Improper Integrals

Practice Problems

Yolymatics Tutorials

www.yolymaticstutorials.com

November 3, 2025

Outline

What are Improper Integrals?

Definition

An **improper integral** is a definite integral that has one or both of the following properties:

- One or both limits of integration are infinite
- The integrand has an infinite discontinuity in the interval of integration

Types of Improper Integrals

1. **Type 1:** Infinite limits of integration
2. **Type 2:** Infinite discontinuity in the integrand

Type 1: Infinite Limits

Upper Limit Infinite

$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

Lower Limit Infinite

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

Both Limits Infinite

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx$$

where c is any real number. Both integrals must converge.

Type 2: Infinite Discontinuity

Discontinuity at Upper Limit

If f is continuous on $[a, b)$ and discontinuous at b:

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

Discontinuity at Lower Limit

If f is continuous on $(a, b]$ and discontinuous at a:

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$

Discontinuity in the Interior

If f is discontinuous at c where $a < c < b$:

Convergence and Divergence

Convergent Improper Integral

An improper integral is **convergent** if the limit defining it exists and is finite.

Divergent Improper Integral

An improper integral is **divergent** if the limit does not exist or is infinite.

Important Note

When an improper integral involves both infinite limits and discontinuities, or multiple discontinuities, each part must be evaluated separately and ALL parts must converge for the integral to be convergent.

Problem 1: Upper Limit Infinite

Problem

Evaluate the improper integral:

$$\int_1^\infty \frac{1}{x^2}\,dx$$

Does it converge or diverge? If it converges, find its value.

Working Space

Problem 2: Upper Limit Infinite

Problem

Evaluate the improper integral:

$$\int_1^\infty \frac{1}{x}\, dx$$

Does it converge or diverge? If it converges, find its value.

Working Space

Problem 3: Upper Limit Infinite with Exponential

Problem

Evaluate the improper integral:

$$\int_0^{\infty} e^{-3x}\, dx$$

Does it converge or diverge? If it converges, find its value.

Working Space

Problem 4: Lower Limit Infinite

Problem

Evaluate the improper integral:

$$\int_{-\infty}^{0} e^{2x}\,dx$$

Does it converge or diverge? If it converges, find its value.

Working Space

Problem 5: Both Limits Infinite

Problem

Evaluate the improper integral:

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx$$

Does it converge or diverge? If it converges, find its value.

Working Space

Problem 6: p-integral

Problem

For what values of p does the following integral converge?

$$\int_2^\infty \frac{1}{x^p}\,dx$$

When it converges, find its value in terms of p.

Working Space

Problem 7: Convergence Test - Polynomial vs Exponential

Problem

Determine whether the following integral converges or diverges:

$$\int_1^\infty \frac{x^3}{e^x}\,dx$$

Hint: Consider the behavior of exponential vs polynomial functions.

Working Space

Problem 8: Discontinuity at Upper Limit

Problem

Evaluate the improper integral:

$$\int_0^1 \frac{1}{\sqrt{1-x}}\, dx$$

Does it converge or diverge? If it converges, find its value.

Working Space

Problem 9: Discontinuity at Lower Limit

Problem

Evaluate the improper integral:

$$\int_0^1 \frac{1}{\sqrt{x}}\, dx$$

Does it converge or diverge? If it converges, find its value.

Working Space

Problem 10: Discontinuity at Interior Point

Problem

Evaluate the improper integral:

$$\int_0^3 \frac{1}{x-1}\,dx$$

Does it converge or diverge? If it converges, find its value.

Hint: The integrand has a discontinuity at $x = 1$.

Working Space

Problem 11: Logarithmic Integrand

Problem

Evaluate the improper integral:

$$\int_0^1 \ln(x)\, dx$$

Does it converge or diverge? If it converges, find its value.

Working Space

Problem 12: Power Function with Discontinuity

Problem

Evaluate the improper integral:

$$\int_0^4 \frac{1}{x^{2/3}}\,dx$$

Does it converge or diverge? If it converges, find its value.

Working Space

Problem 13: Both Infinite Limit and Discontinuity

Problem

Evaluate the improper integral:

$$\int_0^{\infty} \frac{1}{\sqrt{x}(1+x)}\, dx$$

Does it converge or diverge? If it converges, find its value.

Hint: Split at $x = 1$ and evaluate two improper integrals.

Working Space

Problem 14: Comparison Test Application

Problem

Determine whether the following integral converges or diverges:

$$\int_1^\infty \frac{1+\sin^2(x)}{x^2}\,dx$$

Hint: Use the comparison test with an appropriate function.

Working Space

Problem 15: Comparison Test - Rational Function

Problem

Determine whether the following integral converges or diverges:

$$\int_2^\infty \frac{3x^2+5}{x^4-1}\,dx$$

Hint: Compare with $\frac{1}{x^2}$ for large x.

Working Space

Problem 16: Limit Comparison Test

Problem

Determine whether the following integral converges or diverges:

$$\int_1^\infty \frac{\sqrt{x}}{x^3 + 2x + 1}\, dx$$

Hint: Use the limit comparison test.

Working Space

Problem 17: Comparison with Known Divergent

Problem

Determine whether the following integral converges or diverges:

$$\int_2^{\infty} \frac{1}{\sqrt{x}\,\ln(x)}\,dx$$

Hint: Compare with $\frac{1}{x}$.

Working Space

Problem 18: p-integral Variation

Problem

Determine whether the following integral converges or diverges:

$$\int_1^\infty \frac{1}{x(\ln x)^2}\, dx$$

Hint: Use substitution $u = \ln(x)$.

Working Space

Problem 19: Exponential with Polynomial

Problem

Determine whether the following integral converges or diverges:

$$\int_0^\infty \frac{x^{10}}{e^{\sqrt{x}}}\, dx$$

Hint: Consider the comparison test.

Working Space

Problem 20: Trigonometric Bounded Function

Problem

Determine whether the following integral converges or diverges:

$$\int_1^\infty \frac{\cos^2(x)}{x^{3/2}}\, dx$$

Hint: Use the fact that $0 \leq \cos^2(x) \leq 1$.

Working Space

Problem 21: Algebraic Function

Problem

Determine whether the following integral converges or diverges:

$$\int_1^\infty \frac{x+1}{\sqrt{x^5+3x^2}}\,dx$$

Hint: Find the dominant terms for large x.

Working Space

Problem 22: Rational Function - Borderline Case

Problem

Determine whether the following integral converges or diverges:

$$\int_1^{\infty} \frac{1}{x\sqrt{x^2 - 1}}\, dx$$

Working Space

Problem 23: Type 2 - Convergence Test

Problem

Determine whether the following integral converges or diverges:

$$\int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx$$

Working Space

Problem 24: Type 2 - p-integral at Zero

Problem

For what values of p does the following integral converge?

$$\int_0^1 \frac{\ln(x)}{x^p}\,dx$$

Working Space

Test 1: Direct Comparison Test

Statement

Suppose f and g are continuous on $[a, \infty)$ with $0 \leq f(x) \leq g(x)$ for all $x \geq a$.

1. If $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges
2. If $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ diverges

Key Strategy

Compare with known convergent/divergent integrals like $\int_a^\infty \frac{1}{x^p}\,dx$

Test 2: Limit Comparison Test

Statement

Suppose f and g are continuous, positive functions on $[a, \infty)$.

If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$ where $0 < L < \infty$,

then $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ either both converge or both diverge.

When to Use

Use when direct comparison is difficult to establish but functions have similar growth rates.

Test 3: p-integral Test (Type 1)

Statement

The integral $\int_1^\infty \frac{1}{x^p}\,dx$:

- **Converges** if $p > 1$
- **Diverges** if $p \leq 1$

When $p > 1$: $\int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{p-1}$

Extended Form

$\int_a^\infty \frac{1}{(x-c)^p}\,dx$ has the same convergence properties for $a > c$.

Test 4: p-integral Test (Type 2)

Statement

The integral $\int_0^1 \frac{1}{x^p}\,dx$:

- **Converges** if $p < 1$
- **Diverges** if $p \geq 1$

When $p < 1$: $\int_0^1 \frac{1}{x^p}\,dx = \frac{1}{1-p}$

Extended Form

$\int_a^b \frac{1}{(x-a)^p}\,dx$ converges if $p < 1$, diverges if $p \geq 1$.

$\int_a^b \frac{1}{(b-x)^p}\,dx$ converges if $p < 1$, diverges if $p \geq 1$.

Test 5: Logarithmic Integrals

Important Results

1. $\displaystyle\int_2^\infty \frac{1}{x\ln(x)}\,dx$ **diverges**
2. $\displaystyle\int_2^\infty \frac{1}{x(\ln x)^p}\,dx$ **converges** if $p > 1$, **diverges** if $p \leq 1$
3. $\displaystyle\int_0^1 \ln(x)\,dx$ **converges** (equals -1)
4. $\displaystyle\int_0^1 |\ln(x)|^p\,dx$ **converges** for all $p > 0$

Test 6: Exponential vs Polynomial

Fundamental Principle

For any polynomial $P(x)$ and any constant $a > 0$:

- $\int_1^\infty P(x)e^{-ax}\,dx$ **converges**
- $\int_1^\infty \frac{e^{ax}}{P(x)}\,dx$ **diverges** if $\deg(P) \geq 0$
- $\lim_{x\to\infty} \frac{P(x)}{e^{ax}} = 0$ (exponential dominates polynomial)

Key Insight

Exponential decay (e^{-ax}) is stronger than any polynomial growth, so it forces convergence.

Test 7: Bounded Functions

Comparison Strategy

If $|f(x)| \leq M$ (bounded) for all $x \geq a$, and $\int_a^\infty g(x)\,dx$ converges with $g(x) > 0$:

Then $\int_a^\infty f(x)g(x)\,dx$ converges.

Common Applications

- $|\sin(x)| \leq 1$, $|\cos(x)| \leq 1$
- If $\int_a^\infty \frac{1}{x^p}\,dx$ converges (i.e., $p > 1$), then $\int_a^\infty \frac{\sin(x)}{x^p}\,dx$ converges

Test 8: Asymptotic Behavior

Dominant Term Analysis

For rational functions, identify the dominant terms as $x \to \infty$:

$$\int_a^\infty \frac{a_n x^n + \cdots + a_0}{b_m x^m + \cdots + b_0}\, dx \text{ behaves like } \int_a^\infty \frac{a_n}{b_m} x^{n-m}\, dx$$

This converges if $n - m < -1$ (i.e., $m > n + 1$).

Example

$\displaystyle\int_1^\infty \frac{3x^2 + 5x + 1}{x^4 + 2x - 7}\, dx$ behaves like $\displaystyle\int_1^\infty \frac{3}{x^2}\, dx$ which converges.

Test 9: Absolute Convergence

Statement

If $\displaystyle\int_a^\infty |f(x)|\, dx$ converges, then $\displaystyle\int_a^\infty f(x)\, dx$ converges.
We say the integral is **absolutely convergent**.

Important Note

The converse is NOT true. An integral can converge without converging absolutely (conditional convergence).

Example: $\displaystyle\int_1^\infty \frac{\sin(x)}{x}\, dx$ converges, but $\displaystyle\int_1^\infty \left|\frac{\sin(x)}{x}\right| dx$ diverges.

Quick Reference: Common Convergent Integrals

Useful Results to Remember

1. $\int_0^\infty e^{-ax}\, dx = \frac{1}{a}$ for $a > 0$
2. $\int_{-\infty}^\infty \frac{1}{1+x^2}\, dx = \pi$
3. $\int_0^\infty x^n e^{-x}\, dx = n!$ for $n \in \mathbb{N}$
4. $\int_{-\infty}^\infty e^{-x^2}\, dx = \sqrt{\pi}$ (Gaussian)
5. $\int_0^\infty \frac{\sin(x)}{x}\, dx = \frac{\pi}{2}$

Important Integration Formulas

Basic Antiderivatives

1. $$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$
2. $$\int \frac{1}{x}\,dx = \ln|x| + C$$
3. $$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$
4. $$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right) + C$$
5. $\int x^n e^{ax}\,dx$ requires integration by parts repeatedly

Thank You!

Yolymatics Tutorials

www.yolymaticstutorials.com

Keep practicing!