5. Dapatkan luas daerah yang dibatasi oleh kurva $x = y^2 - 1$ dan $x = 1 - y^2$. Sketsa grafiknya
   (ETS 2022/2023, Selasa 28 Maret 2023)

Jawab :

$x = y^2 - 1$ dan $x = 1 - y^2$

① Titik Potong

$X_1 = X_2$

$y^2 - 1 = 1 - y^2$

$2y^2 - 2 = 0$
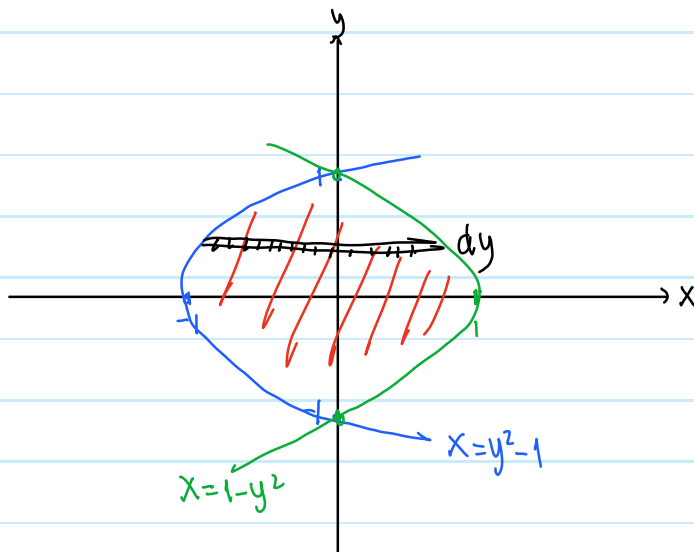
$2(y^2 - 1) = 0$

$2(y+1)(y-1) = 0$

$y = -1 \lor y = 1$

② Gambar

• $X = y^2 - 1$

X	y
0	-1
-1	0
0	1

• $X = 1 - y^2$

X	y
0	-1
1	0
0	1

③ Luas

$dL = (X_1 - X_2)\, dy$

$dL = (1 - y^2 - (y^2 - 1))\, dy$

$L = \int_{-1}^{1} (1 - y^2 - (y^2 - 1))\, dy$

$= \int_{-1}^{1} 2 - 2y^2 \, dy$

$= \left[2y - \frac{2}{3}y^3 \right] \Big|_{-1}^{1}$

$$= \left[2(1) - \frac{2}{3}(1)^3 \right] - \left[2(-1) - \frac{2}{3}(-1)^3 \right]$$

$$= 2 - \frac{2}{3} + 2 - \frac{2}{3}$$

$$= 4 - \frac{4}{3}$$

$$= \frac{8}{3} \quad \text{Satuan Luas.}$$

1. Dapatkan volume benda putar jika daerah yang dibatasi oleh $y = \sqrt{4 - x^2}$, sumbu-x dengan $-1 \le x \le 2$ diputar pada garis $x = 2$, serta sketsa daerahnya. *(EAS 2022/2023, Selasa 13 Juni 2023)*

Jawab $y = 0$

$y = \sqrt{4 - x^2}$, Sb x , $-1 \le x \le 2$ $\to$ $x = 2$
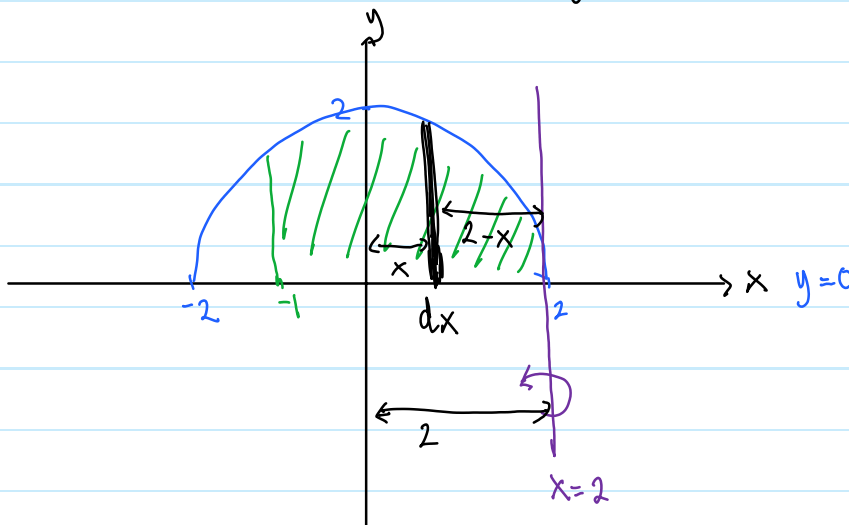
① Gambar

$y = \sqrt{4 - x^2}$

$y^2 = 4 - x^2$

$x^2 + y^2 = 4$

$x^2 + y^2 = 2^2$ (Lingkaran $P(0,0)$ dan $r = 2$)

Karena $y = \sqrt{4 - x^2}$, maka pilih setengah lingkaran bagian atas

② Volume cincin.

$$dv = 2\pi (2 - x)(\sqrt{4 - x^2}) \, dx$$

$$V = \int_{-1}^{2} 2\pi(2-x)\left(\sqrt{4-x^2}\right) dx$$

$$= 2\pi \int_{-1}^{2} 2\sqrt{4-x^2} - x\sqrt{4-x^2}\, dx$$

$$= 4\pi \int_{-1}^{2} \sqrt{4-x^2}\, dx - 2\pi \int_{-1}^{2} x\sqrt{4-x^2}\, dx$$

- untuk $\int_{-1}^{2} \sqrt{4-x^2}\, dx$

Misal   Batas

$x = 2\sin\theta$ $x = -1 \to -1 = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$ $-\frac{1}{2} = \sin\theta$

$\qquad\qquad\qquad\qquad\qquad \theta = -\frac{\pi}{6}$

$\qquad\qquad\qquad x = 2 \to 2 = 2\sin\theta$

$\qquad\qquad\qquad\qquad 1 = \sin\theta$

$\qquad\qquad\qquad\qquad \theta = \frac{\pi}{2}$

$$\int_{-1}^{2} \sqrt{4-x^2}\, dx = \int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \sqrt{4 - 4\sin^2\theta}\ 2\cos\theta\, d\theta$$

$4 - 4\sin^2\theta$

$= 4(1-\sin^2\theta)$

$= 4\cos^2\theta$

$$= \int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \overset{2\cos\theta}{\overbrace{\sqrt{4\cos^2\theta}}}\ 2\cos\theta\, d\theta$$

$$= \int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} 4\cos^2\theta\, d\theta$$

$\int \cos^2\theta\, d\theta$

$= \frac{1}{2}\sin^2\theta + C$

$$= \int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} 4\left[\frac{1}{2} + \frac{1}{2}\cos 2\theta\right] d\theta$$

$$= \int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} 2 + 2\cos 2\theta\, d\theta$$

$$= \left[2\theta + \sin 2\theta\right]\Big|_{-\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \left[\pi + 0\right] - \left[-\frac{\pi}{3} - \frac{1}{2}\sqrt{3}\right]$$

$$= \pi + \frac{\pi}{3} + \frac{1}{2}\sqrt{3} = \frac{4\pi}{3} + \frac{1}{2}\sqrt{3}$$

Untuk $\int_{-1}^{2} x\sqrt{4-x^2}\, dx$

misal   janub

$u = 4 - x^2$ $x = -1 \to u = 3$

$du = -2x \, dx$ $x = 2 \to u = 0$

$-\dfrac{du}{2} = x \, dx$

$\displaystyle \int_{-1}^{2} x \sqrt{4-x^2} \, dx = \int_{-1}^{2} \sqrt{4-x^2} \, x \, dx$

$$= \int_{3}^{0} \sqrt{u} \; -\frac{du}{2}$$

$$= -\frac{1}{2} \int_{3}^{0} u^{1/2} \, du$$

$$= -\frac{1}{2} \left[\frac{2}{3} u^{\frac{3}{2}} \right] \Big|_{3}^{0}$$

$$= 0 + \frac{1}{2} \left[\frac{2}{3} \, 3^{3/2} \right]$$

$$= \frac{1}{3} \, 3\sqrt{3}$$

$$= \sqrt{3}$$

Jadi, $\displaystyle \mathcal{V} = 4\pi \int_{-1}^{2} \sqrt{4-x^2} \, dx - 2\pi \int_{-1}^{2} x \sqrt{4-x^2} \, dx$

$$= 4\pi \left[\frac{4\pi}{3} + \frac{1}{2}\sqrt{3} \right] - 2\pi \left[\sqrt{3} \right]$$

$$= \frac{16\pi^2}{3} + 2\pi\sqrt{3} - 2\pi\sqrt{3}$$

$$= \frac{16\pi^2}{3}$$

2. Dapatkan titik berat daerah yang dibatasi $x = \frac{1}{y}, x = 0, y = 1$ dan $y = e$ dan sketsa daerahnya. *(EAS 2022/2023, Selasa 13 Juni 2023)*

Jawab

2,7.--

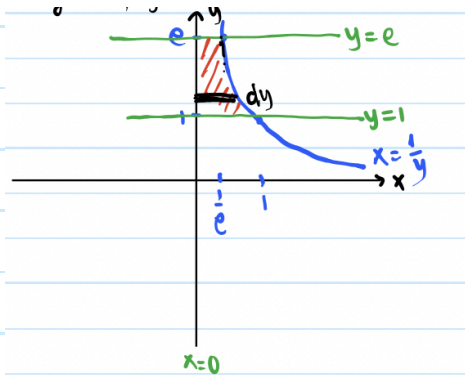$x = \frac{1}{y} \, , \; x = 0 \, , \; y = 1 \, , \; y = e$

① Gambar

$\quad x = \frac{1}{y}$

$\quad x \, | \, y$

$\frac{1}{e}\!\uparrow\!\downarrow\!e$

② titik berat

$$\bar{x} = \frac{My}{M} \quad ; \quad \bar{y} = \frac{Mx}{M}$$

- $M = \int_c^d x\,dy$

  $= \int_1^e \frac{1}{y}\,dy$

  $= \ln y \big|_1^e$

  $= \ln e - \ln 1 = 1 - 0 = 1$

  $\boxed{\ln e^x = x \\ \ln 1 = 0}$

- $My = \frac{1}{2}\int_c^d x^2\,dy$

  $= \frac{1}{2}\int_1^e \left(\frac{1}{y}\right)^2 dy$

  $= \frac{1}{2}\int_1^e y^{-2}\,dy$

  $= \frac{1}{2}\left[-\frac{1}{y}\right]\big|_1^e = \frac{1}{2}\left[-\frac{1}{e}+1\right]$

- $Mx = \int_c^d y\,x\,dy$

  $= \int_1^e y\left(\frac{1}{y}\right)dy$

  $= \int_1^e 1\,dy$

  $= y\big|_1^e = e-1$

Jadi, $\bar{x} = \frac{My}{M} = \dfrac{\frac{1}{2}\left[-\frac{1}{e}+1\right]}{1} = \frac{1}{2}\left[-\frac{1}{e}+1\right]$

$\bar{y} = \frac{Mx}{M} = \frac{e-1}{1} = e-1$

$(\bar{x},\bar{y}) = \left(\frac{1}{2}\left[-\frac{1}{e}+1\right],\ e-1\right)$

3. Diberikan persamaan parametrik $x = t - 1$ dan $y = \frac{1}{2t-3}$ dengan $t > \frac{3}{2}$. Dapatkan persamaan garis singgung kurva pada $t = 2$ serta sketsa. *(EAS 2022/2023, Selasa 13 Juni 2023)*

- Persamaan garis singgung kurva

$$y - y_1 = m(x - x_1)$$

- $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

- $m = \frac{dy}{dx}\Big|_{t=a}$

Jawab.

$$x = t-1 \quad ; \quad y = \frac{1}{2t-3} \quad ; \quad t \geq \frac{3}{2}$$

① Mencari pgs di $t = 2$

a. $x = t-1$

$$\frac{dx}{dt} = 1$$

b. $y = \frac{1}{2t-3} = (2t-3)^{-1}$

$$\frac{dy}{dt} = -(2t-3)^{-2} \cdot 2 = \frac{-2}{(2t-3)^2}$$

c. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{-2}{(2t-3)^2}}{1} = \frac{-2}{(2t-3)^2}$

d. $m = \frac{dy}{dx}\Big|_{t=2} = \frac{-2}{(4-3)^2} = -2$

e. $x_1 = x(2) = 2-1 = 1$

$$y_1 = y(2) = \frac{1}{2(2)-3} = 1$$

Jadi, pgsnya $\quad y - y_1 = m(x - x_1)$

$$y - 1 = -2(x-1)$$

$$y = -2x + 2 + 1$$

$$y = -2x + 3$$

② Gambar kurva.

$x = t - 1 \rightarrow t = x + 1 \cdots (1)$

$y = \dfrac{1}{2t - 3} \cdots (2)$

Untuk mengeliminasi parameter t, substitusi
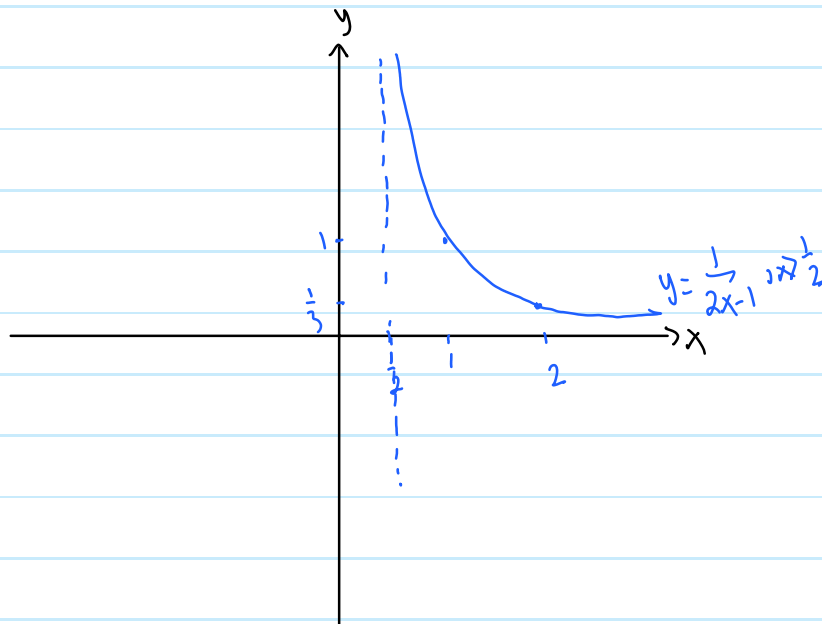persamaan (1) ke (2)

$y = \dfrac{1}{2(x+1) - 3}$

$y = \dfrac{1}{2x - 1} , x > \dfrac{1}{2}$

* Batas
$t > \dfrac{3}{2} \rightarrow x > \dfrac{3}{2} - 1$
$\qquad\qquad x > \dfrac{1}{2}$

x	y
1	1
2	$\frac{1}{3}$
4	$\frac{1}{7}$

$y = \dfrac{1}{2x - 1} , x > \dfrac{1}{2}$

4. Dapatkan luas daerah irisan dari $r = 2\sin\theta$ dan $r = 2\cos\theta$. *(EAS 2022/2023, Senin 12 Juni 2023)*

Jawab

$r = 2\sin\theta , r = 2\cos\theta$

① Titik potong

$r_1 = r_2$

$2\sin\theta = 2\cos\theta$

$\sin\theta = \cos\theta$

$\theta = \dfrac{\pi}{4}$

$\circ$ $r = 2a\sin\theta$
Lingkaran $P(0,a)$ dgn $r = a$
$\bullet$ $r = 2a\cos\theta$
Lingkaran $P(a,0)$ dgn $r = a$

② Gambar

$\bullet$ $r = 2\sin\theta$

$r = 2.1 \sin\theta$ (Lingkaran $P(0,1)$ dan $r = 1$)

- $r = 2\cos\theta$

$r = 2.1\cos\theta$ (Lingkaran $P(1,0)$ dan $r = 1$)

$\theta = \frac{\pi}{4}$

$L_1 = L_2$
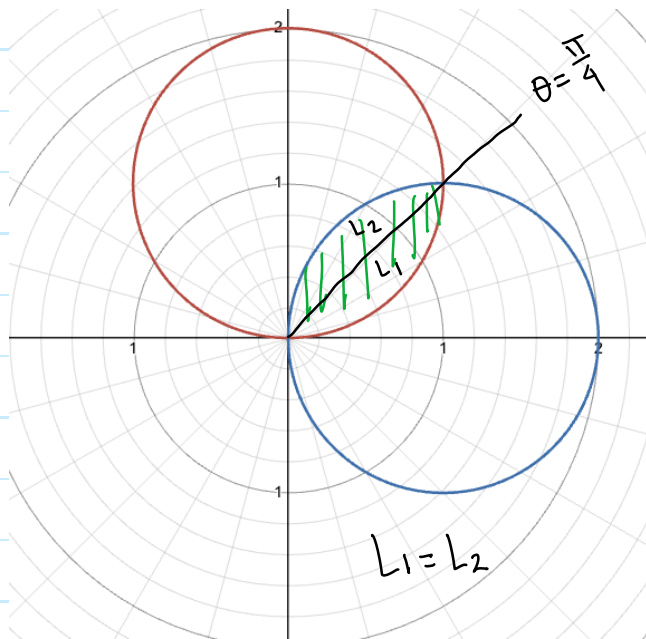
③ Luas

$dL = dL_1 + dL_2$

$dL = 2\, dL_1$

$dL = 2\left[\frac{1}{2} r^2\, d\theta\right]$

$dL = 2\left[\frac{1}{2}(2\sin\theta)^2\, d\theta\right]$

$L = \int_0^{\frac{\pi}{4}} 2\left[\frac{1}{2}(2\sin\theta)^2\right] d\theta$

$= \int_0^{\pi/4} (2\sin\theta)^2\, d\theta$

$= \int_0^{\pi/4} 4\sin^2\theta\, d\theta$

$= \int_0^{\pi/4} 4\left[\frac{1}{2} - \frac{1}{2}\cos 2\theta\right] d\theta$

$= \int_0^{\pi/4} 2 - 2\cos 2\theta\, d\theta$

$= \left[2\theta - \sin 2\theta\right]\Big|^{\pi/4}$

$$= \frac{\pi}{2} - \sin \frac{\pi}{2} - 0$$

$$= \frac{\pi}{2} - 1$$

5. Diberikan fungsi $f(x) = 2 - x + 3x^2 - x^3$. (EAS 2022/2023, Selasa 13 Juni 2023)
   (a) Dapatkan polinomial Taylor derajat 4 dari fungsi tersebut di sekitar $x = -1$.
   (b) Dapatkan deret Taylor fungsi tersebut di sekitar $x = -1$ dan nyatakan dalam notasi sigma.

Polinomial Taylor   disekitar $x = a$

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^n + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Deret dan notasi sigma

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots$$

Jawab
$$f(x) = 2 - x + 3x^2 - x^3$$

5a. $f(x) = 2 - x + 3x^2 - x^3$ → $f(-1) = 2 + 1 + 3 + 1 = 7$
   $f'(x) = -1 + 6x - 3x^2$ → $f'(-1) = -1 - 6 - 3 = -10$
   $f''(x) = 6 - 6x$ → $f''(-1) = 6 + 6 = 12$
   $f'''(x) = -6$ → $f'''(-1) = -6$
   $f^{(4)}(x) = 0$ → $f^{(4)}(-1) = 0$

Jadi
$$P_4(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4$$

$$P_4(x) = 7 - 10(x+1) + \frac{12}{2!}(x+1)^2 - \frac{6}{3!}(x+1)^3 + \frac{0}{4!}(x+1)^4$$

$$P_4(x) = 7 - 10(x+1) + 6(x+1)^2 - (x+1)^3$$

5b. Deret dan notasi sigma
   Deretnya adalah

$$2 - x + 3x^2 - x^3 = 7 - 10(x+1) + 6(x+1)^2 - (x+1)^3$$
$$= 7 - 10x - 10 + 6(x^2 + 2x + 1) - (x^3 + 3x^2 + 3x + 1)$$
$$= 7 - 10x - 10 + 6x^2 + 12x + 6 - x^3 - 3x^2 - 3x - 1$$
$$= -x^3 + 3x^2 - x + 2$$