Linear Transformations

Geometric Transformations in the Plane

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Today's objectives

- Understand what linear transformations are and their properties
- Learn the five fundamental transformations: scaling, reflection, projection, shearing, rotation
- Represent transformations using matrices
- Compose transformations and understand geometric effects
- Apply transformations to solve geometric problems

Foundations

What is a transformation?

Definition

A **transformation** $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a function that takes points (vectors) in the plane and maps them to other points in the plane.

If $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$, then $T(\mathbf{v}) = \begin{bmatrix} x' \\ y' \end{bmatrix}$ is the **image** of $\mathbf{v}$ under T.

Notation:

- $T(\mathbf{v})$ or $T\mathbf{v}$ = image of vector $\mathbf{v}$
- Original object = **pre-image**
- Transformed object = **image**

Linear transformations: Definition

Definition

A transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is **linear** if it satisfies two properties:

1. **Additivity:** $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all vectors $\mathbf{u}, \mathbf{v}$
2. **Homogeneity:** $T(c\mathbf{v}) = cT(\mathbf{v})$ for all scalars c and vectors $\mathbf{v}$

Key consequence

Every linear transformation can be represented by a matrix: $T(\mathbf{v}) = A\mathbf{v}$

Important: Linear transformations always map the origin to itself: $T(\mathbf{0}) = \mathbf{0}$

Matrix representation

How to find the matrix

To find the matrix A for a linear transformation T:

Step 1: Find where T sends the standard basis vectors:

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Step 2: The matrix A has these images as its columns:

$$A = \begin{bmatrix} | & | \\ T(\mathbf{e}_1) & T(\mathbf{e}_2) \\ | & | \end{bmatrix}$$

Properties of linear transformations

Linear transformations preserve:

- The origin
- Straight lines
- Parallelism
- Ratios of distances along lines

They may change:

- Lengths
- Angles
- Areas
- Orientation

Non-examples

These are NOT linear transformations:

- Translation: $(x, y) \mapsto (x + 3, y + 2)$ — doesn't fix origin
- $(x, y) \mapsto (x^2, y)$ — doesn't preserve additivity

Scaling

Scaling transformations

Definition

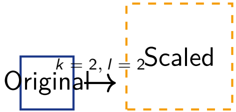
A **scaling** (or dilation) multiplies coordinates by constants:

$$T(x, y) = (kx, ly)$$

where $k, l \in \mathbb{R}$ are the scaling factors.

Matrix form:

$$A = \begin{bmatrix} k & 0 \\ 0 & l \end{bmatrix}$$



Special cases:

Scaling: Examples and effects

Example: Stretch and compress

Matrix $A = \begin{bmatrix} 3 & 0 \\ 0 & 0.5 \end{bmatrix}$

- Stretches by factor 3 in x-direction
- Compresses by factor 0.5 in y-direction

Geometric effects:

- If $|k| > 1$: stretch in x-direction
- If $0 < |k| < 1$: compress in x-direction
- If $k < 0$: also reflects across y-axis
- Area multiplied by $|kl|$

Reflections

Reflection transformations

Common reflections

Across x-axis: $(x, y) \mapsto (x, -y)$ Matrix: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Across y-axis: $(x, y) \mapsto (-x, y)$ Matrix: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

Across line $y = x$: $(x, y) \mapsto (y, x)$ Matrix: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Across line $y = -x$: $(x, y) \mapsto (-y, -x)$ Matrix: $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

Properties:

Reflection across arbitrary lines

Reflection across line through origin at angle θ

The matrix for reflection across a line making angle θ with the positive x-axis:

$$R_\theta = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$$

Verification for common cases

- $\theta = 0$ (x-axis): $R_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ ✓
- $\theta = \pi/4$ (line $y = x$): $R_{\pi/4} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ ✓

Projections

Projection transformations

Definition

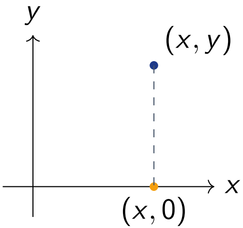
A **projection** maps points onto a line or subspace, collapsing one dimension.

Common projections:

Onto x-axis:

$$(x, y) \mapsto (x, 0)$$

$$P_x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$



Projection onto x-axis

Onto y-axis:

$$(x, y) \mapsto (0, y)$$

Projection onto arbitrary lines

Projection onto line through origin

For a unit vector $\mathbf{u} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ along a line:

The projection matrix is:

$$P = \mathbf{u}\mathbf{u}^T = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

Geometric interpretation:

- Projects every point perpendicularly onto the line
- Points on the line stay fixed
- Points perpendicular to the line map to origin

Shearing

Shear transformations

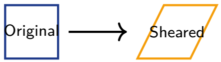
Definition

A **shear** "slides" one coordinate direction by an amount proportional to the other.

Horizontal shear:

$$T(x, y) = (x + ky, y)$$

$$S_x = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$



Horizontal shear with $k = 0.5$

Vertical shear:

$$T(x, y) = (x, kx + y)$$

Rotation

Rotation transformations

Rotation by angle θ counterclockwise

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

This rotates every point by angle θ counterclockwise around the origin.

Common angles:

- 90°: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
- 180°: $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
- 270°: $\begin{bmatrix} 0 & 1 \end{bmatrix}$

Properties:

- Preserves distances
- Preserves angles
- Determinant $= 1$
- $R_{-\theta} = R_\theta^{-1} = R_\theta^T$

Composing transformations

Composition

To apply transformation T_2 after T_1: $(T_2 \circ T_1)(\mathbf{v}) = T_2(T_1(\mathbf{v}))$

If T_1 has matrix A_1 and T_2 has matrix A_2, then:

$$T_2 \circ T_1 \text{ has matrix } A_2 A_1 \quad \text{(note the order!)}$$

Important

Matrix multiplication is NOT commutative: $A_2 A_1 \neq A_1 A_2$ in general.
Order matters! "Rotate then scale" $\neq$ "Scale then rotate"

Working with transformations: A procedure

Step-by-step guide

To find the effect of transformation T:

1. **Find the matrix:** Determine $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$, or use known forms
2. **Apply to point:** Compute $A\mathbf{v}$ using matrix multiplication
3. **Geometric interpretation:** Describe the effect (rotation? stretch? etc.)

To compose transformations:

1. Find matrix for each transformation
2. Multiply matrices in reverse order: A_2A_1 for "T_1 then T_2"
3. Apply the composite matrix

Determining Transformation Matrices

How to find transformation matrices

Method 1: Using standard basis vectors

For a transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$:

Step 1: Find where the standard basis vectors map:

- $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto T(\mathbf{e}_1) = \begin{bmatrix} a \\ c \end{bmatrix}$
- $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto T(\mathbf{e}_2) = \begin{bmatrix} b \\ d \end{bmatrix}$

Step 2: The matrix is:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Why? Because $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$

Determining matrices: Example 1

Example: Reflection across line $y = x$

Find the matrix for reflection across the line $y = x$.

Solution:

- The point $(1, 0)$ reflects to $(0, 1)$: $T(\mathbf{e}_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- The point $(0, 1)$ reflects to $(1, 0)$: $T(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Therefore: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Verify: $(x, y) \mapsto (y, x)$ ✓

Determining matrices: Example 2

Example: Projection onto x-axis

Find the matrix for projection onto the x-axis.

Solution:

- $(1, 0)$ projects to $(1, 0)$: $T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
- $(0, 1)$ projects to $(0, 0)$: $T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Therefore: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Verify: $(x, y) \mapsto (x, 0)$ ✓

Method 2: Using geometric properties

For rotations

Rotation by angle θ counterclockwise about the origin:

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

For reflections across a line through origin

Reflection across line making angle θ with x-axis:

$$\mathrm{Refl}_\theta = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$$

Tip: Memorize these formulas or derive from basis vectors!

Transformation of Lines and Planes

How lines transform

Key principle

Linear transformations map lines to lines (or points).

To find the image of a line L under transformation T:

1. **Parametric method:**
   - Write line in parametric form: $\mathbf{r}(t) = \mathbf{a} + t\mathbf{d}$
   - Apply transformation: $T(\mathbf{r}(t)) = T(\mathbf{a}) + tT(\mathbf{d})$
   - This gives the image line
2. **Two-point method:**
   - Find two points on the line
   - Transform both points
   - Image line passes through these two image points

Transforming lines: Example

Example

Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ (scaling). Find the image of the line $y = 2x + 1$.

Solution using two points:

- Point 1: $(0, 1)$ maps to $A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$
- Point 2: $(1, 3)$ maps to $A \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 9 \end{bmatrix}$

Slope of image line: $\frac{9-3}{2-0} = 3$
Equation: $y - 3 = 3(x - 0)$ gives $y = 3x + 3$

Transforming lines: Parametric approach

Same example using parametric form

Line $y = 2x + 1$ in parametric form:

$$\mathbf{r}(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Apply $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$:

$$T(\mathbf{r}(t)) = A \begin{bmatrix} 0 \\ 1 \end{bmatrix} + tA \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} + t \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

This is the line through $(0, 3)$ with direction $\begin{bmatrix} 2 \\ 6 \end{bmatrix}$.

When lines collapse to points

Special case

If the transformation matrix is singular (determinant $= 0$), some lines may collapse to a single point!

Example: Projection

Projection onto x-axis: $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

The vertical line $x = 2$ (all points $(2, y)$) maps to:

$$P \begin{bmatrix} 2 \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \text{for all } y$$

The entire line collapses to the single point $(2, 0)$!

Invariant Lines and Fixed Points

Definition of invariant lines

Invariant line (or fixed line)

A line L is **invariant** under transformation T if T maps L to itself (though individual points may move).

Mathematically: $T(L) = L$

Fixed point

A point $\mathbf{p}$ is **fixed** if $T(\mathbf{p}) = \mathbf{p}$.

Important distinction

- Invariant line: The line as a set stays the same
- Fixed point: An individual point doesn't move

Finding invariant lines

Method using eigenvectors

For transformation with matrix A:

1. Find eigenvalues λ by solving $\det(A - \lambda I) = 0$
2. For each eigenvalue, find corresponding eigenvector $\mathbf{v}$
3. The line through origin in direction $\mathbf{v}$ is invariant

Why? If $A\mathbf{v} = \lambda\mathbf{v}$, then $\mathbf{v}$ stays on the same line (just scaled by λ).

The line $\{\mathbf{x} : \mathbf{x} = t\mathbf{v}, t \in \mathbb{R}\}$ satisfies:

$$T(t\mathbf{v}) = tT(\mathbf{v}) = t\lambda\mathbf{v} \quad \text{(still on the line!)}$$

Invariant lines: Example 1

Example: Scaling transformation

Find invariant lines for $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

Solution:

- Eigenvalues: $\lambda_1 = 2$, $\lambda_2 = 3$
- Eigenvector for $\lambda_1 = 2$: $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (the x-axis)
- Eigenvector for $\lambda_2 = 3$: $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (the y-axis)

Invariant lines:

- x-axis (scaled by factor 2)

Invariant lines: Example 2

Example: Reflection across x-axis

Find invariant lines for $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Solution:

- Eigenvalues: $\lambda_1 = 1$, $\lambda_2 = -1$
- For $\lambda_1 = 1$: eigenvector $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
- For $\lambda_2 = -1$: eigenvector $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Invariant lines:

- x-axis (all points fixed, since $\lambda = 1$)

Finding invariant lines: Alternative method

Direct approach

A line through origin with direction $\mathbf{d} = \begin{bmatrix} a \\ b \end{bmatrix}$ is invariant if $A\mathbf{d} = \lambda\mathbf{d}$ for some scalar λ.

This gives: $\begin{bmatrix} * & * \\ * & * \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \lambda \begin{bmatrix} a \\ b \end{bmatrix}$

Solve this system to find possible values of a and b.

For lines not through origin

If the line doesn't pass through the origin, check if $T(\mathbf{p})$ lies on the line for points $\mathbf{p}$ on that line.
This requires case-by-case analysis.

Invariant lines: Geometric interpretation

Common cases:

- **Reflections:** The mirror line is invariant (all points fixed)
- **Rotations:** Only the origin is fixed (except 180°)
- **Projections:** The projection target line is invariant
- **Scaling:** Coordinate axes are invariant

Key insight:
Finding invariant lines helps understand the "structure" of a transformation.

Invariant directions are the "natural axes" for the transformation.

Practice Problems

Problem 1: Identify the transformation

Problem

For each matrix, identify the type of transformation and describe its geometric effect:

(a) $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

(b) $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(c) $C = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Workspace:

Problem 1: Workspace (continued)

Problem 2: Apply scaling transformation

Problem

Consider the scaling transformation $T(x, y) = (3x, 2y)$.
(a) Write the matrix for this transformation
(b) Find the image of the point $(2, 4)$
(c) Find the image of the triangle with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$
(d) How does the area change?

Workspace:

Problem 2: Workspace (continued)

Problem 3: Reflection across x-axis

Problem

Let R be reflection across the x-axis.
(a) Write the matrix for R
(b) Find $R(3, -2)$
(c) Find the image of the line segment from $(1, 2)$ to $(3, 4)$
(d) Verify that $R \circ R = I$ (identity) by computing R^2

Workspace:

Problem 3: Workspace (continued)

Problem 4: Reflection across $y = x$

Problem

Consider reflection T across the line $y = x$.

(a) What is the matrix for T?

(b) Find the image of $(5, 2)$ under T

(c) Show that the point $(3, 3)$ is fixed by T

(d) Find a point that maps to $(4, -1)$

Workspace:

Problem 4: Workspace (continued)

Problem 5: Projection onto x-axis

Problem

Let P be the projection onto the x-axis.
(a) Write the matrix for P
(b) Find $P(4, 7)$
(c) Verify that $P \circ P = P$ by computing P^2
(d) What is the image of the circle $x^2 + y^2 = 9$ under P?

Workspace:

Problem 5: Workspace (continued)

Problem 6: Projection onto y-axis

Problem

Consider the transformation $T(x, y) = (0, y)$.
(a) Show that T is linear by verifying additivity and homogeneity
(b) Find the matrix for T
(c) What is the kernel (set of vectors mapped to zero)?
(d) Find the image of the square with vertices $(0, 0), (2, 0), (2, 2), (0, 2)$

Workspace:

Problem 6: Workspace (continued)

Problem 7: Horizontal shear

Problem

Let S be the shear transformation with matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

(a) Write the formula for $S(x, y)$

(b) Find the image of the unit square (vertices at $(0, 0), (1, 0), (1, 1), (0, 1)$)

(c) Show that the area is preserved

(d) What happens to the point $(0, 3)$?

Workspace:

Problem 7: Workspace (continued)

Problem 8: Vertical shear

Problem

Consider the transformation $T(x, y) = (x, -3x + y)$.
(a) Find the matrix for T
(b) Is this a horizontal or vertical shear?
(c) Find $T(2, 5)$
(d) Find all points (x, y) such that $T(x, y) = (x, y)$ (fixed points)

Workspace:

Problem 8: Workspace (continued)

Problem 9: Rotation by 90°

Problem

Let R be counterclockwise rotation by 90° about the origin.

(a) Write the matrix for R

(b) Find the image of $(3, -2)$ under R

(c) Find the image of $(1, 0)$ and $(0, 1)$ under R

(d) Apply R four times to the point $(1, 0)$. What do you observe?

Workspace:

Problem 9: Workspace (continued)

Problem 10: Rotation by 180°

Problem

Consider rotation by 180° about the origin.

(a) Find the rotation matrix

(b) Show this is equivalent to scaling by -1 in both directions

(c) Find the image of the line segment from $(2, 3)$ to $(4, 5)$

(d) Is rotation by 180° the same as reflection through the origin?

Workspace:

Problem 10: Workspace (continued)

Problem 11: Rotation by 45°

Problem

Let R be counterclockwise rotation by $45°$.
(a) Write the matrix for R (leave in terms of $\cos 45°$ and $\sin 45°$, or use $\sqrt{2}/2$)
(b) Find the image of $(1, 0)$ under R
(c) Find the image of $(1, 1)$ under R
(d) Verify that distances are preserved: $\|R\mathbf{v}\| = \|\mathbf{v}\|$

Workspace:

Problem 11: Workspace (continued)

Problem 12: Finding transformation matrices

Problem

For each transformation, find its matrix:

(a) Scaling by factor 5 in the x-direction and factor $1/2$ in the y-direction

(b) Reflection across the line $y = -x$

(c) Counterclockwise rotation by 270°

(d) Horizontal shear that maps $(0, 1)$ to $(3, 1)$

Workspace:

Problem 12: Workspace (continued)

Problem 13: Composition — Reflect then scale

Problem

Let R be reflection across the y-axis and S be scaling by factor 2 (uniform).
(a) Find the matrix for R and the matrix for S
(b) Find the matrix for $S \circ R$ (reflect first, then scale)
(c) Find the image of $(3, 1)$ under $S \circ R$
(d) Is $S \circ R = R \circ S$? Verify by computing $R \circ S$

Workspace:

Problem 13: Workspace (continued)

Problem 14: Composition — Rotate then project

Problem

Let R be rotation by 90° counterclockwise and P be projection onto the x-axis.

(a) Find the matrices for R and P

(b) Find the matrix for $P \circ R$

(c) Describe geometrically what $P \circ R$ does

(d) Find the image of the point $(0, 5)$ under $P \circ R$

Workspace:

Problem 14: Workspace (continued)

Problem 15: Composition — Shear then rotate

Problem

Let S be horizontal shear: $S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, and R be rotation by 90°.

(a) Compute $R \circ S$ (shear first, then rotate)

(b) Compute $S \circ R$ (rotate first, then shear)

(c) Are they equal? What does this demonstrate?

(d) Find the image of $(1, 0)$ under both compositions

Workspace:

Problem 15: Workspace (continued)

Problem 16: Inverse transformations

Problem

Consider the transformation with matrix $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$.

(a) Find A^{-1}

(b) Verify that $AA^{-1} = I$

(c) If $T(\mathbf{v}) = (5, 9)$, find $\mathbf{v}$

(d) Describe geometrically what A and A^{-1} do

Workspace:

Problem 16: Workspace (continued)

Problem 17: Determinants and area

Problem

Let T have matrix $A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$.

(a) Compute $\det(A)$

(b) Find the area of the image of the unit square under T

(c) The triangle with vertices $(0, 0), (1, 0), (0, 1)$ has area $1/2$. What is the area of its image?

(d) In general, if a region has area A_0, what is the area of its image under T?

Workspace:

Problem 17: Workspace (continued)

Problem 18: Fixed points and eigenvectors

Problem

Consider the shear $S = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

(a) Find all vectors $\mathbf{v}$ such that $S\mathbf{v} = \mathbf{v}$ (fixed points)

(b) Find all vectors $\mathbf{v}$ such that $S\mathbf{v} = \lambda\mathbf{v}$ for some scalar λ

(c) Geometrically, what do the fixed points represent?

(d) Does the shear have any directions that are only scaled (not sheared)?

Workspace:

Problem 18: Workspace (continued)

Problem 19: Comprehensive problem

Problem

Start with the unit square S with vertices $(0,0),(1,0),(1,1),(0,1)$.

(a) Apply horizontal shear with $k=1$: $S_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Find the new vertices.

(b) Then apply rotation by 90°. Find the final vertices.

(c) Find a single matrix that performs both transformations.

(d) Sketch the original square and both transformed versions.

Workspace:

Problem 19: Workspace (continued)

Problem 19: Workspace (continued 2)

Problem 20: Application problem

Problem

A computer graphics system needs to:

1. Scale an image by factor 2 in the x-direction
2. Reflect it across the x-axis
3. Rotate it by 45° counterclockwise

(a) Find the matrix for each operation
(b) Find a single matrix that performs all three operations in order
(c) What is the image of the point $(1, 1)$ after all transformations?
(d) Is the order of operations important? Explain.

Workspace:

Problem 20: Workspace (continued)

Problem 20: Workspace (continued 2)

Problem 21: Finding the transformation

Problem

A linear transformation T satisfies:

$$T\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}2\\3\end{bmatrix},\quad T\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}-1\\4\end{bmatrix}$$

(a) Find the matrix for T

(b) Find $T\begin{bmatrix}3\\2\end{bmatrix}$

(c) Find $T\begin{bmatrix}-1\\5\end{bmatrix}$

(d) Can you identify what geometric transformation this is?

Problem 21: Workspace (continued)

Problem 22: Proving linearity

Problem

For each transformation, determine if it is linear. If yes, find its matrix. If no, explain why.

(a) $T(x, y) = (2x - y, x + 3y)$

(b) $T(x, y) = (x + 1, y - 2)$

(c) $T(x, y) = (xy, x + y)$

(d) $T(x, y) = (3x, 0)$

Workspace:

Problem 22: Workspace (continued)

Problem 23: Challenge — Double reflection

Problem

Let R_1 be reflection across the x-axis and R_2 be reflection across the line $y = x$.
(a) Find matrices for R_1 and R_2
(b) Compute $R_2 \circ R_1$ (reflect across x-axis, then across $y = x$)
(c) What single transformation does $R_2 \circ R_1$ represent?
(d) Try $R_1 \circ R_2$. Is it the same?

Workspace:

Problem 23: Workspace (continued)

Problem 24: Challenge — Three transformations

Problem

Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

(a) Identify each transformation geometrically

(b) Compute CBA (apply A, then B, then C)

(c) Compute ABC. Compare with CBA.

(d) Find the image of $(1, 0)$ under both compositions

Workspace:

Problem 24: Workspace (continued)

Problem 24: Workspace (continued 2)

Problem 25: Determining transformation matrix

Problem

Find the matrix for each transformation described geometrically:

(a) Reflection across the line $y = 2x$

(b) Projection onto the line $y = x$

(c) Rotation by 60° counterclockwise about the origin

(d) Stretch by factor 4 in the x-direction, no change in y-direction

Workspace:

Problem 25: Workspace (continued)

Problem 26: Transformation of a line

Problem

Let T have matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ (horizontal shear).

(a) Find the image of the line $y = x$ under T

(b) Find the image of the line $y = -x + 2$ under T

(c) Does T map parallel lines to parallel lines?

(d) Find the image of the line $x = 3$ (vertical line)

Workspace:

Problem 26: Workspace (continued)

Problem 27: Transformation of lines with projection

Problem

Let $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ (projection onto x-axis).

(a) Find the image of the line $y = 2x + 1$ under P

(b) Find the image of the horizontal line $y = 3$ under P

(c) Which lines collapse to a single point?

(d) Which lines remain unchanged?

Workspace:

Problem 27: Workspace (continued)

Problem 28: Finding invariant lines

Problem

Find all invariant lines (lines that map to themselves) for each transformation:

(a) $A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ (scaling)

(b) $B = \begin{bmatrix} \cos(30^\circ) & -\sin(30^\circ) \\ \sin(30^\circ) & \cos(30^\circ) \end{bmatrix}$ (rotation by 30°)

(c) $C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (reflection across x-axis)

(d) $D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ (projection onto x-axis)

Workspace:

Problem 28: Workspace (continued)

Problem 29: Invariant lines using eigenvectors

Problem

Consider the transformation T with matrix $A = \begin{bmatrix} 4 & 1 \\ 0 & 3 \end{bmatrix}$.

(a) Find the eigenvalues of A by solving $\det(A - \lambda I) = 0$
(b) For each eigenvalue, find the corresponding eigenvector
(c) Identify the invariant lines through the origin
(d) What happens to points on each invariant line?

Workspace:

Problem 29: Workspace (continued)

Problem 30: Fixed points vs invariant lines

Problem

For $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (rotation by 90°):

(a) Find all fixed points (points $\mathbf{p}$ where $A\mathbf{p} = \mathbf{p}$)

(b) Are there any invariant lines through the origin?

(c) What is the geometric interpretation of your answers?

(d) Consider $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ (rotation by 180°). How many invariant lines does this have?

Workspace:

Problem 30: Workspace (continued)

Problem 31: Comprehensive — Lines and invariance

Problem

Let T be reflection across the line $y = x$, with matrix $R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(a) Find the image of the line $y = 2x$ under T

(b) Find all invariant lines for this transformation

(c) Which invariant line(s) consist entirely of fixed points?

(d) Verify your answer to (a) using two specific points on $y = 2x$

Workspace:

Problem 31: Workspace (continued)

Problem 32: Design your own

Problem

Design a linear transformation that:

- Maps $(1, 0)$ to $(2, 1)$
- Maps $(0, 1)$ to $(-1, 3)$

(a) Write the matrix for this transformation
(b) Find where it maps the point $(3, 2)$
(c) Describe the geometric effect (scaling? rotation? combination?)
(d) Find the determinant. What does it tell you about area?

Workspace:

Problem 25: Workspace (continued)

Summary

Key takeaways

- **Linear transformations** preserve vector addition and scalar multiplication
- Every linear transformation in $\mathbb{R}^2$ can be represented by a 2×2 matrix
- **Determining matrices:** Use standard basis vectors or geometric formulas
- **Five fundamental transformations:**
  - Scaling, Reflection, Projection, Shearing, Rotation
- **Transformation of lines:** Use parametric form or two-point method
- **Invariant lines:** Find using eigenvectors; lines that map to themselves
- **Compositions:** Multiply matrices (order matters: $A_2 A_1$ for "T_1 then T_2")
- Geometric properties: origin fixed, lines $\to$ lines, parallelism preserved

Thank you!

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Keep transforming and stay curious!