Lessons learned from STEP questions

(Not every question you've done)

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Pure Mathematics

STEP II 2015 Q1 (15-S2-Q1 Analysis)

- Remember that for some summation, the result is related to the sum. E.g, if the total of a sum has a natural log in it, you may need to substitute a natural log into the sum.
- You can use some $f(x) > 0$ to generate a summation inequality by substituting $f(x)$ into the summation.

Repeats: *None*

STEP II 2015 Q2 (15-S2-Q2 Euclidean geometry)

- Don't be afraid to *redraw* a geometry diagram in order to see things more clearly.
- Extend lengths(???).

Repeats: *None*

STEP II 2015 Q4 (15-S2-Q4 Curve Sketching)

- When sketching a curve, you can sometimes try and sketch an inverse function, and reflect along $y = x$.
- Periodic functions can be defined at different values. E.g , $f(x) = \tan^{-1}(x)$ but $f(0) = \pi$.
- *Always* double check with the original equation if you have applied a transformation to draw a curve.

Repeats: *None*

STEP II 2015 Q6 (15-S2-Q6 Integration)

- You can abuse the fact that $\sin(x) = \sin(\pi - x)$ in $\int_0^\pi f(x)dx$ in order to change the equation around a bit, whilst conserving the sine function.
- You can integrating any power of $\sec(x)$ by separating $\sec^2(x)$ by use of $1 + \tan^2()x$.

Repeats: 16-S2-Q7

STEP II 2016 Q7 (16-S2-Q7 Trigonometric integrals)

- It's worth checking if an integral can be done outside of the scaffolding
- Note that you are likely to use $I = \int f(x)dx - I$ if you have something bizarre such as an x outside of a $\sin(x)$.
- Worth noting that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$.

Repeats: 15-S2-Q4

STEP II 2017 Q1 (17-S2-Q1 Arctan integral)

- You can generalise some integrals with a co-efficient of x to any power.
- When doing calculus on an algebraic exponent, try to see where it ends up in the result.

Repeats: 13-S2-Q2

STEP II 2017 Q6 (17-S2-Q5 Parabola and circle)

- When judging whether a point is a minima/maxima, it's worth double checking the context of the equation in order to assess whether or not it is. E.g the width of a parabola always approaches infinity.

Repeats: *None*

STEP II 2017 Q7 (17-S2-Q7 x to the power x to the power x)

- For a 3-way inequality, always double check that you have proven all relations. i.e for $a < b < c$, you need to prove the relationship between every pair of variables
- You can abuse a domain to demonstrate a inequality relationship between a function and its variable
- Whenever you see something along the lines of x^x, test out a log, because linear algebra is relatively easier

Repeats: *None*

STEP II 2017 Q8 (17-S2-Q8 Vector geometry)

- If and only if you're desperate, evaluate vector equations using their components.
- Dot products of perpendicular vectors eliminates scalars.

Repeats: *None*

STEP II 2018 Q4 (17-S2-Q3 Tan integral)

- If you cut and change the orientation of an odd function, you can form a rectangle (easy integration)
- Remember that at the centre of rotational symmetry (at $y = 0$) of an odd-function means that it has 0 area when taking the integral at limits equidistant from it.

Repeats: *None*

STEP II 2018 Q3 (17-S2-Q3 Trigonometric equations)

- Note that if you aren't reusing *techniques* in a question, it's a matter of using *results* from the previous section to prove the next
- Use substitutions to convert from one equation to another, to better match a known result

Repeats: *None*

STEP II 2019 Q1 (*Not Available*)

- Remember that a factorisation doesn't always have to have two terms, and can be spotted with similar looking equations
- Instead of proving both ways, try to prove an *'if and only if'* using bijective ($\Leftrightarrow$) reasoning

Repeats: *None*

STEP II 2019 Q2 (*Not Available*)

- If you use a sketch to prove something, and a particular case utilises it, it may be worth seeing how the sketch differs for the particular case
- If you have $x = f(g(x))$, such as $x = g(x)^2 + 2g(x)^2 + 1$, you can rewrite it as $g^{-1}(x) = f(x)$; the example gives $g^{-1}(x) = x^2 + 2x + 1$

Repeats: *None*

STEP II 2019 Q3 (*Not Available*)

- Note that you can maximise a polynomial by equating all coefficients to the largest, making an easily manipulated inequality.
- Bounds on variables can be combined to form an inequality for an expression.
- Note that you may find $a \leq b + 1$ by first finding $a \leq b$

Repeats: *None*

STEP II 2019 Q4 (*Not Available*)

- Integration paired with logs allows you to convert between $\sum$ & $\prod$
- You can substitute into limits in order to simplify expressions. E.g $\lim_{n \to 0}(\frac{1}{n})$ is the same as $\lim_{\frac{1}{n} \to \infty}(\frac{1}{n})$.

Repeats: *None*

STEP II 2019 Q7 (*Not Available*)

- Draw vector diagrams for vector geometry questions, just to see if some ideas work, and to see if you can arrive at counterexamples etc
- Some geometric properties are more easily arrived at through vectors. Use and abuse the idea that for some vectors that cancel, $\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} = 0$, you can generate equations such as $\mathbf{a}^2 + \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{d} = 0$ and then manipulate them to your preference.

Repeats: *None*

STEP II 2019 Q8 (*Not Available*)

- Keep a note of matrices that generalise the identity matrix in one form or another; it may be the case that the result you prove with the identity matrix is similar to the one you are trying to generalise.

Repeats: *None*

STEP II 2013 Q1 (13-S2-Q1 Curve sketching)

- Note that if a later sketch has similar, but different elements from a previous questions (of which also asks for a sketch), draw everything from that sketch too. For example, if in (i) you drew the case where a line $y = mx$ is tangential to a circle given that it crosses through the origin, and in (ii) you need to sketch some other detail of the same circle but transformed, just see how $y = mx$ now looks with the same circle
- You can 'use the result from (ii)' to prove some other result, even if you haven't actually reached that result.
- When generalising from say $y = mx \to y = mx + c$, be aware that you can translate every other element in a graph to use results. For example, say you prove something about $y = mx$ & $y = \ln(x)$. You can potentially use this result to prove something about $y = mx + c$ & $y = \ln(x)$, by transforming both to $y = mx$ and $y = \ln(x) - c$

Repeats: *None*

STEP II 2013 Q2 (13-S2-Q2 Integration)

- $(1-x)^n - x(1-x) = (1-x)(1-x)^n = 1 - x^{n+1} \to$ Be aware that factorisations like this can occur; you can make it so that $a(x) + b(x) = c(x)$, so don't dismiss it!
- Note that just because you have a 'hence', it does not necessarily mean that what you have just proved is the first thing you start with. In fact it could be last thing! If I'm given $f(x)$ as a function, and show that it can be written as $g(x)$, I may start with $f(x)$ to prove that it can be written as $h(x)$ but *via* g(x)
- Recursive formulae can sometimes be solved by continual substitution: $I_n = f(I_{n-1}) = f(I_{n-2}) = (I_{n-3})...$

Repeats: *17-S2-Q1*

STEP II 2013 Q3 (13-S2-Q3 Cubic equation)

- You can 'use the result from (ii)' to prove some other result, even if you haven't actually reached that result.
- When generalising from say $y = mx \to y = mx + c$, be aware that you can translate every other element in a graph to use results. For example, say you prove something about $y = mx$ & $y = \ln(x)$. You can potentially use this result to prove something about $y = mx + c$ & $y = \ln(x)$, by transforming both to $y = mx$ and $y = \ln(x) - c$

Repeats: *None*

STEP II 2014 Q1 (Q4 14-S2-Q1 Euclidean geometry)

- If you derive a formula from using the adjacents of some triangles, consider the opposites, even if they don't explicitly ask.
- You can use trig identities as a skeleton to derive an equation. i.e if you have $\sin\theta = f(x)$ & $\cos\theta = g(x)$, you can do $(f(x))^2 + (g(x))^2 = 1$. Note how we are can get a quadratic because of the squares!
- Try using the context of the geometry of a question to make arguments about what is and is not possible.

Repeats: *None*

STEP II 2014 Q4 (14-S2-Q4 Integration by substitution)

- You can use a triangle to prove if some trig identities are constant, such as $\arctan(x) + \arctan(\frac{1}{x})$
- To prove some things are constant, their derivative will be equal to 0.
- When choosing a suitable constant, try to assign it a value later, and do at least part of it algebraically; this will save you from having to redo it if you get it wrong.

Repeats: *None*

STEP II 2012 Q8 (12-S2-Q8 Difference equaetions)

- If you want to show that something, say p, does not change in a sequence, you can (by basically induction) show that p_n does not differ from p_{n+1}. not necessarily obvious.

Repeats: *None*

STEP III 2019 Q3 (*Not Available*)

- Note that if you want to show that something, for all x, in the form $mx + c = 0$, you only need to show that $m, c = 0$. Repeats: *None*

STEP III 2019 Q8 (*Not Available*)

- note that for 3D diagrams, avoid drawing diagonals vertically, in case you have an *actual* vertical line.

Repeats: *None*

STEP III 2012 Q3 (12-S3-Q3 Curve sketching)

- Make sure that if you are basing a 'show that' proof by observation (i.e seeing that you need only solve simultaneously or you just need to differentiate), justify why you are doing so, and be careful that it isn't just a co-incidence.

Repeats: *None*

STEP III 2012 Q4 (12-S3-Q4 Sum)

- When simplifying an expression in a summation, make sure you take a note of how to change the start value for the index. For example, $\sum_{i=0}^{\infty} \frac{n}{n!}$ where cancelling the numerator out only works for the $i > 0$ terms, and $i = 1$ leads to a zero term in the sum
- Integrating and immediately differentiating may be a useful tool, especially when trying to simplify sums.

Repeats: *None*

STEP III 2012 Q4 (18-S3-Q6 Complex geometry)

- When trying to produce an equation from another, if you have unwanted variables, see if you can factorise on both sides to produce those variables; that way you can (hopefully) cancel it down to the required equation.

Repeats: *None*

STEP III 2019 Q6 (*Note Available*)

- Sometimes, if you write an equation in one form, and turn it into another, and then apply a transformation. You may also need to rewrite it in that other form

Repeats: *None*

STEP III 2019 Q7 (*Note Available*)

- Remember. If you are analysing how functions behave for small x and/or small y, omit the terms that rush to 0 quickly and for large x and/or y, omit the terms that rush to infinity too slowly
- Always check the validity of points
- you can write an equation by treating the y or x as constant. For example, a quadratic in terms of x but with co-efficients in terms of y.

Repeats: *None*

Mechanics

STEP II 2019 Q9 (*Not available*)

- You can get away with logical reasoning for explaining why things happen when they do. E.g the distance from projection always increases below ground
- It may be worth drawing projections on an axis
- Reference frames are your friend. It's not a de facto tool for STEP questions and can likely act as a shortcut which is a good time-saver
- Differentiate with the square of something if it makes things easier when finding something increasing or decreasing. This works because $\frac{d}{dx}[f(x)f(x)] = 2f'(x)f(x)$ and $f'(x) = 0 \rightarrow 2f'(x)f(x) = 0$. Just watch out for the extraneous solutions given by the original expression.

Repeats: *None*

STEP II 2018 Q9 (18-S2-Q9 Beads on wire)

- You can solve simultaneously sometimes for co-efficient of restitution and conservation of momentum.

Repeats: *None*

STEP II 2018 Q9 (18-S2-Q10 Ant on elastic string)

- Note that when calculating return journeys, it may just be a matter of switching some signs around and reusing the calculations you've made before. Just be cautious about which signs you switch

Repeats: *None*

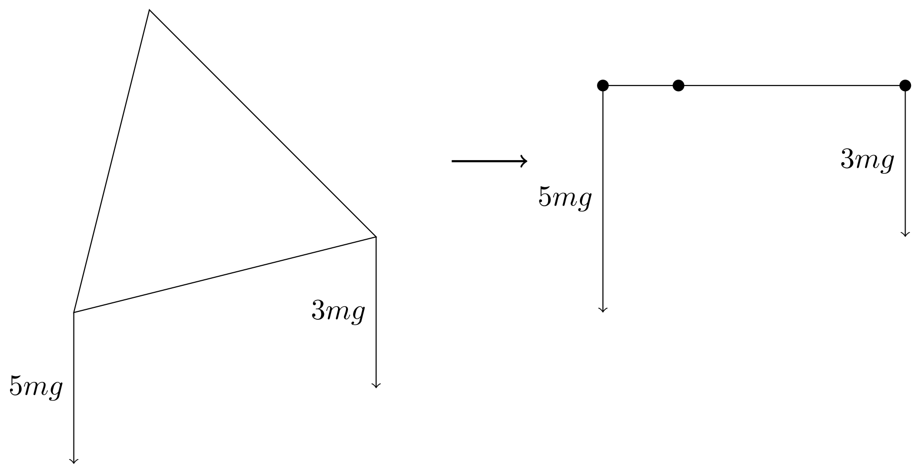
STEP II 2018 Q10 (15-S2-Q10 Pulley)

- Note that when the geometry varies with time, you cannot apply components of speed,acceleration, forces etc. You have to evaluate things in terms of distance, eliminating as many variables as possible using constants

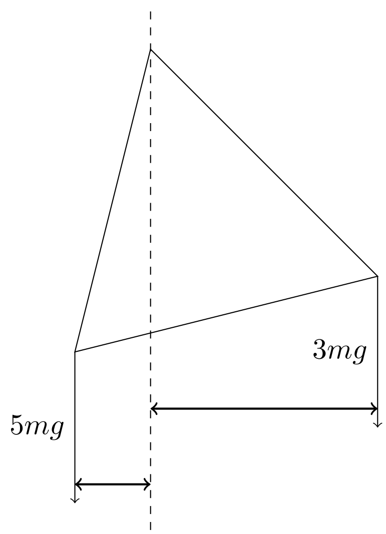
Repeats: *None*

STEP II 2015 Q9 (15-S2-Q9 Rotating rods)

- Project a plane onto a straight line to assess the line in which the centre of mass hangs. This makes things more clear:



- Remember that the force is the perpendicular distance to a pivot; don't always go for the component:



- When you need to find out a condition for something to happen. 9 times out of 10, it's energy conservation

Repeats: 16-S2-Q10

STEP II 2018 Q11 (15-S2-Q11 Collisions)

- Use reference frames to check things. What would a moving system look like if static?
- Not necessarily useful here, but keep in mind that for a symmetrical system, two objects colliding elastically is identical to them passing right through each other
- u^2 implies energy, and u can just imply momentum.
- Note that ‘an impulse is delivered to’ may be used in place of ‘is projected to’. i.e don’t assume a question is force, momentum, impulse etc upon seeing the word ’impulse’.

Repeats: *None*

STEP II 2013 Q10 (13-S2-Q10 Projectile Motion)

- Instead of bothering with long and complicated equations to do with angles of projections, it may be worth seeing if you can answer a question with parabolic equation $y = x(c - x)$, since it’s analogous to the path taken by the particle
- Trig equations don’t always imply trig manipulation.

Repeats: *None*

STEP III 2013 Q11 (12-S3-Q11 Acceleration)

- Consider $h(x) = g(\frac{dx}{dt})$ where x is a function of time. You can do two things to find the acceleration $(\frac{d^2x}{dt^2})$. The first way is differentiating wrt t; $\frac{dx}{dt}h'(x) = \frac{d^2x}{dt^2}g'(\frac{dx}{dt})$. The other way is differentiating wrt x; $h'(x) = \frac{d\left(\frac{dx}{dt}\right)}{dx}g'(\frac{dx}{dt}) = \frac{d^2x}{dt^2}\frac{dt}{dx}g'(\frac{dx}{dt})$.

Repeats: *None*

Statistics

STEP II 2013 Q12 (13-S2-Q12 Poisson distributions)

- Note that coding expected values only affects the value part of the equation. I.e, since $E(x) = \sum P(x)V(x)$, where P and V correspond to the chance and value, $E(x^2) = P(x)V(x^2)$. Consider the fact that $E(x)$ is what you expect to win for some scenario. $E(x^2)$ can be visualised as the scenario where you take your winnings, and invest them to get an x^2 return. This has nothing to with the chance of winning x, but you are getting x^2 the value.
- You can apply $\frac{y}{y+1} = \frac{y+1-1}{y+1}$ in other ways such as $y^2\frac{1}{y!} = (y(y-1)+y)\frac{1}{y!}$.

Repeats: *None*

STEP II 2014 Q12 (14-S2-Q12 Hazard Function)

- When stuck, try to use the result to see what you do next in a calculation.
- Write out the conditional probability formula, even if calculating a chance in terms of functions.

Repeats: *None*

STEP II 2012 Q13 (12-S2-Q13 Geometric Probability)

- If a function ('to show') has something in common with a probability, but is negative, such as $-\frac{x}{2}e^{\frac{x}{2}}$ coming from the derivative of $P(x) = e^{\frac{x}{2}}$, you may need to consider the complement. i.e $\frac{d}{dx}\left[1 - e^{\frac{x}{2}}\right]$
- Be careful about a randomly chosen point; if everything is random relative to each other, then you can set one thing to be fixed. i.e you can set one point of random points to be $(0,0)$

Repeats: *None*

STEP III 2012 Q12 (12-S3-Q13 Geometric Probability)

- It may be worth examining a cumulative probability distribution function as to find a probability density function.
- Note the importance of the complement when calculating probability $P(A < \min(a_1, a_2, a_3))$ vs $1 - P(A > \min(a_1, a_2, a_3))$.

Repeats: *None*

STEP III 2007 Q14 (07-S3-Q14 Geometric Probability)

- For any cumulative probability, $\sum_{i=1}^{n} F_{X_n}(x)) = F_X(x)^n$
- Work out a probability distribution if you can't do so immediately.

Repeats: *None*

Miscellaneous

Variance and Expected Value

Expected value ($\mathrm{E}(x)$), obeys linearity:

$$\mathrm{E}(\lambda x + \mu) = \lambda\,\mathrm{E}(x) + \mu$$

Note that coding expected values only affects the value part of the equation. I.e, since $\mathrm{E}(x) = \sum P(x)V(x)$, where P and V correspond to the chance and value, $\mathrm{E}(x^2) = P(x)V(x^2)$. Consider the fact that $\mathrm{E}(x)$ is what you expect to win for some scenario. $\mathrm{E}(x^2)$ can be visualised as the scenario where you take your winnings, and invest them to get an x^2 return. This has nothing to with the chance of winning x, but you are getting x^2 the value. In equation form:

$$\mathrm{E}(f(x)) = V(f(x))P(x)$$

Variance can be defined in terms of expected value:

$$\mathrm{Var}(x) = \mathrm{E}(x^2) - \mathrm{E}(x)$$

Variance can be combined with the following expression:

$$\begin{aligned}
\mathrm{Var}(x+y) &= \mathrm{Var}(x) + \mathrm{Var}(y) + 2\,\mathrm{Var}(X,Y) \\
&= \mathrm{Var}(x) + \mathrm{Var}(y) + 2\,\mathrm{E}((\mathrm{E}(X) - X)(\mathrm{E}(Y) - Y))) \\
&= \mathrm{Var}(x) + \mathrm{Var}(y) + 2(\mathrm{E}(XY) - \mathrm{E}(X)\,\mathrm{E}(Y))
\end{aligned}$$

Types of STEP Questions

- The iterative. Part 3 is an extension of part 2 which is an extension of part 3.
- The paragraph. Usually one tricky question or a literal step by step guide walk-through.
- Simon Says. Do the first part, and use the same method for a similar but harder second part.
- The combination. Do the first part. Do the completely unrelated second part. Bring the two together in the third
- Sketches. They don't tend to deviate, but it's literally the same graph just continuously generalised