Pertemuan 9 - Juni

Thursday, 20 June 2024
17.00

5. Dapatkan luas daerah yang dibatasi oleh kurva $y = x^3 + 1$, $\underbrace{\text{sumbu-}x}_{y=0}$, dan $\underbrace{\text{sumbu-}y}_{x=0}$ pada kuadran II. Sketsa grafiknya. *(ETS 2022/2023, Selasa 28 Maret 2023)*

Jawab

$y = x^3 + 1$, $y = 0$, $x = 0$

① Tipot

$y_1 = y_2$

$x^3 + 1 = 0$

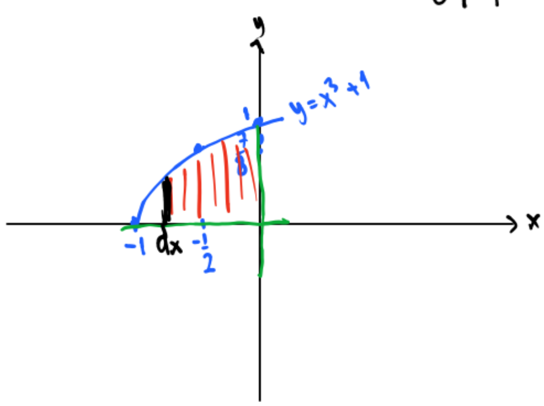
$x^3 = -1$

$x = -1$

② Gambar

$y = x^3 + 1$

x	y
-1	0
$-\frac{1}{2}$	$\frac{7}{8}$
0	1



③ Luas

$dL = (x^3 + 1)\,dx$

$L = \int_{-1}^{0} x^3 + 1 \; dx$

$= \frac{1}{4}x^4 + x \Big|_{-1}^{0}$

$= 0 - \left[\frac{1}{4} - 1\right]$

$= 0 - \left[-\frac{3}{4}\right]$

$= \frac{3}{4}$ Satuan Luas.

1. Dapatkan volume benda putar jika daerah yang dibatasi oleh $x = y^2, y = 1$ dan $x = 0$ diputar pada sumbu-x, serta sketsa daerahnya. (EAS 2022/2023, Senin 12 Juni 2023)

Jawab

$x = y^2,\ y = 1,\ x = 0$ ↻ sb x

① Tipot

$x_1 = x_2$

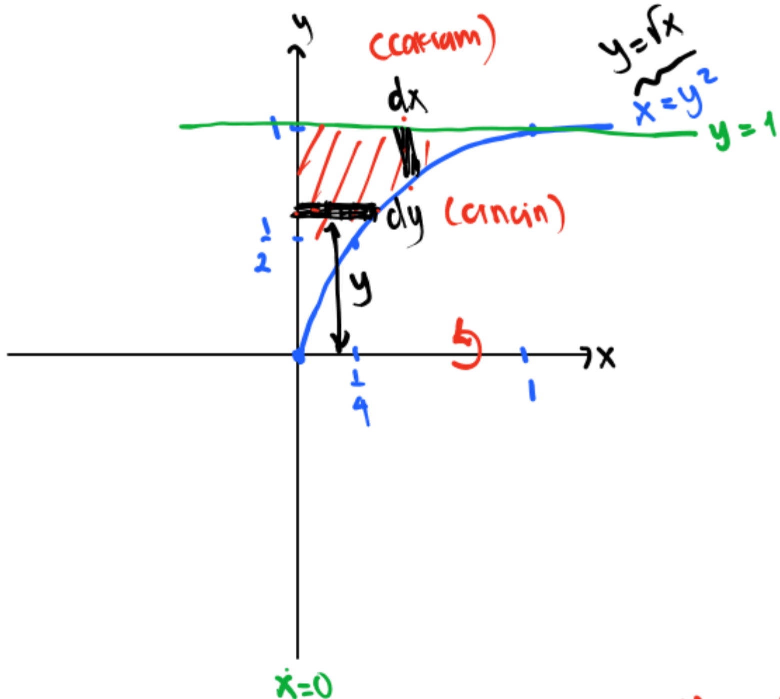
$y^2 = 0$

$y = 0$

② Gambar

$x = y^2$

x	y
0	0
$\frac{1}{4}$	$\frac{1}{2}$
1	1



③ Volume

Cincin //

jarak partisi ke sb putar k.panan - k.kiri

$$dV = 2\pi (y)(y^2 - 0)\, dy$$

Cakram ⊥

jarak terjauh dri sb putar

jarak terdekat dgn sb putar

$$dV = \pi \left(1^2 - (\sqrt{x})^2\right) dx$$

$$V = \int_0^1 2\pi (y)(y^2 - 0)\, dy$$

$$= \int_0^1 2\pi y^3\, dy$$

$$= \frac{2\pi}{4} y^4 \Big|_0^1$$

$$= \frac{2\pi}{4} - 0$$

$$= \frac{\pi}{2} \text{ s.v}$$

$$V = \int_0^1 \pi \left(1^2 - (\sqrt{x})^2\right) dx$$

$$= \int_0^1 \pi (1 - x)\, dx$$

$$= \int_0^1 \pi - \pi x \, dx$$

$$= \pi x - \frac{\pi}{2} x^2 \Big|_0^1$$

$$= \pi - \frac{\pi}{2} - 0$$

$$= \frac{\pi}{2} \text{ SV}$$

2. Use Guldine's Theorem to find the volume of the solid generated when the region enclosed by $y = x, y = 2 - x$, and $x = 0$ is revolved about the line $y = x - 3$. *(EAS 2022/2023, Senin 12 Juni 2023)*

$$-x + y + 3 = 0$$

Volume Guldin

$$V = 2\pi \cdot d \cdot L$$

Luas

d → jarak titik berat ke sb putar

① Gambar

- Tipot
- $y = x$
- $y = 2 - x$
- $y = x - 3$

$y_1 = y_2$

$x = 2-x$

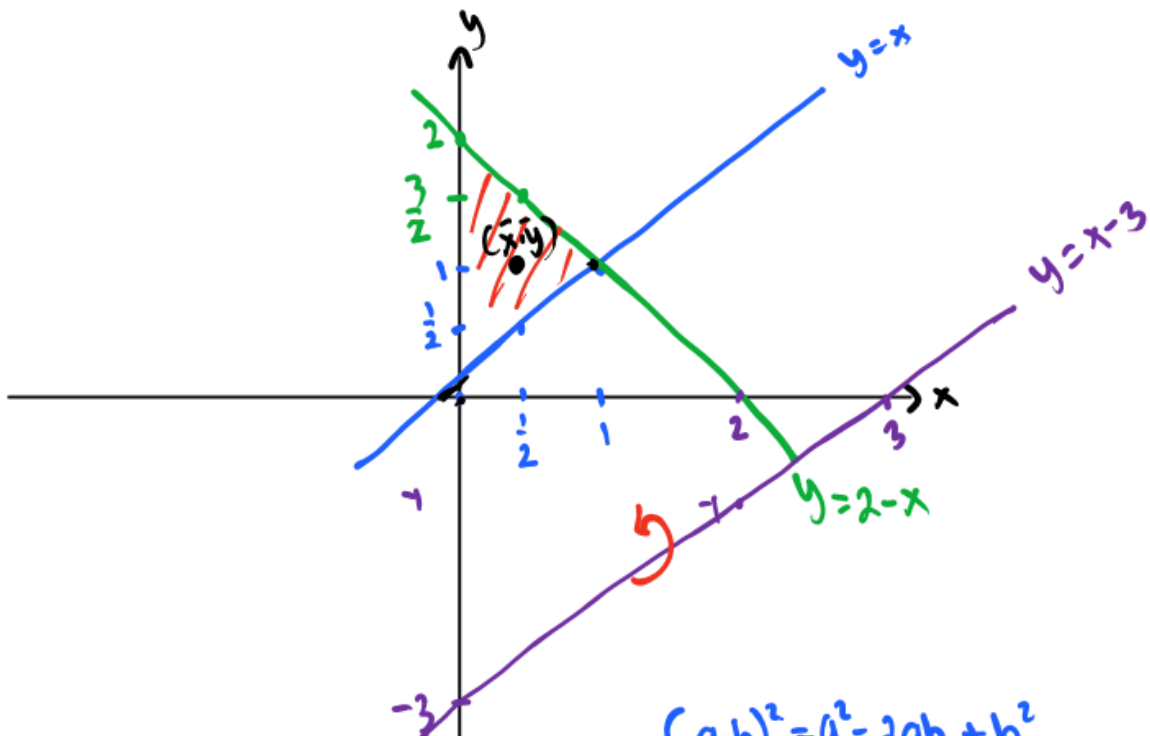
$2x = 2$

$x = 1$

x	y
0	0
$\frac{1}{2}$	$\frac{1}{2}$
1	1

x	y
0	2
$\frac{1}{2}$	$\frac{3}{2}$
1	1

x	y
0	-3
2	-1
3	0



$(a-b)^2 = a^2 - 2ab + b^2$

② Titik berat

$\bar{x} = \frac{M_y}{M}$, $\bar{y} = \frac{M_x}{M}$

- $M = \int_a^b y_1 - y_2 \, dx$

  $= \int_0^1 2-x-x \, dx$

- $M_x = \frac{1}{2} \int_a^b y_1^2 - y_2^2 \, dx$

  $= \frac{1}{2} \int_0^1 (2-x)^2 - (x)^2 \, dx$

$= \int_0^1 2-2x\, dx$

$= 2x - x^2 \Big|_0^1$

$= 2-1-0$

$= 1$

- $M_y = \int_a^b x(y_1-y_2)\, dx$

$= \int_0^1 x(2-\underbrace{x-x}_{-2x})\, dx$

$= \int_0^1 2x - 2x^2\, dx$

$= x^2 - \frac{2}{3}x^3 \Big|_0^1$

$= 1 - \frac{2}{3} - 0$

$= \frac{1}{3}$

$= \frac{1}{2}\int_0^1 4-4x+x^2-x^2\, dx$

$= \frac{1}{2}\int_0^1 4-4x\, dx$

$= \int_0^1 2-2x\, dx$

$= 2x - x^2 \Big|_0^1$

$= 2-1-0$

$= 1$

jadi, $\bar{x} = \frac{M_y}{M} = \frac{(\frac{1}{3})}{1} = \frac{1}{3}$; $\bar{y} = \frac{M_x}{M} = \frac{1}{1} = 1$

Titik berat $(\bar{x}, \bar{y}) = (\frac{1}{3}, 1)$

③ d

jarak titik berat $(\bar{x}, \bar{y})$ ke garis $Ax + By + C = 0$

$$d = \frac{|A\bar{x} + B\bar{y} + C|}{\sqrt{A^2 + B^2}}$$

jarak titik berat $(\bar{x}, \bar{y}) = (\frac{1}{3}, 1)$ ke garis $-x + y + 3 = 0$

$A = -1,\ B = 1,\ C = 3$

$$d = \frac{|-\frac{1}{3} + 1 + 3|}{\sqrt{(-1)^2 + (1)^2}} = \frac{|\frac{11}{3}|}{\sqrt{2}} = \frac{11}{3\sqrt{2}}$$

④ Luas

$L = M = 1$

⑤ Volume

$V = 2\pi \cdot d \cdot L$

$= 2\pi \cdot \frac{11}{3\sqrt{2}} \cdot 1$

$= \frac{22\pi}{3\sqrt{2}}$ S. volume.

3. Dapatkan panjang busur kurva astroida $x = 5\cos^3 t, y = 5\sin^3 t$ untuk $0 \le t \le \frac{\pi}{2}$. Serta sketsa grafik tersebut. *(EAS 2022/2023, Selasa 13 Juni 2023)*

jawab

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$S = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

① $\frac{dx}{dt}$ dan $\frac{dy}{dt}$

$$x = 5\cos^3 t$$

$$\frac{dx}{dt} = 15\cos^2 t \cdot (-\sin t) = -15 \sin t \cos^2 t$$

$$y = 5\sin^3 t$$

$$\frac{dy}{dt} = 15\sin^2 t \cdot (\cos t) = 15 \cos t \sin^2 t$$

② S (Panjang busur)

$$S = \int_0^{\pi/2} \sqrt{(-15 \sin t \cos^2 t)^2 + (15 \cos t \sin^2 t)^2}\, dt$$

$$= \int_0^{\pi/2} \sqrt{225 \sin^2 t \cos^4 t + 225 \cos^2 t \sin^4 t}\ dt$$

$$= \int_0^{\pi/2} \sqrt{225 \sin^2 t \cos^2 t \,(\underbrace{\cos^2 t + \sin^2 t}_{1})}\ dt$$

$$= \int_0^{\pi/2} 15 \sin t \cos t \ dt$$

Misal

$u = \sin t$

$du = \cos t\, dt$

Batas

$t = 0 \rightarrow u = 0$

$t = \frac{\pi}{2} \rightarrow u = 1$

$$S = \int_0^1 15 u\, du$$

$$= \frac{15}{2} u^2 \Big|_0^1$$

$$= \frac{15}{2} \text{ s. panjang.}$$

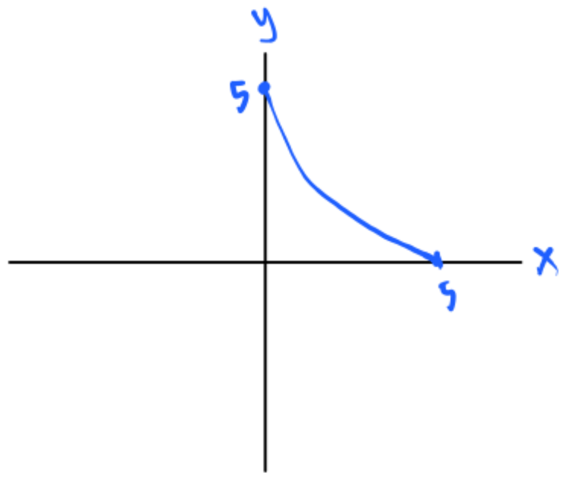
③ Sketsa grafik

$x = 5\cos^3 t$

$y = 5\sin^3 t$; $0 \leq t \leq \frac{\pi}{2}$

$t = 0 \longrightarrow x = 5\cos^3 0 = 5$
$\qquad y = 5 \sin^3 0 = 0$ $\quad (5, 0)$

$t = \frac{\pi}{2} \rightarrow x = 5\cos^3 \frac{\pi}{2} = 0$
$\qquad y = 5\sin^3 \frac{\pi}{2} = 5$ $\quad (0, 5)$



4. Sketsa grafik daerah di dalam kurva kutub $r = 6\sin\theta$ dan diluar kurva kutub $r = 2 + 2\sin\theta$, selanjutnya hitung luas daerah tersebut. *(EAS 2021/2022, Rabu 8 Juni 2022)*

Jawab.

① Tipot

$r_1 = r_2$

$6\sin\theta = 2 + 2\sin\theta$

$4\sin\theta = 2$

$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

② Gambar

$r = 2a\sin\theta$

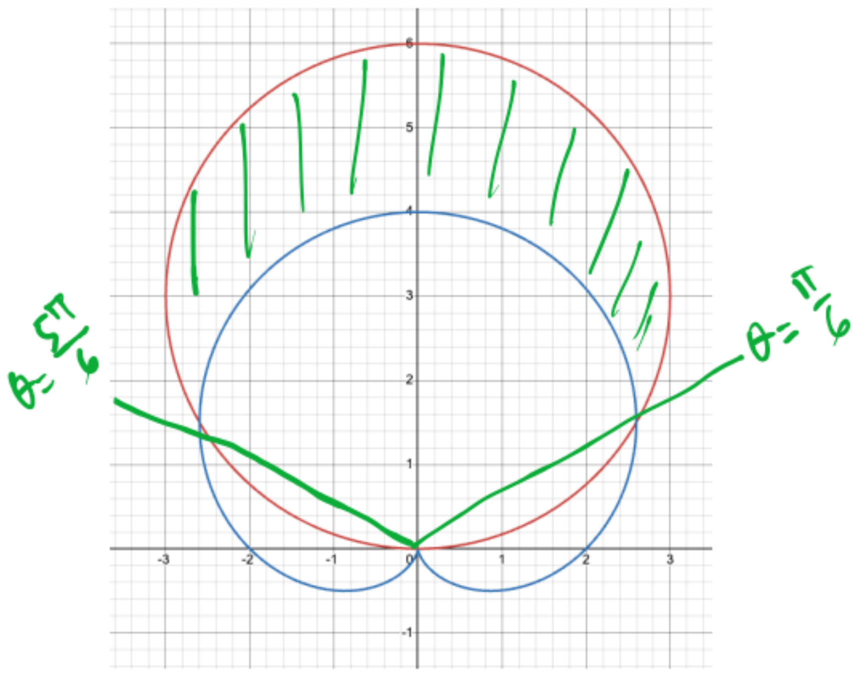
Lingkaran $P(0,a)$ dan $r = a$

• $r = 6\sin\theta$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$3\pi/6$	π
r	0	3	$3\sqrt{2}$	$3\sqrt{3}$	6	$3\sqrt{3}$	$3\sqrt{2}$	3	0

• $r = 2 + 2\sin\theta$

θ	0	$\pi/2$	π	$3\pi/2$	2π
r	2	4	2	0	2



③ Luas.

$$dL = \frac{1}{2}(r_1^2 - r_2)^2 \, d\theta$$

$$dL = \frac{1}{2}\left((6\sin\theta)^2 - (2+2\sin\theta)^2\right) d\theta$$

$$L = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{2}\left((6\sin\theta)^2 - (2+2\sin\theta)^2\right) d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{6}}^{5\pi/6} 36\sin^2\theta - (4+8\sin\theta + 4\sin^2\theta)\, d\theta$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax + C$$

$$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} 32\sin^2\theta - 8\sin\theta - 4\, d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} 32\left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right) - 8\sin\theta - 4\, d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{6}}^{5\pi/6} 12 - 16\cos 2\theta - 8\sin\theta\, d\theta$$

$$= \int_{\pi/6}^{5\pi/6} 6 - 8\cos 2\theta - 4\sin\theta\, d\theta$$

$$= \left[6\theta - \frac{8}{2}\sin 2\theta + 4\cos\theta\right]\Big|_{\pi/6}^{5\pi/6}$$

$= \ldots$ satuan luas.

5. Given a function $f(x) = \frac{1}{x^2}$. *(EAS 2022/2023, Senin 12 Juni 2023)*
   (a) Find the Taylor polynomial of order 4 about $x = 1$.
   (b) Find the Taylor series about $x = 1$. Express your answer as sigma notation.

① Polinomial Taylor derajat-n di $x=a$

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

② Deret taylor dan Notasi sigma

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots$$

Jawab

5a. $f(x) = \frac{1}{x^2} = x^{-2}$ $\rightarrow f(1) = \frac{1}{1^2} = 1 = 1!$

$f'(x) = -2x^{-3}$ $\rightarrow f'(1) = -2 = -(2!)$

$f''(x) = 6x^{-4}$ $\rightarrow f''(1) = 6 = 3!$

$f'''(x) = -24x^{-5}$ $\rightarrow f'''(1) = -24 = -(4!)$

$f^{iv}(x) = 120x^{-6}$ $\rightarrow f^{iv}(1) = 120 = 5!$

Jadi,

$$P_4(x) = 1 - 2(x-1) + \frac{6}{2!}(x-1)^2 - \frac{24}{3!}(x-1)^3 + \frac{120}{4!}(x-1)^4$$

$$= 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + 5(x-1)^4$$

5b Deret taylor dan Notasi sigma

$$f^{(k)}(1) = (-1)^k (k+1)!$$

Deret

$$\frac{1}{x^2} = 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + 5(x-1)^4 + \dots + (-1)^k (k+1)! (x-1)^k + \dots$$

Notasi sigma

$$\frac{1}{x^2} = \sum_{k=0}^{\infty} (-1)^k (k+1)! (x-1)^k$$