



ESCUELA TECNOLÓGICA INSTITUTO TÉCNICO CENTRAL (ETITC)

Facultad de sistemas

Ejercicios \LaTeX : Ecuaciones Diferenciales II

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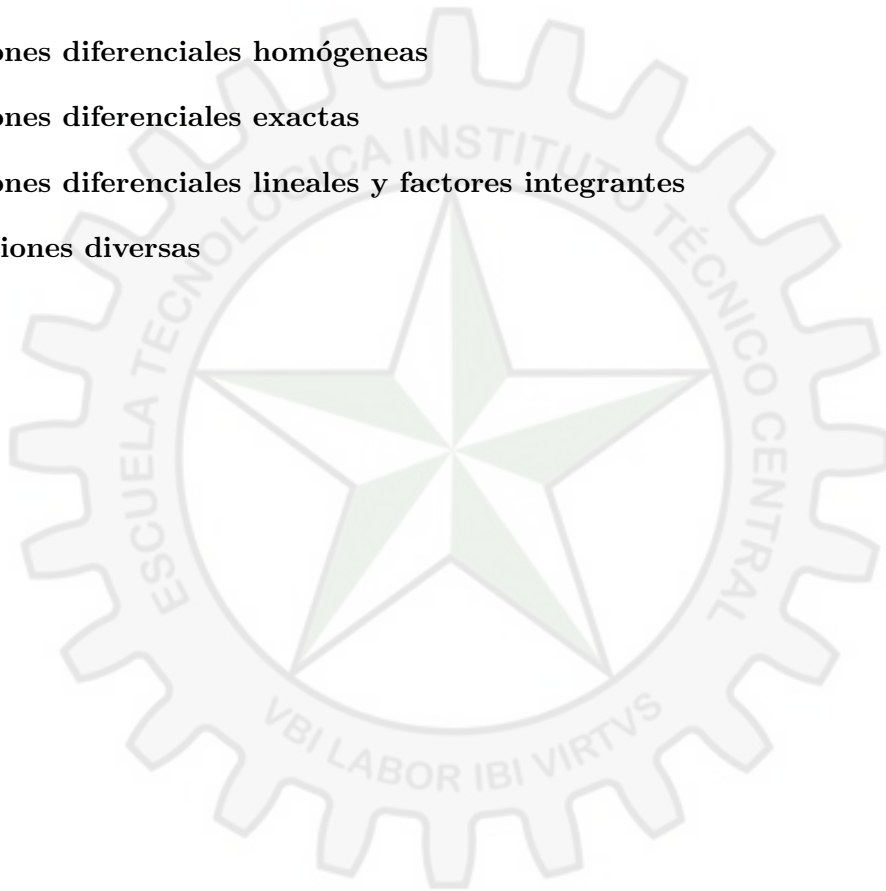
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Capítulo 1

Introducción a las ecuaciones diferenciales

Ejercicio 1.0.1 (Ejercicio 1)

$$y' = 3x^2; \quad \boxed{y(x) = x^3 + 7}$$

Sea $y(x) = x^3 + 7$

$$y'(x) = 3x^2$$

Sustituyendo en la ecuación diferencial

$$\boxed{3x^2 = 3x^2}$$

Ejercicio 1.0.2 (Ejercicio 2)

$$y' + 2xy^2 = 0; \quad \boxed{y(x) = \frac{1}{1+x^2}}$$

Forma 1.

Sea $y(x) = \frac{1}{1+x^2}$

$$y'(x) = \frac{(1)'(1+x^2) - (1)(1+x^2)'}{(1+x^2)^2} \iff \boxed{y'(x) = -\frac{2x}{(1+x^2)^2}}$$

Sustituyendo en la ecuación diferencial

$$-\frac{2x}{(1+x^2)^2} + 2x \left(\frac{1}{(1+x^2)} \right)^2 = 0 \iff -\frac{2x}{(1+x^2)^2} + 2x \left(\frac{1}{(1+x^2)^2} \right) = 0$$

$$-\frac{\cancel{2x}}{\cancel{(1+x^2)^2}} + \frac{\cancel{2x}}{\cancel{(1+x^2)^2}} = 0 \iff \boxed{0 = 0}$$

Forma 2.

$$\text{Sea } y(x) = \frac{1}{1+x^2} \iff \boxed{y(x) = (1+x^2)^{-1}}$$

$$y'(x) = -1(1+x^2)^{-1-1} \times 2x \iff y'(x) = -(1+x^2)^{-2} \times 2x$$

$$y'(x) = -\frac{2x}{(1+x^2)^2}$$

Sustituyendo en la ecuación diferencial

$$-\frac{2x}{(1+x^2)^2} + 2x \left(\frac{1}{(1+x^2)} \right)^2 = 0 \iff -\frac{2x}{(1+x^2)^2} + 2x \left(\frac{1}{(1+x^2)^2} \right) = 0$$

$$-\frac{2x}{\cancel{(1+x^2)^2}} + \frac{2x}{\cancel{(1+x^2)^2}} = 0 \iff \boxed{0 = 0}$$

Ejercicio 1.0.3 (Ejercicio 3)

$$xy' = y + x^2 + y^2 \quad \boxed{y(x) = x \tan x}$$

Sea $y(x) = x \tan x$

$$y'(x) = (x)' \tan x + x(\tan x)' \iff \boxed{y'(x) = \tan x + x \sec^2 x}$$

Sustituyendo en la ecuación diferencial

$$x(\tan x + x \sec^2 x) = x \tan x + x^2 + (x \tan x)^2$$

$$x \tan x + x^2 \sec^2 x = x \tan x + x^2 + x^2 \tan^2 x$$

$$x \tan x + x^2 \sec^2 x = x \tan x + x^2 \underbrace{(1 + \tan^2 x)}_{\sec^2 x}$$

$$\boxed{x \tan x + x^2 \sec^2 x = x \tan x + x^2 \sec^2 x}$$

$$\cos^2 x + \sin^2 x = 1$$

$$\frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \iff 1 + \tan^2 x = \sec^2 x$$

Ejercicio 1.0.4 (Ejercicio 4)

$$y'' = 9y; \quad \boxed{y(x) = C_1 e^{3x} + C_2 e^{-3x}}$$

Sea $y(x) = C_1 e^{3x} + C_2 e^{-3x}$

$$y'(x) = 3e^{3x} - 3e^{-3x}$$

$$y''(x) = 9e^{3x} + 9e^{-3x}$$

Sustituyendo en la ecuación diferencial

$$9e^{3x} + 9e^{-3x} = 9(C_1 e^{3x} + C_2 e^{-3x}) \iff \boxed{9e^{3x} + 9e^{-3x} = 9e^{3x} + 9e^{-3x}}$$

Ejercicio 1.0.5 (Ejercicio 5) Muestre que la función $u(x, t) = (x - at)^2 + (x + at)^3$ satisface la ecuación de calor

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial u}{\partial t} = 2(x - at) \times (-a) + 3(x + at)^2 \times a$$

$$\frac{\partial u}{\partial t} = -2a(x - at) + 3a(x + at)^2$$

$$\frac{\partial^2 u}{\partial t^2} = -2a \times (-a) + 3a \times 2(x + at) \times a$$

$$\frac{\partial^2 u}{\partial t^2} = 2a^2 + 6a^2(x + at)$$

$$\frac{\partial u}{\partial x} = 2(x - at) + 3(x + at)^2$$

$$\frac{\partial^2 u}{\partial x^2} = 2 + 6(x + at)$$

Sustituyendo en la ecuación diferencial

$$2a^2 + 6a^2(x + at) = a^2(2 + 6(x + at))$$

$$2a^2 + 6a^2(x + at) = 2a^2 + 6a^2(x + at)$$

Verifique que la función dada satisface la ecuación indicada

Ejercicio 1.0.6 (Ejercicio 6)

$$\frac{\partial z}{\partial x} + z \frac{\partial z}{\partial y} - z = x; \quad z(x, y) = \frac{1}{2}x^2 + y - 3$$

$$\frac{\partial z}{\partial x} = \frac{1}{2} \cancel{2} x \iff \boxed{\frac{\partial z}{\partial x} = x}$$

$$\boxed{\frac{\partial z}{\partial y} = 1}$$

Sustituyendo en la ecuación diferencial

$$x + \left(\frac{1}{2}x^2 + y - 3\right) \times 1 - \left(\frac{1}{2}x^2 + y - 3\right) = x$$

$$x + \left(\cancel{\frac{1}{2}x^2 + y - 3}\right) - \left(\cancel{\frac{1}{2}x^2 + y - 3}\right) = x \iff \boxed{x = x}$$

Ejercicio 1.0.7 (Ejercicio 7)

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 - 2z = 0; \quad z(x, y) = \frac{(x - y + 2)^2}{4}$$

$$\frac{\partial z}{\partial x} = \boxed{\frac{x - y + 2}{2}}$$

$$\frac{\partial z}{\partial y} = \boxed{\frac{-x + y - 2}{2}}$$

Sustituyendo en la ecuación diferencial

$$\begin{aligned} \left(\frac{x - y + 2}{2}\right)^2 + \left(\frac{-x + y - 2}{2}\right)^2 - 2\left(\frac{(x - y + 2)^2}{4}\right) &= 0 \\ \frac{(x - y + 2)^2}{4} + \frac{(-x + y - 2)^2}{4} - 2\left(\frac{(x - y + 2)^2}{4}\right) &= 0 \end{aligned}$$

Ejercicio 1.0.8 (Ejercicio 8) Muestre que la función $z(x, y) = \sin(x^2 + y^2)$ satisface la ecuación

$$y \frac{\partial^2 z}{\partial x^2} - x \frac{\partial^2 z}{\partial y \partial x} - \frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial x} = \cos(x^2 + y^2) \times 2x \iff \frac{\partial z}{\partial x} = 2x \cos(x^2 + y^2)$$

$$\frac{\partial^2 z}{\partial x^2} = 2 \cos(x^2 + y^2) + 2x [-\sin(x^2 + y^2) \times 2x]$$

$$\boxed{\frac{\partial^2 z}{\partial x^2} = 2 \cos(x^2 + y^2) - 4x^2 \sin(x^2 + y^2)}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right)$$

$$\frac{\partial z}{\partial x} = 2x \cos(x^2 + y^2)$$

$$\frac{\partial}{\partial y} [2x \cos(x^2 + y^2)] = 2x (-\sin(x^2 + y^2)) \times 2y$$

$$\boxed{\frac{\partial^2 z}{\partial y \partial x} = -4xy \sin(x^2 + y^2)}$$

$$\frac{\partial z}{\partial y} = \cos(x^2 + y^2) \times 2y \iff \boxed{\frac{\partial z}{\partial y} = 2y \cos(x^2 + y^2)}$$

Sustituyendo en la ecuación diferencial parcial

$$y [2 \cos(x^2 + y^2) - 4x^2 \sin(x^2 + y^2)] - x [-4xy \sin(x^2 + y^2)] - 2y \cos(x^2 + y^2) = 0$$

$$\cancel{2y \cos(x^2 + y^2)} - \cancel{4x^2 y \sin(x^2 + y^2)} + \cancel{4x^2 y \sin(x^2 + y^2)} - \cancel{2y \cos(x^2 + y^2)} = 0$$

$$\boxed{0 = 0}$$

Capítulo 2

Separación de variables

Ejercicio 2.0.1 (Ejercicio 9)

$$x y' = y(3 - x)$$

$$x \frac{dy}{dx} = y(3 - x) \iff \frac{dy}{y} = \left(\frac{3 - x}{x} \right) dx$$

$$\frac{dy}{y} = \left(\frac{3}{x} - \frac{x}{x} \right) dx \iff \frac{dy}{y} = \left(\frac{3}{x} - 1 \right) dx$$

$$\int \frac{dy}{y} = \int \left(\frac{3}{x} - 1 \right) dx$$

$$\int \frac{dy}{y} = 3 \int \frac{dx}{x} - \int dx \iff \ln y = 3 \ln x - x + C$$

$$\ln y = \ln x^3 - x + C \iff e^{\ln y} = e^{(\ln x^3 - x + C)}$$

$$y = e^{\ln x^3} \cdot e^{-x} \cdot \underbrace{e^C}_{\hat{C}} \iff \boxed{y(x) = \hat{C} x^3 e^{-x}}$$

Prueba

Sea $\boxed{y(x) = \hat{C} x^3 e^{-x}}$

$$y'(x) = 3\hat{C}x^2 e^{-x} + \hat{C}x^3 \cdot e^{-x} \times (-1) \iff \boxed{y'(x) = 3\hat{C}x^2 e^{-x} - \hat{C}x^3 e^{-x}}$$

Sustituyendo en la ecuación diferencial

$$x y' = y(3 - x)$$

$$x(3\hat{C}x^2 e^{-x} - \hat{C}x^3 e^{-x}) = \hat{C}x^3 e^{-x}(3 - x)$$

$$\boxed{3\hat{C}x^3 e^{-x} - \hat{C}x^4 e^{-x} = 3\hat{C}x^3 e^{-x} - \hat{C}x^4 e^{-x}}$$

Ejercicio 2.0.2 (Ejercicio 10)

$$\frac{dy}{dx} = \frac{x}{y} - \frac{x}{1+y}; \quad y(0) = 1$$

$$\frac{dy}{dx} = \frac{x(1+y) - xy}{y(1+y)} \iff \frac{dy}{dx} = \frac{x + xy - xy}{y(1+y)}$$

$$\frac{dy}{dx} = \frac{x + \cancel{xy} - \cancel{xy}}{y(1+y)} \iff \frac{dy}{dx} = \frac{x}{y(1+y)}$$

$$(y(1+y)) dy = x dx \iff (y + y^2) dy = x dx$$

$$\int y dy + \int y^2 dy = \int x dx \iff \boxed{\frac{y^2}{2} + \frac{y^3}{3} = \frac{x^2}{2} + C}$$

Problema de condición inicial: $x = 0, y = 1$

$$\frac{(1)^2}{2} + \frac{(1)^3}{3} = \frac{(0)^2}{2} + C \iff \frac{1}{2} + \frac{1}{3} = C \iff \boxed{C = \frac{5}{6}}$$

Exhibir la solución particular

$$\frac{y^2}{2} + \frac{y^3}{3} = \frac{x^2}{2} + \frac{5}{6} \iff \frac{3y^2 + 2y^3}{6} = \frac{3x^2 + 5}{6} \iff \boxed{2y^3 + 3y^2 = 3x^2 + 5}$$

Ejercicio 2.0.3 (Ejercicio 11)

$$\sin(2x) + \cos(3y) = 0; \quad y\left(\frac{\pi}{2}\right) = \frac{\pi}{3}$$

$$\sin(2x) + \cos(3y) = 0 \iff \cos(3y) = -\sin(2x)$$

$$\int \cos(3y) = \int -\sin(2x)$$

$$\boxed{\frac{\sin(3y)}{3} = \frac{\cos(2x)}{2} + C}$$

Remplazo los valores de x y y

$$\frac{\sin(\pi)}{3} = \frac{\cos(\pi)}{2} + C$$

$$\frac{\sin(\pi)}{3} - \frac{\cos(\pi)}{2} = C \iff \boxed{\frac{1}{2} = C}$$

$$\boxed{\frac{\sin(3y)}{3} = \frac{\cos(2x)}{2} + \frac{1}{2}}$$

Ejercicio 2.0.4 (Ejercicio 12)

$$\sqrt{1-y^2} dx - \sqrt{1-x^2} dy = 0; \quad y(0) = \frac{\sqrt{3}}{2}$$

$$\sqrt{1-y^2} dx = \sqrt{1-x^2} dy$$

$$\sqrt{1-x^2} dy = \sqrt{1-y^2} dx$$

$$\frac{dy}{\sqrt{1-y^2}} = \frac{dx}{\sqrt{1-x^2}}$$

$$\int \frac{dy}{\sqrt{1-y^2}} = \int \frac{dx}{\sqrt{1-x^2}}$$

$$\arcsin(y) = \arcsin(x) + C$$

Remplazando valores de x y y

$$\arcsin\left(\frac{\sqrt{3}}{2}\right) = \arcsin(0) + C$$

$$60 = +C$$

$$\sin^{-1} y = \sin^{-1} x + C \quad \Longleftrightarrow \quad \sin(\sin^{-1} y) = \sin(\sin^{-1} x + C)$$

$$\text{Identidad: } \sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$y = \sin(\sin^{-1} x) \cos C + \cos(\sin^{-1} x) \sin C$$

$$x = \sin \theta \implies \theta = \sin^{-1} x$$

$$y = x \cos C + \cos \theta \sin C$$

$$\boxed{y(x) = \cos C + \sqrt{1-x^2} \sin C}$$

Remplazando C

$$y(x) = \cos(60) + \sqrt{1-x^2} \sin(60)$$

$$\boxed{y(x) = \frac{x}{2} + \frac{\sqrt{3}}{2} \sqrt{1-x^2}}$$

Ejercicio 2.0.5 (Ejercicio 13)

Sea

$$V = \frac{2}{2t^2 + 3t + 1} \quad \Longleftrightarrow \quad \frac{dx}{dt} = \frac{2}{2t^2 + 3t + 1}$$

$$dx = \left(\frac{2}{2t^2 + 3t + 1} \right) dt \iff \int dx = \int_{t=0}^{t=1} \left(\frac{2}{2t^2 + 3t + 1} \right) dt$$

Aparte:

$$2t^2 + 3t + 1 = \frac{\textcolor{red}{2}(2t^2 + 3t + 1)}{\textcolor{red}{2}}$$

$$2t^2 + 3t + 1 = \frac{4t^2 + 2(3t) + 2}{2}$$

$$2t^2 + 3t + 1 = \frac{4t^2 + 3(2t) + 2}{2}$$

$$2t^2 + 3t + 1 = \frac{(2t + 2)(2t + 1)}{2} \iff 2t^2 + 3t + 1 = \frac{2(t + 1)(2t + 1)}{2}$$

$$2t^2 + 3t + 1 = \boxed{(t + 1)(2t + 1)}$$

$$\frac{2}{2t^2 + 3t + 1} = \boxed{\frac{2}{(t + 1)(2t + 1)} = \frac{A}{t + 1} + \frac{B}{2t + 1}}$$

$$A = \frac{2}{2(\textcolor{red}{-1}) + 1} = \boxed{-2}$$

$$B = \frac{2}{\textcolor{red}{-\frac{1}{2}} + 1} = \frac{2}{\frac{1}{2}} = \boxed{4}$$

$$x = \int_0^1 \left(\frac{-2}{t + 1} \right) dt + \int_0^1 \left(\frac{4}{2t + 1} \right) dt \iff x = -2 \underbrace{\int_0^1 \left(\frac{1}{t + 1} \right) dt}_{I_1} + 4 \underbrace{\int_0^1 \left(\frac{1}{2t + 1} \right) dt}_{I_2}$$

I_1

$$\int_0^1 \left(\frac{1}{t + 1} \right) dt$$

$$\boxed{u = t + 1 \implies \frac{du}{dt} = 1 \iff dt = du}$$

$$\int_0^1 \left(\frac{1}{u} \right) du$$

$$\ln |u| \Big|_0^1 \iff \ln |t + 1| \Big|_0^1$$

I_2

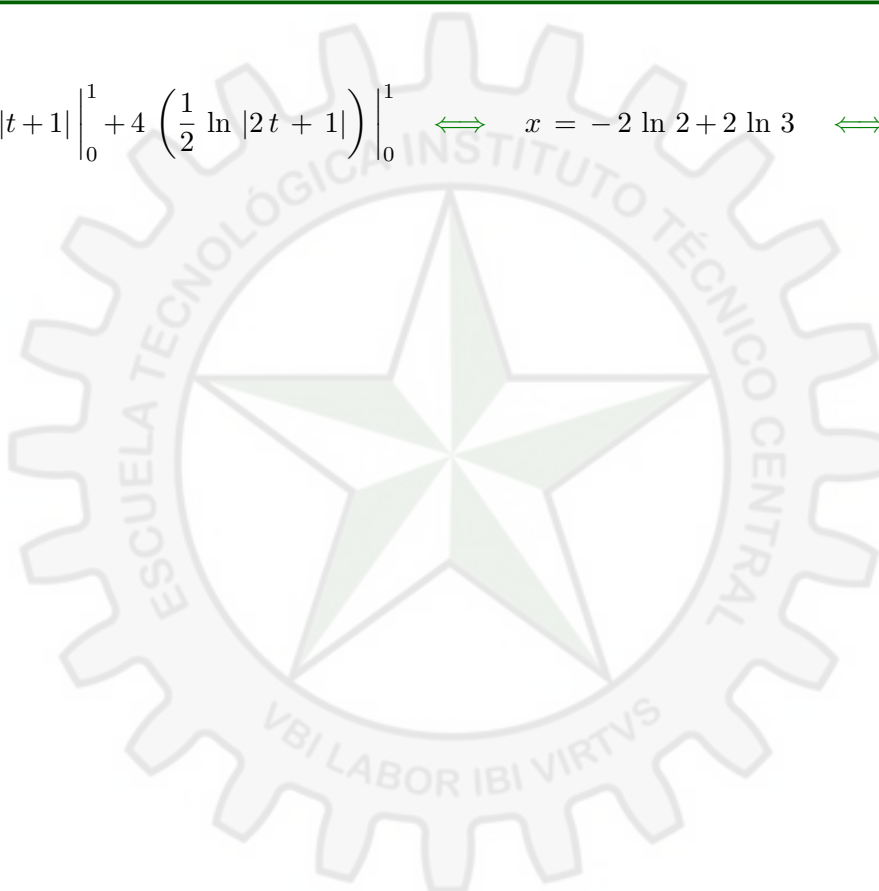
$$\int_0^1 \left(\frac{1}{2t+1} \right) dt$$

$$u = 2t + 1 \implies \frac{du}{dt} = 2 \iff dt = \frac{1}{2} du$$

$$\frac{1}{2} \int_0^1 \left(\frac{1}{u} \right) du$$

$$\frac{1}{2} \ln |u| \Big|_0^1 \iff \frac{1}{2} \ln |t+1| \Big|_0^1$$

$$x = -2 \ln |t+1| \Big|_0^1 + 4 \left(\frac{1}{2} \ln |2t+1| \right) \Big|_0^1 \iff x = -2 \ln 2 + 2 \ln 3 \iff x = 2 \ln \left(\frac{3}{2} \right)$$



Capítulo 3

Ecuaciones diferenciales homogéneas

Ejercicio 3.0.1 (Ejercicio 14)

$$\begin{aligned}
 y' &= \frac{y^2 + x y}{x^2}; \quad y(1) = 1 \\
 \frac{dy}{dx} &= \frac{y^2}{x^2} + \frac{xy}{x^2} \iff \frac{dy}{dx} = \frac{y^2}{x^2} + \frac{xy}{x^2} \\
 \frac{dy}{dx} &= \frac{y^2}{x^2} + \frac{y}{x} \iff \frac{dy}{dx} = \left(\frac{y}{x}\right)^2 + \frac{y}{x} \\
 x \frac{dU}{dx} + U &= U^2 + U \iff x \frac{dU}{dx} + \cancel{U} = U^2 + \cancel{U} \\
 x \frac{dU}{dx} &= U^2 \iff \frac{dU}{U^2} = \frac{1}{x} dx \iff \int U^{-2} dU = \int \left(\frac{1}{x}\right) dx \\
 \frac{U^{-2+1}}{-2+1} &= \ln x + C \iff -U^{-1} = \ln x + C \\
 -\frac{1}{U} &= \ln x + C \iff -\frac{1}{\frac{y}{x}} = \ln x + C \iff -\frac{x}{y} = \ln x + C \\
 -\frac{x}{\ln x + C} &= y \iff \boxed{y(x) = -\frac{x}{\ln x + C}}
 \end{aligned}$$

Hallar la solución particular: $x = 1, y = 1$

$$y(1) = -\frac{1}{\ln 1 + C} \iff -\frac{1}{C} = 1 \iff C = -1$$

Exhibir la solución particular

$$\boxed{y(x) = -\frac{x}{\ln x - 1}}$$

Ejercicio 3.0.2 (Ejercicio 15)

$$x \frac{dy}{dx} = x e^{y/x} + y; \quad y(1) = 0$$

$$x \frac{dy}{dx} = x e^{y/x} + y \iff \frac{dy}{dx} = \frac{(x e^{y/x} + y)}{x} \iff \frac{dy}{dx} = \frac{x e^{y/x}}{x} + \frac{y}{x}$$

$$\frac{dy}{dx} = e^{y/x} + \frac{y}{x} \iff x \frac{dU}{dx} + \cancel{y} = e^U + \cancel{y} \iff x \frac{dU}{dx} = e^U$$

$$\frac{dU}{e^U} = \frac{1}{x} dx \iff e^{-U} dU = \frac{1}{x} dx \iff \int e^{-U} dU = \int \left(\frac{1}{x}\right) dx$$

$$-e^{-U} = \ln x + C \iff -C = e^{-U} + \ln x \iff \boxed{-C = e^{-y/x} + \ln x}$$

Hallar la solución particular $x = 1; y = 0$

$$-C = e^0 + \ln 1 \iff -C = 1 \iff C = -1$$

Exhibir la solución particular

$$-(-1) = e^{-y/x} + \ln x \iff \boxed{e^{-y/x} + \ln x = 1}$$

Ejercicio 3.0.3 (Ejercicio 16)

$$x^2 y' - 3xy - 2y^2 = 0$$

$$\frac{x^2}{x^2} y' - 3 \frac{xy}{x^2} - 2 \frac{y^2}{x^2} = 0 \iff y' - 3 \frac{y}{x} - 2 \left(\frac{y}{x}\right)^2 = 0 \iff \frac{dy}{dx} - 3 \frac{y}{x} - 2 \left(\frac{y}{x}\right)^2 = 0$$

$$x \frac{dU}{dx} + U - 3U - 2U^2 = 0 \iff x \frac{dU}{dx} - 2U - 2U^2 = 0$$

$$x \frac{dU}{dx} = 2U^2 + 2U \iff x \frac{dU}{dx} = 2U(U + 1) \iff \frac{dU}{U(U + 1)} = \frac{2}{x} dx$$

$$\underbrace{\int \left(\frac{dU}{U(U + 1)} \right)}_{I_1} = \int \frac{2}{x} dx$$

I_1

$$\int \left(\frac{dU}{U(U+1)} \right)$$

$$\frac{1}{U(U+1)} = \frac{A}{U} + \frac{B}{U+1}$$

$$A = \frac{1}{\textcolor{red}{0} + 1} = \boxed{1}$$

$$B = \frac{1}{\textcolor{red}{-1}} = \boxed{-1}$$

$$\int \left(\frac{dU}{U(U+1)} \right) = \int \frac{1}{U} dU - \int \frac{1}{U+1} dU$$

$$\int \left(\frac{dU}{U(U+1)} \right) = \ln U - \ln |U+1| = \boxed{\ln \left(\frac{U}{U+1} \right)}$$

$$\ln \left(\frac{U}{U+1} \right) = 2 \ln x + C \iff \ln \left(\frac{\frac{y}{x}}{\frac{y}{x} + \frac{1}{1}} \right) = \ln x^2 + C$$

$$\ln \left(\frac{\frac{y}{x}}{\frac{y}{x} + \frac{1}{1}} \right) = \ln x^2 + C \iff \ln \left(\frac{xy}{x(x+y)} \right) = \ln x^2 + C$$

$$\ln \left(\frac{y}{x+y} \right) = \ln x^2 + C$$

Tomando exponencial a ambos lados

$$\frac{y}{x+y} = e^{(\ln x^2 + C)} \iff \frac{y}{x+y} = e^{\ln x^2} \cdot \underbrace{e^C}_{\hat{C}} \iff \frac{y}{x+y} = \hat{C} x^2$$

$$\boxed{y = \hat{C} x^2 (x+y)}$$

Ejercicio 3.0.4 (Ejercicio 17)

$$-y dx + (x + \sqrt{xy}) dy = 0$$

$$(x + \sqrt{xy}) dy = y dx \iff \frac{dy}{dx} = \frac{y}{x + \sqrt{xy}}$$

$$x \frac{dU}{dx} + U = \frac{Ux}{x + \sqrt{xUx}} \iff x \frac{dU}{dx} + U = \frac{Ux}{x + \sqrt{x^2 U}} \iff x \frac{dU}{dx} + U = \frac{Ux}{x + x\sqrt{U}}$$

$$x \frac{dU}{dx} + U = \frac{U\cancel{x}}{\cancel{x}(1 + \sqrt{U})} \iff x \frac{dU}{dx} = \frac{U}{(1 + \sqrt{U})} - U \iff x \frac{dU}{dx} = \frac{U - U(1 + \sqrt{U})}{(1 + \sqrt{U})}$$

$$\begin{aligned}
x \frac{dU}{dx} &= \frac{\cancel{x} - \cancel{x} - U \sqrt{U}}{(1 + \sqrt{U})} \iff x \frac{dU}{dx} = -\frac{U \sqrt{U}}{(1 + \sqrt{U})} \\
-\frac{(1 + \sqrt{U})}{U \sqrt{U}} dU &= \frac{dx}{x} \iff -\left(\frac{dU}{U U^{1/2}} + \frac{\cancel{\sqrt{U}}}{U \cancel{\sqrt{U}}} dU\right) = \frac{dx}{x} \iff -\frac{dU}{U^{3/2}} - \frac{dU}{U} = \frac{dx}{x} \\
-U^{-3/2} dU - \frac{dU}{U} &= \frac{dx}{x} \iff -\int U^{-3/2} dU - \int \frac{dU}{U} = \int \frac{dx}{x} \\
\cancel{\int} \frac{U^{-1/2}}{\cancel{1/2}} - \ln U &= \ln x + C \iff \frac{2}{\sqrt{U}} - \ln U = \ln x + C \\
\frac{2}{\sqrt{y/x}} - \ln\left(\frac{y}{x}\right) &= \ln x + C \iff \frac{2}{\sqrt{y/x}} - (\ln y - \ln x) = \ln x + C \\
\frac{2}{\sqrt{y/x}} - \ln y + \cancel{\ln x} &= \cancel{\ln x} + C \iff \frac{2}{\sqrt{y/x}} - \ln y = C \iff \frac{2}{\sqrt{y/x}} = C + \ln y \\
\left(\frac{2}{\sqrt{y/x}}\right)^2 &= (C + \ln y)^2 \iff \frac{4}{y/x} = (C + \ln y)^2 \\
\frac{4x}{y} &= (C + \ln y)^2 \iff \boxed{4x = y(C + \ln y)^2}
\end{aligned}$$

Ejercicio 3.0.5 (Ejercicio 18)

$$x y' = y + x^2 \sec(y/x); \quad y(1) = \pi/2$$

$$\begin{aligned}
x y' &= y + x^2 \sec(y/x) \iff y' = \frac{y + x^2 \sec(y/x)}{x} \iff y' = \frac{y}{x} + \frac{x^{\cancel{2}} \sec(y/x)}{\cancel{x}} \\
y' &= \frac{y}{x} + x \sec(y/x) \iff \frac{dy}{dx} = \frac{y}{x} + x \sec(y/x) \\
x \frac{dU}{dx} + \cancel{y} &= \cancel{y} + x \sec U \iff x \frac{dU}{dx} = x \sec U \\
\frac{1}{\sec U} dU &= \frac{x}{x} dx \iff \cos U dU = dx
\end{aligned}$$

$$\int \cos U dU = \int dx \iff \sin U = x + C \iff \boxed{\sin(y/x) = x + C}$$

Hallar la solución particular $x = 1; y = \pi/2$

$$\sin(\pi/2) = \color{red}{1} + C \iff 1 - 1 = C \iff \boxed{C = 0}$$

Exhibir la solución particular

$$\boxed{\sin(y/x) = x}$$

Ejercicio 3.0.6 (Ejercicio 19)

$$y' = \tan(x + y) - 1 \quad (v = x + y)$$

Sustitución

$$\boxed{v = x + y}$$

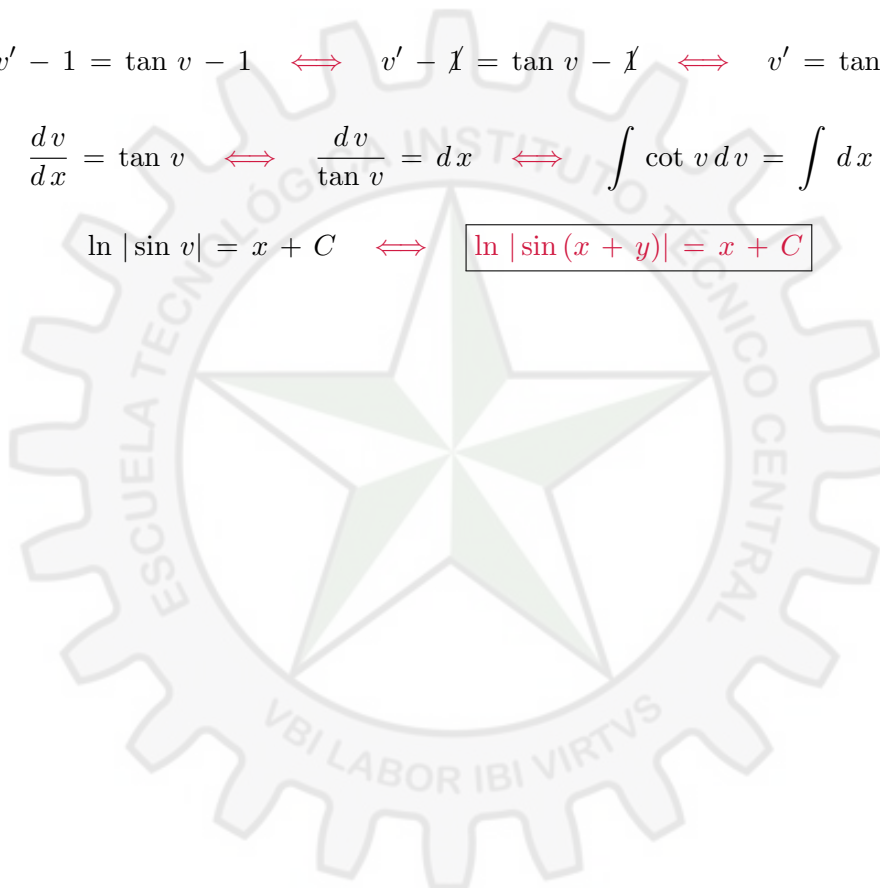
$$\frac{dv}{dx} = \frac{dx}{dx} + \frac{dy}{dx} \iff \frac{dv}{dx} = 1 + \frac{dy}{dx} \iff v' = y' + 1$$

$$\boxed{y' = v' - 1}$$

$$v' - 1 = \tan v - 1 \iff v' - 1 = \tan v - 1 \iff v' = \tan v$$

$$\frac{dv}{dx} = \tan v \iff \frac{dv}{\tan v} = dx \iff \int \cot v \, dv = \int dx$$

$$\ln |\sin v| = x + C \iff \boxed{\ln |\sin(x + y)| = x + C}$$



Capítulo 4

Ecuaciones diferenciales exactas

Ejercicio 4.0.1 (Ejercicio 20)

$$\left(\sin y + y \sin x + \frac{1}{x}\right) dx + \left(x \cos y - \cos x + \frac{1}{y}\right) dy = 0$$

Verificar la exactitud.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\underbrace{\sin y + y \sin x + \frac{1}{x}}_{M(x,y)} dx + \underbrace{x \cos y - \cos x + \frac{1}{y}}_{N(x,y)} dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y} \left(\sin y + y \sin x + \frac{1}{x} \right) = \boxed{\cos y + \sin x}$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x} \left(x \cos y - \cos x + \frac{1}{y} \right) = \cos y - (-\sin x) = \boxed{\cos y + \sin x}$$

Hallar la solución

$$\boxed{M(x, y) = \frac{\partial U}{\partial x}} \quad \wedge \quad \boxed{N(x, y) = \frac{\partial U}{\partial y}}$$

$$\boxed{\frac{\partial U}{\partial x} = \sin y + y \sin x + \frac{1}{x}} \quad \wedge \quad \boxed{\frac{\partial U}{\partial y} = x \cos y - \cos x + \frac{1}{y}}$$

$$U(x, y) = \int \left(x \cos y - \cos x + \frac{1}{y} \right) dy$$

$$U(x, y) = x \int \cos y dy - \cos x \int dy + \int \left(\frac{1}{y} \right) dy$$

$$\boxed{U(x, y) = x \sin y - y \cos x + \ln y + K(x)}$$

Eliminar la dependencia de la variable

$$\frac{\partial U}{\partial x} = \sin y - y(-\sin x) + K'(x) \quad \Longleftrightarrow \quad \frac{\partial U}{\partial x} = \sin y + y \sin x + K'(x)$$

$$\sin y + y \sin x + \frac{1}{x} = \sin y + y \sin x + K'(x) \quad \Longleftrightarrow \quad K'(x) = \frac{1}{x}$$

$$K(x) = \int \left(\frac{1}{x}\right) dx \iff \boxed{K(x) = \ln x + K}$$

Exhibir la solución

$$U(x, y) = x \sin y - y \cos x + \ln y + \ln x + K$$

$$\boxed{U(x, y) = x \sin y - y \cos x + \ln |xy| + K}$$

Ejercicio 4.0.2 (Ejercicio 21)

$$\left(x + \frac{2}{y}\right) dy + y dx = 0$$

Verificar la exactitud.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\underbrace{x + \frac{2}{y}}_{N(x, y)} dy + \underbrace{y dx}_{M(x, y)} = 0$$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y} (y) = \boxed{1}$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x} \left(x + \frac{2}{y}\right) = \boxed{1}$$

Hallar la solución

$$\boxed{M(x, y) = \frac{\partial U}{\partial x}} \quad \wedge \quad \boxed{N(x, y) = \frac{\partial U}{\partial y}}$$

$$U(x, y) = \int (y) dx$$

$$U(x, y) = y \int dx \iff U(x, y) = y \int dx$$

$$\boxed{U(x, y) = yx + K(y)}$$

Eliminar la dependencia de la variable

$$\frac{\partial U}{\partial y} = x + K'(y)$$

$$x + \frac{2}{y} = x + K'(y)$$

$$\cancel{x} + \frac{2}{y} = \cancel{x} + K'(y)$$

$$K(y) = \int \frac{2}{y} dy$$

$$K(y) = 2 \ln y$$

$$K(y) = \ln y^2$$

Exhibir la solución

$$\boxed{U(x, y) = yx + \ln y^2 + K}$$

Ejercicio 4.0.3 (Ejercicio 22)

$$\left(4x^3y^3 + \frac{1}{x}\right) dx + \left(3x^4y^2 - \frac{1}{y}\right) dy = 0; \quad x = e, y = 1$$

Verificar la exactitud.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y} \left(4x^3y^3 + \frac{1}{x}\right) = \boxed{12x^3y^2}$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x} \left(3x^4y^2 - \frac{1}{y}\right) = \boxed{12x^3y^2}$$

Hallar la solución

$$\boxed{M(x, y) = \frac{\partial U}{\partial x}}$$

\wedge

$$\boxed{N(x, y) = \frac{\partial U}{\partial y}}$$

$$U(x, y) = \int \left(4x^3y^3 + \frac{1}{x}\right) dx$$

$$U(x, y) = \int (4x^3y^3) dx + \int \left(\frac{1}{x}\right) dx$$

$$U(x, y) = 4y^3 \int (x^3) dx + \int \left(\frac{1}{x}\right) dx$$

$$U(x, y) = 4y^3 \frac{x^4}{4} + \ln x + K(y) \iff \boxed{U(x, y) = x^4y^3 + \ln x + K(y)}$$

Eliminar la dependencia

$$\frac{\partial U}{\partial y} = 3x^4y^2 + K'(y) \iff 3x^4y^2 - \frac{1}{y} = 3x^4y^2 + K'(y)$$

$$K'(y) = -\frac{1}{y} \iff K(y) = -\int \frac{1}{y} dy \iff \boxed{K(y) = -\ln y + K}$$

Exhibir la solución general

$$U(x, y) = x^4y^3 + \ln x - \ln y + K \iff \boxed{U(x, y) = x^4y^3 + \ln \left(\frac{x}{y}\right) + K}$$

Solución particular sujeta al problema de condiciones iniciales $\boxed{x = e \wedge y = 1}$

$$U = e^4(1)^3 + \ln\left(\frac{e}{1}\right) + K \iff U - K = e^4 + \ln e^1$$

$$\underbrace{U - K}_{\beta} = e^4 + 1$$

$$\boxed{x^4 y^3 + \ln\left(\frac{x}{y}\right) = e^4 + 1}$$

Ejercicio 4.0.4 (Ejercicio 23)

$$\underbrace{(3y)}_{M(x,y)} dx + \underbrace{(2x)}_{N(x,y)} dy = 0; \quad x = -1, y = 1, 4, \quad F = x^2 y$$

Verificar la exactitud.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(3y) = 3$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(2x) = 2$$

Como la ecuación diferencial NO es exacta, multiplique la misma por $F = x^2 y$

$$x^2 y ((3y) dx + (2x) dy) = 0$$

$$\underbrace{(3x^2 y^2)}_{M(x,y)} dx + \underbrace{(2x^3 y)}_{N(x,y)} dy$$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(3x^2 y^2) = 6x^2 y$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(2x^3 y) = 6x^2 y$$

Hallar la solución

$$\boxed{M(x, y) = \frac{\partial U}{\partial x}} \quad \wedge \quad \boxed{N(x, y) = \frac{\partial U}{\partial y}}$$

$$U(x, y) = \int (3x^2 y^2) dx$$

$$U(x, y) = 3y^2 \int (x^2) dx$$

$$\boxed{U(x, y) = x^3 y^2 + K(y)}$$

$$\frac{\partial U}{\partial y} = 2x^3 y + K'(y)$$

$$2x^3y = 2x^3y + K'(y)$$

$$K(y) = 0$$

Exhibir la solución general

$$U(x, y) = x^3 y^2 + K$$

Exhibir la solución particular sujeta a la condición inicial $x = -1, y = 1, 4$

$$\beta = -1, 96$$

Exhibir la solución particular

$$x^3 y^2 = -1, 96$$

Ejercicio 4.0.5 (Ejercicio 24)

$$\underbrace{(2x^{-1}y - 3)}_{M(x, y)} dx + \underbrace{(3 - 2y^{-1}x)}_{N(x, y)} dy = 0; \quad x = 1, y = -1, \quad F = x^2 y^2$$

Verificar la exactitud.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y} (2x^{-1}y - 3) = 2x^{-1}$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x} (3 - 2y^{-1}x) = -2y^{-1}$$

Como la ecuación diferencial NO es exacta, multiplique la misma por $F = x^2 y^2$

$$x^2 y^2 ((2x^{-1}y - 3) dx + (3 - 2y^{-1}x) dy) = 0$$

$$\underbrace{(2xy^3 - 3x^2y^2)}_{M(x, y)} dx + \underbrace{(3x^2y^2 - 2x^3y)}_{N(x, y)} dy$$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y} (2xy^3 - 3x^2y^2) = 6xy^2 - 6x^2y$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x} (3x^2y^2 - 2x^3y) = 6xy^2 - 6x^2y$$

Hallar la solución

$$M(x, y) = \frac{\partial U}{\partial x} \quad \wedge \quad N(x, y) = \frac{\partial U}{\partial y}$$

$$\frac{\partial U}{\partial x} = 2xy^3 - 3x^2y^2 \quad \wedge \quad \frac{\partial U}{\partial y} = 3x^2y^2 - 2x^3y$$

$$U(x, y) = \int (2xy^3 - 3x^2y^2) dx$$

$$U(x, y) = 2y^3 \int x dx - 3y^2 \int x^2 dx$$

$$U(x, y) = 2y^3 \frac{x^2}{2} - 3y^2 \frac{x^3}{3} + K(y) \iff \boxed{U(x, y) = x^2 y^3 - x^3 y^2 + K(y)}$$

Eliminar la dependencia de la variable

$$\frac{\partial U}{\partial y} = 3x^2 y^2 - 2x^3 y + K'(y)$$

$$3x^2 y^2 - 2x^3 y = 3x^2 y^2 - 2x^3 y + K'(y) \iff K'(y) = 0 \iff K(y) = K$$

Exhibir la solución general

$$\boxed{U(x, y) = x^2 y^3 - x^3 y^2 + K}$$

Exhibir la solución particular sujeta a la condición inicial $x = 1, y = -1$

$$U = (1)^2 (-1)^3 - (1)^3 (-1)^2 + K \iff \underbrace{U - K}_{\beta} = -1 - 1 \iff \beta = -2$$

Exhibir la solución particular

$$\boxed{x^2 y^3 - x^3 y^2 = -2}$$

Capítulo 5

Ecuaciones diferenciales lineales y factores integrantes

Ejercicio 5.0.1 (Ejercicio 25)

$$\begin{aligned}
 & x \frac{dy}{dx} - 3y = x^4 \\
 & xy' - 3y = x^4 \iff \frac{xy'}{x} - \frac{3y}{x} = \frac{x^4}{x} \iff \boxed{y' - \underbrace{\frac{3}{x}}_{P(x)} y = x^3} \\
 & \mu(x) = \exp\left(\int P(x) dx\right) \\
 & \int \left(-\frac{3}{x}\right) dx \iff -3 \int \left(\frac{1}{x}\right) dx \iff -3 \ln x \iff \boxed{\ln x^{-3}} \\
 & \mu(x) = e^{\ln x^{-3}} \iff \boxed{\mu(x) = x^{-3} = \frac{1}{x^3}} \\
 & \frac{1}{x^3} \left(y' - \frac{3}{x} y\right) = \frac{1}{x^3} (x^4) \iff \frac{y'}{x^3} - \frac{3}{x^4} y = 1 \iff \left(\frac{1}{x^3} y\right)' = 1 \\
 & \int \left(\frac{1}{x^3} y\right)' = \int dx \iff \frac{1}{x^3} y = x + C \\
 & \boxed{y(x) = x^4 + C x^3}
 \end{aligned}$$

Ejercicio 5.0.2 (Ejercicio 26)

$$x^2 y' + 2xy - x + 1 = 0; \quad y(1) = 0$$

$$x^2 y' + 2xy = x - 1$$

$$\frac{x^2 y'}{x^2} + \frac{2xy}{x^2} = \frac{x-1}{x^2} \iff \boxed{y' + \underbrace{\frac{2}{x}}_{P(x)} y = \frac{1}{x} - \frac{1}{x^2}}$$

$$\mu(x) = \exp\left(\int P(x) dx\right)$$

$$\int \left(\frac{2}{x}\right) dx = 2 \int \left(\frac{1}{x}\right) = 2 \ln x = \boxed{\ln x^2}$$

$$\mu(x) = e^{\ln x^2} \iff \boxed{\mu(x) = x^2}$$

$$x^2 \left(y' + \frac{2}{x} y\right) = x^2 \left(\frac{1}{x} - \frac{1}{x^2}\right)$$

$$x^2 y' + 2xy = x - 1 \iff (x^2 y)' = x - 1$$

$$\int (x^2 y)' = \int x dx - \int dx \iff x^2 y = \frac{x^2}{2} - x + C$$

$$\boxed{y(x) = \frac{1}{2} - \frac{1}{x} + \frac{C}{x^2}}$$

Exhibir la solución particular sujeta a la condición inicial $x = 1, y = 0$

$$y(1) = \frac{1}{2} - \frac{1}{1} + \frac{C}{(1)^2} \iff \frac{1}{2} - 1 + C = 0$$

$$-\frac{1}{2} + C = 0 \iff \boxed{C = \frac{1}{2}}$$

$$\boxed{y(x) = \frac{1}{2} - \frac{1}{x} + \frac{1}{2x^2}}$$

Ejercicio 5.0.3 (Ejercicio 27)

$$y' - y \underbrace{\cot x}_{P(x)} = 2x - x^2 \cot x; \quad y\left(\frac{\pi}{2}\right) = \frac{1}{4}\pi^2 + 1$$

¡CUMPLIDO!

$$\mu(x) = \exp\left(\int P(x) dx\right) \iff \mu(x) = e^{(\ln |\sin x|^{-1})} \iff \boxed{\mu(x) = (\sin x)^{-1} = \csc x}$$

$$\int -\cot x dx = -\int \cot x dx = -\ln |\sin x| = \boxed{\ln |\sin x|^{-1}}$$

$$\csc x (y' - y \cot x) = \csc x (2x - x^2 \cot x)$$

$$(\csc x) y' - y \csc x \cot x = 2x \csc x - x^2 \csc x \cot x$$

$$(y \csc x)' = 2x \csc x - x^2 \csc x \cot x$$

$$\int (y \csc x)' = \int (2x \csc x) dx - \int (x^2 \csc x \cot x) dx$$

$$y \csc x = \underbrace{\int (2x \csc x) dx}_{I_1} - \underbrace{\int (x^2 \csc x \cot x) dx}_{I_2}$$

I_2

$$\int (x^2 \csc x \cot x) dx$$

ILATE

$$U = x^2 \implies dV = \csc x \cot x dx$$

$$dU = 2x dx \implies V = -\csc x$$

Un dia Vi Una vaca Vestida de Uniforme

$$\int U dV = UV - \int V dU$$

$$\int (x^2 \csc x \cot x) dx = -x^2 \csc x - \int (2x(-\csc x)) dx$$

$$\int (x^2 \csc x \cot x) dx = -x^2 \csc x + \int (2x \csc x) dx$$

Sustituyendo

$$y \csc x = \int (2x \csc x) dx - \left(-x^2 \csc x + \int (2x \csc x) dx \right)$$

$$y \csc x = \int (2x \csc x) dx + x^2 \csc x - \int (2x \csc x) dx$$

$$y \csc x = \cancel{\int (2x \csc x) dx} + x^2 \csc x - \cancel{\int (2x \csc x) dx} + C$$

$$y \csc x = x^2 \csc x + C \iff y(x) = \frac{x^2 \csc x}{\csc x} + \frac{C}{\csc x}$$

$$y(x) = x^2 + C \sin x$$

Hallar la solución particular

$$y\left(\frac{\pi}{2}\right) = \left(\frac{\pi}{2}\right)^2 + C \sin\left(\frac{\pi}{2}\right) \iff \frac{\pi^2}{4} + C = \frac{\pi^2}{4} + 1 \iff C = 1$$

$$y(x) = x^2 + \sin x$$

Ejercicio 5.0.4 (Ejercicio 28)

$$y' + y = 2xe^{-x} + x^2$$

¡CUMPLIDO!

$$P(x) = 1$$

$$\mu(x) = \exp\left(\int P(x) dx\right)$$

$$\int dx \iff \boxed{x}$$

$$\boxed{\mu(x) = e^x}$$

$$e^x(y' + y) = e^x(2xe^{-x} + x^2) \iff e^xy' + e^xy = 2x + e^xx^2 \iff (e^xy)' = 2x + x^2e^x$$

$$\int (e^xy)' = \int (2x + x^2e^x) dx \iff \int (e^xy)' = \int 2x dx + \underbrace{\int x^2e^x dx}_{I_1}$$

I_1

$$\int x^2e^x dx$$

$$\boxed{U = x^2 \implies dV = e^x}$$

$$\boxed{dU = 2x dx \implies V = e^x}$$

$$\int U dV = UV - \int V dU$$

$$\int x^2e^x dx = x^2e^x - 2 \underbrace{\int xe^x dx}_{I_{1,1}}$$

$$\int x^2e^x dx = x^2e^x - 2(xe^x - e^x)$$

$I_{1,1}$

$$\int xe^x dx$$

$$\boxed{U = x \implies dV = e^x}$$

$$\boxed{dU = dx \implies V = e^x}$$

$$\int xe^x dx = xe^x - \int e^x dx$$

$$\int xe^x dx = xe^x - e^x$$

$$e^xy = x^2 + x^2e^x - 2(xe^x - e^x) + C \iff e^xy = x^2 + x^2e^x - 2xe^x + 2e^x + C$$

$$y = \frac{x^2 + x^2 e^x - 2x e^x + 2e^x + C}{e^x} \iff \boxed{y(x) = x^2 e^{-x} + x^2 - 2x + 2 + C e^{-x}}$$

Ejercicio 5.0.5 (Ejercicio 29)

$$x y' = y + x^2 \sin x \iff \boxed{x y' - y = x^2 \sin x}$$

$$\frac{x}{x} y' - \frac{1}{x} y = \frac{x^2 \sin x}{x} \iff y' - \underbrace{\frac{1}{x}}_{P(x)} y = x \sin x$$

$$\mu(x) = \exp \left(\int P(x) dx \right) \iff \mu(x) = e^{(\ln x^{-1})} \iff \boxed{\mu(x) = x^{-1} = \frac{1}{x}}$$

$$\int P(x) dx = \int \left(-\frac{1}{x} \right) dx = - \int \frac{1}{x} dx = - \ln x = \boxed{\ln x^{-1}}$$

$$\frac{1}{x} \left(y' - \frac{1}{x} y \right) = \frac{1}{x} (x \sin x) \iff \frac{1}{x} y' - \frac{1}{x^2} y = \sin x$$

$$\left(\frac{1}{x} y \right)' = \sin x$$

$$\int \left(\frac{1}{x} y \right)' = \int \sin x dx \iff \frac{1}{x} y = -\cos x + C \iff \boxed{y(x) = -x \cos x + C x}$$

Prueba

$$y(x) = -x \cos x + C x$$

$$y'(x) = -\cos x + (-x(-\sin x)) + C \iff \boxed{y'(x) = -\cos x + x \sin x + C}$$

Sustituyendo en la ecuación diferencial

$$x y' = y + x^2 \sin x$$

$$x (-\cos x + x \sin x + C) = -x \cos x + C x + x^2 \sin x$$

$$\boxed{-x \cos x + x^2 \sin x + C x = -x \cos x + C x + x^2 \sin x}$$

Capítulo 6

Aplicaciones diversas

Ejercicio 6.0.1 (Ejercicio 30)

Ejercicio 6.0.2 (Ejercicio 31)

Ejercicio 6.0.3 (Ejercicio 32) Suponga que un país tiene una variación en la población $P(t)$ dado por

$$\frac{dP}{dt} = \frac{dB}{dt} - \frac{dD}{dt}$$

Donde dB/dt y dD/dt son las tasas de natalidad y mortalidad respectivamente.

Determine $P(t)$ si $dB/dt = k_1 P$ y $dD/dt = k_2 P$.

$$\begin{aligned}\frac{dP}{dt} &= k_1 P - k_2 P \iff \frac{dP}{dt} = (k_1 - k_2) P \\ \frac{dP}{P} &= (k_1 - k_2) dt \iff \int \frac{dP}{P} = (k_1 - k_2) \int dt \\ \ln P &= (k_1 - k_2)t + C \iff e^{\ln P} = e^{((k_1 - k_2)t + C)} \\ e^{\ln P} &= e^{(k_1 - k_2)t} \cdot \underbrace{e^C}_{P_0} \iff \boxed{P(t) = P_0 e^{(k_1 - k_2)t}}\end{aligned}$$

▢ Primer caso: $k_1 > k_2$

$$P(t) = P_0 e^{(k_1 - k_2)t} \implies P(t) \rightarrow \infty$$

▢ Segundo caso: $k_1 = k_2$

$$P(t) = P_0 e^{(k_1 - k_2)t} \implies P(t) = P_0$$

▢ Tercer caso: $k_1 < k_2$

$$P(t) = P_0 e^{-kt} \iff P(t) = P_0 \frac{1}{e^{kt}} \iff \boxed{P(t) = P_0 \lim_{t \rightarrow \infty} \left(\frac{1}{e^{kt}} \right) \rightarrow 0}$$

Ejercicio 6.0.4 (Ejercicio 33)

Suponga que un cuerpo se mueve a través de un medio con resistencia proporcional a su velocidad v , de tal manera que $dv/dt = -Kv$. Muestre que su velocidad y posición en el tiempo t están dadas por

$$v(t) = v_0 e^{-Kt} \quad y$$

$$x(t) = x_0 + \frac{v_0}{K} (1 - e^{-Kt})$$

Demostrando la velocidad

$$\frac{dv}{dt} = -Kv \iff \frac{dv}{v} = -K dt$$

$$\int \frac{1}{v} dv = -K \int dt \iff \ln v = -Kt + C$$

$$e^{\ln v} = e^{-Kt+C} \iff v = e^{-Kt+C} \iff v = e^{-Kt} e^C \iff v = e^{-Kt} \underbrace{e^C}_{v_0}$$

$$\boxed{v(t) = v_0 e^{-Kt}}$$

Demostrando la posición recordando que $v = dx/dt$

$$\frac{dx}{dt} = v_0 e^{-Kt} \iff dx = (v_0 e^{-Kt}) dt$$

$$\int dx = \int (v_0 e^{-Kt}) dt \iff \int dx = v_0 \underbrace{\int e^{-Kt} dt}_{I_1}$$

I_1

$$\int e^{-Kt} dt$$

$$u = -Kt \quad \wedge \quad \frac{du}{dt} = -K \rightarrow dt = -\frac{1}{K} du$$

$$-\frac{1}{K} \int e^u du \iff -\frac{1}{K} e^u + C \iff -\frac{e^{-Kt}}{K} + C$$

$$x = v_0 \left(-\frac{e^{-Kt}}{K} \right) + \underbrace{C}_{x_0} \iff x = x_0 - \frac{v_0 e^{-Kt}}{K} \iff x = x_0 + \frac{v_0 (v_0 - e^{-Kt})}{K}$$

$$\boxed{x(t) = x_0 + \frac{v_0}{K} (1 - e^{-Kt})}$$

Ejercicio 6.0.5 (Ejercicio 34)

Un cuerpo inicialmente en reposo en x_0 se mueve en línea recta bajo la acción de una fuerza $\vec{F} = -k/x^2$. Demostrar que su velocidad en x es $v^2 = 2(k/m)(1/x - 1/x_0)$.

$$\vec{F} = -k/x^2 \iff m \frac{d\vec{V}}{dt} = -\frac{k}{x^2} \iff \frac{d\vec{V}}{dt} = -\frac{k}{m} \frac{1}{x^2}$$

$$\int \left(\frac{d\vec{V}}{dt} \right) dx = -\frac{k}{m} \int x^{-2} dx \iff \int \vec{V} d\vec{V} = -\frac{k}{m} \int_{x_0}^x x^{-2} dx$$

$$\frac{V^2}{2} = -\frac{k}{m} \left(\frac{x^{-2+1}}{-2+1} \Big|_{x_0}^x \right)$$

$$\frac{V^2}{2} = -\frac{k}{m} \left(\frac{x^{-1}}{-1} \Big|_{x_0}^x \right) \iff \frac{V^2}{2} = -\frac{k}{m} \left(-\frac{1}{x} \Big|_{x_0}^x \right)$$

$$\frac{V^2}{2} = \frac{k}{m} \left(\frac{1}{x} \Big|_{x_0}^x \right) \iff \frac{V^2}{2} = \frac{k}{m} \left(\frac{1}{x} - \frac{1}{x_0} \right)$$

$$\boxed{V^2 = \frac{2k}{m} \left(\frac{1}{x} - \frac{1}{x_0} \right)}$$

Ejercicio 6.0.6 (Ejercicio 35)

La población de un pueblo crece con una razón proporcional a la población en el tiempo t . La población inicial de 500 aumenta 15% en 10 años. ¿Cuál será la población pasados 30 años?

$$P_0 = 500; \beta = 0,15 \text{ en } t_1 = 10; t_2 = 30$$

Hallando el incremento de un año

$$\frac{\beta}{t_1} = \frac{0,15}{10} = \underbrace{0,015}_{\alpha}$$

Hallando la población 30 años

$$P(t) = P_0 e^{\alpha t_2} \iff P(30) = 500 e^{(0,015 \times 30)} \iff P(30) = 500 e^{0,45}$$

$$\boxed{P(30) \approx 784}$$

Este documento fue levantado en \LaTeX .