Nested Summations

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Abstract

As it happens, Fibonacci numbers can be expressed as finitely nested finite sums. And likewise, numbers in Fibonacci sequences that start with natural numbers other than 0 and 1, e.g., Lucas numbers, can be expressed as finitely nested finite sums as well. This article also shows how to also express Jacobstahl numbers as finitely nested finite sums. The construction scales to Jacobstahl sequences with a core multiplicative factor other than the standard one. All of this ought to keep Jens entertained (or at least busy) in one of his legendary proof sessions with his students.

CCS Concepts: • **Theory of computation** → **Automated reasoning**; *Logic and verification*; *Programming logic*; • **Mathematics of computing** → **Discrete mathematics**; **Number-theoretic computations**; • **Computing methodologies** → **Number theory algorithms**.

Keywords: finitely nested finite sums, Fibonacci numbers, Lucas numbers, Jacobstahl numbers, Jacobstahl-Lucas numbers, the Coq Proof Assistant, the On-Line Encyclopedia of Integer Sequences

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1 Introduction

A beginning is the time for taking the most delicate care that the balances are correct.
– Frank Herbert

Definition 1.1 (Standard Fibonacci sequence (A000045 = [0, 1, 1, 2, 3, 5, 8, 13, $\cdots$[)). The standard Fibonacci sequence F^0 is such that $F^0_0 = 0$, $F^0_1 = 1$, and $\forall n \in \mathbb{N}, F^0_{n+2} = F^0_{n+1} + F^0_n$. Its members are called *Fibonacci numbers.*

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Fun fact: Given a function $f \in \{0,1\} \rightarrow \mathbb{N}$ such that $f(0) = 0$ and $f(1) = 1$, Fibonacci numbers can be expressed as nested sums, using two base cases and an induction step with two induction hypotheses:

- $\sum_{i_1=0}^{1} f(1-i_1) = f(1-0) + f(1-1) = f(1) + f(0) = 1 + 0 = F^0_1 + F^0_0 = F^0_2$
- $\sum_{i_2=0}^{1} \sum_{i_1=0}^{1-i_2} f(1-i_1) = \sum_{i_1=0}^{1} f(1-i_1) + \sum_{i_1=0}^{0} f(1-i_1) = F^0_2 + f(1) = F^0_2 + 1 = F^0_2 + F^0_1 = F^0_3$
- $\forall n \in \mathbb{N}, \sum_{i_{n+1}=0}^{1} \sum_{i_n=0}^{1-i_{n+1}} \cdots \sum_{i_1=0}^{1-i_2} f(1-i_1) = F^0_{n+2} \;\wedge\; \sum_{i_{n+2}=0}^{1} \sum_{i_{n+1}=0}^{1-i_{n+2}} \cdots \sum_{i_1=0}^{1-i_2} f(1-i_1) = F^0_{n+3} \;\Rightarrow\; \sum_{i_{n+3}=0}^{1} \sum_{i_{n+2}=0}^{1-i_{n+3}} \sum_{i_{n+1}=0}^{1-i_{n+2}} \cdots \sum_{i_1=0}^{1-i_2} f(1-i_1) = \sum_{i_{n+2}=0}^{1} \sum_{i_{n+1}=0}^{1-i_{n+2}} \cdots \sum_{i_1=0}^{1-i_2} f(1-i_1) + \sum_{i_{n+2}=0}^{0} \sum_{i_{n+1}=0}^{1-i_{n+2}} \cdots \sum_{i_1=0}^{1-i_2} f(1-i_1) = F^0_{n+3} + \sum_{i_{n+1}=0}^{1} \cdots \sum_{i_1=0}^{1-i_2} f(1-i_1) = F^0_{n+3} + F^0_{n+2} = F^0_{n+4}$

This inductive construction is the proof of the following theorem:

Theorem 1.2 (Fibonacci numbers as nested sums). *Given $f \in \{0,1\} \rightarrow \mathbb{N}$ such that $f(0) = 0$ and $f(1) = 1$,*

$$\forall n \in \mathbb{N}, \sum_{i_{n+1}=0}^{1} \sum_{i_n=0}^{1-i_{n+1}} \cdots \sum_{i_1=0}^{1-i_2} f(1-i_1) \quad = \quad F^0_{n+2}$$

1.1 Notations, Terminology, and Formalization

In the On-Line Encyclopedia of Integer Sequences [14], A000045 is the sequence of Fibonacci numbers (Definition 1.1), A000032 is the sequence of Lucas numbers (Definition 2.4), A001045 is the sequence of Jacobstahl numbers (Definition 5.1), and A014551 is the sequence of Jacobstahl-Lucas numbers (Definition 6.6).

In Definition 1.1, [0, 1, 1, 2, 3, 5, 8, 13, $\cdots$[denotes an infinite sequence of integers that starts with 0 and continues with 1, 1, 2, 3, 5, 8, 13, etc.

To reflect the second-order recurrence of Fibonacci numbers, nearly all the induction proofs here use the following second-order induction principle, which is proved in the accompanying Coq file:

$$\frac{P(0) \qquad P(1) \qquad \forall k \in \mathbb{N}, P(k) \wedge P(k+1) \Rightarrow P(k+2)}{\forall n \in \mathbb{N}, P(n)}$$

The term "second-order induction" matches the term "second-order recurrence." Theorem 1.2, for example, is proved by second-order induction, witness its two base cases and its two induction hypotheses. This theorem can also be proved as a corollary of the following lemma, which is also proved by second-order induction on n (the first base case reads $f(1-0) = F^0_{0+1}$):

Lemma 1.3. *Given $f \in \{0,1\} \to \mathbb{N}$ such that $f(0) = 0$ and $f(1) = 1$,*

$$\forall n \in \mathbb{N},\ \sum_{i_n=0}^{0} \sum_{i_{n-1}=0}^{1-i_n} \cdots \sum_{i_1=0}^{1-i_2} f(1-i_1) \quad = \quad F^0_{n+1}$$

In Section 4, we will encounter third- and fourth-order recurrences, and likewise we will use the matching terms "third-order induction" and "fourth-order induction." (The alternative would be to align the terminology "second-order induction" with that of "second-order function" (a function that takes a first-order function as argument) but that would make the second-order induction principle parameterized with a first-order induction principle, which seems like a topic of study in itself. Conversely, the terminology "weak/strong induction" is not reflected by weak/strong functions nor with weak/strong recurrences, so we aligned "nth-order induction" with "nth-order recurrence," opting for terminological consistency.)

Except for the part involving rational numbers in Section 6 (namely Theorem 6.1, Theorem 6.4, Corollary 6.5, and Corollary 6.7) where the author ran out of steam, the entirety of this article is formalized with literate proofs using the Coq Proof Assistant [3]. The narrative, however, only uses ordinary mathematical vernacular. (The narrative and the formalization fueled each other, but as usual in mechanized reasoning, this narrative is only a best effort: in case of doubt, the formalization takes authoritative precedence [5].)

1.2 Related Work

The Online Encyclopedia of Integer Sequences contains an entry for the number of types of binary trees of height n, i.e., to A002449 = [1, 2, 6, 26, 166, 1626, 25510, 664666, $\cdots$ [. In this entry [11], Irwin conjectures that the following nested sums also compute A002449, for $n \geq 1$:

$$\mathrm{A002449}_{n+2} \quad = \quad \sum_{i_1=1}^{2} \sum_{i_2=1}^{2\cdot i_1} \cdots \sum_{i_{n-1}=1}^{2\cdot i_{n-2}} \sum_{i_n=1}^{2\cdot i_{n-1}} 2 \cdot i_n$$

In the material that accompanies this entry in the On-Line Encyclopedia of Integer Sequences [8], Lahlou proves this conjecture using generating trees, i.e., trees that are constructed coinductively [17]. Lahlou also lists a dozen nested summations that compute known integer sequences, including Fibonacci numbers:

Theorem 1.4 (Fibonacci numbers as nested sums (Lahlou))**.** $\forall n \in \mathbb{N}, A_n = F^0_{n+1}$ *where*

$$A_n \;=\; \begin{cases} 1 & \textit{if } n = 0 \\ 1 & \textit{if } n = 1 \\ \sum_{i_2=1}^{1} \sum_{i_3=1}^{3-i_2} \cdots \sum_{i_n=1}^{3-i_{n-1}} (3 - i_n) & \textit{otherwise} \end{cases}$$

This theorem can also be proved by second-order induction on n, as formalized in the accompanying Coq file.

Irwin's conjecture, West's generating trees, Lahlou's note, and Baumann's "k-dimensionale Champagnerpyramide" [2] are the nearest related work here, with Lahlou's work as the closest. The author's own discovery of the expressiveness of nested summations stemmed from studying Moessner's theorem [9] and identifying that its streams are artifacts of dynamic programming [4]. Here is Moessner's theorem without dynamic programming:

$$\forall x, n \in \mathbb{N}, \sum_{i_1=0}^{x} \sum_{i_2=0}^{i_1+\lfloor\frac{i_1}{1}\rfloor} \sum_{i_3=0}^{i_2+\lfloor\frac{i_2}{2}\rfloor} \cdots \sum_{i_n=0}^{i_{n-1}+\lfloor\frac{i_{n-1}}{n-1}\rfloor} 1 = (x+1)^n$$

In this identity, the upper bound of an inner sum depends on the index of an outer sum. Here are two more examples:

$$\begin{aligned} \forall x, n \in \mathbb{N}, \sum_{i_1=0}^{x} \sum_{i_2=0}^{i_1} \sum_{i_3=0}^{i_2} \cdots \sum_{i_n=0}^{i_{n-1}} 1 \quad &= \quad \binom{x+n}{n} \\ \forall n \in \mathbb{N}, \sum_{i_1=0}^{0} \sum_{i_2=0}^{1-i_1} \sum_{i_3=0}^{1-i_2} \cdots \sum_{i_n=0}^{1-i_{n-1}} 1 \quad &= \quad F^0_{n+1} \end{aligned}$$

The second example motivated the present article.

Aspnes, in his lecture notes on Discrete Mathematics [1, Chapter 6], points out that sums are implemented by for loops.

Rosen, in Example 6, page 449 of his textbook about Discrete Mathematics and its applications [16, Section 6.5.3], states the nested for loops that correspond to

$$\sum_{i_1=1}^{x} \sum_{i_2=1}^{i_1} \cdots \sum_{i_n=1}^{i_{n-1}} 1$$

for two given natural numbers x and n. He proves combinatorially that the result is the multiset coefficient $\left(\!\binom{x}{n}\!\right)$, a theorem that can also be proved by nested first-order induction on n and then on x, as formalized in the accompanying Coq file.

Example 27.2, page 222, of Grimaldi's Introduction to Fibonacci and Catalan numbers [7, Chapter 27] and Example 9, page 150, of the second edition of the Handbook of Discrete and Combinatorial Mathematics [15, Section 3.1.3] demonstrate how Catalan numbers (A000108 = [1, 1, 2, 5, 14, 42, 132, 429, 1430, $\cdots$ [in the On-Line Encyclopedia of Integer Sequences [10]) can also be obtained using nested for loops:

$$\forall n \in \mathbb{N}, \sum_{i_1=1}^{1} \sum_{i_2=1}^{i_1+1} \sum_{i_3=1}^{i_2+1} \cdots \sum_{i_n=1}^{i_{n-1}+1} 1 \quad = \quad C_n$$

Grimaldi's justification is based on the same coinductive argument as Lahlou's proof using West's generating trees: the growing indices match the structure of a tree where subtrees have a growing number of subtrees [7, Figure 27.3, page 222]. Since the number of nodes at level n of this tree is C_n, these nested sums compute Catalan numbers.

Catalan numbers can also be obtained via Moessner's theorem (and the initial upper bound 0 can be meaningfully generalized to a natural number x):

$$\forall n : \mathbb{N}, \sum_{i_1=0}^{0} \sum_{i_2=0}^{1+i_1} \sum_{i_3=0}^{1+i_2} \cdots \sum_{i_n=0}^{1+i_{n-1}} 1 \quad = \quad C_n$$

1.3 Roadmap

Section 2 investigates the finitely nested finite sums that compute Fibonacci numbers and Section 3 generalizes these nested sums to Fibonacci sequences that do not necessarily start with 0 and 1. Section 4 describes the nested sums that arise from similar third-order recurrences, fourth-order recurrences, etc. Section 5 took some ingenuity: it states the nested sums that compute Jacobstahl numbers and it generalizes these nested sums to Jacobstahl sequences that do not necessarily start with 0 and 1 and whose core multiplicative factor is not necessarily the standard one. Section 6 describes how to extrapolate the first elements of a sequence whose elements are computed by a nested sum. Section 7 concludes.

2 Fibonacci Numbers

In Theorem 1.2, f is essentially the identity function, but it could continue the summation:

Proposition 2.1. *Given $g \in \{0,1\} \to \mathbb{N}$ such that $g(0) = 0$ and $g(1) = 1$, and given $m \in \mathbb{N}$ and $f \in \{0,1\} \to \mathbb{N}$ such that $f(j) = \sum_{j_m=0}^{1-j} \sum_{j_{m-1}=0}^{1-j_m} \cdots \sum_{j_1=0}^{1-j_2} g(1-j_1)$,*

$$\forall n \in \mathbb{N}, \sum_{i_{n+1}=0}^{1} \sum_{i_n=0}^{1-i_{n+1}} \cdots \sum_{i_1=0}^{1-i_2} f(1-i_1) \quad = \quad F^0_{n+m+2}$$

Proof. By second-order induction on n, using Lemma 1.3. □

Definition 2.2 (Tribonacci sequence (A000073 = [0, 0, 1, 1, 2, 4, 7, 13, 24, 44, . . . [)). The standard tribonacci sequence T is such that $T_0 = 0$, $T_1 = 0$, $T_2 = 1$, and $\forall n \in \mathbb{N}, T_{n+3} = T_{n+2} + T_{n+1} + T_n$. Its members are called *tribonacci numbers.*

Since there are two integers between 0 and 1 (inclusively) and so three integers between 0 and 2, one might conjecture that $\sum_{i_{n+1}=0}^{2} \sum_{i_n=0}^{2-i_{n+1}} \cdots \sum_{i_1=0}^{2-i_2} f(2-i_1)$ is a tribonacci number, for any given natural number n, but that is not the case, as this sum computes members of A006356 in the On-Line Encyclopedia of Integer Sequences [12]:

Theorem 2.3 (Members of A006356 = [1, 3, 6, 14, 31, 70, 157, 353, 793, · · · [as nested sums). *Given $f \in \{0,1,2\} \to \mathbb{N}$ such that $f(0) = 0$, $f(1) = 0$, and $f(2) = 1$, $\forall n \in \mathbb{N}$,*

$$\sum_{i_{n+1}=0}^{2} \sum_{i_n=0}^{2-i_{n+1}} \cdots \sum_{i_1=0}^{2-i_2} f(2-i_1) \quad = \quad \text{A006356}_n \quad = \quad \text{A}_n$$

where $\text{A}_0 = 1$, $\text{A}_1 = 3$, $\text{A}_2 = 6$, and $\forall n \in \mathbb{N}$, $\text{A}_{n+3} = 2 \cdot \text{A}_{n+2} + \text{A}_{n+1} - \text{A}_n$.

Proof. By induction on n, using the following third-order induction principle:

$$\frac{P(0) \quad P(1) \quad P(2) \quad \forall k \in \mathbb{N}, P(k) \wedge P(k+1) \wedge P(k+2) \Rightarrow P(k+3)}{\forall n \in \mathbb{N}, P(n)}$$

□

The recurrence pattern in A006356 is eye catching because replacing A with F^0 in its induction step yields a Lucas number:

$$\begin{aligned} 2 \cdot F^0_{n+2} + F^0_{n+1} - F^0_n &= 2 \cdot F^0_{n+1} + 2 \cdot F^0_n + F^0_{n+1} - F^0_n \\ &= 2 \cdot F^0_{n+1} + F^0_n + F^0_{n+1} \\ &= 2 \cdot F^0_{n+1} + F^0_{n+2} \\ &= F^0_{n+2} + 2 \cdot F^0_{n+1} \\ &= L_{n+2} \end{aligned}$$

Definition 2.4 (Lucas sequence (A000032 = [2, 1, 3, 4, 7, 11, 18, 29, 47, · · · [)). The standard Lucas sequence L is such that $L_0 = 2$, $L_1 = 1$, and $\forall n \in \mathbb{N}, L_{n+2} = L_{n+1} + L_n$. Its members are called *Lucas numbers.*

Theorem 2.5 (Lucas numbers in terms of Fibonacci numbers). $\forall n \in \mathbb{N}, L_{n+1} = F^0_{n+1} + 2 \cdot F^0_n$

Proof. By second-order induction on n. □

Theorem 2.5 is eye catching too because it is reminiscent of the recurrence pattern of Jacobstahl numbers – namely $J_{n+2} = J_{n+1} + 2 \cdot J_n$, as elaborated in Section 5.

We provisionally conclude that tribonacci numbers are a red herring (which is not a good thing here, unlike a marinated red herring from Skagen at a Julefrokost in Denmark, but let's not digress, even though this Festschrift is dedicated to Jens and he loves to talk about food). Section 3 generalizes Theorem 1.2 to also account for Lucas numbers, Section 5 turns to Jacobstahl sequences, and Section 6 revisits the recurrence pattern of A006356.

3 Parameterized Fibonacci Sequences

Definition 3.1 (Fibonacci sequence parameterized with its zero case). For all natural numbers z, the Fibonacci sequence F^z is such that $F^z_0 = z$, $F^z_1 = 1$, and $\forall n \in \mathbb{N}, F^z_{n+2} = F^z_{n+1} + F^z_n$.

For example, F^0 is the standard Fibonacci sequence and F^2 is the Lucas sequence.

The following theorem generalizes Theorem 2.5 from Lucas numbers to Fibonacci numbers parameterized with their zero case.

Theorem 3.2 (Members of Fibonacci sequences parameterized with their zero case in terms of Fibonacci numbers). $\forall z \in \mathbb{N}, \forall n \in \mathbb{N}, F^z_{n+1} = F^0_{n+1} + z \cdot F^0_n$

Proof. By second-order induction on n. □

Theorem 3.3 (Members of Fibonacci sequences parameterized with their zero case as nested sums). *Given $z \in \mathbb{N}$ and $f \in \{0,1\} \to \mathbb{N}$ such that $f(0) = z$ and $f(1) = 1$,*

$$\forall n \in \mathbb{N}, \sum_{i_{n+1}=0}^{1} \sum_{i_n=0}^{1-i_{n+1}} \cdots \sum_{i_1=0}^{1-i_2} f(1-i_1) \quad = \quad F^z_{n+2}$$

Proof. By second-order induction on n, as in the proof of Theorem 1.2. □

Corollary 3.4 (Lucas numbers as nested sums). *Given* $f \in \{0,1\} \to \mathbb{N}$ *such that* $f(0) = 2$ *and* $f(1) = 1$,

$$\forall n \in \mathbb{N}, \sum_{i_{n+1}=0}^{1} \sum_{i_n=0}^{1-i_{n+1}} \cdots \sum_{i_1=0}^{1-i_2} f(1-i_1) \quad = \quad F^2_{n+2} \;=\; L_{n+2}$$

Definition 3.5 (Fibonacci sequence parameterized with both its base cases). For all functions $f \in \{0,1\} \to \mathbb{N}$, the Fibonacci sequence F^f is such that $F^f_0 = f(0)$, $F^f_1 = f(1)$, and $\forall n \in \mathbb{N}, F^f_{n+2} = F^f_{n+1} + F^f_n$.

For example, F^f is the standard Fibonacci sequence if f maps 0 to 0 and 1 to 1, and F^f is the Lucas sequence if f maps 0 to 2 and 1 to 1. The On-Line Encyclopedia of Integer Sequences contains a variety of examples of Fibonacci sequences that start with other pairs of natural numbers. As shown below (Theorem 3.7), their members can all be expressed as nested sums.

The following theorem generalizes Theorem 2.5 from the Lucas sequence to Fibonacci sequences parameterized with both their base cases:

Theorem 3.6 (Members of Fibonacci sequences parameterized with their base cases in terms of Fibonacci numbers). $\forall f \in \{0,1\} \to \mathbb{N}, \forall n \in \mathbb{N}, F^f_{n+1} = f(1) \cdot F^0_{n+1} + f(0) \cdot F^0_n$

Proof. By second-order induction on n. □

Theorem 3.7 (Members of Fibonacci sequences parameterized with their base cases in terms of Fibonacci numbers as nested sums). $\forall f \in \{0,1\} \to \mathbb{N}$,

$$\forall n \in \mathbb{N}, \sum_{i_{n+1}=0}^{1} \sum_{i_n=0}^{1-i_{n+1}} \cdots \sum_{i_1=0}^{1-i_2} f(1-i_1) \quad = \quad F^f_{n+2}$$

Proof. By second-order induction on n. □

4 Higher-Order Recurrences

The pattern of Theorem 1.2 and 2.3 scales to 3, 4, 5, etc. According to the On-Line Encyclopedia of Integer Sequences, it accounts for the number of paths with n turns when light is reflected from 3, 4, 5, etc. glass plates. So A006356 follows a third-order recurrence, A006357 follows a fourth-order recurrence, A006358 follows a fifth-order recurrence, etc., and their members can be computed with finitely nested finite sums.

In each induction step, the coefficients can be calculated but they are already listed in the page about linear recurrences with constant coefficients in the On-Line Encyclopedia of Integer Sequences [13].

The present section details two examples that one would be hard pressed to prove by hand. (For simplicity, in the accompanying Coq file, the second example is formalized as a sequence of integers rather than as a sequence of natural numbers.)

4.1 A Fourth-Order Recurrence

Theorem 4.1 (Members of A006357 = [1, 4, 10, 30, 85, 246, 707, 2037, ⋯ [as nested sums). *Given* $f \in \{0,1,2,3\} \to \mathbb{N}$ *such that* $f(0) = 0$, $f(1) = 0$, $f(2) = 0$, *and* $f(3) = 1$,

$$\forall n \in \mathbb{N}, \sum_{i_{n+1}=0}^{3} \sum_{i_n=0}^{3-i_{n+1}} \cdots \sum_{i_1=0}^{3-i_2} f(3-i_1) \quad = \quad \mathrm{A006357}_n \;=\; \mathrm{A}_n$$

where $\mathrm{A}_0 = 1$, $\mathrm{A}_1 = 4$, $\mathrm{A}_2 = 10$, $\mathrm{A}_3 = 30$, *and* $\forall n \in \mathbb{N}$, $\mathrm{A}_{n+4} = 2 \cdot \mathrm{A}_{n+3} + 3 \cdot \mathrm{A}_{n+2} - \mathrm{A}_{n+1} - \mathrm{A}_n$.

Proof. By fourth-order induction on n. □

4.2 A Fifth-Order Recurrence

Theorem 4.2 (Members of A006358 = [1, 5, 15, 55, 190, 671, 2353, 8272, ⋯ [as nested sums). *Given* $f \in \{0,1,2,3,4\} \to \mathbb{N}$ *such that* $f(0) = 0$, $f(1) = 0$, $f(2) = 0$, $f(3) = 0$, *and* $f(4) = 1$,

$$\forall n \in \mathbb{N}, \sum_{i_{n+1}=0}^{4} \sum_{i_n=0}^{4-i_{n+1}} \cdots \sum_{i_1=0}^{4-i_2} f(4-i_1) \quad = \quad \mathrm{A006358}_n \;=\; \mathrm{A}_n$$

where $\mathrm{A}_0 = 1$, $\mathrm{A}_1 = 5$, $\mathrm{A}_2 = 15$, $\mathrm{A}_3 = 55$, $\mathrm{A}_4 = 190$, *and* $\forall n \in \mathbb{N}$, $\mathrm{A}_{n+5} = 3 \cdot \mathrm{A}_{n+4} + 3 \cdot \mathrm{A}_{n+3} - 4 \cdot \mathrm{A}_{n+2} - \mathrm{A}_{n+1} + \mathrm{A}_n$.

Proof. By fifth-order induction on n. □

5 Jacobstahl Numbers

Definition 5.1 (Standard Jacobstahl sequence (A001045 = [0, 1, 1, 3, 5, 11, 21, 43, 85, 171, ⋯ [)). The standard Jacobstahl sequence J is such that $J_0 = 0$, $J_1 = 1$, and $\forall n \in \mathbb{N}, J_{n+2} = J_{n+1} + 2 \cdot J_n$. Its members are called *Jacobstahl numbers.*

Theorem 5.2 (Jacobstahl numbers as nested sums). *Given* $f \in \{0,1\} \to \mathbb{N}$ *such that* $f(0) = 0$ *and* $f(1) = 1$, $\forall n \in \mathbb{N}$, $J_{n+2} =$

$$\sum_{i_{n+1}=0}^{1} (i_{n+1}{+}1) \cdot \sum_{i_n=0}^{1-i_{n+1}} (i_n{+}1) \cdot \sum_{i_{n-1}=0}^{1-i_n} \cdots (i_2{+}1) \cdot \sum_{i_1=0}^{1-i_2} (i_1{+}1) \cdot f(1-i_1)$$

Proof. By second-order induction on n, as in the proofs of Theorem 1.2 and 3.3. □

Definition 5.3 (Generalized Jacobstahl sequence). For any given natural numbers z, w and k, the Jacobstahl sequence $J^{z,w,k}$ is such that $J^{z,w,k}_0 = z$, $J^{z,w,k}_1 = w$, and $\forall n \in \mathbb{N}, J^{z,w,k}_{n+2} = k \cdot J^{z,w,k}_{n+1} + k \cdot (k+1) \cdot J^{z,w,k}_n$, where k is the core multiplicative factor.

For example, $J^{0,1,1}$ is the standard Jacobstahl sequence. The On-Line Encyclopedia of Integer Sequences contains a variety of examples of Jacobstahl sequences that start with a natural number and then its successor and that have 1 as a core multiplicative factor, and of Jacobstahl sequences that start with 0 and whose core multiplicative factor is not necessarily 1.

Theorem 5.4 (Members of the generalized Jacobstahl sequences that start with a natural number and then its successor and that have 1 as a core multiplicative factor as nested sums)**.** *Given $z \in \mathbb{N}$ and $f \in \{0,1\} \rightarrow \mathbb{N}$ such that $f(0) = z$ and $f(1) = z+1$, $\forall n \in \mathbb{N}$, $J^{z,z+1,1}_{n+2}$ =*

$$\sum_{i_{n+1}=0}^{1}(i_{n+1}{+}1) \cdot \sum_{i_n=0}^{1-i_{n+1}}(i_n{+}1) \cdot \sum_{i_{n-1}=0}^{1-i_n} \cdots (i_2{+}1) \cdot \sum_{i_1=0}^{1-i_2}(i_1{+}1) \cdot f(1-i_1)$$

Proof. By second-order induction on n. □

Theorem 5.5 (Members of the generalized Jacobstahl sequences that start with 0 as nested sums)**.** *Given $w \in \mathbb{N}$ and $f \in \{0,1\} \rightarrow \mathbb{N}$ such that $f(0) = 0$ and $f(1) = w$, $\forall k, n \in \mathbb{N}$, $J^{0,w,k}_{n+2}$ =*

$$\sum_{i_{n+1}=0}^{1}(i_{n+1}{+}k) \cdot \sum_{i_n=0}^{1-i_{n+1}}(i_n{+}k) \cdot \sum_{i_{n-1}=0}^{1-i_n} \cdots \ (i_2{+}k) \cdot \sum_{i_1=0}^{1-i_2}(i_1{+}k) \cdot f(1-i_1)$$

Proof. By second-order induction on n. □

If we let z be 0, w be 1, and k be 1, $J^{0,1,1}_n$ coincides with J_n and Theorem 5.4 and 5.5 coincide with Theorem 5.2. (As Mayer Goldberg puts it in his lecture notes about the lambda calculus [6], w should be pronounced "wone.")

6 Beginnings

Theorems 1.2, 3.3, 3.7, 5.2, 5.4, and 5.5 are about second-order recurrences, and their sums compute numbers that start at index 2 in the corresponding sequences. This section describes how to extrapolate the first elements of the sequences so that the sums compute numbers that start at index 0 in these sequences (making sure that the balances are correct).

6.1 Second-Order Recurrence for the Fibonacci Sequence

Revisiting the standard Fibonacci sequence in Section 2, here is a general solution for expressing integer sequences s that satisfy $\forall n \in \mathbb{N}, s_{n+2} = s_{n+1} + s_n$ as a sum

$$s_n = \sum_{i_{n+1}=0}^{1} \sum_{i_n=0}^{1-i_{n+1}} \cdots \sum_{i_1=0}^{1-i_2} f(1-i_1),$$

given a function $f \in \{0,1\} \rightarrow \mathbb{Q}$.

We want the sequence to start with s_0 and s_1:

$$\begin{cases} s_0 = \sum_{i_1=0}^{1} f(1-i_1) = f(1) + f(0) \\ s_1 = \sum_{i_2=0}^{1} \sum_{i_1=0}^{1-i2} f(1-i_1) = 2 \cdot f(1) + f(0) \end{cases}$$

These equations are satisfied given the following function $f \in \{0,1\} \rightarrow \mathbb{Q}$:

$$f(0) = \frac{2 \cdot s_0 - s_1}{2} \quad \wedge \quad f(1) = \frac{s_1 - s_0}{2}$$

Theorem 6.1. *Given an integer sequence s and a function $f \in \{0,1\} \rightarrow \mathbb{Q}$ such that $f(0) = \frac{2 \cdot s_0 - s_1}{2}$ and $f(1) = \frac{s_1 - s_0}{2}$,*

$$(\forall n \in \mathbb{N},\ s_n = \sum_{i_{n+1}=0}^{1} \sum_{i_n=0}^{1-i_{n+1}} \cdots \sum_{i_1=0}^{1-i_2} f(1-i_1)) \ \Rightarrow \ \forall n \in \mathbb{N}, s_{n+2} = s_{n+1} + 2 \cdot s_n$$

Proof. By second-order induction on n. □

Corollary 6.2 (Fibonacci numbers as nested sums, extrapolated)**.** *Given $f \in \{0,1\} \rightarrow \mathbb{Z}$ such that $f(0) = -1$ and $f(1) = 1$,*

$$\forall n \in \mathbb{N}, \sum_{i_{n+1}=0}^{1} \sum_{i_n=0}^{1-i_{n+1}} \cdots \sum_{i_1=0}^{1-i_2} f(1-i_1) \quad = \quad F^0_n$$

Compared to Theorem 1.2, the index of F^0, on the right-hand side, is n, not $n+2$.

Corollary 6.3 (Lucas numbers as nested sums, extrapolated)**.** *Given $f \in \{0,1\} \rightarrow \mathbb{Z}$ such that $f(0) = 3$ and $f(1) = -1$,*

$$\forall n \in \mathbb{N}, \sum_{i_{n+1}=0}^{1} \sum_{i_n=0}^{1-i_{n+1}} \cdots \sum_{i_1=0}^{1-i_2} f(1-i_1) \quad = \quad L_n$$

Compared to Corollary 3.4, the index of L, on the right-hand side, is n, not $n+2$.

6.2 Second-Order Recurrence for the Jacobstahl Sequence

Revisiting the standard Jacobstahl sequence in Section 5, here is a general solution for expressing integer sequences s that satisfy $\forall n \in \mathbb{N}, s_{n+2} = s_{n+1} + 2 \cdot s_n$ as a sum s_n =

$$\sum_{i_{n+1}=0}^{1}(i_{n+1}{+}1) \cdot \sum_{i_n=0}^{1-i_{n+1}}(i_n{+}1) \cdot \sum_{i_{n-1}=0}^{1-i_n} \cdots (i_2{+}1) \cdot \sum_{i_1=0}^{1-i_2}(i_1{+}1) \cdot f(1{-}i_1),$$

given a function $f \in \{0,1\} \rightarrow \mathbb{Q}$.

We want the sequence to start with s_0 and s_1:

$$\begin{cases} s_0 = \sum_{i_1=0}^{1} (i_1+1) \cdot f(1-i_1) \\ = f(1) + 2 \cdot f(0) \\ s_1 = \sum_{i_2=0}^{1} (i_2+1) \cdot \sum_{i_1=0}^{1-i2} (i_1+1) \cdot f(1-i_1) \\ = 3 \cdot f(1) + 2 \cdot f(0) \end{cases}$$

These equations are satisfied given the following function $f \in \{0,1\} \rightarrow \mathbb{Q}$:

$$f(0) = \frac{3 \cdot s_0 - s_1}{4} \quad \wedge \quad f(1) = \frac{s_1 - s_0}{2}$$

Theorem 6.4. *Given an integer sequence s and a function $f \in \{0,1\} \rightarrow \mathbb{Q}$ such that $f(0) = \frac{3 \cdot s_0 - s_1}{4}$ and $f(1) = \frac{s_1 - s_0}{2}$,*

$$(\forall n \in \mathbb{N}, s_n{=} \sum_{i_{n+1}=0}^{1} (i_{n+1}{+}1) \cdot \sum_{i_n=0}^{1-i_{n+1}} \cdots (i_1{+}1) \cdot f(1{-}i_1)) \Rightarrow \forall n \in \mathbb{N}, s_{n+2} = s_{n+1} + 2 \cdot s_n$$

Proof. By second-order induction on n. □

Corollary 6.5 (Jacobstahl numbers as nested sums, extrapolated)**.** *Given $f \in \{0,1\} \rightarrow \mathbb{Q}$ such that $f(0) = -\frac{1}{4}$ and $f(1) = \frac{1}{2}$,*

$$\forall n \in \mathbb{N}, \sum_{i_{n+1}=0}^{1} \sum_{i_n=0}^{1-i_{n+1}} \cdots \sum_{i_1=0}^{1-i_2} f(1-i_1) \quad = \quad J^{0,1,1}_n$$

Compared to Corollary 5.2, the index of $J^{0,1,1}$, on the right-hand side, is n, not $n+2$.

Definition 6.6 (Jacobstahl-Lucas sequence (A014551 = [2, 1, 5, 7, 17, 31, 65, ... []))**.** The Jacobstahl-Lucas sequence $J^{2,1,1}$ is such that $J_0 = 2$, $J_1 = 1$, and $\forall n \in \mathbb{N}, J^{2,1,1}_{n+2} = J^{2,1,1}_{n+1} + 2 \cdot J^{2,1,1}_n$. Its members are called *Jacobstahl-Lucas numbers*.

Corollary 6.7 (Jacobstahl-Lucas numbers as nested sums)**.** *Given* $f \in \{0, 1\} \to \mathbb{Q}$ *such that* $f(0) = \frac{3}{2}$ *and* $f(1) = \frac{5}{2}$*,*

$$\forall n \in \mathbb{N}, \sum_{i_{n+1}=0}^{1} \sum_{i_n=0}^{1-i_{n+1}} \cdots \sum_{i_1=0}^{1-i_2} f(1-i_1) \quad = \quad J_n^{2,1,1}$$

6.3 A Third-Order Recurrence

Revisiting A006356 in Section 2, here is a general solution for expressing integer sequences s that satisfy $\forall n \in \mathbb{N}, s_{n+3} = 2 \cdot s_{n+2} + s_{n+1} - s_n$ as a sum

$$s_n = \sum_{i_n=0}^{2} \sum_{i_{n-1}=0}^{2-i_n} \cdots \sum_{i_1=0}^{2-i_2} f(i_1),$$

given a function $f \in \{0, 1, 2\} \to \mathbb{Z}$.

We want the sequence to start with s_0, s_1, and s_2:

$$\begin{cases} s_0 &= \sum_{i_1=0}^{2} f(i_1) \\ &= \quad f(0) + \quad f(1) + \quad f(2) \\ s_1 &= \sum_{i_2=0}^{2} \sum_{i_1=0}^{2-i2} f(i_1) \\ &= 3 \cdot f(0) + 2 \cdot f(1) + \quad f(2) \\ s_2 &= \sum_{i_3=0}^{2} \sum_{i_2=0}^{2-i_3} \sum_{i_1=0}^{2-i2} f(i_1) \\ &= 6 \cdot f(0) + 5 \cdot f(1) + 3 \cdot f(2) \end{cases}$$

These equations are satisfied given the following function $f \in \{0, 1, 2\} \to \mathbb{Z}$:

$$\begin{cases} f(0) &= \quad s_0 + 2 \cdot s_1 - \quad s_2 \\ f(1) &= -3 \cdot s_0 - 3 \cdot s_1 + 2 \cdot s_2 \\ f(2) &= \quad 3 \cdot s_0 + \quad s_1 - \quad s_2 \end{cases}$$

Theorem 6.8. *Given an integer sequence* s *and a function* $f \in \{0, 1, 2\} \to \mathbb{Z}$ *such that* $f(0) = s_0 + 2 \cdot s_1 - s_2$, $f(1) = -3 \cdot s_0 - 3 \cdot s_1 + 2 \cdot s_2$, *and* $f(2) = 3 \cdot s_0 + s_1 - s_2$*,*

$$(\forall n \in \mathbb{N},\ s_n = \sum_{i_n=0}^{2} \sum_{i_{n-1}=0}^{2-i_n} \cdots \sum_{i_1=0}^{2-i_2} f(i_1)) \quad \Rightarrow$$
$$\forall n \in \mathbb{N}, s_{n+3} = 2 \cdot s_{n+2} + s_{n+1} - s_n$$

Proof. By third-order induction on n. □

7 Conclusion

Finitely nested finite sums lend themselves to an inductive construction of numbers that satisfy a recurrence relation, e.g., Fibonacci numbers (Section 2), Jacobstahl numbers (Section 5), and variants, which include Lucas numbers (Section 3) and Jacobstahl-Lucas numbers (Section 6.2). These sums can also be used to construct, e.g., binomial coefficients, integral powers, multiset coefficients, and Catalan numbers (Section 1.2). They scale to higher-order recurrence relations (Section 4) but it is currently unclear to the author how to use them to construct, e.g., Pell numbers and tribonacci numbers.

To close, and getting back to Lahlou's work [8], it is a fun exercise to identify the generating trees that correspond to the nested sums that were studied here.

8 Data-Availability Statement

The accompanying material [5] contains an implementation in ANSI C and a formalization in Coq.

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