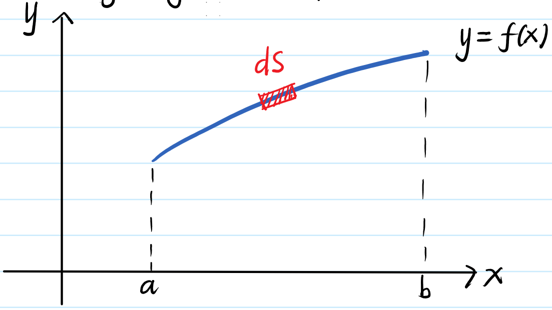
Materi : Aplikasi Integral untuk Menghitung Panjang Busur & Luas Kulit

A. Panjang Busur pada Bidang

(i) Integralnya terhadap x



→ ds = elemen kecil panjang busur

$ds \approx \Delta s$

$dy \approx \Delta y$

$dx \approx \Delta x$

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2$$

$$\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2} \quad (\text{dibagi } \Delta x)$$

$$\frac{\Delta s}{\Delta x} = \sqrt{\frac{(\Delta x)^2 + (\Delta y)^2}{(\Delta x)^2}}$$

$$\frac{\Delta s}{\Delta x} = \sqrt{1 + \frac{(\Delta y)^2}{(\Delta x)^2}}$$

$$\Delta s = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \cdot \Delta x \quad ✓$$

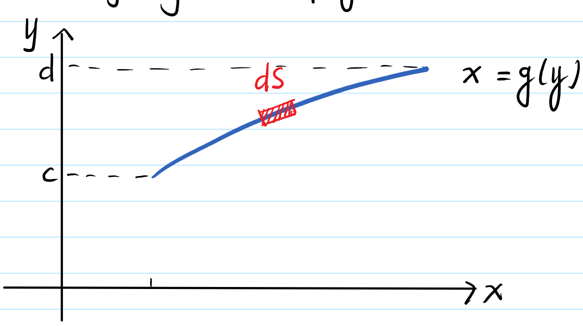
Untuk $\Delta x \to 0$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx$$

$$S = \int ds$$

$$\boxed{S = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx}$$

(ii) Integralnya terhadap y



$$\boxed{S = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy}$$

→ $x = g(y)$

$\frac{dx}{dy}$ → turunan dari fungsi $g(y)$

Contoh Soal :

1. Dapatkan panjang busur kurva $24xy = y^4 + 48$ dari $y = 2$ ke $y = 4$.

→ Menentukan $\frac{dx}{}$

$\therefore$ Menentukan $\frac{dx}{dy}$

$$24xy = y^4 + 48 \longrightarrow x = g(y)$$

$$x = \frac{y^4 + 48}{24y}$$

$$x = \frac{y^3}{24} + \frac{2}{y}$$

$$x = \frac{y^3}{24} + 2y^{-1}$$

$$\frac{dx}{dy} = \frac{\cancel{3}y^2}{\cancel{24}\,8} - 2y^{-2}$$

$$\frac{dx}{dy} = \frac{y^2}{8} - \frac{2}{y^2}$$

$$\frac{dx}{dy} = \frac{y^4 - 16}{8y^2}$$

$$\therefore S = \int_2^4 \sqrt{1 + \left(\frac{y^4-16}{8y^2}\right)^2}\, dy$$

$$= \int_2^4 \sqrt{1 + \left(\frac{y^8 - 32y^4 + 256}{64y^4}\right)}\, dy$$

$$= \int_2^4 \sqrt{\frac{64y^4 + y^8 - 32y^4 + 256}{64y^4}}\, dy$$

$$= \int_2^4 \sqrt{\frac{y^8 + 32y^4 + 256}{64y^4}}\, dy$$

$$\sqrt{\frac{(y^4+16)^2}{(8y^2)^2}} \qquad \rightarrow \frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$$

$$= \int_2^4 \sqrt{\left(\frac{y^4+16}{8y^2}\right)^2}\, dy$$

$$\sqrt{a^2} = a$$

$$= \int_2^4 \frac{y^4+16}{8y^2}\, dy$$

$$= \int_2^4 \frac{y^2}{8} + 2y^{-2}\, dy$$

$$= \frac{1}{8} \cdot \frac{1}{3} y^3 + 2 \cdot -y^{-1} \Big|_2^4$$

$$= \frac{1}{24} y^3 - \frac{2}{y} \Big|_2^4$$

$$= [\ldots$$

$\overline{24}\;0 \qquad \overline{y}\;|_2$

$$= \left[\frac{1}{24}\cdot 64 - \frac{2}{4}\right] - \left[\frac{1}{24}\cdot 8 - \frac{2}{2}\right]$$

$$= \frac{64}{24} - \frac{1}{2} - \frac{8}{24} + 1$$

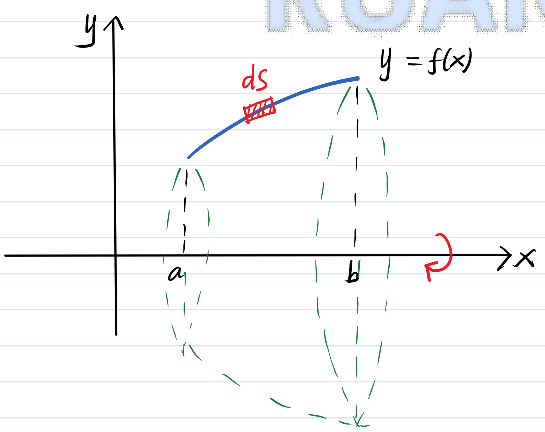
$$= \frac{56}{24} + \frac{1}{2}$$

$$= \frac{56+12}{24}$$

$$= \frac{68}{24} = \frac{17}{6} \text{ satuan panjang}$$

B. Luas Permukaan Benda Putar

(i). Luas permukaan apabila busur diputar pada sumbu x.



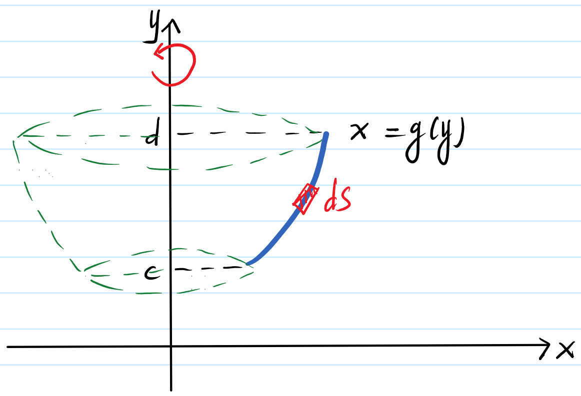
$\Rightarrow$ dK = elemen kecil luas permukaan

$$dK = 2\pi \cdot y \cdot ds$$

$$dK = 2\pi\, f(x) \cdot \sqrt{1+\left(\frac{dy}{dx}\right)^2}\; dx$$

$$\boxed{K = \int_a^b 2\pi \cdot f(x) \cdot \sqrt{1+\left(\frac{dy}{dx}\right)^2}\; dx}$$

(ii). Luas permukaan apabila busur diputar pada sumbu y.



$$dK = 2\pi \cdot x \cdot ds$$

$$dK = 2\pi \cdot g(y) \cdot \sqrt{1+\left(\frac{dx}{dy}\right)^2}\; dy$$

$$\boxed{K = \int_c^d 2\pi \cdot g(y) \cdot \sqrt{1+\left(\frac{dx}{dy}\right)^2} \cdot dy}$$

Contoh Soal

1. Dapatkan luas permukaan ; $x = |y-11|$; $0 \le y \le 2$; diputar thd sb. y.

$x = g(y)$

$\Rightarrow x = |y|$

$x = g(y-11)$

$\Rightarrow x = |y-11|$

$x = g(y)$

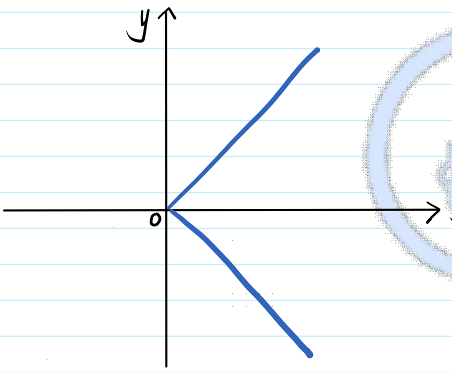
$x = |y|$

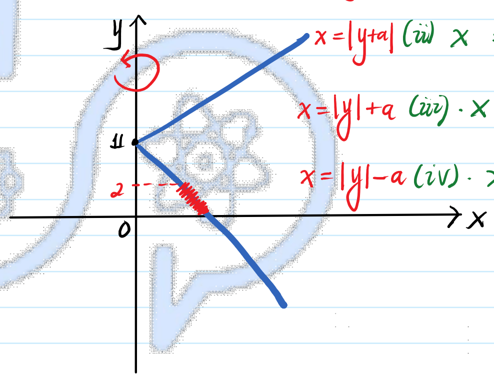
$x = |y-a|$ (i) $x = g(y-a) \rightarrow$ digeser keatas sejauh a

$x = |y+a|$ (ii) $x = g(y+a) \rightarrow$ digeser kebawah

y

y





$x = |y-a|$ (i) $x = g(y-a) \rightarrow$ digeser keatas sejauh a

$x = |y+a|$ (ii) $x = g(y+a) \rightarrow$ digeser kebawah sejauh a

$x = |y|+a$ (iii) $x = g(y)+a \rightarrow$ digeser kekanan sejauh a

$x = |y|-a$ (iv) $x = g(y)-a \rightarrow$ digeser kekiri sejauh a

$\cdot\!> \ x = |y - 11|$

$$|y-11| = \begin{cases} y - 11 & ; \ y - 11 \geq 0 \\ & \quad\ y \geq 11 \\ -y + 11 & ; \ y - 11 < 0 \\ & \quad\ y < 11 \end{cases}$$

$$x = \begin{cases} y - 11 & ; \ y \geq 11 \\ -y + 11 & ; \ \boxed{y < 11} \end{cases} \qquad (0 \leq y \leq 2)$$

$\cdot\!> \ x = -y + 11 \longrightarrow \dfrac{dx}{dy} = -1$

$$K = \int_0^2 2\pi \cdot x \cdot \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \int_0^2 2\pi \cdot (-y + 11) \cdot \underbrace{\sqrt{1 + (-1)^2}}_{\sqrt{2}}\, dy$$

$$= \int_0^2 2\sqrt{2}\,\pi\, (-y + 11)\, dy$$

$$= 2\sqrt{2}\,\pi \int_0^2 -y + 11\, dy$$

$$= 2\sqrt{2}\,\pi \left[-\frac{1}{2}y^2 + 11y\right]_0^2$$

$$= 2\sqrt{2}\,\pi \left[\left(-\frac{1}{2} \cdot 4^2 + 11 \cdot 2\right) - (0 + 0)\right]$$

$$= 2\sqrt{2}\,\pi\, (-2 + 22)$$

$$= 2\sqrt{2}\,\pi\, (20)$$

$$= 40\sqrt{2}\,\pi \text{ satuan luas}$$

(2) Tunjukkan luas permukaan bola dengan jari-jari r adalah $4\pi r^2$.

$\cdot\!>$ Persamaan lingkaran pusat $(0,0)$ & berjari-jari r.

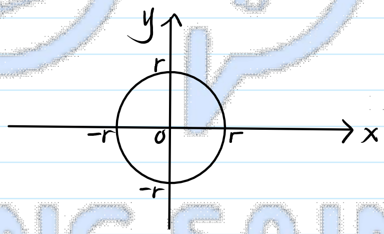
$x^2 + y^2 = r^2 \longrightarrow$

$x^2 + y^2 = r^2$

$x^2 = \sqrt{r^2 - y^2}$

• Persamaan lingkaran pusat (0,0) & berjari-jari r

$x^2 + y^2 = r^2$



$x^2 + y^2 = r^2$

$x^2 = \sqrt{r^2 - y^2}$

$x = \pm\sqrt{r^2 - y^2}$

• $x = \sqrt{r^2 - y^2}$ ($\frac{1}{2}$ ⊙ kanan)

• $x = -\sqrt{r^2 - y^2}$ ($\frac{1}{2}$ ⊙ kiri)

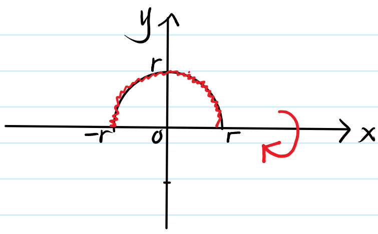
Setengah lingkaran

$y^2 = r^2 - x^2$

$y = \pm\sqrt{r^2 - x^2}$

$y = -\sqrt{r^2 - x^2}$ ($\frac{1}{2}$ ⊙ dibawah sb. x)

• $y = \sqrt{r^2 - x^2}$ ($\frac{1}{2}$ ⊙ diatas sb. x)



$$dK = 2\pi \cdot y \cdot dS$$

$$dK = 2\pi y \cdot \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$K = \int_{-r}^{r} 2\pi \cdot \sqrt{r^2 - y^2} \cdot \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \quad \checkmark$$

> Menentukan $\frac{dy}{dx}$

$$y = \sqrt{r^2 - x^2} = (r^2 - x^2)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{\cancel{2}} (r^2 - x^2)^{-1/2} \cdot -\cancel{2}x = \frac{-x}{(r^2-x^2)^{1/2}} = \frac{-x}{\sqrt{r^2-x^2}}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{x^2}{r^2 - x^2}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{x^2}{r^2-x^2}}$$

$$= \sqrt{\frac{r^2 \cancel{-x^2} \cancel{+x^2}}{r^2-x^2}}$$

$$= \sqrt{\frac{r^2}{r^2-x^2}} = \frac{\sqrt{r^2}}{\sqrt{r^2-x^2}} = \frac{r}{\sqrt{r^2-x^2}} \quad \checkmark$$

$$K = \int_{-r}^{r} 2\pi \cdot \cancel{\sqrt{r^2-x^2}} \cdot \frac{r}{\cancel{\sqrt{r^2-x^2}}}\, dx$$

$$= \int_{-r}^{r} 2\pi r \, dx$$

$$= 2\pi r . x \Big|_{-r}^{r}$$

$$= 2\pi r \left[r - (-r) \right]$$

$$= 2\pi r \cdot (2r)$$

$$= 4\pi r^2 \text{ satuan luas} \qquad \text{(Terbukti)}$$