Pertemuan 6 - Juni

Wednesday, 12 June 2024
19.32

3. Buatlah sketsa dan dapatkan panjang kurva yang dibentuk oleh kurva: *(EAS 2020/2021, Rabu 30 Juni 2021)*

$$r = \frac{2a}{1+\cos\theta} \text{ dan } r = 2a(1+\cos\theta) \text{ di } 0 \le \theta \le \frac{\pi}{2}$$

$x^2+y^2 = r^2$

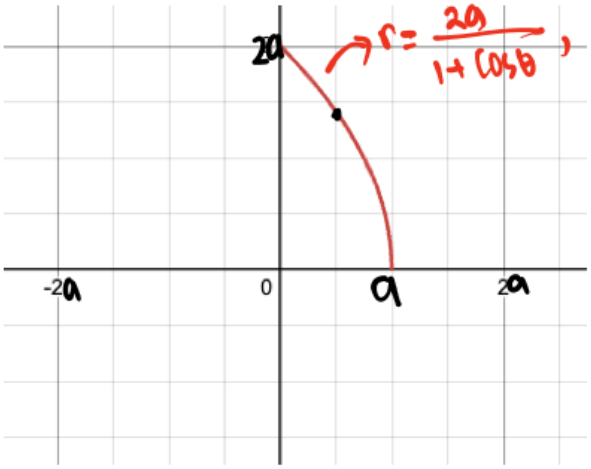
$x = r\cos\theta$

$y = r\sin\theta$

$(a-b)^2 = a^2 - 2ab + b^2$

3a. $r = \frac{2a}{1+\cos\theta}$ di $0 \le \theta \le \frac{\pi}{2}$

θ	0	$\frac{\pi}{3}$	$\pi/2$
r	a	$4a/3$	$2a$



$r = \frac{2a}{1+\cos\theta}$, $0 \le \theta \le \frac{\pi}{2}$

Panjang kurva

$r = \frac{2a}{1+\cos\theta}$

Misal

$u = 2a \rightarrow u' = 0$

$v = 1+\cos\theta \rightarrow v' = -\sin\theta$

$r = u/v$

$\frac{dr}{d\theta} = \frac{u'v - uv'}{v^2}$

$= \frac{0 + 2a\sin\theta}{(1+\cos\theta)^2} = \frac{2a\sin\theta}{(1+\cos\theta)^2}$

$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

$S = \int_0^{\frac{\pi}{2}} \sqrt{\left(\frac{2a}{1+\cos\theta}\right)^2 + \left(\frac{2a\sin\theta}{(1+\cos\theta)^2}\right)^2}\, d\theta$

$= \int_0^{\frac{\pi}{2}} \sqrt{\frac{4a^2}{(1+\cos\theta)^2} + \frac{4a^2\sin^2\theta}{(1+\cos\theta)^4}}\, d\theta$

$= \int_0^{\frac{\pi}{2}} \sqrt{\frac{4a^2(1+\cos\theta)^2 + 4a^2\sin^2\theta}{(1+\cos\theta)^4}}\, d\theta$

$= \int_0^{\pi/2} \frac{1}{(1+\cos\theta)^2} \sqrt{4a^2(1+2\cos\theta+\cos^2\theta) + 4a^2\sin^2\theta}\, d\theta$

$= \int_0^{\pi/2} \frac{1}{(1+\cos\theta)^2} \sqrt{4a^2 + 8a^2\cos\theta + 4a^2\cos^2\theta + 4a^2\sin^2\theta}\, d\theta$

$= \int_0^{\pi/2} \frac{1}{(1+\cos\theta)^2} \sqrt{4a^2 + 8a^2\cos\theta + 4a^2(\cos^2\theta + \sin^2\theta)}\, d\theta$

$= \int_0^{\pi/2} \frac{1}{(1+\cos\theta)^2} \sqrt{8a^2(1-\cos\theta)}\, d\theta$

$= \int_0^{\pi/2} \frac{1}{(1+\cos\theta)^2} \sqrt{8a^2 \cdot 2\sin^2\frac{1}{2}\theta}\, d\theta$

$$= \int_0^{\pi/2} \frac{1}{(1+\cos\theta)^2} \, 4a \sin\tfrac{1}{2}\theta \, d\theta$$

$$= \int_0^{\pi/2} \frac{1}{(2\cos^2\frac{1}{2}\theta)^2} \, 4a \sin\tfrac{1}{2}\theta \, d\theta$$

Misal

$u = \cos\frac{1}{2}\theta$

$du = -\frac{1}{2}\sin\frac{1}{2}\theta \, d\theta$

$-2du = \sin\frac{1}{2}\theta \, d\theta$

Batas

$\theta = 0 \rightarrow u = 1$

$\theta = \frac{\pi}{2} \rightarrow u = \frac{1}{2}\sqrt{2}$

$$S = \int_1^{\frac{1}{2}\sqrt{2}} \frac{1}{(2u)^2} \, 4a \, (-2du)$$

$$= \int_1^{\frac{1}{2}\sqrt{2}} \frac{-8a}{4u^2} \, du$$

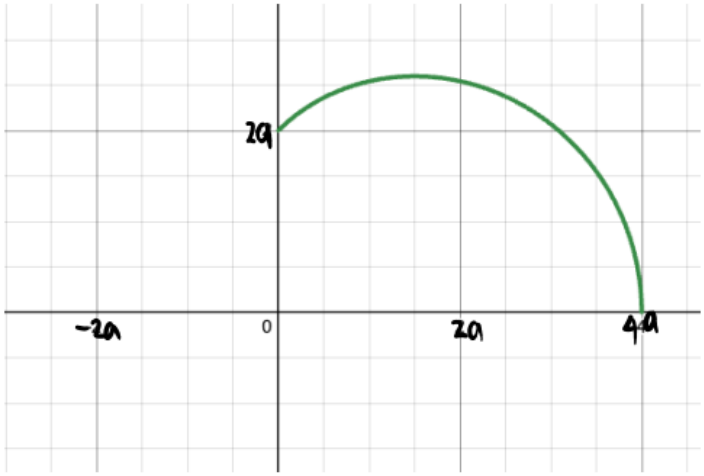
$$= \int_1^{\frac{1}{2}\sqrt{2}} -au^{-2} \, du$$

$$= \frac{a}{u} \Big|_1^{\frac{1}{2}\sqrt{2}}$$

$$= \frac{a}{\frac{1}{2}\sqrt{2}} - a = \cdots \text{ satuan panjang.}$$

3b. $r = 2a(1+\cos\theta)$, $0 \le \theta \le \frac{\pi}{2}$

θ	0	$\frac{\pi}{3}$	$\pi/2$
r	$4a$	$3a$	$2a$



Panjang busur

$r = 2a(1+\cos\theta) = 2a + 2a\cos\theta$

$\frac{dr}{d\theta} = -2a\sin\theta$ $\qquad (a+b)^2 = a^2 + 2ab + b^2$

$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

$S = \int_0^{\frac{\pi}{2}} \sqrt{(2a+2a\cos\theta)^2 + (-2a\sin\theta)^2}\, d\theta$

$$= \int_0^{\pi/2} \sqrt{4a^2 + 8a^2\cos\theta + 4a^2\cos^2\theta + 4a^2\sin^2\theta}\, d\theta$$

$$= \int_0^{\pi/2} \sqrt{4a^2 + 8a^2\cos\theta + 4a^2(\cos^2\theta + \sin^2\theta)}\, d\theta$$

$$= \int_0^{\pi/2} \sqrt{8a^2 + 8a^2\cos\theta}\, d\theta$$

$$= \int_0^{\pi/2} \sqrt{8a^2(1+\cos\theta)}\, d\theta$$

$$= \int_0^{\pi/2} \sqrt{8a^2(2\cos^2\tfrac{1}{2}\theta)}\, d\theta$$

$$= \int_0^{\pi/2} \sqrt{16a^2\cos^2\tfrac{1}{2}\theta}\, d\theta$$

$$= \int_0^{\pi/2} 4a\cos\tfrac{1}{2}\theta\, d\theta$$

$$= 4a \cdot 2\sin\tfrac{1}{2}\theta \Big|_0^{\pi/2}$$

$$= [8a\sin\tfrac{\pi}{4}) - [8a\sin 0)$$

$$= 8a \cdot \tfrac{1}{2}\sqrt{2} \text{ satuan panjang}$$

$\frac{1}{2} + \frac{1}{2}\cos\theta = \cos^2\frac{1}{2}\theta$

$1 + \cos\theta = 2\cos^2\frac{1}{2}\theta$

$\int \cos ax\, dx = \frac{1}{a}\sin ax + C$

3. Diberikan kurva kutub $r = 2(1 + \cos\theta), 0 \le \theta \le 2\pi$. *(EAS 2021/2022, Rabu 8 Juni 2022)*
   (a) Dapatkan kemiringan garis singgung pada kurva kutub tersebut di titik $\theta = \frac{\pi}{3}$.
   (b) Dapatkan semua titik (r, θ) pada kurva kutub tersebut dimana garis singgungnya vertikal.

$x = r\cos\theta \rightarrow \frac{dx}{d\theta} = \cos\theta \frac{dr}{d\theta} + r(-\sin\theta)$

$\frac{dx}{d\theta} = -r\sin\theta + \cos\theta \frac{dr}{d\theta}$

$y = r\sin\theta \rightarrow \frac{dy}{d\theta} = \sin\theta \frac{dr}{d\theta} + r\cos\theta$

$= r\cos\theta + \sin\theta \frac{dr}{d\theta}$

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r\cos\theta + \sin\theta \, dr/d\theta}{-r\sin\theta + \cos\theta \, dr/d\theta}$

3a. Kemiringan di $\theta = \frac{\pi}{3}$

$r = 2 + 2\cos\theta$

- $\frac{dr}{d\theta} = -2\sin\theta$
- $\frac{dy}{d\theta} = r\cos\theta + \sin\theta \frac{dr}{d\theta}$

$= (2 + 2\cos\theta)\cos\theta + \sin\theta \cdot -2\sin\theta$

$= 2\cos\theta + 2\cos^2\theta - 2\sin^2\theta$

$= 2\cos\theta + 2(\cos^2\theta - \sin^2\theta)$

$= 2\cos\theta + 2\cos 2\theta$

- $\frac{dx}{d\theta} = -r\sin\theta + \cos\theta \frac{dr}{d\theta}$

$= -(2+2\cos\theta)\sin\theta + \cos\theta \cdot -2\sin\theta$

$= -2\sin\theta - 2\cos\theta\sin\theta - 2\cos\theta\sin\theta$

$= -2\sin\theta - 4\cos\theta\sin\theta$

- $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$

$= \frac{2\cos\theta + 2\cos 2\theta}{-2\sin\theta - 4\cos\theta\sin\theta}$

- $m = \frac{dy}{dx}\Big|_{\theta=\frac{\pi}{3}} = \frac{2\cos\frac{\pi}{3} + 2\cos\frac{2\pi}{3}}{-2\sin\pi/3 - 4\cos\frac{\pi}{3}\sin\frac{\pi}{3}}$

$= \frac{2(\frac{1}{2}) + 2(-\frac{1}{2})}{-2(\frac{1}{2}\sqrt{3}) - 4(\frac{1}{2})(\frac{1}{2}\sqrt{3})}$

$= \underline{\underline{0}}$

3b. Garis singgung vertikal

↳ $\frac{dx}{d\theta} = 0$ dan $\frac{dy}{d\theta} \neq 0$

(i) $\frac{dx}{d\theta} = 0$

$-2\sin\theta - 4\cos\theta\sin\theta$

$-2\sin\theta\,(1 + 2\cos\theta) = 0$

- $-2\sin\theta = 0$

  $\theta = \{0, \pi, 2\pi\}$

- $1 + 2\cos\theta = 0$

  $\cos\theta = -\frac{1}{2}$

  $\theta = \{\frac{2\pi}{3}, \frac{4\pi}{3}\}$

(ii) Cek apakah $\theta = \{0, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, 2\pi\}$ memenuhi $\frac{dy}{d\theta} \neq 0$

$\frac{dy}{d\theta} = 2\cos\theta + 2\cos 2\theta$

$\left.\frac{dy}{d\theta}\right|_{\theta=0} = 2\cos 0 + 2\cos 0 = 4 \neq 0$ ✓

$\left.\frac{dy}{d\theta}\right|_{\theta=\frac{2\pi}{3}} = 2\cos\frac{2\pi}{3} + 2\cos\frac{4\pi}{3} = 0$ ✓

$\left.\frac{dy}{d\theta}\right|_{\theta=\pi} = 2\cos\pi + 2\cos 2\pi = 0$ ✗

$\frac{dy}{d\theta}\Big|_{\theta=\frac{4\pi}{3}} = 2\cos\frac{4\pi}{3} + 2\cos\frac{8\pi}{3} \neq 0$ ✓

$\frac{dy}{d\theta}\Big|_{\theta=2\pi} = 2\cos \quad + 2\cos 4\pi \neq 0$ ✓

θ yg memenuhi $\theta = \{0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi\}$

$\theta = 0 \rightarrow r = 2 + 2\cos \quad 4$

$\theta = \frac{2\pi}{3} \rightarrow r = 2 + 2\cos\frac{2\pi}{3} = 1$

$\theta = \frac{4\pi}{3} \rightarrow r = 2 + 2\cos\frac{4\pi}{3} = 1$

$\theta = 2\pi \rightarrow r = 2 + 2\cos 2\pi = 4$

Jadi, titik di $(4,0), (1, \frac{4\pi}{3}), (1, \frac{2\pi}{3}), (4, \frac{8\pi}{3})$

4. Sketch the graph of the region that is outside of the polar curve $r = 3$ and inside of the polar curve $r = 2 - 2\cos\theta$, and find the area of that region. *(EAS 2021/2022, Rabu 8 Juni 2022)*

① Tipot

$r_1 = r_2$

$3 = 2 - 2\cos\theta$

$1 = -2\cos\theta$

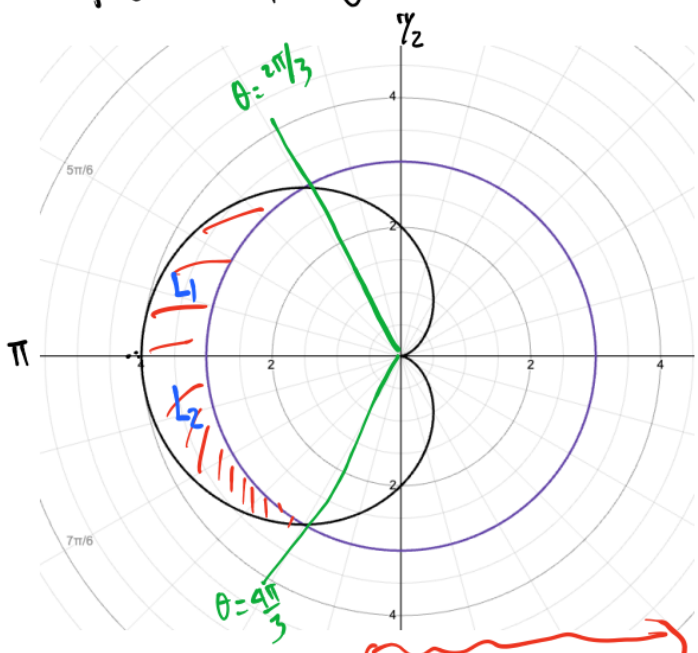
$-\frac{1}{2} = \cos\theta$

$\theta = \{\frac{2\pi}{3}, 4\pi/3\}$

② $r = 3$ (lingkaran $P(0,0)$ dan jari ② 3)

$r = 2 - 2\cos\theta$

θ	0	$\frac{\pi}{2}$	$\frac{3\pi}{2}$	2π
r	0	2	4	0



$$L = \frac{1}{2}\int_{\alpha}^{\beta} r_1^2 - r_2^2 \, d\theta$$

$L_1 = L_2$

$L = L_1 + L_2$

$= 2L_1$

$= 2\int_{\frac{2\pi}{3}}^{\pi} \frac{1}{2}\left((2-2\cos\theta)^2 - (3)^2\right) d\theta$

$= \int_{\frac{2\pi}{3}}^{\pi} 4 - 8\cos\theta + 4\cos^2\theta - 9 \, d\theta$

$= \int_{\frac{2\pi}{3}}^{\pi} -5 - 8\cos\theta + 4\left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta$

$= \int_{\frac{2\pi}{3}}^{\pi} -3 - 8\cos\theta + 2\cos 2\theta \, d\theta$

$= -3\theta - 8\sin\theta + \sin 2\theta \Big|_{\frac{2\pi}{3}}^{\pi}$

$=$ ... (Satuan luas)