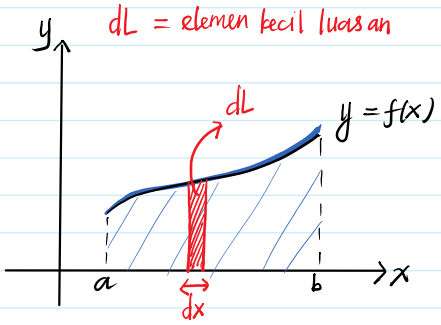
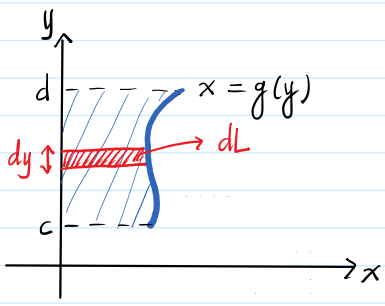
Materi : Aplikasi Integral untuk Menghitung Luasan & Volume

A Luas yang dibatasi 1 Fungsi

(I) $y = f(x)$; $a \le x \le b$; sb. x (II) $x = g(y)$; $c \le y \le d$; sb. y





$dL = f(x) \cdot dx$

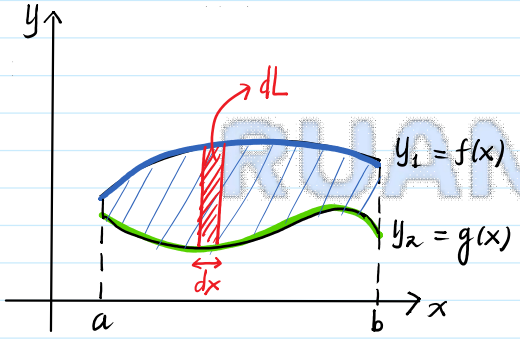
$L = \int_a^b f(x)\, dx$

$dL = g(y) \cdot dy$

$L = \int_c^d g(y)\, dy$

B Luas yang dibatasi 2 Fungsi

(I) Luas antara $y_1 = f(x)$ dan $y_2 = g(x)$; $a \le x \le b$

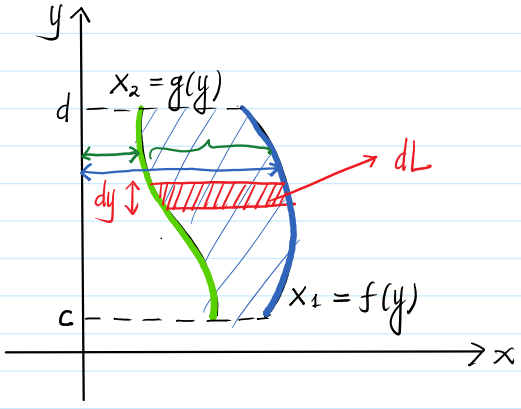


kurva atas − kurva bawah

$dL = [f(x) - g(x)] \cdot dx$

$L = \int_a^b [f(x) - g(x)]\, dx$

(II) Luas antara $x_1 = f(y)$ dan $x_2 = g(y)$; $c \le y \le d$



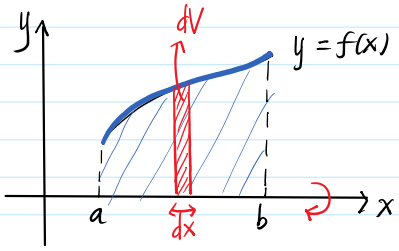
kurva kanan − kurva kiri

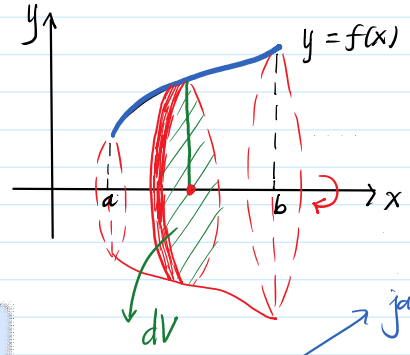
$dL = [f(y) - g(y)] \cdot dy$

$L = \int_c^d [f(y) - g(y)] \cdot dy$

C Volume Benda Putar Metode Cakram (1 Fungsi)

(I) $y = f(x)$; $a \le x \le b$; sb. x ; diputar terhadap sb. x

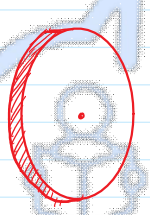




dV = elemen kecil volume

jarak grafik ke sb. putar (panjang dari dV)

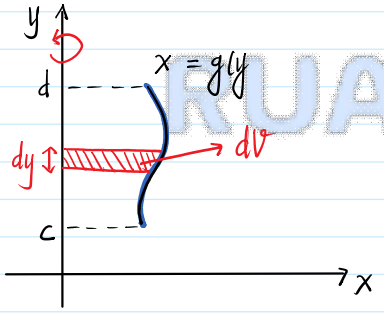
Metode Cakram :
Posisi dV ⊥ sb. putar

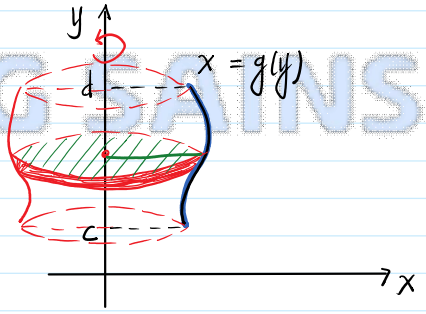


$$dV = \pi \cdot [f(x)]^2 \cdot dx$$

$$V = \int_a^b \pi \cdot [f(x)]^2 \, dx$$

(II) $x = g(y)$; $c \le y \le d$; sb. y ; diputar terhadap sb. y.



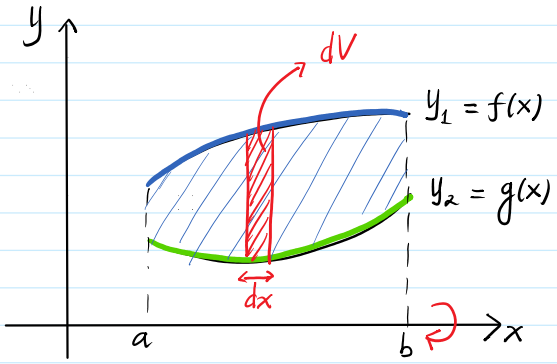


$$dV = \pi [g(y)]^2 \cdot dy$$

$$V = \int_c^d \pi [g(y)]^2 \, dy$$

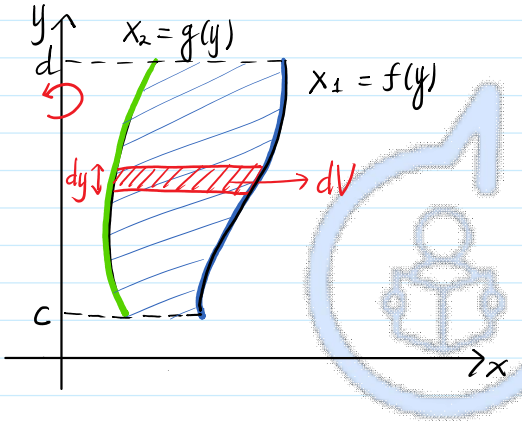
D Volume Benda Putar Metode Cakram (2 Fungsi)

(I) $y_1 = f(x)$; $y_2 = g(x)$; diputar terhadap sb. x



$$dV = \pi \cdot [f(x)]^2 \cdot dx - \pi \cdot [g(x)]^2 \cdot dx$$
$$= \pi \left([f(x)]^2 - [g(x)]^2 \right) \cdot dx$$
$$V = \int_a^b \pi \left([f(x)]^2 - [g(x)]^2 \right) dx$$

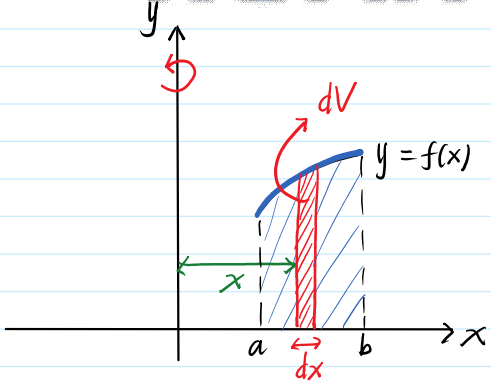
(II) $x_1 = f(y)$; $x_2 = g(y)$; diputar terhadap sb. y

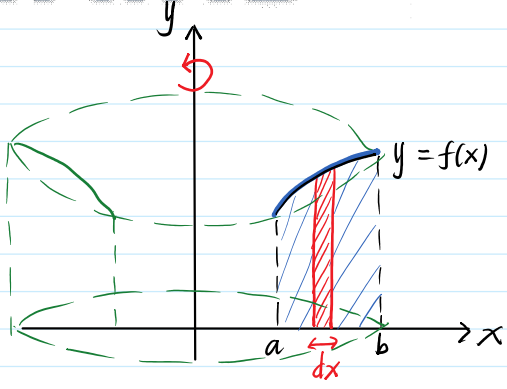


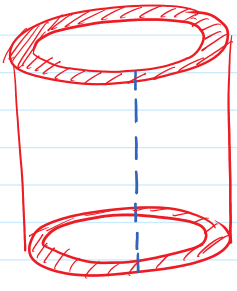
$$dV = \pi \cdot [f(y)]^2 \cdot dy - \pi \cdot [g(y)]^2 \cdot dy$$
$$= \pi \left([f(y)]^2 - [g(y)]^2 \right) dy$$
$$V = \int_c^d \pi \cdot \left([f(y)]^2 - [g(y)]^2 \right) dy$$

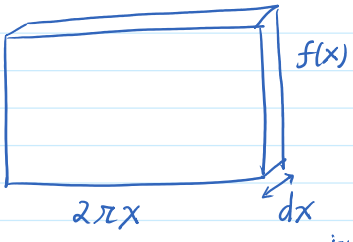
E Volume Benda Putar Metode Cincin Silinder (1 Fungsi)

(I) $y = f(x)$; $a \le x \le b$; sb. x ; diputar terhadap sb. y









Metode Cincin Silinder : posisi dV // sb. putar

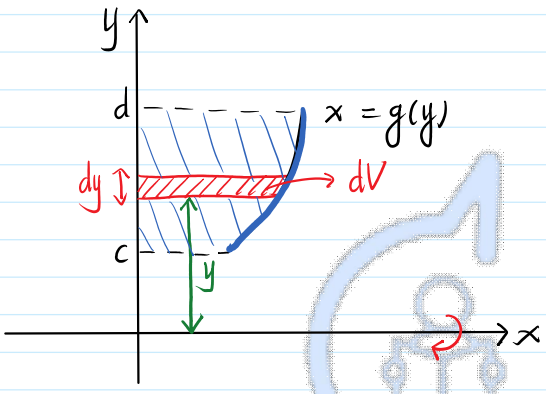
jarak dV ke sb. putar

$$dV = 2\pi x \cdot f(x)\, dx$$

$$V = \int_a^b 2\pi x \cdot f(x)\, dx$$

$$V = \int_a^b 2\pi x \cdot f(x)\, dx$$

(II) $x = g(y)$; $c \le y \le d$; sb. y ; diputar terhadap sb. x

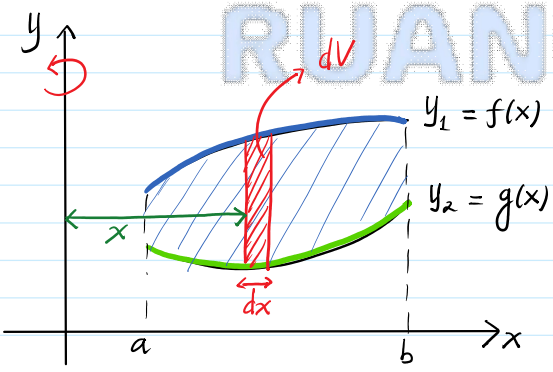


$$dV = 2\pi y \cdot g(y) \cdot dy$$

$$V = \int_c^d 2\pi y \cdot g(y) \cdot dy$$

F. Volume Benda Putar Metode Cincin Silinder (2 Fungsi)

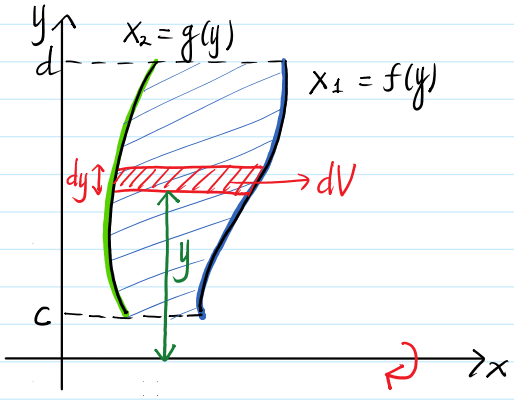
(I) $y_1 = f(x)$; $y_2 = g(x)$; diputar terhadap sb. y



$$dV = 2\pi x\,[f(x) - g(x)]\,dx$$

$$V = \int_a^b 2\pi x\,[f(x) - g(x)]\,dx$$

(II) $x_1 = f(y)$; $x_2 = g(y)$; diputar terhadap sb. x



$$dV = 2\pi y\,[f(y) - g(y)] \cdot dy$$

$$V = \int_c^d 2\pi y\,[f(y) - g(y)]\,dy$$

Contoh Soal

Dapatkan luas daerah yg dibatasi oleh kurva $y^2 = 2x$ dan $y = 2x - 2$;

Dapatkan luas daerah yg dibatasi oleh kurva $y^2 = 2x$ dan $y = 2x - 2$;
dengan mengintegralkan ① terhadap x
② terhadap y

$\Rightarrow y = ax^2$ $a>0$ ∪ $a<0$ ∩

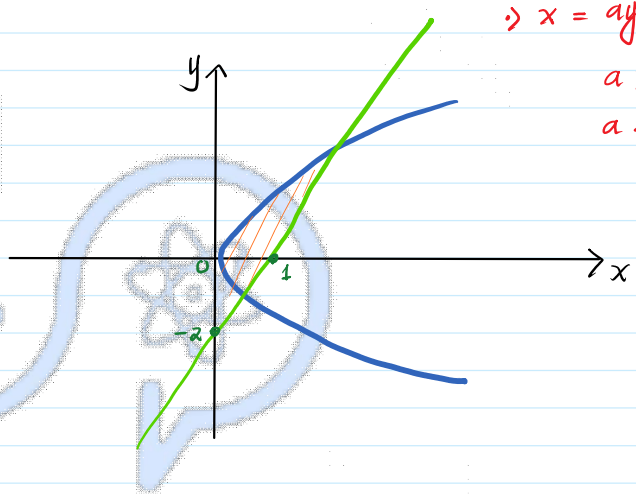
$\Rightarrow x = ay^2$ $a>0$ ⊂ $a<0$ ⊃

Grafik :

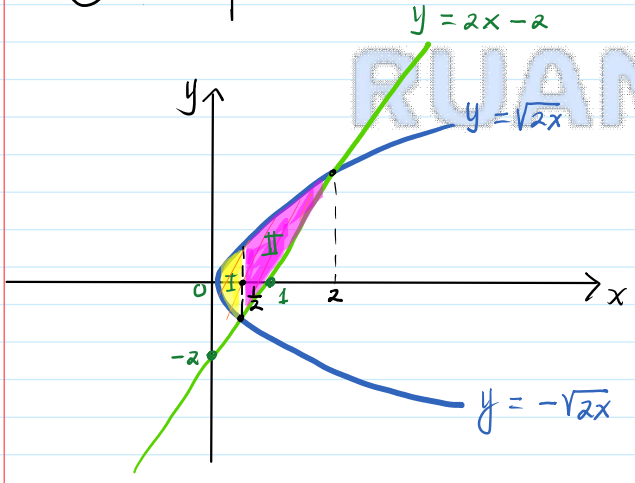
$\Rightarrow y^2 = 2x \longrightarrow x = \frac{1}{2}y^2$

$\Rightarrow y = 2x - 2$

x	0	1
y	-2	0



① terhadap x



$\Rightarrow y^2 = 2x$

$y_1 = \pm\sqrt{2x}$

$\Rightarrow y_2 = 2x - 2$

$\Rightarrow$ Menentukan titik potong :

$y_1 = y_2$

$\sqrt{2x} = 2x - 2$

$2x = 4x^2 - 8x + 4$

$4x^2 - 10x + 4 = 0$

$2(2x^2 - 5x + 2) = 0$

$2(2x - 1)(x - 2) = 0$

$x = \frac{1}{2}$; $x = 2$

$L = L_I + L_{II}$

$\Rightarrow L_I = \int_0^{1/2} \left(\sqrt{2x} - (-\sqrt{2x})\right) dx$

$\sqrt{2}\cdot\sqrt{x} = \sqrt{2}\cdot x^{1/2}$

$= \int_0^{1/2} 2\sqrt{2x}\, dx$

$= 2\sqrt{2} \cdot \frac{2}{3} x^{3/2} \Big|_0^{1/2}$

$\sqrt{\left(\frac{1}{2}\right)^3} = \sqrt{\frac{1}{8}} = \frac{\sqrt{1}}{\sqrt{8}} = \frac{1}{2\sqrt{2}}$

$= \frac{4\sqrt{2}}{3}\left[\left(\frac{1}{2}\right)^{3/2} - 0\right]$

$= \frac{4\sqrt{2}}{3} \cdot \frac{1}{2\sqrt{2}}$

$= \frac{2}{3}$

$\Rightarrow L_{II} = \int_{1/2}^{2} \left(\sqrt{2x} - (2x - 2)\right) dx$

$\sqrt{2}\cdot\sqrt{x} = \sqrt{2}\cdot x^{1/2}$

$= \int_{1/2}^{2} \sqrt{2x} - 2x + 2\, dx$

$= \sqrt{2} \cdot \frac{2}{3} x^{3/2} - x^2 + 2x \Big|_{1/2}^{2}$

$\sqrt{2^3} = \sqrt{8} = 2\sqrt{2}$

$= \left[\frac{2\sqrt{2}}{3} \cdot 2^{3/2} - 2^2 + 2\cdot 2\right]$

$- \left[\frac{2\sqrt{2}}{3} \cdot \left(\frac{1}{2}\right)^{3/2} - \left(\frac{1}{2}\right)^2 + 2\cdot\frac{1}{2}\right]$

$= \frac{8}{3} - \left[\frac{2\sqrt{2}}{3} \cdot \frac{1}{2\sqrt{2}} - \frac{1}{4} + 2\right]$

$= \frac{8}{3} - \frac{1}{3} + \frac{1}{4} - 2$

$= \frac{7}{3} - \frac{3}{4}$

$L = L_I + L_{II}$

$\frac{1}{3} - \frac{1}{3} \cdot \frac{1}{4} - a$

$= \frac{7}{3} - \frac{3}{4}$

$= \frac{28 - 9}{12}$

$= \frac{19}{12}$

$L = L_I + L_{II}$

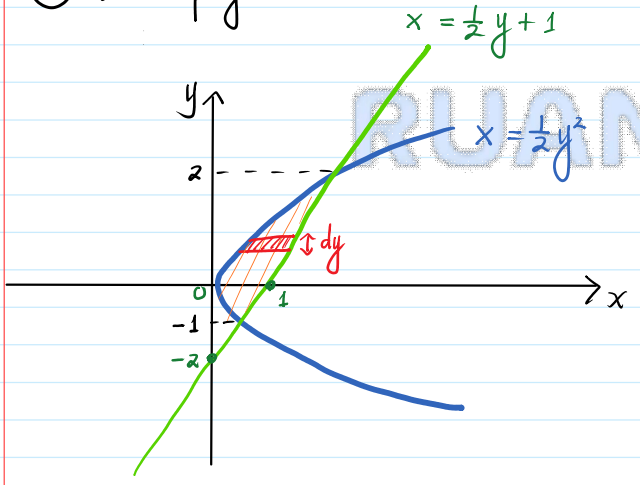
$= \frac{2}{3} + \frac{19}{12}$

$= \frac{8 + 19}{12}$

$= \frac{27}{12}$

$= \frac{9}{4}$ satuan luas

② terhadap y



$\rightarrow y^2 = 2x$

$x_1 = \frac{1}{2}y^2$

$\rightarrow y = 2x - 2$

$y + 2 = 2x$

$x_2 = \frac{1}{2}y + 1$

→ Menentukan titik potong

$x_1 = x_2$

$\frac{1}{2}y^2 = \frac{1}{2}y + 1$ (kedua ruas x2)

$y^2 = y + 2$

$y^2 - y - 2 = 0$

$(y + 1)(y - 2) = 0$

$y = -1 \;,\; y = 2$

$L = \int_{-1}^{2} \left(\frac{1}{2}y + 1\right) - \frac{1}{2}y^2 \, dy$

$= \frac{1}{2} \cdot \frac{1}{2}y^2 + y - \frac{1}{2} \cdot \frac{1}{3}y^3 \Big|_{-1}^{2}$

$= \frac{1}{4}y^2 + y - \frac{1}{6}y^3 \Big|_{-1}^{2}$

$= \left(\frac{1}{4} \cdot 2^2 + 2 - \frac{1}{6} \cdot 2^3\right) - \left(\frac{1}{4}(-1)^2 - 1 - \frac{1}{6}(-1)^3\right)$

$= \left(3 - \frac{8}{6}\right) - \left(\frac{1}{4} - 1 + \frac{1}{6}\right)$

$= 3 - \frac{8}{6} - \frac{1}{4} + 1 - \frac{1}{6}$

$= 4 - \frac{9}{6} - \frac{1}{4}$

$= \frac{48 - 18 - 3}{12}$

$= \frac{27}{12}$

$= \frac{9}{4}$ satuan luas

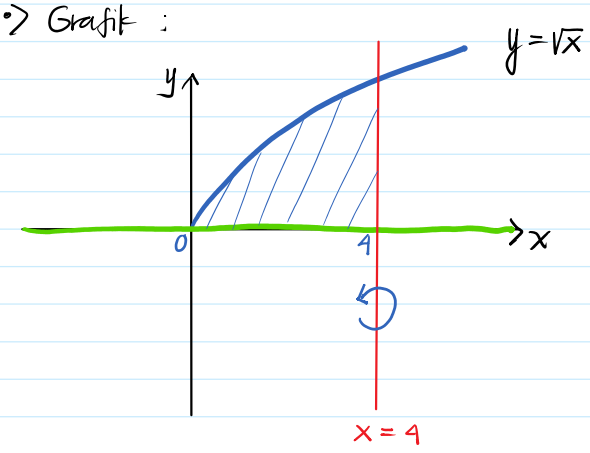
$= \frac{9}{4}$ satuan luas

Contoh Soal

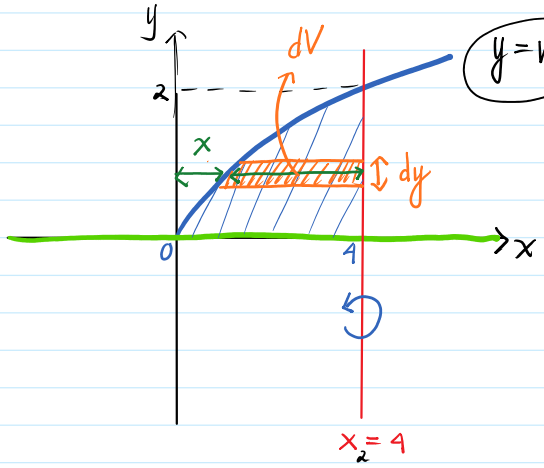
Dapatkan volume benda putar yg dihasilkan bila daerah yg dibatasi $y = \sqrt{x}$; $y = 0$; dan $x = 4$ diputar terhadap garis $x = 4$. ✓

① Metode Cakram
② Metode Cincin Silinder

⇒ Grafik :



① Metode Cakram



$y = \sqrt{x}$ → $x_1 = y^2$

⇒ $dL = \pi (4 - y^2)^2 \, dy$

$L = \int_0^2 \pi (4 - y^2)^2 \, dy$

$= \pi \int_0^2 16 - 8y^2 + y^4 \, dy$

$= \pi \left[16y - \frac{8}{3} y^3 + \frac{1}{5} y^5 \right]_0^2$

$= \pi \left[\left(32 - \frac{64}{3} + \frac{32}{5} \right) - (0 - 0 + 0) \right]$

$= \pi \left[\frac{480 - 320 + 96}{15} \right]$

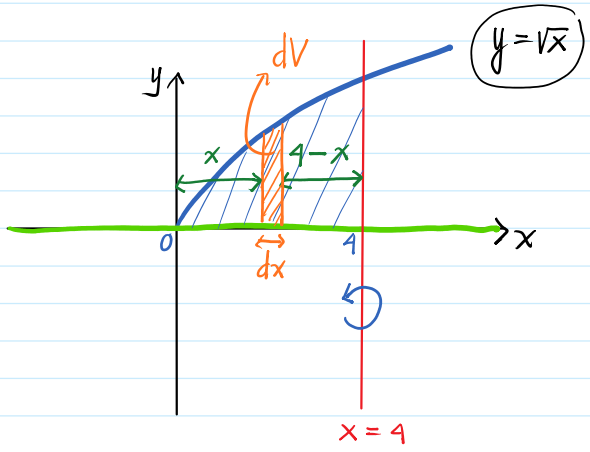
$= \pi \left(\frac{256}{15} \right)$ satuan volume

Titik potong :

$x_1 = x_2$
$y^2 = 4$
$y^2 - 4 = 0$

$(y+2)(y-2) = 0$
$y = -2$, $y = 2$ ✓

② Metode Cincin Silinder



$$\therefore dV = 2\pi (4-x)\cdot \sqrt{x}\ dx$$

$$V = \int_0^4 2\pi (4-x)\, x^{1/2}\, dx$$

$$= 2\pi \int_0^4 4x^{1/2} - x^{3/2}\, dx$$

$$= 2\pi \left[4\cdot \frac{2}{3} x^{3/2} - \frac{2}{5} x^{5/2} \right]_0^4$$

$$= 2\pi \left[\left(\frac{8}{3} \cdot (2^2)^{3/2} - \frac{2}{5} (2^2)^{5/2} \right) - (0-0) \right]$$

$$= 2\pi \left[\frac{8}{3} \cdot 2^3 - \frac{2}{5} \cdot 2^5 \right]$$

$$= 2\pi \left[\frac{64}{3} - \frac{64}{5} \right]$$

$$= 2\pi \left(\frac{320 - 192}{15} \right)$$

$$= 2\pi \left(\frac{128}{15} \right)$$

$$= \frac{256}{15}\pi \text{ satuan volume}$$