Numerical Methods for Differential Equations
Homework 4

Due by 2pm Thurs 8th November. ***Please submit your hardcopy at the start of lecture.*** *The answers you submit should be attractive, brief, complete, and should include program listings and plots where appropriate. The use of "publish" in* MATLAB*/Octave is one option; another is* `ubc.syzygy.ca`.

1. **Of Jupiter and Moons.** Login to `https://ubc.syzygy.ca`. Start a new Python notebook. Start by typing "`import numpy`" or maybe "`from numpy import *`" and pressing "Shift-Enter". Do some reading. Figure how do build the "1 -2 1" L matrix for approximating the second derivative. Perform a convergence study for $u = \sin(2\pi x)$ for $x \in [0, 1)$, displaying your results in a table of values.

   If you prefer, you could also try Julia (also at `https://ubc.syzygy.ca`). Or for something more familiar, login to `https://sfu-os.syzygy.ca`, start a new Octave notebook and do the above exercise using familiar Octave syntax but in a Jupyter notebook.[1]

2. **AB2 stability analysis.** Consider the Adams–Bashforth-2 linear multistep method.
   (a) Complete a zero-stability analysis of the method.
   (b) Perform an absolute stability analysis of the scheme.
   (c) Plot the region of absolute stability in the complex plane.
   (d) Now consider a fixed λ and take limit $k \to 0$: comment on any similarities/differences to your results in (2a).

3. **Temporal midpoint rule: "Leap Frog".** Consider the multistep scheme constructed from 2nd-order centered differences in time for the ODE problem $u_t = f(u)$.

   Perform an absolute stability analysis for this time-stepping scheme. What is different about the region of absolute stability, compared to other methods we have looked at? Hint: you may need to numerically experiment with the complex roots of a quadratic to determine what is included in the stability region.

4. **von Neumann analysis.** Consider the Leap Frog method for the advection problem $u_t + au_x = 0$. That is, 2nd-order centered differences in both space and time.

   Perform a von Neumann stability analysis. Do the algebra in terms of the "*Courant number* $\nu = \frac{ka}{h}$" where k is the time-step and h is the spatial step. What restrictions on ν do there seem to be? What does this mean for k?

   Hint: you might want to look at [1] or another book on von Neumann analysis for details.

5. **Leap frog numerical experiments.** Write a code for 1D advection using the leap frog scheme from the previous exercise to solve $u_t + u_x = 0$.

   Use the domain $0 \leq x \leq 25$ with periodic boundary conditions and $T_f = 17$. For an initial condition, take:

   $$u(x, 0) = \eta(x) = \exp(-20(x-2)^2) + \exp(-(x-5)^2)$$

   (Hint: you will need a second starting value: use the exact value of $u(x, k)$)

   Use $h = 0.05$ and $\nu = 1.1$. What happens?

   Use $h = 0.05$ and $\nu = 0.8$. What happens?[2]

[1] If you take this route, you will need a `github.com` for authentication. Note `github.com` is not the same thing as `gitlab.math.ubc.ca`; Git, Github.com and `gitlab.math.ubc.ca` are related in the same way as "email", "`@gmail.com`", and "`@math.ubc.ca`".

[2] FYI: von Neumann analysis can also give us inside into the "dispersive" properties of a scheme, telling us whether waves of different wavenumbers move at different speeds. You may have seen the dispersion relation and "group velocity versus phase velocity" for continuous problems: this is analogous.

6. **Respecting Boundaries.** Consider the 1-d "shifted Poisson equation" $u_{xx} - u = f(x)$ where f is some given function. In each case below, make appropriate modifications to the "1 -2 1" L matrix, and perform a convergence study.
   (a) Homogeneous Dirichlet boundary conditions. Let $u(0) = 0$ and $u(1) = 0$. I suggest using your grid as `x = linspace(0, 1, N)` but there are various possible approaches. To design a convergence study, start by choosing $u = \sin(\pi x)$ and determine (by hand) what f should be.
   (b) Nonhomogeneous Dirichlet boundary conditions. Let $u(0) = 1$ and $u(1) = 3$. Start by choosing $u = 3^x$.
   (c) Periodic boundary conditions: $u(0) = u(1)$. Choose $u = \exp(\sin(2\pi x))$. Your grid should be `x = 0:h:(1-h)`.
   (d) Neumann boundary conditions: $\frac{\mathrm{d}u}{\mathrm{d}x}(0) = \frac{\mathrm{d}u}{\mathrm{d}x}(1) = 0$. Include the end points in your grid. Use $u = \cos(\pi x)$.
   (e) For the periodic case, repeat this for the Poisson equation $u_{xx} = f(x)$: what goes wrong? Comment on this with respect to unique solutions.

7. **2-d Poisson.** Get ("git clone") the code from https://gitlab.math.ubc.ca/cbm/fdmatrices. This automates some of "kron black magic" mentioned in lecture. Using this code (or alternatively a different code of your own making), setup the problem $-u_{xx} - u_{yy} = f(x, y)$, on a rectangular domain $R = [0, 2] \times [0, 1]$ with Dirichlet boundary conditions $u(x, y) = g(x, y) \equiv 0$ on the boundary of R. You might find `ex_laplace2d.m` helpful.
   (a) Choose $f = \sin(\pi x)\sin(2\pi y)$ and $g \equiv 0$. Find the exact solution and do a convergence study to clearly show that you get second-order accuracy.
   (b) Your friend mistakenly uses "`0:h:(2-h)`" (like in the periodic case) but still imposes zero boundary data: what happens to the convergence rate?
   (c) Again using homogeneous Dirichlet boundary conditions of $u = 0$ on the boundary of R, consider $f = \frac{1}{a}e^{-\|\vec{x}-\vec{x}_s\|^2/\varepsilon}$ where $\vec{x}$ means $(x, y)^T$ and $\vec{x}_s = (1.5, 0.6)^T$. Suppose $\varepsilon = 1/100$. Find a value for a such that the double integral of f over R is 1 (you can do this approximately if you want).
   (d) Solve the problem above numerically using $h = 1/128$. Plot the function and find the discrete maximum value (over the grid values).

References

[1] R. J. LeVeque. *Finite Difference Methods for Ordinary and Partial Differential Equations: Steady-State and Time-Dependent Problems*. SIAM, 2007.