Complement - Times Series Analysis

* Complement to Part 1 - covariance of a random walk

- We remind that a random walk time series S_t is defined by the equation

  $$S_t = X_1 + X_2 + \ldots + X_t$$

  where the X_i are IID random variables, with $\mathbb{E}(X_i) = 0$, $\mathbb{V}(X_i) = \sigma^2$).

- We have stated in the course that a random walk time series is not stationary.

Proof :

For the time series S_t to be (weakly) stationary, we need $\mathbb{E}(S_t) = 0$ and $\gamma_S(t+h,t)$ independant of t.

- $\mathbb{E}(S_t) = \mathbb{E}(X_1+X_2+\ldots+X_t) = \mathbb{E}(X_1)+\mathbb{E}(X_2)+\ldots+\mathbb{E}(X_t)$ as the expectation is a linear operator.

  As $\mathbb{E}(X_1) = \mathbb{E}(X_2) = \ldots = \mathbb{E}(X_t) = 0$, we have $\mathbb{E}(S_t) = 0$. In other words, if we average on a large number of steps, the mean of a random walk process is null.

- By definition, $\gamma_S(t+h,t) = \mathbb{E}[(S_t - \mu_S)(S_{t+h} - \mu_S)]$. Here $\mu_S = \mathbb{E}(S_t) = 0$ so

  $$\gamma_S(t+h,t) = \mathbb{E}(S_t.S_{t+h}) = \mathbb{E}[(X_1 + \ldots + X_t)(X_1 + \ldots + X_t + \ldots + X_{t+h})]$$

  When we develop the product, we obtain

  $$\gamma_S(t+h,t) = \mathbb{E}\left(\sum_{i=1}^{t}\sum_{j=1}^{t+h} X_i.X_j\right) = \sum_{i=1}^{t}\sum_{j=1}^{t+h}\mathbb{E}(X_i.X_j)$$

  In this sum, the terms $\mathbb{E}(X_i.X_j)$ with $i \neq j$ are 0, because X_i and X_j are independant so $\mathbb{E}(X_i.X_j) = \mathbb{E}(X_i).\mathbb{E}(X_j) = 0$.

  This leaves us with

  $$\gamma_S(t+h,t) = \sum_{i=1}^{t}\mathbb{E}(X_i.X_i)$$

  As $\mathbb{V}(X_i)$, the variance of X_i is σ^2 and using that $\mathbb{V}(X_i) = \mathbb{E}(X_i.X_i) - \mathbb{E}(X_i).\mathbb{E}(X_i) = \mathbb{E}(X_i.X_i)$, we obtain finally that

  $$\gamma_S(t+h,t) = \sum_{i=1}^{t}\sigma^2 = t\sigma^2$$

  So $\gamma_S(t+h,t)$ is a function of t and the process is not stationary.