Linear Algebra Comprehensive Worksheet

Yolymatics Tutorials

Instructions

This worksheet covers key topics in linear algebra. Show all your working clearly. Use additional paper if needed.

1 Cross Product (Section 3.5)

Problem 1. Calculate the cross product $\vec{a} \times \vec{b}$ for the following vectors:

(a) $\vec{a} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}, \vec{b} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$

(b) $\vec{a} = \begin{pmatrix} 5 \\ 0 \\ -3 \end{pmatrix}, \vec{b} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$

Problem 2. Given vectors $\vec{u} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix}$:

(a) Find $\vec{u} \times \vec{v}$

(b) Verify that $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$

(c) Find the area of the parallelogram spanned by $\vec{u}$ and $\vec{v}$

Problem 3. Prove that for any vectors $\vec{a}$ and $\vec{b}$ in $\mathbb{R}^3$:

(a) $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$ (anti-commutativity)

(b) $\vec{a} \times \vec{a} = \vec{0}$

Problem 4. Find the volume of the parallelepiped determined by the vectors:

$$\vec{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}, \quad \vec{c} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

2 Eigenvalues and Eigenvectors (Section 5.1)

Problem 5. Find the eigenvalues and corresponding eigenvectors for each matrix:

(a) $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

(b) $B = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

(c) $C = \begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix}$

Problem 6. Consider the matrix $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 2 & -4 & 2 \end{pmatrix}$

(a) Find the characteristic polynomial of A

(b) Find all eigenvalues of A

(c) Find an eigenvector for each eigenvalue

Problem 7. Let $\lambda = 5$ be an eigenvalue of $A = \begin{pmatrix} 5 & 2 \\ 2 & 5 \end{pmatrix}$ with eigenvector $\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(a) Verify that $\vec{v}$ is indeed an eigenvector with eigenvalue $\lambda = 5$

(b) Find the other eigenvalue and eigenvector

(c) What is the trace and determinant of A? How do they relate to the eigenvalues?

Problem 8. Prove that if λ is an eigenvalue of an invertible matrix A, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

3 Diagonalization (Section 5.2)

Problem 9. Determine whether each matrix is diagonalizable. If so, find matrices P and D such that $A = PDP^{-1}$.

(a) $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$

(b) $B = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$

(c) $C = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$

Problem 10. Given $A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$ has eigenvalues $\lambda_1 = 6$ and $\lambda_2 = 1$ with corresponding eigenvectors $\vec{v}_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$:

(a) Write A in the form PDP^{-1}

(b) Use this to compute A^5

(c) Find A^{10}

Problem 11. Let $A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}$.

(a) Find the eigenvalues of A

(b) Determine the geometric multiplicity of each eigenvalue

(c) Is A diagonalizable? Justify your answer

4 Differential Equations (Section 5.4)

Problem 12. Solve the system of differential equations:

$$\frac{d\vec{x}}{dt} = A\vec{x}, \quad \text{where} \quad A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad \vec{x}(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Problem 13. Consider the system $\vec{x}' = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} \vec{x}$ with initial condition $\vec{x}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

(a) Find the general solution

(b) Find the particular solution satisfying the initial condition

(c) Sketch the phase portrait

Problem 14. Solve the decoupled system:

$$\begin{cases} \frac{dy_1}{dt} = 3y_1 \\ \frac{dy_2}{dt} = -2y_2 \end{cases}$$

with initial conditions $y_1(0) = 2$ and $y_2(0) = 4$.

Problem 15. A dynamical system is described by $\vec{x}' = A\vec{x}$ where $A = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$.

(a) Find the eigenvalues of A

(b) Classify the equilibrium point at the origin (stable/unstable, node/spiral/saddle)

(c) Describe the long-term behavior of solutions

5 Inner Product (Section 6.1)

Problem 16. Compute the inner product $\langle \vec{u}, \vec{v} \rangle$ for:

(a) $\vec{u} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix}$ (standard inner product)

(b) $\vec{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ with $\langle \vec{u}, \vec{v} \rangle = 2u_1v_1 + 3u_2v_2$

Problem 17. Let $V = P_2$ be the space of polynomials of degree at most 2, with inner product $\langle p, q\rangle = \int_0^1 p(x)q(x)\,dx$.

(a) Compute $\langle 1, x\rangle$, $\langle 1, x^2\rangle$, and $\langle x, x^2\rangle$

(b) Find $\|x\|$ and $\|x^2\|$

(c) Are $\{1, x, x^2\}$ orthogonal?

Problem 18. Verify that the following defines an inner product on $\mathbb{R}^2$:

$$\langle \vec{u}, \vec{v}\rangle = 3u_1v_1 + 2u_2v_2$$

by checking all four axioms of an inner product.

Problem 19. Given $\vec{u} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$:

(a) Find $\|\vec{u}\|$

(b) Find a unit vector in the direction of $\vec{u}$

(c) Find all vectors orthogonal to $\vec{u}$

6 Angle and Orthogonality (Section 6.2)

Problem 20. Find the angle between the vectors:

(a) $\vec{u} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

(b) $\vec{u} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$

(c) $\vec{u} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

Problem 21. Determine which of the following pairs of vectors are orthogonal:

(a) $\vec{a} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 1 \\ -2 \\ -4 \end{pmatrix}$

(b) $\vec{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

(c) $\vec{a} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$

Problem 22. Find the orthogonal projection of $\vec{b} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$ onto $\vec{a} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$.

(a) Find $\text{proj}_{\vec{a}}\vec{b}$

(b) Find the component of $\vec{b}$ orthogonal to $\vec{a}$

(c) Verify that these two components are orthogonal

Problem 23. Use the Cauchy-Schwarz inequality to prove that for any vectors $\vec{u}$ and $\vec{v}$ in an inner product space:

$$|\langle \vec{u}, \vec{v} \rangle| \leq \|\vec{u}\| \cdot \|\vec{v}\|$$

Then verify this inequality for $\vec{u} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

7 Gram-Schmidt Process (Section 6.3)

Problem 24. Apply the Gram-Schmidt process to find an orthogonal basis for the subspace spanned by:

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Then normalize to obtain an orthonormal basis.

Problem 25. Find an orthonormal basis for the column space of:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}$$

Problem 26. Consider the polynomials $p_1(x) = 1$, $p_2(x) = x$, $p_3(x) = x^2$ on the interval $[0, 1]$ with inner product $\langle p, q \rangle = \int_0^1 p(x)q(x)\,dx$.

(a) Apply Gram-Schmidt to obtain an orthogonal set $\{q_1, q_2, q_3\}$

(b) Normalize to get an orthonormal set

Problem 27. Given vectors $\vec{u}_1 = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$ and $\vec{u}_2 = \begin{pmatrix} -1 \\ 0 \\ 7 \end{pmatrix}$:

(a) Find an orthogonal basis $\{\vec{v}_1, \vec{v}_2\}$ for $\text{span}\{\vec{u}_1, \vec{u}_2\}$

(b) Extend this to an orthogonal basis for $\mathbb{R}^3$

8 Orthogonal Matrices (Section 7.1)

Problem 28. Determine which of the following matrices are orthogonal:

(a) $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

(b) $B = \frac{1}{3}\begin{pmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{pmatrix}$

(c) $C = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

Problem 29. Let $Q = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}$

(a) Verify that Q is orthogonal

(b) Find Q^{-1}

(c) Show that Q preserves the length of vectors

Problem 30. Prove that if Q is an orthogonal matrix, then:

(a) $\det(Q) = \pm 1$

(b) Q^T is also orthogonal

(c) The rows of Q form an orthonormal set

Problem 31. Find an orthogonal matrix Q whose first column is $\frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$.

9 Orthogonal Diagonalization (Section 7.2)

Problem 32. Determine whether each symmetric matrix is orthogonally diagonalizable. If so, find an orthogonal matrix P and diagonal matrix D such that $A = PDP^T$.

(a) $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

(b) $B = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$

Problem 33. Let $A = \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}$.

(a) Show that A is symmetric

(b) Find the eigenvalues and eigenvectors

(c) Find an orthogonal matrix P that diagonalizes A

(d) Verify that $P^T AP$ is diagonal

Problem 34. Prove the Spectral Theorem: Every symmetric matrix is orthogonally diagonalizable.

Problem 35. Given $A = \begin{pmatrix} 6 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$:

(a) Find an orthogonal diagonalization of A

(b) Use this to compute A^{10}

10 Quadratic Forms (Section 7.3)

Problem 36. Write each quadratic form in matrix notation $\vec{x}^T A\vec{x}$:

(a) $Q(x_1, x_2) = 3x_1^2 + 4x_1x_2 + 5x_2^2$

(b) $Q(x_1, x_2, x_3) = x_1^2 - 2x_2^2 + 3x_3^2 + 4x_1x_2 - 6x_2x_3$

Problem 37. For the quadratic form $Q(x_1, x_2) = 5x_1^2 + 8x_1x_2 + 5x_2^2$:

(a) Write Q in the form $\vec{x}^T A\vec{x}$

(b) Find an orthogonal change of variables that eliminates the cross-product term

(c) Classify the quadratic form (positive definite, negative definite, or indefinite)

Problem 38. Determine whether each quadratic form is positive definite, negative definite, indefinite, or positive/negative semidefinite:

(a) $Q(\vec{x}) = 2x_1^2 + 4x_1x_2 + 3x_2^2$

(b) $Q(\vec{x}) = -x_1^2 + 2x_1x_2 - x_2^2$

(c) $Q(\vec{x}) = x_1^2 + x_2^2 + x_3^2$

Problem 39. Find the principal axes and classify the conic section:

$$5x^2 - 4xy + 2y^2 = 6$$

11 Optimization using Quadratic Forms (Section 7.4)

Problem 40. Find the maximum and minimum values of $Q(\vec{x}) = 3x_1^2 + 2x_2^2 + 2x_1x_2$ subject to $\|\vec{x}\| = 1$.

Problem 41. Consider the function $f(x, y) = 4x^2 + 4xy + y^2$.

(a) Find the maximum value of f on the unit circle $x^2 + y^2 = 1$

(b) Find the minimum value of f on the unit circle

(c) At what points are these extreme values attained?

Problem 42. Use quadratic forms to find the extreme values of:

$$f(x, y, z) = x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$$

subject to the constraint $x^2 + y^2 + z^2 = 1$.

Problem 43. A company's profit function is given by $P(x_1, x_2) = -2x_1^2 - x_2^2 + x_1x_2 + 8x_1 + 6x_2$.

(a) Find the critical points

(b) Use the second derivative test (Hessian matrix) to classify each critical point

(c) Find the maximum profit

12 Rank, Nullity and Matrix Spaces (Section 4.9)

Problem 44. For each matrix, find the rank, nullity, a basis for the column space, and a basis for the null space:

(a) $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 2 \end{pmatrix}$

(b) $B = \begin{pmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 1 & 5 \\ -1 & -2 & 1 & 1 \end{pmatrix}$

Problem 45. Verify the Rank-Nullity Theorem for the matrix:

$$A = \begin{pmatrix} 1 & 3 & 4 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & -4 & 0 \end{pmatrix}$$

Problem 46. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation with matrix representation:

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(a) Find $\text{rank}(A)$ and $\text{nullity}(A)$

(b) Find a basis for $\text{Col}(A)$ and $\text{Nul}(A)$

(c) Is T one-to-one? Is T onto?

Problem 47. Find the dimension of the row space, column space, and null space of:

$$A = \begin{pmatrix} 2 & 4 & -2 & 1 \\ 1 & 2 & -1 & 0 \\ 3 & 6 & -3 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Then verify that $\dim(\text{Row}(A)) = \dim(\text{Col}(A))$.

Problem 48. Show that the set of all 2×2 symmetric matrices forms a subspace of $M_{2\times 2}$. What is its dimension? Find a basis.

13 Geometry of Matrix Operators (Section 8.6)

Problem 49. Describe the geometric action of each linear transformation:

(a) $T(\vec{x}) = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \vec{x}$

(b) $T(\vec{x}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \vec{x}$

(c) $T(\vec{x}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \vec{x}$

Problem 50. Let $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ where $\theta = \frac{\pi}{4}$.

(a) Describe the geometric transformation represented by A

(b) Find the image of $\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ under this transformation

(c) Find A^4. What transformation does it represent?

Problem 51. Consider the shear transformation $S = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$.

(a) Describe how S transforms the unit square with vertices $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$ for $k = 2$

(b) Find the eigenvalues and eigenvectors of S

(c) Does a shear transformation preserve area? Justify your answer

Problem 52. The matrix $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ has eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 2$ with eigenvectors $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

(a) Describe geometrically what A does to vectors in the directions of $\vec{v}_1$ and $\vec{v}_2$

(b) Sketch the image of the unit circle under transformation by A

(c) What is the area magnification factor of this transformation?

Problem 53. Decompose the transformation $T(\vec{x}) = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} \vec{x}$ into:

(a) A rotation followed by scaling along coordinate axes

(b) Describe the geometric effect on the unit circle

End of Worksheet