NDA: Time Series Analysis - part 1

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Bibliography

Formal content:

- Peter Brockwell and Richard Davis
  Introduction to Time Series and Forecasting
- William Thistleton and Tural Sadigov
  MOOC Coursera: *Practical Time Series Analysis*

Informal guide in python:

- `www.machinelearningplus.com/time-series/`

Illustrative datasets:

- `data.world/datasets/time-series`
- `www.kaggle.com/tags/time-series`

Outline

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What is time series analysis?

Definition

Set of observations $\{x_t\}$, recorded at time $t \in T_0$

Think of each x_t as a realization from a distribution

Specificities of the problem

A unique realization of the process

$\Rightarrow$ necessary to make assumptions

- observe time series, identify particularities
- choose a family of models X_t to represent data
- check the goodness of the model

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Assumptions for this course

Restrictions to a subfamily of problems

- discrete time series (discrete time set)
- fixed time steps (time resolution)
- univariate (one single variable over time) $\neq$ multivariate
  $\rightarrow$ processes have values in $\mathbb{R}$

And only a few approaches (Box-Jenkins)

- *e.g.* no Fourier analysis

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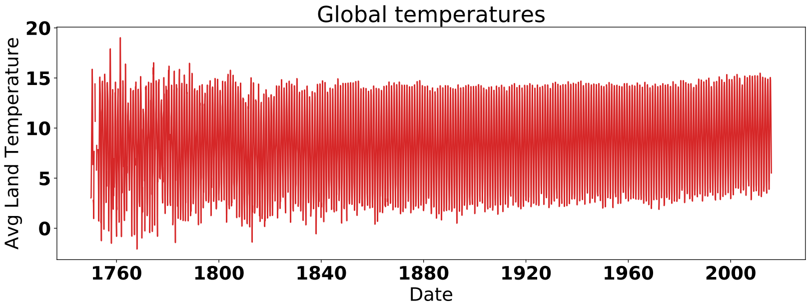
A few examples

Average global land temperature (per month)



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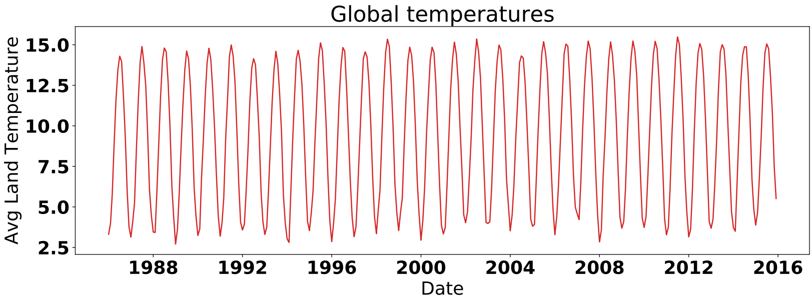
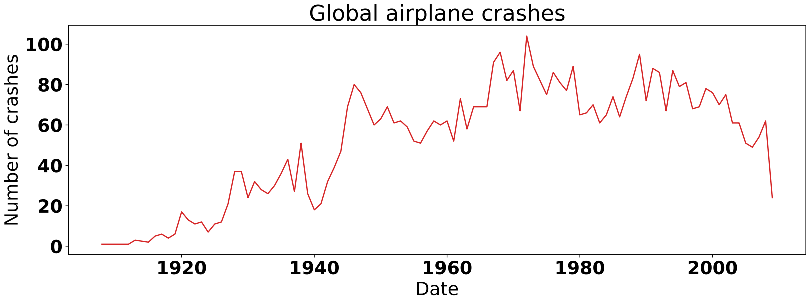


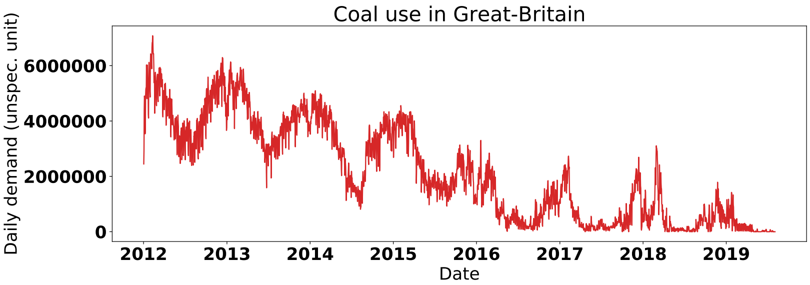
A few examples

Number of civil airplane crashes (per year)



A few examples

Daily demand of power obtained with coal in GB



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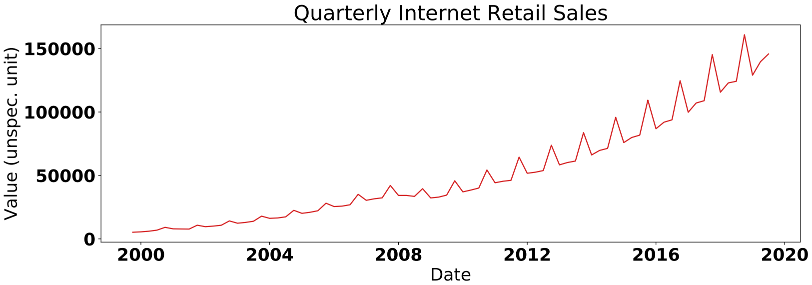


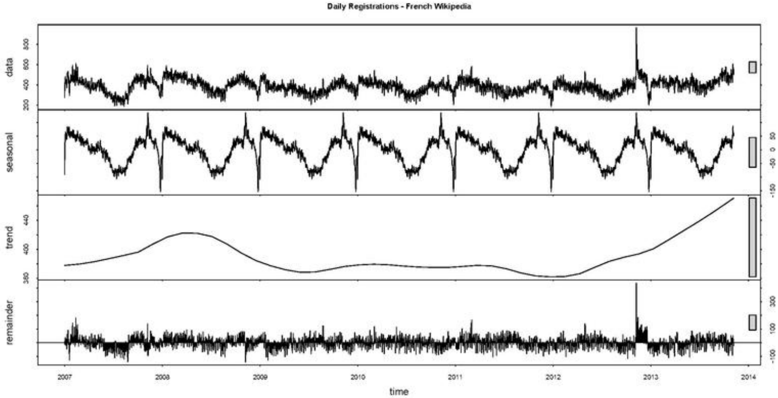
Goals of time series analysis

- Have a simplified description of the data
  $\rightarrow$ improve our understanding (*ex: climate data*)
- Test an assumption
  ex: is there a significant measurable global warming?
- Filter: separate signal from noise
  ex: known physical signal broadcast $\rightarrow$ *filter noise*
- Predict future values
  ex: predict the future demand for a product
- Simulate a process in a complex model
  ex: expectation for the GDP to predict economic activity

How to analyze a time series? (1)

Analyze from Greek *análusis* $\sim$ unravel $\Rightarrow$ decompose

Decompose the time series into parts, for example:

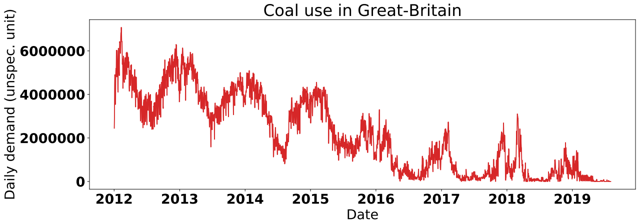


How to analyze a time series? (1)

First step

Plot the time series to:

- identify the existence of a trend *(tendance)*
- uncover seasonal variations *(variations saisonnières)*
- detect changes of behavior
- spot outliers *(valeurs aberrantes)*



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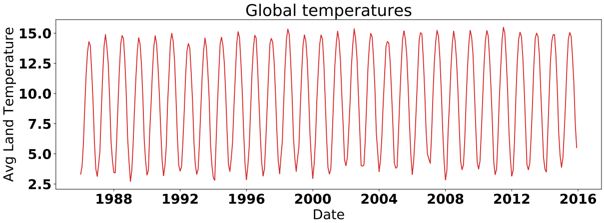

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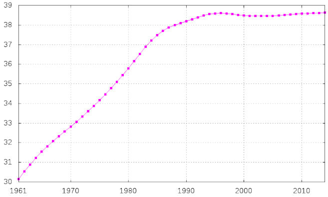
Poland population



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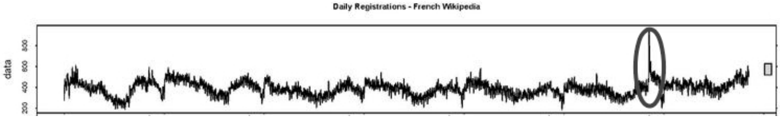

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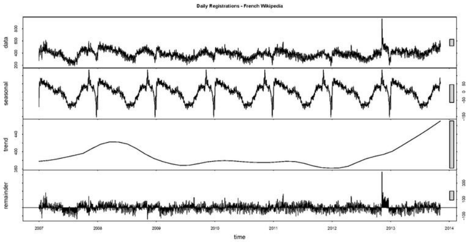
$\rightarrow$ subjective components in this analysis

The classical decomposition

Classical decomposition of the time series

$$X_t = s_t + m_t + r_t$$

- seasonality s_t
- trend m_t
- remainder r_t



Outline

Mean and covariance of a time series

Two fundamental definitions

Let $\{X_t\}$ a time series with $\mathbb{E}[X_t^2] < \infty$ (finite variance)
rk: here we consider X_t as a model

- **mean function** of X_t, defined for all t:

$$\mu_X(t) = \mathbb{E}[X_t]$$

- **covariance function** of X_t, defined for all r, s:

$$\gamma_X(r,s) = Cov(X_r, X_s) = \mathbb{E}[(X_r - \mu_X(r))(X_s - \mu_X(s))]$$

Stationarity

Informal definition

A process is **stationary** *(stationnaire)* if its statistical properties are similar when shifted in time

Remarks:

- stationarity is a property of a model (not of data)
- stationary processes are simpler to investigate
  $\Rightarrow$ usual to transform a TS to obtain a stationary process

Stationarity

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A process is **stationary** *(stationnaire)* if its statistical properties are similar when shifted in time

Formal definition

A process is said to be **weakly stationary** if

- the mean function $\mu_X(t)$ is independent of $t \Rightarrow \mu_X$
- $\gamma_X(t+h, t)$ is independent of t for any h (including $h = 0$)
  h is called the *lag* *(décalage)*

$$\gamma_X(t+h, t) = \mathbb{E}[(X_{t+h} - \mu_X)(X_t - \mu_X)] = \gamma_X(h)$$

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A process is **stationary** *(stationnaire)* if its statistical properties are similar when shifted in time

Formal definition

A process is said to be **strictly (or strongly) stationary** if

- $\forall n$ and $\forall h$

$$P(X_1 = x_1, \dots, X_n = x_n) = P(X_{1+h} = x_1, \dots, X_{n+h} = x_n)$$

Unless specified otherwise, we talk about weak stationarity in the following

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Autocorrelation function

Notice that for a stationary time series: $\gamma_X(t+h,t) = \gamma_X(h)$
$\Rightarrow$ the covariance function γ_X has one variable (the lag)

Definition

For a stationary time series:

- the **autocovariance function** at lag h is:

$$\gamma_X(h) = Cov(X_{t+h}, X_t)$$

- the **autocorrelation function** (ACF) at lag h is:

$$\rho_X(h) = \frac{\gamma_X(h)}{\gamma_X(0)}$$

Equivalent on real data

Concepts well defined on models, but what about real data?
Let $\{x_1, \ldots, x_n\}$ be a series of observations

Sample mean

- the **sample mean** estimator is

$$\overline{x} = \frac{1}{n}\sum_{t=1}^{n} x_t$$

Equivalent on real data

Concepts well defined on models, but what about real data?
Let $\{x_1, \ldots, x_n\}$ be a series of observations

Sample autocovariance function

- the **sample autocovariance function** estimator is

$$\hat{\gamma}(h) = \frac{1}{n} \sum_{t=1}^{n-|h|} (x_{t+|h|} - \overline{x}).(x_t - \overline{x}) \ , \ -n < h < n$$

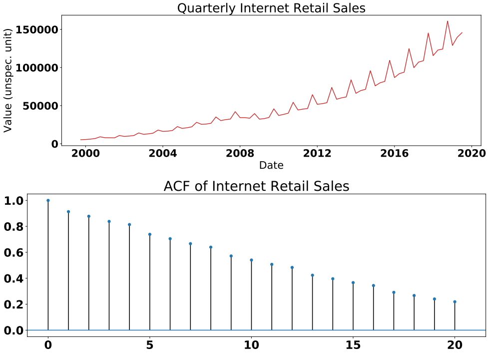
remark: notice the denominator (because of mathematical properties)

- the **sample autocorrelation function** estimator is

$$\hat{\rho}(h) = \frac{\hat{\gamma}(h)}{\hat{\gamma}(0)} \ , \ -n < h < n. \text{ Note that } \hat{\rho}(h) \in [-1; 1]$$

Equivalent on real data

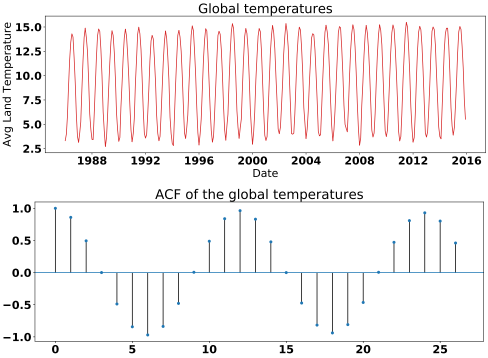
Data with strong trend:



slow decay of correlations with h

Equivalent on real data

Data with strong seasonality:



periodicity on the ACF (here monthly measures $\Rightarrow$ period = 12)

Outline

What is a time series model?

Definition

Time series model: specification of the joint distributions of a sequence of random variables X_t of which the observed data is supposed to be the realization

Remarks:

- suppose to know $\forall t$ the distribs $P(x_1, \ldots, x_t, \ldots, x_n)$
  $\Rightarrow$ in most case too many parameters to handle
- in practice, we focus on first and second order moments:
  - expected values $\mathbb{E}[X_t]$
  - and expected products $\mathbb{E}[X_{t+h}X_t]$, $h = 1, 2, \ldots$

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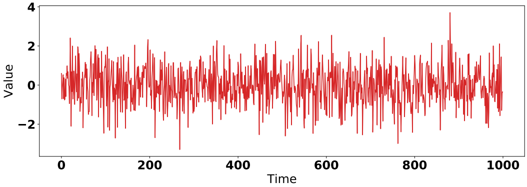
Independent Identically Distributed noise model

IID noise

- independant:
  $P(X_1 = x_1, \ldots, X_n = x_n) = P(X_1 = x_1) \cdot \ldots \cdot P(X_n = x_n)$
- identically distributed: $P(X_t = x) = P(X_{t'} = x)$

IID noise is obviously stationary

ex: repeated coin flipping with heads=1, tails=-1 should be IID noise



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White noise *(bruit blanc)*

Special case IID noise with

- 0 mean: $E[X_t] = 0$
- autocovariance function:
  $\gamma_X(h) = \sigma^2$ if $h = 0$ and $\gamma_X(h) = 0$ if $h \neq 0$

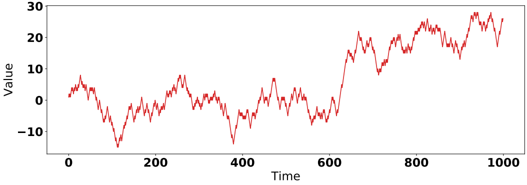
Random Walk model

How to build a random walk? *(marche aléatoire)*

Suppose $\{X_t\}$ is IID noise, then $\{S_t\}$ defined as:

$$S_t = X_1 + \ldots + X_t$$

is a random walk



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- and conversely $X_t = S_t - S_{t-1}$, or $S_t = S_{t-1} + X_t$

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Outline

How to analyze a time series? (2)

Second step:

- (if necessary) transform data
- remove the trend and seasonal components to get stationary residuals *(résidus)*

Residual time series obtained (remainder) should be stationary, but not necessarily IID noise…

Back to the classical decomposition

Classical decomposition of the time series

$$X_t = s_t + m_t + r_t$$

- seasonality s_t
- trend m_t
- remainder r_t

What is the difference between seasonality and trend?

- seasonality is periodic $s_{t+d} = s_t$
- seasonality is null on average $\sum_{j=1}^{d} s_j = 0$

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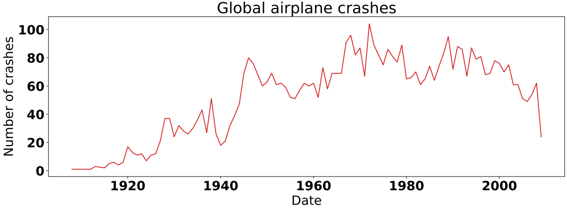
Isolate the trend component

Model and regression

→ cf. course *Regression*

eg. 2^{nd} order polynomial model with least squares regression



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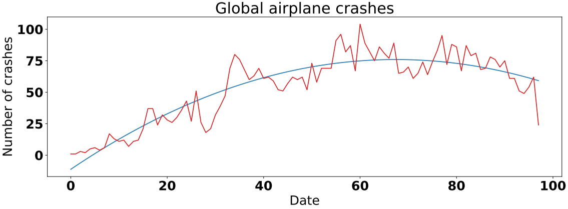
Model and regression

$\rightarrow$ cf. course *Regression*

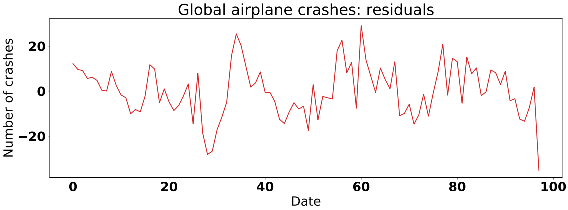
eg. 2nd order polynomial model with least squares regression

$$\text{Minimize } \sum_{t=1}^{n}(x_t - m_t)^2, \text{ with } m_t = a_0 + a_1 t + a_2 t^2$$



Isolate the trend component

Then we plot the residuals $\{x_t - m_t\}$



Questions to ask oneself:

- Is there a perceptible trend? Is it smooth? Do we see stretches (séquences) of values of the same sign?
- Does it look stationary? Does it look like noise?

Isolate seasonal component

Model and regression

Which model?

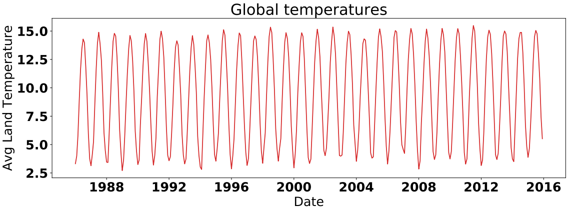
Isolate seasonal component

Model and regression

Which model? Harmonic regression

$$s_t = (a_0 +) \sum_{j=1}^{k} a_j cos\left(\frac{2\pi t}{T_j}\right) + b_j sin\left(\frac{2\pi t}{T_j}\right)$$

where T_j are the expected periods of the process



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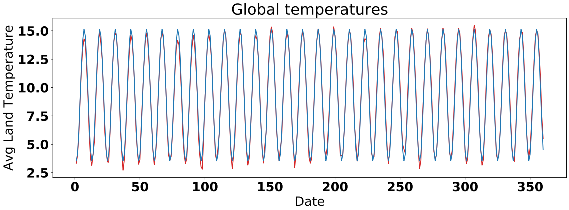

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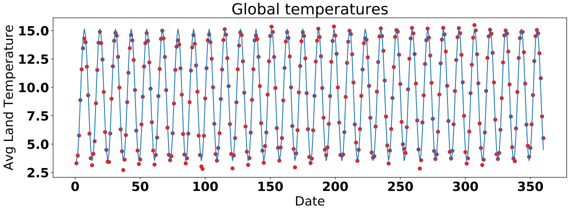

About pre-processing

Second step:

- (if necessary) transform data
- remove the trend and seasonal components to get stationary residuals

When is it necessary to transform data?

Some cases

- if outliers → if justified, discard them
  ex: external stimulus, mistake in data acquisition, ...
- if obvious different regimes
  → break data into homogeneous segments
- if noise or seasonality increase with trend level
  → logarithmic transformation of the data

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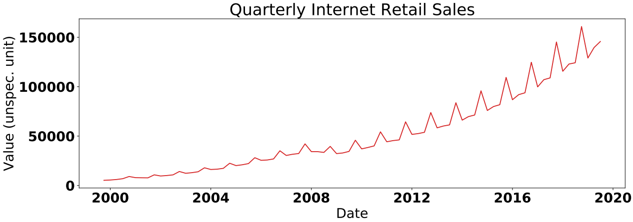
Logarithmic transformation

If fluctuations (seasonality, noise) grow with magnitude...



...after logarithmic transform

→ c.f. course *Regression (heteroscedasticity)*

Conduct similar analysis on the transformed time series and reverse the transformations in the end to model the original data

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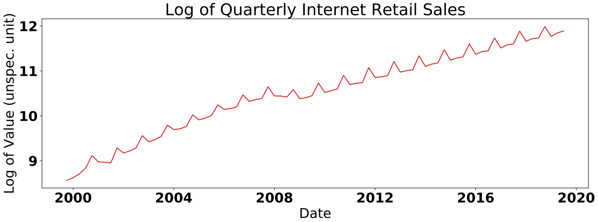

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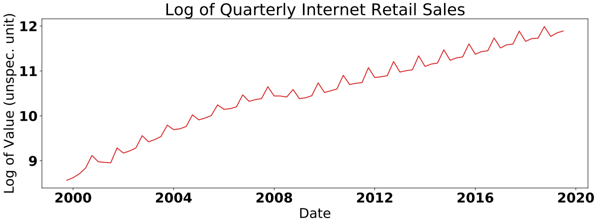

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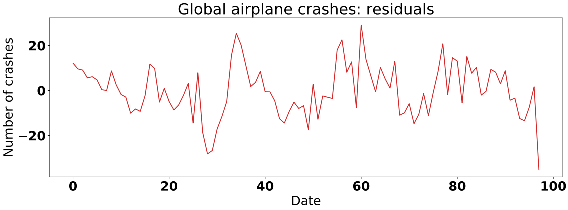
Stationarity of the residuals

Trend and seasonal components modeled $\rightarrow$ residuals



Visual evaluation of stationarity

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Stationarity of the residuals

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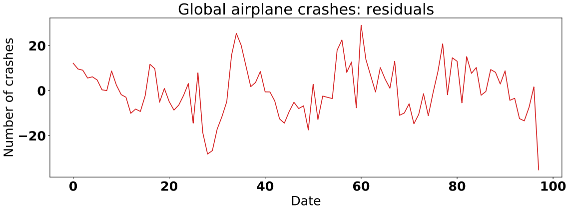

Quantitative evaluation of stationarity?

Testing stationarity of a time series

A general method in Time Series Analysis:

- assume a model
- compute consequences of this model
- test if observations are compatible

$\rightarrow$ c.f. course *Hypothesis testing*

We illustrate this logic with the Dickey-Fuller test

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Illustration: RW and AR(1) processes

Auto-regressive processes

Consider the random walk model (not stationary):

$$X_t = X_{t-1} + W_t \text{ where } W_t \text{ is the “error”, some white noise}$$

A specific kind of auto-regressive models: AR(1)

$$X_t = \phi X_{t-1} + W_t$$

$\Rightarrow$ Random Walk is AR(1) with $\phi = 1$

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AR(1) processes

$$X_t = \phi X_{t-1} + W_t$$

W_t is a white noise (0 means, σ^2 variance)

Stationary? (necessary conditions)

- $\mathbb{E}[X_t] = 0$
- $\gamma_X(h) = \phi^{|h|}\gamma_X(0) = \phi^{|h|}\frac{\sigma^2}{1-\phi^2} \to$ problem if $\phi = 1$

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Testing for unit roots

Notion of characteristic equation

$$X_t - \phi X_{t-1} - W_t = 0$$

$$\longrightarrow x - \phi = 0$$

1 is a root of this equation if $\phi = 1$

More generally, if we have a process modeled by the equation:

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} + \ldots - \phi_p X_{t-p} - W_t = 0$$

$$\longrightarrow x^p - \phi_1 x^{p-1} - \phi_2 x^{p-2} + \ldots - \phi_p = 0$$

if 1 is a root, the process is not stationary (admitted)

Stationarity tests look for unit roots

Testing for unit roots

Notion of characteristic equation

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Dickey-Fuller test

Do we have $X_t = X_{t-1} + W_t$?

(and actually some other models that we ignore in this course)

We suppose that we follow this model and we test for $\phi = 1$

- H_0: $\phi = 1$ and process is not stationary
- H_1: $|\phi| < 1$ and process is stationary

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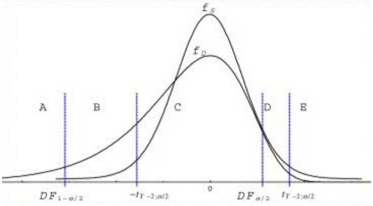
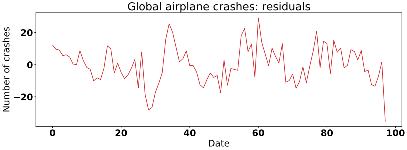
Estimate (ϕ -1) from data

compute $\frac{\hat{\phi}-1}{\hat{\sigma}/\sqrt{N}}$ and compare to Dickey-Fuller distribution



Testing in practice (airplane crashes residuals)



$$\frac{\hat{\phi}-1}{\sigma/\sqrt{N}} = -2.39 \text{ from data}$$

critical value from table (for $p = 0.05$): $\frac{\hat{\phi}-1}{\sigma/\sqrt{N}} < -1.944$

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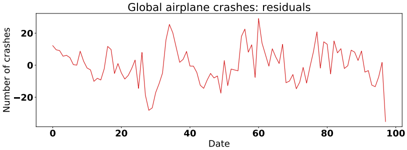

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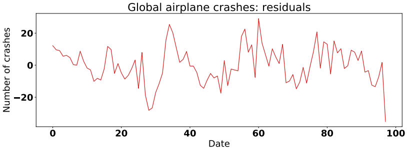

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Remarks

- other tests available, with other models
  Augmented Dickey-Fuller, Phillips-Perron, ...
- in general implemented in statistic software

Toward ARMA models

A very standard process model:

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots + \phi_p X_{t-p} + W_t$$

autoregressive with a memory of length $p \rightarrow$ AR(p)

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$\longrightarrow$ family of ARMA models (next course)

Studying time series in python

Among several options, `pandas` library

A few useful functions:

- Load data as dataframe:
  `read_csv` from `pandas` library
- Fitting:
  `curve_fit` in `scipy.optimize` library
- Autocorrelation function:
  `plot_ACF` in `statsmodels` library