

Multisets

Combinations with Repetition

Yolymatics Tutorials

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What is a Multiset?

Definition

A **multiset** (or **bag**) is a collection of objects where repetition of elements is allowed, but order does not matter.

Comparison with Sets

- **Set:** $\{a, b, c\}$ - each element appears at most once
- **Multiset:** $\{a, a, b, c, c, c\}$ - elements can repeat
- **Notation:** Sometimes written as $\{a^2, b, c^3\}$ to show multiplicities

Key Property

The order of elements does NOT matter: $\{a, a, b\} = \{a, b, a\} = \{b, a, a\}$

Multisets vs Combinations vs Permutations

Key Distinctions

Concept	Order?	Repetition?	Example
Permutations	Matters	No	$ABC \neq BAC$
Combinations	Doesn't	No	$\{A, B, C\} = \{C, A, B\}$
Multisets	Doesn't	Yes	$\{A, A, B\} = \{B, A, A\}$
Permutations w/ Rep	Matters	Yes	$AAB \neq ABA$

Real-World Examples

- Selecting fruits: 2 apples, 1 banana, 3 oranges
- Distributing identical items to distinct boxes
- Choosing toppings for pizza (can repeat)

The Fundamental Multiset Formula

Problem Statement

How many ways can we select r objects from n types of objects, where repetition is allowed and order does not matter?

Formula: Combinations with Repetition

$$\left(\left(\begin{matrix} n \\ r \end{matrix}\right)\right) = \binom{n+r-1}{r} = \binom{n+r-1}{n-1} = \frac{(n+r-1)!}{r!(n-1)!}$$

where:

- n = number of types (distinct objects to choose from)
- r = number of selections (objects to be selected)

Notation

$\left(\left(\begin{matrix} n \\ r \end{matrix}\right)\right)$ is read as "n multichoose r" or "multiset coefficient"

Stars and Bars Method

Visual Representation

The multiset problem is equivalent to placing r identical stars into n distinct bins using $n - 1$ bars as separators.

Example: Choose 5 items from 3 types

Select 5 objects from types $\{A, B, C\}$

Representation: $\star\star | \star\star\star |$ means 2 of type A, 3 of type B, 0 of type C

Total arrangements: We have 5 stars and 2 bars to arrange

$$\binom{5+2}{2} = \binom{7}{2} = 21 \text{ ways}$$

Key Insight

We arrange r stars and $(n - 1)$ bars, giving $\binom{n+r-1}{r}$ or $\binom{n+r-1}{n-1}$

Problem 1: Simple Multiset Selection

Problem

A bakery offers 4 types of cookies: chocolate chip, oatmeal, sugar, and peanut butter. You want to buy 6 cookies. How many different selections can you make?

Note: You can choose multiple cookies of the same type.

Problem 2: Distributing Identical Objects

Problem

In how many ways can 10 identical candies be distributed among 3 children?

Note: A child can receive 0, 1, 2, ..., or all 10 candies.

Problem 3: Non-negative Integer Solutions

Problem

Find the number of non-negative integer solutions to the equation:

$$x_1 + x_2 + x_3 + x_4 = 12$$

Problem 4: Coin Selection

Problem

A vending machine accepts pennies, nickels, dimes, and quarters. How many different ways can you put 8 coins into the machine?

Note: The coins need not have the same value.

Problem 5: Letter Repetition

Problem

How many 5-letter "words" can be formed using only the letters A, B, and C, where order does not matter?

Example: AAABC is the same as AABCA and BAAAC.

Problem 6: Positive Integer Solutions

Problem

Find the number of **positive** integer solutions to:

$$x_1 + x_2 + x_3 = 10$$

Hint: Each variable must be at least 1.

Problem 7: Minimum Constraints

Problem

How many ways can 15 identical books be distributed among 4 students if each student must receive at least 2 books?

Problem 8: Upper Bound Constraints

Problem

Find the number of non-negative integer solutions to:

$$x_1 + x_2 + x_3 = 8$$

where $x_1 \leq 3$, $x_2 \leq 4$, and $x_3 \leq 5$.

Hint: Use inclusion-exclusion principle.

Problem 9: Mixed Constraints

Problem

Find the number of integer solutions to:

$$x_1 + x_2 + x_3 + x_4 = 20$$

where $x_1 \geq 2$, $x_2 \geq 0$, $x_3 \geq 1$, and $x_4 \geq 3$.

Problem 10: Even Distribution Constraint

Problem

How many ways can 12 identical items be distributed among 3 boxes such that each box contains an even number of items (possibly 0)?

Hint: Let $x_i = 2y_i$ where $y_i \geq 0$.

Problem 11: Generating Functions Application

Problem

In how many ways can you make change for \$1.00 using only pennies, nickels, dimes, and quarters?

Note: Find non-negative integer solutions to $p + 5n + 10d + 25q = 100$.

Problem 12: Partition with Distinct Parts

Problem

How many ways can you select a multiset of size 6 from the set $\{1, 2, 3, 4, 5\}$?

Note: Each number can be selected multiple times.

Problem 13: Polynomial Coefficient

Problem

Find the coefficient of x^{10} in the expansion of:

$$(1 + x + x^2 + x^3 + \dots)^4$$

Hint: This is equivalent to a multiset problem.

Problem 14: Dice Combinations

Problem

How many different unordered outcomes are possible when rolling 5 distinguishable dice?

Example: Rolling $(1,1,3,4,6)$ is the same as $(1,3,1,6,4)$ in an unordered collection.

Problem 15: Bounded Selections

Problem

A fruit stand has apples, bananas, and cherries. You want to buy exactly 8 pieces of fruit. However, the stand only has 3 bananas available. How many different selections are possible?

Problem 16: Identify the Method

Problem

For each scenario, determine whether it is a multiset problem, regular combination, or permutation:

- 1 Selecting 4 cards from a standard deck (52 cards)
- 2 Arranging 5 books on a shelf
- 3 Choosing 10 donuts from 6 varieties
- 4 Forming a 3-digit PIN code

Problem 17: Ice Cream Scoops

Problem

An ice cream shop has 12 flavors. You order a bowl with 4 scoops. How many different bowls are possible if:

- 1 Order matters and repetition is not allowed
- 2 Order doesn't matter and repetition is not allowed
- 3 Order doesn't matter and repetition is allowed

Problem 18: Committee with Repeated Members

Problem

A committee of 5 people is to be formed from 8 candidates, where a person can be selected multiple times (representing multiple votes or roles). How many different committees can be formed?

Formula 1: Basic Multiset Coefficient

Statement

The number of multisets of size r chosen from n types is:

$$\left(\binom{n}{r}\right) = \binom{n+r-1}{r} = \binom{n+r-1}{n-1}$$

Alternative Forms

- $\frac{(n+r-1)!}{r!(n-1)!}$ - factorial form
- $\frac{(n+r-1)(n+r-2)\cdots(n)}{r!}$ - product form
- Number of non-negative integer solutions to $x_1 + x_2 + \cdots + x_n = r$

Formula 2: Positive Integer Solutions

Statement

The number of **positive** integer solutions to:

$$x_1 + x_2 + \cdots + x_n = r$$

where each $x_i \geq 1$, is:

$$\binom{r-1}{n-1} = \binom{r-1}{r-n}$$

Derivation

Substitute $y_i = x_i - 1$ where $y_i \geq 0$:

$$y_1 + y_2 + \cdots + y_n = r - n$$

This gives $\binom{(r-n)+n-1}{n-1} = \binom{r-1}{n-1}$ non-negative solutions.

Formula 3: General Minimum Constraints

Statement

The number of integer solutions to $x_1 + x_2 + \cdots + x_n = r$ where $x_i \geq m_i$ for each i :

$$\binom{r - (m_1 + m_2 + \cdots + m_n) + n - 1}{n - 1}$$

Method

- 1 Substitute $y_i = x_i - m_i$ so that $y_i \geq 0$
- 2 The equation becomes: $y_1 + y_2 + \cdots + y_n = r - \sum m_i$
- 3 Apply the basic multiset formula to the y_i variables

Formula 4: Inclusion-Exclusion for Upper Bounds

Statement

For solutions to $x_1 + x_2 + \cdots + x_n = r$ with upper bounds $x_i \leq u_i$:

Use the Inclusion-Exclusion Principle:

$$\sum_{\text{all}} - \sum_{\text{violate 1}} + \sum_{\text{violate 2}} - \cdots$$

Example: $x_1 + x_2 = 10$ with $x_1 \leq 6$, $x_2 \leq 7$

$$\text{Total} = \binom{11}{1} = 11$$

$$\text{Subtract } x_1 \geq 7 = \binom{4}{1} = 4$$

$$\text{Subtract } x_2 \geq 8 = \binom{3}{1} = 3$$

Formula 5: Stars and Bars Visualization

Interpretation

To distribute r identical objects into n distinct bins:

- Represent objects as r stars: $\star\star\star\cdots\star$
- Use $n - 1$ bars to create n compartments: $|$
- Arrange r stars and $n - 1$ bars in a line

Total arrangements: $\binom{r+(n-1)}{r} = \binom{r+n-1}{n-1}$

Example

Distribute 5 items into 3 bins: $\star\star|\star|\star\star$

This represents: Bin 1 gets 2, Bin 2 gets 1, Bin 3 gets 2

Total ways: $\binom{5+3-1}{3-1} = \binom{7}{2} = 21$

Formula 6: Generating Functions

Statement

The generating function for selecting r objects from n types with repetition is:

$$(1 + x + x^2 + x^3 + \cdots)^n = \frac{1}{(1 - x)^n}$$

The coefficient of x^r gives the number of multisets of size r .

Power Series Expansion

$$\frac{1}{(1 - x)^n} = \sum_{r=0}^{\infty} \binom{n + r - 1}{r} x^r$$

The coefficient of x^r is $\binom{n+r-1}{r}$, confirming the multiset formula.

Property 1: Multiset Symmetry

Statement

$$\left(\binom{n}{r}\right) = \binom{n+r-1}{r} = \binom{n+r-1}{n-1}$$

This symmetry shows that selecting r items from n types is equivalent to distributing r items into n bins.

Interpretation

- $\binom{n+r-1}{r}$ - choose positions for r stars among r stars and $n-1$ bars
- $\binom{n+r-1}{n-1}$ - choose positions for $n-1$ bars among r stars and $n-1$ bars

Property 2: Relationship to Regular Combinations

Comparison

- **Regular combinations:** $\binom{n}{r}$ - choose r from n distinct objects, no repetition
- **Multisets:** $\left(\!\!\binom{n}{r}\!\!\right) = \binom{n+r-1}{r}$ - choose r from n types, repetition allowed

Example Comparison

- $\binom{4}{3} = 4$ - choose 3 from 4 distinct items: $\{A, B, C\}, \{A, B, D\}, \{A, C, D\}, \{B, C, D\}$
- $\binom{4+3-1}{3} = \binom{6}{3} = 20$ - choose 3 from 4 types with repetition allowed

Property 3: Special Cases

Important Special Values

- ① $\binom{n}{0} = \binom{n-1}{0} = 1$ - one way to select nothing
- ② $\binom{n}{1} = \binom{n}{1} = n$ - n ways to select one item
- ③ $\binom{1}{r} = \binom{r}{r} = 1$ - one way to select r items from 1 type
- ④ $\binom{2}{r} = \binom{r+1}{r} = r + 1$ - partitions of r into at most 2 parts

Common Problem Types Summary

Problem Type Recognition

Problem Type	Formula
Distribute r identical to n distinct	$\binom{n+r-1}{r}$
Non-negative solutions: $\sum x_i = r$	$\binom{n+r-1}{n-1}$
Positive solutions: $\sum x_i = r$	$\binom{r-1}{n-1}$
With minimums: $x_i \geq m_i$	$\binom{r - \sum_{i=1}^n m_i + n - 1}{n-1}$
With maximums: $x_i \leq u_i$	Inclusion-Exclusion
Coefficient in $(1 + x + x^2 + \dots)^n$	$\binom{n+r-1}{r}$

Essential Formulas

- ① **Basic Multiset:** $\left(\!\!\binom{n}{r}\!\!\right) = \binom{n+r-1}{r} = \frac{(n+r-1)!}{r!(n-1)!}$
- ② **Positive Integer Solutions:** $x_1 + \cdots + x_n = r$ with $x_i \geq 1$: $\binom{r-1}{n-1}$
- ③ **With Minimums:** $x_i \geq m_i$: Let $S = \sum m_i$, answer is $\binom{r-S+n-1}{n-1}$
- ④ **Stars and Bars:** r stars, $n-1$ bars: $\binom{r+n-1}{n-1}$
- ⑤ **Regular Combination:** $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ (no repetition)
- ⑥ **Permutation:** $P(n, r) = \frac{n!}{(n-r)!}$ (order matters, no repetition)

Thank You!

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Keep counting!