases (True Positive).
- ▶ It happens that in 2% of scans there is a False Alarm (False Positive).
- ▶ We know: a real target appears every 1000 time units.

Question: What is the probability that an alarm by the radar corresponds to a true target?

Solution 1: RADAR

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- ▶ $P(\text{Alarm} \mid \text{Target}) = 0.99$
- ▶ $P(\text{Alarm} \mid \text{Nothing}) = 0.02$
- ▶ $P(\text{Target}) = 0.001$

$$\begin{aligned} P(\text{Target} \mid \text{Alarm}) &= \frac{P(\text{Alarm} \mid \text{Target}) \cdot P(\text{Target})}{P(\text{Alarm})} \\ &= \frac{0.99 \cdot 0.001}{0.99 \cdot 0.001 + 0.02 \cdot 0.999} \\ &\approx 0.05. \end{aligned}$$

Exercise 2: WIFI

Users want to use a public WiFi shared with others. At different times per day, users have different access probability:

1. When there are few users connected (GOOD):

$$P(\text{Access}) = 99/100.$$

2. When there are many users online (BAD):

$$P(\text{Access}) = 50/100.$$

☞ We do not know how many users are connected. Suppose a first user requests access and he receives it!

Question: What is the probability that a second user will receive access as well? (i.i.d.)

Solution 2: WIFI

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The user does not know if the channel is GOOD or BAD, so let us choose a prior distribution

$$P(GOOD) = P(BAD) = 0.5.$$

We want to compute:

$$P(X_2 = 1 \mid X_1 = 1) = \frac{P(X_2 = 1, X_1 = 1)}{P(X_1 = 1)}.$$

We have the following information:

$$P(X_i = 1 \mid GOOD) = 0.99 \quad P(X_i = 1 \mid BAD) = 0.5.$$

Solution 2: WIFI cont'd

$$\begin{aligned}P(X_2 = 1, X_1 = 1) &= P(X_2 = 1|GOOD)P(X_1 = 1|GOOD)P(GOOD) \\&+ P(X_2 = 1|BAD)P(X_1 = 1|BAD)P(BAD) \\&= (0.99)^2 \frac{1}{2} + (0.50)^2 \frac{1}{2}\end{aligned}$$

Also,

$$\begin{aligned}P(X_1 = 1) &= P(X_1 = 1|GOOD)P(GOOD) + P(X_1 = 1|BAD)P(BAD) \\&= 0.99 \frac{1}{2} + 0.50 \frac{1}{2}\end{aligned}$$

Altogether,

$$\begin{aligned}P(X_2 = 1 \mid X_1 = 1) &= \frac{(0.99)^2 \frac{1}{2} + (0.50)^2 \frac{1}{2}}{0.99 \frac{1}{2} + 0.50 \frac{1}{2}} \\&= \frac{(0.99)^2 + (0.50)^2}{0.99 + 0.50} \approx 82,6\%\end{aligned}$$

Solution 2: WIFI cont'd

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The states of the two efforts to access are **not independent!**

$$82.6\% \approx P(X_2 = 1 \mid X_1 = 1) \neq P(X_2 = 1) = \frac{1}{2}(0.99 + 0.50) \approx 75\%$$

- ☞ The fact that the first user got access, gives extra information in order to infer the probability that the second user gets also access.

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END