Kelas Februari - Materi 2

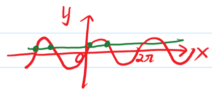
18 February 2023 10:00

RUANG SAINS

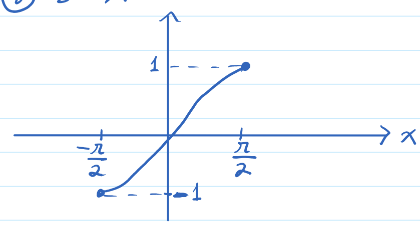
Materi : Fungsi Transenden : Fungsi Invers Trigonometri, Fungsi Hiperbolik, Fungsi Invers Hiperbolik.

1. Fungsi Invers Trigonometri

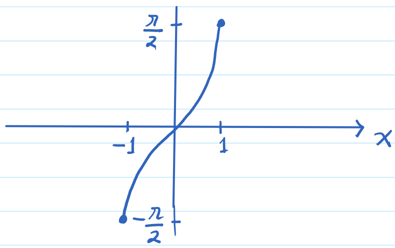
① $y = \sin^{-1} x$ ekivalen dengan $\sin y = x$ jika $\begin{cases} -1 \le x \le 1 \\ -\frac{\pi}{2} \le y \le \frac{\pi}{2} \end{cases}$



(i) $\sin x$



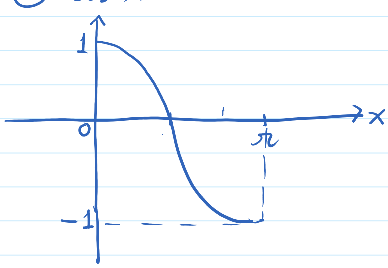
(ii) $\sin^{-1} x$



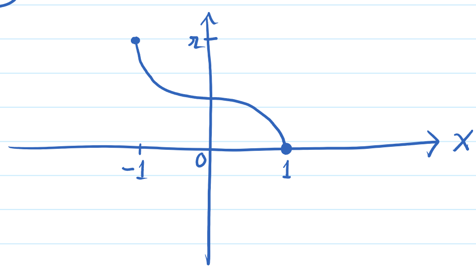
Grafik $f(x)$ & $f^{-1}(x)$ satu sama lain merupakan pencerminan terhadap garis $y = x$

② $y = \cos^{-1} x$ ekivalen dengan $\cos y = x$ jika $\begin{cases} -1 \le x \le 1 \\ 0 \le y \le \pi \end{cases}$

(i) $\cos x$

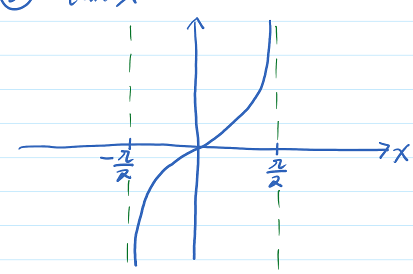


(ii) $\cos^{-1} x$

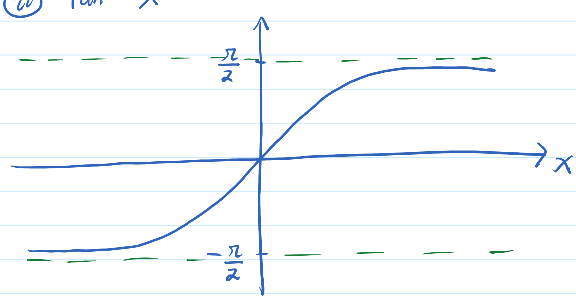


③ $y = \tan^{-1} x$ ekivalen dengan $\tan y = x$ jika $\begin{cases} -\infty \le x \le +\infty \\ -\frac{\pi}{2} \le y \le \frac{\pi}{2} \end{cases}$

(i) $\tan x$



(ii) $\tan^{-1} x$



Soal Latihan :

① $y = \sin^{-1}\left(\frac{1}{2}\right)$; $y = \ldots ?$

artinya $\sin y = \frac{1}{2}$; $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

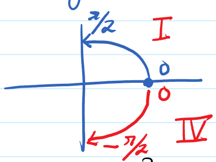
② $y = \tan^{-1}(\sqrt{3})$; $y = \ldots ?$

artinya $\tan y = \sqrt{3}$; $-\frac{\pi}{2} \le y \; \frac{\pi}{2}$

① $y = \sin^{-1}\left(\frac{1}{2}\right)$; $y = \ldots ?$

artinya $\sin y = \frac{1}{2}$; $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$y = \frac{\pi}{6}$



② $y = \tan^{-1}(\sqrt{3})$; $y = \ldots ?$

artinya $\tan y = \sqrt{3}$; $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$y = \frac{\pi}{3}$

③ $y = \sin^{-1}\left(-\frac{1}{2}\sqrt{3}\right)$; $y = \ldots ?$

artinya $\sin y = -\frac{1}{2}\sqrt{3}$; $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$y = -\frac{\pi}{3}$

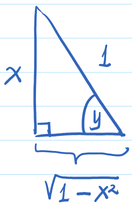
✱ Penyederhanaan Ekspresi yang Memuat Fungsi Invers Trigonometri

① Sederhanakan fungsi $\cos(\sin^{-1}x)$!

Misalkan $y = \sin^{-1}x$

maka artinya $\sin y = \frac{x}{1}$; $\rightarrow \sin y = \frac{\text{sisi depan}}{\text{sisi miring}}$

Gambarkan pd segitiga siku^{2}



$\therefore \cos(\sin^{-1}x) = \cos y$

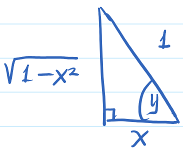
$= \frac{\sqrt{1-x^2}}{1}$

$= \sqrt{1-x^2}$

② Sederhanakan fungsi $\tan(\cos^{-1}x)$!

Misalkan $y = \cos^{-1}x$

maka artinya $\cos y = \frac{x}{1}$



$\therefore \tan(\cos^{-1}x) = \tan y$

$= \frac{\sqrt{1-x^2}}{x}$

2 Turunan dan Integral Fungsi Invers Trigonometri

Ⓐ Rumus – rumus Turunan

① $\frac{d}{dx}\left[\sin^{-1}x\right] = \frac{1}{\sqrt{1-x^2}}$

② $\frac{d}{dx}\left[\cos^{-1}x\right] = -\frac{1}{\sqrt{1-x^2}}$

③ $\frac{d}{dx}\left[\tan^{-1}x\right] = \frac{1}{1+x^2}$

④ $\frac{d}{dx}\left[\cot^{-1}x\right] = -\frac{1}{1+x^2}$

⑤ $\frac{d}{dx}\left[\sec^{-1}x\right] = \frac{1}{x\sqrt{x^2-1}}$

⑥ $\frac{d}{dx}\left[\csc^{-1}x\right] = -\frac{1}{x\sqrt{x^2-1}}$

Cara Mendapatkan Turunan Fungsi Invers Trigonometri

① $y = \sin^{-1}x \longrightarrow \frac{dy}{dx} = \ldots ?$

Cara Mendapatkan Turunan Fungsi Invers Trigonometri

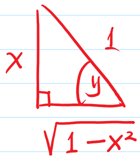
① $y = \sin^{-1} x \longrightarrow \dfrac{dy}{dx} = \ldots ?$

artinya $\underbrace{\sin y = x}$ $\longrightarrow$ $\sin y = \dfrac{x}{1}$

$$\frac{d}{dx}[\sin y] = \frac{d}{dx}[x]$$

$$\cos y \cdot 1 \cdot \frac{dy}{dx} = 1$$

$$\cos y \cdot \frac{dy}{dx} = 1$$



$$\cos y = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$$

Ⓑ Rumus-rumus Turunan yang diperumum

① $\dfrac{d}{dx}[\sin^{-1} u] = \dfrac{1}{\sqrt{1-u^2}} \cdot \dfrac{du}{dx}$; ② $\dfrac{d}{dx}[\cos^{-1} u] = -\dfrac{1}{\sqrt{1-u^2}} \cdot \dfrac{du}{dx}$

③ $\dfrac{d}{dx}[\tan^{-1} u] = \dfrac{1}{1+u^2} \cdot \dfrac{du}{dx}$; ④ $\dfrac{d}{dx}[\cot an^{-1} u] = -\dfrac{1}{1+u^2} \cdot \dfrac{du}{dx}$

⑤ $\dfrac{d}{dx}[\sec^{-1} u] = \dfrac{1}{u\sqrt{u^2-1}} \cdot \dfrac{du}{dx}$; ⑥ $\dfrac{d}{dx}[\operatorname{cosec}^{-1} u] = -\dfrac{1}{u\sqrt{u^2-1}} \cdot \dfrac{du}{dx}$

Soal Latihan.

Dapatkan $\dfrac{dy}{dx}$ jika diberikan

① $y = \sin^{-1}(x^4)$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(x^4)^2}} \cdot \frac{d}{dx}[x^4] = \frac{1}{\sqrt{1-x^8}} \cdot 4x^3 = \frac{4x^3}{\sqrt{1-x^8}}$$

② $y = \tan^{-1}(e^{2x})$

$$\frac{dy}{dx} = \frac{1}{1+(e^{2x})^2} \cdot \frac{d}{dx}[e^{2x}] = \frac{1}{1+e^{4x}} \cdot e^{2x} \cdot 2 = \frac{2e^{2x}}{1+e^{4x}}$$

③ $y = \tan^{-1}\left(\dfrac{2-x}{2+x}\right)$

$$\frac{dy}{dx} = \frac{1}{1+\left(\frac{2-x}{2+x}\right)^2} \cdot \frac{d}{dx}\underbrace{\left[\frac{2-x}{2+x}\right]}$$

$$\frac{u}{v} = \frac{u'v - uv'}{v^2}$$

$$= \frac{1}{\frac{(2+x)^2+(2-x)^2}{(2+x)^2}} \cdot \left[\frac{-1(2+x)-(2-x)1}{(2+x)^2}\right] \checkmark$$

$$= \frac{\cancel{(2+x)^2}}{(4+\cancel{4x}+x^2)+(4-\cancel{4x}+x^2)} \cdot \left(\frac{-2\cancel{-x}-2\cancel{+x}}{\cancel{(2+x)^2}}\right)$$

$$= \frac{-4}{8+2x^2}$$

$$= \frac{-4}{8 + 2x^2}$$

$$= \frac{-\cancel{4}^{\,2}}{\cancel{2}(4+x^2)}$$

$$= -\frac{2}{4+x^2}$$

c) Rumus Integrasi

1. $\int \frac{1}{\sqrt{1-u^2}}\, du = \sin^{-1} u + C$ ✓

2. $\int \frac{1}{1+u^2}\, du = \tan^{-1} u + C$

3. $\int \frac{1}{u\sqrt{u^2-1}}\, du = \sec^{-1} u + C$

Soal Latihan

1. $\int \frac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1} x + C$

2. $\int \frac{1}{\sqrt{1-9x^2}}\, dx = \ldots ?$ $\qquad \int \frac{1}{\sqrt{1-u^2}}\, du = \sin^{-1} u + C$

$$\int \frac{1}{\sqrt{1-9x^2}}\, dx = \int \frac{1}{\sqrt{1-(3x)^2}}\, dx \quad ✓$$

Substitusi : $u = 3x$

$du = 3\, dx$

$\frac{1}{3}\, du = dx$

$$= \int \frac{1}{\sqrt{1-u^2}} \cdot \frac{1}{3}\, du$$

$$= \frac{1}{3} \int \frac{1}{\sqrt{1-u^2}}\, du$$

$$= \frac{1}{3} \sin^{-1} u + C \qquad = \frac{1}{3} \sin^{-1}(3x) + C$$

3. $\int \frac{1}{\sqrt{4-9x^2}}\, dx = \ldots ?$ $\qquad \int \frac{1}{\sqrt{1-u^2}}\, du = \sin^{-1} u + C$

$$\int \frac{1}{\sqrt{4-9x^2}}\, dx = \int \frac{1}{\sqrt{4\left(1-\frac{9x^2}{4}\right)}}\, dx$$

$$= \int \frac{1}{\sqrt{4}} \cdot \frac{1}{\sqrt{1-\frac{9x^2}{4}}}\, dx$$

$$= \frac{1}{2} \int \frac{1}{\sqrt{1-\frac{9x^2}{4}}}\, dx$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{1-\frac{9x^2}{4}}}\,dx$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{1-\left(\frac{3x}{2}\right)^2}}\,dx \quad \checkmark \quad \rightarrow \quad = \frac{1}{\cancel{2}}\int \frac{1}{\sqrt{1-u^2}}\cdot\frac{\cancel{2}}{3}\,du$$

$$= \frac{1}{3}\int \frac{1}{\sqrt{1-u^2}}\,du$$

$$= \frac{1}{3}\sin^{-1} u + C$$

$$= \frac{1}{3}\sin^{-1}\left(\frac{3}{2}x\right) + C$$

Substitusi : $u = \frac{3}{2}x$

$du = \frac{3}{2}\,dx$

$\frac{2}{3}\,du = dx$

$\boxed{3}$ Fungsi Hiperbolik

(i). $\sinh x = \dfrac{e^x - e^{-x}}{2}$

dibaca sin hiperbolik x

(ii). $\cosh x = \dfrac{e^x + e^{-x}}{2}$

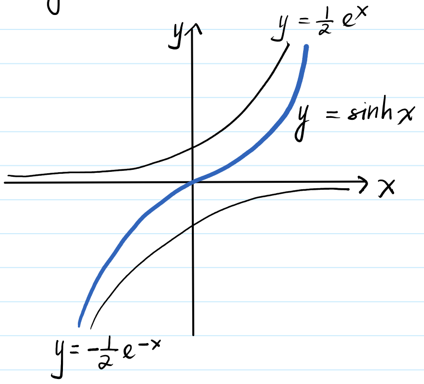
iii). $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

iv). $\operatorname{cotanh} x = \dfrac{1}{\tanh x} = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

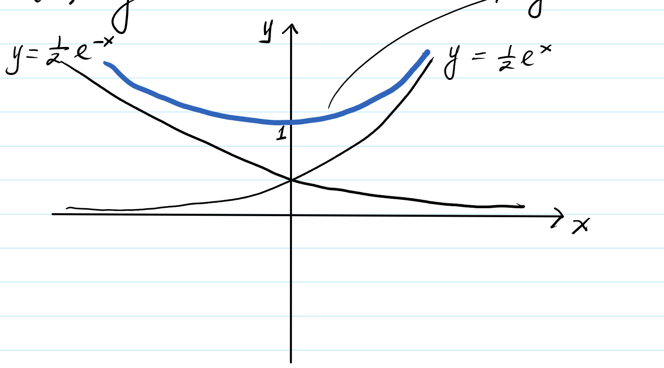
v). $\operatorname{sech} x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$

vi). $\operatorname{cosech} x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}}$

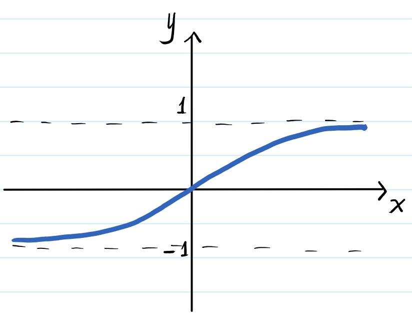
(i). $y = \sinh x$



(ii). $y = \cosh x$

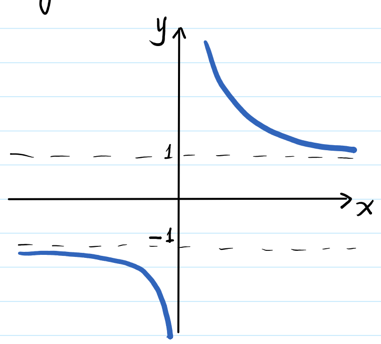


(iii) $y = \tanh$

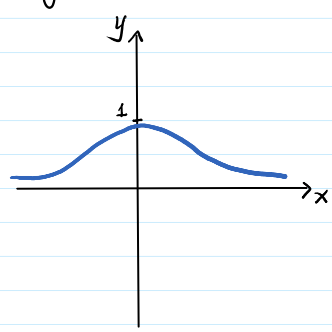


csch ✓

(iv) $y = \operatorname{cotanh} x$

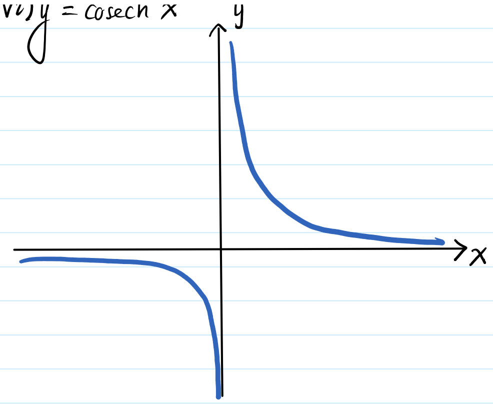


(v). $y = \operatorname{sech} x$



(vi) $y = \operatorname{cosech} x$

(vi) $y = \text{cosech } x$



※ Kesamaan Hiperbolik :

1. $\cosh^2 x - \sinh^2 x = 1$
2. $1 - \tanh^2 x = \text{sech}^2 x$
3. $\cosh x + \sinh x = e^x$
4. $\cosh x - \sinh x = e^{-x}$

※ Rumus Turunan Fungsi Hiperbolik

1. $\frac{d}{dx}[\sinh u] = \cosh u \cdot \frac{du}{dx}$
2. $\frac{d}{dx}[\cosh u] = \sinh u \cdot \frac{du}{dx}$
3. $\frac{d}{dx}[\tanh u] = \text{sech}^2 u \cdot \frac{du}{dx}$
4. $\frac{d}{dx}[\text{sech } u] = -\text{sech } u \cdot \tanh u \cdot \frac{du}{dx}$
5. $\frac{d}{dx}[\text{cotanh } u] = -\text{cosech}^2 u \cdot \frac{du}{dx}$
6. $\frac{d}{dx}[\text{cosech } u] = -\text{cosech } u \cdot \text{cotanh } u \cdot \frac{du}{dx}$

Contoh Soal :

1. Diberikan $y = \tanh(x^3+2x)$; $\frac{dy}{dx} = \ldots ?$

$$\frac{dy}{dx} = \text{sech}^2(x^3+2x) \cdot \frac{d}{dx}(x^3+2x)$$
$$= \text{sech}^2(x^3+2x) \cdot [3x^2+2]$$
$$= (3x^2+2) \cdot \text{sech}^2(x^3+2x)$$

※ Rumus Integral Fungsi Hiperbolik

1. $\int \sinh u \, du = \cosh u + C$
2. $\int \cosh u \, du = \sinh u + C$
3. $\int \text{sech}^2 u \, du = \tanh u + C$
4. $\int \text{sech } u \cdot \tanh u \, du = -\text{sech } u + C$
5. $\int \text{cosech}^2 u \, du = -\text{cotanh } u + C$
6. $\int \text{cosech } u \cdot \text{cotanh } u \, du = -\text{cosech } u + C$

Soal Latihan :

1. Dapatkan $\frac{dy}{dx}$ dari :

(1) Dapatkan $\frac{dy}{dx}$

(a) $y = \text{sech}\,(e^{3x})$

$$\frac{dy}{dx} = -\text{sech}\,(e^{3x}) \cdot \tanh(e^{3x}) \cdot \frac{d}{dx}[e^{3x}]$$

$$= -\text{sech}\,(e^{3x}) \cdot \tanh(e^{3x}) \cdot (3 \cdot e^{3x})$$

(b) $y = \sinh(\cos 5x)$

$$\frac{dy}{dx} = \cosh(\cos 5x) \cdot \frac{d}{dx}[\cos 5x]$$

$$= \cosh(\cos 5x) \cdot [-\sin 5x \cdot 5]$$

$$= -5 \sin 5x \cdot \cosh(\cos 5x)$$

(2) Hitung integral dari :

(a) $\int \sinh^8 x \cdot \underbrace{\cosh x \; dx} = \cdots ?$

Substitusi : $u = \sinh x$
$\underbrace{du} = \cosh x \; dx$

$$= \int u^8 \, du$$

$$= \frac{1}{9} u^9 + C$$

$$= \frac{1}{9} \sinh^9 x + C$$

(b) $\int \frac{\sinh 2x}{3 + 5\cosh 2x} dx = \cdots ?$

Substitusi : $u = 3 + 5\cosh 2x$
$du = 5 \sinh 2x \cdot 2 \; dx$
$du = 10 \sinh 2x \; dx \longrightarrow \frac{1}{10} du = \sinh 2x \; dx$

$$= \int \frac{1}{3 + 5\cosh 2x} \cdot \underbrace{\sinh 2x \; dx}$$

$$= \int \frac{1}{u} \cdot \underbrace{\frac{1}{10} du}$$

$$= \frac{1}{10} \int \frac{1}{u} du$$

$$= \frac{1}{10} \ln |u| + C$$

$$= \frac{1}{10} \ln |3 + 5\cosh 2x| + C$$

4 Fungsi Invers Hiperbolik

$u = \sinh^{-1} x$ ekivalen dengan $\sinh u = x$ untuk semua x, u.

4 Fungsi Invers Hiperbolik

$y = \sinh^{-1} x$ ekivalen dengan $\sinh y = x$ untuk semua x, y.

$y = \cosh^{-1} x$ ekivalen dengan $\cosh y = x$ jika $\begin{cases} x \geq 1 \\ y \geq 0 \end{cases}$

$y = \tanh^{-1} x$ ekivalen dengan $\tanh y = x$ jika $\begin{cases} -1 < x < 1 \\ -\infty < y < +\infty \end{cases}$

※ Bentuk Logaritmik Fungsi Invers Hiperbolik

1. $\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$
2. $\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right)$
3. $\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$
4. $\operatorname{cotanh}^{-1} x = \frac{1}{2} \ln\left(\frac{x+1}{x-1}\right)$
5. $\operatorname{sech}^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$
6. $\operatorname{cosech}^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right)$

※ Turunan Fungsi Invers Hiperbolik.

1. $\frac{d}{dx}\left[\sinh^{-1} x\right] = \frac{1}{\sqrt{1+x^2}}$
2. $\frac{d}{dx}\left[\cosh^{-1} x\right] = \frac{1}{\sqrt{x^2-1}}$; $x > 1$
3. $\frac{d}{dx}\left[\tanh^{-1} x\right] = \frac{1}{1-x^2}$; $|x| < 1$
4. $\frac{d}{dx}\left[\operatorname{cotanh}^{-1} x\right] = \frac{1}{1-x^2}$; $|x| > 1$
5. $\frac{d}{dx}\left[\operatorname{sech}^{-1} x\right] = -\frac{1}{x\sqrt{1-x^2}}$; $0 < x < 1$
6. $\frac{d}{dx}\left[\operatorname{cosech}^{-1} x\right] = -\frac{1}{|x| \cdot \sqrt{1+x^2}}$; $x \neq 0$

※ Contoh Soal

1. $y = \sinh^{-1}(2x^2)$; $\frac{dy}{dx} = \ldots$?

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+(2x^2)^2}} \cdot \frac{d}{dx}\left[2x^2\right]$$

$$= \frac{1}{\sqrt{1+4x^4}} \cdot 4x$$

$$= \frac{4x}{\quad}$$

$\sqrt{1+4x^4}$

$$= \frac{4x}{\sqrt{1+4x^4}}$$

\# Integral Fungsi Invers Hiperbolik

① $\displaystyle\int \frac{1}{\sqrt{1+u^2}}\,du = \sinh^{-1} u + C$

② $\displaystyle\int \frac{1}{\sqrt{u^2-1}} = \cosh^{-1} u + C \qquad (u > 1)$

③ $\displaystyle\int \frac{1}{1-u^2}\,du = \begin{cases} \tanh^{-1} u + C & ; \ |u| < 1 \\ \operatorname{cotanh}^{-1} u + C & ; \ |u| > 1 \end{cases}$

④ $\displaystyle\int \frac{1}{u\sqrt{1-u^2}} = -\operatorname{sech}^{-1}|u| + C$

⑤ $\displaystyle\int \frac{1}{u\sqrt{1+u^2}} = -\operatorname{cosech}^{-1}|u| + C$

\# Contoh Soal

① $\displaystyle\int \frac{1}{\sqrt{1+9t^2}}\,dt = \ldots ?$ $\qquad \displaystyle\int \frac{1}{\sqrt{1+u^2}}\,du = \sinh^{-1} u + C$

$$\int \frac{1}{\sqrt{1+9t^2}}\,dt = \int \frac{1}{\sqrt{1+(3t)^2}}\,dt$$

Substitusi : $u = 3t$

$du = 3\,dt$

$\frac{1}{3}\,du = dt$

$$= \int \frac{1}{\sqrt{1+u^2}} \cdot \frac{1}{3}\,du$$

$$= \frac{1}{3} \sinh^{-1} u + C$$

$$= \frac{1}{3} \sinh^{-1}(3t) + C$$

Soal Latihan Integral Tertentu :

① $\displaystyle\int_{\ln 2}^{\ln \frac{2}{\sqrt{3}}} \frac{e^{-x}}{\sqrt{1-e^{-2x}}}\,dx = \ldots ?$

$$= \int_{\ln 2}^{\ln \frac{2}{\sqrt{3}}} \frac{e^{-x}}{\sqrt{1-(e^{-x})^2}}\,dx \longrightarrow \int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1} u$$

$$a^{-n} = \frac{1}{a^n}$$

Substitusi : $u = e^{-x}$ ✓

$du = -e^{-x}\,dx$

$\underbrace{-du} = e^{-x}\,dx$

Batas integral :

$\Rightarrow x = \ln 2 \longrightarrow u = e^{-\ln 2}$

$= e^{-1 \cdot \ln 2}$

$= e^{\ln 2^{-1}}$ $\qquad \ln a^n = n \cdot \ln a$

$= e^{\ln \frac{1}{2}}$ $\qquad 2^{-1} = \frac{1}{2}$

$= e$

$$\underbrace{-du} = e^{-x}\,dx$$

$$= e^{\ln 2^{-1}}$$
$$= e^{\ln \frac{1}{2}} \qquad 2^{-1} = \frac{1}{2}$$
$$\underline{u = \frac{1}{2}} \qquad \boxed{e^{\ln x} = x}$$

$$x = \ln \frac{2}{\sqrt{3}} \longrightarrow u = e^{-\ln \frac{2}{\sqrt{3}}}$$
$$= e^{\ln \left(\frac{2}{\sqrt{3}}\right)^{-1}}$$
$$= e^{\ln \frac{\sqrt{3}}{2}}$$
$$= \frac{\sqrt{3}}{2}$$
$$= \underline{\frac{1}{2}\sqrt{3}}$$

$$= \int_{\frac{1}{2}}^{\frac{1}{2}\sqrt{3}} \frac{1}{\sqrt{1-(e^{-x})^2}} \cdot \underbrace{e^{-x}\,dx}$$

$$= \int_{\frac{1}{2}}^{\frac{1}{2}\sqrt{3}} \frac{1}{\sqrt{1-u^2}} \cdot -du$$

$$= - \int_{1/2}^{1/2\sqrt{3}} \frac{1}{\sqrt{1-u^2}}\,du$$

$$= \underbrace{- \sin^{-1} u}\Big|_{1/2}^{1/2\sqrt{3}}$$

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$ Teorema Fundamental Kalkulus I

$$= -\sin^{-1}\left(\frac{1}{2}\sqrt{3}\right) - \left(-\sin^{-1}\left(\frac{1}{2}\right)\right)$$

$$= -\sin^{-1}\left(\frac{1}{2}\sqrt{3}\right) + \sin^{-1}\left(\frac{1}{2}\right)$$

$$= -\frac{\pi}{3} + \frac{\pi}{6}$$

$$= \frac{-2\pi + \pi}{6}$$

$$= -\frac{\pi}{6}$$