

2019 年数学(二) 真题解析

一、选择题

(1) 【答案】 (C).

【解】 方法一

$$\lim_{x \rightarrow 0} \frac{x - \tan x}{x^3} = \lim_{x \rightarrow 0} \frac{1 - \sec^2 x}{3x^2} = -\frac{1}{3} \text{ 得 } x - \tan x \sim -\frac{1}{3}x^3,$$

故 $x - \tan x$ 为 3 阶无穷小, 即 $k=3$, 应选(C).

方法二

$$\text{由 } \tan x = x + \frac{1}{3}x^3 + o(x^3) \text{ 得 } x - \tan x \sim -\frac{1}{3}x^3,$$

故 $k=3$, 应选(C).

(2) 【答案】 (B).

$$\text{【解】 } y' = x \cos x - \sin x, \quad y'' = -x \sin x,$$

$$\text{令 } y'' = -x \sin x = 0 \text{ 得 } x = 0, x = \pi,$$

当 $x \in (-\frac{\pi}{2}, 0)$ 时, $y'' < 0$, 当 $x \in (0, \pi)$ 时, $y'' < 0$, 则 $(0, 2)$ 不是拐点;

当 $x \in (\pi, \frac{3\pi}{2})$ 时, $y'' > 0$, 故 $(\pi, -2)$ 为拐点, 应选(B).

(3) 【答案】 (D).

【解】 方法一

$$\text{由 } \int_0^{+\infty} x e^{-x} dx = \Gamma(2) = 1 \text{ 得 } \int_0^{+\infty} x e^{-x} dx \text{ 收敛};$$

$$\text{由 } \int_0^{+\infty} x e^{-x^2} dx = -\frac{1}{2} e^{-x^2} \Big|_0^{+\infty} = \frac{1}{2} \text{ 得 } \int_0^{+\infty} x e^{-x^2} dx \text{ 收敛};$$

$$\text{由 } \int_0^{+\infty} \frac{\arctan x}{1+x^2} dx = \frac{1}{2} (\arctan x)^2 \Big|_0^{+\infty} = \frac{\pi^2}{8} \text{ 得 } \int_0^{+\infty} \frac{\arctan x}{1+x^2} dx \text{ 收敛},$$

$$\text{故 } \int_0^{+\infty} \frac{x}{1+x^2} dx \text{ 发散, 应选(D).}$$

方法二

$$\text{由 } \lim_{x \rightarrow +\infty} x \cdot \frac{x}{1+x^2} = 1 \text{ 且 } \alpha = 1 \leq 1 \text{ 得广义积分 } \int_0^{+\infty} \frac{x}{1+x^2} dx \text{ 发散, 应选(D).}$$

(4) 【答案】 (D).

【解】 由 $y = (C_1 + C_2 x)e^{-x} + e^x$ 为微分方程 $y'' + ay' + by = 0$ 的通解得 $\lambda_1 = \lambda_2 = -1$, 则 $a=2, b=1$;

再由 $y' = e^x$ 为微分方程 $y'' + ay' + by = ce^x$ 的特解得 $c=4$, 应选(D).

(5) 【答案】 (A).

【解】 由 $t \geq 0$ 时, $\sin t \leq t$ 得 $\sin \sqrt{x^2 + y^2} \leq \sqrt{x^2 + y^2}$, 从而 $I_2 < I_1$;

$$\text{又 } 1 - \cos \sqrt{x^2 + y^2} = 2 \sin^2 \frac{\sqrt{x^2 + y^2}}{2} = 2 \sin \frac{\sqrt{x^2 + y^2}}{2} \cdot \sin \frac{\sqrt{x^2 + y^2}}{2},$$

$$\sin \sqrt{x^2 + y^2} = 2 \sin \frac{\sqrt{x^2 + y^2}}{2} \cdot \cos \frac{\sqrt{x^2 + y^2}}{2},$$

$$\text{由 } x^2 + y^2 \leq \left(\frac{\pi}{2}\right)^2 \text{ 得 } \frac{\sqrt{x^2 + y^2}}{2} \in \left[0, \frac{\pi}{4}\right], \text{ 从而 } \sin \frac{\sqrt{x^2 + y^2}}{2} \leq \cos \frac{\sqrt{x^2 + y^2}}{2},$$

于是 $\sin \sqrt{x^2 + y^2} \geq 1 - \cos \sqrt{x^2 + y^2}$, 故 $I_3 < I_2$, 应选(A).

(6) 【答案】 (A).

【解】 若 $\lim_{x \rightarrow a} \frac{f(x) - g(x)}{(x - a)^2} = 0$, 得 $f(a) = g(a)$;

由 $\lim_{x \rightarrow a} \frac{f(x) - g(x)}{(x - a)^2} = 0$ 得 $\lim_{x \rightarrow a} \frac{f'(x) - g'(x)}{2(x - a)} = 0$, 从而 $f'(a) = g'(a)$;

$$\text{由 } 0 = \frac{1}{2} \lim_{x \rightarrow a} \frac{f'(x) - g'(x)}{x - a} = \frac{1}{2} \lim_{x \rightarrow a} \left[\frac{f'(x) - f'(a)}{x - a} - \frac{g'(x) - g'(a)}{x - a} \right] = \frac{1}{2} [f''(a) - g''(a)]$$

得 $f''(a) = g''(a)$,

即由 $\lim_{x \rightarrow a} \frac{f(x) - g(x)}{(x - a)^2} = 0$ 可得 $f(x), g(x)$ 在 $x = a$ 处相切且曲率相等;

反之, 若 $f(a) = g(a), f'(a) = g'(a), |f''(a)| = |g''(a)|$ 则不能保证 $\lim_{x \rightarrow a} \frac{f(x) - g(x)}{(x - a)^2} = 0$,

故 $\lim_{x \rightarrow a} \frac{f(x) - g(x)}{(x - a)^2} = 0$ 是 $y = f(x), y = g(x)$ 在 $x = a$ 对应点处相切且曲率相等的充分

但不必要条件, 应选(A).

(7) 【答案】 (A).

【解】 因为 $AX = 0$ 的基础解系中含 2 个解向量, 所以 $r(A) = 2 < 4$,

故 $r(A^*) = 0$, 应选(A).

(8) 【答案】 (C).

【解】 令 $AX = \lambda X (X \neq 0)$,

由 $A^2 + A = 2E$ 得 $(A^2 + A - 2E)X = (\lambda^2 + \lambda - 2)X = 0$,

从而有 $\lambda^2 + \lambda - 2 = 0$, 即 $\lambda = -2$ 或 $\lambda = 1$,

因为 $|A| = 4$, 所以 $\lambda_1 = 1, \lambda_2 = \lambda_3 = -2$,

故二次型 $X^T AX$ 的规范形为 $y_1^2 - y_2^2 - y_3^2$, 应选(C).

二、填空题

(9) 【答案】 $4e^2$.

$$\begin{aligned} \lim_{x \rightarrow 0} (x + 2^x)^{\frac{2}{x}} &= \lim_{x \rightarrow 0} \left[(1 + x + 2^x - 1)^{\frac{1}{x + 2^x - 1}} \right]^{\frac{2(x + 2^x - 1)}{x}} = e^{2 \lim_{x \rightarrow 0} \left(1 + \frac{2^x - 1}{x} \right)} \\ &= e^{2(1 + \ln 2)} = e^{\ln 4 e^2} = 4e^2. \end{aligned}$$

(10) 【答案】 $\frac{3\pi}{2} + 2$.

【解】 $t = \frac{3\pi}{2}$ 对应曲线上的点为 $\left(\frac{3\pi}{2} + 1, 1\right)$,

$$\frac{dy}{dx} = \frac{\sin t}{1 - \cos t}, \text{ 斜率为 } \left. \frac{dy}{dx} \right|_{t = \frac{3\pi}{2}} = -1,$$

切线方程为

$$y - 1 = -\left(x - \frac{3\pi}{2} - 1\right),$$

令 $x = 0$ 得切线在 y 轴上的截距为 $y = \frac{3\pi}{2} + 2$.

(11) 【答案】 $yf\left(\frac{y^2}{x}\right)$.

【解】 $\frac{\partial z}{\partial x} = -\frac{y^3}{x^2}f'\left(\frac{y^2}{x}\right), \quad \frac{\partial z}{\partial y} = f\left(\frac{y^2}{x}\right) + \frac{2y^2}{x}f'\left(\frac{y^2}{x}\right),$

则 $2x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = -\frac{2y^3}{x}f'\left(\frac{y^2}{x}\right) + yf\left(\frac{y^2}{x}\right) + \frac{2y^3}{x}f'\left(\frac{y^2}{x}\right) = yf\left(\frac{y^2}{x}\right).$

(12) 【答案】 $\frac{1}{2}\ln 3$.

【解】 曲线段的长度为

$$\begin{aligned} s &= \int_0^{\frac{\pi}{6}} \sqrt{1+y'^2} dx = \int_0^{\frac{\pi}{6}} \sqrt{1+\tan^2 x} dx = \int_0^{\frac{\pi}{6}} \sec x dx \\ &= \ln |\sec x + \tan x| \Big|_0^{\frac{\pi}{6}} = \ln\left(\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right) = \ln\sqrt{3} = \frac{1}{2}\ln 3. \end{aligned}$$

(13) 【答案】 $\frac{\cos 1 - 1}{4}$.

【解】 $\int_0^1 f(x) dx = \int_0^1 \left(\int_1^x \frac{\sin t^2}{t} dt \right) d\left(\frac{x^2}{2}\right) = \left(\frac{x^2}{2} \int_1^x \frac{\sin t^2}{t} dt\right) \Big|_0^1 - \int_0^1 \frac{x^2}{2} \cdot \frac{\sin x^2}{x} dx$
 $= -\frac{1}{2} \int_0^1 x \sin x^2 dx = -\frac{1}{4} \int_0^1 \sin x^2 d(x^2) = \frac{1}{4} \cos x^2 \Big|_0^1 = \frac{\cos 1 - 1}{4}.$

(14) 【答案】 -4 .

【解】

$$\begin{aligned} A_{11} - A_{12} &= 1 \times A_{11} - 1 \times A_{12} + 0A_{13} + 0A_{14} = |A| = \begin{vmatrix} 1 & -1 & 0 & 0 \\ -2 & 1 & -1 & 1 \\ 3 & -2 & 2 & -1 \\ 0 & 0 & 3 & 4 \end{vmatrix} \\ &= \begin{vmatrix} 1 & 0 & 0 & 0 \\ -2 & -1 & -1 & 1 \\ 3 & 1 & 2 & -1 \\ 0 & 0 & 3 & 4 \end{vmatrix} = \begin{vmatrix} -1 & -1 & 1 \\ 1 & 2 & -1 \\ 0 & 3 & 4 \end{vmatrix} = \begin{vmatrix} -1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 3 & 4 \end{vmatrix} = -4. \end{aligned}$$

三、解答题

(15) 【解】 当 $x > 0$ 时, $f'(x) = (x^{2x})' = (e^{2x \ln x})' = x^{2x} \cdot (2 \ln x + 2)$;

当 $x < 0$ 时, $f'(x) = (x+1)e^x$,

由 $\lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0^+} \frac{e^{2x \ln x} - 1}{x} = \lim_{x \rightarrow 0^+} 2 \ln x = -\infty$ 得

$f(x)$ 在 $x = 0$ 处不可导, 于是有

$$f'(x) = \begin{cases} x^{2x}(2 \ln x + 2), & x > 0, \\ (x+1)e^x, & x < 0. \end{cases}$$

$f'(x)=0$ 或 $f(x)$ 的不可导的点为 $x=-1, x=0, x=\frac{1}{e}$,

当 $x < -1$ 时, $f'(x) < 0$; 当 $-1 < x < 0$ 时, $f'(x) > 0$;

当 $0 < x < \frac{1}{e}$ 时, $f'(x) < 0$; 当 $x > \frac{1}{e}$ 时, $f'(x) > 0$,

故 $x=-1$ 为极小值点, 极小值为 $f(-1)=1-\frac{1}{e}$;

$x=0$ 为极大值点, 极大值为 $f(0)=1$;

$x=\frac{1}{e}$ 为极小值点, 极小值为 $f\left(\frac{1}{e}\right)=\left(\frac{1}{e}\right)^{\frac{2}{e}}$.

(16) 【解】 令
$$\frac{3x+6}{(x-1)^2(x^2+x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+x+1},$$

由 $A(x-1)(x^2+x+1)+B(x^2+x+1)+(Cx+D)(x-1)^2=3x+6$ 得

$$\begin{cases} A+C=0, \\ B-2C+D=0, \\ B+C-2D=3, \\ -A+B+D=6, \end{cases}$$

解得 $A=-2, B=3, C=2, D=1$, 故

$$\begin{aligned} \int \frac{3x+6}{(x-1)^2(x^2+x+1)} dx &= \int \left[\frac{-2}{x-1} + \frac{3}{(x-1)^2} + \frac{2x+1}{x^2+x+1} \right] dx \\ &= -2\ln|x-1| - \frac{3}{x-1} + \ln(x^2+x+1) + C. \end{aligned}$$

(17) 【解】 (I) $y = \left(\int \frac{1}{2\sqrt{x}} e^{\frac{x^2}{2}} \cdot e^{\int -x dx} dx + C \right) e^{\int -x dx} = (\sqrt{x} + C) e^{\frac{x^2}{2}},$

由 $y(1)=\sqrt{e}$ 得 $C=0$, 即 $y(x)=\sqrt{x} e^{\frac{x^2}{2}}$.

(II) $V = \int_1^2 \pi y^2 dx = \pi \int_1^2 x e^{x^2} dx = \frac{\pi}{2} e^{x^2} \Big|_1^2 = \frac{\pi}{2} (e^4 - e).$

(18) 【解】 由对称性得

$$\iint_D \frac{x+y}{\sqrt{x^2+y^2}} dx dy = \iint_D \frac{y}{\sqrt{x^2+y^2}} dx dy,$$

令 $\begin{cases} x=r\cos\theta, \\ y=r\sin\theta \end{cases} \left(\frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}, 0 \leq r \leq \sin^2\theta \right)$, 则

$$\begin{aligned} \iint_D \frac{x+y}{\sqrt{x^2+y^2}} dx dy &= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} d\theta \int_0^{\sin^2\theta} r \sin\theta dr = \frac{1}{2} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin^5\theta d\theta \\ &= -\frac{1}{2} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (1-\cos^2\theta)^2 d(\cos\theta) \stackrel{\cos\theta=t}{=} -\frac{1}{2} \int_{\frac{\sqrt{2}}{2}}^{-\frac{\sqrt{2}}{2}} (1-t^2)^2 dt = \int_0^{\frac{\sqrt{2}}{2}} (1-t^2)^2 dt \\ &= \int_0^{\frac{\sqrt{2}}{2}} (1-2t^2+t^4) dt = \left(\frac{\sqrt{2}}{2} - \frac{2}{3} \times \frac{\sqrt{2}}{4} + \frac{1}{5} \times \frac{\sqrt{2}}{8} \right) = \frac{43}{120} \sqrt{2}. \end{aligned}$$

(19) 【解】 $S_n = \sum_{k=0}^{n-1} (-1)^k \int_{k\pi}^{(k+1)\pi} e^{-x} \sin x dx$

$$\begin{aligned}
&= \sum_{k=0}^{n-1} (-1)^k \left[-\frac{1}{2} e^{-x} (\sin x + \cos x) \right] \Big|_{k\pi}^{(k+1)\pi} \\
&= \frac{1}{2} \sum_{k=0}^{n-1} (-1)^{k+1} [e^{-(k+1)\pi} (-1)^{k+1} - e^{-k\pi} (-1)^k] \\
&= \frac{1}{2} \sum_{k=0}^{n-1} [e^{-(k+1)\pi} + e^{-k\pi}] = \frac{1}{2} \left[1 + 2 \sum_{k=1}^n e^{-k\pi} - e^{-n\pi} \right] = \frac{1}{2} \left[1 + \frac{2e^{-\pi}(1-e^{-n\pi})}{1-e^{-\pi}} - e^{-n\pi} \right], \\
\text{故 } \lim_{n \rightarrow \infty} S_n &= \lim_{n \rightarrow \infty} \frac{1}{2} \left[1 + \frac{2e^{-\pi}(1-e^{-n\pi})}{1-e^{-\pi}} - e^{-n\pi} \right] = \frac{1}{2} + \frac{1}{e^{\pi}-1}.
\end{aligned}$$

(20) 【解】 $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} e^{ax+by} + av e^{ax+by}, \quad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} e^{ax+by} + bv e^{ax+by},$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x^2} e^{ax+by} + 2a \frac{\partial v}{\partial x} e^{ax+by} + a^2 v e^{ax+by},$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial y^2} e^{ax+by} + 2b \frac{\partial v}{\partial y} e^{ax+by} + b^2 v e^{ax+by},$$

代入已知等式得

$$\begin{aligned}
2 \frac{\partial^2 v}{\partial x^2} e^{ax+by} + 4a \frac{\partial v}{\partial x} e^{ax+by} + 2a^2 v e^{ax+by} - 2 \frac{\partial^2 v}{\partial y^2} e^{ax+by} - 4b \frac{\partial v}{\partial y} e^{ax+by} - 2b^2 v e^{ax+by} \\
+ 3 \frac{\partial v}{\partial x} e^{ax+by} + 3av e^{ax+by} + 3 \frac{\partial v}{\partial y} e^{ax+by} + 3bv e^{ax+by} = 0,
\end{aligned}$$

整理得

$$2 \frac{\partial^2 v}{\partial x^2} - 2 \frac{\partial^2 v}{\partial y^2} + (4a+3) \frac{\partial v}{\partial x} + (3-4b) \frac{\partial v}{\partial y} + (2a^2-2b^2+3a+3b)v = 0,$$

由题意得 $\begin{cases} 4a+3=0, \\ 3-4b=0, \end{cases}$ 解得 $a = -\frac{3}{4}, b = \frac{3}{4}.$

(21) 【证明】(I) 令 $F(x) = \int_0^x f(t) dt$, 则 $F'(x) = f(x)$,

由拉格朗日中值定理得

$$1 = \int_0^1 f(x) dx = F(1) - F(0) = F'(c)(1-0) = f(c)(0 < c < 1),$$

因为 $f(c) = f(1) = 1$, 所以由罗尔定理, 存在 $\xi \in (c, 1) \subset (0, 1)$, 使得 $f'(\xi) = 0$.

(II) 令 $\varphi(x) = f(x) + x^2$,

$$\varphi(0) = 0, \quad \varphi(c) = f(c) + c^2 = 1 + c^2, \quad \varphi(1) = 2,$$

由拉格朗日中值定理, 存在 $\eta_1 \in (0, c), \eta_2 \in (c, 1)$, 使得

$$\varphi'(\eta_1) = \frac{\varphi(c) - \varphi(0)}{c} = \frac{1+c^2}{c} = c + \frac{1}{c},$$

$$\varphi'(\eta_2) = \frac{\varphi(1) - \varphi(c)}{1-c} = \frac{2-1-c^2}{1-c} = 1+c,$$

再由拉格朗日中值定理, 存在 $\eta \in (\eta_1, \eta_2) \subset (0, 1)$, 使得

$$\varphi''(\eta) = \frac{\varphi'(\eta_2) - \varphi'(\eta_1)}{\eta_2 - \eta_1} = \frac{1+c-c-\frac{1}{c}}{\eta_2 - \eta_1} = \frac{1-\frac{1}{c}}{\eta_2 - \eta_1} < 0,$$

而 $\varphi''(x) = f''(x) + 2$, 即 $f''(\eta) + 2 < 0$, 故 $f''(\eta) < -2$.

$$(22) \text{【解】} (\alpha_1, \alpha_2, \alpha_3) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 4 & 4 & a^2+3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & a^2-1 \end{pmatrix},$$

当 $a = -1$ 时, 向量组 $\alpha_1, \alpha_2, \alpha_3$ 的秩为 2,

$$\text{由 } (\beta_1, \beta_2, \beta_3) = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 3 \\ 2 & 2 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ 得 } \beta_1, \beta_2, \beta_3 \text{ 的秩为 2,}$$

$$(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 2 & 1 & 2 & 3 \\ 4 & 4 & 4 & 2 & 2 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 & 2 & 2 \\ 0 & 0 & 0 & -2 & 2 & 0 \end{pmatrix},$$

因为 $r(\alpha_1, \alpha_2, \alpha_3) \neq r(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$, 所以两个向量组不等价;

$$\text{当 } a = 1 \text{ 时, } (\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 2 & 1 & 2 & 3 \\ 4 & 4 & 4 & 4 & 0 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

因为 $r(\alpha_1, \alpha_2, \alpha_3) = r(\beta_1, \beta_2, \beta_3) = r(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) = 2$,

所以两个向量组等价.

令 $x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3 = \beta_3$,

$$\text{再由 } (\alpha_1, \alpha_2, \alpha_3, \beta_3) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 3 \\ 4 & 4 & 4 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ 得}$$

方程组 $x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3 = \beta_3$ 的通解为

$$X = k \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} -2k+3 \\ k-2 \\ k \end{pmatrix} (k \text{ 为任意常数}),$$

故 $\beta_3 = (-2k+3)\alpha_1 + (k-2)\alpha_2 + k\alpha_3$ (k 为任意常数).

当 $a \neq \pm 1$ 时, 向量组 $\alpha_1, \alpha_2, \alpha_3$ 的秩为 3,

$$\text{由 } (\beta_1, \beta_2, \beta_3) = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 3 \\ a+3 & 1-a & a^2+3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1-a & a^2-a \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & a^2-1 \end{pmatrix} \text{ 得}$$

向量组 $\beta_1, \beta_2, \beta_3$ 的秩为 3,

$$\begin{aligned} \text{再由 } (\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) &= \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 2 & 1 & 2 & 3 \\ 4 & 4 & a^2+3 & a+3 & 1-a & a^2+3 \end{pmatrix} \\ &\rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 & 2 & 2 \\ 0 & 0 & a^2-1 & a-1 & 1-a & a^2-1 \end{pmatrix} \text{ 得} \end{aligned}$$

$r(\alpha_1, \alpha_2, \alpha_3) = r(\beta_1, \beta_2, \beta_3) = r(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) = 3$,

故两个向量组等价.

令 $x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3 = \beta_3$,

$$\begin{aligned} \text{由 } (\alpha_1, \alpha_2, \alpha_3, \beta_3) &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 3 \\ 4 & 4 & a^2+3 & a^2+3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & a^2-1 & a^2-1 \end{pmatrix} \\ &\rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \text{ 得} \\ &\beta_3 = \alpha_1 - \alpha_2 + \alpha_3. \end{aligned}$$

(23) 【解】(I) 因为 $A \sim B$, 所以 $\text{tr}(A) = \text{tr}(B)$, 即 $x - 4 = y + 1$, 或 $y = x - 5$,

再由 $|A| = |B|$ 得 $-2(-2x + 4) = -2y$, 即 $y = -2x + 4$,

解得 $x = 3, y = -2$.

$$(II) A = \begin{pmatrix} -2 & -2 & 1 \\ 2 & 3 & -2 \\ 0 & 0 & -2 \end{pmatrix}, B = \begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

显然矩阵 A, B 的特征值为 $\lambda_1 = -2, \lambda_2 = -1, \lambda_3 = 2$,

$$\text{由 } 2E + A \rightarrow \begin{pmatrix} 0 & -2 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & \frac{1}{4} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} \text{ 得 } A \text{ 的属于特征值 } \lambda_1 = -2 \text{ 的线性无关的特}$$

$$\text{征向量为 } \alpha_1 = \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix};$$

$$\text{由 } E + A \rightarrow \begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ 得 } A \text{ 的属于特征值 } \lambda_2 = -1 \text{ 的线性无关的特征向}$$

$$\text{量为 } \alpha_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix};$$

$$\text{由 } 2E - A \rightarrow \begin{pmatrix} 2 & 1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ 得 } A \text{ 的属于特征值 } \lambda_3 = 2 \text{ 的线性无关的特征向}$$

$$\text{量为 } \alpha_3 = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix},$$

$$\text{令 } P_1 = \begin{pmatrix} -1 & -2 & -1 \\ 2 & 1 & 2 \\ 4 & 0 & 0 \end{pmatrix}, \text{ 则 } P_1^{-1}AP_1 = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix};$$

$$\text{由 } 2E + B \rightarrow \begin{pmatrix} 4 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ 得 } B \text{ 的属于特征值 } \lambda_1 = -2 \text{ 的线性无关的特征向量}$$

为 $\beta_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$;

由 $E + B = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ 得 B 的属于特征值 $\lambda_2 = -1$ 的线性无关的特征向

量为 $\beta_2 = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}$;

由 $2E - B = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ 得 B 的属于特征值 $\lambda_2 = 2$ 的线性无关的特征向量

为 $\beta_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$,

令 $P_2 = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, 则 $P_2^{-1}BP_2 = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$,

由 $P_1^{-1}AP_1 = P_2^{-1}BP_2$ 得 $(P_1P_2^{-1})^{-1}A(P_1P_2^{-1}) = B$,

故 $P = P_1P_2^{-1} = \begin{pmatrix} -1 & -1 & -1 \\ 2 & 1 & 2 \\ 0 & 0 & 4 \end{pmatrix}$.