Analysis

Comprehensive Worksheet

Improper Integrals, Series, and Differential Equations

Yolymatics Tutorials

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Student Name: ______________________________

Date: ______________________________

Show all work clearly. Use additional paper if needed.

Section 7.8 | Improper Integrals

Problem 1: Type I — Infinite interval

Determine whether the following improper integrals converge or diverge. If convergent, evaluate.

(a) $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$

(b) $\displaystyle\int_0^\infty e^{-3x}\,dx$

(c) $\displaystyle\int_{-\infty}^0 \frac{1}{1+x^2}\,dx$

Problem 2: Type II — Discontinuous integrand

Evaluate the following improper integrals or show that they diverge.

(a) $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$

(b) $\displaystyle\int_0^3 \frac{1}{(x-1)^{2/3}}\,dx$

(c) $\displaystyle\int_{-1}^1 \frac{1}{x^2}\,dx$

Problem 3: Comparison test for improper integrals

Use the comparison test to determine whether the integral converges or diverges.

(a) $\displaystyle\int_1^\infty \frac{1+\sin^2 x}{x^2}\,dx$

(b) $\displaystyle\int_2^\infty \frac{1}{\sqrt{x^3-1}}\,dx$

Problem 4: Challenge problem

Find the value of p for which the integral $\displaystyle\int_e^{\infty} \frac{1}{x(\ln x)^p}\,dx$ converges.

Beta and Gamma Functions

Recall:

- Gamma function: $\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}\,dx$, with $\Gamma(n+1) = n\Gamma(n)$ and $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$
- Beta function: $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

Problem 5: Gamma function properties

(a) Show that $\Gamma(1) = 1$ and find $\Gamma(5)$.

(b) Prove that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ by evaluating $\int_0^\infty x^{-1/2}e^{-x}\,dx$.

(c) Evaluate $\Gamma\left(\frac{7}{2}\right)$.

Problem 6: Beta function evaluation

(a) Evaluate $B(3, 4)$ using the definition.

(b) Verify your answer using the relationship $B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

(c) Use the beta function to evaluate $\int_0^1 x^5(1-x)^3\,dx$.

Problem 7: Applications

Express the following integrals in terms of gamma or beta functions, then evaluate:

(a) $\displaystyle\int_0^\infty x^4 e^{-x}\,dx$

(b) $\displaystyle\int_0^{\pi/2} \sin^3\theta\cos^5\theta\,d\theta$ (Hint: Use substitution $x = \sin^2\theta$)

Section 11.1 | Sequences

Problem 8: Finding limits of sequences

Find the limit of each sequence or show that it diverges.

(a) $a_n = \frac{3n^2+5n}{2n^2-1}$

(b) $a_n = \frac{n^2}{e^n}$

(c) $a_n = \frac{\ln n}{n}$

(d) $a_n = \left(1 + \frac{2}{n}\right)^n$

Problem 9: Monotonicity and boundedness

For the sequence $a_n = \frac{n}{n+1}$:

(a) Prove that the sequence is increasing.

(b) Prove that the sequence is bounded above by 1.

(c) Find $\lim_{n\to\infty} a_n$.

Problem 10: Recursive sequences

Consider the sequence defined by $a_1 = 1$ and $a_{n+1} = \sqrt{2 + a_n}$ for $n \geq 1$.

(a) Calculate a_2, a_3, a_4, a_5.

(b) Prove that the sequence is increasing and bounded above by 2.

(c) Find the limit of the sequence.

Section 11.2 | Series

Problem 11: Geometric series

Determine whether each geometric series converges or diverges. If it converges, find the sum.

(a) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$

(b) $\displaystyle\sum_{n=0}^{\infty}\frac{3^n}{4^{n-1}}$

(c) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\left(\frac{5}{4}\right)^n$

Problem 12: Telescoping series

Find the sum of the series by writing out several terms and identifying the pattern.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

(b) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)$

Problem 13: Divergence test

Use the divergence test to show that the following series diverge.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^2+1}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{2n+1}$

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Section 11.3 | Integral Test

Problem 14: Using the integral test

Use the integral test to determine whether the series converges or diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

Problem 15: The p-series

(a) State the p-series test and explain when a p-series converges.

(b) Determine convergence or divergence: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

(c) For what values of p does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge?

Problem 16: Estimating sums

Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

(a) Use the integral test remainder estimate to find bounds for the error if we approximate the sum by S_{10} (sum of first 10 terms).

(b) How many terms are needed to ensure the error is less than 0.0001?

Section 11.4 | Comparison Test

Problem 17: Direct comparison test

Use the direct comparison test to determine whether the series converges or diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n + n}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n^3 - n}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^2}$

Problem 18: Limit comparison test

Use the limit comparison test to determine convergence or divergence.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2+3n}{5n^4+n^2-1}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n^2}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+2n}}$

Section 11.5 | Alternating Series Test and Absolute Convergence

Problem 19: Alternating series test

Determine whether the alternating series converges or diverges.

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ (alternating harmonic series)

(b) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n}\frac{n}{n^2+1}$

(c) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+1}$

Problem 20: Absolute vs conditional convergence

Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^2}$

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n n}{n+1}$

Problem 21: Error estimation

For the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$:

(a) Estimate the sum using the first 5 terms.

(b) Use the alternating series estimation theorem to find a bound for the error.

Section 11.6 | Ratio and Root Test

Problem 22: Ratio test

Use the ratio test to determine whether the series converges or diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

Problem 23: Root test

Use the root test to determine convergence or divergence.

(a) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{2n+3}{3n+2}\right)^n$

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{n^n}{(n+1)^{2n}}$

Problem 24: Choosing the right test

For each series, state which test you would use and determine convergence/divergence.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 3^n}{n!}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

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Section 11.7 | Strategy for Testing Series (Extra Practice)

Problem 25: Mixed practice I

Determine whether each series converges or diverges. State which test you use.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{n^4+n}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n^3}$

Problem 26: Mixed practice II

Determine convergence or divergence. Justify your answer.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{2 \cdot 5 \cdot 8 \cdots (3n-1)}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{n+1}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 + 2^n}{n^2 + 3^n}$

Problem 27: Challenging series

Determine whether the following series converge or diverge.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\ln n)^n}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n!\, e^n}{n^n}$

Section 11.8 | Power Series

Problem 28: Radius and interval of convergence

Find the radius of convergence and interval of convergence for each power series.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n \cdot 2^n}$

Problem 29: Power series with factorials

Find the radius and interval of convergence.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{n!\,x^n}{(2n)!}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(x+3)^n}{n \cdot 5^n}$

Problem 30: Testing endpoints

For the power series $\sum_{n=1}^{\infty} \frac{x^n}{n^3}$:

(a) Find the radius of convergence.

(b) Test convergence at both endpoints.

(c) State the interval of convergence.

Section 11.9 | Representations of Functions as Power Series

Problem 31: Power series representation

Find a power series representation for each function and determine the radius of convergence.

(a) $f(x) = \frac{1}{1+x}$

(b) $f(x) = \frac{1}{1-x^2}$

(c) $f(x) = \frac{x}{(1-x)^2}$

Problem 32: Integration and differentiation

(a) Starting with the geometric series, find a power series for $f(x) = \ln(1 + x)$.

(b) Find a power series representation for $f(x) = \arctan x$.

(c) Use part (b) to find a series for π (Hint: evaluate at $x = 1$).

Problem 33: Application

Use a power series to evaluate the integral $\int_0^{0.5} \frac{\sin x}{x}\, dx$ accurate to three decimal places.

Section 11.10 | Taylor and Maclaurin Series

Problem 34: Finding Maclaurin series

Find the Maclaurin series for each function.

(a) $f(x) = e^x$

(b) $f(x) = \sin x$

(c) $f(x) = \cos x$

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Problem 35: Taylor series at x

Find the Taylor series for $f(x) = \ln x$ centered at $a = 1$.

Problem 36: Using known series

Use known Maclaurin series to find the Maclaurin series for:

(a) $f(x) = x\cos(x^2)$

(b) $f(x) = e^{-x^2}$

(c) $f(x) = \frac{1-\cos x}{x^2}$

Problem 37: Taylor polynomial approximation

(a) Find the third-degree Taylor polynomial $T_3(x)$ for $f(x) = \sqrt{x}$ at $a = 4$.

(b) Use Taylor's inequality to estimate the error in using $T_3(x)$ to approximate $f(5)$.

Problem 38: Applications of Taylor series

(a) Use the Maclaurin series for e^x to evaluate $\lim_{x\to 0} \frac{e^x-1-x}{x^2}$.

(b) Approximate $\sqrt{e}$ using the first four terms of the Maclaurin series for $e^{x/2}$.

Second-Order Linear Differential Equations

Recall: A second-order linear differential equation has the form:

$$a(x)y'' + b(x)y' + c(x)y = f(x)$$

If $f(x) = 0$, the equation is **homogeneous**. Otherwise, it is **nonhomogeneous**.

Problem 39: Homogeneous equations with constant coefficients

Solve the following differential equations.

(a) $y'' - 5y' + 6y = 0$

(b) $y'' + 4y' + 4y = 0$

(c) $y'' + 2y' + 5y = 0$

Problem 40: Initial value problems

Solve the initial value problem.

(a) $y'' - 3y' - 4y = 0, \quad y(0) = 2, \quad y'(0) = -1$

(b) $y'' + 9y = 0, \quad y(0) = 1, \quad y'(0) = 3$

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Problem 41: Characteristic equation analysis

For each differential equation, find the characteristic equation, classify the roots, and write the general solution.

(a) $y'' - 6y' + 9y = 0$

(b) $y'' + y' - 6y = 0$

(c) $y'' - 2y' + 10y = 0$

Nonhomogeneous Linear Equations

Method: The general solution is $y = y_h + y_p$, where:

- y_h is the general solution to the homogeneous equation
- y_p is a particular solution to the nonhomogeneous equation

Problem 42: Method of undetermined coefficients

Solve the differential equation using the method of undetermined coefficients.

(a) $y'' - 3y' + 2y = e^{3x}$

(b) $y'' + 4y = 8\sin(2x)$

Problem 43: Polynomial and exponential forcing

Find the general solution.

(a) $y'' - y' - 2y = 4x^2$

(b) $y'' + y = xe^x$

Problem 44: Variation of parameters

Use the method of variation of parameters to solve:

(a) $y'' + y = \sec x, \quad 0 < x < \pi/2$

(b) $y'' - 2y' + y = \frac{e^x}{x}, \quad x > 0$

Problem 45: Initial value problem — Nonhomogeneous

Solve the initial value problem.
$y'' + 4y' + 4y = e^{-2x}, \quad y(0) = 1, \quad y'(0) = 0$

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Problem 46: Application — Spring-mass system

A mass on a spring satisfies the differential equation:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t)$$

For $m = 1$ kg, $c = 4$ kg/s, $k = 5$ N/m, and external force $F(t) = 10\cos(2t)$ N:

(a) Write the differential equation.

(b) Find the general solution.

(c) If $x(0) = 0$ and $x'(0) = 0$, find the particular solution.

Excellent Work!

You have completed a comprehensive review of Analysis topics.

Remember to review any problems you found challenging.

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