NDA: Time Series Analysis - Part 2

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January 2021

Outline

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1. What to do with the residuals?
2. Autoregressive models
3. Moving Average models
4. ARMA models

How to analyze a time series? (3)

Third step:

- fit the residuals

If the residuals is an IID time series, nothing else to model...

Simple method to test if a time series is IID

General idea: suppose IID random variables
What should we observe? Is it the case?

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Sample ACF criterion

Because of the Central Limit Theorem:

- suppose x_t IID with mean 0 and variance 1 (white noise)
- if n large enough, $\hat{\rho}_x(h)$ is approx. distributed as $\mathcal{N}(0, \frac{1}{\sqrt{n}})$

In practice, consider the 95% confidence interval:
How many values fall out of $\left[\frac{-1.96}{\sqrt{n}}, \frac{+1.96}{\sqrt{n}}\right]$? By how much?

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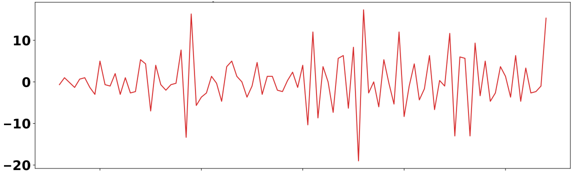

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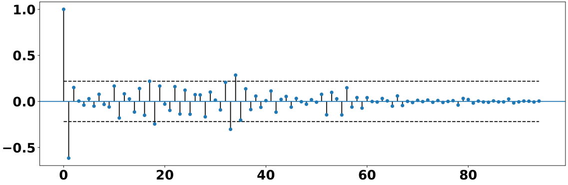

not IID noise

How to analyze a time series? (3)

Third step:

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If the residuals is an IID time series, nothing else to model...
Otherwise, we use ARMA models

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AR(1) process

What is autoregression?

auto means self $\Rightarrow$ regression from itself

The most basic AR model: 1^{st} order regression or AR(1)

$\{X_t\}$ is a series satisfying:

$$X_t = \phi X_{t-1} + W_t \ , \quad |\phi| < 1$$

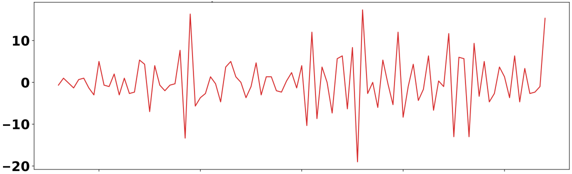
where W_t is a white noise (mean 0, variance σ^2)

Reminder: stationary

- $\mathbb{E}[X_t] = 0$
- $\gamma_X(h) = \phi^{|h|}\gamma_X(0) = \phi^{|h|}\frac{\sigma^2}{1-\phi^2}$

AR(1) process

Consider the following time series r_t:



Suppose AR(1) model for a time series r_t, how to compute ϕ?

- plot r_t as a function of r_{t-1} *(lag–1 plot)*
- linear fit, slope is an estimation of ϕ

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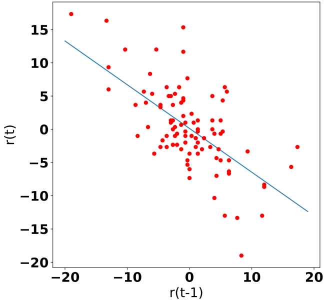
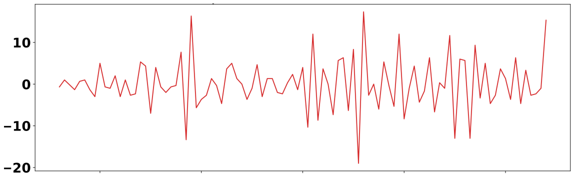


on the example $\phi \simeq -0.659$

AR(1) process

Now, we compute the residuals of the TS – AR(1) model:

$$r_t - \phi r_{t-1}$$



Is it closer to an IID noise?

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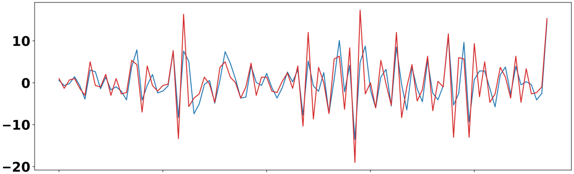

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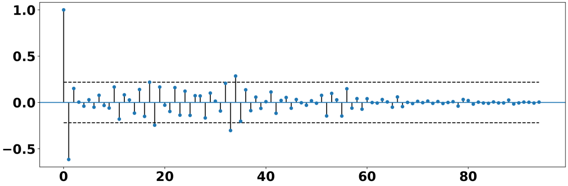

before AR(1): not IID

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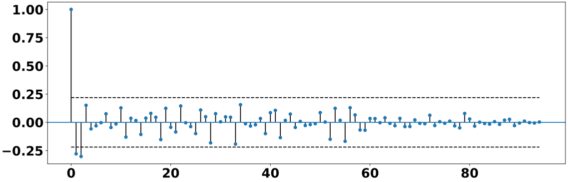

after AR(1): closer to IID $\longrightarrow$ other tests?

Generalized AR models

AR(1) model

W_t is white noise
signal = noise and (weighted) influence of the signal at $t - 1$

- $X_t = \phi X_{t-1} + W_t$
- $X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots + \phi_p X_{t-p} + W_t$

Generalized AR models

AR(p) model

W_t is white noise
signal = noise and influence of the signal at p previous steps

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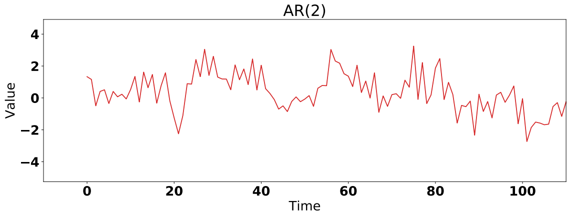
Generalized AR models

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} - \ldots - \phi_p X_{t-p} = W_t$$

Some characteristics

- Stationary process? see characteristic polynomial
  $P(x) = 1 - \phi_1 x - \phi_2 x^2 - \ldots - \phi_p x^p \rightarrow$ if no unit root AR(p) stationary
- typical shape: smooth decay, no cut-off



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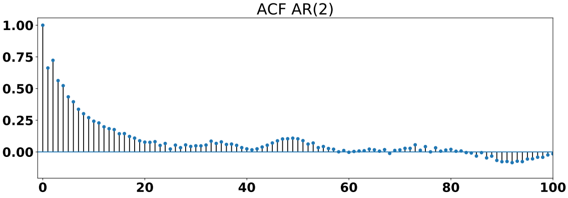

How to find AR(p) ACF coefficients?

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots + W_t$$

Yule-Walker equations

We have seen that for AR(1):

$$\gamma(h) = \phi^{|h|}\gamma(0) = \phi^{|h|}\frac{\sigma^2}{1-\phi^2}$$

What about the general case?

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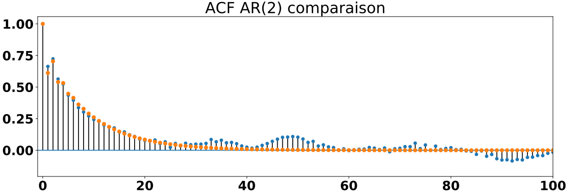
How to find AR(p) ACF coefficients?

Illustration on a practical case

$$X_t = \frac{1}{3}X_{t-1} + \frac{1}{2}X_{t-2} + W_t$$

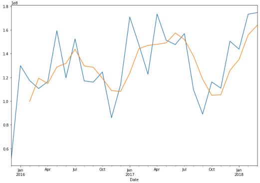


Sample ACF (blue) vs Yule-Walker coefficients (orange)

Outline

MA(q) processes

Moving average to smooth a signal:



Original signal $f(x)$ smoothed over a window size w, typically:

ex1: $g_1(x) = \frac{f(x)+f(x-1)+f(x+1)}{3}$ (no weight, two-sided, $w = 3$)

ex2: $g_2(x) = \frac{f(x)+f(x-1)+f(x-2)}{3}$ (no weight, one-sided, $w = 3$)

ex3: $g_3(x) = \frac{f(x)+a.f(x-1)+b.f(x-2)+c.f(x-3)}{(1+a+b+c)}$ (weighted, one-sided, $w = 4$)

MA(q) processes

MA(1) model

W_t is white noise
signal = weighted average of noise at t and of noise at $t-1$

- $X_t = \beta_0 W_t + \beta_1 W_{t-1}$
- $X_t = \beta_0 W_t + \beta_1 W_{t-1} + \ldots + \beta_q W_{t-q}$

MA(q) processes

MA(q) model

W_t is white noise
signal = weighted average of noise at t and q previous steps

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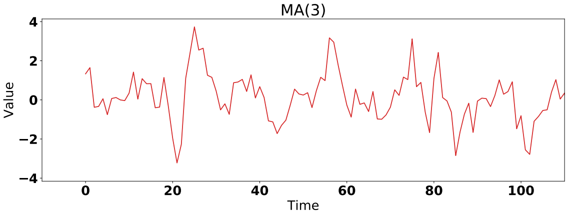
Some characteristics

- Stationary process
  mean and autocovariance at lag h do not depend on time
  Ex: prove if $h \leq q$, $Cov(X_t, X_{t+h}) = \sigma^2 \sum_{i=0}^{q-h} \beta_i \beta_{i+h}$
- ACF cuts off at lag q



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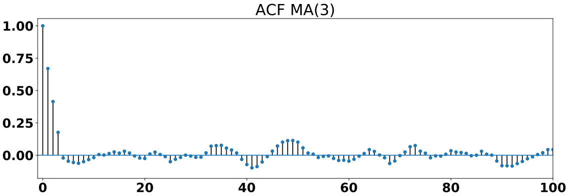

Connection between MA and AR

From MA(1) to AR

Considering the MA(1) process:

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\begin{aligned}
X_t &= W_t + \beta W_{t-1} \\
\Rightarrow W_t &= X_t - \beta W_{t-1} \\
\Rightarrow W_t &= X_t - \beta(X_{t-1} - \beta W_{t-2}) \\
\Rightarrow W_t &= X_t - \beta X_{t-1} + \beta^2 X_{t-2} - \beta^3 X_{t-3} + \dots \\
\Rightarrow X_t &= W_t + \beta X_{t-1} - \beta^2 X_{t-2} + \beta^3 X_{t-3} - \dots
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In other words, MA(1) is an AR(∞) process

More generally, MA(q) can (often) be seen as AR(∞) process

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ARMA models

ARMA(p,q) model

ARMA(p,q) model is a combination of AR(p) and MA(q) model:

$$X_t = \phi_1 X_{t-1} + \ldots + \phi_p X_{t-p} + W_t + \beta_1 W_{t-1} + \ldots + \beta_q W_{t-q}$$

As for AR(p) and MA(q) parameters can be found from the ACF

In practice

- fit the residuals with several (low) values of p and q
- select what is the *best* model
- $\Rightarrow$ complete model:

  trend + seasonality + ARMA(p,q) residuals

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A few questions that we have left aside ...

- How to choose p and q of an ARMA process?
  $\rightarrow$ *useful to define partial ACF*
- How to find the coefficients of an ARMA process?
  $\rightarrow$ *transform it in AR(∞) (or MA(∞)) process*
- How do we select the best model?
  $\rightarrow$ *complexity criteria and the overfitting problem (AIC, BIC)*
- How to deal more systematically with the trend?
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Studying time series in python

Among several options, `pandas` library

A few useful functions:

- Load data as dataframe:
  `read_csv` from `pandas` library
- Moving average:
  `rolling` from `pandas` library
- Fitting:
  `curve_fit` in `scipy.optimize` library
- Autocorrelation function:
  `plot_ACF` in `statsmodels` library
- ARMA model fit:
  `ARMA.fit` in `statsmodels` library