M344 Discrete Mathematics

Exam Preparation Problems

Yolymatics Tutorials

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Outline

Question 1 [3 marks]: Ring Arithmetic

Problem

In the ring of integers modulo 3, whose elements are $0, 1, 2$, compute:

$$((2+2) \times (1+1))^5$$

Working Space

Question 2 [3 marks]: Modular Equations

Problem

Consider the following modular equation:

$$5x \equiv 8 \pmod{7}$$

What can you say about existence and uniqueness of the solution? Justify your answer.

Working Space

Question 3 [3 marks]: Euler's Theorem

Problem

Let a and n be natural numbers. Under which assumption does Euler's Theorem assert that:

$$a^{\varphi(n)} \equiv 1 \pmod{n}?$$

Describe $\varphi(n)$ in this formula.

Working Space

Question 4 [3 marks]: Chinese Remainder Theorem

Problem

Can we apply the Chinese Remainder Theorem to conclude that there exists a natural number x which is divisible by both 5 and 7?

What can we say about uniqueness of x?

Working Space

Question 5: GCD and Linear Combinations [3 marks]

Problem

Let $p = 2^n 3^m$ and $q = 3^m 5^k$, where n, m, k are natural numbers.
Consider the set $S = \{px + qy \mid x, y \in \mathbb{Z}\}$.
For which values of m is 3^m an element of this set? Justify your answer.

Working Space

Question 6 [3 marks]: Euler's Totient Function

Problem

Compute 120 mod $\varphi(120)$ and show calculation.

Working Space

Question 7 [3 marks]: Distribution Problem

Problem

An absent minded postman has to deliver 7 different letters to 3 different addresses. However, out of being absent minded, the postman did not look at the addresses on the letters upon delivery. He does remember correctly though that the first two addresses must receive 2 letters each.

In how many different ways could the absent minded postman deliver?

Working Space

Question 8 [3 marks]: Outcomes with Constraints

Problem

How many outcomes can there be if 3 people are donating their towels to 2 different charities, where each person donates up to 10 towels in total?
Note that an outcome is determined by the total number of towels donated to each charity.

Working Space

Question 9 [3 marks]: Complement Counting

Problem

There is a scientific experiment with 10 vials. Each vial either contains water or a special clear chemical that looks like water. Exactly 6 of the vials contain the special chemical.
A researcher must choose a set of 4 vials to test. In how many scenarios will at least one of the chosen vials contain water?

Working Space

Question 10 [3 marks]: Graph Isomorphism

Problem

Write down an example of a graph as an ordered pair of sets, having 3 vertices and 2 edges. Write down another graph that is isomorphic to the first one, but not equal to the first one, although it has the same set of vertices.

Working Space

Question 11 [5 marks]: Combinatorial Word Problem

Sample Problem

A university club has 12 members: 7 undergraduate students and 5 graduate students. The club needs to form a committee of 6 members to organize an event.

1. How many different committees can be formed if there are no restrictions?
2. How many committees contain exactly 2 graduate students?
3. How many committees contain at least 3 undergraduate students?

Show all calculations.

Working Space

Working Space (continued)

Question 12 [5 marks]: Constrained Sequences

Problem

Consider the set $X \times Y$, where X has 3 distinct elements and Y has 4 distinct elements. In how many ways can we create a list

$$(x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$$

of 5 elements of $X \times Y$, where the order matters, so that:

- $x_{2i} = x_{2i+1}$ for every $i \in \{0, 1\}$
- $y_{2i+1} = y_{2i+2}$ for every $i \in \{0, 1\}$
- The list contains at least two distinct elements of $X \times Y$

Include proof in your answer.

Working Space

Working Space (continued)

Question 13 [5 marks]: Number Theory Proof

Sample Problem

Let a, b, c be integers such that $\gcd(a, b) = 1$.
Prove: If $a \mid c$ and $b \mid c$, then $ab \mid c$.
Your proof should:

- Use properties of divisibility
- Apply the coprimality condition
- Be rigorous and complete

Working Space

Working Space (continued)

Question 14 [7 marks]: Graph Theory Analysis

Problem

Consider the graph

$$G = (\{1,2,3\}, \{\{x,y\} \subseteq \{1,2,3\} \mid x - y \in \{1,2\}\})$$

Answer the following questions.

Question 14(a) [1 mark]: Graph Classification

Problem

Is G a complete graph, a bipartite graph, a cycle, or a path graph?

Working Space

Question 14(b) [1 mark]: Isomorphic Graphs

Problem

What are all possible graphs which are isomorphic to G and at the same time, whose set V of vertices is the same as that of G, i.e., $V = \{1, 2, 3\}$?

Working Space

Question 14(c) [5 marks]: Connected Subgraphs

Problem

Which are all connected subgraphs of G and which of these connected subgraphs are isomorphic to each other?

Working Space

Working Space (continued)

Question 15 [8 marks]: Combined Concepts

Sample Problem

Let $G = (V, E)$ be a graph where $V = \{1, 2, \ldots, n\}$ and two vertices i and j are adjacent if and only if $\gcd(i, j) = 1$ (i.e., i and j are coprime).

Part (a) [2 marks]: Draw the graph for $n = 6$.

Part (b) [3 marks]: For which values of n is this graph connected? Justify your answer.

Part (c) [3 marks]: Prove that if p is a prime number and $n = p$, then the graph is complete.

Working Space

Working Space (continued)

Working Space (continued)

Number Theory Reference

Key Theorems

1. **Euclidean Algorithm**: $\gcd(a, b) = \gcd(b, a \bmod b)$
2. **Bézout's Identity**: $\gcd(a, b) = ax + by$ for some $x, y \in \mathbb{Z}$
3. **Euler's Theorem**: If $\gcd(a, n) = 1$, then $a^{\varphi(n)} \equiv 1 \pmod{n}$
4. **Chinese Remainder Theorem**: If $\gcd(m, n) = 1$, then the system
$$x \equiv a \pmod{m}, \quad x \equiv b \pmod{n}$$
has a unique solution modulo mn
5. **Euler's Totient**: $\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$

Combinatorics Reference

Key Formulas

1. **Permutations**: $P(n, r) = \frac{n!}{(n-r)!}$
2. **Combinations**: $\binom{n}{r} = \frac{n!}{r!(n-r)!}$
3. **Multisets**: $\left(\!\binom{n}{r}\!\right) = \binom{n+r-1}{r}$
4. **Inclusion-Exclusion**: $|A \cup B| = |A| + |B| - |A \cap B|$
5. **Complement Principle**: $|A^c| = |U| - |A|$
6. **Product Rule**: If task 1 has m outcomes and task 2 has n outcomes, total is $m \times n$

Graph Theory Reference

Key Definitions

- **Graph**: $G = (V, E)$ where V is vertex set, E is edge set
- **Complete Graph** K_n: Every pair of vertices is adjacent
- **Bipartite Graph**: Vertices can be partitioned into two sets with edges only between sets
- **Cycle** C_n: Closed path with n vertices
- **Path** P_n: Connected graph with n vertices, no cycles
- **Isomorphism**: Bijection $f : V_1 \to V_2$ preserving adjacency
- **Subgraph**: $H = (V', E')$ where $V' \subseteq V$ and $E' \subseteq E$
- **Connected**: Path exists between any two vertices

Problem-Solving Strategies

Approach for Section A (3-mark problems)

- Read carefully and identify the concept being tested
- Write down relevant definitions or theorems
- Show all calculation steps
- State your final answer clearly
- Time: 5-6 minutes per question

Approach for Section B (5-mark problems)

- Break problem into smaller parts
- Define variables clearly
- Include justifications for each step
- For proofs: state what you're proving, show logical steps, conclude
- Time: 10-12 minutes per question

Problem-Solving Strategies (continued)

Approach for Section C (7-8 mark problems)

- Read all parts before starting
- Draw diagrams when helpful (especially for graphs)
- Each part may build on previous parts
- Allocate time: Part (a) 2-3 min, Part (b) 4-5 min, Part (c) proof 5-6 min
- Check your work if time permits

Common Mistakes to Avoid

- Not checking if conditions for theorems are satisfied (e.g., coprimality)
- Confusing "at least" with "exactly" in counting problems
- Forgetting to justify existence vs uniqueness
- Missing edge cases in graph problems

Good Luck!

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You've got this!