Materi : Integral Fungsi Rasional dan Teknik Integrasi Lain

A Fungsi Rasional & Pecahan Parsial.

※ Fungsi rasional $\longrightarrow f(x) = \dfrac{P(x)}{Q(x)}$; $P(x)$ & $Q(x)$ merupakan polinomial

Contoh : $f(x) = \dfrac{1}{x^2-1}$

※ Mendapatkan Bentuk Dekomposisi Pecahan Parsial

(I) Faktor - Faktor Linear

$$\frac{A}{(ax+b)^n} = \frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_{n-1}}{(ax+b)^{n-1}} + \frac{A_n}{(ax+b)^n}$$

Contoh :

Dapatkan bentuk dekomposisi parsial dari

(1) $\dfrac{1}{x^2+3x-10} = \ldots ?$

$$\frac{1}{x^2+3x-10} = \frac{-1/7}{x+5} + \frac{1/7}{x-2}$$

$$\frac{1}{(x+5)(x-2)} = \frac{A}{x+5} + \frac{B}{x-2}$$

$$\frac{1}{(x+5)(x-2)} = \frac{A(x-2)+B(x+5)}{(x+5)(x-2)}$$

$$1 = A(x-2)+B(x+5)$$
$$1 = Ax - 2A + Bx + 5B$$
$$1 = (A+B)x + (-2A+5B) \checkmark$$

› Koefisien $x \longrightarrow A+B = 0 \longrightarrow A = -B \longrightarrow A = -\frac{1}{7}$

› Konstanta $\longrightarrow -2A+5B = 1$

$$-2(-B)+5B = 1$$
$$7B = 1$$
$$B = \frac{1}{7}$$

(2) $\dfrac{2t-3}{t^3-t^2} = \ldots ?$

$$\frac{2t-3}{t^2(t-1)} = \frac{A}{t} + \frac{B}{t^2} + \frac{C}{t-1} = \frac{1}{t} + \frac{3}{t^2} + \frac{-1}{t-1}$$

$$\frac{2t-3}{t^2(t-1)} = \frac{At(t-1)+B(t-1)+C\cdot t^2}{t^2(t-1)}$$

$$2t-3 = At^2 - At + Bt - B + C\cdot t^2$$
$$2t-3 = (A+C)t^2 + (-A+B)t - B$$

› Koefisien $t^2 \longrightarrow A+C = 0 \longrightarrow C = -A = -1$

› Koefisien $t \longrightarrow -A+B = 2 \longrightarrow A = B-2$

› Konstanta $\longrightarrow -B = -3$ $\quad A = 1$

$B = 3$

- Koefisien $t \rightarrow -A+B = 2 \rightarrow A = B-2$
- Konstanta $\rightarrow -B = -3$ $\quad A = 1$

$B = 3$

II Faktor - Faktor Kuadratik

$$\frac{Ax+B}{(ax^2+bx+c)^m} = \frac{A_1x+B_1}{(ax^2+bx+c)} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_{m-1}x+B_{m-1}}{(ax^2+bx+c)^{m-1}} + \frac{A_mx+B_m}{(ax^2+bx+c)^m}$$

Contoh :

Dapatkan bentuk dekomposisi parsial dari

1. $\frac{2t^3-t}{(t^2+5)^2} = \cdots ?$

$$\frac{2t^3-t}{(t^2+5)^2} = \frac{At+B}{(t^2+5)} + \frac{Ct+D}{(t^2+5)^2} = \frac{2t}{t^2+5} + \frac{-11t}{(t^2+5)^2}$$

$$\frac{2t^3-t}{(t^2+5)^2} = \frac{(At+B)(t^2+5)+(Ct+D)}{(t^2+5)^2}$$

$$2t^3-t = At^3+5At+Bt^2+5B+Ct+D$$

$$2t^3-t = A\cdot t^3+Bt^2+(5A+C)t+(5B+D)$$

- Koef $\cdot t^3 \rightarrow A = 2$
- Koef $\cdot t^2 \rightarrow B = 0$
- Koef $\cdot t \rightarrow 5A+C = -1$

$10+C = -1$

$C = -11$

- Konstanta $\rightarrow 5B+D = 0$

$0+D = 0$

$D = 0$

B Integral Fungsi Rasional $\rightarrow$ Derajat Pembilang < Derajat Penyebut

$\hookrightarrow$ diselesaikan dengan pecah parsial

1). Tentukan bentuk dekomposisi parsial
2). Integralkan hasil ①

Contoh :

1. $\int \frac{1}{x^2-8x+7}\,dx = \cdots ?$

$$\int \frac{1}{(x-1)(x-7)}\,dx = \cdots$$

$$\frac{1}{(x-1)(x-7)} = \frac{A}{x-1} + \frac{B}{x-7}$$

$$\frac{1}{(x-1)(x-7)} = \frac{A(x-7)+B(x-1)}{(x-1)(x-7)}$$

$1 = Ax - 7A + Bx - B$

$$= \int \frac{-1/6}{x-1} + \frac{1/6}{x-7}\,dx$$

$$= \int \frac{-1/6}{x-1}\,dx + \int \frac{1/6}{x-7}\,dx$$

$$= -\frac{1}{6}\int \underbrace{\frac{1}{x-1}}\,dx + \frac{1}{6}\int \frac{1}{x-7}\,dx$$

Substitusi $u = x-1$

$du = dx$

$$\frac{1}{(x-1)(x-7)} = \frac{A(x-7)+B(x-1)}{(x-1)(x-7)}$$

$$1 = Ax - 7A + Bx - B$$
$$1 = (A+B)x + (-7A - B)$$

- Koef. $x \rightarrow A+B = 0 \rightarrow A = -B \rightarrow A = -\frac{1}{6}$
- Konstanta $\rightarrow -7A - B = 1$

$$-7(-B) - B = 1$$
$$6B = 1$$
$$B = \frac{1}{6}$$

Substitusi $u = x-1$
$du = dx$

$$= \int \frac{1}{u}\,du$$
$$= \ln|u|$$
$$= \ln|x-1|$$

$$= -\frac{1}{6}\ln|x-1| + \frac{1}{6}\ln|x-7| + C$$

- $\int \frac{1}{ax+b}\,dx = \frac{1}{a}.\ln|ax+b| + C$ ✓

$\boxed{C}$ Integral Fungsi Rasional : ① . Derajat Pembilang = Derajat Penyebut

② . Derajat Pembilang > Derajat Penyebut

Teknik Penyelesaian Integral .

Karena pangkat pembilang > pangkat penyebut $\rightarrow$ maka lakukan pembagian dulu ,

sehingga : $f(x) = \underbrace{\text{hasil bagi}}_{\text{polinomial}} + \underbrace{\text{sisa bagi}}_{\text{bentuknya pecahan}}$ $\rightarrow$ pecahan parsial

$$\int x^n\,dx = \frac{1}{n+1}.x^{n+1}$$

Contoh :

① $\int \frac{x^5+2x^2+1}{x^3+x}\,dx = \dots ?$

$$\begin{array}{r|l} & x^2 - 1 \quad \text{hasil bagi} \\ x^3+x & x^5+2x^2+1 \\ & \underline{x^5+x^3} \quad - \\ & -x^3+2x^2+1 \\ & \underline{-x^3-x} \quad - \\ & 2x^2+x+1 \quad \text{sisa} \end{array}$$

$$\int \frac{x^5+2x^2+1}{x^3+x}\,dx = \int \underbrace{(x^2-1)}_{\frac{1}{3}x^3 - x} + \underbrace{\frac{2x^2+x+1}{x^3+x}}_{\text{Pecah parsial } ⊛}\,dx$$

$$\frac{2x^2+x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$\frac{2x^2+x+1}{x(x^2+1)} = \frac{A(x^2+1)+(Bx+C)x}{x(x^2+1)}$$

$$2x^2+x+1 = Ax^2+A+Bx^2+Cx$$
$$2x^2+x+1 = (A+B)x^2+Cx+A$$

- Koef. $x^2 \rightarrow A+B = 2 \rightarrow B = 2 - A$
- Koef. $x \rightarrow C = 1 \qquad B = 1$
- Konstanta $\rightarrow A = 1$

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•) Koef. $x \longrightarrow C = 1$

•) Konstanta $\longrightarrow A = 1$

$$= \int \frac{1}{x} + \frac{x+1}{x^2+1}\, dx$$

$$= \int \frac{1}{x}\, dx + \int \frac{x+1}{x^2+1}\, dx$$

$$= \underbrace{\int \frac{1}{x}\, dx}_{\ln|x|} + \underbrace{\int \frac{x}{x^2+1}\, dx}_{\text{Substitusi}: u = x^2+1} + \underbrace{\int \frac{1}{x^2+1}\, dx}_{\tan^{-1} x}$$

Substitusi : $u = x^2+1$

$du = 2x\, dx$

$\frac{1}{2} du = x\, dx$

$$= \int \frac{1}{x^2+1} \cdot x\, dx$$

$$= \int \frac{1}{u} \cdot \frac{1}{2}\, du$$

$$= \frac{1}{2} \ln|u|$$

$$= \frac{1}{2} \ln|x^2+1|$$

$$\int \frac{x^5+2x^2+1}{x^3+x}\, dx = \int (x^2-1)\, dx + \int \frac{1}{x}\, dx + \int \frac{x}{x^2+1}\, dx + \int \frac{1}{x^2+1}\, dx$$

$$= \frac{1}{3}x^3 - x + \ln|x| + \frac{1}{2}\ln|x^2+1| + \tan^{-1} x + C$$

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D. Teknik-Teknik Integrasi yang lain.

I. Substitusi Trigonometri

(i) $\sqrt{a^2 - x^2} \longrightarrow$ substitusi : $x = a \sin\theta$

(ii) $\sqrt{a^2 + x^2} \longrightarrow$ substitusi : $x = a \tan\theta$

(iii) $\sqrt{x^2 - a^2} \longrightarrow$ substitusi : $x = a \sec\theta$

※ $\sin^2\theta + \cos^2\theta = 1$

※ $1 + \tan^2\theta = \sec^2\theta$

Contoh :

① $\int \sqrt{1 - 9t^2}\, dt = \ldots ?$ $\quad \sqrt{a^2 - \boxed{x^2}}$

$$= \int \sqrt{9\left(\frac{1}{9} - t^2\right)}\, dt$$

$$= 3 \int \sqrt{\frac{1}{9} - t^2}\, dt \longrightarrow a^2 = \frac{1}{9}, \quad a = \frac{1}{3}$$

Substitusi : $t = \frac{1}{3}\sin\theta$

$dt = \frac{1}{3}\cos\theta \cdot d\theta$

$$= \cancel{3} \int \sqrt{\frac{1}{9} - \frac{1}{9}\sin^2\theta} \cdot \frac{1}{\cancel{3}}\cos\theta\, d\theta$$

$$= \int \sqrt{\frac{1}{9}(1 - \sin^2\theta)} \cdot \cos\theta\, d\theta$$

$$= \frac{1}{3} \int \sqrt{\cos^2\theta} \cdot \cos\theta\, d\theta$$

$$= \frac{1}{3} \int \cos\theta \cdot \cos\theta\, d\theta$$

$\frac{1}{3}\cos^3\theta$ ✗

$$= \int \cos^2\theta\, d\theta$$

$$= \frac{1}{3} \int \frac{1}{2} + \frac{1}{2}\cos 2\theta\, d\theta$$

$$= \frac{1}{3}\left[\frac{1}{2}\theta + \frac{1}{2} \cdot \sin 2\theta \cdot \frac{1}{2}\right] + C$$

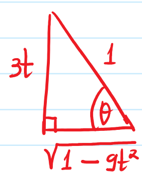
$$= \frac{1}{6}\theta + \frac{1}{12}\sin 2\theta + C$$

$t = \frac{1}{3}\sin\theta$

$\sin\theta = \frac{3t}{1} \longrightarrow \theta = \sin^{-1}(3t)$

$\rightarrow \sin 2\theta = 2\sin\theta \cdot \cos\theta$

$= 2 \cdot 3t \cdot \sqrt{1 - 9t^2}$



$$= \frac{1}{6}\sin^{-1}(3t) + \frac{t\sqrt{1 - 9t^2}}{2} + C$$

E. Integral yang Mencakup $ax^2 + bx + c$

Langkah penyelesaian : (i). Membuat bentuk kuadrat sempurna dari $ax^2 + bx + c$

(ii). Melakukan substitusi yang sesuai

Contoh :

① Selesaikan $\int \frac{x}{x^2 - 4x + 8}\, dx = \ldots ?$

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$$\int \frac{\ldots}{x^2-4x+8} \ldots ?$$

$$\int \frac{x}{x^2-4x\boxed{+8}}\,dx = \int \frac{x}{\underbrace{(x-2)^2}_{x^2-4x\boxed{+4}}+4}\,dx$$

$$\int \frac{1}{u^2\boxed{+1}}\,du = \tan^{-1} u \ \checkmark$$

Substitusi : $u = x-2 \rightarrow x = u+2$

$du = dx$

$$= \int \frac{u+2}{u^2+4}\,du$$

$$= \underbrace{\int \frac{u}{u^2+4}\,du}_{*} + \underbrace{\int \frac{2}{u^2+4}\,du}_{\#}$$

$$= \frac{1}{2}\ln|(x-2)^2+4| + \tan^{-1}\left(\frac{x-2}{2}\right) + C$$

$$\int \frac{u}{u^2+4}\,du =$$

Substitusi : $t = u^2+4$

$dt = 2u\,du$

$\frac{1}{2}dt = u\,du$

$$= \int \frac{1}{2}\cdot\frac{1}{t}\,dt$$

$$= \frac{1}{2}\ln|t| = \frac{1}{2}\ln|u^2+4| = \frac{1}{2}\ln|(x-2)^2+4|$$

$\#$ $$\int \frac{2}{u^2+4}\,du = 2\int \frac{1}{u^2\boxed{+4}}\,du$$

$$= 2\int \frac{1}{4\left(\frac{u^2}{4}\boxed{+1}\right)}\,du$$

$$= \frac{1}{2}\int \frac{1}{\left(\frac{u}{2}\right)^2+1}\,du$$

Substitusi : $t = \frac{u}{2}$

$dt = \frac{1}{2}du$

$2\,dt = du$

$$= \cancel{\frac{1}{2}}\int \frac{1}{t^2+1}\cdot\cancel{2}\,dt$$

$= \tan^{-1} t$

$= \tan^{-1}\left(\frac{u}{2}\right)$

$= \tan^{-1}\left(\frac{x-2}{2}\right)$

F. Integral yang Memuat Pangkat Rasional

Contoh :

① Selesaikan $\int \frac{1}{\sqrt{t}-\sqrt[3]{t}}\,dt = \ldots ?$

$\rightarrow t^{3/5} \rightarrow \frac{3}{5}$

$\rightarrow t^{1/3} \rightarrow \frac{1}{3}$

Substitusi : $u = t^{\frac{1}{5\times 3}}$

$u = t^{\frac{1}{15}}$

$$\int \frac{1}{t^{1/2}-t^{1/3}}\,dt \longrightarrow$$

$\rightarrow t^{1/2} \rightarrow \frac{1}{2}$

$\rightarrow t^{1/3} \rightarrow \frac{1}{3}$

$\boxed{\text{Substitusi : } u = t^{\frac{1}{n}}} \rightarrow$ n adalah KPK dari penyebut

$u = t^{\frac{1}{2\times 3}}$

$u = t^{1/6}$

$u^6 = t$

$6u^5\,du = dt$

$\rightarrow t^{1/2} = (t^{1/6})^3 = u^3$

$\rightarrow t^{1/3} = (t^{1/6})^2 = u^2$

$$= \int \frac{1}{u^3-u^2}\cdot 6u^5\,du$$

$$= \int \frac{6u^5}{u^3-u^2}\,du = \int 6u^2+6u+6\,du + \int \frac{6\cancel{u^2}}{\cancel{u^2}(u-1)}\,du$$

$$= \int 6u^2+6u+6\,du + 6\int \frac{1}{u-1}\,du$$

$$= 2u^3+3u^2+6u+6\ln|u-1| + C$$

$$= 2t^{1/2}+3t^{1/3}+6t^{1/6}+6\ln|t^{1/6}-1| + C$$

$$\begin{array}{r} 6u^2+6u+6 \\ u^3-u^2 \overline{\smash{)}\,6u^5} \\ 6u^5-6u^4 \quad - \\ \hline 6u^4 \\ 6u^4-6u^3 \quad - \\ \hline 6u^3 \\ 6u^3-6u^2 \end{array}$$

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$$\begin{array}{l} 6u^4 - 6u^3 \\ \hline 6u^3 \\ 6u^3 - 6u^2 \\ \hline 6u^2 \end{array}$$

$\boxed{G}$ Integral yang Memuat Fungsi – Fungsi Rasional dalam $\sin x$ dan $\cos x$

$$\int R(x)\,dx = \ldots ?$$

$R(x)$ adalah fungsi rasional ($\sin x$ & $\cos x$).

- Substitusi : $u = \tan\left(\frac{x}{2}\right)$
  - $\sin x = \dfrac{2u}{1+u^2}$
  - $\cos x = \dfrac{1-u^2}{1+u^2}$
  - $dx = \dfrac{2}{1+u^2}\,du$

Contoh :

$$\int \frac{1}{1+\sin x+\cos x}\,dx = \ldots ?$$

Substitusi : $u = \tan\left(\frac{x}{2}\right)$ ✓

$$= \int \frac{1}{1+\frac{2u}{1+u^2}+\frac{1-u^2}{1+u^2}} \cdot \frac{2}{1+u^2}\,du$$

$$= \int \frac{1}{\frac{1+\cancel{u^2}+2u+1-\cancel{u^2}}{1+u^2}} \cdot \frac{2}{1+u^2}\,du$$

$$= \int \frac{\cancel{1+u^2}}{\cancel{2}(u+1)} \cdot \frac{\cancel{2}}{\cancel{1+u^2}}\,du$$

$$= \int \frac{1}{u+1}\,du$$

$$= \ln|u+1| + C$$

$$= \ln\left|\tan\left(\frac{x}{2}\right)+1\right| + C$$

- $\displaystyle\int \frac{\cos x}{1+\sin x}\,dx = \ldots ?$

Substitusi : $u = 1+\sin x$ ✓

$du = \cos x\,dx$

$$= \int \frac{1}{u}\,du$$

$$= \ln|u| + C$$

$$= \ln|1+\sin x| + C$$