Improper Integrals

Practice Problems

Yolymatics Tutorials

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Outline

What are Improper Integrals?

Definition

An **improper integral** is a definite integral that has one or both of the following properties:

- One or both limits of integration are infinite
- The integrand has an infinite discontinuity in the interval of integration

Types of Improper Integrals

1. **Type 1:** Infinite limits of integration
2. **Type 2:** Infinite discontinuity in the integrand

Type 1: Infinite Limits

Upper Limit Infinite

$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

Lower Limit Infinite

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

Both Limits Infinite

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx$$

where c is any real number. Both integrals must converge.

Type 2: Infinite Discontinuity

Discontinuity at Upper Limit

If f is continuous on $[a, b)$ and discontinuous at b:

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

Discontinuity at Lower Limit

If f is continuous on $(a, b]$ and discontinuous at a:

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$

Discontinuity in the Interior

If f is discontinuous at c where $a < c < b$:

Convergence and Divergence

Convergent Improper Integral

An improper integral is **convergent** if the limit defining it exists and is finite.

Divergent Improper Integral

An improper integral is **divergent** if the limit does not exist or is infinite.

Important Note

When an improper integral involves both infinite limits and discontinuities, or multiple discontinuities, each part must be evaluated separately and ALL parts must converge for the integral to be convergent.

Problem 1: Upper Limit Infinite

Problem

Evaluate the improper integral:

$$\int_1^{\infty} \frac{1}{x^2}\, dx$$

Does it converge or diverge? If it converges, find its value.

Working Space

Problem 2: Upper Limit Infinite

Problem

Evaluate the improper integral:

$$\int_1^{\infty} \frac{1}{x}\,dx$$

Does it converge or diverge? If it converges, find its value.

Working Space

Problem 3: Upper Limit Infinite with Exponential

Problem

Evaluate the improper integral:

$$\int_0^{\infty} e^{-3x}\, dx$$

Does it converge or diverge? If it converges, find its value.

Working Space

Problem 4: Lower Limit Infinite

Problem

Evaluate the improper integral:

$$\int_{-\infty}^{0} e^{2x}\,dx$$

Does it converge or diverge? If it converges, find its value.

Working Space

Problem 5: Both Limits Infinite

Problem

Evaluate the improper integral:

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx$$

Does it converge or diverge? If it converges, find its value.

Working Space

Problem 6: p-integral

Problem

For what values of p does the following integral converge?

$$\int_2^\infty \frac{1}{x^p}\,dx$$

When it converges, find its value in terms of p.

Working Space

Problem 7: Trigonometric Function

Problem

Evaluate the improper integral:

$$\int_0^\infty \sin(x)\,dx$$

Does it converge or diverge? Explain your reasoning.

Working Space

Problem 8: Discontinuity at Upper Limit

Problem

Evaluate the improper integral:

$$\int_0^1 \frac{1}{\sqrt{1-x}}\,dx$$

Does it converge or diverge? If it converges, find its value.

Working Space

Problem 9: Discontinuity at Lower Limit

Problem

Evaluate the improper integral:

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx$$

Does it converge or diverge? If it converges, find its value.

Working Space

Problem 10: Discontinuity at Interior Point

Problem

Evaluate the improper integral:

$$\int_0^3 \frac{1}{x-1}\,dx$$

Does it converge or diverge? If it converges, find its value.

Hint: The integrand has a discontinuity at $x = 1$.

Working Space

Problem 11: Logarithmic Integrand

Problem

Evaluate the improper integral:

$$\int_0^1 \ln(x)\, dx$$

Does it converge or diverge? If it converges, find its value.

Working Space

Problem 12: Power Function with Discontinuity

Problem

Evaluate the improper integral:

$$\int_0^4 \frac{1}{x^{2/3}}\, dx$$

Does it converge or diverge? If it converges, find its value.

Working Space

Problem 13: Both Infinite Limit and Discontinuity

Problem

Evaluate the improper integral:

$$\int_0^\infty \frac{1}{\sqrt{x}(1+x)}\,dx$$

Does it converge or diverge? If it converges, find its value.

Hint: Split at $x = 1$ and evaluate two improper integrals.

Working Space

Problem 14: Comparison Test Application

Problem

Determine whether the following integral converges or diverges:

$$\int_1^{\infty} \frac{1+\sin^2(x)}{x^2}\, dx$$

Hint: Use the comparison test with an appropriate function.

Working Space

Problem 15: Exponential with Polynomial

Problem

Evaluate the improper integral:

$$\int_0^{\infty} x^2 e^{-x}\,dx$$

Does it converge or diverge? If it converges, find its value.

Hint: Use integration by parts twice.

Working Space

Problem 16: Rational Function

Problem

Evaluate the improper integral:

$$\int_2^\infty \frac{1}{x^2 - 1}\, dx$$

Does it converge or diverge? If it converges, find its value.

Hint: Use partial fractions first.

Working Space

Important Results for Improper Integrals

p-integrals (Type 1)

$$\int_1^\infty \frac{1}{x^p}\,dx \text{ converges if } p > 1 \text{ and diverges if } p \leq 1$$

When $p > 1$: $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{p-1}$

p-integrals (Type 2)

$$\int_0^1 \frac{1}{x^p}\,dx \text{ converges if } p < 1 \text{ and diverges if } p \geq 1$$

When $p < 1$: $\displaystyle\int_0^1 \frac{1}{x^p}\,dx = \frac{1}{1-p}$

Common Convergent Integrals

Useful Results

1. $\displaystyle\int_0^\infty e^{-ax}\,dx = \frac{1}{a}$ for $a > 0$
2. $\displaystyle\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx = \pi$
3. $\displaystyle\int_0^\infty \frac{\sin(x)}{x}\,dx = \frac{\pi}{2}$
4. $\displaystyle\int_0^\infty x^n e^{-x}\,dx = n!$ for $n \in \mathbb{N}$
5. $\displaystyle\int_{-\infty}^\infty e^{-x^2}\,dx = \sqrt{\pi}$ (Gaussian integral)

Comparison Tests

Direct Comparison Test

If $0 \leq f(x) \leq g(x)$ for all $x \geq a$:

- If $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges
- If $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ diverges

Limit Comparison Test

If $f(x) \geq 0$ and $g(x) > 0$ for $x \geq a$, and:

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = L \text{ where } 0 < L < \infty$$

Then $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ either both converge or both diverge.

Thank You!

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