Advanced Calculus

Comprehensive Worksheet

Partial Fractions, Parametric Equations, Optimization & Volumes

Yolymatics Tutorials

November 2, 2025

Student Name: ______________________________

Date: ______________________________

Show all work clearly. Use proper mathematical notation.

Part 1: Partial Fractions for Linear Factors

Key Points

Remember:

- For distinct linear factors: $\frac{P(x)}{(ax+b)(cx+d)} = \frac{A}{ax+b} + \frac{B}{cx+d}$
- For repeated linear factors: $\frac{P(x)}{(ax+b)^2} = \frac{A}{ax+b} + \frac{B}{(ax+b)^2}$
- Always ensure the degree of numerator $<$ degree of denominator
- If not, perform polynomial division first

Problem 1: Basic partial fractions

Express each of the following in partial fractions.

(a) $\dfrac{7x+5}{(x+1)(x+2)}$

(b) $\dfrac{3x-1}{(x-2)(x+3)}$

(c) $\dfrac{5x+11}{x^2+5x+6}$ *(Hint: Factorize the denominator first)*

Problem 2: Three linear factors

Express in partial fractions:

(a) $\dfrac{2x^2+3x+1}{(x+1)(x-1)(x+2)}$

(b) $\dfrac{x^2-2x+3}{x(x+1)(x-2)}$

Problem 3: Repeated linear factors

Express in partial fractions:

(a) $\dfrac{5x+7}{(x+1)^2}$

(b) $\dfrac{3x^2+5x+2}{(x+2)^2(x-1)}$

(c) $\dfrac{2x+1}{x(x+1)^2}$

Problem 4: Improper fractions (require polynomial division)

Express in partial fractions:

(a) $\dfrac{x^3 + 2x^2 - x + 1}{(x-1)(x+2)}$

(Hint: Perform polynomial division first since degree of numerator $\geq$ degree of denominator)

(b) $\dfrac{2x^3 - 3x^2 + 4x - 5}{x^2 - x - 2}$

Problem 5: Using partial fractions in integration

Use partial fractions to evaluate the following integrals:

(a) $\displaystyle\int \frac{5x+1}{(x+1)(x-2)}\,dx$

(b) $\displaystyle\int \frac{3x-4}{x^2-5x+6}\,dx$

(c) $\displaystyle\int_1^2 \frac{2x+3}{(x+1)(x+2)}\,dx$

Part 2: Parametric Equations

Key Points

Remember:

- Parametric form: $x = f(t)$, $y = g(t)$
- To find $\frac{dy}{dx}$: Use $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$
- To find Cartesian equation: Eliminate the parameter t
- Arc length: $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

Problem 6: Converting to Cartesian form

Find the Cartesian equation of the curve given parametrically:

(a) $x = 2t$, $y = t^2$

(b) $x = t + 1$, $y = t^2 - 2t$

(c) $x = 3\cos t$, $y = 3\sin t$, where $0 \leq t \leq 2\pi$

(d) $x = 2t - 1$, $y = \frac{1}{t}$, where $t \neq 0$

Problem 7: Finding $\frac{dy}{dx}$ for parametric curves

For each parametric curve, find $\frac{dy}{dx}$ in terms of t:

(a) $x = t^3$, $y = t^2$

(b) $x = 2t + 1$, $y = t^2 - 3t$

(c) $x = e^t$, $y = e^{2t}$

(d) $x = \sin t$, $y = \cos 2t$

Problem 8: Tangents and normals to parametric curves

A curve is given parametrically by $x = t^2 - 1$, $y = 2t + 3$.

(a) Find $\frac{dy}{dx}$ in terms of t.

(b) Find the equation of the tangent to the curve at the point where $t = 2$.

(c) Find the equation of the normal to the curve at the point where $t = 1$.

Problem 9: Area under parametric curves

A curve is defined parametrically by $x = t^2$, $y = 2t$, where $0 \leq t \leq 2$.

(a) Sketch the curve for the given range of t.

(b) Find the area under the curve from $t = 0$ to $t = 2$ using the formula:

$$A = \int_a^b y \frac{dx}{dt}\, dt$$

Problem 10: Second derivative of parametric curves

For the parametric equations $x = 2t - 1$, $y = t^3 + t$:
(a) Find $\frac{dy}{dx}$ in terms of t.

(b) Find $\frac{d^2y}{dx^2}$ using: $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dt}\left(\dfrac{dy}{dx}\right) \div \dfrac{dx}{dt}$

(c) Find the point(s) where the curve has a horizontal tangent.

Part 3: Optimization Problems

Key Points

Steps for Optimization:

1. Draw a diagram if appropriate
2. Identify the quantity to be maximized or minimized
3. Express this quantity as a function of one variable
4. Find the derivative and set it equal to zero
5. Verify it's a maximum or minimum (second derivative test)
6. Check endpoints if the domain is restricted

Problem 11: Maximizing area

A farmer has 200 metres of fencing and wants to fence a rectangular field that borders a straight river. He does not need to fence the side along the river.

(a) Let x be the width of the field (perpendicular to the river). Express the length of the field in terms of x.

(b) Write an expression for the area A of the field in terms of x.

(c) Find the value of x that maximizes the area.

(d) What is the maximum area of the field?

(e) Verify that this is indeed a maximum using the second derivative test.

Problem 12: Minimizing cost

A cylindrical can is to be designed to hold 1000 cm^3 of liquid. The material for the top and bottom costs \$0.05 per cm^2, while the material for the side costs \$0.03 per cm^2. Let r be the radius and h be the height of the cylinder.

(a) Write the volume constraint equation.

(b) Express h in terms of r using the volume constraint.

(c) Write an expression for the total cost C in terms of r only.

(d) Find the radius r that minimizes the cost.

(e) Find the corresponding height h and the minimum cost.

Problem 13: Maximizing volume

A box with an open top is to be constructed from a square piece of cardboard, 12 inches on each side, by cutting out equal squares from each corner and folding up the sides.

(a) Let x be the side length of the squares cut from each corner. Draw a diagram showing the dimensions.

(b) Express the volume V of the box as a function of x.

(c) What is the domain of x? (Consider physical constraints)

(d) Find the value of x that maximizes the volume.

(e) What is the maximum volume?

Problem 14: Minimizing distance

Find the point on the parabola $y = x^2$ that is closest to the point $(0, 3)$.

(a) Let (x, x^2) be a point on the parabola. Write an expression for the distance D from this point to $(0, 3)$.

(b) To simplify, minimize D^2 instead. Write the expression for D^2.

(c) Find $\frac{d(D^2)}{dx}$ and set it equal to zero.

(d) Solve for x and find the corresponding y coordinate.

(e) Verify this is a minimum and find the minimum distance.

Problem 15: Optimization with constraints

A rectangular poster is to contain 150 cm^2 of printed material with margins of 2 cm at the top and bottom and 3 cm on each side. What dimensions of the poster will minimize the total area?

(a) Let x and y be the dimensions of the printed area. Write the constraint equation.

(b) Express the total width and height of the poster (including margins) in terms of x and y.

(c) Write the expression for the total area A of the poster.

(d) Express A as a function of x only.

(e) Find the dimensions that minimize the total area.

Part 4: Volumes of Revolution

Key Points

Key Formulas:

- **Rotation about x-axis:** $V = \pi \int_a^b y^2 \, dx$
- **Rotation about y-axis:** $V = \pi \int_c^d x^2 \, dy$
- **Disk method:** Use when rotating a region between curve and axis
- **Washer method:** $V = \pi \int_a^b (R^2 - r^2) \, dx$ for hollow solids
- Always sketch the region and visualize the solid

Problem 16: Basic volumes about the x-axis

Find the volume of the solid generated by rotating the given region about the x-axis.

(a) The region bounded by $y = x^2$, the x-axis, and the lines $x = 0$ and $x = 2$.

(b) The region bounded by $y = \sqrt{x}$, the x-axis, and the line $x = 4$.

(c) The region bounded by $y = e^x$, the x-axis, $x = 0$, and $x = 1$.

Problem 17: Volumes about the y-axis

Find the volume of the solid generated by rotating the given region about the y-axis.

(a) The region bounded by $x = y^2$, the y-axis, and the line $y = 2$.

(b) The region bounded by $x = \sqrt{y}$, the y-axis, and the line $y = 4$.

Problem 18: Volumes between two curves

Find the volume generated when the region between the two curves is rotated about the x-axis.

(a) The region bounded by $y = x$ and $y = x^2$ for $0 \leq x \leq 1$.

(Hint: Use washer method with outer radius $R = x$ and inner radius $r = x^2$)

(b) The region bounded by $y = 4 - x^2$ and $y = 0$ for $-2 \leq x \leq 2$.

Problem 19: Parametric volumes of revolution

A curve is defined parametrically by $x = t^2$, $y = 2t$ for $0 \leq t \leq 2$.
(a) Sketch the curve.

(b) Find the volume generated when this curve is rotated about the x-axis using:

$$V = \pi \int_a^b y^2 \frac{dx}{dt}\, dt$$

Problem 20: Application — Sphere volume

(a) The circle $x^2 + y^2 = r^2$ can be written as $y = \sqrt{r^2 - x^2}$ for the upper semicircle. Rotate this curve about the x-axis from $x = -r$ to $x = r$ to generate a sphere.
Find the volume and verify that it equals $\frac{4}{3}\pi r^3$.

(b) A sphere of radius 6 cm has a cylindrical hole of radius 2 cm drilled through its center. Find the volume of the remaining solid.
(Hint: Use the washer method)

Problem 21: Challenge — Cone volume

A right circular cone has height h and base radius r. The cone can be generated by rotating the line segment from $(0, 0)$ to (h, r) about the x-axis.

(a) Find the equation of the line in the form $y = f(x)$.

(b) Set up and evaluate the integral to find the volume of the cone.

(c) Verify your answer matches the formula $V = \frac{1}{3}\pi r^2 h$.

Part 5: Mixed Practice — Comprehensive Problems

Problem 22: Integration using partial fractions

Evaluate: $\displaystyle\int_0^1 \frac{4x+2}{(x+1)(2x+1)}\,dx$

Problem 23: Parametric curve analysis

A curve is given by $x = 3\cos t$, $y = 2\sin t$ for $0 \le t \le 2\pi$.

(a) Find the Cartesian equation of the curve.

(b) Find the area enclosed by the curve.

Problem 24: Optimization with calculus

A wire of length 100 cm is cut into two pieces. One piece is bent into a circle and the other into a square. How should the wire be cut to minimize the total area enclosed?

(a) Let x be the length of wire used for the circle. Set up the area function $A(x)$.

(b) Find the value of x that minimizes the total area.

(c) What if we want to maximize the area instead?

Problem 25: Volume of revolution application

The region bounded by $y = \sin x$, $y = 0$, $x = 0$, and $x = \pi$ is rotated about the x-axis.
(a) Sketch the region.

(b) Find the volume of the solid generated.

(c) Find the volume if the same region is rotated about the line $y = -1$ instead.
(Hint: Use the washer method with outer radius $R = \sin x + 1$ and inner radius $r = 1$)

Summary — Key Formulas

Partial Fractions:

- Distinct factors: $\frac{P(x)}{(ax+b)(cx+d)} = \frac{A}{ax+b} + \frac{B}{cx+d}$
- Repeated factors: $\frac{P(x)}{(ax+b)^n} = \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}$
- Always factorize denominators completely

Parametric Equations:

- Derivative: $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$
- Second derivative: $\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \div \frac{dx}{dt}$
- Area: $A = \int_a^b y \frac{dx}{dt}\, dt$

Optimization:

- Find critical points: $f'(x) = 0$
- Second derivative test: $f''(x) < 0$ (max), $f''(x) > 0$ (min)
- Check endpoints for constrained domains

Volumes of Revolution:

- Disk method (about x-axis): $V = \pi \int_a^b y^2\, dx$
- Disk method (about y-axis): $V = \pi \int_c^d x^2\, dy$
- Washer method: $V = \pi \int_a^b (R^2 - r^2)\, dx$
- Parametric: $V = \pi \int_a^b y^2 \frac{dx}{dt}\, dt$

Excellent Work!

You have mastered essential advanced calculus techniques.

Yolymatics Tutorials
www.yolymaticstutorials.com
yolymatics007@gmail.com

Keep practicing and exploring the beauty of calculus!