# Multisets in Discrete Mathematics

## Combinations with Repetition - M344

Yolymatics Tutorials

www.yolymaticstutorials.com

November 3, 2025

# Outline

# What is a Multiset?

## Formal Definition

A **multiset** (or **bag**) is a generalization of a set where elements can appear more than once. Formally, a multiset $M$ over a set $S$ is a function $m : S \to \mathbb{N}_0$ where $m(x)$ denotes the **multiplicity** of element $x$ in $M$.

## Notation and Examples

- **Multiset notation**: $\{a, a, b, c, c, c\}$ or $\{a^2, b, c^3\}$
- **Cardinality**: $|M| = \sum_{x \in S} m(x)$ (total number of elements counting multiplicities)
- **Example**: $M = \{1^3, 2^2, 5\}$ means: 1 appears 3 times, 2 appears 2 times, 5 appears once

## Key Property

Order does NOT matter: $\{a, a, b\} = \{a, b, a\} = \{b, a, a\}$

# Multisets in Discrete Mathematics

## Applications in Discrete Math

- **Combinatorics**: Counting with repetition allowed
- **Number Theory**: Integer partitions and compositions
- **Generating Functions**: Coefficient extraction in power series
- **Graph Theory**: Degree sequences (multisets of vertex degrees)
- **Algorithm Analysis**: Analyzing data structures with duplicates
- **Recurrence Relations**: Solutions to linear homogeneous recurrences

# Multisets vs Combinations vs Permutations

## Counting Classification

| Type | Order? | Repetition? | Formula | Example |
|---|---|---|---|---|
| Permutations | Yes | No | $P(n,r) = \frac{n!}{(n-r)!}$ | $n = 4, r = 2 : 12$ |
| Combinations | No | No | $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ | $n = 4, r = 2 : 6$ |
| Perm. w/ Rep | Yes | Yes | $n^r$ | $n = 4, r = 2 : 16$ |
| **Multisets** | **No** | **Yes** | $\binom{n+r-1}{r}$ | $n = 4, r = 2 : 10$ |

## Discrete Math Context

- **Problem**: Choose $r$ objects from $n$ types with repetition, order irrelevant
- **Notation**: $\left(\!\binom{n}{r}\!\right)$ called "n multichoose r"
- **Read as**: "Combinations of $n$ types taken $r$ at a time with repetition"

# The Fundamental Multiset Formula

## Problem Statement

How many ways can we select $r$ objects from $n$ types of objects, where repetition is allowed and order does not matter?

## Formula: Combinations with Repetition

$$\left(\!\!\binom{n}{r}\!\!\right) = \binom{n+r-1}{r} = \binom{n+r-1}{n-1} = \frac{(n+r-1)!}{r!(n-1)!}$$

where:

- $n =$ number of types (distinct objects to choose from)
- $r =$ number of selections (objects to be selected)

## Notation

$\left(\!\binom{n}{r}\!\right)$ is read as "n multichoose r" or "multiset coefficient"

# Stars and Bars Method

## Visual Representation

The multiset problem is equivalent to placing $r$ identical stars into $n$ distinct bins using $n-1$ bars as separators.

## Example: Choose 5 items from 3 types

Select 5 objects from types $\{A, B, C\}$

Representation: $\star\star|\star\star\star|$ means 2 of type A, 3 of type B, 0 of type C

Total arrangements: We have 5 stars and 2 bars to arrange

$$\binom{5+2}{2} = \binom{7}{2} = 21 \text{ ways}$$

## Key Insight

We arrange $r$ stars and $(n-1)$ bars, giving $\binom{n+r-1}{\cdot}$ or $\binom{n+r-1}{\cdot}$

# Problem 1: Multiset Cardinality and Subsets

## Problem

Let $M = \{a^3, b^2, c^4\}$ be a multiset.

1. What is the cardinality $|M|$?
2. How many multisets can be formed from $M$ (including the empty multiset and $M$ itself)?
3. How many sub-multisets of $M$ have cardinality exactly 5?

# Working Space

# Problem 2: Basic Counting with Repetition

## Problem

Using the multiset formula $\left(\!\binom{n}{r}\!\right) = \binom{n+r-1}{r}$:

1. Calculate $\left(\!\binom{5}{3}\!\right)$
2. How many multisets of cardinality 6 can be formed from a set of 4 elements?
3. Verify that $\left(\!\binom{n}{r}\!\right) = \binom{n}{r}$ is NOT generally equal

# Working Space

# Problem 3: Non-negative Integer Solutions

## Problem

Find the number of non-negative integer solutions to:

$$x_1 + x_2 + x_3 + x_4 = 12$$

Explain why this is equivalent to finding multisets of cardinality 12 from a set of 4 elements.

# Working Space

# Problem 4: Weak Compositions

## Problem

A **weak composition** of $n$ into $k$ parts is a sequence $(a_1, a_2, \ldots, a_k)$ where $a_i \geq 0$ and $\sum a_i = n$.

1. How many weak compositions of 10 into 4 parts exist?
2. Compare this with the number of compositions (where each $a_i > 0$)

# Working Space

# Problem 5: Distribution and Stars-Bars

## Problem

1. In how many ways can 8 identical objects be distributed into 3 distinct boxes?
2. Express this problem using stars and bars notation
3. What if each box must contain at least one object?

# Working Space

# Problem 6: Positive Integer Solutions and Compositions

## Problem

Find the number of **positive** integer solutions to:

$$x_1 + x_2 + x_3 = 10$$

1. Use the substitution method ($y_i = x_i - 1$)
2. Explain why this counts the number of **compositions** of 10 into 3 parts
3. Compare with weak compositions

# Working Space

# Problem 6: Positive Integer Solutions

## Problem

Find the number of **positive** integer solutions to:

$$x_1 + x_2 + x_3 = 10$$

*Hint: Each variable must be at least 1.*

# Working Space

# Problem 7: General Minimum Constraints

## Problem

Find the number of integer solutions to:

$$x_1 + x_2 + x_3 + x_4 = 20$$

where $x_1 \geq 3$, $x_2 \geq 1$, $x_3 \geq 2$, and $x_4 \geq 0$.
Use the substitution method and express using multiset notation.

# Working Space

# Problem 8: Inclusion-Exclusion Principle

## Problem

Find the number of non-negative integer solutions to:

$$x_1 + x_2 + x_3 = 10$$

where $x_1 \leq 4$, $x_2 \leq 5$, and $x_3 \leq 6$.

1. Calculate the total without upper bounds
2. Apply inclusion-exclusion systematically

# Working Space

# Problem 9: Bijection and Correspondence

## Problem

Prove that there is a bijection between:

1. Multisets of cardinality $r$ from an $n$-element set
2. Non-negative integer solutions to $x_1 + x_2 + \cdots + x_n = r$

Describe the correspondence explicitly.

# Working Space

# Problem 10: Recurrence Relations

## Problem

Let $M(n, r)$ denote the number of multisets of cardinality $r$ from an $n$-element set.

1. Prove that $M(n, r) = M(n-1, r) + M(n, r-1)$ for $n, r \geq 1$
2. Give an initial conditions for this recurrence
3. Explain the combinatorial meaning of this recurrence

# Working Space

# Problem 11: Generating Functions

## Problem

Consider the generating function:

$$G(x) = (1 + x + x^2 + x^3 + \cdots)^n = \frac{1}{(1-x)^n}$$

1. Explain why the coefficient of $x^r$ in $G(x)$ gives $\left(\!\binom{n}{r}\!\right)$
2. Use the binomial series to verify: $\frac{1}{(1-x)^n} = \sum_{r=0}^{\infty} \binom{n+r-1}{r} x^r$
3. Find the coefficient of $x^{10}$ in $(1 + x + x^2 + \cdots)^5$

# Working Space

# Problem 12: Integer Partitions

## Problem

An **integer partition** of $n$ is a multiset of positive integers that sum to $n$.

1. How many partitions of 7 exist?
2. How many partitions of 10 into at most 4 parts exist?
3. Express "partitions of $n$ into at most $k$ parts" as a multiset problem

*Note: Order doesn't matter, so* $\{3, 2, 2\}$ *is the same partition as* $\{2, 3, 2\}$

# Working Space

# Problem 13: Multinomial Coefficients

## Problem

The multinomial coefficient is defined as:

$$\binom{n}{k_1, k_2, \ldots, k_m} = \frac{n!}{k_1!k_2!\cdots k_m!}$$

where $k_1 + k_2 + \cdots + k_m = n$.

1. Calculate $\binom{10}{3,3,4}$
2. How does this relate to multisets?
3. Find $\sum \binom{10}{k_1,k_2,k_3}$ where the sum is over all non-negative $k_i$ with $k_1 + k_2 + k_3 = 10$

# Working Space

# Problem 14: Ferrer's Diagram and Conjugates

## Problem

A **Ferrer's diagram** represents integer partitions graphically using dots.

1. Draw the Ferrer's diagram for the partition $\{5, 3, 3, 1\}$ of 12
2. The **conjugate partition** is obtained by reflecting the diagram
3. Find the conjugate of $\{5, 3, 3, 1\}$
4. Prove that partitions of $n$ into at most $k$ parts bijectively correspond to partitions of $n$ with largest part at most $k$

## Working Space

# Problem 15: Constrained Generating Functions

## Problem

Find the number of non-negative integer solutions to:

$$x_1 + x_2 + x_3 = 15$$

where $x_1$ is even, $x_2$ is odd, and $x_3 \leq 5$.
Use generating functions or direct counting with appropriate substitutions.

# Working Space

# Problem 16: Combinatorial Identity

## Problem

Prove the following identity:

$$\sum_{k=0}^{r} \binom{n}{k}\binom{m}{r-k} = \binom{n+m}{r}$$

1. Give a combinatorial proof using multisets
2. Relate this to the generating function $(1+x)^n(1+x)^m = (1+x)^{n+m}$

# Working Space

# Problem 17: Vandermonde's Identity for Multisets

## Problem

Prove that:

$$\left(\!\!\binom{n}{r}\!\!\right) = \sum_{k=0}^{r} \binom{n-1}{k} \left(\!\!\binom{n-1}{r-k}\!\!\right)$$

Give a combinatorial interpretation of this identity.

*Hint: Consider how many times the first element appears in a multiset.*

# Working Space

# Problem 18: Stars and Bars - Formal Proof

## Problem

Give a rigorous proof of the stars and bars theorem:
"The number of ways to place $r$ indistinguishable balls into $n$ distinguishable bins is $\binom{r+n-1}{n-1}$"

1. Explain the bijection between distributions and binary strings
2. Formalize the correspondence
3. Extend to handle minimum constraints

# Working Space

# Formula 1: Basic Multiset Coefficient

## Statement

The number of multisets of size $r$ chosen from $n$ types is:

$$\left(\!\!\binom{n}{r}\!\!\right) = \binom{n+r-1}{r} = \binom{n+r-1}{n-1}$$

## Alternative Forms

- $\dfrac{(n+r-1)!}{r!(n-1)!}$ - factorial form
- $\dfrac{(n+r-1)(n+r-2)\cdots(n)}{r!}$ - product form
- Number of non-negative integer solutions to $x_1 + x_2 + \cdots + x_n = r$

# Formula 2: Positive Integer Solutions

## Statement

The number of **positive** integer solutions to:

$$x_1 + x_2 + \cdots + x_n = r$$

where each $x_i \geq 1$, is:

$$\binom{r-1}{n-1} = \binom{r-1}{r-n}$$

## Derivation

Substitute $y_i = x_i - 1$ where $y_i \geq 0$:

$$y_1 + y_2 + \cdots + y_n = r - n$$

This gives $\binom{(r-n)+n-1}{n-1} = \binom{r-1}{n-1}$ non-negative solutions.

# Formula 3: General Minimum Constraints

## Statement

The number of integer solutions to $x_1 + x_2 + \cdots + x_n = r$ where $x_i \geq m_i$ for each $i$:

$$\binom{r - (m_1 + m_2 + \cdots + m_n) + n - 1}{n - 1}$$

## Method

1. Substitute $y_i = x_i - m_i$ so that $y_i \geq 0$
2. The equation becomes: $y_1 + y_2 + \cdots + y_n = r - \sum m_i$
3. Apply the basic multiset formula to the $y_i$ variables

# Formula 4: Inclusion-Exclusion for Upper Bounds

## Statement

For solutions to $x_1 + x_2 + \cdots + x_n = r$ with upper bounds $x_i \leq u_i$:
Use the Inclusion-Exclusion Principle:

$$\sum_{\text{all}} - \sum_{\text{violate 1}} + \sum_{\text{violate 2}} - \cdots$$

## Example: $x_1 + x_2 = 10$ with $x_1 \leq 6$, $x_2 \leq 7$

$$\text{Total} = \binom{11}{1} = 11$$

$$\text{Subtract } x_1 \geq 7 = \binom{4}{1} = 4$$

$$\text{Subtract } x_2 \geq 8 = \binom{3}{1} = 3$$

# Formula 5: Stars and Bars Visualization

## Interpretation

To distribute $r$ identical objects into $n$ distinct bins:

- Represent objects as $r$ stars: $\star \star \star \cdots \star$
- Use $n - 1$ bars to create $n$ compartments: $|$
- Arrange $r$ stars and $n - 1$ bars in a line

Total arrangements: $\binom{r+(n-1)}{r} = \binom{r+n-1}{n-1}$

## Example

Distribute 5 items into 3 bins: $\star \star | \star | \star \star$

This represents: Bin 1 gets 2, Bin 2 gets 1, Bin 3 gets 2

Total ways: $\binom{5+3-1}{3-1} = \binom{7}{2} = 21$

# Formula 6: Generating Functions

## Statement

The generating function for selecting $r$ objects from $n$ types with repetition is:

$$(1 + x + x^2 + x^3 + \cdots)^n = \frac{1}{(1-x)^n}$$

The coefficient of $x^r$ gives the number of multisets of size $r$.

## Power Series Expansion

$$\frac{1}{(1-x)^n} = \sum_{r=0}^{\infty} \binom{n+r-1}{r} x^r$$

The coefficient of $x^r$ is $\binom{n+r-1}{r}$, confirming the multiset formula.

# Property 1: Multiset Symmetry

## Statement

$$\left(\!\!\binom{n}{r}\!\!\right) = \binom{n+r-1}{r} = \binom{n+r-1}{n-1}$$

This symmetry shows that selecting $r$ items from $n$ types is equivalent to distributing $r$ items into $n$ bins.

## Interpretation

- $\binom{n+r-1}{r}$ - choose positions for $r$ stars among $r$ stars and $n-1$ bars
- $\binom{n+r-1}{n-1}$ - choose positions for $n-1$ bars among $r$ stars and $n-1$ bars

# Property 2: Relationship to Regular Combinations

## Comparison

- **Regular combinations**: $\binom{n}{r}$ - choose $r$ from $n$ distinct objects, no repetition
- **Multisets**: $\left(\!\binom{n}{r}\!\right) = \binom{n+r-1}{r}$ - choose $r$ from $n$ types, repetition allowed

## Example Comparison

- $\binom{4}{3} = 4$ - choose 3 from 4 distinct items: $\{A, B, C\}, \{A, B, D\}, \{A, C, D\}, \{B, C, D\}$
- $\binom{4+3-1}{3} = \binom{6}{3} = 20$ - choose 3 from 4 types with repetition allowed

# Property 3: Special Cases

## Important Special Values

1. $\left(\!\binom{n}{0}\!\right) = \binom{n-1}{0} = 1$ - one way to select nothing
2. $\left(\!\binom{n}{1}\!\right) = \binom{n}{1} = n$ - $n$ ways to select one item
3. $\left(\!\binom{1}{r}\!\right) = \binom{r}{r} = 1$ - one way to select $r$ items from 1 type
4. $\left(\!\binom{2}{r}\!\right) = \binom{r+1}{r} = r + 1$ - partitions of $r$ into at most 2 parts

# Common Problem Types Summary

## Problem Type Recognition

| **Problem Type** | **Formula** |
|---|---|
| Distribute $r$ identical to $n$ distinct | $\binom{n+r-1}{r}$ |
| Non-negative solutions: $\sum x_i = r$ | $\binom{n+r-1}{n-1}$ |
| Positive solutions: $\sum x_i = r$ | $\binom{r-1}{n-1}$ |
| With minimums: $x_i \geq m_i$ | $\binom{r-\sum m_i+n-1}{n-1}$ |
| With maximums: $x_i \leq u_i$ | Inclusion-Exclusion |
| Coefficient in $(1 + x + x^2 + \cdots)^n$ | $\binom{n+r-1}{r}$ |

# Quick Formula Reference

## Essential Formulas

1. **Basic Multiset**: $\left(\!\binom{n}{r}\!\right) = \binom{n+r-1}{r} = \frac{(n+r-1)!}{r!(n-1)!}$
2. **Positive Integer Solutions**: $x_1 + \cdots + x_n = r$ with $x_i \geq 1$: $\binom{r-1}{n-1}$
3. **With Minimums**: $x_i \geq m_i$: Let $S = \sum m_i$, answer is $\binom{r-S+n-1}{n-1}$
4. **Stars and Bars**: $r$ stars, $n-1$ bars: $\binom{r+n-1}{n-1}$
5. **Regular Combination**: $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ (no repetition)
6. **Permutation**: $P(n, r) = \frac{n!}{(n-r)!}$ (order matters, no repetition)

# Thank You!

Yolymatics Tutorials

www.yolymaticstutorials.com

Keep counting!