Yolymatics

TUTORIALS

CMB & Early Universe

Comprehensive Study Notes & Practice Problems

November 6, 2025

Contents

1 Introduction to the Early Universe

> **What is the Early Universe?**
>
> The **early universe** refers to the period from the Big Bang (approximately 13.8 billion years ago) to roughly 380,000 years after, when the universe underwent rapid expansion, extreme temperatures, and fundamental physical processes that shaped the cosmos we observe today.

The study of the early universe combines cosmology, particle physics, and general relativity to understand:

- The origin and evolution of matter and energy

- The formation of fundamental particles

- The emergence of the first atoms

- The seeds of structure formation (galaxies, stars, planets)

- The observable imprints left on the cosmic microwave background

1.1 Timeline of the Early Universe

Key Epochs in Cosmic History

1. **Planck Epoch** ($t < 10^{-43}$ s): The earliest moment, where quantum gravity effects dominate. Our current physics theories break down here.

2. **Grand Unification Epoch** (10^{-43} s to 10^{-36} s): The strong nuclear force separates from the electroweak force. Temperature $T \sim 10^{29}$ K.

3. **Inflationary Epoch** (10^{-36} s to 10^{-32} s): The universe undergoes exponential expansion, growing by a factor of at least 10^{26} in a fraction of a second.

4. **Electroweak Epoch** (10^{-32} s to 10^{-12} s): The electromagnetic and weak nuclear forces separate. The Higgs mechanism gives particles mass.

5. **Quark Epoch** (10^{-12} s to 10^{-6} s): Quarks and gluons exist freely in a quark-gluon plasma at temperatures $T > 10^{13}$ K.

6. **Hadron Epoch** (10^{-6} s to 1 s): Quarks combine to form hadrons (protons and neutrons). The universe cools to $T \sim 10^{13}$ K.

7. **Lepton Epoch** (1 s to 10 s): Leptons (electrons, positrons, neutrinos) dominate. Matter-antimatter annihilation occurs.

8. **Photon Epoch/Radiation Era** (10 s to 380,000 years): Photons dominate the energy density. Nucleosynthesis begins at $t \sim 3$ minutes.

9. **Recombination Era** ($t \sim 380,000$ years): Electrons combine with nuclei to form neutral atoms. The universe becomes transparent to radiation.

1.2 The Big Bang Theory

Fundamental Principles

The Big Bang theory is supported by three major observational pillars:

1. **Hubble's Law**: Galaxies are receding from us with velocities proportional to their distance ($v = H_0 d$), indicating universal expansion.

2. **Cosmic Microwave Background (CMB)**: Remnant thermal radiation from the early universe, observed at $T = 2.725$ K today.

3. **Primordial Nucleosynthesis**: The observed abundances of light elements (H, He, Li, Be) match predictions from Big Bang nucleosynthesis (BBN).

The universe began in an extremely hot, dense state and has been expanding and cooling ever since. This expansion is described by the Friedmann equations derived from General Relativity:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3} \tag{1}$$

where:

- H is the Hubble parameter

- $a(t)$ is the scale factor (describes the relative expansion)

- ρ is the energy density

- k is the curvature parameter ($k = 0$ for flat universe)

- Λ is the cosmological constant (dark energy)

2 The Cosmic Microwave Background (CMB)

2.1 What is the CMB?

The **Cosmic Microwave Background (CMB)** is the relic thermal radiation from the early universe, emitted approximately 380,000 years after the Big Bang during the epoch of recombination. It represents the oldest light we can observe and provides a snapshot of the universe when it first became transparent.

2.2 Discovery and Historical Context

The CMB was accidentally discovered in 1965 by Arno Penzias and Robert Wilson at Bell Laboratories, earning them the Nobel Prize in Physics in 1978. They detected a persistent background noise at microwave wavelengths (around 7.35 cm) that was:

- Uniform across the entire sky

- Isotropic (same in all directions to about 1 part in 10^5)

- Consistent with blackbody radiation at $T \approx 2.7$ K

This discovery provided compelling evidence for the Big Bang theory and revolutionized cosmology.

2.3 Formation of the CMB: Recombination

The Recombination Process

Before recombination ($t < 380,000$ years):

- The universe was a hot, dense plasma of electrons, protons, and photons

- Temperature: $T > 3000$ K

- Photons constantly scattered off free electrons (Thomson scattering)

- The universe was *opaque* to radiation

During recombination ($t \approx 380,000$ years):

- Temperature dropped to $T \approx 3000$ K

- Electrons combined with protons to form neutral hydrogen atoms:

$$e^- + p^+ \rightarrow H + \gamma \tag{2}$$

- Free electron density decreased dramatically

- Photons could travel freely without scattering

- The universe became *transparent*

The photons released during recombination have been traveling through space ever since. Due to the expansion of the universe, their wavelengths have been stretched (redshifted) from the visible/infrared range to microwave wavelengths today.

2.4   Redshift and Temperature Evolution

The temperature of the CMB has cooled due to cosmic expansion. The relationship between redshift z, scale factor a, and temperature is:

$$T(z) = T_0(1 + z) = T_0 \frac{a_0}{a} \tag{3}$$

where:

- $T_0 = 2.725$ K (current CMB temperature)

- At recombination: $z \approx 1090$, so $T_{rec} \approx 2970$ K

- The scale factor has increased by a factor of 1091 since recombination

2.5   Blackbody Spectrum of the CMB

The CMB has the most perfect blackbody spectrum ever measured in nature. The Planck function describes the intensity:

$$I(\nu, T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/k_B T} - 1} \tag{4}$$

COBE and Planck Measurements

COBE (1989-1993) satellite measured:

- Peak wavelength: $\lambda_{max} = 1.9$ mm

- Temperature: $T = 2.725 \pm 0.002$ K

- Perfect blackbody fit (agreement to 1 part in 10^5)

Planck satellite (2009-2013) improved precision:

- Temperature: $T_0 = 2.7255 \pm 0.0006$ K

- Mapped temperature fluctuations to $\Delta T/T \sim 10^{-5}$

- Provided detailed angular power spectrum

2.6   CMB Anisotropies: Temperature Fluctuations

While the CMB is remarkably uniform, it contains tiny temperature fluctuations at the level of $\Delta T/T \sim 10^{-5}$. These anisotropies arise from:

2.6.1   Primary Anisotropies

1. **Quantum Fluctuations during Inflation**

   - Primordial density perturbations generated during inflation
   - Stretched to cosmic scales by exponential expansion
   - Seeded structure formation (galaxies, galaxy clusters)

2. **Acoustic Oscillations (Baryon-Photon Plasma)**

   - Before recombination, photons and baryons were tightly coupled
   - Density perturbations created pressure waves (sound waves)
   - These oscillations froze at recombination
   - Observed as peaks in the angular power spectrum

3. **Sachs-Wolfe Effect**

   - Photons climbing out of gravitational potential wells lose energy (gravitational redshift)
   - Creates temperature variations: $\frac{\Delta T}{T} \approx \frac{\Delta \Phi}{c^2}$
   - Dominant on large angular scales (> 1)

2.6.2   Secondary Anisotropies

1. **Integrated Sachs-Wolfe Effect (ISW)**

   - Time-varying gravitational potentials along the photon path
   - Important in universes with dark energy

2. **Sunyaev-Zel'dovich Effect (SZ)**

   - CMB photons scattered by hot electrons in galaxy clusters
   - Creates temperature distortions
   - Used to detect and study galaxy clusters

3. **Gravitational Lensing**

   - CMB photons deflected by intervening mass distributions
   - Smooths out small-scale anisotropies
   - Provides information about dark matter distribution

2.7   Angular Power Spectrum

The CMB temperature fluctuations are analyzed using spherical harmonics:

$$\frac{\Delta T(\theta, \phi)}{T} = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(\theta, \phi) \tag{5}$$

The angular power spectrum C_ℓ is defined as:

$$C_\ell = \frac{1}{2\ell + 1} \sum_{m=-\ell}^{\ell} |a_{\ell m}|^2 \tag{6}$$

Physical Interpretation of Power Spectrum Features

Key Features:

- **First Acoustic Peak** ($\ell \approx 220$): Corresponds to compressions at recombination. Location determines spatial curvature ($\Omega_k \approx 0$, flat universe).

- **Second Peak** ($\ell \approx 540$): First rarefaction. Ratio to first peak constrains baryon density $\Omega_b h^2 \approx 0.022$.

- **Third Peak** ($\ell \approx 800$): Second compression. Peak heights constrain dark matter density $\Omega_c h^2 \approx 0.12$.

- **Damping Tail** ($\ell > 1000$): Photon diffusion (Silk damping) smooths small-scale fluctuations.

- **Large Scale** ($\ell < 100$): Sachs-Wolfe plateau constrains primordial power spectrum.

3 Cosmic Inflation

3.1 The Horizon and Flatness Problems

> **Problems with Standard Big Bang**
>
> 1. **Horizon Problem:**
>
>   - CMB is uniform to 1 part in 10^5 across the entire sky
>
>   - Regions separated by more than 2° were never in causal contact before recombination
>
>   - How did they reach the same temperature?
>
> 2. **Flatness Problem:**
>
>   - Universe is spatially flat: $\Omega_{total} = 1.000 \pm 0.001$
>
>   - Requires extreme fine-tuning of initial conditions ($|\Omega - 1| < 10^{-60}$ at Planck time)
>
>   - Why is the universe so precisely flat?
>
> 3. **Magnetic Monopole Problem:**
>
>   - Grand Unified Theories predict copious production of magnetic monopoles
>
>   - None have been observed

3.2 Inflationary Solution

Inflation is a period of exponential expansion in the very early universe ($t \sim 10^{-36}$ to 10^{-32} s) driven by a scalar field (the *inflaton*) with negative pressure.

During inflation, the scale factor grows exponentially:

$$a(t) = a_i e^{Ht} \tag{7}$$

where H is approximately constant during inflation.

> **How Inflation Solves the Problems**
>
> 1. **Horizon Problem**: The observable universe originated from a tiny causally connected region that was stretched to cosmic scales.
>
> 2. **Flatness Problem**: Exponential expansion drives $\Omega \to 1$ regardless of initial conditions (like blowing up a balloon makes its surface appear flat).
>
> 3. **Monopole Problem**: Monopoles are diluted to negligibly low densities by the enormous expansion.

3.3 Quantum Fluctuations and Structure Formation

Quantum fluctuations in the inflaton field during inflation are:

- Stretched to macroscopic scales by exponential expansion

- Frozen as classical density perturbations when they exit the horizon

- Nearly scale-invariant (approximately same amplitude on all scales)

- Gaussian distributed with small non-Gaussianities

The power spectrum of primordial perturbations is:

$$P(k) = A_s \left(\frac{k}{k_0} \right)^{n_s - 1} \tag{8}$$

where $n_s \approx 0.96$ is the scalar spectral index (slightly less than 1 indicates "red" tilt). These perturbations:

- Imprint temperature fluctuations on the CMB

- Seed gravitational collapse of matter

- Lead to formation of galaxies, galaxy clusters, and large-scale structure

4 Big Bang Nucleosynthesis (BBN)

4.1 Formation of Light Elements

> **BBN Timeline**
>
> At $t \approx 1$ second after the Big Bang:
>
> - Temperature: $T \approx 10^{10}$ K (~ 1 MeV)
>
> - The universe consists of photons, neutrinos, electrons, positrons, protons, and neutrons
>
> - Weak interactions keep neutrons and protons in equilibrium:
>
> $$n + \nu_e \leftrightarrow p + e^- \tag{9}$$
> $$n + e^+ \leftrightarrow p + \bar{\nu}_e \tag{10}$$
> $$n \leftrightarrow p + e^- + \bar{\nu}_e \tag{11}$$
>
> At $t \approx 3$ minutes:
>
> - Temperature drops to $T \approx 10^9$ K (~ 0.1 MeV)
>
> - Nuclear reactions begin forming light elements
>
> - Neutron-to-proton ratio freezes at $n/p \approx 1/7$
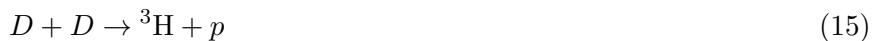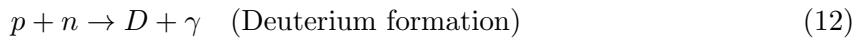
4.2 Nuclear Reaction Network

Key reactions in BBN:

$$p + n \to D + \gamma \quad \text{(Deuterium formation)} \tag{12}$$
$$D + p \to {}^3\text{He} + \gamma \tag{13}$$
$$D + D \to {}^3\text{He} + n \tag{14}$$
$$D + D \to {}^3\text{H} + p \tag{15}$$
$${}^3\text{He} + n \to {}^4\text{He} + \gamma \tag{16}$$
$${}^3\text{H} + p \to {}^4\text{He} + \gamma \tag{17}$$

4.3 Predicted Abundances

> **BBN Predictions vs. Observations**
>
> | Element | Predicted | Observed |
> |---|---|---|
> | ^{4}He (by mass) | 24-25% | 24% |
> | D/H (by number) | 2.5×10^{-5} | $(2.5 - 3) \times 10^{-5}$ |
> | ^{3}He/H | 10^{-5} | 10^{-5} |
> | ^{7}Li/H | 5×10^{-10} | 1.5×10^{-10} |
>
> The agreement between predictions and observations is remarkable, especially for helium-4 and deuterium. The **lithium problem** (discrepancy in ^{7}Li) remains an open question in cosmology.

4.4 Constraints on Physics

BBN provides constraints on:

1. **Baryon Density**: $\Omega_b h^2 = 0.022 \pm 0.002$ (consistent with CMB measurements)

2. **Number of Neutrino Species**: $N_\nu = 3.0 \pm 0.3$ (confirms three generations of Standard Model neutrinos)

3. **Neutron Lifetime**: $\tau_n = 880.2 \pm 1.0$ s

4. **Early Universe Expansion Rate**: Consistent with standard cosmology

5 Dark Matter and Dark Energy

5.1 Evidence for Dark Matter

> **What is Dark Matter?**
>
> Dark matter is a form of matter that does not emit, absorb, or reflect electromagnetic radiation. It interacts primarily through gravity and possibly the weak nuclear force. It constitutes approximately 27% of the universe's total mass-energy content.

Evidence for dark matter comes from multiple independent sources:

1. **Galaxy Rotation Curves**

   - Orbital velocities of stars remain constant at large radii
   - Expected to decrease as $v \propto r^{-1/2}$ for visible matter alone
   - Requires extended dark matter halo: $M(r) \propto r$

2. **Gravitational Lensing**

   - Light from distant galaxies bent by intervening mass
   - Mass distribution inferred from lensing exceeds visible matter by factor of 6-7
   - Bullet Cluster shows dark matter separated from baryonic matter

3. **CMB Acoustic Peaks**

   - Ratio of peak heights depends on baryon-to-photon ratio
   - Total matter density from peak locations
   - Difference gives dark matter density: $\Omega_c h^2 \approx 0.12$

4. **Large-Scale Structure**

   - Galaxy distribution and clustering
   - Formation of structure requires dark matter to seed gravitational collapse
   - Simulations with dark matter match observations

5.2 Dark Matter Candidates

> **Possible Dark Matter Particles**
>
> **Cold Dark Matter (CDM):**
>
> - WIMPs (Weakly Interacting Massive Particles): $m \sim 10 - 1000 \text{ GeV/c}^2$
> - Axions: $m \sim 10^{-6} - 10^{-3} \text{ eV/c}^2$
> - Sterile neutrinos: $m \sim \text{keV/c}^2$
>
> **Properties:**
>
> - Non-relativistic when structure formation begins
> - Allows bottom-up structure formation (small scales first)
> - Consistent with observed large-scale structure

5.3 Dark Energy

What is Dark Energy?

Dark energy is a mysterious form of energy that permeates all of space and drives the accelerated expansion of the universe. It constitutes approximately 68% of the universe's total mass-energy content.

5.3.1 Discovery

In 1998, observations of Type Ia supernovae showed:

- Distant supernovae are dimmer than expected

- Implies universe is accelerating, not decelerating

- Requires a component with negative pressure ($w = p/\rho < -1/3$)

5.3.2 Properties

Dark energy can be characterized by the equation of state parameter:

$$w = \frac{p}{\rho c^2} \tag{18}$$

For a cosmological constant: $w = -1$ (constant in time and space)
Current measurements: $w = -1.03 \pm 0.03$ (consistent with cosmological constant)

5.3.3 Candidates

1. **Cosmological Constant (Λ)**

   - Vacuum energy with constant density
   - Simplest explanation, fits data well
   - Fine-tuning problem: predicted value 10^{120} times larger than observed

2. **Quintessence**
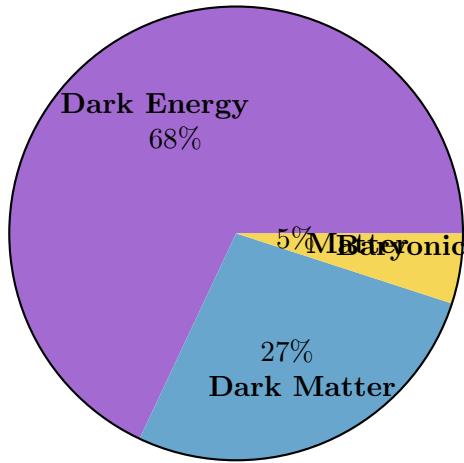
   - Dynamic scalar field with time-varying energy density
   - w can vary with time
   - Many models, none strongly favored

3. **Modified Gravity**

   - Alternative to General Relativity on large scales
   - Examples: f(R) gravity, DGP model
   - Most models ruled out by observations

6 Current Composition of the Universe

Based on Planck 2018 results, the universe today consists of:

Cosmological Parameters (Planck 2018)

Energy Densities:

- Total density parameter: $\Omega_{tot} = 1.0000 \pm 0.0008$ (flat universe)

- Dark energy: $\Omega_\Lambda = 0.6847 \pm 0.0073$

- Dark matter: $\Omega_c h^2 = 0.1200 \pm 0.0012$

- Baryonic matter: $\Omega_b h^2 = 0.02237 \pm 0.00015$

Expansion and Age:

- Hubble constant: $H_0 = 67.4 \pm 0.5$ km/s/Mpc

- Age of universe: $t_0 = 13.787 \pm 0.020$ billion years

Primordial Fluctuations:

- Scalar spectral index: $n_s = 0.9649 \pm 0.0042$

- Amplitude: $A_s = 2.1 \times 10^{-9}$

- Optical depth (reionization): $\tau = 0.054 \pm 0.007$

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7   Future Observations and Open Questions

7.1   Next-Generation CMB Experiments

1. CMB-S4 (2030s)

- Ground-based observatory with 500,000 detectors
- Goals: Constrain neutrino masses, test inflation models, measure gravitational lensing

2. LiteBIRD (Launch: 2028)

- Japanese space mission
- Search for B-mode polarization from primordial gravitational waves
- Target sensitivity: $r < 0.001$ (tensor-to-scalar ratio)

3. Simons Observatory (Operating)

- Ground-based in Atacama Desert, Chile
- High-resolution maps of CMB temperature and polarization

7.2   Open Questions in Cosmology

Major Unsolved Problems

1. **What is dark matter?**

   - Direct detection experiments ongoing
   - Particle physics beyond Standard Model required

2. **What is dark energy?**

   - Cosmological constant or dynamical field?
   - Why is its value so small compared to predictions?

3. **Did inflation occur?**

   - What is the inflaton field?
   - Detection of primordial gravitational waves would confirm

4. **What happened before the Big Bang?**

   - Quantum gravity needed
   - String theory, loop quantum gravity candidates

5. **Why is there more matter than antimatter?**

   - Baryon asymmetry problem
   - CP violation in early universe

6. **What is the fate of the universe?**

   - Big Rip, Heat Death, or Big Crunch?
   - Depends on evolution of dark energy

Practice Problems for Students

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Instructions for Practice Problems

These problems are designed to test your understanding of the CMB and early universe concepts. Work through them systematically, showing all calculations. Discuss your answers with your instructor.

Difficulty levels: $\star$ = Basic, $\star\star$ = Intermediate, $\star\star\star$ = Advanced

</div>

Section 1: Basic Concepts and Calculations

Problem 1 ($\star$)**.** The current temperature of the CMB is measured to be $T_0 = 2.725$ K.

(a) Calculate the peak wavelength of the CMB using Wien's displacement law: $\lambda_{max} = \frac{b}{T}$ where $b = 2.898 \times 10^{-3}$ m·K.

(b) In what part of the electromagnetic spectrum does this wavelength fall?

(c) Calculate the photon number density using $n_\gamma = \frac{2\zeta(3)}{\pi^2} \left(\frac{k_B T}{\hbar c} \right)^3$ where $\zeta(3) = 1.202$.

Working space:

    Answer:

Problem 2 ($\star$)**.** At recombination, the universe had a temperature of approximately $T_{rec} = 3000$ K and redshift $z_{rec} \approx 1090$.

(a) Using $T(z) = T_0(1 + z)$, verify that these values are consistent with today's CMB temperature.

(b) By what factor has the universe expanded since recombination?

(c) If a photon had wavelength $\lambda = 500$ nm at recombination, what is its wavelength today?

Working space:

Answer:

Problem 3 ($\star\star$). The Hubble constant is measured as $H_0 = 67.4$ km/s/Mpc.

(a) Convert H_0 to SI units (s^{-1}). (Note: 1 Mpc $= 3.086 \times 10^{22}$ m)

(b) Calculate the Hubble time: $t_H = 1/H_0$. Express your answer in years.

(c) The actual age of the universe is 13.8 billion years. Why is this different from the Hubble time?

Working space:

Answer:

Section 2: CMB Anisotropies and Power Spectrum

Problem 4 (★★)**.** CMB temperature fluctuations are of order $\Delta T/T \sim 10^{-5}$.

(a) If $T_0 = 2.725$ K, what is the typical magnitude of temperature fluctuations ΔT in microkelvin (μK)?

(b) The first acoustic peak in the angular power spectrum occurs at $\ell \approx 220$. Calculate the corresponding angular scale $\theta \approx 180/\ell$.

(c) This angular scale corresponds to the sound horizon at recombination. If the sound speed in the baryon-photon plasma was $c_s \approx c/\sqrt{3}$, estimate the comoving size of the sound horizon.

Working space:

Answer:

Problem 5 (★ ★ ★)**.** The acoustic oscillations in the early universe can be modeled as:

$$\frac{\partial^2 \delta}{\partial t^2} + \frac{\dot{a}}{a}\frac{\partial \delta}{\partial t} = c_s^2 \nabla^2 \delta + (\text{source terms})$$

where $\delta = \delta\rho/\rho$ is the density contrast.

(a) Explain physically what each term represents.

(b) In the radiation-dominated era with scale factor $a(t) \propto t^{1/2}$, show that the damping term is $\dot{a}/a = 1/(2t)$.

(c) Why do we observe multiple peaks in the CMB power spectrum?

Working space:

Answer:

Section 3: Cosmic Inflation

Problem 6 (★★)**.** During inflation, the scale factor grows as $a(t) = a_i e^{Ht}$ where H is approximately constant.

(a) If inflation lasts for time $\Delta t = 10^{-32}$ s and $H \approx 10^{36}$ s^{-1}, by what factor does the universe expand during inflation?

(b) A quantum fluctuation of wavelength $\lambda_i = 10^{-35}$ m at the start of inflation is stretched by this expansion. What is its final wavelength?

(c) Compare this to the current Hubble radius $c/H_0 \approx 10^{26}$ m. Could this fluctuation be observable today?

Working space:

Answer:

Problem 7 (★★★)**.** The horizon problem can be quantified by calculating the particle horizon at recombination.

(a) In a radiation-dominated universe, $a(t) \propto t^{1/2}$. The particle horizon is:

$$d_H(t) = a(t) \int_0^t \frac{c\, dt'}{a(t')} = 2ct$$

At recombination ($t_{rec} \approx 380,000$ years), calculate $d_H(t_{rec})$.

(b) The proper distance to the CMB surface today is about $d_{CMB} \approx 45$ billion light-years. Calculate the angular size of the horizon at recombination as seen today: $\theta_H = d_H(t_{rec})/d_{CMB}$.

(c) Regions separated by more than this angle were never in causal contact. Why is the CMB uniform over larger angular scales?

Working space:

Answer:

Section 4: Big Bang Nucleosynthesis

Problem 8 ($\star$). The neutron-to-proton ratio at the start of nucleosynthesis (T $\approx$ 0.8 MeV) is determined by thermal equilibrium:

$$\frac{n}{p} = e^{-\Delta mc^2/k_B T}$$

where $\Delta m = m_n - m_p = 1.293$ MeV/c^2.

(a) Calculate the neutron-to-proton ratio at $T = 0.8$ MeV. (Use $k_B T = 0.8$ MeV)

(b) Free neutrons decay with half-life $t_{1/2} = 10.2$ minutes. If nucleosynthesis begins at $t \approx 3$ minutes, what fraction of neutrons have decayed?

(c) Almost all remaining neutrons are incorporated into ^{4}He. If the final n/p ratio is 1/7, calculate the mass fraction of helium-4.

Working space:

Answer:

Problem 9 ($\star\star$). The abundance of deuterium is very sensitive to the baryon density. Consider the reaction $p + n \leftrightarrow D + \gamma$.

(a) The deuterium binding energy is $B_D = 2.22$ MeV. At what temperature does deuterium formation become favorable?

(b) The "deuterium bottleneck" delays nucleosynthesis. Explain why deuterium must form before heavier elements.

(c) Observed deuterium abundance: D/H $\approx 2.5 \times 10^{-5}$. Higher baryon density leads to more deuterium burning. Is this consistent with Planck's measurement of $\Omega_b h^2 = 0.022$?

Working space:

Answer:

Section 5: Dark Matter and Dark Energy

Problem 10 (⋆⋆). Galaxy rotation curves provide evidence for dark matter. Consider a galaxy with flat rotation curve $v(r) = v_0 = 200$ km/s for $r > 10$ kpc.

(a) Using Newton's law of gravity, $v^2 = GM(r)/r$, show that the enclosed mass grows linearly with radius: $M(r) \propto r$.

(b) Calculate the total mass enclosed within $r = 50$ kpc. (Use $G = 4.3 \times 10^{-3}$ pc $(km/s)^2$ $M_\odot^{-1}$)

(c) If visible matter accounts for only 1/6 of this mass, what is the dark matter mass within 50 kpc?

Working space:

Answer:

Problem 11 (⋆ ⋆ ⋆). The Friedmann equation for a flat universe ($k = 0$) is:

$$H^2 = \frac{8\pi G}{3}(\rho_m + \rho_\Lambda)$$

where $\rho_m = \rho_{m,0}a^{-3}$ (matter) and $\rho_\Lambda = $ constant (dark energy).

(a) Show that $H^2 = H_0^2[\Omega_m a^{-3} + \Omega_\Lambda]$ where $\Omega_m + \Omega_\Lambda = 1$.

(b) The universe transitions from deceleration to acceleration when $\ddot{a} = 0$. Using the acceleration equation:
$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho_m + 3p_\Lambda)$$
and $p_\Lambda = -\rho_\Lambda c^2$, find the scale factor a_{trans} at transition.

(c) For $\Omega_m = 0.3$ and $\Omega_\Lambda = 0.7$, at what redshift did acceleration begin?

Working space:

Answer:

Section 6: Advanced Integration Problems

Problem 12 ($\star\star\star$). The energy density of the CMB is given by the Stefan-Boltzmann law:

$$\rho_{CMB} = \frac{4\sigma T^4}{c^3} = \frac{\pi^2 k_B^4}{15\hbar^3 c^3} T^4$$

(a) Calculate the current energy density of the CMB for $T_0 = 2.725$ K. Express in J/m^3 and eV/cm^3.

(b) At what redshift was the energy density of CMB photons equal to the critical density today ($\rho_c = 9.5 \times 10^{-27}$ kg/m^3)?

(c) Matter density scales as $\rho_m \propto a^{-3}$ while radiation scales as $\rho_r \propto a^{-4}$. At what redshift were matter and radiation densities equal (matter-radiation equality)?

Working space:

Answer:

Problem 13 ($\star\star\star$). The comoving distance to an object at redshift z is:

$$\chi(z) = \frac{c}{H_0} \int_0^z \frac{dz'}{E(z')}$$

where $E(z) = \sqrt{\Omega_m (1+z)^3 + \Omega_\Lambda}$.

(a) For a flat universe with $\Omega_m = 0.3$ and $\Omega_\Lambda = 0.7$, set up (but don't solve) the integral for the comoving distance to the CMB surface at $z = 1090$.

(b) The angular diameter distance is $d_A = \chi/(1+z)$. Calculate d_A for the CMB. (You may use numerical integration or approximation)

(c) The sound horizon at recombination subtends angle $\theta = 0.6$. What is the physical size of the sound horizon in Mpc?

Working space:

Answer:

Section 7: Conceptual Questions

Problem 14 ($\star\star$). Answer the following conceptual questions with clear explanations:

(a) Why is the CMB a blackbody spectrum? What process established thermal equilibrium in the early universe?

(b) Explain why the CMB is redshifted, but still maintains a blackbody spectrum (hint: discuss adiabatic expansion).

(c) The CMB is nearly isotropic, but there's a dipole anisotropy of order $\Delta T/T \sim 10^{-3}$. What causes this dipole?

Working space:

Answer:

Problem 15 ($\star\star\star$). Consider the following statements and explain whether each is true or false:

(a) "The CMB represents the surface of last scattering, so we are looking at a 2D shell around us."

(b) "If inflation never occurred, we would still observe the CMB, but it would have different properties."

(c) "Dark energy and dark matter are both 'dark' because they don't emit light, so they might be related."

(d) "The temperature fluctuations in the CMB are caused by quantum fluctuations that occurred during inflation."

Working space:

Answer:

Section 8: Research-Based Questions

Problem 16 ($\star\star\star$)**. Project Question:** Research and write a short report (2-3 pages) on one of the following topics:

(a) The Planck satellite mission: instruments, observations, and key discoveries

(b) The search for primordial gravitational waves (B-mode polarization)

(c) The Hubble tension: different measurements of H_0 and implications

(d) The lithium problem in Big Bang nucleosynthesis

Your report should include:

- Historical background

- Current observational status

- Theoretical implications

- Future prospects

- At least 5 references

Working space:

Additional Resources

Recommended Textbooks:

- Ryden, B. "Introduction to Cosmology" (2nd Edition)

- Dodelson, S. & Schmidt, F. "Modern Cosmology" (2nd Edition)

- Weinberg, S. "Cosmology"

Online Resources:

- Planck Mission Results: http://www.cosmos.esa.int/web/planck

- Particle Data Group: http://pdg.lbl.gov

- NASA's WMAP Tutorial: http://map.gsfc.nasa.gov

Video Lectures:

- Leonard Susskind's Cosmology Lectures (Stanford)

- Sean Carroll's "The Big Picture" Lectures

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"Understanding the universe begins with curiosity"