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4. Bayesian Inference

Network Data Analysis - NDA'22
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Sorbonne-LIP6

Octobre 03, 2022

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B.1 Christopher M. Bishop, "Pattern Recognition and Machine Learning", Springer 2006.

B.2 H. Pishro-Nik, "Introduction to probability, statistics, and random processes", available at https://www.probabilitycourse.com, Kappa Research LLC, 2014.

☞ Chapter 8.3, 8.4

Bayesian Art

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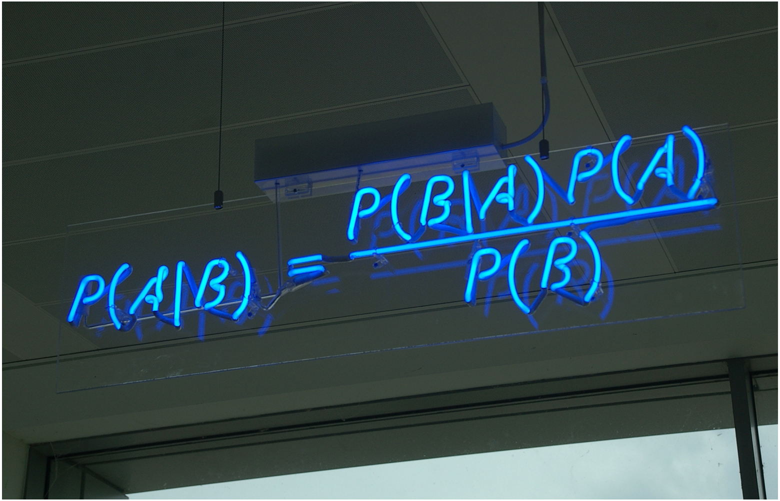

Heads or Tails?

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Suppose we toss a coin three times: (H, H, H)

What can we say about the probability to get heads (H) in the next toss?

Probability of Heads

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We remind the frequentist estimation (Sample Mean):

$$\hat{\Theta} = \overline{X} = \frac{1+1+1}{3} = 1$$

☞ The estimated probability for heads (H) is 1, thus we expect surely to get heads next time we throw the coin.

Is this a good estimate?

Probability of Heads

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We remind the frequentist estimation (Sample Mean):

$$\hat{\Theta} = \overline{X} = \frac{1+1+1}{3} = 1$$

☞ The estimated probability for heads (H) is 1, thus we expect surely to get heads next time we throw the coin.

Is this a good estimate?

This is the best we can do, given the information we have.

Limited experience

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In the "Heads or Tails" game, we can repeat the experiment several times, until we get a good "frequentist" estimate of the chance to fall Heads (H).

If the coin is fair, the unknown parameter will obviously be $1/2$. The sample mean will "eventually" converge to this value because of zero bias.

But, there are also other events that cannot be repeated many times:

Will the Arctic ice cap have disappeared by the end of the century?

Revise Uncertainty

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☞ By obtaining fresh data, we can revise every year our opinion on the rate of ice loss, given some previous idea that we had.

Thomas Bayes (1701-1761)

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- Theologist, scientist, mathematician.
- Inverse Probability "Essay towards solving a problem in the doctrine of chances" (1764)
- The name "Bayes Theorem" was given by Poincaré.

Pierre-Simon Laplace (1749-1827)

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- "Best mathematician in France" at that time.
- "Théorie Analytique des Probabilités" (1812)

Bayes rule

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Back to our estimation problem. Suppose that we observe data $\mathcal{D} = \{x_1, \ldots, x_n\}$, and we want to estimate θ.

In the Heads-Tails example, the estimate was the probability of Heads.

Bayes rule, assumes a prior distribution $f_\Theta(\theta)$ over the value of θ.

$$f_{\Theta|\mathcal{D}}(\theta|\mathcal{D}) = \frac{P(\mathcal{D}|\theta) \cdot f_\Theta(\theta)}{P(\mathcal{D})}$$

The posterior density $f_{\Theta|\mathcal{D}}(\theta|\mathcal{D})$ can be used to infer Θ.

Bayes rule assumes that the unknown is a random variable Θ rather than fixed and deterministic.

Prior and Posterior distributions

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- $P(\mathcal{D}|\theta)$ is just the likelihood function ! How probable is the observed data given the parameter θ and the distribution.
- $P(\mathcal{D})$ is the overall probability to observe the data

$$P(\mathcal{D}) \quad = \quad \int P(\mathcal{D}|\theta) f_{\Theta}(\theta) d\theta.$$

Note: It is a normalisation constant.

Bayes theorem in simple words

$$\text{posterior} \;\; \propto \;\; \text{likelihood} \times \text{prior}$$

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Bayes theorem in simple words

$$\text{posterior} \propto \text{likelihood} \times \text{prior}$$

☞ The prior distribution summarises our initial **uncertainty** over the parameter value θ, and the posterior, how this uncertainty is updated after the data is taken into account.

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Application: Wireless Communications

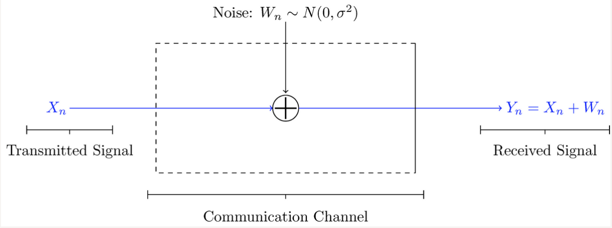

Figure: Source H. Pishro-Nik (B.2)

Application: Wireless Communications

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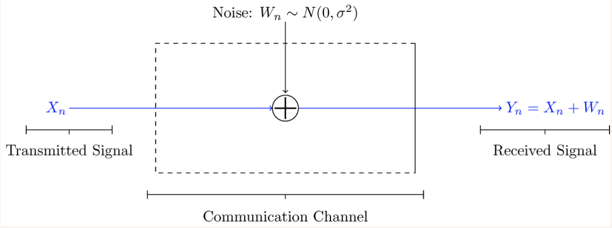

Figure: Source H. Pishro-Nik (B.2)

☞ We want to estimate X_n based on the received Y_n, and assuming we know the prior distribution. Then, the posterior (pdf) is

$$f_{X_n|Y_n}(x|y) \quad = \quad \frac{f_{Y_n|X_n}(y|x) \cdot f_X(x)}{f_Y(y)}.$$

Application: Spam filter

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Given that a certain word W appears in an email, is it Spam or Ham?

The Software applies Bayes' theorem (PMF):

$$P(S|W) = \frac{P(W|S) \cdot P(S)}{P(W|S) \cdot P(S) + P(W|H) \cdot P(H)}$$

$Pr(S\|W)$	probability that a message is a spam, given it contains "W"
$Pr(S)$	overall probability that any message is spam
$Pr(W\|S)$	probability that the word "W" appears in spam messages
$Pr(H)$	overall probability that any given message is not spam
$Pr(W\|H)$	probability that the word "W" appears in "ham" messages.

Example: Inference

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☞ 3 coins in my pocket

1. Biased 3:1 in favour of Tails
2. Fair coin
3. Biased 3:1 in favour of Heads

I randomly pick one coin, flip it and get Heads (H). What is the probability that I have chosen coin No.3?

INPUT

$X = 1$ means Heads, $X = 0$ means Tails, θ is the mean.
Prior: $P(\theta = 0.25) = P(\theta = 0.5) = P(\theta = 0.75) = \frac{1}{3}$.

Example: Inference cont'd

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Coin	θ	Prior $P(\theta)$	Likelihood $P(X=1\|\theta)$	Posterior $P(X=1\|\theta)P(\theta)$	Posterior Norm. $\frac{P(X=1\|\theta)P(\theta)}{P(X=1)}$
No.1	1/4	1/3	1/4	1/12	1/6
No.2	1/2	1/3	1/2	1/6	1/3
No.3	3/4	1/3	3/4	1/4	1/2

where, the normalising constant is $P(X=1) = 1/12 + 1/6 + 1/4 = 1/2$.

☞ Answer: I have chosen No.3 with probability 50%, No.2 with probability 33.3% and No.1 with probability 16.7%.

Coin No.3 is both the ML estimate as well as the "MAP" estimate.

Example: Inference (variation)

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☞ 3 coins in my pocket

1. Fair coin (x2)
2. Biased 3:2 in favour of Heads (x1)

I randomly pick one coin, flip it and get Heads (H). What is the probability that I have chosen coin No.2?

INPUT
$X = 1$ means Heads, $X = 0$ means Tails, θ is the mean.
Prior: $P(\theta = 0.25) = 2/3$ and $P(\theta = 0.75) = 1/3$.

Inference (variation) cont'd

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		Prior	Likelihood	Posterior	Posterior Norm.
Coin	θ	$P(\theta)$	$P(X=1\vert\theta)$	$P(X=1\vert\theta)P(\theta)$	$\frac{P(X=1\vert\theta)P(\theta)}{P(X=1)}$
No.1 (x2)	0.500	2/3	1/2	1/3	5/8
No.2 (x1)	0.600	1/3	3/5	1/5	3/8

where, the normalising constant is $P(X=1) = 1/3 + 1/5 = 8/15$.

☞ Answer: I have chosen No.1 with probability 62.5%, and No.2 with probability 37.5%.

Coin No.1 is the "MAP" estimate.

☞ But, coin No.2 has higher Likelihood than coin No.1. The maximum likelihood estimator (ML) is coin No.2! (why?)

MAP Estimator

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Maximum Likelihood (ML) estimator

$$\theta_{ML} = \arg\max_{\theta} P(\mathcal{D} \mid \theta)$$

Maximum A Posteriori (MAP) estimator

$$\begin{aligned}\theta_{MAP} &= \arg\max_{\theta} P(\theta \mid \mathcal{D}) \\ &= \arg\max_{\theta} P(\mathcal{D} \mid \theta) \cdot P_{\Theta}(\theta)\end{aligned}$$

Note: When the prior is a uniform distribution, then $P_{\Theta}(\theta)$ is a constant and $\theta_{ML} = \theta_{MAP}$.
☞ The MAP is a summary statistic of the posterior distribution, which corresponds to the mode (arg max).

MMSE

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We saw that the **MAP** corresponds to the estimator that maximizes the posterior distribution.
Are there other possibilities?

The posterior mean

$$\hat{\theta} \quad = \quad \mathbb{E}\,[\theta \mid \mathcal{D}\,]\,.$$

is called the **Minimum Mean Squared Error Estimate (MMSE)**.

☞ It is the best estimate, in terms of the mean squared error.

✑ It is an unbiased estimator!

Minimise the MSE

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Let a general estimate for θ, given data $\mathcal{D}$ be a function of the data

$$\hat{\Theta} := g(\mathcal{D}).$$

The mean squared error (MSE) is given by

$$\mathbb{E}\left[(\theta - g(\mathcal{D}))^2 \mid \mathcal{D}\right].$$

By developing this we get

$$\mathbb{E}\left[\theta^2 - 2\theta g(\mathcal{D}) + g(D)^2 \mid \mathcal{D}\right] \quad = \quad \mathbb{E}\left[\theta^2\right] - 2g(\mathcal{D})\mathbb{E}\left[\theta \mid \mathcal{D}\right] + g(D)^2.$$

To minimize, we differentiate over $g(\mathcal{D})$ and set to 0

$$-2\mathbb{E}\left[\theta \mid \mathcal{D}\right] + 2g(D) \quad = \quad 0.$$

Normal distribution

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Consider a single real-valued variable x that follows a Gaussian distribution

$$\mathcal{N}\left(x|\mu,\sigma^2\right) = \frac{1}{(2\pi\sigma^2)^{1/2}}\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\},$$

- with mean μ,
- variance σ^2,
- standard deviation σ (derived as $\sqrt{Var(X)}$),
- (Sometimes we use precision $\beta = 1/\sigma^2$ instead of variance).

Gaussian PDF: properties

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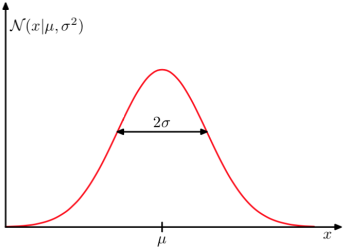
- Positive: $\mathcal{N}\left(x|\mu,\sigma^2\right) > 0$,
- Valid probability density: $\int_{-\infty}^{+\infty} \mathcal{N}\left(x|\mu,\sigma^2\right) dx = 1$
- Mean: $\mathbb{E}[X] = \int_{-\infty}^{\infty} \mathcal{N}\left(x|\mu,\sigma^2\right) x dx = \mu$,
- Second moment: $\mathbb{E}[X^2] = \mu^2 + \sigma^2$,
- Variance: $\mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \mu^2 + \sigma^2 - \mu^2 = \sigma^2$.



Source: Bishop (B.2)

Gaussian inference

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- Data

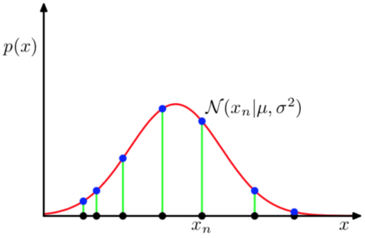
$$\mathcal{D} = \{x_1, \ldots, x_N\}$$

- Data i.i.d. from Gaussian PDF

$$\mathcal{N}\left(x|\mu,\sigma^2\right) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

- Unknown parameters

$$\theta \quad = \quad \{\mu, \sigma^2\}$$



Source: Bishop (B.2)

Gaussian ML

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- Likelihood

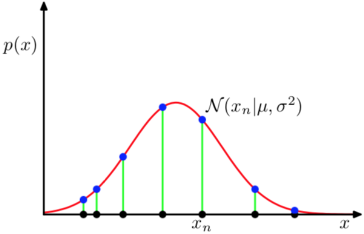
$$P(\mathcal{D}|\theta) \quad = \quad \prod_{n=1}^{N} \mathcal{N}\left(x_n|\mu,\sigma^2\right)$$

- Maximum likelihood

$$\theta_{ML} = \arg\max_{\theta} P(\mathcal{D}|\theta)$$

- equivalent problem

$$\theta_{ML} = \arg\max_{\theta} \log P(\mathcal{D}|\theta)$$



Source: Bishop (B.2)

Gaussian ML solution

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$$(\mu_{ML}, \sigma_{ML}) \quad = \quad \arg\max_{\mu,\sigma} \left\{ -\frac{1}{2\sigma^2} \sum_{n=1}^{N} (x_n - \mu)^2 - \frac{N}{2} \log \sigma^2 - \frac{N}{2} \log(2\pi) \right\}$$

- Maximise first over μ

$$\frac{\partial \log P(\mathcal{D}|\theta)}{\partial \mu} = 0 \quad \Rightarrow \quad \mu_{ML} = \frac{1}{N} \sum_{n=1}^{N} x_n =: \overline{X}_n$$

- Then, maximise over σ^2

$$\frac{\partial \log P(\mathcal{D}|\theta)}{\partial \sigma^2} = 0 \quad \Rightarrow \quad \sigma^2_{ML} = \frac{1}{N} \sum_{n=1}^{N} (x_n - \mu_{ML})^2 .$$

Problems related to ML solution

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The available dataset, could be a result of i.i.d Gaussian realisations, but the available values contain uncertainty. Let us observe the extreme case for $N = 1$

- ML estimate of μ

$$\mu_{ML} = x_1$$

- and ML estimate of σ^2

$$\sigma^2_{ML} = (x_1 - \mu_{ML})^2 = (x_1 - x_1)^2 = 0.$$

How about the Bayesian approach?

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Gaussian posterior

☞ Assume for simplicity known $\{\sigma^2\}$ variance.

Unknown parameter $\theta = \{\mu\}$.

- Likelihood

$$P(\mathcal{D}|\mu, \sigma^2) \quad = \quad \prod_{n=1}^{N} \mathcal{N}\left(x_n|\mu, \sigma^2\right)$$

- Combine likelihood with a Gaussian prior over μ

$$P(\mu) \quad = \quad \mathcal{N}\left(\mu \mid m_0, s_0^2\right)$$

- The posterior is proportional to

$$P(\mu \mid \mathcal{D}, \sigma^2) \quad \propto \quad P(\mathcal{D}|\mu, \sigma^2) \cdot P(\mu)$$

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Bayesian update

$$
\begin{aligned}
P(\mu \mid \mathcal{D}, \sigma^2) &\propto P(\mathcal{D}|\mu, \sigma^2) \cdot P(\mu) \\
&= \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x_i-\mu)^2}{2\sigma^2}\right) \cdot \frac{1}{\sqrt{2\pi s_0^2}} \exp\left(-\frac{(\mu-m_0)^2}{2s_0^2}\right) \\
&= \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \frac{1}{\sqrt{2\pi s_0^2}} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{N}(x_i-\mu)^2 - \frac{1}{2s_0^2}(\mu-m_0)^2\right) \\
&= C_1 \cdot \exp(-\frac{1}{2\sigma^2}\sum_{i=1}^{N}(x_i^2+\mu^2-2\mu x_i) - \frac{1}{2s_0^2}(\mu^2+m_0^2-2\mu m_0)) \\
&= C_2 \cdot \exp\left(-\frac{1}{2\hat{\sigma}_N^2}\left[\mu^2 - 2\mu\hat{\sigma}_N^2(\frac{N\mu_{ML}}{\sigma^2}+\frac{m_0}{s_0^2}) + C_3\right]\right).
\end{aligned}
$$

C_2 and C_3 are such that $\frac{P(\mathcal{D}|\mu,\sigma^2)\cdot P(\mu)}{Z}$ is a probability density function.
For the Gaussian pdf the max coincides with the mean due to symmetry!

Gaussian MAP for (μ, σ^2)

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The posterior distribution $P(\mu \mid \mathcal{D}, \sigma^2) \sim \mathcal{N}\left(\mu \mid \hat{\mu}_N, \hat{\sigma}_N^2\right)$:

- $\frac{1}{\hat{\sigma}_N^2} = \frac{1}{s_0^2} + \frac{N}{\sigma^2} \Rightarrow \hat{\sigma}_N^2 = \frac{\sigma^2 s_0^2}{N s_0^2 + \sigma^2}$ **(post-variance)**
- $\hat{\mu}_N = \frac{\sigma^2}{N s_0^2 + \sigma^2} m_0 + \frac{N s_0^2}{N s_0^2 + \sigma^2} \mu_{ML}$. **(post-mean)**

where $\hat{\mu}_N, \hat{\sigma}_N^2$ are the Bayesian MAP estimates, $\mu_{ML} = \frac{1}{N} \sum_{i=1}^N x_i$.

Limiting cases

	$N = 0$	$N \to \infty$
$\hat{\sigma}_N^2$	s_0^2	0
$\hat{\mu}_N$	m_0	μ_{ML}

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Posterior for the mean

- Posterior of the Gaussian mean for increasing data size N

(The variance reduces with N.)

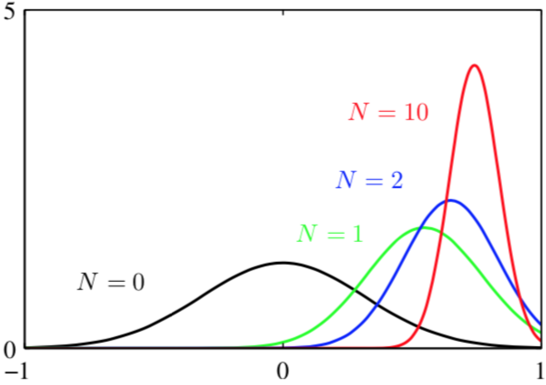

Figure: Bishop (B.2), p.99

Conjugate Priors

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☞ In the above case:
Observe already that the posterior distribution has the same shape (Gaussian) as the prior!

$P(\theta)$ is a **conjugate prior** for a particular likelihood $P(\mathcal{D} \mid \theta)$ if the posterior is of the same functional form as the prior.

*For all members of the exponential family it is possible to construct a conjugate prior

$$P(\mathbf{x} \mid \theta) = h(\mathbf{x})g(\theta)\exp\left(\theta^T u(\mathbf{x})\right).$$

Conjugate Priors for Gaussian variance*

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- Likelihood

$$
\begin{aligned}
P(\mathcal{D}|\mu,\sigma^2) &= \prod_{n=1}^{N} \mathcal{N}(x_n|\mu,\sigma^2) \\
&\overset{\beta:=1/\sigma^2}{=} \left(\frac{\beta}{2\pi}\right)^{N/2} \exp\left\{-\frac{\beta}{2}\sum_{i=1}^{N}(x_i-\mu)^2\right\}
\end{aligned}
$$

- For known mean, the suitable prior is:

$$
P(\beta) = \mathrm{Gam}(\beta \mid a, b) = \frac{1}{\Gamma(a)} b^a \beta^{a-1} \exp(-b\beta).
$$

Note: $\Gamma(x) := \int_0^\infty u^{x-1} e^{-u} du$. Also $\Gamma(x+1) = x\Gamma(x)$. It holds: $\Gamma(1) = 1$ and hence $\Gamma(x+1) = x!$ when x is integer.

Properties: $\mathbb{E}[\lambda] = \frac{a}{b}$, and $Var[\lambda] = \frac{a}{b^2}$.

Conjugate Priors for Gaussian (general)*

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Likelihood reformulated

$$
\begin{aligned}
P(\mathcal{D}|\mu,\sigma^2) &\overset{\beta:=1/\sigma^2}{=} \left(\frac{\beta}{2\pi}\right)^{N/2} \exp\left\{-\frac{\beta}{2}\sum_{i=1}^{N}(x_i-\mu)^2\right\} \\
&\propto \left[\beta^{1/2}\exp\left(-\frac{\beta\mu^2}{2}\right)\right]^N \exp\left\{\beta\mu\sum_{i=1}^{N}x_i - \frac{\beta}{2}\sum_{i=1}^{N}x_i^2\right\}
\end{aligned}
$$

☞ For unknown mean and variance the conjugate prior is

$$
P(\mu,\beta) = \mathcal{N}\left(\mu \mid \mu_0, (\beta)^{-1}\right) \cdot \mathrm{Gam}(\beta \mid a, b).
$$

Normal-Gamma distribution (coupling between μ and β)

Bernoulli inference

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Consider a number N of Bernoulli realisations with parameter θ

$$P(x = 1 \mid \theta) \quad = \quad \theta.$$

The probability distribution for $\mathrm{Bernoulli}$ is given by

$$\mathrm{Bernoulli}\,(x \mid \theta) \quad = \quad \theta^{x}(1-\theta)^{1-x},$$

which has variance $Var[x] = \theta(1-\theta)$.

The Likelihood function, given i.i.d. observations from $P(x = 1 \mid \theta)$

$$P\,(\mathcal{D} \mid \theta) \quad = \quad \prod_{i=1}^{N} \theta^{x_i}(1-\theta)^{1-x_i}.$$

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Bernoulli ML

From the Maximum Likelihood estimate, we get:

$$\begin{aligned} \theta_{ML} &= \arg\max_{\mu} \log P(\mathcal{D} \mid \theta) \\ &= \arg\max_{\theta} \sum_{i=1}^{N} \log P(x_i \mid \theta) \\ &= \arg\max_{\theta} \sum_{i=1}^{N} \{x_i \log(\theta) + (1 - x_i) \log(1 - \theta)\} \end{aligned}$$

$$\frac{d}{d\theta} \log P(\mathcal{D} \mid \theta) = 0 \quad \Rightarrow \quad \sum_{i=1}^{N} \frac{x_i}{\theta} = \sum_{i=1}^{N} \frac{1 - x_i}{1 - \theta}$$

$$\theta_{ML} = \frac{1}{N} \sum_{i=1}^{N} x_i .$$

Binomial Heads

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Equivalently, we see that

$$\theta_{ML} \quad = \quad \frac{m(N)}{N},$$

where $m(N)$ is the number of Heads (H) in a Heads-Tails experiment of size N.

The number $m(N)$ follows the Binomial distribution

$$m(N) \sim \mathrm{Binomial}(m \mid N, \theta) = \left(\begin{array}{c} N \\ m \end{array} \right) \theta^m (1-\theta)^{N-m},$$

where $\left(\begin{array}{c} N \\ m \end{array} \right) = \frac{N!}{(N-m)!m!}$ is the number of choosing m objects out of N identical ones.

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Binomial Distribution

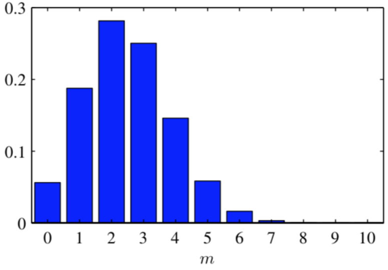

Figure: Bishop (B.2), p.70

- $\mathbb{E}[m] := \sum_{m=0}^{N} m\text{Binomial}(m \mid N, \theta) = N\theta$.
- $Var[m] := \sum_{m=0}^{N}(m - \mathbb{E}[m])^2\text{Binomial}(m \mid N, \theta) = N\theta(1-\theta)$.

Bernoulli ML issues

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☞ The ML estimator for the Bernoulli is based strongly on the available data, and tends to severely overfit the estimated value for small data-sets.

Remember the Heads-Tails example $\{H, \ H, \ H\}$.

$$\theta_{ML} \quad = \quad \frac{1}{3}\sum_{i=1}^{3} x_i = \frac{1+1+1}{3} = 1.$$

Prediction: From the above the coin should always (a.s.) give Heads !

Bayesian approach

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• We will use the Bayesian approach and will propose a conjugate prior that keeps the same shape when multiplied by the Likelihood function.

- We saw that the Likelihood function is

$$P(\mathcal{D} \mid \theta) = \prod_{i=1}^{N} \theta^{x_i}(1-\theta)^{1-x_i} = \theta^{m}(1-\theta)^{\ell},$$

where m is the count of (H), ℓ is the count of (T) and $\ell = N - m$.

Bayesian approach

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$$P(\mathcal{D} \mid \theta) = \prod_{i=1}^{N} \theta^{x_i}(1-\theta)^{1-x_i} = \theta^{m}(1-\theta)^{\ell},$$

  where m is the count of (H), ℓ is the count of (T) and $\ell = N - m$.

- The Beta function has the conjugate property

$$\mathrm{Beta}(\theta \mid a, b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\theta^{a-1}(1-\theta)^{b-1}.$$

Beta Moments

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The mean and variance of the Beta distribution are given by

$$\begin{aligned} \mathbb{E}[\theta] &= \frac{a}{a+b}, \\ Var[\theta] &= \frac{ab}{(a+b)^2(a+b+1)}. \end{aligned}$$

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Beta plots

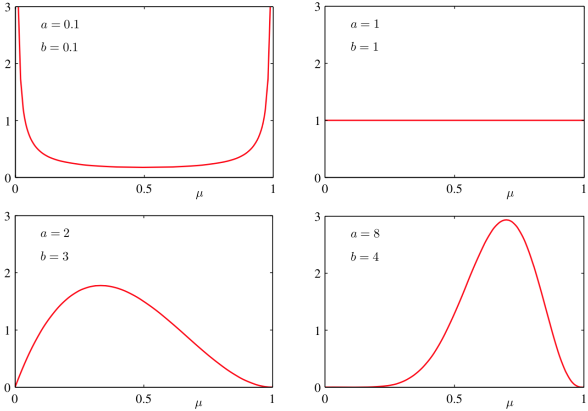

Figure 2.2 Plots of the beta distribution $\text{Beta}(\mu|a, b)$ given by (2.13) as a function of μ for various values of the hyperparameters a and b.

Figure: Bishop (B.2), p.72

Bernoulli posterior

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• Suppose $\mathrm{Beta}(\theta|a, b)$ **is the prior distribution for** θ and multiply with the binomial likelihood function.

☞ Posterior distribution $\mathrm{Beta}(\theta|a+m, b+\ell)$:

$$P(\theta \mid m, \ell, a, b) \quad \propto \quad \theta^{m+a-1}(1-\theta)^{\ell+b-1}.$$

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Bernoulli posterior cont'd

Taking into account the normalisation, we have:

$$P(\theta \mid m, \ell, a, b) \quad = \quad \frac{\Gamma(m+a+\ell+b)}{\Gamma(m+a)\Gamma(\ell+b)}\theta^{m+a-1}(1-\theta)^{\ell+b-1}.$$

❏ Observing m Heads in data, adds m to a. Similarly, observing ℓ Tails in data adds ℓ to b. $\rightarrow$ These hyperparameters can be seen as the effective number of observations of $x = 1$ and $x = 0$.

❏ The posterior probability distribution has an **updated mean**

$$P(x=1 \mid \mathcal{D}) \quad = \quad \frac{m+a}{m+a+\ell+b}$$

When $m, \ell \rightarrow \infty$: $P(x=1 \mid \mathcal{D}) \approx \frac{m}{m+\ell} = \frac{m}{N} = \theta_{ML}$.

Sequential Learning

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- Very often we do not have the whole dataset $\mathcal{D}$ available.
- Data arrives sequentially, and we need to update our estimates using the new info.
- e.g. $\mathcal{D}_1$, $\mathcal{D}_2$, ..., $\mathcal{D}_t$,...
- In the simplest case, each data-set consists of 1 single new data (measurement)

Question: How do the ML and MAP (Bayesian) estimators update sequentially?

Sequential ML

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Consider we have observed data from $\mathcal{D}' = \{x_1, \dots, x_{N-1}\}$, estimate $\theta_{ML}^{(N-1)}$, and then observe x_N and update the estimate.

- Gaussian, Bernoulli:

$$\begin{aligned} \theta_{ML}^{(N)} &= \frac{1}{N}\sum_{i=1}^{N} x_i \\ &= \dots \\ &= \theta_{ML}^{(N-1)} + \frac{1}{N}\left(x_N - \theta_{ML}^{(N-1)}\right) \end{aligned}$$

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Robbins-Monro algorithm

In the more general case, we can use following sequential algorithm:

$$\theta^{(N)} = \theta^{(N-1)} - a_{N-1} \frac{\partial}{\partial \theta^{(N-1)}} \left[-\log P(x_N \mid \theta^{(N-1)}) \right]$$

where

- $\lim_{N\to\infty} a_N = 0$,
- $\sum_{N=1}^{\infty} a_N = \infty$,
- $\sum_{N=1}^{\infty} a_N^2 < \infty$

Note that in the case of Gaussian $-\log P(x|\mu,\sigma^2) = \frac{1}{2\sigma^2}(x-\mu)^2$,

$$\mu^{(N)} = \mu^{(N-1)} + a_{N-1} \frac{1}{\sigma^2} \left(x_N - \mu^{(N-1)} \right).$$

What is a_{N-1} for the Gaussian ML?

Sequential MAP

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- We consider again a data set $\mathcal{D}'$ of $N-1$ data ponts, and observation x_N.
- Posterior distribution:

$$\begin{aligned} P(\theta \mid \{\mathcal{D}', x_N\}) &\propto \prod_{i=1}^{N} P(x_i \mid \theta)) \cdot P(\theta) \\ &= \left[\prod_{i=1}^{N-1} P(x_i \mid \theta) \cdot P(\theta)\right] P(x_N \mid \theta) \\ &= P(x_N \mid \theta) \cdot P(\theta \mid \mathcal{D}'). \end{aligned}$$

The posterior distribution after $N-1$ observations, becomes the new prior!

Exercise 1: RADAR

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A radar scans a surface for dangerous targets every time unit [hour].

- The detection mechanism of the radar can detect a real target in 99% of all cases (True Positive).
- It happens that in 2% of scans there is a False Alarm (False Positive).
- We know: a real target appears every 1000 time units.

Question: What is the probability that an alarm by the radar corresponds to a true target?

Solution 1: RADAR

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- $P(Alarm \mid Target) = 0.99$
- $P(Alarm \mid Nothing) = 0.02$
- $P(Target) = 0.001$

$$\begin{aligned} P(Target \mid Alarm) &= \frac{P(Alarm \mid Target) \cdot P(Target)}{P(Alarm)} \\ &= \frac{0.99 \cdot 0.001}{0.99 \cdot 0.001 + 0.02 \cdot 0.999} \\ &\approx 0.05. \end{aligned}$$

Exercise 2: WIFI

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Users want to use a public WiFi shared with others. At different times per day, users have different access probability:

1. When there are few users connected (GOOD): $P(Access) = 99/100$.
2. When there are many users online (BAD): $P(Access) = 50/100$.

☞ We do not know how many users are connected. Suppose a first user requests access and he receives it!

Question: What is the probability that a second user will receive access as well? (i.i.d.)

Solution 2: WIFI

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The user does not know if the channel is GOOD or BAD, so let us choose a prior distribution

$$P(GOOD) = P(BAD) = 0.5.$$

We want to compute:

$$P(X_2 = 1 \mid X_1 = 1) = \frac{P(X_2 = 1, X_1 = 1)}{P(X_1 = 1)}.$$

We have the following information:

$$P(X_i = 1 \mid GOOD) = 0.99 \qquad P(X_i = 1 \mid BAD) = 0.5.$$

Solution 2: WIFI cont'd

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$$\begin{aligned}
P(X_2 = 1, X_1 = 1) &= P(X_2 = 1 | GOOD) P(X_1 = 1 | GOOD) P(GOOD) \\
&+ P(X_2 = 1 | BAD) P(X_1 = 1 | BAD) P(BAD) \\
&= (0.99)^2 \frac{1}{2} + (0.50)^2 \frac{1}{2}
\end{aligned}$$

Also,

$$\begin{aligned}
P(X_1 = 1) &= P(X_1 = 1 | GOOD) P(GOOD) + P(X_1 = 1 | BAD) P(BAD) \\
&= 0.99 \frac{1}{2} + 0.50 \frac{1}{2}
\end{aligned}$$

Altogether,

$$\begin{aligned}
P(X_2 = 1 \mid X_1 = 1) &= \frac{(0.99)^2 \frac{1}{2} + (0.50)^2 \frac{1}{2}}{0.99 \frac{1}{2} + 0.50 \frac{1}{2}} \\
&= \frac{(0.99)^2 + (0.50)^2}{0.99 + 0.50} \approx 82,6\%
\end{aligned}$$

Solution 2: WIFI cont'd

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The states of the two efforts to access are **not independent**!

$$82.6\% \approx P(X_2 = 1 \mid X_1 = 1) \quad \neq \quad P(X_2 = 1) = \frac{1}{2}(0.99 + 0.50) \approx 75\%$$

☞ The fact that the first user got access, gives extra information in order to infer the probability that the second user gets also access.

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END