Pertemuan 3

Sunday, 09 June 2024
19.36

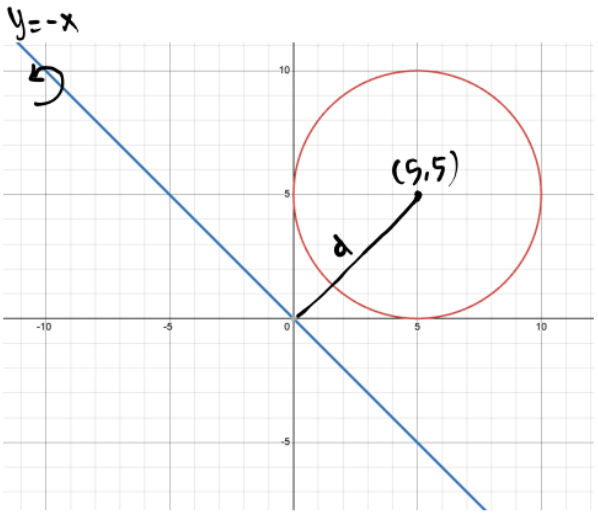
1. Hitung luas permukaan benda putar dari $y^2 - 10x + x^2 - 10y + 25 = 0$ diputar terhadap garis $y = -x$. *(EAS 2020/2021, Rabu 30 Juni 2021)*

Dari Guldin II

$K = 2\pi d \cdot S$

K = luas permukaan
d = jarak titik berat busur ke sb putar
S = Panjang busu

$y^2 - 10x + x^2 - 10y + 25 = 0$
$x^2 - 10x + y^2 - 10y + 25 = 0$
$(x-5)^2 - 25 + (y-5)^2 - 25 + 25 = 0$
$(x-5)^2 + (y-5)^2 = 25$
Lingkaran $P(5,5)$, $r = 5$
diputar $y = -x \rightarrow x + y = 0$



Titik berat busur di $(\bar{x}, \bar{y}) = (5,5)$

(i) jarak titik berat $(\bar{x}, \bar{y})$ ke sb putar $ax + by + c = 0$

$$d = \frac{|a\bar{x} + b\bar{y} + c|}{\sqrt{a^2 + b^2}}$$

jarak titik berat $(\bar{x}, \bar{y}) = (5, 5)$ ke sb putar $y = -x$ $\quad x + y = 0$

$a = 1, \; b = 1, \; c = 0$

$$d = \frac{|5 + 5 + 0|}{\sqrt{1^2 + 1^2}}$$

$$= \frac{10}{\sqrt{2}}$$

(ii) panjang busur lingkaran

$$S = 2\pi r$$

$$= 2\pi \cdot 5$$

$$= 10\pi$$

(iii) $K = 2\pi \cdot d \cdot S$

$$= 2\pi \cdot \frac{10}{\sqrt{2}} \cdot 10\pi$$

$$= \frac{200\pi^2}{\sqrt{2}}$$

2. Diberikan daerah yang dibatasi oleh kurva $y = x^2$ dan $y = 4$. *(EAS 2021/2022, Rabu 8 Juni 2022)*
   (a) Sketsa daerah tersebut
   (b) Dapatkan titik berat daerah tersebut
   (c) Dapatkan volume daerah tersebut jika diputar terhadap garis $y = 8 - 2x$.

Jawab :

2a. $y = x^2$, $y = 4$

Titik potong

$y_1 = y_2$

$x^2 = 4$

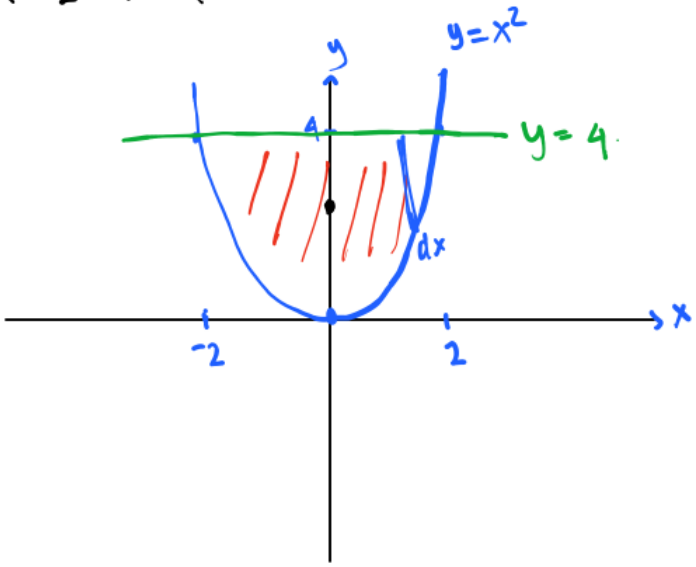
$x^2 - 4 = 0$

$(x+2)(x-2) = 0$

$x = -2 \vee x = 2$

• $y = x^2$

x	y
-2	4
0	0
2	4



2b. Titik berat

$$\bar{x} = \frac{M_y}{M} \quad ; \quad \bar{y} = \frac{M_x}{M}$$

(i) $M = \int_a^b y_1 - y_2 \, dx$

$= \int_{-2}^{2} 4 - x^2 \, dx$

$= 4x - \frac{1}{3}x^3 \Big|_{-2}^{2}$

$= \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right)$

$= 16 - \frac{16}{3}$

$= \frac{32}{3}$

(ii) $M_y = \int_a^b x(y_1 - y_2) \, dx$

$= \int_{-2}^{2} x(4 - x^2) \, dx$

$= \int_{-2}^{2} 4x - x^3 \, dx$

$= 2x^2 - \frac{1}{4}x^4 \Big|_{-2}^{2}$

$= [8 - 8] - [8 - 8]$

$= 0$

(iii) $M_x = \frac{1}{2} \int_a^b y_1^2 - y_2^2 \, dx$

$= \frac{1}{2} \int_{-2}^{2} 4^2 - (x^2)^2 \, dx$

$= \frac{1}{2} \int_{-2}^{2} 16 - x^4 \, dx$

$= \frac{1}{2} \left[16x - \frac{1}{5}x^5 \right]\Big|_{-2}^{2}$

$= \frac{1}{2} \left[32 - \frac{32}{5} \right] - \frac{1}{2} \left[-32 + \frac{32}{5} \right]$

$= 16 - \frac{16}{5} + 16 - \frac{16}{5}$

$= 32 - \frac{32}{5}$

$= \frac{128}{5}$

Jadi,

$$\bar{x} = \frac{M_y}{M} = \frac{0}{\left(\frac{32}{3}\right)} = 0$$

$$\bar{y} = \frac{M_x}{M} = \frac{\left(\frac{12}{5}\right)}{\left(\frac{32}{3}\right)} = \frac{128}{5} \times \frac{3}{32}$$

$$= \frac{12}{5}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{12}{5}\right)$$

2c. Volume yg diputar di sb $y = 8 - 2x$

$2x + y - 8 = 0$

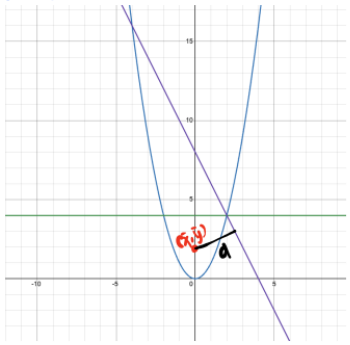
$V = 2\pi \cdot d \cdot L$

d = jarak titik berat ke sb putar

L = Luas.

$y = 8 - 2x$

x	y
0	8
-1	10
1	6
2	4



(i) $L = M$

$= \frac{32}{3}$

(ii) jarak titik berat $(\bar{x}, \bar{y}) = \left(0, \frac{12}{5}\right)$ kesb putar $2x + y - 8 = 0$

$a = 2, b = 1, c = -8$

$$d = \frac{|a\bar{x} + b\bar{y} + c|}{\sqrt{a^2 + b^2}}$$

$$= \frac{\left|0 + \frac{12}{5} + (-8)\right|}{\sqrt{2^2 + 1^2}}$$

$$= \frac{\left|\frac{13}{5} - \frac{40}{5}\right|}{\sqrt{5}}$$

$$= \frac{\left|\frac{-28}{5}\right|}{\sqrt{5}}$$

$$= \frac{28}{5} \times \frac{1}{\sqrt{5}} = \frac{28}{5\sqrt{5}}$$

(iii) $V = 2\pi \cdot d \cdot L$

$$= 2\pi \cdot \frac{28}{5\sqrt{5}} \cdot \frac{32}{3}$$

= ... Satuan Volume.

2. Given the region between the curves $y = -x^2 + 4$ and x-axis. *(EAS 2021/2022, Rabu 8 Juni 2022)*
   (a) Sketch the graph of the region
   (b) Find the centroid of the region
   (c) Find the volume that is generated by revolving the region about a line $x = 3$

$y=0$

2a. $y = -x^2 + 4$ dan $y = 0$

Titik

$y = -x^2 + 4$

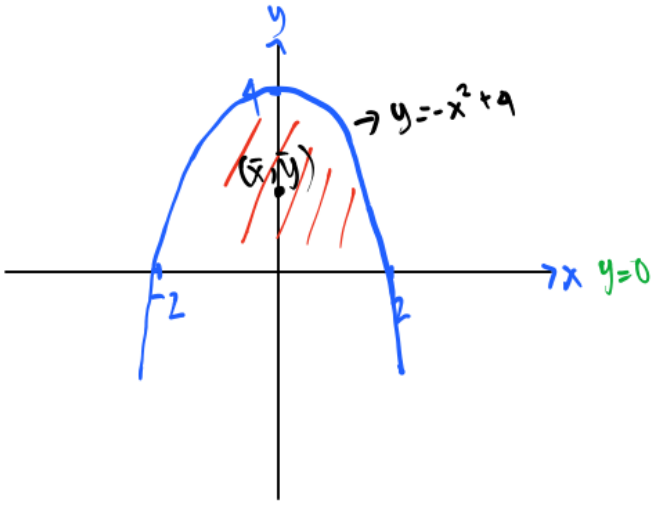
$y_1 = y_2$

$-x^2 + 4 = 0$

$(-x+2)(x+2) = 0$

$x = -2 \vee x = 2$

x	y
-2	0
0	4
2	0



2b. Titik berat

$$\bar{x} = \frac{My}{M} \; ; \; \bar{y} = \frac{Mx}{M}$$

- $\bar{x} = 0$ (karena simetri thdp $x = 0$)
- $M = \int_a^b y \, dx$

  $= \int_{-2}^{2} -x^2 + 4 \, dx$

$$= -\frac{1}{3}x^3 + 4x \Big|_{-2}^{2}$$

$$= \left[\frac{-8}{3} + 8\right] - \left[\frac{8}{3} - 8\right]$$

$$= \frac{-8}{3} + 8 - \frac{8}{3} + 8$$

$$= 16 - \frac{16}{3}$$

$$= \frac{32}{3} \qquad (a+b)^2 = a^2 + 2ab + b^2$$

- $M_x = \frac{1}{2}\int_a^b y^2\,dx$

$$= \frac{1}{2}\int_{-2}^{2} (-x^2+4)^2\,dx$$

$$= \frac{1}{2}\int_{-2}^{2} x^4 - 8x^2 + 16\,dx$$

$$= \frac{1}{2}\left[\frac{1}{5}x^5 - \frac{8}{3}x^3 + 16x\right]\Big|_{-2}^{2}$$

$$= \frac{1}{2}\left[\frac{32}{5} - \frac{64}{3} + 32\right] - \frac{1}{2}\left[\frac{-32}{5} + \frac{64}{3} - 32\right]$$

$$= \frac{16}{5} - \frac{32}{3} + 16 + \frac{16}{5} - \frac{32}{3} + 16$$

$$= 32 + \frac{32}{5} - \frac{64}{3}$$

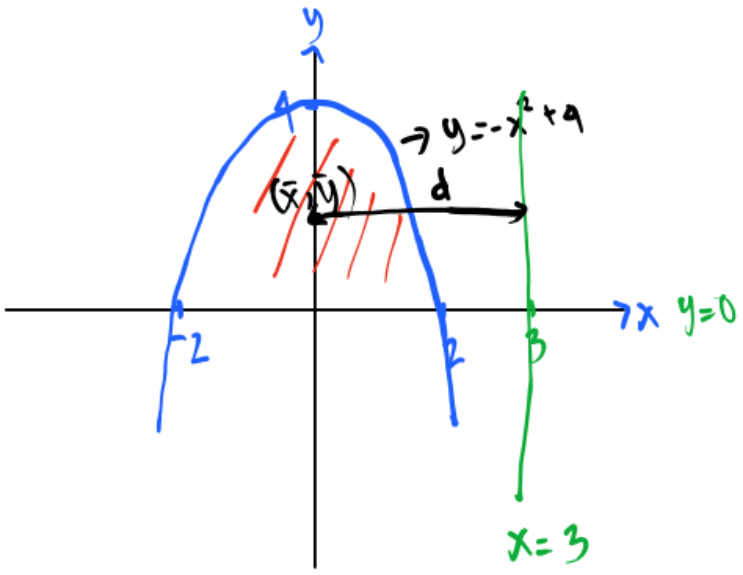
$$= \frac{256}{15}$$

Jadi, $\bar{x} = 0$

$$\bar{y} = \frac{Mx}{M} = \frac{256/15}{32/3} = \frac{\cancel{256}^{8}}{\cancel{15}_{5}} \times \frac{\cancel{3}}{\cancel{32}} = \frac{8}{5}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{8}{5}\right)$$

2c. Volume $\circlearrowleft$ $x = 3$



(i) $L = M = \frac{32}{3}$

(ii) $d = 3$

(iii) $V = 2\pi \cdot d \cdot L$

$= 2\pi \cdot 3 \cdot \frac{32}{3}$

$= 64\pi$ satuan volume.

2. Diberikan daerah yang dibatasi oleh kurva $y = \sqrt{16 - x^2}$ dan sumbu-x. *(EAS 2021/2022, Rabu 8 Juni 2022)*
   (a) Sketsa daerah tersebut
   (b) Dapatkan titik berat daerah tersebut
   (c) Dapatkan volume daerah tersebut jika diputar terhadap garis $y = x - 5$.

$y = 0$

2a. $y = \sqrt{16 - x^2}$, $y = 0$

Tipot

$y_1 = y_2$

$\sqrt{16 - x^2} = 0$

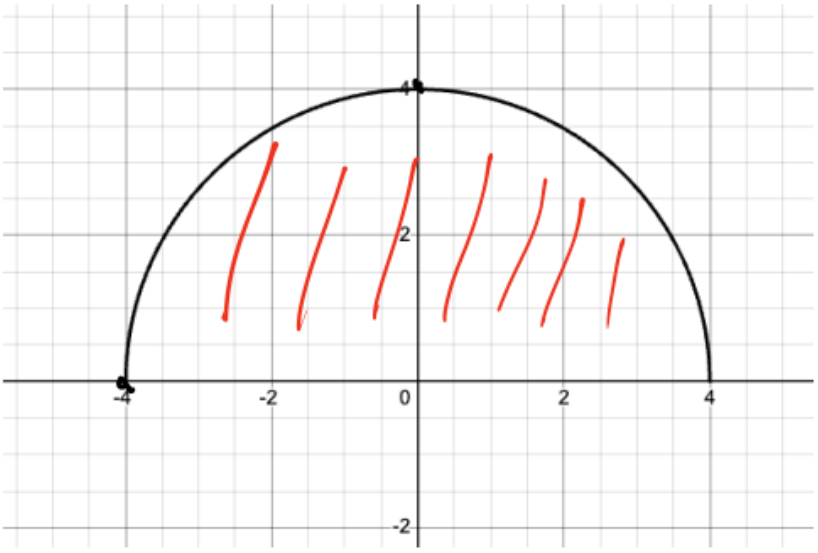
$16 - x^2 = 0$

$(4 - x)(4 + x) = 0$

$x = -4 \vee x = 4$

$y = \sqrt{16 - x^2}$

x	y
-4	0
0	4
4	0



$L\Omega = \pi r^2 \cdot \frac{1}{2}$

$= \pi (4)^2 \cdot \frac{1}{2}$

$= 8\pi$

2b. Titik berat

$\bar{x} = \frac{My}{M}$; $\bar{y} = \frac{Mx}{M}$

- $\bar{x} = 0$ (karena simetri dgn sb x=0)
- $M = \int_a^b y \, dx$

$= \int_{-4}^{4} \sqrt{16 - x^2} \, dx$

($16 = 4^2$)

Misal

$x = 4 \sin\theta$

$dx = 4\cos\theta \, d\theta$

Batas

$x = -4 \rightarrow -4 = 4 \sin\theta$

$-1 = \sin\theta$

$\sin -\alpha = -\sin\alpha$

$x = \quad \rightarrow 4 = 4\sin\theta$

$\theta = -\frac{\pi}{2}$

$1 = \sin\theta$

$\theta = \frac{\pi}{2}$

$$M = \int_{-\frac{\pi}{2}}^{\pi/2} \sqrt{16 - 16\sin^2\theta}\ 4\cos\theta\, d\theta$$

$16(1-\sin^2\theta)$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{16\cos^2\theta} \cdot 4\cos\theta\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 16\cos^2\theta\, d\theta \qquad \cos^2\theta = \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta\right)$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 16\left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta\right) d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 8 + 8\cos 2\theta\, d\theta \qquad \int \cos a\theta\, d\theta = \tfrac{1}{a}\sin a\theta + C$$

$$= 8\theta + 4\sin 2\theta \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$-4(\sin\pi)$

$$= \left[4\pi + 4\sin\pi\right] - \left[-4\pi + 4\sin -\pi\right]$$

$= 4\pi + 0 + 4\pi + 0$

$= 8\pi$

- $M_x = \frac{1}{2}\int_a^b y^2\,dx$

$= \frac{1}{2}\int_{-4}^{4} \left(\sqrt{16-x^2}\right)^2 dx$

$= \frac{1}{2}\int_{-4}^{4} 16 - x^2\,dx$

$= \frac{1}{2}\left[16x - \frac{1}{3}x^3\right]\Big|_{-4}^{4}$

$= \frac{1}{2}\left[64 - \frac{64}{3}\right) - \frac{1}{2}\left[-64 + \frac{64}{3}\right]$

$= 32 - \frac{32}{3} + 32 - \frac{32}{3}$

$= 64 - \frac{64}{3}$

$= \frac{128}{3}$

jadi, $\bar{x} = 0$

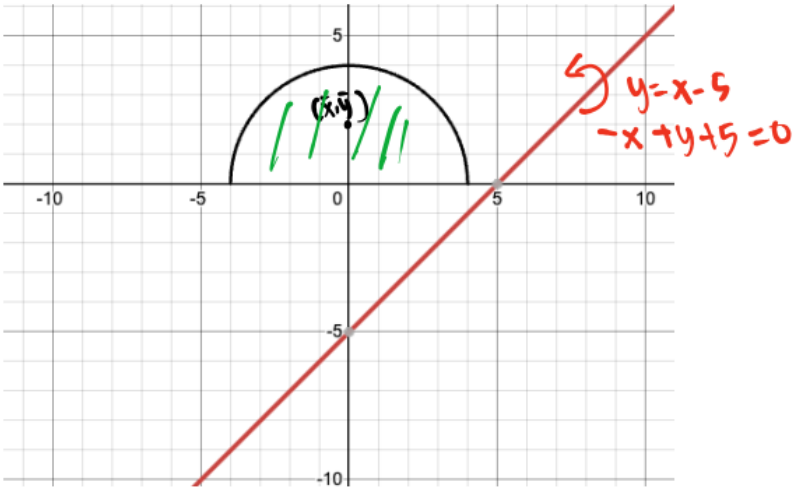
$$\bar{y} = \frac{My}{M} = \frac{128/3}{8\pi} = \frac{16}{\frac{128}{3}} \times \frac{1}{8\pi} = \frac{16}{3\pi}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{16}{3\pi}\right)$$

2c. Volume S $y = x - 5$

$y = x - 5$

x	y
0	-5
1	-4
4	-1



(i) $L = M = 8\pi$

(ii) jarak titik berat $(\bar{x}, \bar{y}) = \left(0, \frac{16\pi}{3}\right)$ ke $-x + y + 5 = 0$

$a = -1, b = 1, c = 5$

$$d = \frac{|a\bar{x} + b\bar{y} + c|}{\sqrt{a^2 + b^2}}$$

$$= \frac{\left| 0 + \frac{16\pi}{3} + 5 \right|}{\sqrt{(-1)^2 + (1)^2}}$$

$$= \frac{\frac{16\pi}{3} + 5}{\sqrt{2}} = \frac{16\pi + 15}{3} \times \frac{1}{\sqrt{2}}$$

$$= \frac{16\pi + 15}{3\sqrt{2}}$$

(iii) $V = 2\pi \cdot d \cdot L$

$= 2\pi \cdot \left(\frac{16\pi + 15}{3\sqrt{2}} \right) \cdot 8\pi$

$=$ .. Satuan Volume