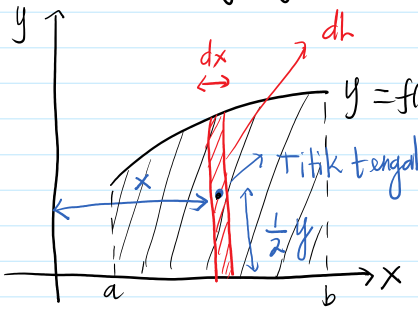
Kelas April - Materi 1

08 April 2023 10:00

Materi : Aplikasi Integral untuk Menghitung Titik Berat & Dalil Guldin

A. Titik Berat Keping Datar Homogen

I. Keping Datar yang dibatasi oleh kurva $y = f(x)$.



- elemen kecil luas : $\boxed{dL = y \cdot dx}$
- elemen kecil massa : $dM = \delta \cdot dL$

$$\boxed{dM = \delta \cdot y \cdot dx}$$

- Momen statis = jarak × massa
- Elemen kecil momen statis thd sb. y

$$dM_y = x \cdot dM$$

$$\boxed{dM_y = x \cdot \delta \cdot y \cdot dx}$$

- Elemen kecil momen statis thd sb. x

$$dM_x = \frac{1}{2}y \cdot dM$$

$$dM_x = \frac{1}{2}y \cdot \delta\, y \cdot dx$$

$$\boxed{dM_x = \frac{1}{2}y^2 \delta\, dx}$$

- $M = \int_{x=a}^{b} dM$
- $M_y = \int_{x=a}^{b} dM_y$
- $M_x = \int_{x=a}^{b} dM_x$

$$\boxed{M = \int_{x=a}^{b} \delta\, y\, dx}$$

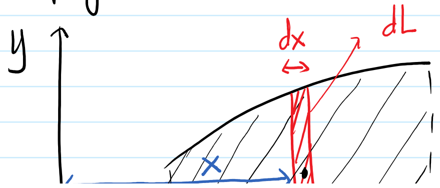
$$\boxed{M_y = \int_{x=a}^{b} \delta \cdot xy\, dx}$$

$$\boxed{M_x = \int_{x=a}^{b} \frac{1}{2}y^2 \delta\, dx}$$

Titik Berat : $Z(\bar{x}, \bar{y})$

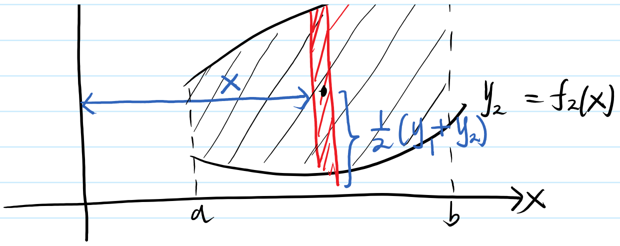
$$\bar{x} = \frac{M_y}{M} \quad \& \quad \bar{y} = \frac{M_x}{M}$$

II. Keping Datar dibatasi oleh kurva $y_1 = f_1(x)$ dan $y_2 = f_2(x)$.



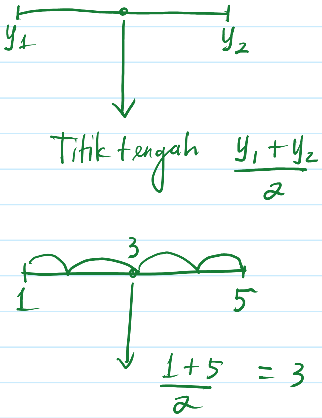
kurva atas − kurva bawah

- elemen kecil luas : $dL = (y_1 - y_2)\, dx$
- elemen kecil massa : $dM = \delta \cdot dL$



•) elemen kecil massa : $dM = \delta \cdot dL$

$$\boxed{\begin{aligned} dM &= \delta\,(y_1 - y_2)\,dx \\ M &= \int_{x=a}^{b} \delta\,(y_1 - y_2)\,dx \end{aligned}}$$



•) elemen kecil momen statis thd sb . y

$$dM_y = x \cdot dM$$

$$\boxed{\begin{aligned} dM_y &= \delta \cdot x\,(y_1 - y_2)\,dx \\ M_y &= \int_{x=a}^{b} \delta \cdot x\,(y_1 - y_2)\,dx \end{aligned}}$$

•) elemen kecil momen statis thd sb . x

$$dM_x = \tfrac{1}{2}(y_1 + y_2)\,dM$$

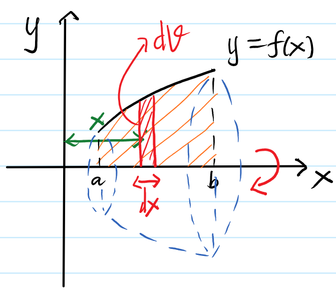
$$dM_x = \tfrac{1}{2}(y_1 + y_2) \cdot \delta\,(y_1 - y_2)\,dx$$

$$\boxed{\begin{aligned} dM_x &= \tfrac{1}{2}\delta\,(y_1^2 - y_2^2)\,dx \\ M_x &= \int_{x=a}^{b} \tfrac{1}{2}\delta\,(y_1^2 - y_2^2)\,dx \end{aligned}}$$

$$\boxed{\begin{aligned} &\text{Titik berat : } Z(\bar{x}, \bar{y}) \\ &\bar{x} = \frac{M_y}{M} \quad ; \quad \bar{y} = \frac{M_x}{M} \end{aligned}}$$

B. Titik Berat Volume Benda Putar

(I) Bidang yang dibatasi kurva $y = f(x)$, sb . x , garis $x = a$ dan $x = b$, diputar thd sb . x . Maka titik beratnya di titik $Z(\bar{x}, 0)$



•) $dV = \pi y^2\,dx$

•)
$$\boxed{\begin{aligned} dM &= \delta\, dV \\ &= \delta \cdot \pi y^2\,dx \\ M &= \delta \pi \int_a^b y^2\,dx \end{aligned}}$$

•) Elemen kecil momen statis thd sb . y

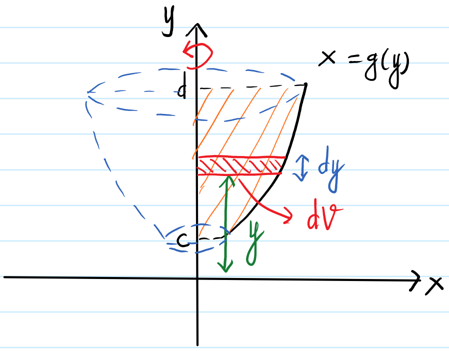
$$\begin{aligned} dM_y &= x \cdot dM \\ &= x \cdot \delta \pi y^2 \cdot dx \end{aligned}$$

$$dM_y = x \cdot dM$$
$$= x \cdot \delta \pi y^2 \cdot dx$$
$$M_y = \delta \pi \int_a^b x y^2 \, dx$$

Titik Berat : $Z(\bar{x}, 0)$

$$\bar{x} = \frac{M_y}{M}$$

(ii) Bidang yang dibatasi kurva $x = g(y)$, sb. y, garis $y = c$ dan $y = d$, diputar thd sb. y. Maka titik beratnya di titik $Z(0, \bar{y})$



- $dV = \pi x^2 \, dy$
- $dM = \delta \, dV$
$$= \delta \cdot \pi x^2 \, dy$$
$$M = \delta \pi \int_c^d x^2 \, dy$$

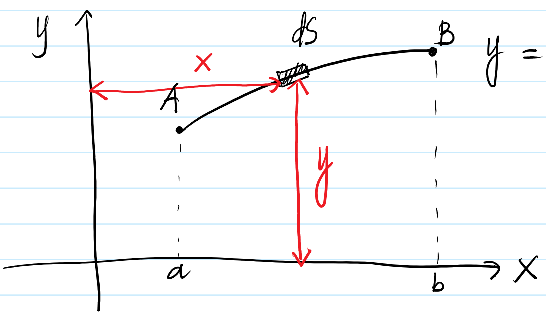
- Elemen kecil momen statis thd sb. x

$$dM_x = y \cdot dM$$
$$= y \cdot \delta \pi x^2 \, dy$$
$$M_x = \delta \pi \int_c^d y x^2 \, dy$$

Titik Berat : $Z(0, \bar{y})$

$$\bar{y} = \frac{M_x}{M}$$

C. Titik berat busur homogen $y = f(x)$.



$y = f(x)$

- elemen kecil panjang busur : $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$
- elemen kecil massa : $dM = \delta \cdot ds$

$$dM = \delta \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$
$$M = \int_a^b \delta \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

- elemen kecil momen statis terhadap sb $\cdot y$

$$dM_y = x \cdot dM$$

$$\boxed{\begin{aligned} dM_y &= x \cdot \delta \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \\ M_y &= \int_a^b \delta \cdot x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \end{aligned}}$$

- elemen kecil momen statis terhadap sb $\cdot x$

$$dM_x = y \cdot dM$$

$$\boxed{\begin{aligned} dM_x &= y \cdot \delta \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \\ M_x &= \int_a^b \delta\, y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \end{aligned}}$$

- $$\bar{x} = \frac{M_y}{M} = \frac{\int_a^b x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx}{\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx}$$

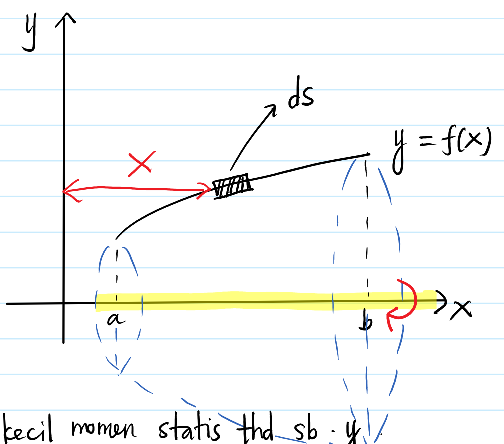
- $$\bar{y} = \frac{M_x}{M} = \frac{\int_a^b y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx}{\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx}$$

Titik Berat :

$Z(\bar{x}, \bar{y})$

D. Titik Berat Kulit Benda Putar

Jika elemen busur ds diputar pd sb $\cdot x$, maka diperoleh luas kulit dK.



- $dK = 2\pi y \cdot ds$
- $$\boxed{\begin{aligned} dM &= \delta\, dK \\ &= \delta \cdot 2\pi y \cdot ds \\ M &= 2\pi\delta \int_a^b y \cdot \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \end{aligned}}$$

- Elemen kecil momen statis thd sb $\cdot y$

$$\boxed{\begin{aligned} dM_y &= x \cdot dM \\ &= x \cdot \delta \cdot 2\pi y \cdot ds \\ M_y &= 2\pi\delta \int_a^b xy \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \end{aligned}}$$

Titik Berat : $Z(\bar{x}, 0)$ ✓

$$\bar{x} = \frac{M_y}{M}$$

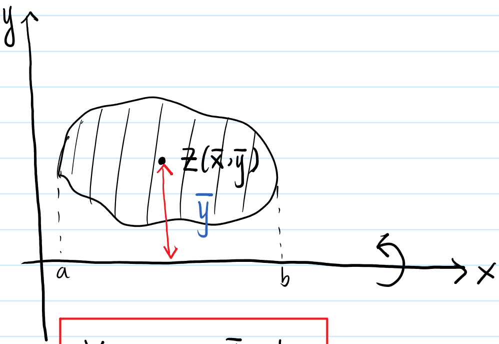
※ Titik Berat : ① Keping Datar : $dM = \delta \cdot dL$

② Busur : $dM = \delta \cdot ds$

③ Luas Permukaan (Kulit) : $dM = \delta \cdot dK$

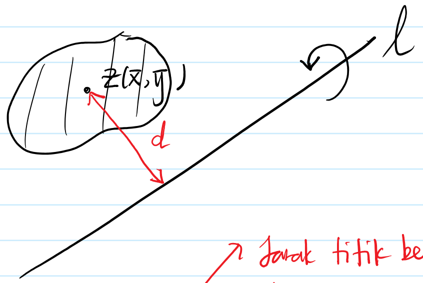
④ Volume / Isi Benda Putar : $dM = \delta \cdot dV$

E. Dalil Guldin I



$$V = 2\pi \cdot \bar{y} \cdot L$$

Secara Umum



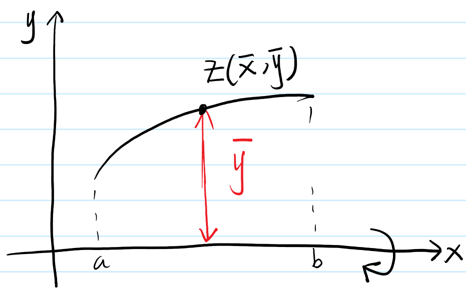
$$V = 2\pi \cdot d \cdot L$$

Jarak titik berat $Z(\bar{x}, \bar{y})$ ke pers. garis sbg sumbu putarnya.

∘> Jarak titik $A(x_0, y_0)$ ke garis $ax + by + c = 0$

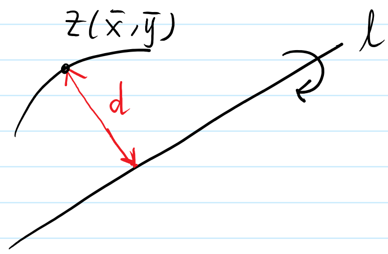
$$d = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

F. Dalil Guldin II



$$K = 2\pi \cdot \bar{y} \cdot S$$

Secara Umum

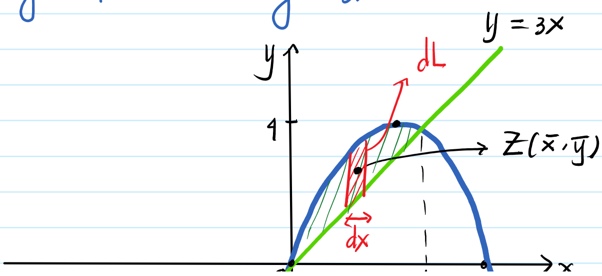


$$K = 2\pi \cdot d \cdot S$$

Contoh Soal

① Dapatkan titik berat keping datar homogen yang dibatasi oleh

$y = 4x - x^2$ dan $y = 3x$



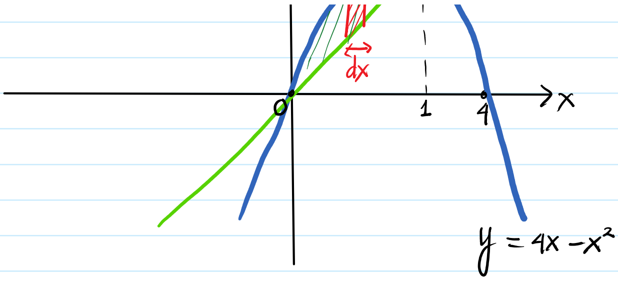
∘> $y = 4x - x^2$ ✓ ⟶ ∩

∘> Titik potong thd sb. x ⟶ $y = 0$

$4x - x^2 = 0$

$x(4 - x) = 0$

$x = 0$, $x = 4$



$x = 0 \quad , x = 4$

> Sumbu simetri : $y = ax^2 + bx + c$

$x = \frac{-b}{2a} = \frac{-4}{2(-1)} = 2$

> Titik Puncak : (sumbu simetri , nilai optimum)

nilai optimum $= f(2)$

$= 4(2) - 2^2$

$= 4$

$(2, 4)$

> Menentukan titik potong (batas integral).

$y = y$

$4x - x^2 = 3x$

$x^2 - x = 0$

$x(x-1) = 0$

$x = 0 \quad , x = 1$

> $M = \int_0^1 \delta \, (y_1 - y_2) \, dx$

$= \delta \int_0^1 ((4x - x^2) - 3x) \, dx$

$= \delta \int_0^1 x - x^2 \, dx$

$= \delta \left[\frac{1}{2}x^2 - \frac{1}{3}x^3 \right]_0^1$

$= \delta \left[\left(\frac{1}{2} - \frac{1}{3}\right) - (0 - 0) \right]$

$= \delta \left(\frac{3-2}{6} \right)$

$= \frac{1}{6}\delta$

> $M_y = \int_0^1 \delta \, x \, (y_1 - y_2) \, dx$

$= \delta \int_0^1 x \, (x - x^2) \, dx$

$= \delta \int_0^1 x^2 - x^3 \, dx$

$= \delta \left[\frac{1}{3}x^3 - \frac{1}{4}x^4 \right]_0^1$

$= \delta \left[\left(\frac{1}{3} - \frac{1}{4}\right) - (0 - 0) \right]$

$= \delta \left(\frac{4-3}{12} \right)$

$= \frac{1}{12}\delta$

> $M_x = \int_0^1 \delta \, (y_1^2 - y_2^2) \, dx$

$= \delta \int_0^1 \left[(4x - x^2)^2 - (3x)^2 \right] dx$

$= \delta \int_0^1 \left[16x^2 - 8x^3 + x^4 - 9x^2 \right] dx$

$= \delta \int_0^1 7x^2 - 8x^3 + x^4 \, dx$

$= \delta \left[\frac{7}{3}x^3 - 2x^4 + \frac{1}{5}x^5 \right]_0^1$

$= \delta \left[\left(\frac{7}{3} - 2 + \frac{1}{5}\right) - (0 - 0 + 0) \right]$

$= \delta \left(\frac{35 - 30 + 3}{15} \right)$

$= \frac{8}{15}\delta$

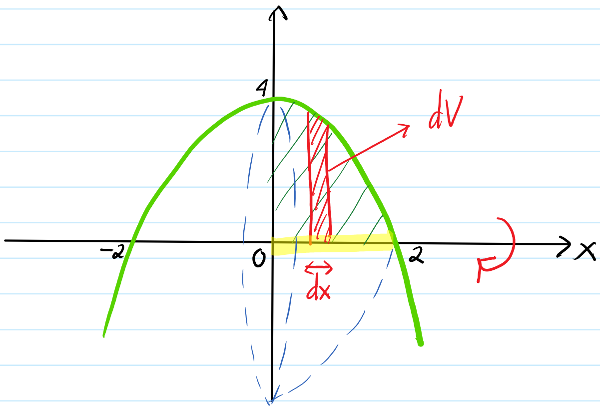
$$= \frac{8}{15}\delta$$

$$\Rightarrow \bar{x} = \frac{M_y}{M} = \frac{\frac{1}{12}\cancel{\delta}}{\frac{1}{6}\cancel{\delta}} = \frac{1}{\cancel{12}_2} \times \cancel{6} = \frac{1}{2}$$

$$\Rightarrow \bar{y} = \frac{M_x}{M} = \frac{\frac{8}{15}\cancel{\delta}}{\frac{1}{6}\cancel{\delta}} = \frac{8}{\cancel{15}_5} \times \cancel{6}^{2} = \frac{16}{5}$$

Titik berat : $Z\left(\frac{1}{2}, \frac{16}{5}\right)$

(2) Dapatkan titik berat volume benda putar yg terjadi jika keping datar yg dibatasi oleh $y = 4 - x^2$ dan sumbu x (di kuadran I) diputar pd sumbu x.



Titik berat : $Z(\bar{x}, 0)$

$$\Rightarrow M = \int_0^2 \delta\pi\,(4 - x^2)^2\,dx$$

$$= \delta\pi \int_0^2 16 - 8x^2 + x^4\,dx$$

$$= \delta\pi \left[16x - \frac{8}{3}x^3 + \frac{1}{5}x^5\right]_0^2$$

$$= \delta\pi\left[\left(32 - \frac{64}{3} + \frac{32}{5}\right) - (0 - 0 + 0)\right]$$

$$= \delta\pi\left(\frac{480 - 320 + 96}{15}\right)$$

$$= \frac{256}{15}\delta\pi$$

$$\Rightarrow M_y = \int_0^2 \delta\pi\, x\cdot(4 - x^2)^2\,dx$$

$$= \delta\pi \int_0^2 x\,(16 - 8x^2 + x^4)\,dx$$

$$= \delta\pi \int_0^2 16x - 8x^3 + x^5\,dx$$

$$= \delta\pi\left[8x^2 - 2x^4 + \frac{1}{6}x^6\right]_0^2$$

$$= \delta\pi\left[\left(\cancel{32} - \cancel{32} + \frac{64}{6}\right) - (0 - 0 + 0)\right]$$

$$= \frac{64}{6}\delta\pi$$

$$\Rightarrow \bar{x} = \frac{M_y}{M} = \frac{\frac{64}{6}\cancel{\delta\pi}}{\frac{256}{15}\cancel{\delta\pi}} = \frac{\cancel{64}}{\cancel{6}_2} \times \frac{\cancel{15}^{5}}{\cancel{256}_4} = \frac{5}{8}$$

$M \qquad \frac{256}{15}\delta\pi \qquad \cancel{6}_2 \qquad \cancel{256}\ 4 \qquad 8$

Titik berat : $Z\left(\frac{5}{8}, 0\right)$

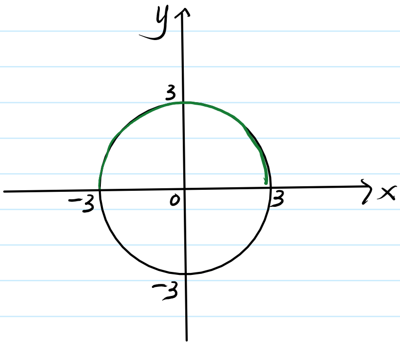
(3) Dapatkan titik berat kulit benda putar yg terjadi jika busur $y = \sqrt{9 - x^2}$; $(0 \le x \le 3)$, diputar pada sumbu x.

$\Rightarrow y = \sqrt{9 - x^2}$ ✓

$y^2 = 9 - x^2$

$x^2 + y^2 = 9$ ✓ $\longrightarrow x^2 + y^2 = r^2$

$\odot$, $P(0,0)$; $r = 3$



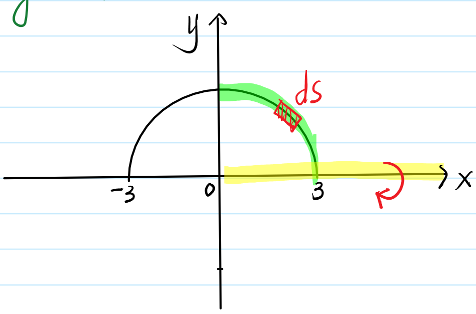
$\Rightarrow y = \sqrt{9 - x^2} \rightarrow \frac{1}{2}\odot$ atas

$\Rightarrow y = -\sqrt{9 - x^2} \rightarrow \frac{1}{2}\odot$ bawah

$\Rightarrow x = \sqrt{9 - y^2} \rightarrow \frac{1}{2}\odot$ kanan

$\Rightarrow x = -\sqrt{9 - y^2} \rightarrow \frac{1}{2}\odot$ kiri

$\Rightarrow y = \sqrt{9 - x^2}$



$y = (9 - x^2)^{\frac{1}{2}}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\cancel{2}}(9 - x^2)^{-\frac{1}{2}} \cdot -\cancel{2}x = \frac{-x}{\sqrt{9 - x^2}}$

$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{x^2}{9 - x^2}$

$\Rightarrow \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{x^2}{9 - x^2}}$

$= \sqrt{\frac{9 - \cancel{x^2} + \cancel{x^2}}{9 - x^2}}$

$= \frac{3}{\sqrt{9 - x^2}}$

$\Rightarrow M = 2\pi\delta \int_0^3 y \cdot \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$= 2\pi\delta \int_0^3 \cancel{\sqrt{9 - x^2}} \cdot \frac{3}{\cancel{\sqrt{9 - x^2}}}\, dx$

$= 6\pi\delta \int_0^3 dx$

$= 6\pi\delta\ x \Big|_0^3$

$= 6\pi\delta(3 - 0)$

$= 18\pi\delta$

$\Rightarrow M_y = 2\pi\delta \int_0^3 xy \cdot \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$= 2\pi\delta \int_0^3 x\cancel{\sqrt{9 - x^2}} \cdot \frac{3}{\cancel{\sqrt{9 - x^2}}}\, dx$

$= 6\pi\delta \int_0^3 x\, dx$

$= 6\pi\delta\ \frac{1}{2}x^2 \Big|_0^3$

$= \cancel{6}^3\pi\delta\left(\frac{9}{\cancel{2}} - 0\right)$

$= 27\pi\delta$

$\Rightarrow \bar{x} = \frac{M_y}{M} = \frac{27\cancel{\pi\delta}}{18\cancel{\pi\delta}} = \frac{3}{2}$

M $\overline{18\ \cancel{18}}$ $\overline{2}$

Titik berat : $Z\left(\frac{3}{2}, 0\right)$