1.4 Nested Quantifiers

Translating from Nested Quantifiers into English

Expressions with nested quantifiers can be complicated. The first step in translating such an expression is to write out what the quantifiers and predicates in the expression mean.

Example 9: Translate the statement

$\forall x\, (C(x) \vee \exists y (C(y) \wedge F(x,y)))$

into English, where C(x) is "x has a computer", F(x,y) is "x and y are friends", and the domain for both x and y consists of all students in the school.

Solution: Every student in the school has a computer or has friend who has a computer.

Example 10: Translate the statement

$\exists x\, \forall y \forall z\, ((F(x,y) \wedge F(x,z) \wedge (y \neq z)) \rightarrow \neg F(y,z))$

into English, where F(a, b) means a and b are friends and the domain for x and y and z consists of all students in school.

Solution: There is a student none of whose friends are also friends with each other.

Translating English Sentences into Logical Expressions

Example 11: Express the statement "If a person is female and is a parent, then this person is someone's mother" as a logical expression involving logical expression involving predicates, quantifiers with a domain consisting of all people and logical connectives.

Solution: equivalently "For every person x, if x is female and a parent, then there is a person y such that x is the mother for this person y." So let F(x): x is female; P(x): x is parent; M(x, y): x is mother of y. Then the statement is:

$\forall x\, ((F(x) \wedge P(x)) \rightarrow \exists y\, M(x, y))$.

Example 12: Express the statement "Everyone has exactly one best friend" as a logical expression involving logical expression involving predicates, quantifiers with a domain consisting of all people and logical connectives.

Solution: equivalently "For every person x, x has exactly one best friend." Attention "exactly one"! . Let B(x, y): y is the best friend of x. So if a person z is not the person y, then z is not the best friend of x. Thus

$$\forall x\exists y(B(x, y)\wedge(z\neq y)\rightarrow\neg B(x, z)).$$

With the uniqueness quantifier $\exists!$

$$\forall x\exists! y B(x, y).$$

Example 13: Use quantifiers to express the statement "There is a woman who has taken a flight on every airline in the world."

Solution: domain for w is all women, for f is all flights, for a is all airlines. Let T(w, f): w has taken f; Q(f, a): f is a flight on a. Then

$$\exists w\forall a\exists f(T(w, f)\wedge Q(f, a)).$$

Or

$$\exists w\forall a\exists f\, R(w, f, a)$$

where R(w, f, a) is "w taken f on a."

Negating Nested Quantifiers

Example 14: Express the negation of the statement $\forall x\exists y\ (xy=1)$ so that no negation precedes a quantifier.

Solution: apply the negation rule step by step.

$$\neg\forall x\exists y\ (xy=1) \equiv\exists x\neg\exists y\ (xy=1) \equiv\exists x\ \forall y\ \neg\ (xy=1) \equiv\exists x\ \forall y\ (xy\neq 1)$$

Example 15: Express the negation of the statement of the Example 13.

Solution:

$$\neg\exists w\forall a\exists f(T(w, f)\wedge Q(f, a)) \text{ is trivial.}$$

Example 16: Express the negation of the statement of the Example 8.

Solution: If $D=[0, +\infty)$ for ε and δ, D=R for x, then

$\forall\varepsilon\exists\delta\forall x(0<|x-a|<\delta\rightarrow|f(x)-L| <\varepsilon)$.