Materi : Integral Parsial dan Trigonometri

A Integrasi Parsial

$\int f(x)\,dx = \cdots ?$

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

integral yg ingin diselesaikan

Pilih suku mana yg menjadi u dan dv.
Lalu tentukan du (turunan dari u).
tentukan v ($v = \int dv$)

① $\int x^n \cdot f(x)\,dx = \cdots ?$

adalah fungsi trigonometri (cos & sin)

$u = x^n$; $dv = f(x)\,dx$

$x^3 \rightarrow 3x^2$ (parsial I)
$6x$ (parsial II)
6 (parsial III)

Contoh :

$\int x \cdot \cos x\,dx = \cdots ?$

$\int u\,dv = u \cdot v - \int v\,du$ ✓

$\rightarrow u = x \quad du = dx$

$\rightarrow dv = \cos x\,dx \quad v = \int \cos x\,dx = \sin x$

$$\int x \cdot \cos x\,dx = x \cdot \sin x - \int \sin x\,dx$$
$$= x \cdot \sin x - (-\cos x) + C$$
$$= x \cdot \sin x + \cos x + C$$

② $\int x^n \cdot e^{mx}\,dx = \cdots ?$

$u = x^n$; $dv = e^{mx}\,dx$

$\int e^{ax}\,dx = \frac{1}{a} \cdot e^{ax}$

Contoh :

$\int x \cdot e^{2x}\,dx = \cdots ?$

$\rightarrow u = x \quad du = dx$

$\rightarrow dv = e^{2x}\,dx \quad v = \int e^{2x}\,dx = \frac{1}{2} e^{2x}$

$\int u\,dv = u \cdot v - \int v\,du$

$= \frac{1}{2} e^{2x}$

$$\int x \cdot e^{2x}\, dx = x \cdot \tfrac{1}{2} e^{2x} - \int \tfrac{1}{2} e^{2x}\, dx$$

$$= \tfrac{1}{2} x \cdot e^{2x} - \tfrac{1}{2} \cdot \tfrac{1}{2} e^{2x} + C$$

$$= \tfrac{1}{2} x \cdot e^{2x} - \tfrac{1}{4} e^{2x} + C$$

③ $\int e^{mx} \cdot f(x)\, dx = \cdots ?$

$f(x)$ adalah fungsi trigonometri (cos & sin)

$u = e^{mx}$; $dv = f(x)\, dx$

Contoh :

$\int e^x \cdot \cos x\, dx = \cdots ?$

$\int u\, dv = u \cdot v - \int v\, du$ ✓

$\therefore u = e^x$ $\quad du = e^x\, dx$ $\quad \therefore dv = \cos x\, dx$ $\quad v = \int \cos x\, dx = \sin x$

$$\int e^x \cdot \cos x\, dx = e^x \cdot \sin x - \int \sin x \cdot e^x\, dx$$

$$= e^x \cdot \sin x - \underbrace{\int e^x \cdot \sin x\, dx}_{\text{diparsialkan lagi}}$$

$\therefore u = e^x$ $\quad du = e^x\, dx$ $\quad \therefore dv = \sin x\, dx$ $\quad v = \int \sin x\, dx = -\cos x$

$$\int e^x \cdot \sin x\, dx = e^x \cdot -\cos x - \int -\cos x \cdot e^x\, dx$$

$$= -e^x \cdot \cos x + \int e^x \cdot \cos x\, dx$$

$$\int e^x \cdot \cos x\, dx = e^x \cdot \sin x - \left[-e^x \cdot \cos x + \int e^x \cdot \cos x\, dx \right]$$

$$\int e^x \cdot \cos x\, dx = e^x \cdot \sin x + e^x \cdot \cos x - \int e^x \cdot \cos x\, dx$$

$$2 \int e^x \cdot \cos x\, dx = e^x \cdot \sin x + e^x \cdot \cos x$$

$$\int e^x \cdot \cos x\, dx = \frac{e^x \cdot \sin x + e^x \cdot \cos x}{2}$$

$$\int e^x \cdot \cos x\, dx = \frac{e^x (\sin x + \cos x)}{2} + C$$ ✓

$$2\int e^x \cdot \cos x \, dx = e^x \cdot \sin x + e^x \cdot \cos x$$

$$\int e^x \cdot \cos x \, dx = \frac{e^x \cdot \sin x + e^x \cdot \cos x}{2}$$

$$\int e^x \cdot \cos x \, dx = \frac{e^x (\sin x + \cos x)}{2} + C \quad \checkmark$$

④ Tipe – Tipe Khusus

$\int f(x)\, dx = \text{---} ?$

$u = f(x) \quad ; \quad dv = dx$

Contoh :

$\int \sin(\ln t)\, dt = \text{---} ?$

$\int u\, dv = u \cdot v - \int v \cdot du$ ✓

$\therefore u = \sin(\ln t) \qquad \therefore dv = dt$

$du = \cos(\ln t) \cdot \frac{1}{t}\, dt \qquad v = \int 1\, dt = t$

$$\int \sin(\ln t)\, dt = \sin(\ln t) \cdot t - \int t \cdot \cos(\ln t) \cdot \frac{1}{t}\, dt$$

$$= t \cdot \sin(\ln t) - \int \cos(\ln t)\, dt$$

dipersialkan kembali

$\therefore u = \cos(\ln t) \qquad \therefore dv = dt$

$du = -\sin(\ln t) \cdot \frac{1}{t}\, dt \qquad v = \int dt = t$

$$\int \cos(\ln t)\, dt = \cos(\ln t) \cdot t - \int t \cdot -\sin(\ln t) \cdot \frac{1}{t}\, dt$$

$$= t \cdot \cos(\ln t) + \int \sin(\ln t)\, dt$$

$$\int \sin(\ln t)\, dt = t \cdot \sin(\ln t) - \left[t \cdot \cos(\ln t) + \int \sin(\ln t)\, dt \right]$$

$$\int \sin(\ln t)\, dt = t \cdot \sin(\ln t) - t \cdot \cos(\ln t) - \int \sin(\ln t)\, dt$$

$$2\int \sin(\ln t)\, dt = t \cdot \sin(\ln t) - t \cdot \cos(\ln t)$$

$$\int \sin(\ln t)\, dt = \frac{t \cdot \sin(\ln t) - t \cdot \cos(\ln t)}{2}$$

$$\int \sin(\ln t)\, dt = \frac{t[\sin(\ln t) - \cos(\ln t)]}{2} + C$$

[R] Rumus Rekursi untuk Sinus & Cosinus

B. Rumus Reduksi untuk Sinus & Cosinus

$$(i)\cdot \int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1}x\cdot\cos x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$$

$$(ii)\cdot \int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x\cdot\sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx$$

Ingat !

- $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$
- $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$

Contoh :

① $\int \sin^4 x\,dx = \ldots ?$

$$\int \sin^4 x\,dx = -\frac{1}{4}\cdot\sin^3 x\cdot\cos x + \frac{3}{4}\int \sin^2 x\,dx$$

$$= -\frac{1}{4}\sin^3 x\cdot\cos x + \frac{3}{4}\int\left[\frac{1}{2} - \frac{1}{2}\cos 2x\right]dx$$

$\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + C$

$\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + C$

$$= -\frac{1}{4}\sin^3 x\cdot\cos x + \frac{3}{4}\left[\frac{1}{2}x - \frac{1}{2}\cdot\sin 2x\cdot\frac{1}{2}\right] + C$$

$$= -\frac{1}{4}\sin^3 x\cdot\cos x + \frac{3}{8}x - \frac{3}{16}\sin 2x + C$$

C. Pengintegralan Perpangkatan Sinus & Cosinus

$$\int \sin^m x\cdot\cos^n x\,dx$$

$\sin^2 x + \cos^2 x = 1$

① Jika m ganjil :
- Pilah faktor dari $\sin x$
- Gunakan kesamaan terkait : $\sin^2 x = 1 - \cos^2 x$
- Substitusi $u = \cos x$

② Jika n ganjil :
- Pilah faktor dari $\cos x$
- Gunakan kesamaan terkait : $\cos^2 x = 1 - \sin^2 x$
- Substitusi $u = \sin x$

③ m genap, n genap :
- Gunakan kesamaan terkait untuk mereduksi pangkat $\sin x$ dan $\cos x$

$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$

$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$

Contoh :

① $\int \sin^3 x\cdot\cos^2 x\,dx = \ldots ?$

pangkat dari $\sin x$ → ganjil (Cara 1).

pangkat dari sin x → ganjil (Cara 1).

$$\int \sin^3 x \cdot \cos^2 x \, dx = \int \sin^2 x \cdot \sin x \cdot \cos^2 x \, dx$$
$$= \int \sin^2 x \cdot \cos^2 x \cdot \sin x \, dx$$
$$= \int (1 - \cos^2 x) \cdot \cos^2 x \cdot \sin x \, dx \quad \checkmark$$

Substitusi : $u = \cos x$

$du = -\sin x \, dx$

$-du = \sin x \, dx$

$$= \int (1 - u^2) u^2 \cdot -du$$
$$= -\int u^2 - u^4 \, du$$
$$= -\left[\frac{1}{3}u^3 - \frac{1}{5}u^5\right] + C$$
$$= -\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C$$

② $\int \sin^4 x \cdot \cos^5 x \, dx = \cdots ?$

pangkat dari cos x → ganjil (Cara 2)

$$\int \sin^4 x \cdot \cos^5 x \, dx = \int \sin^4 x \cdot \cos^4 x \cdot \cos x \, dx$$
$$= \int \sin^4 x \cdot (\cos^2 x)^2 \cdot \cos x \, dx$$
$$= \int \sin^4 x \, (1 - \sin^2 x)^2 \cdot \cos x \, dx \quad \checkmark$$

Substitusi : $u = \sin x$

$du = \cos x \, dx$

$$= \int u^4 (1 - u^2)^2 \cdot du$$
$$= \int u^4 (1 - 2u^2 + u^4) \, du$$
$$= \int u^4 - 2u^6 + u^8 \, du$$

$$= \frac{1}{5}u^5 - \frac{2}{7}u^7 + \frac{1}{9}u^9 + C$$
$$= \frac{1}{5}\sin^5 x - \frac{2}{7}\sin^7 x + \frac{1}{9}\sin^9 x + C$$

Integral dengan Bentuk : $\int \sin mx \cdot \cos nx \, dx$ & $\int \cos mx \cdot \cos nx \, dx$

(i). $\sin mx \cdot \cos nx = \dfrac{\sin(m+n)x + \sin(m-n)x}{2}$

(ii). $\cos mx \cdot \cos nx = \dfrac{\cos(m+n)x - \cos(m-n)x}{2}$

Contoh :

① $\int \sin 3\theta \cdot \cos 5\theta \, d\theta = \cdots ?$

$$\sin 3\theta \cdot \cos 5\theta = \frac{\sin(3\theta + 5\theta) + \sin(3\theta - 5\theta)}{2}$$
$$= \sin(8\theta) + \sin(-2\theta)$$

⇒ $\sin(-\theta) = -\sin\theta$

⇒ $\cos(-\theta) = \cos\theta$

$$\sin 3\theta \cdot \cos 5\theta = \frac{\sin(3\theta+5\theta) + \sin(3\theta-5\theta)}{2}$$

$$= \frac{\sin(8\theta) + \sin(-2\theta)}{2}$$

$$= \frac{1}{2}\left[\sin 8\theta - \sin 2\theta\right]$$

$$\int \sin 3\theta \cdot \cos 5\theta \, d\theta = \frac{1}{2}\int \sin 8\theta - \sin 2\theta \, d\theta$$

$$= \frac{1}{2}\left[-\frac{1}{8}\cos 8\theta - \left(-\frac{1}{2}\cos 2\theta\right)\right] + C$$

$$= -\frac{1}{16}\cos 8\theta + \frac{1}{4}\cos 2\theta + C$$

[D] Integrasi Perpangkatan Secan & Tangen

Ingat !

- $\int \tan x \, dx = \ln|\sec x| + C$
- $\int \sec x \, dx = \ln|\sec x + \tan x| + C$

※ Rumus Reduksi

(i) $\int \sec^n x \, dx = \frac{\sec^{n-2} x \cdot \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x \, dx$

(ii) $\int \tan^m x \, dx = \frac{\tan^{m-1} x}{m-1} - \int \tan^{m-2} x \, dx$

Contoh :

① $\int \tan^3 x \, dx = \ldots ?$

$$= \frac{\tan^2 x}{2} - \int \tan x \, dx$$

$$= \frac{\tan^2 x}{2} - \ln|\sec x| + C$$

$\int \tan^m x \cdot \sec^n x \, dx$

① Jika m ganjil :
- Pilah faktor dari $\tan x$
- Gunakan kesamaan terkait : $\tan^2 x = \sec^2 x - 1$
- Substitusi $u = \sec x$

② Jika n genap :
- Pilah faktor dari $\sec x$
- Gunakan kesamaan terkait : $\sec^2 x = \tan^2 x + 1$
- Substitusi $u = \tan x$

③ m genap, n ganjil :
- Gunakan kesamaan terkait untuk mereduksi pangkat $\tan x$: $\tan^2 x = \sec^2 x - 1$
- Gunakan rumus reduksi

n ganjil ∫ ... tan x

⇒ Gunakan rumus reduksi untuk pangkat sec x

Contoh :

① $\int \tan^4 x \cdot \sec x \, dx = \cdots ?$

pangkat dari tan x → genap
pangkat dari sec x → ganjil
} (Cara 3)

$$\int \tan^4 x \cdot \sec x \, dx = \int (\tan^2 x)^2 \cdot \sec x \, dx$$
$$= \int (\sec^2 x - 1)^2 \cdot \sec x \, dx$$
$$= \int (\sec^4 x - 2\sec^2 x + 1) \sec x \, dx$$
$$= \int \sec^5 x - 2\sec^3 x + \sec x \, dx$$
$$= \underbrace{\int \sec^5 x \, dx}_{C} - 2\underbrace{\int \sec^3 x \, dx}_{B} + \underbrace{\int \sec x \, dx}_{A}$$
$$= a + b + c + C$$

A) $\int \sec x \, dx = \underbrace{\ln|\sec x + \tan x|}_{a}$

B) $\int \sec^3 x \, dx = \frac{\sec x \cdot \tan x}{2} + \frac{1}{2}\int \sec x \, dx$
$$= \underbrace{\frac{\sec x \cdot \tan x}{2} + \frac{1}{2}\ln|\sec x + \tan x|}_{b} \checkmark$$

C) $\int \sec^5 x \, dx = \frac{\sec^3 x \cdot \tan x}{4} + \frac{3}{4}\int \sec^3 x \, dx$
$$= \frac{\sec^3 x \cdot \tan x}{4} + \frac{3}{4}\left[\frac{\sec x \cdot \tan x}{2} + \frac{1}{2}\ln|\sec x + \tan x|\right]$$
$$= \underbrace{\frac{\sec^3 x \cdot \tan x}{4} + \frac{3}{8} \cdot \sec x \cdot \tan x + \frac{3}{8}\ln|\sec x + \tan x|}_{c}$$