Daya's STEP Mark Scheme

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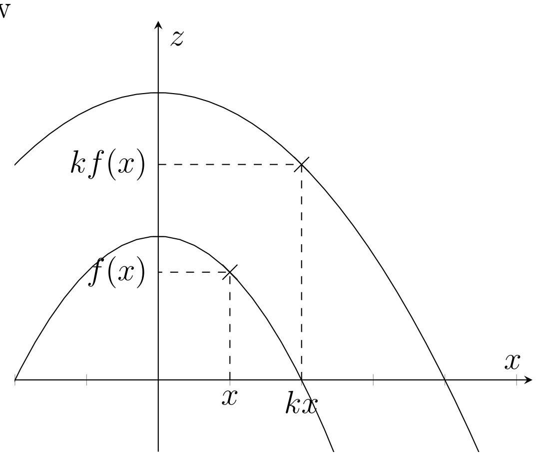
Question 1

M1M1 Correct graph below



M1A1 Let $g(x)$ be the enlarged graph of $f(x)$.

$$g(kx) = kf(x)$$
$$g(x') = kf\left(\frac{x'}{k}\right)$$

where $x' = \dfrac{x}{k}$

(4)

M1A1 (i)

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$
$$\text{Let } x' = \frac{x}{a}$$
$$h(x') = \frac{1}{a}(a(ax')^2 + b(ax') + c)$$
$$h(x') = (ax')^2 + bx' + \frac{c}{a}$$
$$h(x) = (ax)^2 + bx + \frac{c}{a}$$

(2)

M1A1 (ii) Completing the square:

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

E1A1 The equation can be translated to generalise a parabola further:

$$g(x) = ax^2$$

E1 Hence any enlargement k maps onto a parabola with a different value for $a = a'$:

M1A1

$$\begin{aligned} kg\left(\frac{x}{k}\right) &= ak\left(\frac{x}{a}\right)^2 \\ &= \frac{k}{a}x^2 \\ \therefore a' &= \frac{k}{a} \end{aligned}$$

(7)

M1A1 (iii) Applying a general enlargement to map the equation with a to the one to b:

$$\begin{aligned} f(x) &= x^3 + ax \\ kf\left(\frac{x}{k}\right) &= \left(\frac{x}{k}\right)^3 + a\left(\frac{x}{k}\right) \end{aligned}$$

E1A1

$$\begin{aligned} k &= 1 \\ \therefore b &= a \end{aligned}$$

M1A1 Applying an enlargement to $h(x) = ax^n$

$$\begin{aligned} kg\left(\frac{x}{k}\right) &= ak\left(\frac{x}{a}\right)^n \\ &= \frac{k}{a}x^n \\ \therefore a' &= \frac{k}{a^{n-1}} \end{aligned}$$

(7)

Question 2

M1M1

$$P(t-n) = a(t-n)^3 + b(t-n)^2 + c(t-n) + d$$
$$P(t-n) = at^3 + (-3an+b)t^2 + (3an^2 - 2bn + c)t + d - nc - an^3 + bn^2$$
$$\therefore n = \frac{b}{3a}$$

A1A1

$$\begin{aligned}\therefore p &= \frac{1}{a}\left(\frac{b^2}{3a} - \frac{2b^2}{3a} + c\right)\\ &= \frac{3ac - b^2}{3a^2}\\ \therefore q &= \frac{1}{a}\left(d - \frac{bc}{3a} - \frac{ab^3}{27a^3} + \frac{b^3}{9a^2}\right)\\ &= \frac{27a^2d - 9abc + 2b^3}{27a^3}\end{aligned}$$

(4)

M1 (i)

$$\begin{aligned}0 &= (u+v)^3 + p(u+v) + q\\ &= u^3 + 3u^2v + 3uv^2 + v^3 + pu + pv + q\\ &= u^3 + v^3 + 3uv(u+v) + (u+v)p + q\end{aligned}$$

M1 Generating a pair of equation by comparing coefficients

$$\begin{aligned}u^3 + v^3 &= -q\\ 3uv &= -p\end{aligned}$$

M1A1 Solving simultaneously:

$$u^3 - \left(\frac{p}{3u}\right)^3 = -q$$
$$u^6 + u^3q - \frac{p^3}{27} = 0$$
$$u^3 = \frac{-q \pm \sqrt{q^2 + \frac{4p^3}{27}}}{2}$$
$$u = \sqrt[3]{\frac{-q - \sqrt{q^2 + \frac{4p^3}{27}}}{2}} = \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$
$$v = \sqrt[3]{\frac{-q + \sqrt{q^2 + \frac{4p^3}{27}}}{2}} = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

(4)

M1 (ii) A) Generating a geometric series:

$$x^5 + x^4 + x^3 + x^2 + x + 1 = x^3 + x^2 + x$$
$$\frac{x^6 - 1}{x - 1} = x(x^2 + x + 1)$$

B1

$$x \neq 1$$

A1A1

$$\begin{aligned} x^6 - 1 &= x(x^3 - 1) \\ (x^3 + 1)(x^3 - 1) - x(x^3 - 1) &= 0 \\ (x^3 - 1)(x^3 - x + 1) &= 0 \\ x_1 &= e^{\frac{\pi}{3}i} \\ x_2 &= e^{-\frac{\pi}{3}i} \\ x_3 &= \sqrt[3]{-\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{27}}} + \sqrt[3]{-\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{27}}} \end{aligned}$$

M1A1

$$\begin{aligned} |x_4| = |x_5| &= 1 \\ x_4 x_5 x_3 &= 1 \\ x_4 &= x_5^* \\ \therefore x_4 &= \sqrt{x_3} + \sqrt{1 - x_3^2}i \\ x_5 &= \sqrt{x_3} - \sqrt{1 - x_3^2}i \end{aligned}$$

(6)

M1 (ii) B) Generating a geometric series:

$$\begin{aligned} x^5 + x^4 + x^3 + x^2 + x + 1 &= x^4 + x^3 + x^2 \\ \frac{x^6 - 1}{x - 1} &= x^2(x^2 + x + 1) \\ x &\neq 1 \\ x^6 - 1 &= x^2(x^3 - 1) \\ (x^3 + 1)(x^3 - 1) - x^2(x^3 - 1) &= 0 \\ (x^3 - 1)(x^3 - x^2 + 1) &= 0 \\ x_1 &= e^{\frac{\pi}{3}i} \\ x_2 &= e^{-\frac{\pi}{3}i} \end{aligned}$$

M1

$$x^3 - x^2 + 1 = t^3 - \frac{1}{3}t + \frac{25}{27}, \quad \text{where } t = x - \frac{1}{3}$$

A1A1

$$\begin{aligned}
x_3 &= \sqrt[3]{-\frac{\frac{25}{27}}{2} + \sqrt{\frac{\left(\frac{25}{27}\right)^2}{4} + \frac{\left(\frac{1}{3}\right)^3}{27}}} + \sqrt[3]{-\frac{\frac{25}{27}}{2} - \sqrt{\frac{\left(\frac{25}{27}\right)^2}{4} + \frac{\left(\frac{1}{3}\right)^3}{27}}} \\
|t_4| = |t_5| &= 1 \\
t_4 t_5 t_3 &= 1 \\
t_4 &= t_5^* \\
\therefore x_4 &= \sqrt{x_3 - \frac{1}{3}} + \frac{1}{3} + \sqrt{1 - \left(x_3 - \frac{1}{3}\right)^2} i \\
\therefore x_5 &= \sqrt{x_3 - \frac{1}{3}} + \frac{1}{3} - \sqrt{1 - \left(x_3 - \frac{1}{3}\right)^2} i
\end{aligned}$$

(6)

Question 3

M1M1 (i) Consider a has factors a_1, a_2 a_3... and b has factors b_1, b_2 b_3...
The factors of ab can be written as such:

$\times$	a_1	a_2	a_3	...
b_1	b_1a_1	b_1a_2	b_1a_3	...
b_2	b_2a_1	b_2a_2	b_2a_3	...
b_3	b_3a_1	b_3a_2	b_3a_3	...
...	...	...	...	...

E1A2 Each cell must be unique as all a_i is unique and no factor of b is common with a.
Hence the number of cells of the table is ab hence $\sigma_0(ab) = \sigma_0(a)\sigma_0(b)$

(5)

E1A2 (ii) p^k only has the factor 1, and p^n for $n \leq k$. Hence there are $k+1$ factors.

(3)

M1M1 (iii)

$$\sigma_0(n^k) = \sigma_0\left(\prod_{i=1}^{\infty}(p_i^{a_i})^k\right)$$

E1M1M1A2 Every term is coprime in the product hence we can apply the result in (i):

$$\begin{aligned}\sigma_0(n^k) &= \prod_{i=1}^{\infty}\sigma_0(p_i^{a_ik}) \\ &= \prod_{i=1}^{\infty}(a_ik+1)\end{aligned}$$

(6)

M1 (iv) Correct prime factorisation $720 = 2^4 \times 3^2 \times 5$

M2 Correct factor Equation

$$\sigma_0(720^k) = (4k+1)(3k+1)(k+1)$$

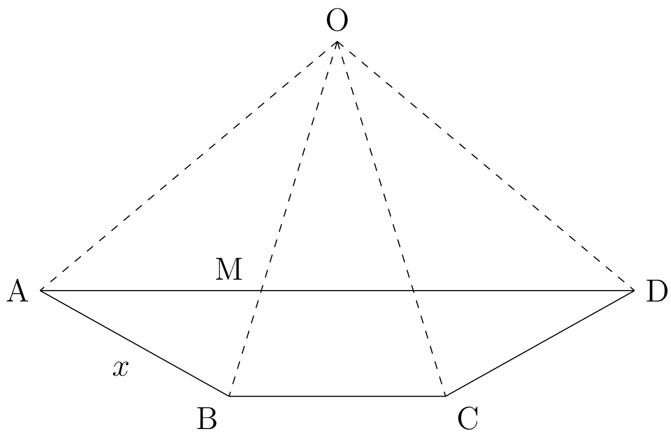
M2A1 Substitution

$$\begin{aligned}\sigma_0(720^3) &= (4(3)+1)(2(3)+1)(3+1) \\ &= 13 \times 7 \times 4 \\ &= 364\end{aligned}$$

(6)

Question 4

M1M1 Correct diagram below



M1M1

$$
\begin{aligned}
O\hat{B}C = O\hat{C}D &= \pi - \frac{1}{2} \cdot \frac{2\pi}{10} \\
&= \frac{2\pi}{5} \\
\therefore D\hat{C}B &= \frac{4\pi}{5} \\
2\pi - 2\left(\frac{4\pi}{5}\right) &= 2A\hat{D}C \\
M\hat{A}B &= \frac{\pi}{5} \\
B\hat{M}D &= 2\pi - \frac{4\pi}{5} - \frac{2\pi}{5} - \frac{\pi}{5} \\
&= \frac{3\pi}{5} \\
M\hat{A}B = A\hat{M}B &= \frac{2\pi}{5}
\end{aligned}
$$

E1 Hence AMB is isosceles therefore $AM = x$.

E1 OAB and AMB are similar triangles. Hence, let $\cos\frac{\pi}{5} = y$

M1 $$OAB: \quad x^2 = 1^2 + 1^2 - 2(1)(1)y \tag{1}$$

M1
$$A\hat{M}O = \frac{3\pi}{5}$$
$$M\hat{A}O = \frac{\pi}{5}$$

E1 Hence AMO is isosceles and $OM = x$, and $BM = 1 - x$

M1
$$AMO: \quad (1-x)^2 = x^2 + x^2 - 2(x)(x)y \tag{2}$$

M1M1 Solving simultaneously:

$$\begin{aligned} x^2 &= 2 - 2y && (1) \\ x^4 - 2x^2 &= -2x^2y \\ x^2 - 2x + 1 &= 2x^2(1-y) && (2) \\ 1 - 2x - x^2 &= -2x^2y \\ 1 - 2x - x^2 &= x^4 - 2x^2 \\ x^4 - (x^2 - 2x + 1) &= 0 \\ x^2 - (x-1)^2 &= 0 \\ (x^2 - x + 1)(x^2 + x - 1) &= 0 \end{aligned}$$

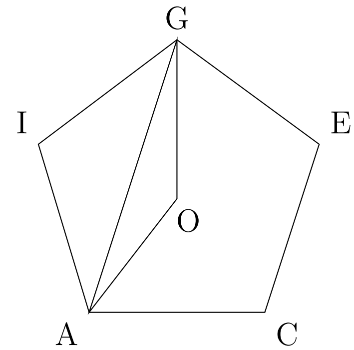
A1
$$x = \frac{-1 \pm \sqrt{5}}{2}$$
$$x = \frac{1 \pm \sqrt{-3}}{2}$$

B1A1 Only valid solution for $x \geq 0, x \in \mathbb{R}$ is $x = \frac{-1+\sqrt{5}}{2}$.

M1A1 Solving for y:

$$\begin{aligned} 2y &= 2 - x^2 \\ &= 2 - \left(\frac{-1+\sqrt{5}}{2}\right)^2 \\ &= 2 - \frac{1 + 5 - 2\sqrt{5}}{4} \\ &= \frac{2\sqrt{5} + 4}{4} \\ y &= \frac{\sqrt{5} + 1}{4} \end{aligned}$$

M1 Correct diagram below



M1A1

$$\begin{aligned}
A\hat{O}G &= \frac{4\pi}{5} \\
\cos(A\hat{O}G) &= -\cos\frac{\pi}{5} \\
AG^2 &= 1^2 + 1^2 - 2(1)(1)\left(-\cos\frac{\pi}{5}\right) \\
AG &= \sqrt{2 + 2\left(\frac{1+\sqrt{5}}{4}\right)} \\
&= \sqrt{\frac{\sqrt{5}+5}{2}}
\end{aligned}$$

Question 5

M1M1A1 (i)

$$\begin{aligned}\int \sec(\arctan(x))dx &= \int \sqrt{1+\tan^2(\arctan(x))}dx \\ &= \int \sqrt{1+x^2}dx\end{aligned}$$

M1M1A1

$$\begin{aligned}u &= \sinh(x) \\ \int \sqrt{1+x^2}dx &= \int \cosh^2(u)du \\ &= \int \frac{\cosh(2u)-1}{2}du \\ &= \frac{\sinh(2u)}{2} - \frac{u}{2} + C \\ &= \frac{\sinh(2\operatorname{arsinh}(x))}{2} - \frac{\operatorname{arsinh}(x)}{2} + C\end{aligned}$$

M1M1M1A1 Attempt to evaluate $\mathrm{gd}(x)$:

$$\begin{aligned}\mathrm{gd}(x) &= \int_0^x \operatorname{sech}(t)dt \\ &= \int_0^x \frac{\cosh(t)}{\cosh^2(t)}dt \\ &= \int_0^x \frac{\cosh(t)}{1+\sinh^2(t)}dt \\ &= \arctan(\sinh(x)) - 0 \\ &= \arctan(\sinh(x))\end{aligned}$$

M1M1M1A1

$$\begin{aligned}\int \sec(\mathrm{gd}(x))dx &= \int \sec(\arctan(\sinh(x)))dx \\ &= \int \sqrt{1+\sinh^2(x)}dx \\ &= \int \cosh(x)dx \\ &= \sinh(x) + C\end{aligned}$$

M1A1 (ii) Correct substitution

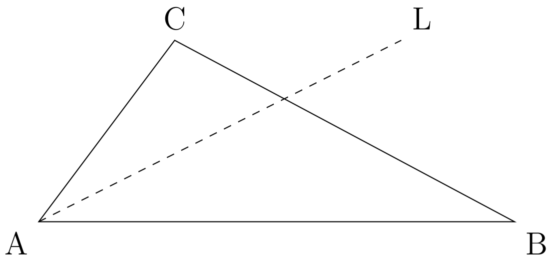
$$\begin{aligned}\mathrm{gd}(u) &= x \\ dx &= \operatorname{sech}(u)du\end{aligned}$$

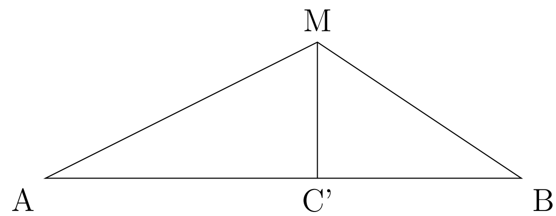
M1M1A1A1

$$\begin{aligned}
\int \sec(x)\operatorname{arsinh}(\tan(x))dx &= \int u\operatorname{sech}(u)\sec(\operatorname{gd}(u)))du \\
&= \int u\operatorname{sech}(u)\sec(\arctan(\sinh(u)))du \\
&= \int u\operatorname{sech}(u)\sqrt{1+\tan^2(\arctan(\sinh(u)))}du \\
&= \int u\operatorname{sech}(u)\sqrt{1+\sinh^2(u)}du \\
&= \int u\operatorname{sech}(u)\cosh(u)du \\
&= \int udu \\
&= \frac{u^2}{2}+C \\
&= \frac{(\operatorname{gd}^{-}1(x))^2}{2}+C
\end{aligned}$$

Question 6

M1M1M1 Correct transformation

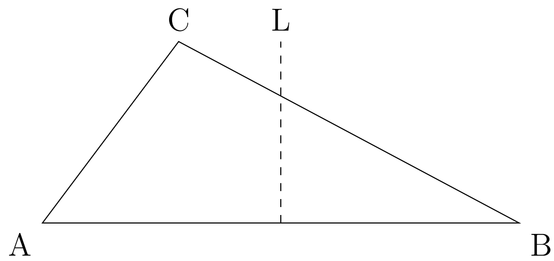


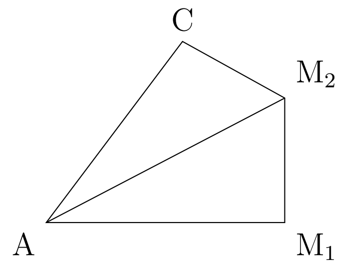


M1M1A1

$$\begin{aligned}
B\hat{A}M &= \frac{A}{2} \\
\tan\frac{A}{2} &= \frac{MC'}{C'A} \\
\sin B &= \frac{MC'}{BM} \\
a &= MC' + BM \\
&= \frac{MC'}{\sin B} + C'A\tan\frac{A}{2} \\
&= \frac{C'A\tan\frac{A}{2}}{\sin B} + b\tan\frac{A}{2} \\
&= \frac{b\tan\frac{A}{2}}{\sin B} + b\tan\frac{A}{2} \\
&= b\tan\frac{A}{2}\left(1 + \frac{1}{\sin B}\right)
\end{aligned}$$

M1M1M1 (ii) reflect along the perpendicular bisector of AB

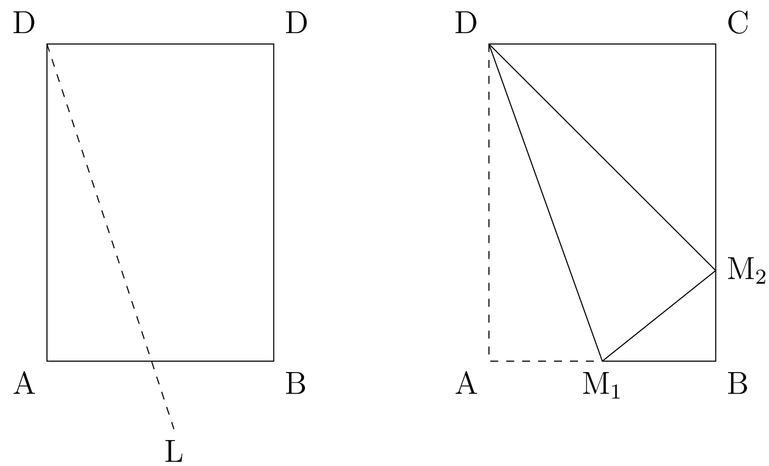




M1M1A1

$$
\begin{aligned}
M_2\hat{A}C &= A - B \\
a &= AM_2 + CM_2 \\
CM_2 &= b\tan(M_2\hat{A}C) \\
CM_2 &= b\tan(A - B) \\
AM_2 &= \frac{b}{\cos(M_2\hat{A}C)} \\
AM_2 &= \frac{b}{\cos(A - B)} \\
a &= \frac{b}{\cos(A - B)} + b\tan(A - B)
\end{aligned}
$$

M1M1M1 Reflect along the perpendicular bisector of AB



M1M1M1A1 Let $\alpha = D\hat{M_1}M_2 \qquad \beta = B\hat{M_1}M_2$

$$
\begin{aligned}
D\hat{M_1}A &= \pi - \alpha - \beta \\
DA = DM_1 \sin(\pi - \alpha - \beta) &= DM_1 \sin(\alpha + \beta) \\
DM_2 &= DM_1 \sin(\alpha) \\
C\hat{M_2}D &= \beta \\
CM_1 &= DM_2 \cos(\beta) \\
&= DM_1 \sin(\alpha)\cos(\beta) \\
M_1M_2 &= DM_1 \sin(\alpha) \\
BM_2 &= M_1M_2 \sin(\beta) \\
&= DM_1 \cos(\alpha)\sin(\beta) \\
DA &= CM_2 + M_2B \\
DM_1 \sin(\alpha + \beta) &= DM_1 \cos(\alpha)\sin(\beta) + DM_1 \sin(\alpha)\cos(\beta) \\
\sin(\alpha + \beta) &= \cos(\alpha)\sin(\beta) + \sin(\alpha)\cos(\beta)
\end{aligned}
$$

B1 Because it's a reflection of a rectangle, $A + B = \pi$. However, the same argument could be made for any value of α, β by varying the height of the rectangle, and changing the dimensions of the right angled triangle on the LHS. This further extends to any α, β due to the periodicity of the sine and cosine function

Question 7

M1M1A1 This transformation is the same as a transformation mapping $y = x\tan(\theta)$ to the x-axis, reflecting, and then rotating back again:

$$\begin{aligned}
\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} &= \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix} \\
&= \begin{pmatrix} \cos^2(\theta) - \sin^2(\theta) & 2\sin(\theta)\cos(\theta) \\ 2\sin(\theta)\cos(\theta) & \sin^2(\theta) - \cos^2(\theta) \end{pmatrix} \\
&= \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}
\end{aligned}$$

A2A2A2 (i) Considering these three cases:

Case 1: $\theta = 2\pi k$	Any curve works as it is the identity matrix
Case 2: $\theta = 2\pi k + \pi$	$y = mx$ for all m.
Case 3: $\theta \in \mathbb{R}$	$y^2 + x^2 = e^2$ for all r.

M1M1A1 (ii) Using a general matrix:

$$\begin{aligned}
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ x^2 \end{pmatrix} &= \begin{pmatrix} ax + bx^2 \\ cx + dx^2 \end{pmatrix} \\
(ax + bx^2)^2 &= cx + dx^2 \\
b^2x^4 + 2abx^3 + (a^2 - d)x^2 - cx &= 0
\end{aligned}$$

The equation must be independent of x:

$$\begin{aligned}
c &= 0 \\
b &= 0 \\
a^2 &= d
\end{aligned}$$

$$\begin{pmatrix} \pm r & 0 \\ 0 & r^2 \end{pmatrix}$$

M1M1A1 Using a general matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ x^n \end{pmatrix} = \begin{pmatrix} ax + bx^n \\ cx + dx^n \end{pmatrix}$$
$$(ax + bx^n)^n = cx + dx^n$$

Comparing co-efficients $b = c = 0$ as before:

$$a^n x^n + dx^n a^n \qquad = d$$

$$\begin{pmatrix} \pm r & 0 \\ 0 & r^n \end{pmatrix} \text{For even n}$$
$$\begin{pmatrix} r & 0 \\ 0 & r^n \end{pmatrix} \text{For odd n}$$

M1A1 The general co-ordinate of a point of the curve can be written as a vector:

$$\begin{pmatrix} t\cos\left(\frac{\pi}{3}\right) - t^2\sin\left(\frac{\pi}{3}\right) \\ t\cos\left(\frac{\pi}{3}\right) + t^2\sin\left(\frac{\pi}{3}\right) \end{pmatrix} = \begin{pmatrix} \cos\left(\frac{\pi}{3}\right) & -\sin\left(\frac{\pi}{3}\right) \\ \sin\left(\frac{\pi}{3}\right) & \cos\left(\frac{\pi}{3}\right) \end{pmatrix} \begin{pmatrix} t \\ t^2 \end{pmatrix}$$

E1A1 This is a rotation of $\frac{\pi}{3}$ radians of an $y = x^2$ curve hence the invariant lines follows:

$$\begin{pmatrix} \pm t & 0 \\ 0 & t^2 \end{pmatrix} \begin{pmatrix} \cos\left(\frac{\pi}{3}\right) & -\sin\left(\frac{\pi}{3}\right) \\ \sin\left(\frac{\pi}{3}\right) & \cos\left(\frac{\pi}{3}\right) \end{pmatrix} = \begin{pmatrix} \pm t\cos\left(\frac{\pi}{3}\right) & \mp t\sin\left(\frac{\pi}{3}\right) \\ t^2\sin\left(\frac{\pi}{3}\right) & t^2\cos\left(\frac{\pi}{3}\right) \end{pmatrix}$$

Question 8

M1A1

$$\begin{aligned} ij &= k \\ ij \times j &= k \times j \\ -i &= kj \end{aligned}$$

M1A1

$$\begin{aligned} ji &= -k \\ k \times ji &= k \times -k \\ kji \times i &= -i \\ -kj &= -i \\ kj &= i \end{aligned}$$

A1A1 Any evidence of having calculated below:

$$\begin{aligned} ik &= j \\ ki &= -j \end{aligned}$$

M2M2M2 i) Attempt to evaluate each term:

$$\begin{aligned} iqi &= iai - ib + icji + idki \\ &= iia - ib - ikc - ijd \\ &= -a - ib - jc - kd \\ jqj &= jaj + jbij - jc + jdkj \\ &= -a + jkb - jc - jid \\ &= -a + ib - jc + kd \\ kqk &= kak + kbik + kcjk - kd \\ &= -a + kjb - kic - kd \\ &= -a + ib + jc - kd \end{aligned}$$

M1A1 Summing them up and multiplying by $-\frac{1}{2}$

$$\begin{aligned} -\frac{1}{2}(q + iqi + jqj + kqk) &= -\frac{1}{2}(-2a + 2bi + 2cj + 2dk) \\ &= a - b - bj - ck \\ &= q^* \end{aligned}$$

M1A1 (ii) Correct factorisation

$$\begin{aligned} a + bi + cj + dk &= a + bi + (cj + dij) \\ &= a + bi + (c + di)j \end{aligned}$$

M1M1A1A1

$$
\begin{aligned}
qq^* &= (z + wj)(z^* - wj) \\
&= zz^* + w^2 - zwj + wjz^* \\
(qq^*)^* &= zz^* + (w^*)^2 + z^*w^*j - w^*jz \\
a^2 + b^2 + c^2 + d^2 &= qq^* \\
&= zz^* + (w^*)^2 + z^*w^*j - w^*jz \\
&= a^2 + b^2 + (w^*)^2 + z^*w^*j - w^*jz \\
c^2 + d^2 &= (w^*)^2 + z^*w^*j - w^*jz
\end{aligned}
$$

Question 9

M1M1M1A1

$$\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}dx = \int_{x^{-1}(a)}^{x^{-1}(b)} \sqrt{1+\left(\frac{\frac{dy}{dr}}{\frac{dx}{dr}}\right)^2}\frac{dx}{dr}dr$$
$$= \int_{x^{-1}(a)}^{x^{-1}(b)} \sqrt{\left(\frac{dx}{dr}\right)^2\left(1+\left(\frac{\frac{dy}{dr}}{\frac{dx}{dr}}\right)^2\right)}dr$$
$$= \int_{x^{-1}(a)}^{x^{-1}(b)} \sqrt{\left(\frac{dy}{dr}\right)^2+\left(\frac{dx}{dr}\right)^2}dr$$

M1M1

$$\frac{dx}{dr} = 1-\cos(\theta)$$
$$\frac{dy}{dr} = -\sin(\theta)$$

M2M2M2A1 Any evidence of having calculated below:

$$\int_0^\pi \sqrt{\sin^2(\theta)+(1-\cos(\theta))^2}d\theta = \int_0^\pi \sqrt{\sin^2(\theta)+\cos(\theta)^2+1-2\cos(\theta)}d\theta$$
$$= \int_0^\pi \sqrt{2-2\cos(\theta)}d\theta$$
$$= \sqrt{2}\int_0^\pi \sqrt{1-\cos(\theta)}d\theta$$
$$= \sqrt{2}\int_0^\pi \sqrt{1-\left(1-2\sin^2\left(\frac{\theta}{2}\right)\right)}d\theta$$
$$= 2\int_0^\pi \sin\left(\frac{\theta}{2}\right)d\theta$$
$$= -4\cos\left(\frac{\theta}{2}\right)\Big|_0^\pi$$
$$= 4$$

M2M2M2A1 Setting y to be the zero-point for gravitational potential energy.

$$\begin{aligned}
\frac{1}{2}mv^2 &= mg(-y) \\
v = \frac{ds}{dt} &= \sqrt{-2gy} \\
t &= \int_0^4 \frac{1}{\sqrt{-2gy}} ds \\
&= \frac{1}{\sqrt{-2g}} \int_0^4 \frac{1}{\sqrt{y}} ds \\
&= \frac{1}{\sqrt{-2g}} \int_0^\pi \frac{1}{\sqrt{y}} \frac{ds}{d\theta} d\theta \\
&= \frac{1}{\sqrt{-2g}} \int_0^\pi \frac{\sqrt{2}\sqrt{1-\cos(\theta)}}{\sqrt{\cos(\theta)-1}} d\theta \\
&= \frac{}{\sqrt{-2g}} \int_0^\pi \sqrt{-2} d\theta \\
&= \pi\sqrt{\frac{1}{g}}
\end{aligned}$$

Question 10

M1A1A1 Any evidence of having calculated below:

$$\begin{aligned} e &= \frac{c}{a} = c \\ e &= \sqrt{1 - \frac{b^2}{a^2}} \\ c &= \sqrt{1 - b^2} \\ \sqrt{1 - c^2} &= b \end{aligned}$$

E1M1M1A1 i) Horizontal component of velocity is the same. Let the angle of approach be α and of deflection β. For a co-efficient of restitution e':

$$\begin{aligned} \text{NLR:} \qquad e' &= \frac{u \sin \beta}{v \sin \alpha} \\ \text{COE:}\ \frac{1}{2}mu^2 &= \frac{1}{2}mv^2 \\ u &= v \\ \text{COM:}\ u \sin \alpha &= v \sin \beta \\ \therefore e' &= 1 \\ \therefore \alpha &= \beta \end{aligned}$$

M1M1A1 ii) Let (x, y) be a point on the ellipse during the collision. The direction vector of the gradient:

$$\begin{aligned} x^2 + \frac{y^2}{b^2} &= 1 \\ 2x + 2y\frac{dy}{dx}\frac{1}{1 - c^2} &= 0 \\ \frac{dy}{dx} &= \frac{x}{y}(c^2 - 1) \\ &\rightarrow \begin{pmatrix} y \\ x(c^2 - 1) \end{pmatrix} \end{aligned}$$

M1M1 The direction vector of the line connecting the two foci can be written as such:

$$\begin{pmatrix} x - c \\ y \end{pmatrix} \qquad \begin{pmatrix} -c - x \\ -y \end{pmatrix}$$

M1M1A1 Use of the dot product to find α:

$$\begin{aligned}
\left|\begin{pmatrix} y \\ x(c^2-1) \end{pmatrix}\right| \cos\alpha &= \frac{\begin{pmatrix} y \\ x(c^2-1) \end{pmatrix} \cdot \begin{pmatrix} x-c \\ y \end{pmatrix}}{\sqrt{(x-c)^2+y^2}} \\
&= \frac{y(x(c^2-1)+x-c)}{\sqrt{x^2-2cx+c^2+(1-x^2)(1-c^2)}} \\
&= \frac{y(xc^2-x+x-c)}{\sqrt{x^2c^2-2cx+1}} \\
&= \frac{yc(xc-1)}{xc-1} \\
&= yc
\end{aligned}$$

M1A1 Use of the dot product to find β:

$$\begin{aligned}
\left|\begin{pmatrix} y \\ x(c^2-1) \end{pmatrix}\right| \cos\beta &= \frac{\begin{pmatrix} y \\ x(c^2-1) \end{pmatrix} \cdot \begin{pmatrix} -c-x \\ -y \end{pmatrix}}{\sqrt{(-x-c)^2+y^2}} \\
&= \frac{-y(x(c^2-1)+(x+c))}{\sqrt{x^2+2cx+c^2+(1-x^2)(1-c^2)}} \\
&= \frac{-y(xc^2-x+x+c)}{\sqrt{x^2c^2+2cx+1}} \\
&= \frac{-yc(xc+1)}{xc+1} \\
&= -yc
\end{aligned}$$

M1E1A1

$$\begin{aligned}
\left|\begin{pmatrix} y \\ x(c^2-1) \end{pmatrix}\right| \cos\alpha &= -\left|\begin{pmatrix} y \\ x(c^2-1) \end{pmatrix}\right| \cos\beta \\
\cos\alpha &= -\cos\beta
\end{aligned}$$

Cosine is an even function, hence $\alpha = \beta$ hence this the path given by an elastic collision.

Question 11

M1M1A1

$$\begin{aligned}\int_0^\infty xe^{-x}dx &= \int_0^\infty x\frac{d}{dx}[-e^{-x}]dx \\ &= -xe^{-x} - \int -e^{-x}dx\Big|_0^\infty \\ &= e^{-x}dx\Big|_0^\infty \\ &= 1\end{aligned}$$

M1A1 i) The probability that a relationship will be a success:

$$\int_1^\infty e^{-x}dx = -e^{-x}\Big|_1^\infty = \frac{1}{e}$$

E1A1 The number of relationships, N, is geometrically distributed by $N \sim \text{Geo}\left(\frac{1}{e}\right)$, hence the expected number of relationships is e

M1M1A1 The expected time given a relationship has failed:

$$\begin{aligned}\int_0^1 xe^{-x}dx &= \int_0^1 x\frac{d}{dx}[-e^{-x}]dx \\ &= -xe^{-x} - \int -e^{-x}dx\Big|_0^1 \\ &= -(1+x)e^{-x}dx\Big|_0^1 \\ &= 1 - \frac{2}{e}\end{aligned}$$

M1A1 Multiplying it by $E(N)$ yields $e-2$. Including the additional successful year $E(t) = e-1$.

E1A1 The expected number of relationships required will remain unchanged as the probability for each trial is the same, hence it remains e.

M1M1A1 Finding the expected time for a new relationship to start:

$$\begin{aligned}\int_0^\infty x^2e^{-x}dx &= \int_0^\infty x^2\frac{d}{dx}[-e^{-x}]dx \\ &= -x^2e^{-x} - \int -2xe^{-x}dx\Big|_0^\infty \\ &= -x^2e^{-x} + 2\left(-xe^{-x} - \int -e^{-x}dx\right)dx\Big|_0^\infty \\ &= 2\end{aligned}$$

E2A1 There are $N-1$ waiting periods for the next relationship hence the expected time waiting is $2(e-1)$. The overall waiting time is $e-2+2(e-1)=3e-4$. With the additional year for the last relationship to be a year, the overall time is $3e-3$.

Question 12

M1M1M1M1
$$\frac{\sin(x)}{x} = \left(1-\frac{x}{\pi}\right)\left(1+\frac{x}{\pi}\right)\left(1-\frac{x}{2\pi}\right)\left(1+\frac{x}{2\pi}\right)\left(1-\frac{x}{3\pi}\right)\left(1+\frac{x}{3\pi}\right)...$$
$$= \left(1-\left(\frac{x}{\pi}\right)^2\right)\left(1-\left(\frac{x}{2\pi}\right)^2\right)\left(1-\left(\frac{x}{3\pi}\right)^2\right)$$

The mclaurin expansion of $\frac{\sin(x)}{x}$:

$$\frac{\sin(x)}{x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!}...}{x}$$
$$= 1 - \frac{x^2}{6} + \frac{x^4}{120}...$$

M2A1 Comparing co-efficients of the x^2 term

$$-\frac{1}{\pi^2} - \frac{1}{4\pi^2} - \frac{1}{9\pi^2}... = -\frac{1}{6}$$
$$1 + \frac{1}{4} + \frac{1}{9}... = \frac{\pi^2}{6}$$

E1A1 i) The probability that a random integer is a multiple of 2 is $\frac{1}{2}$. The chance that 2 is $\frac{1}{4}$. Hence the chance that neither is $1 - \frac{1}{4}$

E1A1 The probability that a random integer is a multiple of p is $\frac{1}{p}$. The chance that 2 is $\frac{1}{p^2}$. Hence the chance that neither is $1 - \frac{1}{p}$

E2M2 Let p_n be the nth prime. The probability that they have no prime factors in common is given by:

$$\left(1-\frac{1}{p_1^2}\right)\left(1-\frac{1}{p_2^2}\right)\left(1-\frac{1}{p_3^2}\right)... = \left[\left(\frac{1}{1-\frac{1}{p_1^2}}\right)\left(\frac{1}{1-\frac{1}{p_2^2}}\right)\left(\frac{1}{1-\frac{1}{p_3^2}}\right)...\right]^{-1}$$

M2 By a geometric series:

$$\left[\left(\frac{1}{1-\frac{1}{p_1^2}}\right)\left(\frac{1}{1-\frac{1}{p_2^2}}\right)\left(\frac{1}{1-\frac{1}{p_3^2}}\right)...\right]^{-1} = \left[\left(1+\frac{1}{p_1^2}+\frac{1}{p_1^4}...\right)\left(1+\frac{1}{p_2^2}+\frac{1}{p_2^4}...\right)\left(1+\frac{1}{p_3^2}+\frac{1}{p_3^4}...\right)\right]^{-1}$$

E2A1 You can generate any number squared by multiplying it's prime numbers in the expansion. The expansion is unique to each number hence:

$$\left[\left(1+\frac{1}{p_1^2}+\frac{1}{p_1^4}...\right)\left(1+\frac{1}{p_2^2}+\frac{1}{p_2^4}...\right)\left(1+\frac{1}{p_3^2}+\frac{1}{p_3^4}...\right)\right]^{-1}=\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}...\right]^{-1}=\frac{6}{\pi^2}$$