Materi : Fungsi Transenden : Fungsi Logaritma, Fungsi Eksponensial, dan Fungsi Inversnya.

1 Fungsi Logaritma

Dalam aljabar, logaritma didefinisikan sebagai pangkat.

Lebih tepatnya jika $b > 0$ dan $b \neq 1$, maka untuk x positif, didefinisikan

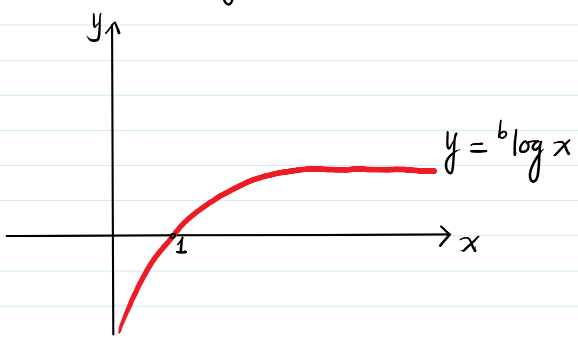
$\underbrace{{}^{b}\log x}_{\text{dibaca}}$: logaritma berbasis b dari x.

※ $y = {}^{b}\log x \implies b^{y} = x$ ✓ $\implies 3 = {}^{2}\log 8 \implies 2^{3} = 8$

Sehingga diperoleh hubungan

$${}^{b}\log b^{y} = y \quad \text{dan} \quad b^{{}^{b}\log x} = x$$

※ Grafik fungsi logaritma



∘> Domain : $\overbrace{(0, +\infty)}^{x > 0}$

∘> Range : $\underbrace{(-\infty, +\infty)}_{y \in \mathbb{R}}$

2 Sifat - sifat Operasi Logaritma

(i). ${}^{b}\log 1 = 0 \implies b^{0} = 1$

(ii). ${}^{b}\log b = 1 \implies b^{1} = b$

(iii). ${}^{b}\log ac = {}^{b}\log a + {}^{b}\log c$

(iv). ${}^{b}\log \frac{a}{c} = {}^{b}\log a - {}^{b}\log c$

(v). ${}^{b}\log a^{n} = n \cdot {}^{b}\log a$

(vi). ${}^{b}\log \frac{1}{c} = -{}^{b}\log c$ $= {}^{b}\log c^{-1} = -1 \cdot {}^{b}\log c = -{}^{b}\log c$

Latihan Soal

1. ${}^{3}\log 27 = \dots ?$

${}^{3}\log 27 = {}^{3}\log 3^{3} = 3 \cdot {}^{3}\log 3 = 3 \cdot 1 = 3$

$${}^{3}\log 27 = {}^{3}\log 3^{3} = 3 \cdot \underbrace{{}^{3}\log 3}_{1} = 3 \cdot 1 = 3$$

② Uraikan logaritma berikut

$$\log \frac{x \cdot \sin^2 x}{x+2} = \ldots ?$$

$$\log \frac{x \cdot \sin^2 x}{x+2} = \log x + \overbrace{\log \sin^2 x}^{\log (\sin x)^2} - \log (x+2)$$

$$= \log x + 2 \cdot \log \sin x - \log (x+2)$$

③ Tuliskan sebagai logaritma tunggal

$$\frac{1}{3} \log x - 2 \log (\sin 3x) + 2 = \ldots ?$$

$$2 = {}^{10}\log a \Longrightarrow 10^2 = a$$
$$100 = a$$

$$= \frac{1}{3} \log x - 2 \log (\sin 3x) + \log 100$$

$$= \log x^{\frac{1}{3}} - \log (\sin 3x)^2 + \log 100$$

$$= \log \frac{x^{\frac{1}{3}} \cdot 100}{(\sin 3x)^2} = \log \frac{100 \cdot \sqrt[3]{x}}{\sin^2 3x}$$

${}^{b}\log x$

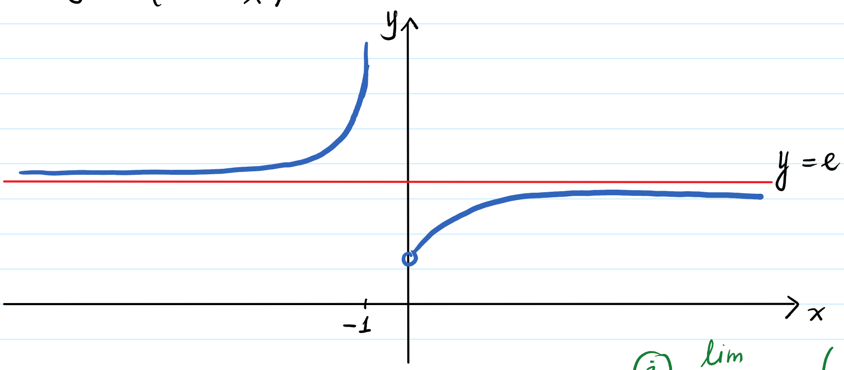
Jika basis logaritma adalah 10 $(b = 10)$, maka b tidak perlu ditulis.

$${}^{10}\log x \longrightarrow \boxed{\log x}$$

3 Bilangan Natural e

e = bilangan euler $\longrightarrow$ $e \approx 2{,}718282$

$$\text{※} \quad y = \left(1 + \frac{1}{x}\right)^{x}$$



(i). $\lim\limits_{x \to +\infty} \left(1 + \frac{1}{x}\right)^{x} = e$

(ii). $\lim\limits_{x \to -\infty} \left(1 + \frac{1}{x}\right)^{x} = e$

(iii). $\lim\limits_{x \to 0} (1 + x)^{\frac{1}{x}} = e$

(i) $\lim\limits_{x \to +\infty} \left(1 + \frac{1}{x}\right)^{x} = e$

Misalkan $t = \frac{1}{x} \longrightarrow x = \frac{1}{t}$

Untuk $x \to +\infty$, maka $t \to ?$

$$t = \frac{1}{+\infty} = 0$$

$$t \to 0$$

$$= \lim_{t \to 0} (1 + t)^{\frac{1}{t}} = e$$

$$= \lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e$$

Latihan Soal

① $\lim\limits_{x \to +\infty} \left(1 + \frac{7}{x}\right)^{x} = \ldots ?$

① $\lim_{x \to +\infty} \left(1 + \frac{7}{x}\right)^x = \dots ?$

Misalkan : $t = \frac{7}{x} \to x = \frac{7}{t}$

Untuk $x \to +\infty$, maka $t = \frac{7}{x} = \frac{7}{+\infty} = 0$ } $t \to 0$

$= \lim_{t \to 0} (1 + t)^{\frac{7}{t}}$

$(a^{mn}) = (a^m)^n$

$= \lim_{t \to 0} \left[(1+t)^{\frac{1}{t}} \right]^7$

$= \left[\lim_{t \to 0} (1+t)^{\frac{1}{t}} \right]^7$

$= e^7$

※ Logaritma dengan basis e, disebut sebagai logaritma natural (ln).

${}^e\log x \longrightarrow \ln x$

(i). $\ln 1 = 0$

(ii). $\ln e = 1$

Sifat - sifat :

1). $\ln ac = \ln a + \ln c$

2). $\ln \frac{1}{c} = -\ln c$

3). $\ln \frac{a}{c} = \ln a - \ln c$

4). $\ln a^n = n \cdot \ln a$

∘ $(\ln a)^n \neq n \cdot \ln a$

∘ $\ln(a+b) \neq \ln a + \ln b$

∘ $\frac{\ln a}{\ln b} \neq \ln a - \ln b$

Pernyataan : $y = \ln x$ dan $x = e^y$, ekivalen.

Selanjutnya, $\underbrace{\ln e^x}_{x \cdot \ln e} = x$ dan $e^{\ln x} = x$

$x \cdot \ln e = x$

4. Turunan Fungsi Logaritma

$\boxed{(i).\ \frac{d}{dx}\left[{}^b\log u\right] = \frac{1}{u \cdot \ln b} \cdot \frac{du}{dx}}$ ✓

Latihan Soal

① $\frac{d}{dx}\left[{}^3\log(5x^2+1)\right] = \dots ?$

$= \frac{1}{(5x^2+1) \cdot \ln 3} \cdot \frac{d}{dx}\left[5x^2+1\right]$

$= \frac{1}{(5x^2+1) \cdot \ln 3} \cdot (10x)$

$= \frac{10x}{(5x^2+1) \cdot \ln 3}$

$\boxed{(ii).\ \frac{d}{dx}\left[\ln u\right] = \frac{1}{u} \cdot \frac{du}{dx}}$

Latihan Soal

② $\frac{d}{dx}\left[\ln\left(\frac{x}{1+x^2}\right)\right] = \dots ?$

$= \frac{1}{\frac{x}{1+x^2}} \cdot \frac{d}{dx}\underbrace{\left(\frac{x}{1+x^2}\right)}$

$\frac{u}{v} = \frac{u'v - uv'}{v^2}$

$= \frac{1+x^2}{x} \cdot \left(\frac{1 \cdot (1+x^2) - x(2x)}{(1+x^2)^2}\right)$

$= \frac{\cancel{1+x^2}}{x} \cdot \left(\frac{1+x^2-2x^2}{(1+x^2)^{\cancel{2}\,1}}\right)$

$= \frac{1-x^2}{x(1+x^2)}$

$\frac{d}{dx}\left[\ln\left(\frac{x}{1+x^2}\right)\right] = \frac{d}{dx}\left[\ln x - \ln(1+x^2)\right]$

$= \frac{1}{x} - \frac{1}{(1+x^2)} \cdot 2x = \frac{1}{x} - \frac{2x}{(1+x^2)}$

$= \frac{(1+x^2) - 2x \cdot x}{x(1+x^2)}$

$= \frac{1-x^2}{x(1+x^2)}$

$\boxed{\frac{a}{b} - \frac{c}{d} = \frac{ad - cb}{bd}}$

$\boxed{5}$ Integral Fungsi Logaritmik (ln)

$$\boxed{\int \frac{1}{u}\, du = \ln|u| + C}$$

Hati – hati :

$$\int \frac{1}{2x}\, dx \neq \ln|2x| + C$$

Contoh :

(i) $\int \frac{1}{2x}\, dx = \ldots ?$

Substitusi : $u = 2x$

$\frac{du}{dx} = 2$

$\frac{1}{2} \cdot du = dx$

$$= \int \frac{1}{u} \cdot \frac{1}{2}\, du$$

$$= \frac{1}{2} \int \frac{1}{u}\, du$$

$$= \frac{1}{2} \ln|u| + C$$

$$= \frac{1}{2} \ln|2x| + C$$

(ii) $\int \frac{x^2}{x^3 - 4}\, dx = \ldots ?$

Substitusi : $u = x^3 - 4$

$\frac{du}{dx} = 3x^2$

$\frac{1}{3x^2} \cdot du = dx$

$$= \int \frac{\cancel{x^2}}{u} \cdot \frac{1}{3\cancel{x^2}} \cdot du$$

$$= \frac{1}{3} \int \frac{1}{u}\, du$$

$$= \frac{1}{3} \ln|u| + C$$

$$= \frac{1}{3} \ln|x^3 - 4| + C$$

$\boxed{6}$ Fungsi Eksponensial

Bentuk b^x adalah bilangan berpangkat (eksponen), dengan x adalah bilangan real.

$$b^x = \underbrace{b \times b \times b \times \cdots \times b}_{x \text{ faktor}}$$

b : basis / bilangan pokok

x : bilangan pangkat

※ Sifat – Sifat Operasi Eksponen

1). $b^x \times b^y = b^{x+y}$

2). $\dfrac{b^x}{b^y} = b^{x-y}$

3). $b^0 = 1$

4). $\dfrac{1}{b^x} = b^{-x}$

5). $(b^x)^y = b^{xy}$

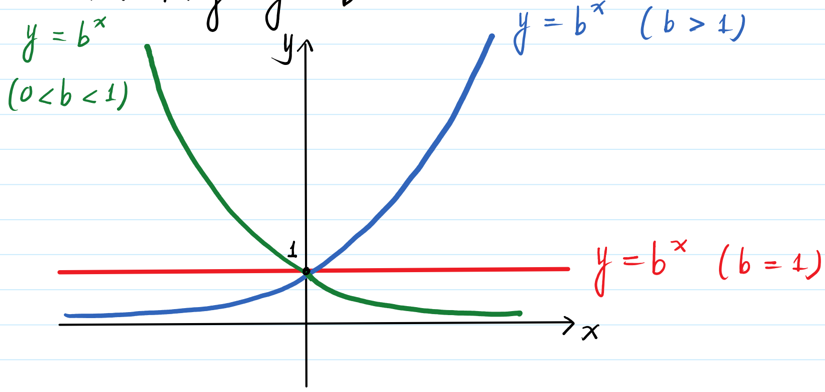
6). $(ab)^x = a^x \times b^x$

7). $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

8). $b^{\frac{x}{y}} = \sqrt[y]{b^x}$

9). $\sqrt[y]{\sqrt[x]{b^k}} = b^{\frac{k}{x \cdot y}}$

※ Grafik Fungsi $y = b^x$

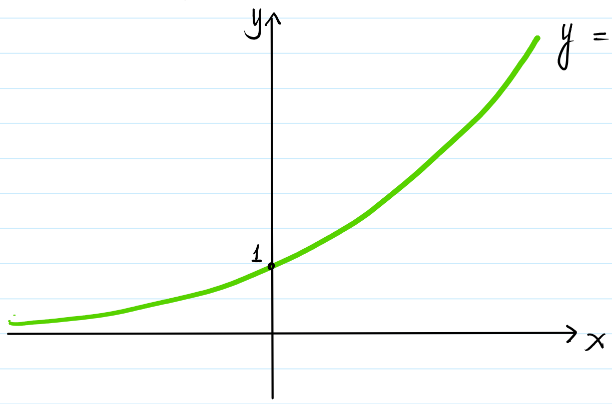


•> $y = b^x$

Domain : $(-\infty, +\infty)$

Range : $(0, +\infty)$

※ Grafik Fungsi $y = e^x$



•> $y = e^x$

Domain : $(-\infty, +\infty)$

Range : $(0, +\infty)$

7 Turunan Fungsi Eksponensial

$\boxed{(i).\ \dfrac{d}{dx}\left[b^u\right] = b^u \cdot \ln b \cdot \dfrac{du}{dx}}$ ✓ $3^x, 4^x$, dst $\Big\}$ $x^n \longrightarrow n \cdot x^{n-1}$

Contoh :

① $\dfrac{d}{dx}\left[2^{x^2}\right] = \ldots ?$

$= 2^{x^2} \cdot \ln 2 \cdot \dfrac{d}{dx}\left[x^2\right] = 2^{x^2} \cdot \ln 2 \cdot 2x$

② $\dfrac{d}{dx}\left[5^{x^3 \cdot \sin 2x}\right] = \ldots ?$

$= 5^{x^3 \cdot \sin 2x} \cdot \ln 5 \cdot \dfrac{d}{dx}\left[x^3 \cdot \sin 2x\right]$

$$dx \quad \underbrace{\qquad}_{u \cdot v \,=\, u'v + uv'}$$

$$= 5^{x^3 \cdot \sin 2x} \cdot \ln 5 \cdot \left[3x^2 \cdot \sin 2x + x^3 \cdot \cos 2x \cdot 2 \right]$$

$$= 5^{x^3 \cdot \sin 2x} \cdot \ln 5 \left(3x^2 \cdot \sin 2x + 2x^3 \cdot \cos 2x \right)$$

$$\boxed{(ii).\ \frac{d}{dx}\left[e^{u} \right] = e^{u} \cdot \frac{du}{dx}}$$

Contoh :

① $\frac{d}{dx}\left[e^{-3x} \right] = \ldots ?$

$$= e^{-3x} \cdot -3 = -3 \cdot e^{-3x}$$

② $\frac{d}{dx}\left[e^{\frac{x^2}{x+2}} \right] = \ldots ?$

$$= e^{\frac{x^2}{x+2}} \cdot \frac{d}{dx}\underbrace{\left[\frac{x^2}{x+2} \right]}_{\frac{u}{v} = \frac{u'v - uv'}{v^2}}$$

$$= e^{\frac{x^2}{x+2}} \left(\frac{2x(x+2) - x^2(1)}{(x+2)^2} \right)$$

$$= e^{\frac{x^2}{x+2}} \cdot \left[\frac{x^2 + 4x}{(x+2)^2} \right]$$

8 Integral Fungsi Eksponensial

$$\boxed{(i).\ \int b^{u}\, du = \frac{b^{u}}{\ln b} + C}$$

Contoh :

① $\int 2^{x}\, dx = \ldots ?$

$$= \frac{2^{x}}{\ln 2} + C$$

$$\boxed{(ii).\ \int e^{u}\, du = e^{u} + C}$$

$$\boxed{\int e^{ax}\, dx = \frac{1}{a} \cdot e^{ax} + C}$$

Contoh :

① $\int e^{-3x}\, dx = \ldots ?$

Substitusi : $u = -3x$

$\frac{du}{dx} = -3$

$-\frac{1}{3}du = dx$

$$\to \int e^{-3x}\, dx = \int e^{u} \cdot -\frac{1}{3}\, du$$

$$= -\frac{1}{3}\int e^{u}\, du$$

$$= -\frac{1}{3} e^{u} + C$$

$$= -\frac{1}{3} e^{-3x} + C$$

② $\int x^2 \cdot e^{-2x^3}\, dx = \ldots ?$

Substitusi : $u = -2x^3$

$\frac{du}{dx} = -6x^2$

$$= -\frac{1}{6}\int e^{u}\, du$$

Substitusi : $u = -2x^3$

$\frac{du}{dx} = -6x^2$

$-\frac{1}{6x^2}\,du = dx$

$= \int \cancel{x^2} \cdot e^u \cdot -\frac{1}{\cancel{6x^2}}\,du$

$= -\frac{1}{6}\int e^u\,du$

$= -\frac{1}{6} e^u + C$

$= -\frac{1}{6} e^{-2x^3} + C$

$e^{\ln x} = x$

③ $\int_{-\ln 3}^{\ln 3} \frac{e^x}{e^x+4}\,dx = \dots ?$ $\rightarrow$ $= \int_{13/3}^{7} \frac{\cancel{e^x}}{u} \cdot \frac{1}{\cancel{e^x}} \cdot du$

Substitusi $u = e^x + 4$ ✓

$\frac{du}{dx} = e^x$

$\frac{1}{e^x}\,du = dx$

$= \int_{13/3}^{7} \frac{1}{u}\,du$

$= \ln|u| \Big|_{13/3}^{7}$

$= \ln 7 - \ln \frac{13}{3}$

(*) $e^{-\ln 3}$
$= e^{-1\cdot\ln 3}$
$= e^{\ln 3^{-1}}$
$= e^{\ln\frac{1}{3}}$
$= \frac{1}{3}$

Batas integral :

› $x = -\ln 3 \rightarrow u = e^{-\ln 3} + 4$ (*) $\frac{1}{3} + 4$

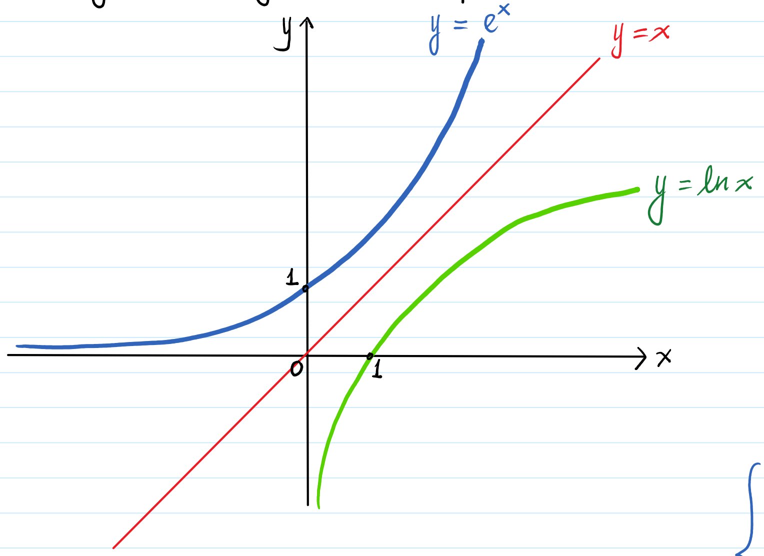
$= \frac{13}{3}$

› $x = \ln 3 \rightarrow u = e^{\ln 3} + 4$

$= 3 + 4$

$= 7$

$\boxed{9}$ Fungsi Invers Logaritma dan Eksponensial



› e^x dan $\ln x$ merupakan fungsi yang saling invers.

(i). $y = e^x$

Domain : $(-\infty, +\infty)$ ✓

Range : $(0, +\infty)$ ✓

(ii). $y = \ln x$

Domain : $(0, +\infty)$ ✓

Range : $(-\infty, +\infty)$ ✓

f^{-1} adalah invers $\underbrace{\text{fungsi } f}_{D_f \;\&\; R_f}$

$D_{f^{-1}} = R_f$

$R_{f^{-1}} = D_f$

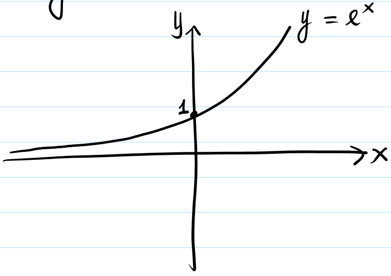
Contoh :

$f(x) = e^x$

Contoh :

(1) Sketsa grafik $y = e^x - 1$

(i) $y = e^x$



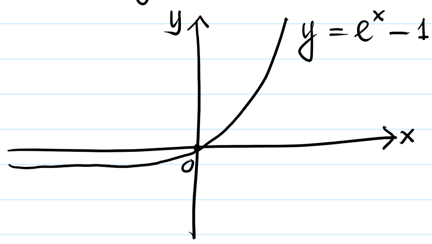
$f(x) = e^x$

$f(x) - 1$

(ii) $y = \boxed{e^x - 1}$

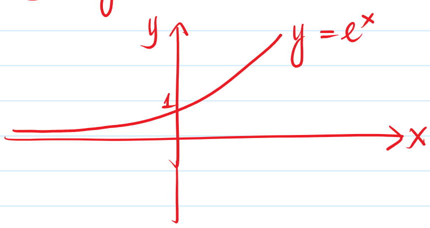
$f(x) + a \longrightarrow$ keatas
$f(x) - a \longrightarrow$ kebawah

Grafik (i) digeser kebawah sejauh 1 satuan



(2) $y = e^{x-1}$

(i) $y = e^x$



$f(x) = e^x$

$f(x-1)$

(ii) $y = \boxed{e^{x-1}}$

$f(x+a) \longrightarrow$ kekiri
$f(x-a) \longrightarrow$ kekanan