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4. Confidence Intervals / Hypothesis Tests

Data Analysis for Networks - NDA'22
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Bibliography

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B.1 H. Pishro-Nik, "Introduction to probability, statistics, and random processes", available at https://www.probabilitycourse.com, Kappa Research LLC, 2014.

☞ Chapter 8.3, 8.4

Intro

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We will discuss in this course two main themes:

- **Confidence Intervals**
- **Hypothesis Tests**

Applications

- Anomaly detection: Sensors observe the network ingress traffic periodically. When the network is healthy, the mean flow rate is R $[bits/sec]$. How can one decide both fast and correctly that an anomaly appears?
- Signal detection: An RF antenna needs to decide the presence or not of a signal (e.g. radar detects target)

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Confidence Intervals

Interval Estimation

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- Let $X_1, \ldots, X_n$ be a random sample from a distribution, with a parameter θ to be estimated.
- We have observed $x_1, \ldots, x_n$.
- We can use $\hat{\Theta} = h(X_1, \ldots, X_n)$ to estimate θ.
- Although $\hat{\Theta}$ can be asymptotically consistent, we don't know how close we are to the real θ.

☞ Introducing interval estimation: instead of giving just one estimate value $\hat{\theta}$, we produce an interval that is likely to include the true value of θ.

$$\hat{\theta} \quad \in \quad \left[\hat{\theta}_\ell,\ \hat{\theta}_h\right].$$

e.g. instead of saying $\hat{\theta} = 34.25$, we report the interval $[30.96,\ 37.81]$.

Confidence Intervals

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There are two important concepts, related:

- the length of the reported interval $\hat{\theta}_h - \hat{\theta}_\ell$.
- the level of confidence about the interval.

☞ The smaller the interval, the higher the precision we estimate θ .
☞The confidence level is the probability that the constructed interval includes the real value of θ. High confidence levels are desirable.

General framework

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An interval estimator with confidence level $1-\alpha$ consists of two estimators $\hat{\Theta}_\ell\,(X_1,\ldots,X_n)$ and $\hat{\Theta}_h\,(X_1,\ldots,X_n)$ such that

$$P\left(\hat{\Theta}_\ell \leq \theta \leq \hat{\Theta}_h\right) \quad \geq \quad 1-\alpha,$$

for every possible value of θ. Equivalently, we say that $\left[\hat{\Theta}_\ell,\ \hat{\Theta}_h\right]$ is a $(1-\alpha)100\%$ confidence interval for θ.

☞ The randomness is due to $\hat{\Theta}_\ell\,(X_1,\ldots,X_n)$ and $\hat{\Theta}_h\,(X_1,\ldots,X_n)$ and not θ.

Finding estimators

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Let X be a continuous random variable with CDF $F_X(x) = P(X \leq x)$.
How can we find x_ℓ and x_h such that

$$P(x_\ell \leq X \leq x_h) \;\; = \;\; 1 - \alpha.$$

☞ Choose x_ℓ and x_h such that:

$$P(X \leq x_\ell) = \frac{\alpha}{2}, \quad \textit{and} \quad P(X \geq x_h) = \frac{\alpha}{2}.$$

✍ This can be re-written as:

$$x_\ell = F_X^{-1}\left(\frac{\alpha}{2}\right), \quad \textit{and} \quad x_h = F_X^{-1}\left(1 - \frac{\alpha}{2}\right).$$

Then $[x_\ell,\; x_h]$ is a $(1-\alpha)$ interval for X.

CDF and PDF

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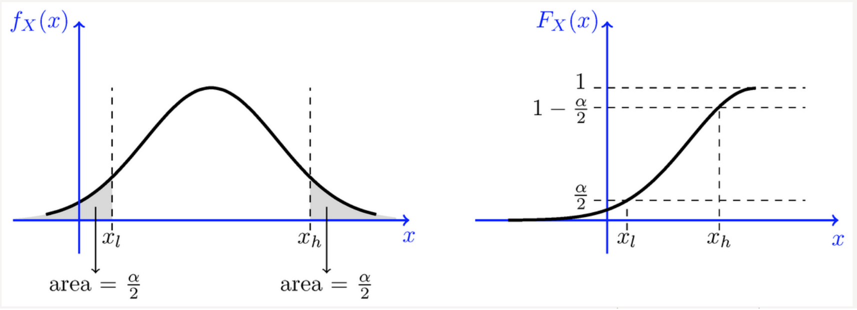

Special case: Normal r.v.

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Let $Z \sim N(0,1)$, find x_ℓ and x_h such that

$$P(x_\ell \leq Z \leq x_h) \;=\; 0.95.$$

As we showed above,

$$x_\ell = \Phi^{-1}\left(\frac{0.05}{2}\right) = -1.96, \quad \textit{and} \quad x_h = \Phi^{-1}\left(1 - \frac{0.05}{2}\right) = +1.96.$$

☞ For the Normal distribution, we denote these values by $z_{\frac{\alpha}{2}} := x_h$ and $z_{1-\frac{\alpha}{2}} := x_\ell$, and we can easily see that $z_{1-\frac{\alpha}{2}} = -z_{\frac{\alpha}{2}}$, so that

$$P\left(-z_{\frac{\alpha}{2}} \leq Z \leq z_{\frac{\alpha}{2}}\right) \;=\; 1-\alpha.$$

Normal interval

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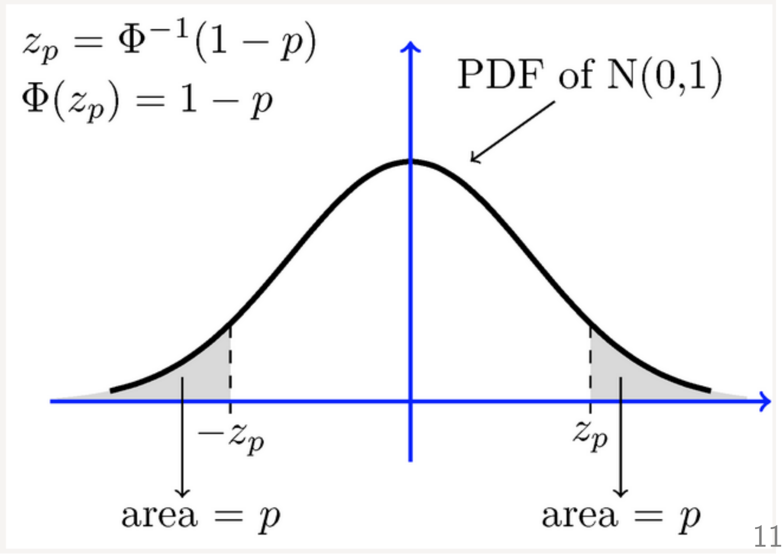

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Sample Mean (from Normal)

Let $(X_1, \ldots, X_n)$ be a random sample of size n from a normal distribution $N(\theta, 1)$. Find a 95% confidence interval for θ.

$$\hat{\Theta} = \overline{X} \quad = \quad \frac{X_1 + \ldots + X_n}{n}.$$

Since $X_i \sim N(\theta, 1)$ and the X_is are i.i.d., we conclude that $\overline{X} \sim N\left(\theta, \frac{1}{n}\right)$. By normalising $\overline{X}$, we conclude that the new random variable

$$\frac{\overline{X} - \theta}{\frac{1}{\sqrt{n}}} \sim N(0, 1).$$

Note here that the above probability distribution does **not** depend on θ! We call the above random variable, a pivotal quantity. Therefore,

$$P\left(\overline{X} - \frac{1.96}{\sqrt{n}} \leq \theta \leq \overline{X} + \frac{1.96}{\sqrt{n}}\right) \quad = \quad 0.95.$$

Sample Mean (known variance)

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✎ Question: Let $(X_1, \ldots, X_n)$ be a random sample of size n from a distribution with known $Var(X_i) = \sigma^2$, and unknown mean $\mathbb{E}[X_i] = \theta$. Find a $1 - \alpha$ confidence interval for θ. Assume n large.

Answer Sample Mean (known variance)

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- We choose as point estimator the sample mean:

$$\hat{\Theta} = \overline{X} \quad = \quad \frac{X_1 + \ldots + X_n}{n}.$$

Since n is large, and the samples are i.i.d, we can apply the Central Limit Theorem (CLT) and conclude that

$$Q := \frac{\overline{X} - \theta}{\frac{\sigma}{\sqrt{n}}}$$

approximately follows $N(0,1)$. Again, Q is a function of the sample and the θ, and its distribution does not depend on θ (pivotal quantity).

$$P\left(-z_{\frac{\alpha}{2}} \leq Q \leq +z_{\frac{\alpha}{2}}\right) \quad = \quad 1 - \alpha.$$

Then, $\left[\overline{X} - z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}},\ \overline{X} + z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}\right]$ is $(1-\alpha)100\%$ confidence interval for θ.

Exercise

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✎ Exercise: We wish to measure a quantity θ, but there is a random error in each measurement (noise). We take n measurements $(X_1, \ldots, X_n)$ and report the average of the measurements as the estimated value of θ. Then, measurement i is

$$X_i \quad = \quad \theta + W_i,$$

W_i being the error in the i-th measurement and all W_is are i.i.d, with $\mathbb{E}[W_i] = 0$ and $Var(W_i) = 4$ [units].

Q: How many measurements n do we need to make until we are 90% sure that the final estimation error is less than 0.25 units?

$$P(\overline{X} - 0.25 \leq \theta \leq \overline{X} + 0.25) \quad \geq \quad 0.90.$$

Solve Exercise

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☞ We will use the estimator $\overline{X}$ for θ, because $\mathbb{E}[X_i] = \theta$.

• We know than the CLT applies for large n. From the above analysis, we have the formula, for a $(1-\alpha)100\%$ confidence interval

$$\left[\overline{X} - z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}},\ \overline{X} + z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}\right].$$

Then we have the equality

$$z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \quad = \quad 0.25.$$

Here, $\alpha = 0.1$, $\sigma = \sqrt{4} = 2$, so that

$$n \quad = \quad (2z_{0.05}/0.25)^2 = \left(8 \cdot \Phi^{-1}(0.95)\right)^2 = (8 \cdot 1.645)^2 \approx 174 \textit{ samples}.$$

Sample Mean (unknown variance)

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✎ Question: Let $(X_1, \ldots, X_n)$ be a random sample of size n from a distribution with unknown $Var(X_i) = \sigma^2$, and unknown mean $\mathbb{E}[X_i] = \theta$. Find a $1 - \alpha$ confidence interval for θ. Assume n large.

☞ We can not use the above discussion, because we do not know σ!

Sample Mean (unknown variance)

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Two approaches:

1. Use an upper bound for σ, so that $\sigma \leq \sigma_{\text{max}}$, (larger interval)

$$\left[\overline{X} - z_{\frac{\alpha}{2}} \frac{\sigma_{\text{max}}}{\sqrt{n}},\ \overline{X} + z_{\frac{\alpha}{2}} \frac{\sigma_{\text{max}}}{\sqrt{n}}\right].$$

2. Use an estimate $\hat{\sigma}$ for σ, and we get

$$\left[\overline{X} - z_{\frac{\alpha}{2}} \frac{\hat{\sigma}}{\sqrt{n}},\ \overline{X} + z_{\frac{\alpha}{2}} \frac{\hat{\sigma}}{\sqrt{n}}\right],$$

which should be relatively good for n large.

Exercise (Voters' polling)

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✎ Exercise: We wish to estimate the percentage of voters that will vote for a certain candidate A in the coming elections. Let the sample size be n (large) and the unknown percentage of supporters θ.

We randomly select (with replacement) a voter and mark $X_i = 1$ if she will vote in favour of candidate A, otherwise $X_i = 0$. $X_i \sim \mathrm{Bernoulli}(\theta)$.

Q1: Find a $(1-\alpha)100\%$ confidence interval for θ based on $X_1, \dots, X_n$.

Q2: Estimate θ such that the margin of error is 3%. Assume a 95% confidence level. That is, we would like to choose n such that

$$P\left(\overline{X} - 0.03 \leq \theta \leq \overline{X} + 0.03\right) \quad \geq \quad 0.95,$$

where $\overline{X}$ is the portion of people in our random sample that say they plan to vote for the Candidate. How large does n need to be?

Answer Q1

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- Since n large, we assume that the CLT holds. The interval is given by:

$$\left[\overline{X} - z_{\frac{\alpha}{2}} \frac{\sigma_{\max}}{\sqrt{n}},\ \overline{X} + z_{\frac{\alpha}{2}} \frac{\sigma_{\max}}{\sqrt{n}}\right],$$

and we need to find an appropriate $\sigma_{\max}$.

Note, however, that

$$Var(X_i) = \theta(1-\theta) \leq \frac{1}{4} \quad \Rightarrow \quad \sigma_{\max} = \frac{1}{2}.$$

If the real θ is too small, or too large, this interval is very conservative.

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Answer Q1

- • Alternatively, we can now use that

$$Var(X_i) = \theta(1-\theta) \quad \Rightarrow \quad \hat{\sigma} \overset{estim.}{=} \sqrt{\overline{X}(1-\overline{X})}.$$

This estimate can be replaced in the expression for the interval.

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Answer Q2

- Using the expression with $\sigma_{\max}$ we get

$$z_{0.025}\frac{1}{2\sqrt{n}} = 0.03 \quad \Rightarrow \quad n = \left(\frac{1.96}{0.06}\right)^2 \approx 1068.$$

This is why most polls need a sample size of $n \approx 1000$.

Sample Mean (general)

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In the most general case, when the variance is unknown, in order to find the confidence interval of the sample mean, we can use the sample standard deviation! (remember?)

$$S = \sqrt{\frac{1}{n-1}\sum_{k=1}^{n}(X_k - \overline{X})^2} = \sqrt{\frac{1}{n-1}\left(\sum_{k=1}^{n} X_k^2 - n\overline{X}^2\right)},$$

then the interval

$$\left[\overline{X} - z_{\frac{\alpha}{2}}\frac{S}{\sqrt{n}},\ \overline{X} + z_{\frac{\alpha}{2}}\frac{S}{\sqrt{n}}\right],$$

is approximately a $(1-\alpha)100\%$ confidence interval for θ.

✎ Exercise: If $n = 100$, $\overline{X} = 15.6$, $S^2 = 8.4$, find an approximate 99% interval for $\theta = \mathbb{E}[X_i]$.

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Hypothesis Testing

Intro

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☞ We need to decide whether some hypothesis is true or not.

RADAR

H_0: No aircraft is present.

H_1: An aircraft is present.

H_0 is the null hypothesis (default to be true), and
H_1 is the alternative hypothesis.

Is the coin fair?

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Let θ be the probability of heads $\theta = P(HEADS)$.

H_0: The coin is fair, i.e. $\theta = \theta_0 = \frac{1}{2}$ (simple hypothesis).

H_1: The coin is not fair, i.e. $\theta \neq \frac{1}{2}$ (composite hypothesis).

☞ Given some measurements (sequential coin tosses) how can we decide whether the coin is fair or not? Let $n = 100$ tosses. Then,

$$X \sim Binomial(100, \theta)$$

is the total number of heads in the 100 tosses.

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Decision criterion for coin

☞ If X is around 50 we accept H_0, because $X/100$ is close to $1/2$.

Let's use the notion of a threshold (for now unknown).

If $|X - 50| \leq t$, accept H_0,

if $|X - 50| > t$, accept H_1.

Some issues here:
- What should be the value of t?
- What guarantees does the choice of t offer to our decision?

Error Probability Types

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Type I Error (False Positive)

$$P(type\ I\ error) = P\left(accept\ H_1 \mid H_0\right) \leq \alpha$$

Type II Error (False Negative)

$$P(type\ II\ error) = P\left(accept\ H_0 \mid H_1\right) \leq \beta$$

α, β are the levels of significance.

☞ Exercise: Calculate the threshold in the coin-toss example, so that

$$P(type\ I\ error) = P\left(|X - 50| > t \mid H_0\right) = \alpha = 0.05$$

Solve Exercise

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Use the CLT to approximate the distribution of the mean by the $\mathcal{N}(0,1)$:

$$Y = \frac{\frac{X}{n} - \theta_0}{\frac{\sqrt{\theta_0(1-\theta_0)}}{\sqrt{n}}} = \frac{X - n\theta_0}{\sqrt{n\theta_0(1-\theta_0)}} = \frac{X-50}{5}$$

$$\begin{aligned} P(type\ I\ error) = P\left(|X-50| > t \mid H_0\right) &= P\left(\left|\frac{X-50}{5}\right| > \frac{t}{5} \mid H_0\right) \\ &= P\left(|Y| > \frac{t}{5} \mid H_0\right) = 0.05. \end{aligned}$$

We can write (due to symmetry of the Normal):

$$2 \cdot P(Y > \frac{t}{5} \mid H_0) = 2 - 2\Phi\left(t/5\right) = 0.05$$

$$\Rightarrow t = 5\Phi^{-1}(0.975) = 5 \cdot 1.96 = 9.8$$

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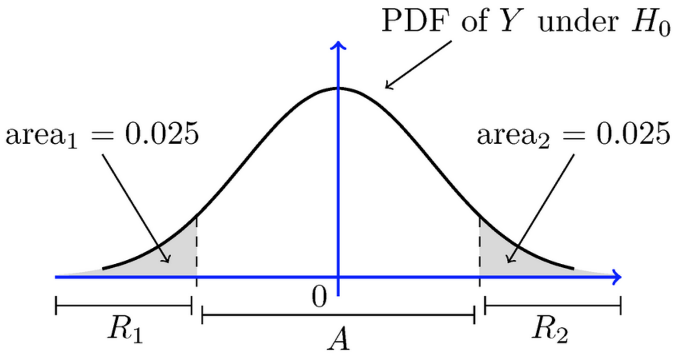
Accept and Reject region



A = Acceptance Region

$R = R_1 \cup R_2$ = Rejection Region

$\alpha = \text{P(type I error)} = \text{area}_1 + \text{area}_2 = 0.05$

Solve Exercise cont'd

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The decision criterion for the coin becomes:

If $40.2 \leq X \leq 59.8$, accept H_0,

if $X > 59.8$ or $X < 40.2$, accept H_1.

To present this result better:

- The acceptance region is $A = \{41, 42, \ldots, 59\}$.
- The rejection region is $R = \{0, \ldots, 40\} \cup \{60, \ldots, 100\}$

Test Statistic

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DEFINITION

Let $X_1, X_2, \ldots, X_n$ be a random sample of interest. A statistic is a real-valued function of the data. e.g. the sample mean,

$$W(X_1, X_2, \ldots, X_n) \quad = \quad \frac{X_1 + X_2 + \ldots + X_n}{n},$$

is a statistic. A test statistic is a statistic based on which we build our test.

In the above example the statistic was X and :

$$Y = \frac{X - n\theta_0}{\sqrt{n\theta_0(1-\theta_0)}}.$$

Test for the mean

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Consider a random sample $X_1, \ldots, X_n$ from a distribution, and make an inference for the mean

H_0: $\mu = \mu_0$,

H_1: $\mu \neq \mu_0$.

☞: Test statistic is the normalised sample mean

$$W(X_1, \ldots, X_n) \;=\; \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}}$$

and use S instead of σ if the standard deviation is unknown.

Threshold for the mean

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If we assume H_0, then the test statistic $W \sim \mathcal{N}(0,1)$.
☞ Choose a threshold, c:
If $|W| \leq c$ we accept H_0 and if $|W| > c$ we accept H_1.
Type I error:

$$\alpha = P(|W| > c \mid H_0) = 2 \cdot P(W > c \mid H_0).$$

Thus, we conclude $P(W > c \mid H_0) = \alpha/2$.
Therefore,

$$c \quad = \quad \Phi^{-1}\left(1 - \frac{\alpha}{2}\right) = z_{\frac{\alpha}{2}}$$

and we accept H_0 if $\left|\frac{\overline{X}-\mu_0}{\sigma/\sqrt{n}}\right| \leq z_{\frac{\alpha}{2}}$, and we reject it otherwise.

Relation to confidence intervals

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The condition to accept the H_0 for the mean μ_0

$$\left|\frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}}\right| \quad \leq \quad z_{\frac{\alpha}{2}}$$

can be rewritten as

$$\mu_0 \quad \in \quad \left[\overline{X} - z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}, \overline{X} + z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right]$$

☞ This is the $(1-\alpha)100\%$ confidence interval for μ_0. (slide 14/43)

Exercise Radar

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RADAR

H_0: No aircraft is present.

H_1: An aircraft is present.

The received signal is

$$X = \theta + W = \begin{cases} W, & \text{no aircraft} \\ 1 + W, & \text{if aircraft is present} \end{cases}$$

where $\theta \in \{0, 1\}$ and $W \sim \mathcal{N}\left(0, \sigma^2 = 1/9\right)$.

Solution Radar*

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Under H_0: $X \sim \mathcal{N}(0, 1/9)$, while, under H_1: $X \sim \mathcal{N}(1, 1/9)$.

RADAR Decision. Choose a threshold c

If $X \leq c$: accept H_0.

If $X > c$: accept H_1.

$$\begin{aligned} P(\textit{type I error}) &= P\left(\textit{Reject } H_0 | \ H_0\right) = P(X > c \mid H_0) \\ &= P\left(W > c\right) = 1 - \Phi(3c) \end{aligned}$$

Letting $\alpha = 0.05$ we obtain

$$c = \frac{1}{3}\Phi^{-1}(1-\alpha) = \frac{1}{3}\Phi^{-1}(1-0.05) = 0.548$$

Solution Radar cont'd*

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$$\begin{aligned} P(type\ II\ error) &= P\left(Accept\ H_0 |\ H_1\right) = P(X \leq c \mid H_1) \\ &= P\left(1 + W \leq c\right) = \Phi(3(c-1)) \end{aligned}$$

Since we found $c = 0.548$ we get

$$\beta = \Phi(-1.356) = 0.088.$$

Solution Radar cont'd II*

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If we want to choose $\alpha = 0.01$, then the threshold takes the value:

$$c = \frac{1}{3}\Phi^{-1}(1-\alpha) = \frac{1}{3}\Phi^{-1}(1-0.01) = 0.775$$

☞ Suppose we measure $X = 0.6$. Then for $\alpha = 0.05$ we get $0.6 > 0.548$ and we reject H_0 (an airplane is detected). But, for $\alpha = 0.01$ we get $0.6 < 0.775$ and we accept H_0 (no airplane).

Solution Radar cont'd III*

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If we want the probability of missing a present aircraft to be less than 5%,

$$\begin{aligned} 0.05 &= \Phi(3(c-1)) \Rightarrow \\ c &= 1 + \frac{1}{3}\Phi^{-1}(0.05) = 0.452 \end{aligned}$$

☞ Thus for type I error significance level, we get:

$$\alpha = 1 - \Phi(3c) = 1 - \Phi(3 \cdot 0.452) = 0.0875.$$

Trade-off between α and β

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☞ We cannot make both α and β small simultaneously: trade-off.

Take a look at the RADAR example:

$$\begin{aligned} \alpha &= 1 - \Phi(3c) \\ \beta &= \Phi(3(c-1)) \end{aligned}$$

Since $\Phi(y)$ is increasing with y, we see that when the c threshold increases, then α decreases, but β increases!

Trade-off cont'd

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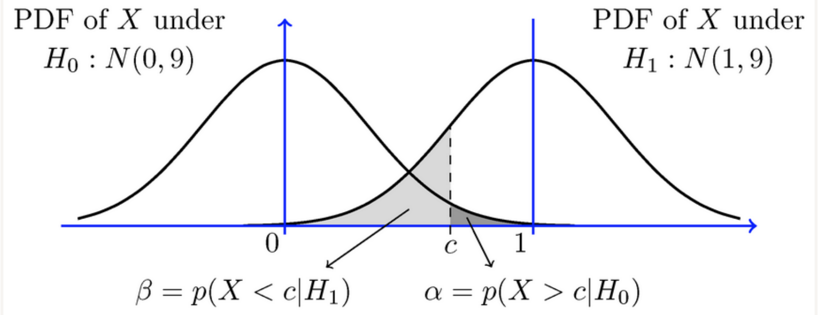

P-values

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> The P-value is the lowest significance level α that results in rejecting the null hypothesis.

Given a threshold c we reject the H_0 if the test statistic has a value larger than the threshold. The smaller the required significance level, the larger the threshold.

If the P-value is small, then the threshold is quite high, and the test statistic even higher, so it is very unlikely to have occurred under H_0, and we are more confident in rejecting the null hypothesis.

☞How do we find P-values? Let's look at an example.

Finding P-values

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The H_0 hypothesis is rejected when $W > c$ for the test statistic.
Let w be the realisation of the test statistic W of a given sample.

> The P-value is the type I error for $c = w$

Example: coin toss. Let us use $W = \frac{X-50}{5}$, so for $X = 60$, $w = 2$

$$\begin{aligned} P - value &= P(type\ I\ error\ for\ c = 2) \\ &= P\left(W > 2\right) = 1 - \Phi(2) = 0.023. \end{aligned}$$

Likelihood Ratio Tests (LRT)

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Let $X_1, \ldots, X_n$ be a random sample from a distribution with parameter θ. The likelihood function is defined for discrete and continuous variables:

$$
\begin{aligned}
L(x_1, \ldots, x_n; \theta) &= P_{X_1,\ldots,X_n}(x_1, \ldots, x_n; \theta) \\
L(x_1, \ldots, x_n; \theta) &= f_{X_1,\ldots,X_n}(x_1, \ldots, x_n; \theta)
\end{aligned}
$$

To decide between two hypotheses:
$H_0 : \theta = \theta_0$ and $H_1 : \theta = \theta_1$
we define the likelihood ratio test

$$
\lambda(x_1, \ldots, x_n) = \frac{L(x_, \ldots, x_n; \theta_0)}{L(x_, \ldots, x_n; \theta_1)},
$$

and we decide for H_0 if $\lambda \geq c$ else for H_1 if $\lambda < c$. The c is chosen based on the desired α.

Exercise LRT

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☞ Exercise: We look again at the RADAR problem. We observe the random variable:

$$X = \theta + W,$$

where $W \sim \mathcal{N}\left(0, \sigma^2 = 1/9\right)$. We need to decide between
$H_0 : \theta = \theta_0 = 0$,
$H_1 : \theta = \theta_1 = 1$.
Let a single observation $X = x$.

Design a level $\alpha = 0.05$ test to decide between H_0 and H_1.

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END