Escuela Tecnológica Instituto Técnico Central (ETITC)

Facultad de sistemas

Ejercicios LaTeX: Cálculo integral

Autores

Sergio Alejandro Enrrique Caballero Leon
Cristian Eduardo Rodriguez Castañeda
Johan Alejandro Sogamoso Camacho
Brian Ferney Rojas Garcia

Presentado a:

Andrés Garzón Mayorga

Bogotá, Septiembre de 2021.

Índice general

Capítulo 1

Introducción a las ecuaciones diferenciales

Ejercicio 1.0.1 (Ejercicio 1)

$$y' \;=\; 3\,x^2; \qquad \boxed{y\,(x) \;=\; x^3 \;+\; 7}$$

Sea $y\,(x) \;=\; x^3 \;+\; 7$

$$y'\,(x) \;=\; 3\,x^2$$

Sustituyendo en la ecuación diferencial

$$\boxed{3\,x^2 \;=\; 3\,x^2}$$

Ejercicio 1.0.2 (Ejercicio 2)

$$y' \;+\; 2\,x\,y^2 \;=\; 0; \qquad \boxed{y\,(x) \;=\; \frac{1}{1 \;+\; x^2}}$$

Forma 1.

Sea $y\,(x) \;=\; \dfrac{1}{1 \;+\; x^2}$

$$y'(x) \;=\; \frac{(1)'\,(1 \;+\; x^2) \;-\; (1)\,(1 \;+\; x^2)'}{(1 \;+\; x^2)^2} \quad \Longleftrightarrow \quad \boxed{y'\,(x) \;=\; -\,\frac{2\,x}{(1 \;+\; x^2)^2}}$$

Sustituyendo en la ecuación diferencial

$$-\,\frac{2\,x}{(1 \;+\; x^2)^2} \;+\; 2\,x\left(\frac{1}{(1 \;+\; x^2)}\right)^2 \;=\; 0 \quad \Longleftrightarrow \quad -\,\frac{2\,x}{(1 \;+\; x^2)^2} \;+\; 2\,x\left(\frac{1}{(1 \;+\; x^2)^2}\right) \;=\; 0$$

$$-\,\cancel{\frac{2\,x}{(1 \;+\; x^2)^2}} \;+\; \cancel{\frac{2\,x}{(1 \;+\; x^2)^2}} \;=\; 0 \quad \Longleftrightarrow \quad \boxed{0 \;=\; 0}$$

Forma 2.

Sea $y\,(x) \;=\; \dfrac{1}{1 \;+\; x^2} \quad \Longleftrightarrow \quad \boxed{y\,(x) \;=\; (1 \;+\; x^2)^{-\,1}}$

$$y'(x) \;=\; -\,1\,(1 \;+\; x^2)^{-\,1\,-\,1} \times 2\,x \quad \Longleftrightarrow \quad y'\,(x) \;=\; -\,(1 \;+\; x^2)^{-\,2} \times 2\,x$$

$$\boxed{y'(x) = -\frac{2x}{(1+x^2)^2}}$$

Sustituyendo en la ecuación diferencial

$$-\frac{2x}{(1+x^2)^2} + 2x\left(\frac{1}{(1+x^2)}\right)^2 = 0 \iff -\frac{2x}{(1+x^2)^2} + 2x\left(\frac{1}{(1+x^2)^2}\right) = 0$$

$$-\cancel{\frac{2x}{(1+x^2)^2}} + \cancel{\frac{2x}{(1+x^2)^2}} = 0 \iff \boxed{0 = 0}$$

Ejercicio 1.0.3 (Ejercicio 3)

$$x\,y' = y + x^2 + y^2 \qquad \boxed{y(x) = x\tan x}$$

Sea $y(x) = x\tan x$

$$y'(x) = (x)'\tan x + x(\tan x)' \iff \boxed{y'(x) = \tan x + x\sec^2 x}$$

Sustituyendo en la ecuación diferencial

$$x(\tan x + x\sec^2 x) = x\tan x + x^2 + (x\tan x)^2$$

$$x\tan x + x^2\sec^2 x = x\tan x + x^2 + x^2\tan^2 x$$

$$x\tan x + x^2\sec^2 x = x\tan x + x^2\,(\underbrace{1 + \tan^2 x}_{\sec^2 x})$$

$$\boxed{x\tan x + x^2\sec^2 x = x\tan x + x^2\sec^2 x}$$

$$\cos^2 x + \sin^2 x = 1$$

$$\frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \iff 1 + \tan^2 x = \sec^2 x$$

Ejercicio 1.0.4 (Ejercicio 4)

$$y'' = 9\,y; \qquad \boxed{y(x) = C_1 e^{3x} + C_2 e^{-3x}}$$

Sea $y(x) = C_1 e^{3x} + C_2 e^{-3x}$

$$y'(x) = 3\,e^{3x} - 3\,e^{-3x}$$

$$y''(x) = 9\,e^{3x} + 9\,e^{-3x}$$

Sustituyendo en la ecuación diferencial

$$9\,e^{3x} + 9\,e^{-3x} = 9\,(C_1 e^{3x} + C_2 e^{-3x}) \iff \boxed{9\,e^{3x} + 9\,e^{-3x} = 9\,e^{3x} + 9\,e^{-3x}}$$

Ejercicio 1.0.5 (Ejercicio 5) *Muestre que la función $u(x,t) = (x - a\,t)^2 + (x + a\,t)^3$ satisface la ecuación de calor*

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial u}{\partial t} = 2\,(x - a\,t) \times (-a) + 3\,(x + a\,t)^2 \times a$$

$$\frac{\partial u}{\partial t} = -2\,a\,(x - a\,t) + 3\,a\,(x + a\,t)^2$$

$$\frac{\partial^2 u}{\partial t^2} = -2\,a \times (-a) + 3\,a \times 2\,(x + a\,t) \times a$$

$$\boxed{\frac{\partial^2 u}{\partial t^2} = 2\,a^2 + 6\,a^2\,(x + a\,t)}$$

$$\frac{\partial u}{\partial x} = 2\,(x - a\,t) + 3\,(x + a\,t)^2$$

$$\boxed{\frac{\partial^2 u}{\partial x^2} = 2 + 6\,(x + a\,t)}$$

Sustituyendo en la ecuación diferencial

$$2\,a^2 + 6\,a^2\,(x + a\,t) = a^2\,(2 + 6\,(x + a\,t))$$

$$\boxed{2\,a^2 + 6\,a^2\,(x + a\,t) = 2\,a^2 + 6\,a^2\,(x + a\,t)}$$

Verifique que la función dada satisface la ecuación indicada

Ejercicio 1.0.6 (Ejercicio 6)

$$\frac{\partial z}{\partial x} + z\,\frac{\partial z}{\partial y} - z = x; \quad z(x,y) = \frac{1}{2}x^2 + y - 3$$

$$\frac{\partial z}{\partial x} = \frac{\cancel{1}}{\cancel{2}}\,\cancel{2}\,x \iff \boxed{\frac{\partial z}{\partial x} = x}$$

$$\boxed{\frac{\partial z}{\partial y} = 1}$$

Sustituyendo en la ecuación diferencial

$$x + \left(\frac{1}{2}x^2 + y - 3\right) \times 1 - \left(\frac{1}{2}x^2 + y - 3\right) = x$$

$$x + \cancel{\left(\frac{1}{2}x^2 + y - 3\right)} - \cancel{\left(\frac{1}{2}x^2 + y - 3\right)} = x \iff \boxed{x = x}$$

Ejercicio 1.0.7 (Ejercicio 7)

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 - 2z = 0; \qquad z(x, y) = \frac{(x - y + 2)^2}{4}$$

$$\frac{\partial z}{\partial x} = \boxed{\frac{x - y + 2}{2}}$$

$$\frac{\partial z}{\partial y} = \boxed{\frac{-x + y - 2}{2}}$$

Sustituyendo en la ecuación diferencial

$$\left(\frac{x - y + 2}{2}\right)^2 + \left(\frac{-x + y - 2}{2}\right)^2 - 2\left(\frac{(x - y + 2)^2}{4}\right) = 0$$

$$\frac{(x - y + 2)^2}{4} + \frac{(-x + y - 2)^2}{4} - 2\left(\frac{(x - y + 2)^2}{4}\right) = 0$$

Ejercicio 1.0.8 (Ejercicio 8) *Muestre que la función* $z(x, y) = \sin(x^2 + y^2)$ *satisface la ecuación*

$$y\frac{\partial^2 z}{\partial x^2} - x\frac{\partial^2 z}{\partial y\,\partial x} - \frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial x} = \cos(x^2 + y^2) \times 2x \iff \frac{\partial z}{\partial x} = 2x\cos(x^2 + y^2)$$

$$\frac{\partial^2 z}{\partial x^2} = 2\cos(x^2 + y^2) + 2x\left[-\sin(x^2 + y^2) \times 2x\right]$$

$$\boxed{\frac{\partial^2 z}{\partial x^2} = 2\cos(x^2 + y^2) - 4x^2\sin(x^2 + y^2)}$$

$$\frac{\partial^2 z}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

$$\frac{\partial z}{\partial x} = 2x\cos(x^2 + y^2)$$

$$\frac{\partial}{\partial y}\left[2x\cos(x^2 + y^2)\right] = 2x\left(-\sin(x^2 + y^2)\right) \times 2y$$

$$\boxed{\frac{\partial^2 z}{\partial y\,\partial x} = -4xy\sin(x^2 + y^2)}$$

$$\frac{\partial z}{\partial y} = \cos(x^2 + y^2) \times 2y \iff \boxed{\frac{\partial z}{\partial y} = 2y\cos(x^2 + y^2)}$$

Sustituyendo en la ecuación diferencial parcial

$$y\left[2\cos(x^2 + y^2) - 4x^2\sin(x^2 + y^2)\right] - x\left[-4xy\sin(x^2 + y^2)\right] - 2y\cos(x^2 + y^2) = 0$$

$$\cancel{2y\cos(x^2 + y^2)} - \cancel{4x^2y\sin(x^2 + y^2)} + \cancel{4x^2y\sin(x^2 + y^2)} - \cancel{2y\cos(x^2 + y^2)} = 0$$

$$\boxed{0 = 0}$$

Capítulo 2

Separación de variables

Ejercicio 2.0.1 (Ejercicio 9)

$$x\,y' = y\,(3 - x)$$

$$x\,\frac{d\,y}{d\,x} = y\,(3 - x) \iff \frac{d\,y}{y} = \left(\frac{3 - x}{x}\right) d\,x$$

$$\frac{d\,y}{y} = \left(\frac{3}{x} - \frac{x}{x}\right) d\,x \iff \frac{d\,y}{y} = \left(\frac{3}{x} - 1\right) d\,x$$

$$\int \frac{d\,y}{y} = \int \left(\frac{3}{x} - 1\right) d\,x$$

$$\int \frac{d\,y}{y} = 3 \int \frac{d\,x}{x} - \int d\,x \iff \ln\,y = 3\,\ln\,x - x + C$$

$$\ln\,y = \ln\,x^3 - x + C \iff e^{\ln\,y} = e^{(\ln\,x^3 - x + C)}$$

$$y = e^{\ln\,x^3} \cdot e^{-x} \cdot \underbrace{e^C}_{\hat{C}} \iff \boxed{y\,(x) = \hat{C}\,x^3\,e^{-x}}$$

Prueba

Sea $\boxed{y\,(x) = \hat{C}\,x^3\,e^{-x}}$

$$y'\,(x) = 3\,\hat{C}\,x^2\,e^{-x} + \hat{C}\,x^3 \cdot e^{-x} \times (-1) \iff \boxed{y'\,(x) = 3\,\hat{C}\,x^2\,e^{-x} - \hat{C}\,x^3\,e^{-x}}$$

Sustituyendo en la ecuación diferencial

$$x\,y' = y\,(3 - x)$$

$$x\,(3\,\hat{C}\,x^2\,e^{-x} - \hat{C}\,x^3\,e^{-x}) = \hat{C}\,x^3\,e^{-x}\,(3 - x)$$

$$\boxed{3\,\hat{C}\,x^3\,e^{-x} - \hat{C}\,x^4\,e^{-x} = 3\,\hat{C}\,x^3\,e^{-x} - \hat{C}\,x^4\,e^{-x}}$$

Ejercicio 2.0.2 (Ejercicio 10)

$$\frac{dy}{dx} = \frac{x}{y} - \frac{x}{1+y}; \quad y(0) = 1$$

$$\frac{dy}{dx} = \frac{x(1+y) - xy}{y(1+y)} \quad \Longleftrightarrow \quad \frac{dy}{dx} = \frac{x + xy - xy}{y(1+y)}$$

$$\frac{dy}{dx} = \frac{x + \cancel{xy} - \cancel{xy}}{y(1+y)} \quad \Longleftrightarrow \quad \frac{dy}{dx} = \frac{x}{y(1+y)}$$

$$(y(1+y))\,dy = x\,dx \quad \Longleftrightarrow \quad (y + y^2)\,dy = x\,dx$$

$$\int y\,dy + \int y^2\,dy = \int x\,dx \quad \Longleftrightarrow \quad \boxed{\frac{y^2}{2} + \frac{y^3}{3} = \frac{x^2}{2} + C}$$

Problema de condición inicial: $x = 0,\ y = 1$

$$\frac{(1)^2}{2} + \frac{(1)^3}{3} = \frac{(0)^2}{2} + C \quad \Longleftrightarrow \quad \frac{1}{2} + \frac{1}{3} = C \quad \Longleftrightarrow \quad \boxed{C = \frac{5}{6}}$$

Exhibir la solución particular

$$\frac{y^2}{2} + \frac{y^3}{3} = \frac{x^2}{2} + \frac{5}{6} \quad \Longleftrightarrow \quad \frac{3y^2 + 2y^3}{6} = \frac{3x^2 + 5}{6} \quad \Longleftrightarrow \quad \boxed{2y^3 + 3y^2 = 3x^2 + 5}$$

Ejercicio 2.0.3 (Ejercicio 11)

$$sin\,(2x) + cos(3y) = 0; \quad y(\frac{\pi}{2}) = \frac{\pi}{3}$$

$$sin\,(2x) + cos(3y) = 0 \quad \Longleftrightarrow \quad cos(3y) = -sin\,(2x)$$

$$\int cos\,(3y) = \int -sin\,(2x)$$

$$\boxed{\frac{sin(3y)}{3} = \frac{cos(2x)}{2} + C}$$

Remplazo los valores de x y y

$$\frac{sin(\pi)}{3} = \frac{cos(\pi)}{2} + C$$

$$\frac{sin(\pi)}{3} - \frac{cos(\pi)}{2} = C \Longleftrightarrow \boxed{\frac{1}{2} = C}$$

$$\boxed{\frac{sin(3y)}{3} = \frac{cos(2x)}{2} + \frac{1}{2}}$$

Ejercicio 2.0.4 (Ejercicio 12)

$$\sqrt{1-y^2}\,dx - \sqrt{1-x^2}\,dy = 0; \quad y(0) = \frac{\sqrt{3}}{2}$$

$$\sqrt{1-y^2}\,dx = \sqrt{1-x^2}\,dy$$

$$\sqrt{1-x^2}\,dy = \sqrt{1-y^2}\,dx$$

$$\frac{dy}{\sqrt{1-y^2}} = \frac{dx}{\sqrt{1-x^2}}$$

$$\int \frac{dy}{\sqrt{1-y^2}} = \int \frac{dx}{\sqrt{1-x^2}}$$

$$\arcsin(y) = \arcsin(x) + C$$

Remplazando valores de x y y

$$\arcsin\left(\frac{\sqrt{3}}{2}\right) = \arcsin(0) + C$$

$$60 = +C$$

$$\sin^{-1} y = \sin^{-1} x + C \quad \Longleftrightarrow \quad \sin(\sin^{-1} y) = \sin(\sin^{-1} x + C)$$

$$\textit{Identidad:} \quad \sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$y = \sin(\sin^{-1} x)\cos C + \cos(\sin^{-1} x)\sin C$$

$$x = 1\sin\theta \quad \Longrightarrow \quad \theta = \sin^{-1} x$$

$$y = x\cos C + \cos\theta\sin C$$

$$\boxed{y(x) = \cos C + \sqrt{1-x^2}\,sin\,C}$$

Remplazando C

$$y(x) = \cos(60) + \sqrt{1-x^2}\,sin(60)$$

$$\boxed{y(x) = \frac{x}{2} + \frac{\sqrt{3}}{2}\sqrt{1-x^2}}$$

Ejercicio 2.0.5 (Ejercicio 13)

Sea

$$V = \frac{2}{2t^2+3t+1} \quad \Longleftrightarrow \quad \frac{dx}{dt} = \frac{2}{2t^2+3t+1}$$

$$d\,x = \left(\frac{2}{2\,t^2 + 3\,t + 1}\right) d\,t \iff \int d\,x = \int_{t=0}^{t=1} \left(\frac{2}{2\,t^2 + 3\,t + 1}\right) d\,t$$

Aparte:

$$2\,t^2 + 3\,t + 1 = \frac{2\,(2\,t^2 + 3\,t + 1)}{2}$$

$$2\,t^2 + 3\,t + 1 = \frac{4\,t^2 + 2\,(3\,t) + 2}{2}$$

$$2\,t^2 + 3\,t + 1 = \frac{4\,t^2 + 3\,(2\,t) + 2}{2}$$

$$2\,t^2 + 3\,t + 1 = \frac{(2\,t + 2)\,(2\,t + 1)}{2} \iff 2\,t^2 + 3\,t + 1 = \frac{2\,(t + 1)\,(2\,t + 1)}{2}$$

$$2\,t^2 + 3\,t + 1 = \boxed{(t + 1)\,(2\,t + 1)}$$

$$\frac{2}{2\,t^2 + 3\,t + 1} = \boxed{\frac{2}{(t + 1)\,(2\,t + 1)} = \frac{A}{t + 1} + \frac{B}{2\,t + 1}}$$

$$A = \frac{2}{2\,(-1) + 1} = \boxed{-2}$$

$$B = \frac{2}{-\frac{1}{2} + 1} = \frac{2}{\frac{1}{2}} = \boxed{4}$$

$$x = \int_0^1 \left(\frac{-2}{t + 1}\right) d\,t + \int_0^1 \left(\frac{4}{2\,t + 1}\right) d\,t \iff x = -2 \underbrace{\int_0^1 \left(\frac{1}{t + 1}\right) d\,t}_{I_1} + 4 \underbrace{\int_0^1 \left(\frac{1}{2\,t + 1}\right) d\,t}_{I_2}$$

I_1

$$\int_0^1 \left(\frac{1}{t + 1}\right) d\,t$$

$$\boxed{u = t + 1 \implies \frac{d\,u}{d\,t} = 1 \iff d\,t = d\,u}$$

$$\int_0^1 \left(\frac{1}{u}\right) d\,u$$

$$\ln |u| \Big|_0^1 \iff \ln |t + 1| \Big|_0^1$$

I_2

$$\int_0^1 \left(\frac{1}{2t+1}\right) dt$$

$$\boxed{u = 2t + 1 \quad \Longrightarrow \quad \frac{du}{dt} = 2 \quad \Longleftrightarrow \quad dt = \frac{1}{2} du}$$

$$\frac{1}{2} \int_0^1 \left(\frac{1}{u}\right) du$$

$$\frac{1}{2} \ln |u| \Big|_0^1 \quad \Longleftrightarrow \quad \frac{1}{2} \ln |t+1| \Big|_0^1$$

$$x = -2 \ln |t+1| \Big|_0^1 + 4 \left(\frac{1}{2} \ln |2t+1|\right) \Big|_0^1 \quad \Longleftrightarrow \quad x = -2 \ln 2 + 2 \ln 3 \quad \Longleftrightarrow \quad \boxed{x = 2 \ln \left(\frac{3}{2}\right)}$$

Capítulo 3

Ecuaciones diferenciales homógeneas

Ejercicio 3.0.1 (Ejercicio 14)

$$y' = \frac{y^2 + x\,y}{x^2}; \quad y\,(1) = 1$$

$$\frac{d\,y}{d\,x} = \frac{y^2}{x^2} + \frac{x\,y}{x^2} \iff \frac{d\,y}{d\,x} = \frac{y^2}{x^2} + \frac{\cancel{x}\,y}{x^{\cancel{2}}}$$

$$\frac{d\,y}{d\,x} = \frac{y^2}{x^2} + \frac{y}{x} \iff \frac{d\,y}{d\,x} = \left(\frac{y}{x}\right)^2 + \frac{y}{x}$$

$$x\,\frac{d\,U}{d\,x} + U = U^2 + U \iff x\,\frac{d\,U}{d\,x} + \cancel{U} = U^2 + \cancel{U}$$

$$x\,\frac{d\,U}{d\,x} = U^2 \iff \frac{d\,U}{U^2} = \frac{1}{x}\,d\,x \iff \int U^{-2}\,d\,U = \int \left(\frac{1}{x}\right)\,d\,x$$

$$\frac{U^{-2+1}}{-2+1} = \ln x + C \iff -U^{-1} = \ln x + C$$

$$-\frac{1}{U} = \ln x + C \iff -\frac{1}{\frac{1}{\frac{y}{x}}} = \ln x + C \iff -\frac{x}{y} = \ln x + C$$

$$-\frac{x}{\ln x + C} = y \iff \boxed{y\,(x) = -\frac{x}{\ln x + C}}$$

Hallar la solución particular: $x = 1,\ y = 1$

$$y\,(1) = -\frac{1}{\ln 1 + C} \iff -\frac{1}{C} = 1 \iff C = -1$$

Exhibir la solución particular

$$\boxed{y\,(x) = -\frac{x}{\ln x - 1}}$$

Ejercicio 3.0.2 (Ejercicio 15)

$$x\,\frac{d\,y}{d\,x} = x\,e^{y/x} + y; \quad y\,(1) = 0$$

$$x\,\frac{d\,y}{d\,x} = x\,e^{y/x} + y \quad \Longleftrightarrow \quad \frac{d\,y}{d\,x} = \frac{(x\,e^{y/x} + y)}{x} \quad \Longleftrightarrow \quad \frac{d\,y}{d\,x} = \frac{\cancel{x}\,e^{y/x}}{\cancel{x}} + \frac{y}{x}$$

$$\frac{d\,y}{d\,x} = e^{y/x} + \frac{y}{x} \quad \Longleftrightarrow \quad x\,\frac{d\,U}{d\,x} + \cancel{U} = e^{U} + \cancel{U} \quad \Longleftrightarrow \quad x\,\frac{d\,U}{d\,x} = e^{U}$$

$$\frac{dU}{e^{U}} = \frac{1}{x}\,dx \quad \Longleftrightarrow \quad e^{-U}\,dU = \frac{1}{x}\,dx \quad \Longleftrightarrow \quad \int e^{-U}\,dU = \int \left(\frac{1}{x}\right)\,dx$$

$$-\,e^{-U} = \ln\,x + C \quad \Longleftrightarrow \quad -\,C = e^{-U} + \ln\,x \quad \Longleftrightarrow \quad \boxed{-\,C = e^{-\,y/x} + \ln\,x}$$

Hallar la solución particular $x = 1;\ y = 0$

$$-\,C = e^{0} + \ln\,1 \quad \Longleftrightarrow \quad -\,C = 1 \quad \Longleftrightarrow \quad C = -\,1$$

Exhibir la solución particular

$$-\,(-\,1) = e^{-\,y/x} + \ln\,x \quad \Longleftrightarrow \quad \boxed{e^{-\,y/x} + \ln\,x = 1}$$

Ejercicio 3.0.3 (Ejercicio 16)

$$x^2\,y' - 3\,x\,y - 2\,y^2 = 0$$

$$\frac{x^2}{x^2}\,y' - 3\,\frac{x\,y}{x^2} - 2\,\frac{y^2}{x^2} = 0 \quad \Longleftrightarrow \quad y' - 3\,\frac{y}{x} - 2\,\left(\frac{y}{x}\right)^2 = 0 \quad \Longleftrightarrow \quad \frac{d\,y}{d\,x} - 3\,\frac{y}{x} - 2\,\left(\frac{y}{x}\right)^2 = 0$$

$$x\,\frac{d\,U}{d\,x} + U - 3\,U - 2\,U^2 = 0 \quad \Longleftrightarrow \quad x\,\frac{d\,U}{d\,x} - 2\,U - 2\,U^2 = 0$$

$$x\,\frac{d\,U}{d\,x} = 2\,U^2 + 2\,U \quad \Longleftrightarrow \quad x\,\frac{d\,U}{d\,x} = 2\,U\,(U + 1) \quad \Longleftrightarrow \quad \frac{d\,U}{U\,(U + 1)} = \frac{2}{x}\,d\,x$$

$$\underbrace{\int \left(\frac{d\,U}{U\,(U + 1)}\right)}_{I_1} = \int \frac{2}{x}\,d\,x$$

I_1

$$\int \left(\frac{dU}{U\,(U+1)} \right)$$

$$\frac{1}{U\,(U+1)} = \frac{A}{U} + \frac{B}{U+1}$$

$$A = \frac{1}{0+1} = \boxed{1}$$

$$B = \frac{1}{-1} = \boxed{-1}$$

$$\int \left(\frac{dU}{U\,(U+1)} \right) = \int \frac{1}{U}\,dU - \int \frac{1}{U+1}\,dU$$

$$\int \left(\frac{dU}{U\,(U+1)} \right) = \ln U - \ln |U+1| = \boxed{\ln \left(\frac{U}{U+1} \right)}$$

$$\ln \left(\frac{U}{U+1} \right) = 2 \ln x + C \iff \ln \left(\frac{\frac{y}{x}}{\frac{y}{x} + \frac{1}{1}} \right) = \ln x^2 + C$$

$$\ln \left(\frac{\frac{y}{x}}{\frac{x+y}{x}} \right) = \ln x^2 + C \iff \ln \left(\frac{x\,y}{x\,(x+y)} \right) = \ln x^2 + C$$

$$\ln \left(\frac{y}{x+y} \right) = \ln x^2 + C$$

Tomando exponencial a ambos lados

$$\frac{y}{x+y} = e^{(\ln x^2 + C)} \iff \frac{y}{x+y} = e^{\ln x^2} \cdot \underbrace{e^{C}}_{\hat{C}} \iff \frac{y}{x+y} = \hat{C}\,x^2$$

$$\boxed{y = \hat{C}\,x^2\,(x+y)}$$

Ejercicio 3.0.4 (Ejercicio 17)

$$-y\,dx + (x + \sqrt{x\,y})\,dy = 0$$

$$(x + \sqrt{x\,y})\,dy = y\,dx \iff \frac{dy}{dx} = \frac{y}{x + \sqrt{x\,y}}$$

$$x\frac{dU}{dx} + U = \frac{U\,x}{x + \sqrt{x\,U\,x}} \iff x\frac{dU}{dx} + U = \frac{U\,x}{x + \sqrt{x^2\,U}} \iff x\frac{dU}{dx} + U = \frac{U\,x}{x + x\sqrt{U}}$$

$$x\frac{dU}{dx} + U = \frac{U\,\cancel{x}}{\cancel{x}(1 + \sqrt{U})} \iff x\frac{dU}{dx} = \frac{U}{(1 + \sqrt{U})} - U \iff x\frac{dU}{dx} = \frac{U - U\,(1 + \sqrt{U})}{(1 + \sqrt{U})}$$

$$x\,\frac{dU}{dx} = \frac{\cancel{U} - \cancel{U} - U\sqrt{U}}{(1+\sqrt{U})} \iff x\,\frac{dU}{dx} = -\,\frac{U\sqrt{U}}{(1+\sqrt{U})}$$

$$-\,\frac{(1+\sqrt{U})}{U\sqrt{U}}\,dU = \frac{dx}{x} \iff -\left(\frac{dU}{U\,U^{1/2}} + \frac{\cancel{\sqrt{U}}}{U\cancel{\sqrt{U}}}\,dU\right) = \frac{dx}{x} \iff -\,\frac{dU}{U^{3/2}} - \frac{dU}{U} = \frac{dx}{x}$$

$$-\,U^{-3/2}\,dU - \frac{dU}{U} = \frac{dx}{x} \iff -\int U^{-3/2}\,dU - \int \frac{dU}{U} = \int \frac{dx}{x}$$

$$\cancel{-}\frac{U^{-1/2}}{\cancel{-}1/2} - \ln U = \ln x + C \iff \frac{2}{\sqrt{U}} - \ln U = \ln x + C$$

$$\frac{2}{\sqrt{y/x}} - \ln\left(\frac{y}{x}\right) = \ln x + C \iff \frac{2}{\sqrt{y/x}} - (\ln y - \ln x) = \ln x + C$$

$$\frac{2}{\sqrt{y/x}} - \ln y + \cancel{\ln x} = \cancel{\ln x} + C \iff \frac{2}{\sqrt{y/x}} - \ln y = C \iff \frac{2}{\sqrt{y/x}} = C + \ln y$$

$$\left(\frac{2}{\sqrt{y/x}}\right)^2 = (C+\ln y)^2 \iff \frac{4}{y/x} = (C+\ln y)^2$$

$$\frac{4x}{y} = (C+\ln y)^2 \iff \boxed{4x = y\,(C+\ln y)^2}$$

Ejercicio 3.0.5 (Ejercicio 18)

$$x\,y' = y + x^2\sec(y/x); \quad y(1) = \pi/2$$

$$x\,y' = y + x^2\sec(y/x) \iff y' = \frac{y + x^2\sec(y/x)}{x} \iff y' = \frac{y}{x} + \frac{x^{\cancel{2}}\sec(y/x)}{\cancel{x}}$$

$$y' = \frac{y}{x} + x\sec(y/x) \iff \frac{dy}{dx} == \frac{y}{x} + x\sec(y/x)$$

$$x\,\frac{dU}{dx} + \cancel{U} = \cancel{U} + x\sec U \iff x\,\frac{dU}{dx} = x\sec U$$

$$\frac{1}{\sec U}\,dU = \frac{x}{x}\,dx \iff \cos U\,dU = dx$$

$$\int \cos U\,dU = \int dx \iff \sin U = x + C \iff \boxed{\sin(y/x) = x + C}$$

Hallar la solución particular $x = 1;\ y = \pi/2$

$$\sin(\pi/2) = 1 + C \iff 1 - 1 = C \iff \boxed{C = 0}$$

Exhibir la solución particular

$$\boxed{sin\,(y/x) = x}$$

Ejercicio 3.0.6 (Ejercicio 19)

$$y' = \tan(x + y) - 1 \qquad (v = x + y)$$

Sustitución

$$\boxed{v = x + y}$$

$$\frac{dv}{dx} = \frac{dx}{dx} + \frac{dy}{dx} \iff \frac{dv}{dx} = 1 + \frac{dy}{dx} \iff v' = y' + 1$$

$$\boxed{y' = v' - 1}$$

$$v' - 1 = \tan v - 1 \iff v' - \cancel{1} = \tan v - \cancel{1} \iff v' = \tan v$$

$$\frac{dv}{dx} = \tan v \iff \frac{dv}{\tan v} = dx \iff \int \cot v \, dv = \int dx$$

$$\ln|\sin v| = x + C \iff \boxed{\ln|\sin(x + y)| = x + C}$$

Capítulo 4

Ecuaciones diferenciales exactas

Ejercicio 4.0.1 (Ejercicio 20)

$$\left(\sin\, y \,+\, y\,\sin\, x \,+\, \frac{1}{x}\right)\, d\,x \,+\, \left(x\,\cos\, y \,-\, \cos\, x \,+\, \frac{1}{y}\right)\, d\,y \,=\, 0$$

Verificar la exactitud.

$$\frac{\partial\, M}{\partial\, y} \,=\, \frac{\partial\, N}{\partial\, x}$$

$$\underbrace{\sin\, y \,+\, y\,\sin\, x \,+\, \frac{1}{x}}_{M\,(x,\,y)}\, d\,x \,+\, \underbrace{x\,\cos\, y \,-\, \cos\, x \,+\, \frac{1}{y}}_{N\,(x,\,y)}\, d\,y \,=\, 0$$

$$\frac{\partial\, M}{\partial\, y} \,=\, \frac{\partial}{\partial\, y}\left(\sin\, y \,+\, y\,\sin\, x \,+\, \frac{1}{x}\right) \,=\, \boxed{\cos\, y \,+\, \sin\, x}$$

$$\frac{\partial\, N}{\partial\, x} \,=\, \frac{\partial}{\partial\, x}\left(x\,\cos\, y \,-\, \cos\, x \,+\, \frac{1}{y}\right) \,=\, \cos\, y \,-\, (-\,\sin\, x) \,=\, \boxed{\cos\, y \,+\, \sin\, x}$$

Hallar la solución

$$\boxed{M\,(x,\, y) \,=\, \frac{\partial\, U}{\partial\, x}} \quad \wedge \quad \boxed{N\,(x,\, y) \,=\, \frac{\partial\, U}{\partial\, y}}$$

$$\boxed{\frac{\partial\, U}{\partial\, x} \,=\, \sin\, y \,+\, y\,\sin\, x \,+\, \frac{1}{x}} \quad \wedge \quad \boxed{\frac{\partial\, U}{\partial\, y} \,=\, x\,\cos\, y \,-\, \cos\, x \,+\, \frac{1}{y}}$$

$$U\,(x,\, y) \,=\, \int \left(x\,\cos\, y \,-\, \cos\, x \,+\, \frac{1}{y}\right)\, d\,y$$

$$U\,(x,\, y) \,=\, x\int \cos\, y\, d\,y \,-\, \cos\, x\int d\,y \,+\, \int \left(\frac{1}{y}\right)\, d\,y$$

$$\boxed{U\,(x,\, y) \,=\, x\,\sin\, y \,-\, y\,\cos\, x \,+\, \ln\, y \,+\, K\,(x)}$$

Eliminar la dependencia de la variable

$$\frac{\partial\, U}{\partial\, x} \,=\, \sin\, y \,-\, y\,(-\,\sin\, x) \,+\, K'\,(x) \quad \Longleftrightarrow \quad \frac{\partial\, U}{\partial\, x} \,=\, \sin\, y \,+\, y\,\sin\, x \,+\, K'\,(x)$$

$$\sin\, y \,+\, y\,\sin\, x \,+\, \frac{1}{x} \,=\, \sin\, y \,+\, y\,\sin\, x \,+\, K'\,(x) \quad \Longleftrightarrow \quad K'\,(x) \,=\, \frac{1}{x}$$

$$K\,(x)\;=\;\int\left(\frac{1}{x}\right)\,d\,x\quad\Longleftrightarrow\quad\boxed{K\,(x)\;=\;\ln\,x\;+\;K}$$

Exhibir la solución

$$U\,(x,\,y)\;=\;x\,\sin\,y\;-\;y\,\cos\,x\;+\;\ln\,y\;+\;\ln\,x\;+\;K$$

$$\boxed{U\,(x,\,y)\;=\;x\,\sin\,y\;-\;y\,\cos\,x\;+\;\ln\,|x\,y|\;+\;K}$$

Ejercicio 4.0.2 (Ejercicio 21)

$$\left(x\;+\;\frac{2}{y}\right)\,d\,y\;+\;y\,d\,x\;=\;0$$

Verificar la exactitud.

$$\frac{\partial\,M}{\partial\,y}\;=\;\frac{\partial\,N}{\partial\,x}$$

$$\underbrace{x\;+\;\frac{2}{y}}_{N\,(x,y)}\,d\,y\;+\;\underbrace{y\,d\,x}_{M\,(x,y)}\;\;d\,x\;=\;0$$

$$\frac{\partial\,M}{\partial\,y}\;=\;\frac{\partial}{\partial\,y}\,(y\,)\;=\;\boxed{1}$$

$$\frac{\partial\,N}{\partial\,x}\;=\;\frac{\partial}{\partial\,x}\left(x\;+\;\frac{2}{y}\right)\;=\boxed{1}$$

Hallar la solución

$$\boxed{M\,(x,\,y)\;=\;\frac{\partial\,U}{\partial\,x}}\qquad\wedge\qquad\boxed{N\,(x,\,y)\;=\;\frac{\partial\,U}{\partial\,y}}$$

$$U\,(x,\,y)\;=\;\int\,(y)\;d\,x$$

$$U\,(x,\,y)\;=\;y\int\;d\,x\Longleftrightarrow U\,(x,\,y)\;=\;y\int\;d\,x$$

$$\boxed{U\,(x,\,y)\;=\;yx+K(y)}$$

Eliminar la dependencia de la variable

$$\frac{\partial\,U}{\partial\,y}\;=\;x\;+\;K'\,(y)$$

$$x\;+\;\frac{2}{y}\;=\;x\;+\;K'\,(y)$$

$$\cancel{x}+\;\frac{2}{y}\;=\cancel{x}+\;K'\,(y)$$

$$K\,(y)=\int\frac{2}{y}\,d\,y$$

$$K(y) = 2\ln y$$

$$K(y) = \ln y^2$$

Exhibir la solución

$$\boxed{U(x,\,y) \;=\; yx \;+ \ln\, y^2 \;+\; K}$$

Ejercicio 4.0.3 (Ejercicio 22)

$$\left(4\,x^3\,y^3 \;+\; \frac{1}{x}\right)\,d\,x \;+\; \left(3\,x^4\,y^2 \;-\; \frac{1}{y}\right)\,d\,y \;=\; 0; \qquad x \;=\; e,\; y \;=\; 1$$

Verificar la exactitud.

$$\frac{\partial\,M}{\partial\,y} \;=\; \frac{\partial\,N}{\partial\,x}$$

$$\frac{\partial\,M}{\partial\,y} \;=\; \frac{\partial}{\partial\,y}\left(4\,x^3\,y^3 \;+\; \frac{1}{x}\right) \;=\; \boxed{12\,x^3\,y^2}$$

$$\frac{\partial\,N}{\partial\,x} \;=\; \frac{\partial}{\partial\,x}\left(3\,x^4\,y^2 \;-\; \frac{1}{y}\right) \;=\; \boxed{12\,x^3 y^2}$$

Hallar la solución

$$\boxed{M(x,\,y) \;=\; \frac{\partial\,U}{\partial\,x}} \qquad \wedge \qquad \boxed{N(x,\,y) \;=\; \frac{\partial\,U}{\partial\,y}}$$

$$U(x,\,y) \;=\; \int\left(4\,x^3\,y^3 \;+\; \frac{1}{x}\right)\,d\,x$$

$$U(x,\,y) \;=\; \int\,(4\,x^3\,y^3)\;d\,x \;+\; \int\left(\frac{1}{x}\right)\,d\,x$$

$$U(x,\,y) \;=\; 4\,y^3 \int\,(x^3)\;d\,x \;+\; \int\left(\frac{1}{x}\right)\,d\,x$$

$$U(x,\,y) \;=\; \cancel{4}\,y^3\,\frac{x^4}{\cancel{4}} \;+\; \ln\,x \;+\; K(y) \quad\Longleftrightarrow\quad \boxed{U(x,\,y) \;=\; x^4\,y^3 \;+\; \ln\,x \;+\; K(y)}$$

Eliminar la dependencia

$$\frac{\partial\,U}{\partial\,y} \;=\; 3\,x^4\,y^2 \;+\; K'(y) \quad\Longleftrightarrow\quad 3\,x^4\,y^2 \;-\; \frac{1}{y} \;=\; 3\,x^4\,y^2 \;+\; K'(y)$$

$$K'(y) \;=\; -\frac{1}{y} \quad\Longleftrightarrow\quad K(y) \;=\; -\int\frac{1}{y}\,d\,y \quad\Longleftrightarrow\quad \boxed{K(y) \;=\; -\,\ln\,y \;+\; K}$$

Exhibir la solución general

$$U(x,\,y) \;=\; x^4\,y^3 \;+\; \ln\,x \;-\; \ln\,y \;+\; K \quad\Longleftrightarrow\quad \boxed{U(x,\,y) \;=\; x^4\,y^3 \;+\; \ln\left(\frac{x}{y}\right) \;+\; K}$$

Solución particular sujeta al problema de condiciones iniciales $\boxed{x = e \wedge y = 1}$

$$U = e^4 (1)^3 + \ln\left(\frac{e}{1}\right) + K \iff U - K = e^4 + \ln e^1$$

$$\underbrace{U - K}_{\beta} = e^4 + 1$$

$$\boxed{x^4 y^3 + \ln\left(\frac{x}{y}\right) = e^4 + 1}$$

Ejercicio 4.0.4 (Ejercicio 23)

$$\underbrace{(3\,y)}_{M\,(x,y)}\;d\,x + \underbrace{(2\,x)}_{N\,(x,y)}\;d\,y = 0; \qquad x = -1,\, y = 1,4, \quad F = x^2\,y$$

Verificar la exactitud.

$$\frac{\partial\,M}{\partial\,y} = \frac{\partial\,N}{\partial\,x}$$

$$\frac{\partial\,M}{\partial\,y} = \frac{\partial}{\partial\,y}\,(3\,y) = 3$$

$$\frac{\partial\,N}{\partial\,x} = \frac{\partial}{\partial\,x} = (2\,x) = 2$$

Como la ecuación diferencial NO es exacta, multiplique la misma por $F = x^2\,y$

$$x^2\,y\;((3\,y)\,d\,x + (2\,x)\,d\,y) = 0$$

$$\underbrace{(3\,x^2\,y^2\,)}_{M\,(x,y)}\,d\,x + \underbrace{(2\,x^3\,y)}_{N\,(x,y)}\,d\,y$$

$$\frac{\partial\,M}{\partial\,y} = \frac{\partial}{\partial\,y}\,(3\,x^2\,y^2) = 6\,x^2\,y$$

$$\frac{\partial\,N}{\partial\,x} = \frac{\partial}{\partial\,x} = (2\,x^3\,y) = 6\,x^2\,y$$

Hallar la solución

$$\boxed{M\,(x,\,y) = \frac{\partial\,U}{\partial\,x}} \quad \wedge \quad \boxed{N\,(x,\,y) = \frac{\partial\,U}{\partial\,y}}$$

$$U\,(x,\,y) = \int (3\,x^2\,y^2)\;d\,x$$

$$U\,(x,\,y) = 3\,y^2 \int (\,x^2\,)\;d\,x$$

$$\boxed{U\,(x,\,y) = x^3\;y^2 + K(y)}$$

$$\frac{\partial\,U}{\partial\,y} = 2\,x^3\,y + K'\,(y)$$

$$2\,x^3\,y\;=\;2\,x^3\,y+K'\,(y)$$

$$K\,(y)\;=0$$

Exhibir la solución general

$$\boxed{U\,(x,\,y)\;=x^3\;y^2\;+\;K}$$

Exhibir la solución particular sujeta a la condición inicial $x\;=\;-1,\;y\;=\;1,4$

$$\beta\;=\;-\,1,96$$

Exhibir la solución particular

$$\boxed{x^3\;y^2\;=\;-\,1,96}$$

Ejercicio 4.0.5 (Ejercicio 24)

$$\underbrace{(2\,x^{-1}\,y\;-\;3)}_{M\,(x,y)}\;d\,x\;+\;\underbrace{(3\;-\;2\,y^{-1}x)}_{N\,(x,y)}\;d\,y\;=\;0;\qquad x\;=\;1,\;y\;=\;-\,1,\qquad F\;=\;x^2\,y^2$$

Verificar la exactitud.

$$\frac{\partial\,M}{\partial\,y}\;=\;\frac{\partial\,N}{\partial\,x}$$

$$\frac{\partial\,M}{\partial\,y}\;=\;\frac{\partial}{\partial\,y}\,(2\,x^{-1}\,y\;-\;3)\;=\;2\,x^{-1}$$

$$\frac{\partial\,N}{\partial\,x}\;=\;\frac{\partial}{\partial\,x}\;=\;(3\;-\;2\,y^{-1}x)\;=\;-\,2\,y^{-1}$$

Como la ecuación diferencial NO es exacta, multiplique la misma por $F\;=\;x^2\,y^2$

$$x^2\,y^2\;\left((2\,x^{-1}\,y\;-\;3)\,d\,x\;+\;(3\;-\;2\,y^{-1}x)\,d\,y\right)\;=\;0$$

$$\underbrace{(2\,x\,y^3\;-\;3\,x^2\,y^2)}_{M\,(x,y)}\;d\,x\;+\;\underbrace{(3\,x^2\,y^2\;-\;2\,x^3\,y)}_{N\,(x,y)}\;d\,y$$

$$\frac{\partial\,M}{\partial\,y}\;=\;\frac{\partial}{\partial\,y}\,(2\,x\,y^3\;-\;3\,x^2\,y^2)\;=\;\boxed{6\,x\,y^2\;-\;6\,x^2\,y}$$

$$\frac{\partial\,N}{\partial\,x}\;=\;\frac{\partial}{\partial\,x}\,(3\,x^2\,y^2\;-\;2\,x^3\,y)\;=\;\boxed{6\,x\,y^2\;-\;6\,x^2\,y}$$

Hallar la solución

$$\boxed{M\,(x,\,y)\;=\;\frac{\partial\,U}{\partial\,x}}\qquad\wedge\qquad\boxed{N\,(x,\,y)\;=\;\frac{\partial\,U}{\partial\,y}}$$

$$\boxed{\frac{\partial\,U}{\partial\,x}\;=\;2\,x\,y^3\;-\;3\,x^2\,y^2}\qquad\wedge\qquad\boxed{\frac{\partial\,U}{\partial\,y}\;=\;3\,x^2\,y^2\;-\;2\,x^3\,y}$$

$$U\,(x,\,y)\;=\;\int\;\left(2\,x\,y^3\;-\;3\,x^2\,y^2\right)\;d\,x$$

$$U\,(x,\,y) \,=\, 2\,y^3 \int x\,d\,x \,-\, 3\,y^2 \int x^2\,d\,x$$

$$U\,(x,\,y) \,=\, \cancel{2}\,y^3\,\frac{x^2}{\cancel{2}} \,-\, \cancel{3}\,y^2\,\frac{x^3}{\cancel{3}} \,+\, K(y) \quad \Longleftrightarrow \quad \boxed{U\,(x,\,y) \,=\, x^2\,y^3 \,-\, x^3\,y^2 \,+\, K\,(y)}$$

Eliminar la dependencia de la variable

$$\frac{\partial\,U}{\partial\,y} \,=\, 3\,x^2\,y^2 \,-\, 2\,x^3\,y \,+\, K'\,(y)$$

$$3\,x^2\,y^2 \,-\, 2\,x^3\,y \,=\, 3\,x^2\,y^2 \,-\, 2\,x^3\,y \,+\, K'\,(y) \quad \Longleftrightarrow \quad K'\,(y) \,=\, 0 \quad \Longleftrightarrow \quad K\,(y) \,=\, K$$

Exhibir la solución general

$$\boxed{U\,(x,\,y) \,=\, x^2\,y^3 \,-\, x^3\,y^2 \,+\, K}$$

Exhibir la solución particular sujeta a la condición inicial $x \,=\, 1,\, y \,=\, -\,1$

$$U \,=\, (1)^2\,(-\,1)^3 \,-\, (1)^3\,(-\,1)^2 \,+\, K \quad \Longleftrightarrow \quad \underbrace{U \,-\, K}_{\beta} \,=\, -\,1 \,-\, 1 \quad \Longleftrightarrow \quad \beta \,=\, -\,2$$

Exhibir la solución particular

$$\boxed{x^2\,y^3 \,-\, x^3\,y^2 \,=\, -\,2}$$

Capítulo 5

Ecuaciones diferenciales lineales y factores integrantes

Ejercicio 5.0.1 (Ejercicio 25)

$$x\,\frac{d\,y}{d\,x} - 3\,y = x^4$$

$$x\,y' - 3\,y = x^4 \iff \frac{x\,y'}{x} - \frac{3\,y}{x} = \frac{x^4}{x} \iff \boxed{y' - \underbrace{\frac{3}{x}}_{P\,(x)}\,y = x^3}$$

$$\mu\,(x) = exp\left(\int P\,(x)\,dx\right)$$

$$\int\left(-\frac{3}{x}\right)\,d\,x \iff -\,3\int\left(\frac{1}{x}\right)\,d\,x \iff -\,3\,\ln\,x \iff \boxed{\ln\,x^{-\,3}}$$

$$\mu\,(x) = e^{\ln\,x^{-\,3}} \iff \boxed{\mu\,(x) = x^{-\,3} = \frac{1}{x^3}}$$

$$\frac{1}{x^3}\left(y' - \frac{3}{x}\,y\right) = \frac{1}{\cancel{x^3}}(\cancel{x^3}) \iff \frac{y'}{x^3} - \frac{3}{x^4}\,y = 1 \iff \left(\frac{1}{x^3}\,y\right)' = 1$$

$$\int\left(\frac{1}{x^3}\,y\right)' = \int d\,x \iff \frac{1}{x^3}\,y = x + C$$

$$\boxed{y\,(x) = x^4 + C\,x^3}$$

Ejercicio 5.0.2 (Ejercicio 26)

$$x^2\,y' + 2\,x\,y - x + 1 = 0; \quad y\,(1) = 0$$

$$x^2\,y' + 2\,x\,y = x - 1$$

$$\frac{x^2\,y'}{x^2} + \frac{2\,x\,y}{x^2} = \frac{x - 1}{x^2} \iff \boxed{y' + \underbrace{\frac{2}{x}}_{P\,(x)}\,y = \frac{1}{x} - \frac{1}{x^2}}$$

$$\mu\,(x) = exp\left(\int P\,(x)\,dx\right)$$

$$\int \left(\frac{2}{x}\right) d\,x \;=\; 2 \int \left(\frac{1}{x}\right) \;=\; 2\,\ln\,x \;=\; \boxed{\ln\,x^2}$$

$$\mu\,(x) \;=\; e^{\ln\,x^2} \qquad \Longleftrightarrow \qquad \boxed{\mu\,(x) \;=\; x^2}$$

$$x^2\,\left(y' \,+\, \frac{2}{x}\,y\right) \;=\; x^2\,\left(\frac{1}{x} \,-\, \frac{1}{x^2}\right)$$

$$x^2\,y' \,+\, 2\,x\,y \;=\; x \,-\, 1 \qquad \Longleftrightarrow \qquad \left(x^2\,y\right)' \;=\; x \,-\, 1$$

$$\int \left(x^2\,y\right)' \;=\; \int x\,d\,x \,-\, \int d\,x \qquad \Longleftrightarrow \qquad x^2\,y \;=\; \frac{x^2}{2} \,-\, x \,+\, C$$

$$\boxed{y\,(x) \;=\; \frac{1}{2} \,-\, \frac{1}{x} \,+\, \frac{C}{x^2}}$$

Exhibir la solución particular sujeta a la condición inicial $x \,=\, 1,\; y \,=\, 0$

$$y\,(1) \;=\; \frac{1}{2} \,-\, \frac{1}{1} \,+\, \frac{C}{(1)^2} \qquad \Longleftrightarrow \qquad \frac{1}{2} \,-\, 1 \,+\, C \;=\; 0$$

$$-\,\frac{1}{2} \,+\, C \;=\; 0 \qquad \Longleftrightarrow \qquad \boxed{C \;=\; \frac{1}{2}}$$

$$\boxed{y\,(x) \;=\; \frac{1}{2} \,-\, \frac{1}{x} \,+\, \frac{1}{2\,x^2}}$$

Ejercicio 5.0.3 (Ejercicio 27)

$$y' \,-\, y\,\underbrace{\cot\,x}_{P\,(x)} \;=\; 2\,x \,-\, x^2\,\cot x; \qquad y\,\left(\frac{\pi}{2}\right) \;=\; \frac{1}{4}\,\pi^2 \,+\, 1$$

¡CUMPLIDO!

$$\mu\,(x) \;=\; \exp\left(\int P\,(x)\,d\,x\right) \quad \Longleftrightarrow \quad \mu\,(x) \;=\; e^{\left(\ln\,|\sin\,x|^{-\,1}\right)} \;\Longleftrightarrow\; \boxed{\mu\,(x) \;=\; (\sin\,x)^{-\,1} \;=\; \csc\,x}$$

$$\int \,-\,\cot\,x\,d\,x \;=\; -\,\int \cot\,x\,d\,x \;=\; -\,\ln\,|\sin\,x| \;=\; \boxed{\ln\,|\sin\,x|^{-\,1}}$$

$$\csc\,x\,\left(y' \,-\, y\,\cot\,x\right) \;=\; \csc\,x\,(2\,x \,-\, x^2\,\cot\,x)$$

$$(\csc\,x)\,y' \,-\, y\,\csc\,x\,\cot\,x \;=\; 2\,x\,\csc\,x \,-\, x^2\,\csc\,x\,\cot\,x$$

$$(y\,\csc\,x)' \;=\; 2\,x\,\csc\,x \,-\, x^2\,\csc\,x\,\cot\,x$$

$$\int (y\,\csc\,x)' \;=\; \int \left(2\,x\,\csc\,x\right)\,d\,x \,-\, \int \left(x^2\,\csc\,x\,\cot\,x\right)\,d\,x$$

$$y\,\csc\,x \;=\; \underbrace{\int \left(2\,x\,\csc\,x\right)\,d\,x}_{I_1} \,-\, \underbrace{\int \left(x^2\,\csc\,x\,\cot\,x\right)\,d\,x}_{I_2}$$

I_2

$$\int \left(x^2 \csc x \cot x\right) dx$$

ILATE

$$\boxed{U = x^2 \implies dV = \csc x \cot x \, dx}$$

$$\boxed{dU = 2\,x\,dx \implies V = -\csc x}$$

Un dia Vi Una vaca Vestida de Uniforme

$$\int U\,dV = U\,V - \int V\,dU$$

$$\int \left(x^2 \csc x \cot x\right) dx = -x^2 \csc x - \int \left(2\,x\left(-\csc x\right)\right) dx$$

$$\boxed{\int \left(x^2 \csc x \cot x\right) dx = -x^2 \csc x + \int \left(2\,x \csc x\right) dx}$$

Sustituyendo

$$y \csc x = \int \left(2\,x \csc x\right) dx - \left(-x^2 \csc x + \int \left(2\,x \csc x\right) dx\right)$$

$$y \csc x = \int \left(2\,x \csc x\right) dx + x^2 \csc x - \int \left(2\,x \csc x\right) dx$$

$$y \csc x = \cancel{\int \left(2\,x \csc x\right) dx} + x^2 \csc x - \cancel{\int \left(2\,x \csc x\right) dx} + C$$

$$y \csc x = x^2 \csc x + C \iff y\left(x\right) = \frac{x^2 \csc x}{\csc x} + \frac{C}{\csc x}$$

$$\boxed{y\left(x\right) = x^2 + C \sin x}$$

Hallar la solución particular

$$y\left(\frac{\pi}{2}\right) = \left(\frac{\pi}{2}\right)^2 + C \sin\left(\frac{\pi}{2}\right) \iff \cancel{\frac{\pi^2}{4}} + C = \cancel{\frac{\pi^2}{4}} + 1 \iff \boxed{C = 1}$$

$$\boxed{y\left(x\right) = x^2 + \sin x}$$

Ejercicio 5.0.4 (Ejercicio 28)

$$y' + y = 2\,x\,e^{-x} + x^2$$

¡CUMPLIDO!

$$P\left(x\right) = 1$$

$$\mu\left(x\right) = exp\left(\int P\left(x\right) dx\right)$$

$$\int d\,x \quad \Longleftrightarrow \quad \boxed{x}$$

$$\boxed{\mu\,(x) \;=\; e^{x}}$$

$$e^{x}\,(y'+y) \;=\; e^{x}\,(2\,x\,e^{-\,x}+x^{2}) \quad \Longleftrightarrow \quad e^{x}\,y'+e^{x}\,y \;=\; 2\,x+e^{x}\,x^{2} \quad \Longleftrightarrow \quad (e^{x}\,y)' \;=\; 2\,x+x^{2}\,e^{x}$$

$$\int (e^{x}\,y)' \;=\; \int (2\,x \;+\; x^{2}\,e^{x})\,d\,x \quad \Longleftrightarrow \quad \int (e^{x}\,y)' \;=\; \int 2\,x\,d\,x \;+\; \underbrace{\int x^{2}\,e^{x}\,d\,x}_{I_1}$$

I_1

$$\int x^{2}\,e^{x}\,d\,x$$

$$\boxed{U \;=\; x^{2} \quad \Longrightarrow \quad d\,V \;=\; e^{x}}$$

$$\boxed{d\,U \;=\; 2\,x\,d\,x \quad \Longrightarrow \quad V \;=\; e^{x}}$$

$$\int U\,d\,V \;=\; U\,V \;-\; \int V\,d\,U$$

$$\int x^{2}\,e^{x}\,d\,x \;=\; x^{2}\,e^{x} \;-\; 2\,\underbrace{\int x\,e^{x}\,d\,x}_{I_{1,1}}$$

$$\int x^{2}\,e^{x}\,d\,x \;=\; x^{2}\,e^{x} \;-\; 2\,(x\,e^{x} \;-\; e^{x})$$

$I_{1,1}$

$$\int x\,e^{x}\,d\,x$$

$$\boxed{U \;=\; x \quad \Longrightarrow \quad d\,V \;=\; e^{x}}$$

$$\boxed{d\,U \;=\; d\,x \quad \Longrightarrow \quad V \;=\; e^{x}}$$

$$\int x\,e^{x}\,d\,x \;=\; x\,e^{x} \;-\; \int e^{x}\,d\,x$$

$$\int x\,e^{x}\,d\,x \;=\; x\,e^{x} \;-\; e^{x}$$

$$e^{x}\,y \;=\; x^{2} \;+\; x^{2}\,e^{x} \;-\; 2\,(x\,e^{x} \;-\; e^{x}) \;+\; C \quad \Longleftrightarrow \quad e^{x}\,y \;=\; x^{2} \;+\; x^{2}\,e^{x} \;-\; 2\,x\,e^{x} \;+\; 2\,e^{x} \;+\; C$$

$$y = \frac{x^2 + x^2\, e^x - 2\,x\,e^x + 2\,e^x + C}{e^x} \iff \boxed{y\,(x) = x^2\, e^{-x} + x^2 - 2\,x + 2 + C\,e^{-x}}$$

Ejercicio 5.0.5 (Ejercicio 29)

$$x\,y' = y + x^2 \sin x \iff \boxed{x\,y' - y = x^2 \sin x}$$

$$\frac{x}{x}\,y' - \frac{1}{x}\,y = \frac{x^2 \sin x}{x} \iff y' - \underbrace{\frac{1}{x}}_{P\,(x)}\,y = x \sin x$$

$$\mu\,(x) = \exp\left(\int P\,(x)\,dx\right) \iff \mu\,(x) = e^{(\ln x^{-1})} \iff \boxed{\mu\,(x) = x^{-1} = \frac{1}{x}}$$

$$\int P\,(x)\,d\,x = \int \left(-\frac{1}{x}\right)\,d\,x = -\int \frac{1}{x}\,d\,x = -\ln x = \boxed{\ln x^{-1}}$$

$$\frac{1}{x}\left(y' - \frac{1}{x}\,y\right) = \frac{1}{x}\,(x \sin x) \iff \frac{1}{x}\,y' - \frac{1}{x^2}\,y = \sin x$$

$$\left(\frac{1}{x}\,y\right)' = \sin x$$

$$\int \left(\frac{1}{x}\,y\right)' = \int \sin x\,d\,x \iff \frac{1}{x}\,y = -\cos x + C \iff \boxed{y\,(x) = -x \cos x + C\,x}$$

Prueba

$$y\,(x) = -x \cos x + C\,x$$

$$y'\,(x) = -\cos x + (-x\,(-\sin x)) + C \iff \boxed{y'\,(x) = -\cos x + x \sin x + C}$$

Sustituyendo en la ecuación diferencial

$$x\,y' = y + x^2 \sin x$$

$$x\,(-\cos x + x \sin x + C) = -x \cos x + C\,x + x^2 \sin x$$

$$\boxed{-x \cos x + x^2 \sin x + C\,x = -x \cos x + C\,x + x^2 \sin x}$$

Capítulo 6

Aplicaciones diversas

Ejercicio 6.0.1 (Ejercicio 30)

Ejercicio 6.0.2 (Ejercicio 31)

Ejercicio 6.0.3 (Ejercicio 32) *Suponga que un país tiene una variación en la población $P(t)$ dado por*

$$\boxed{\frac{dP}{dt} = \frac{dB}{dt} - \frac{dD}{dt}}$$

Donde dB/dt y dD/dt son las tasas de natalidad y mortalidad respectivamente.

Determine $P(t)$ si $dB/dt = k_1 P$ y $dD/dt = k_2 P$.

$$\frac{dP}{dt} = k_1 P - k_2 P \iff \frac{dP}{dt} = (k_1 - k_2) P$$

$$\frac{dP}{P} = (k_1 - k_2)\, dt \iff \int \frac{dP}{P} = (k_1 - k_2) \int dt$$

$$\ln P = (k_1 - k_2)\, t + C \iff e^{\ln P} = e^{((k_1 - k_2)\, t + C)}$$

$$e^{\ln P} = e^{(k_1 - k_2)\, t} \cdot \underbrace{e^C}_{P_0} \iff \boxed{P(t) = P_0\, e^{(k_1 - k_2)\, t}}$$

✍ *Primer caso:* $\boxed{k_1 > k_2}$

$$P(t) = P_0\, e^{(k_1 - k_2)\, t} \implies P(t) \longrightarrow \infty$$

✍ *Segundo caso:* $\boxed{k_1 = k_2}$

$$P(t) = P_0\, e^{(k_1 - k_2)\, t} \implies P(t) = P_0$$

✍ *Tercer caso:* $\boxed{k_1 < k_2}$

$$P(t) = P_0\, e^{-k\, t} \iff P(t) = P_0 \frac{1}{e^{k\, t}} \iff \boxed{P(t) = P_0 \lim_{t \to \infty} \left(\frac{1}{e^{k\, t}}\right) \longrightarrow 0}$$

Ejercicio 6.0.4 (Ejercicio 33)

Suponga que un cuerpo se mueve a través de un medio con resistencia proporcional a su velocidad v, de tal manera que $d\,v/d\,t\;=\;-K\,v$. Muestre que su velocidad y posición en el tiempo t están dadas por

$$v\,(t)\;=\;v_0\,e^{-K\,t}\quad y$$

$$x\,(t)\;=\;x_0\;+\;\frac{v_0}{K}\,(1\;-\;e^{-k\,t})$$

Demostrando la velocidad

$$\frac{d\,v}{d\,t}\;=\;-\,K\,v\quad\Longleftrightarrow\quad\frac{d\,v}{v}\;=\;-\,K\,d\,t$$

$$\int\frac{1}{v}\,d\,v\;=\;-\,K\int d\,t\quad\Longleftrightarrow\quad\ln\,v=\;-\,K\,t\;+\;C$$

$$e^{\ln\,v}\;=\;e^{-\,K\,t+C}\quad\Longleftrightarrow\quad v\;=\;e^{-\,K\,t+C}\quad\Longleftrightarrow\quad v\;=\;e^{-\,K\,t}\,e^{C}\quad\Longleftrightarrow\quad v\;=\;e^{-\,K\,t}\;\underbrace{\hat{C}}_{v_0}$$

$$\boxed{v\,(t)\;=\;v_0\,e^{-\,K\,t}}$$

Demostrando la posición recordando que $v\;=\;d\,x/d\,t$

$$\frac{d\,x}{d\,t}\;=\;v_0\,e^{-\,K\,t}\quad\Longleftrightarrow\quad d\,x\;=\;(v_0\,e^{-\,K\,t})\,d\,t$$

$$\int d\,x\;=\;\int(v_0\,e^{-\,K\,t})\,d\,t\quad\Longleftrightarrow\quad\int d\,x\;=\;v_0\underbrace{\int e^{-\,K\,t}\,d\,t}_{I_1}$$

I_1

$$\int e^{-\,K\,t}\,d\,t$$

$$u\;=\;-\,K\,t\quad\wedge\quad\frac{d\,u}{d\,t}\;=\;-\,K\;\to\;d\,t\;=\;-\,\frac{1}{K}\,d\,u$$

$$-\,\frac{1}{K}\int e^{u}\,d\,u\quad\Longleftrightarrow\quad-\,\frac{1}{K}\,e^{u}\;+\;C\quad\Longleftrightarrow\quad-\;\frac{e^{-\,K\,t}}{K}\;+\;C$$

$$x\;=\;v_0\,(-\,\frac{e^{-\,K\,t}}{K})\;+\;\underbrace{C}_{x_0}\quad\Longleftrightarrow\quad x\;=\;x_0\;-\;\frac{v_0e^{-\,K\,t}}{K}\Longleftrightarrow x\;=\;x_0\;+\;\frac{v_0\,\left(v_0\;-\;e^{-\,K\,t}\right)}{K}$$

$$\boxed{x\,(t)\;=\;x_0\;+\;\frac{v_0}{K}\,(1\;-\;e^{-\,K\,t})}$$

Ejercicio 6.0.5 (Ejercicio 34)

Un cuerpo inicialmente en reposo en x_0 se mueve en línea recta bajo la acción de una fuerza $\vec{F} = -k/x^2$. Demostrar que su velocidad en x es $v^2 = 2(k/m)(1/x - 1/x_0)$.

$$\vec{F} = -k/x^2 \iff m\,\frac{d\vec{V}}{dt} = -\frac{k}{x^2} \iff \frac{d\vec{V}}{dt} = -\frac{k}{m}\,\frac{1}{x^2}$$

$$\int \left(\frac{d\vec{V}}{dt}\right) dx = -\frac{k}{m}\int x^{-2}\,dx \iff \int \vec{V}\,d\vec{V} = -\frac{k}{m}\int_{x_0}^{x} x^{-2}\,dx$$

$$\frac{V^2}{2} = -\frac{k}{m}\left(\frac{x^{-2+1}}{-2+1}\Bigg|_{x_0}^{x}\right)$$

$$\frac{V^2}{2} = -\frac{k}{m}\left(\frac{x^{-1}}{-1}\Bigg|_{x_0}^{x}\right) \iff \frac{V^2}{2} = -\frac{k}{m}\left(-\frac{1}{x}\Bigg|_{x_0}^{x}\right)$$

$$\frac{V^2}{2} = \frac{k}{m}\left(\frac{1}{x}\Bigg|_{x_0}^{x}\right) \iff \frac{V^2}{2} = \frac{k}{m}\left(\frac{1}{x} - \frac{1}{x_0}\right)$$

$$\boxed{V^2 = \frac{2k}{m}\left(\frac{1}{x} - \frac{1}{x_0}\right)}$$

Ejercicio 6.0.6 (Ejercicio 35)

La población de un pueblo crece con una razón proporcional a la población en el tiempo t. La población inicial de 500 aumenta 15 % en 10 años. ¿Cuál será la población pasados 30 años?

$P_0 = 500$; $\beta = 0,15$ *en* $t_1 = 10$; $t_2 = 30$

Hallando el incremento de un año

$$\frac{\beta}{t_1} = \frac{0,15}{10} = \underbrace{0,015}_{\alpha}$$

Hallando la poblacion 30 años

$$P(t) = P_0\,e^{\alpha\,t_2} \iff P(30) = 500\,e^{(0{,}015\,\times\,30)} \iff P(30) = 500\,e^{0{,}45}$$

$$\boxed{P(30) \approx 784}$$

Este documento fue levantado en LaTeX.