# NDA: Time Series Analysis - part 1

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# Bibliography

Formal content:

- Peter Brockwell and Richard Davis
  *Introduction to Time Series and Forecasting*
- William Thistleton and Tural Sadigov
  MOOC Coursera: *Practical Time Series Analysis*

Informal guide in python:

- www.machinelearningplus.com/time-series/

Illustrative datasets:

- data.world/datasets/time-series
- www.kaggle.com/tags/time-series

# Outline

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# What is time series analysis?

**Definition**

Set of observations $\{x_t\}$, recorded at time $t \in T_0$

Think of each $x_t$ as a realization from a distribution

**Specificities of the problem**

A unique realization of the process

$\Rightarrow$ necessary to make assumptions

- observe time series, identify particularities
- choose a family of models $X_t$ to represent data
- check the goodness of the model

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# Assumptions for this course

## Restrictions to a subfamily of problems

- discrete time series (discrete time set)
- fixed time steps (time resolution)
- univariate (one single variable over time) $\neq$ multivariate
  $\rightarrow$ processes have values in $\mathbb{R}$

## And only a few approaches (Box-Jenkins)

- *e.g.* no Fourier analysis

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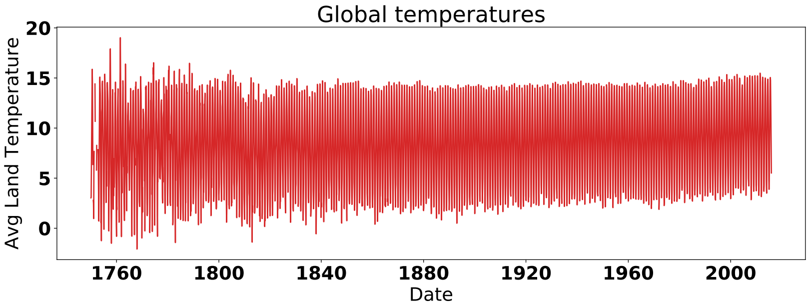
# A few examples

Average global land temperature (per month)



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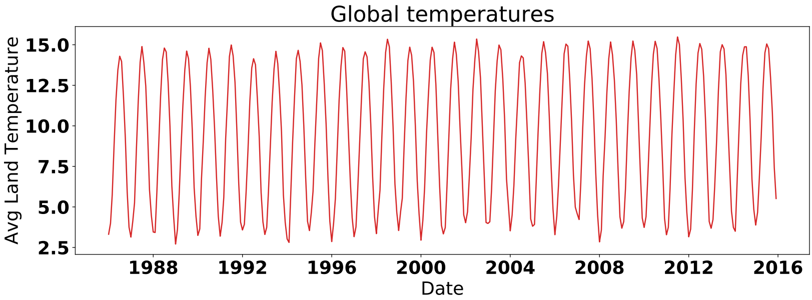
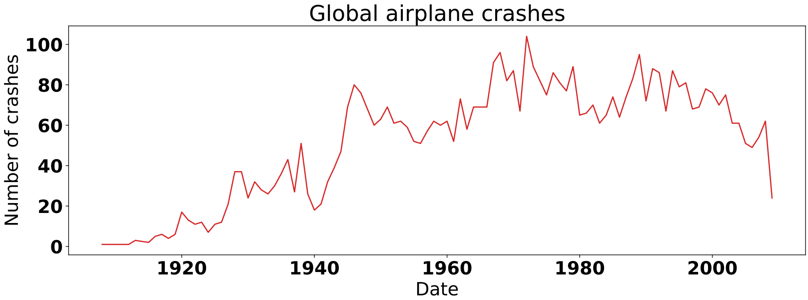


# A few examples

Number of civil airplane crashes (per year)



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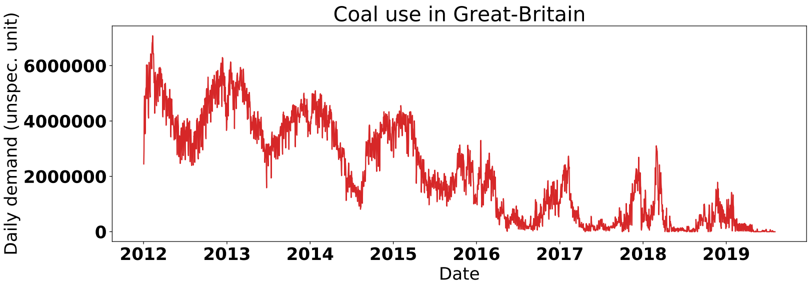
Daily demand of power obtained with coal in GB



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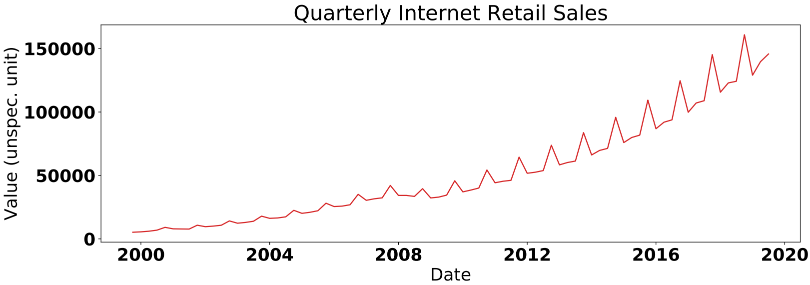


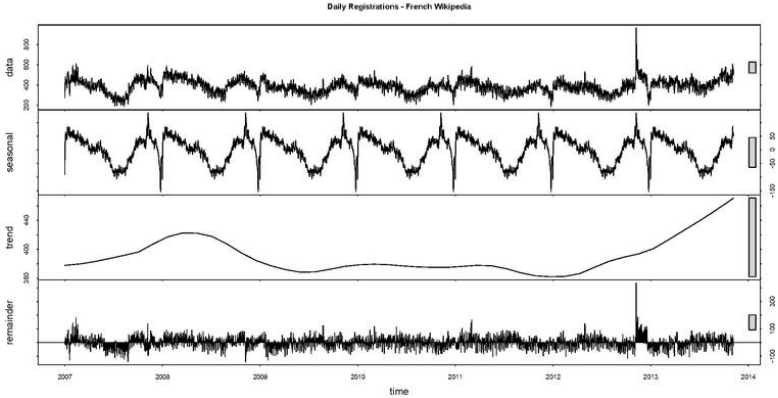
# Goals of time series analysis

- Have a simplified description of the data
  $\rightarrow$ improve our understanding (*ex: climate data*)
- Test an assumption
  *ex: is there a significant measurable global warming?*
- Filter: separate signal from noise
  *ex: known physical signal broadcast* $\rightarrow$ *filter noise*
- Predict future values
  *ex: predict the future demand for a product*
- Simulate a process in a complex model
  *ex: expectation for the GDP to predict economic activity*

# How to analyze a time series? (1)

Analyze from Greek *análusis* $\sim$ unravel $\Rightarrow$ decompose

Decompose the time series into parts, for example:

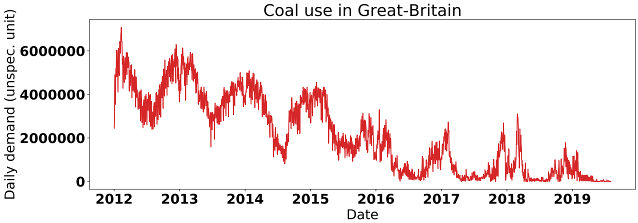


# How to analyze a time series? (1)

## First step

Plot the time series to:

- identify the existence of a trend *(tendance)*
- uncover seasonal variations *(variations saisonnières)*
- detect changes of behavior
- spot outliers *(valeurs aberrantes)*



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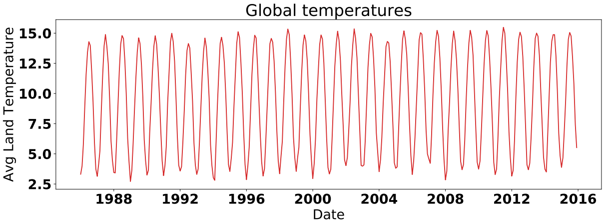

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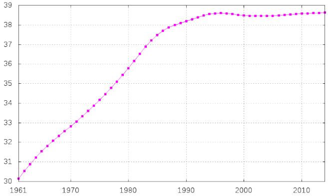
Poland population



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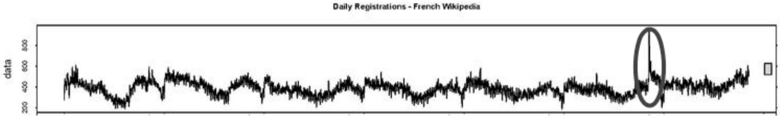

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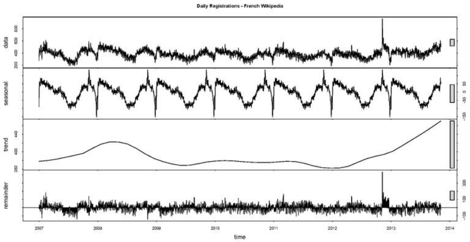
$\rightarrow$ subjective components in this analysis

# The classical decomposition

## Classical decomposition of the time series

$$X_t = s_t + m_t + r_t$$

- seasonality $s_t$
- trend $m_t$
- remainder $r_t$



# Outline

# Mean and covariance of a time series

## Two fundamental definitions

Let $\{X_t\}$ a time series with $\mathbb{E}[X_t^2] < \infty$ (finite variance)
*rk: here we consider $X_t$ as a model*

- **mean function** of $X_t$, defined for all $t$:

$$\mu_X(t) = \mathbb{E}[X_t]$$

- **covariance function** of $X_t$, defined for all $r, s$:

$$\gamma_X(r, s) = Cov(X_r, X_s) = \mathbb{E}[(X_r - \mu_X(r))(X_s - \mu_X(s))]$$

# Stationarity

**Informal definition**

A process is **stationary** *(stationnaire)* if its statistical properties are similar when shifted in time

Remarks:

- stationarity is a property of a model (not of data)
- stationary processes are simpler to investigate
  $\Rightarrow$ usual to transform a TS to obtain a stationary process

# Stationarity

## Informal definition

A process is **stationary** *(stationnaire)* if its statistical properties are similar when shifted in time

## Formal definition

A process is said to be **weakly stationary** if

- the mean function $\mu_X(t)$ is independent of $t \Rightarrow \mu_X$
- $\gamma_X(t+h, t)$ is independent of $t$ for any $h$ (including $h = 0$)

$h$ is called the *lag* *(décalage)*

$$\gamma_X(t+h, t) = \mathbb{E}[(X_{t+h} - \mu_X)(X_t - \mu_X)] = \gamma_X(h)$$

# Stationarity

## Informal definition

A process is **stationary** *(stationnaire)* if its statistical properties are similar when shifted in time

## Formal definition

A process is said to be **strictly (or strongly) stationary** if

- $\forall n$ and $\forall h$

$$P(X_1 = x_1, \ldots, X_n = x_n) = P(X_{1+h} = x_1, \ldots, X_{n+h} = x_n)$$

*Unless specified otherwise, we talk about weak stationarity in the following*

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# Autocorrelation function

Notice that for a stationary time series: $\gamma_X(t+h,t) = \gamma_X(h)$
$\Rightarrow$ the covariance function $\gamma_X$ has one variable (the lag)

**Definition**

For a stationary time series:

- the **autocovariance function** at lag $h$ is:

$$\gamma_X(h) = Cov(X_{t+h}, X_t)$$

- the **autocorrelation function** (ACF) at lag $h$ is:

$$\rho_X(h) = \frac{\gamma_X(h)}{\gamma_X(0)}$$

# Equivalent on real data

Concepts well defined on models, but what about real data?
Let $\{x_1, \ldots, x_n\}$ be a series of observations

## Sample mean

- the **sample mean** estimator is

$$\overline{x} = \frac{1}{n}\sum_{t=1}^{n} x_t$$

# Equivalent on real data

Concepts well defined on models, but what about real data?
Let $\{x_1, \dots, x_n\}$ be a series of observations

## Sample autocovariance function

- the **sample autocovariance function** estimator is

$$\hat{\gamma}(h) = \frac{1}{n} \sum_{t=1}^{n-|h|} (x_{t+|h|} - \overline{x}).(x_t - \overline{x}) \ , \ -n < h < n$$

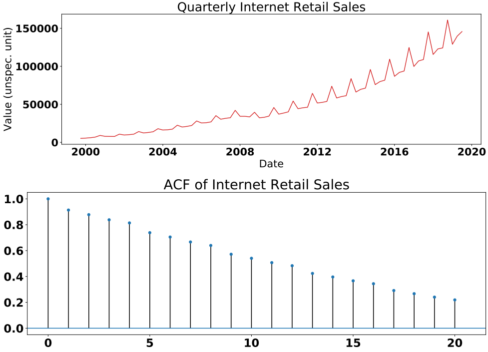
*remark: notice the denominator (because of mathematical properties)*

- the **sample autocorrelation function** estimator is

$$\hat{\rho}(h) = \frac{\hat{\gamma}(h)}{\hat{\gamma}(0)} \ , \ -n < h < n. \text{ Note that } \hat{\rho}(h) \in [-1; 1]$$

# Equivalent on real data

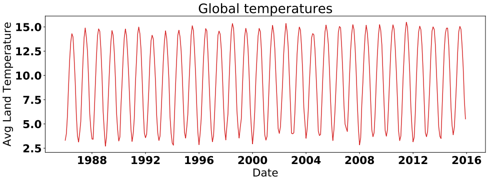
Data with strong trend:

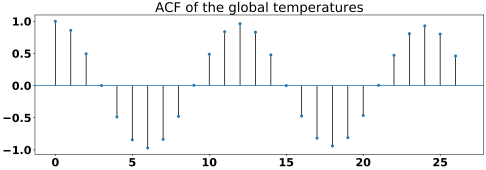



slow decay of correlations with $h$

# Equivalent on real data

Data with strong seasonality:




periodicity on the ACF (here monthly measures $\Rightarrow$ period = 12)

# Outline

# What is a time series model?

## Definition

**Time series model**: specification of the joint distributions of a sequence of random variables $X_t$ of which the observed data is supposed to be the realization

Remarks:

- suppose to know $\forall t$ the distribs $P(x_1, \dots, x_t, \dots, x_n)$
  $\Rightarrow$ in most case too many parameters to handle
- in practice, we focus on first and second order moments:
  - expected values $\mathbb{E}[X_t]$
  - and expected products $\mathbb{E}[X_{t+h}X_t]$, $h = 1, 2, \dots$

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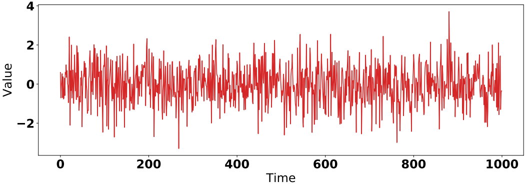
# Independent Identically Distributed noise model

## IID noise

- independant:
  $P(X_1 = x_1, \ldots, X_n = x_n) = P(X_1 = x_1) \cdot \ldots \cdot P(X_n = x_n)$
- identically distributed: $P(X_t = x) = P(X_{t'} = x)$

IID noise is obviously stationary

*ex: repeated coin flipping with heads=1, tails=-1 should be IID noise*



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## White noise *(bruit blanc)*

Special case IID noise with

- 0 mean: $E[X_t] = 0$
- autocovariance function:
$\gamma_X(h) = \sigma^2$ if $h = 0$ and $\gamma_X(h) = 0$ if $h \neq 0$

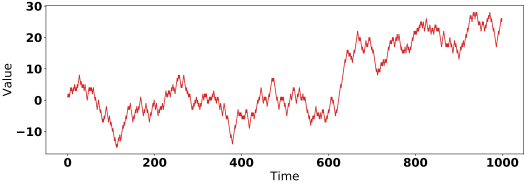
# Random Walk model

**How to build a random walk?** *(marche aléatoire)*

Suppose $\{X_t\}$ is IID noise, then $\{S_t\}$ defined as:

$$S_t = X_1 + \ldots + X_t$$

is a random walk



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- and conversely $X_t = S_t - S_{t-1}$, or $S_t = S_{t-1} + X_t$

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Remarks:

- is a random walk stationary? No
- it's a summation of an IID process
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# Outline

# How to analyze a time series? (2)

**Second step**:

- (if necessary) transform data
- remove the trend and seasonal components to get stationary residuals *(résidus)*

Residual time series obtained (remainder) should be stationary, but not necessarily IID noise...

# Back to the classical decomposition

**Classical decomposition of the time series**

$$X_t = s_t + m_t + r_t$$

- seasonality $s_t$
- trend $m_t$
- remainder $r_t$

What is the difference between seasonality and trend?

- seasonality is periodic $s_{t+d} = s_t$
- seasonality is null on average $\sum_{j=1}^{d} s_j = 0$

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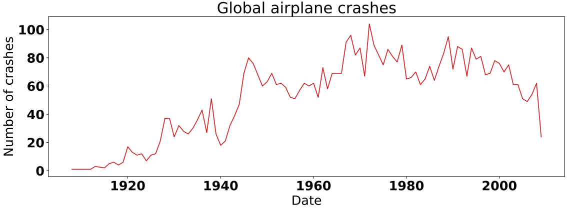
# Isolate the trend component

## Model and regression

$\rightarrow$ cf. course *Regression*

*eg. 2nd order polynomial model with least squares regression*



# Isolate the trend component

## Model and regression

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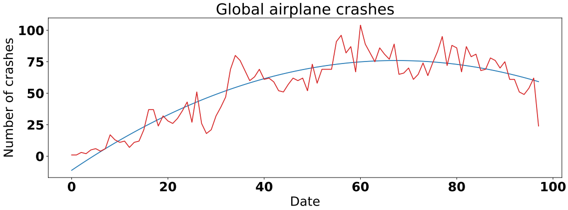
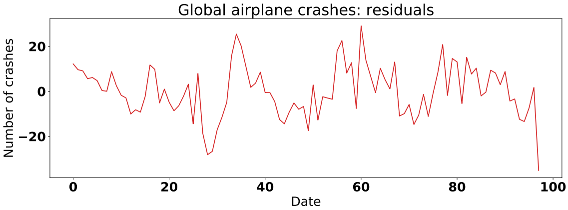
$$\text{Minimize } \sum_{t=1}^{n}(x_t - m_t)^2, \text{ with } m_t = a_0 + a_1 t + a_2 t^2$$



# Isolate the trend component

Then we plot the residuals $\{x_t - m_t\}$




Questions to ask oneself:

- Is there a perceptible trend? Is it smooth? Do we see stretches (séquences) of values of the same sign?
- Does it look stationary? Does it look like noise?

# Isolate seasonal component

**Model and regression**

Which model?

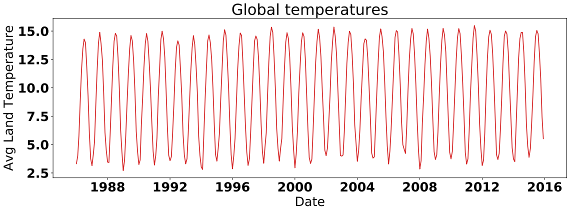
# Isolate seasonal component

## Model and regression

Which model? Harmonic regression

$$s_t = (a_0 +) \sum_{j=1}^{k} a_j cos\left(\frac{2\pi t}{T_j}\right) + b_j sin\left(\frac{2\pi t}{T_j}\right)$$

where $T_j$ are the expected periods of the process



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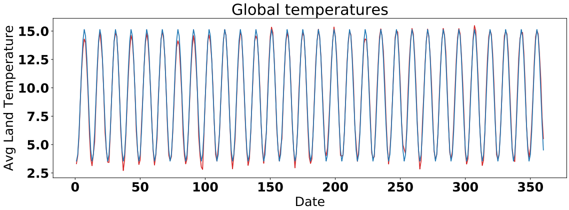

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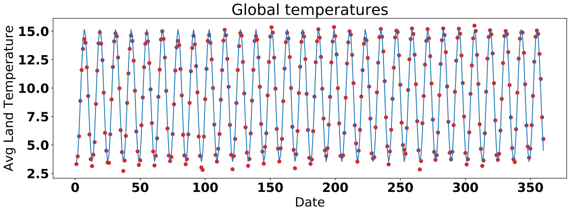

# About pre-processing

**Second step**:

- (if necessary) transform data
- remove the trend and seasonal components to get stationary residuals

When is it necessary to transform data?

**Some cases**

- if outliers $\rightarrow$ if justified, discard them
  *ex: external stimulus, mistake in data acquisition, ...*
- if obvious different regimes
  $\rightarrow$ break data into homogeneous segments
- if noise or seasonality increase with trend level
  $\rightarrow$ logarithmic transformation of the data

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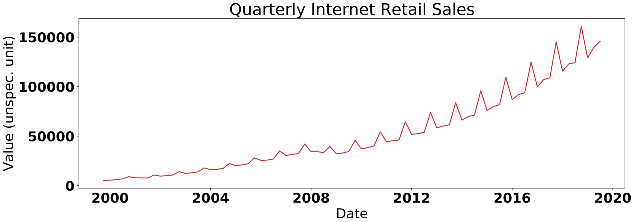
# Logarithmic transformation

If fluctuations (seasonality, noise) grow with magnitude…

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→ c.f. course *Regression (heteroscedasticity)*

Conduct similar analysis on the transformed time series and reverse the transformations in the end to model the original data

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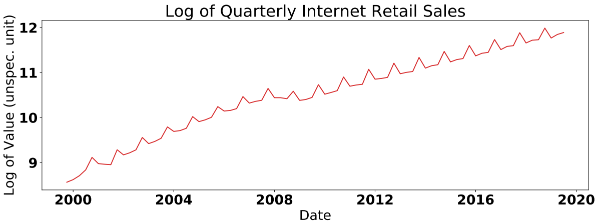


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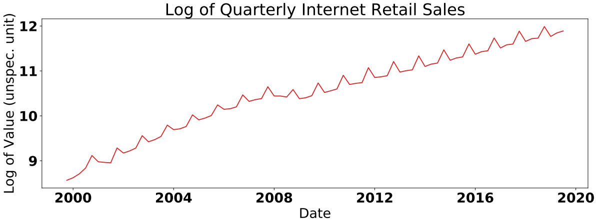


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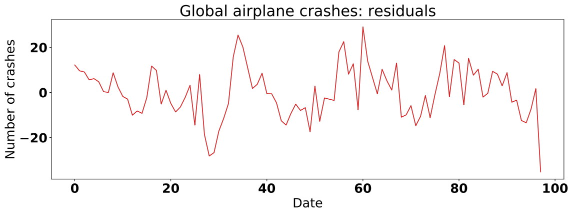
# Stationarity of the residuals

Trend and seasonal components modeled $\rightarrow$ residuals




Visual evaluation of stationarity

- Is there a perceptible trend? Is it smooth? Do we see stretches (séquences) of values of the same sign?
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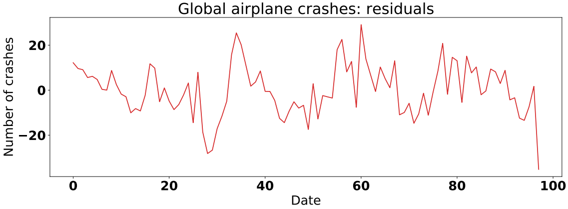
# Stationarity of the residuals

Trend and seasonal components modeled → residuals




Quantitative evaluation of stationarity?

# Testing stationarity of a time series

A general method in Time Series Analysis:

- assume a model
- compute consequences of this model
- test if observations are compatible

$\rightarrow$ c.f. course *Hypothesis testing*

We illustrate this logic with the Dickey-Fuller test

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# Illustration: RW and AR(1) processes

## Auto-regressive processes

Consider the random walk model (not stationary):

$$X_t = X_{t-1} + W_t \text{ where } W_t \text{ is the “error”, some white noise}$$

A specific kind of auto-regressive models: AR(1)

$$X_t = \phi X_{t-1} + W_t$$

$\Rightarrow$ Random Walk is AR(1) with $\phi = 1$

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# Illustration: RW and AR(1) processes

## AR(1) processes

$$X_t = \phi X_{t-1} + W_t$$

$W_t$ is a white noise (0 means, $\sigma^2$ variance)

Stationary? (necessary conditions)

- $\mathbb{E}[X_t] = 0$
- $\gamma_X(h) = \phi^{|h|}\gamma_X(0) = \phi^{|h|}\frac{\sigma^2}{1-\phi^2} \rightarrow$ problem if $\phi = 1$

# Illustration: RW and AR(1) processes

**AR(1) processes**

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# Testing for unit roots

## Notion of characteristic equation

$$X_t - \phi X_{t-1} - W_t = 0$$

$$\longrightarrow x - \phi = 0$$

1 is a root of this equation if $\phi = 1$

More generally, if we have a process modeled by the equation:

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} + \ldots - \phi_p X_{t-p} - W_t = 0$$

$$\longrightarrow x^p - \phi_1 x^{p-1} - \phi_2 x^{p-2} + \ldots - \phi_p = 0$$

if 1 is a root, the process is not stationary (admitted)

Stationarity tests look for unit roots

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# Dickey-Fuller test

Do we have $X_t = X_{t-1} + W_t$?

(and actually some other models that we ignore in this course)

We suppose that we follow this model and we test for $\phi = 1$

- $H_0$: $\phi = 1$ and process is not stationary
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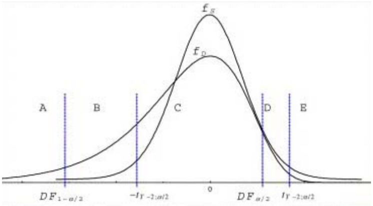
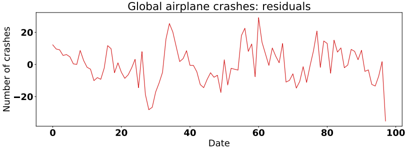
Estimate ($\phi$ -1) from data, compare to Dickey-Fuller distribution



# Testing in practice (airplane crashes residuals)




$$\hat{\phi} - 1 = -0.488 \text{ from data}$$

critical value from table (for $p = 0.05$): $-1.944 < \hat{\phi} - 1$

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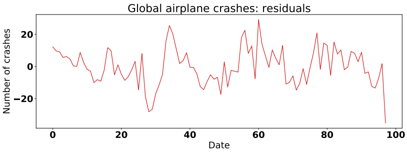


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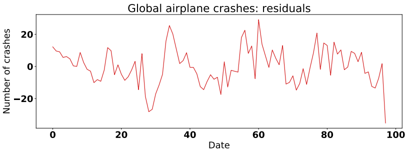


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## Remarks

- other tests available, with other models
  *Augmented Dickey-Fuller, Phillips-Perron, . . .*
- in general implemented in statistic software

# Toward ARMA models

A very standard process model:

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots + \phi_p X_{t-p} + W_t$$

autoregressive with a memory of length $p \to$ AR(p)

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$\longrightarrow$ family of ARMA models (next course)

# Studying time series in python

Among several options, `pandas` library

A few useful functions:

- Load data as dataframe:
  `read_csv` from `pandas` library
- Fitting:
  `curve_fit` in `scipy.optimize` library
- Autocorrelation function:
  `plot_ACF` in `statsmodels` library