Linear Algebra
Comprehensive Worksheet

Yolymatics Tutorials
Professional Mathematics Education

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Instructions for Students

- Show all your working clearly in the spaces provided
- Write your final answers in the designated answer boxes
- Use additional paper if you need more space
- Each section corresponds to a specific textbook chapter
- Remember to label all diagrams and sketches

1 Cross Product (Section 3.5)

Problem 1. Calculate the cross product $\vec{a} \times \vec{b}$ for the following vectors:

(a) $\vec{a} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}, \vec{b} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$

Working:

Answer:

(b) $\vec{a} = \begin{pmatrix} 5 \\ 0 \\ -3 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$

Working:

Answer:

Problem 2. Given vectors $\vec{u} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix}$:

(a) Find $\vec{u} \times \vec{v}$

Working:

Answer:

(b) Verify that $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$

Working:

Answer:

(c) Find the area of the parallelogram spanned by $\vec{u}$ and $\vec{v}$

Working:

Answer:

Problem 3. Prove that for any vectors $\vec{a}$ and $\vec{b}$ in $\mathbb{R}^3$:

(a) $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$ (anti-commutativity)

Working:

(b) $\vec{a} \times \vec{a} = \vec{0}$

Working:

Problem 4. Find the volume of the parallelepiped determined by the vectors:

$$\vec{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}, \quad \vec{c} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

Working:

Answer:

2 Eigenvalues and Eigenvectors (Section 5.1)

Problem 5. Find the eigenvalues and corresponding eigenvectors for each matrix:

(a) $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

Working:

Answer:

(b) $B = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

Working:

Answer:

(c) $C = \begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix}$

Working:

Answer:

Problem 6. Consider the matrix $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 2 & -4 & 2 \end{pmatrix}$

(a) Find the characteristic polynomial of A

Working:

Answer:

(b) Find all eigenvalues of A

Working:

Answer:

(c) Find an eigenvector for each eigenvalue

Working:

Answer:

Problem 7. Let $\lambda = 5$ be an eigenvalue of $A = \begin{pmatrix} 5 & 2 \\ 2 & 5 \end{pmatrix}$ with eigenvector $\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(a) Verify that $\vec{v}$ is indeed an eigenvector with eigenvalue $\lambda = 5$

Working:

(b) Find the other eigenvalue and eigenvector

Working:

Answer:

(c) What is the trace and determinant of A? How do they relate to the eigenvalues?

Working:

Answer:

Problem 8. Prove that if λ is an eigenvalue of an invertible matrix A, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

Working:

3 Diagonalization (Section 5.2)

Problem 9. Determine whether each matrix is diagonalizable. If so, find matrices P and D such that $A = PDP^{-1}$.

(a) $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$

Working:

Answer:

(b) $B = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$

Working:

Answer:

(c) $C = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$

Working:

Answer:

Problem 10. Given $A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$ has eigenvalues $\lambda_1 = 6$ and $\lambda_2 = 1$ with corresponding eigenvectors $\vec{v}_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$:

(a) Write A in the form PDP^{-1}

Working:

Answer:

(b) Use this to compute A^5

Working:

Answer:

(c) Find A^{10}

Working:

Answer:

Problem 11. Let $A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}$.

(a) Find the eigenvalues of A

Working:

Answer:

(b) Determine the geometric multiplicity of each eigenvalue

Working:

Answer:

(c) Is A diagonalizable? Justify your answer

Working:

Answer:

4 Differential Equations (Section 5.4)

Problem 12. Solve the system of differential equations:

$$\frac{d\vec{x}}{dt} = A\vec{x}, \quad \text{where} \quad A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad \vec{x}(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Working:

Answer:

Problem 13. Consider the system $\vec{x}' = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} \vec{x}$ with initial condition $\vec{x}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

(a) Find the general solution

Working:

Answer:

(b) Find the particular solution satisfying the initial condition

Working:

Answer:

(c) Sketch the phase portrait

Working:

Problem 14. Solve the decoupled system:

$$\begin{cases} \frac{dy_1}{dt} = 3y_1 \\ \frac{dy_2}{dt} = -2y_2 \end{cases}$$

with initial conditions $y_1(0) = 2$ and $y_2(0) = 4$.

Working:

Answer:

Problem 15. A dynamical system is described by $\vec{x}' = A\vec{x}$ where $A = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$.

(a) Find the eigenvalues of A

Working:

Answer:

(b) Classify the equilibrium point at the origin (stable/unstable, node/spiral/saddle)

Working:

Answer:

(c) Describe the long-term behavior of solutions

Working:

Answer:

5 Inner Product (Section 6.1)

Problem 16. Compute the inner product $\langle \vec{u}, \vec{v} \rangle$ for:

(a) $\vec{u} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix}$ (standard inner product)

Working:

Answer:

(b) $\vec{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ with $\langle \vec{u}, \vec{v} \rangle = 2u_1v_1 + 3u_2v_2$

Working:

Answer:

Problem 17. Let $V = P_2$ be the space of polynomials of degree at most 2, with inner product $\langle p, q\rangle = \int_0^1 p(x)q(x)\,dx$.

(a) Compute $\langle 1, x\rangle$, $\langle 1, x^2\rangle$, and $\langle x, x^2\rangle$

Working:

Answer:

(b) Find $\|x\|$ and $\|x^2\|$

Working:

Answer:

(c) Are $\{1, x, x^2\}$ orthogonal?

Working:

Answer:

Problem 18. Verify that the following defines an inner product on $\mathbb{R}^2$:

$$\langle \vec{u}, \vec{v} \rangle = 3u_1v_1 + 2u_2v_2$$

by checking all four axioms of an inner product.

Working:

Problem 19. Given $\vec{u} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$:

(a) Find $\|\vec{u}\|$

Working:

Answer:

(b) Find a unit vector in the direction of $\vec{u}$

Working:

Answer:

(c) Find all vectors orthogonal to $\vec{u}$

Working:

Answer:

6 Angle and Orthogonality (Section 6.2)

Problem 20. Find the angle between the vectors:

(a) $\vec{u} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

Working:

Answer:

(b) $\vec{u} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$

Working:

Answer:

(c) $\vec{u} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

Working:

Answer:

Problem 21. Determine which of the following pairs of vectors are orthogonal:

(a) $\vec{a} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 1 \\ -2 \\ -4 \end{pmatrix}$

Working:

Answer:

(b) $\vec{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

Working:

Answer:

(c) $\vec{a} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$

Working:

Answer:

Problem 22. Find the orthogonal projection of $\vec{b} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$ onto $\vec{a} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$.

(a) Find $\text{proj}_{\vec{a}}\vec{b}$

Working:

Answer:

(b) Find the component of $\vec{b}$ orthogonal to $\vec{a}$

Working:

Answer:

(c) Verify that these two components are orthogonal

Working:

Problem 23. Use the Cauchy-Schwarz inequality to prove that for any vectors $\vec{u}$ and $\vec{v}$ in an inner product space:

$$|\langle \vec{u}, \vec{v} \rangle| \leq \|\vec{u}\| \cdot \|\vec{v}\|$$

Then verify this inequality for $\vec{u} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Working:

7 Gram-Schmidt Process (Section 6.3)

Problem 24. Apply the Gram-Schmidt process to find an orthogonal basis for the subspace spanned by:

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Then normalize to obtain an orthonormal basis.

Working:

Answer:

Problem 25. Find an orthonormal basis for the column space of:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}$$

Working:

Answer:

Problem 26. Consider the polynomials $p_1(x) = 1$, $p_2(x) = x$, $p_3(x) = x^2$ on the interval $[0, 1]$ with inner product $\langle p, q \rangle = \int_0^1 p(x)q(x)\,dx$.

(a) Apply Gram-Schmidt to obtain an orthogonal set $\{q_1, q_2, q_3\}$

Working:

Answer:

(b) Normalize to get an orthonormal set

Working:

Answer:

Problem 27. Given vectors $\vec{u}_1 = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$ and $\vec{u}_2 = \begin{pmatrix} -1 \\ 0 \\ 7 \end{pmatrix}$:

(a) Find an orthogonal basis $\{\vec{v}_1, \vec{v}_2\}$ for $\text{span}\{\vec{u}_1, \vec{u}_2\}$

Working:

Answer:

(b) Extend this to an orthogonal basis for $\mathbb{R}^3$

Working:

Answer:

8 Orthogonal Matrices (Section 7.1)

Problem 28. Determine which of the following matrices are orthogonal:

(a) $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Working:

Answer:

(b) $B = \frac{1}{3}\begin{pmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{pmatrix}$

Working:

Answer:

(c) $C = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

Working:

Answer:

Problem 29. Let $Q = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}$

(a) Verify that Q is orthogonal

Working:

(b) Find Q^{-1}

Working:

Answer:

(c) Show that Q preserves the length of vectors

Working:

Problem 30. Prove that if Q is an orthogonal matrix, then:

(a) $\det(Q) = \pm 1$

Working:

(b) Q^T is also orthogonal

Working:

(c) The rows of Q form an orthonormal set

Working:

Problem 31. Find an orthogonal matrix Q whose first column is $\frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$.

Working:

Answer:

9 Orthogonal Diagonalization (Section 7.2)

Problem 32. Determine whether each symmetric matrix is orthogonally diagonalizable. If so, find an orthogonal matrix P and diagonal matrix D such that $A = PDP^T$.

(a) $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

Working:

Answer:

(b) $B = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$

Working:

Answer:

Problem 33. Let $A = \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}$.

(a) Show that A is symmetric

Working:

(b) Find the eigenvalues and eigenvectors

Working:

Answer:

(c) Find an orthogonal matrix P that diagonalizes A

Working:

Answer:

(d) Verify that $P^T AP$ is diagonal

Working:

Problem 34. Prove the Spectral Theorem: Every symmetric matrix is orthogonally diagonalizable.

Working:

Problem 35. Given $A = \begin{pmatrix} 6 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$:

(a) Find an orthogonal diagonalization of A

Working:

Answer:

(b) Use this to compute A^{10}

Working:

Answer:

10 Quadratic Forms (Section 7.3)

Problem 36. Write each quadratic form in matrix notation $\vec{x}^T A\vec{x}$:

(a) $Q(x_1, x_2) = 3x_1^2 + 4x_1x_2 + 5x_2^2$

Working:

Answer:

(b) $Q(x_1, x_2, x_3) = x_1^2 - 2x_2^2 + 3x_3^2 + 4x_1x_2 - 6x_2x_3$

Working:

Answer:

Problem 37. For the quadratic form $Q(x_1, x_2) = 5x_1^2 + 8x_1x_2 + 5x_2^2$:

(a) Write Q in the form $\vec{x}^T A\vec{x}$

Working:

Answer:

(b) Find an orthogonal change of variables that eliminates the cross-product term

Working:

Answer:

(c) Classify the quadratic form (positive definite, negative definite, or indefinite)

Working:

Answer:

Problem 38. Determine whether each quadratic form is positive definite, negative definite, indefinite, or positive/negative semidefinite:

(a) $Q(\vec{x}) = 2x_1^2 + 4x_1x_2 + 3x_2^2$

Working:

Answer:

(b) $Q(\vec{x}) = -x_1^2 + 2x_1x_2 - x_2^2$

Working:

Answer:

(c) $Q(\vec{x}) = x_1^2 + x_2^2 + x_3^2$

Working:

Answer:

Problem 39. Find the principal axes and classify the conic section:

$$5x^2 - 4xy + 2y^2 = 6$$

Working:

Answer:

11 Optimization using Quadratic Forms (Section 7.4)

Problem 40. Find the maximum and minimum values of $Q(\vec{x}) = 3x_1^2 + 2x_2^2 + 2x_1x_2$ subject to $\|\vec{x}\| = 1$.

Working:

Answer:

Problem 41. Consider the function $f(x, y) = 4x^2 + 4xy + y^2$.

(a) Find the maximum value of f on the unit circle $x^2 + y^2 = 1$

Working:

Answer:

(b) Find the minimum value of f on the unit circle

Working:

Answer:

(c) At what points are these extreme values attained?

Working:

Answer:

Problem 42. Use quadratic forms to find the extreme values of:

$$f(x, y, z) = x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$$

subject to the constraint $x^2 + y^2 + z^2 = 1$.

Working:

Answer:

Problem 43. A company's profit function is given by $P(x_1, x_2) = -2x_1^2 - x_2^2 + x_1 x_2 + 8x_1 + 6x_2$.

(a) Find the critical points

Working:

Answer:

(b) Use the second derivative test (Hessian matrix) to classify each critical point

Working:

Answer:

(c) Find the maximum profit

Working:

Answer:

12 Rank, Nullity and Matrix Spaces (Section 4.9)

Problem 44. For each matrix, find the rank, nullity, a basis for the column space, and a basis for the null space:

(a) $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 2 \end{pmatrix}$

Working:

Answer:

(b) $B = \begin{pmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 1 & 5 \\ -1 & -2 & 1 & 1 \end{pmatrix}$

Working:

Answer:

Problem 45. Verify the Rank-Nullity Theorem for the matrix:

$$A = \begin{pmatrix} 1 & 3 & 4 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & -4 & 0 \end{pmatrix}$$

Working:

Answer:

Problem 46. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation with matrix representation:

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(a) Find $\operatorname{rank}(A)$ and $\operatorname{nullity}(A)$

Working:

Answer:

(b) Find a basis for Col(A) and Nul(A)

Working:

Answer:

(c) Is T one-to-one? Is T onto?

Working:

Answer:

Problem 47. Find the dimension of the row space, column space, and null space of:

$$A = \begin{pmatrix} 2 & 4 & -2 & 1 \\ 1 & 2 & -1 & 0 \\ 3 & 6 & -3 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Then verify that $\dim(\text{Row}(A)) = \dim(\text{Col}(A))$.

Working:

Answer:

Problem 48. Show that the set of all 2×2 symmetric matrices forms a subspace of $M_{2\times 2}$. What is its dimension? Find a basis.

Working:

Answer:

13 Geometry of Matrix Operators (Section 8.6)

Problem 49. Describe the geometric action of each linear transformation:

(a) $T(\vec{x}) = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \vec{x}$

Working:

Answer:

(b) $T(\vec{x}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \vec{x}$

Working:

Answer:

(c) $T(\vec{x}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \vec{x}$

Working:

Answer:

Problem 50. Let $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ where $\theta = \frac{\pi}{4}$.

(a) Describe the geometric transformation represented by A

Working:

Answer:

(b) Find the image of $\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ under this transformation

Working:

Answer:

(c) Find A^4. What transformation does it represent?

Working:

Answer:

Problem 51. Consider the shear transformation $S = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$.

(a) Describe how S transforms the unit square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, $(0, 1)$ for $k = 2$

Working:

(b) Find the eigenvalues and eigenvectors of S

Working:

Answer:

(c) Does a shear transformation preserve area? Justify your answer

Working:

Answer:

Problem 52. The matrix $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ has eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 2$ with eigenvectors $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

(a) Describe geometrically what A does to vectors in the directions of $\vec{v}_1$ and $\vec{v}_2$

Working:

Answer:

(b) Sketch the image of the unit circle under transformation by A

Working:

(c) What is the area magnification factor of this transformation?

Working:

Answer:

Problem 53. Decompose the transformation $T(\vec{x}) = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} \vec{x}$ into:

(a) A rotation followed by scaling along coordinate axes

Working:

Answer:

(b) Describe the geometric effect on the unit circle

Working:

Answer:

End of Worksheet

Well done for completing this comprehensive worksheet!

Review your work and check your answers with your instructor.