Introduction to Data Processing and Representation - HW1

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1. Solving the L^p problem using the L^2 norm

(a) $p=1$: for each interval I_i take the median of f in it, means: for $x \in I_i$: $\hat{f}(x) = y_i$ where y_i holds that $\int_{\substack{a \le x \le b \\ f(x) \le y_i}} dx = \int_{\substack{a \le x \le b \\ y_i < f(x)}} dx$.

$p=2$: now we take the average in each interval. For $x \in I_i$: $\hat{f}(x) = \frac{1}{|I_i|} \int_{I_i} f(x)\,dx$.

(b) As we are looking for $\hat{f}$ which is constant on every interval I_i, we can reduce the problem to finding a value y_i that minimize $\int_{I_i} (f(x)-y_i)^2 w(x)\,dx$ for each $i=1,\dots,N$ (see part (d) of this question).
To find that y_i we will derive the target function with respect to y_i, and find where it equals zero:

$$\frac{d}{dy_i} \int_{I_i} (f(x)-y_i)^2 w(x)\,dx = \int_{I_i} \frac{d}{dy_i} (f(x)-y_i)^2 w(x)\,dx$$

$$= -\int_{I_i} 2w(x) \cdot (f(x)-y_i)\,dx \underset{\text{wanted}}{=} 0$$

$$\Rightarrow \int_{I_i} w(x) f(x)\,dx = y_i \cdot \int_{I_i} w(x)\,dx$$

$$\Rightarrow \boxed{y_i = \frac{1}{\int_{I_i} w(x)dx} \cdot \int_{I_i} w(x) f(x)\, dx}$$

ⓒ We will use the same approach - we will look for y_i that minimiz $\int_{I_i} w(x)|f(x) - y_i| dx$.

$$\frac{d}{dy_i} \int_{I_i} w(x)|f(x) - y_i| dx = \frac{d}{dy_i}\left(\int_{\substack{x\in I_i \\ f(x)<y_i}} w(x)(y_i - f(x)) dx + \int_{\substack{x\in I_i \\ y_i \le f(x)}} w(x)(f(x) - y_i) dx \right)$$

$$= \left(\int_{\substack{x\in I_i \\ f(x)<y_i}} w(x)\, dx - \int_{\substack{x\in I_i \\ y_i \le f(x)}} w(x)\, dx \right) \underset{\substack{\uparrow \\ \text{wanted}}}{=} 0$$

$$\Rightarrow \int_{\substack{x\in I_i \\ f(x)<y_i}} w(x)\, dx = \int_{\substack{x\in I_i \\ y_i \le f(x)}} w(x)\, dx \qquad (*)$$

We will have to define $\hat{f}(x) = y_i$ for a number y_i that satisfies $(*)$.

ⓓ In the next page we prove a claim that states that

$$\min_{\hat{f}} \int_0^1 |f(x) - \hat{f}(x)|^p w(x)\, dx = \sum_{i=1}^{N} \min_{\hat{f}_i} \int_{I_i} |f_i(x) - \hat{f}_i(x)|^p w(x)\, dx.$$

Using it, we can define

$$\varepsilon_i^p(f_i, \hat{f}_i) = \min_{\hat{f}_i} \int_{I_i} |f_i(x) - \hat{f}_i(x)|^p w(x)\, dx$$

and get that $\varepsilon^p(f, \hat{f}) = \sum_{i=1}^{N} \varepsilon_i^p(f_i, \hat{f}_i)$.

claim:

$$\min_{\hat{f}} \int_0^1 |f(x)-\hat{f}(x)|^p w(x)\,dx = \sum_{i=1}^{N} \min_{\hat{f}_i} \int_{I_i} |f_i(x)-\hat{f}_i(x)|^p w(x)\,dx$$

proof:

$\geq$:

$$\min_{\hat{f}} \int_0^1 |f(x)-\hat{f}(x)|^p w(x)\,dx = \min_{\hat{f}} \sum_{i=1}^{N} \int_{I_i} |f_i(x) - \underbrace{\hat{f}|_{I_i}(x)}_{\hat{f} \text{ reduced to } I_i}|^p w(x)\,dx .$$

$$\geq \min_{\hat{f}} \sum_{i=1}^{N} \min_{\hat{f}_i} \int_{I_i} |f_i(x) - \hat{f}_i(x)|^p w(x)\,dx \underset{*}{=} \sum_{i=1}^{N} |f_i(x) - \hat{f}_i(x)|^p w(x)\,dx$$

$*$ note that $\hat{f}$ does not participate in the term that we minimize with respect to it.

$\leq$:

Assume towards a contradiction that

$$\min_{\hat{f}} \int_0^1 |f(x)-\hat{f}(x)|^p w(x)\,dx > \sum_{i=1}^{N} \min_{\hat{f}_i} \int_{I_i} |f_i(x)-\hat{f}_i(x)|^p w(x)\,dx .$$

This means that there is a set $\{\hat{g}_i : I_i \to \mathbb{R}\}_{i=1}^{N}$ such that

$$\min_{\hat{f}} \int_0^1 |f(x) - \hat{f}(x)|^p w(x)\,dx > \sum_{i=1}^{N} |f_i(x) - \hat{g}_i(x)|^p w(x)\,dx .$$

Define $\hat{g} : [0,1] \to \mathbb{R}$ with $\hat{g}(x) = \hat{g}_i(x)$ if $x \in I_i$ (well defined as I_i's are disjoint and covers $[0,1]$).

Then

$$\min_{\hat{f}} \int_0^1 |f(x) - \hat{f}(x)|^p w(x)\,dx > \sum_{i=1}^{N} |f_i(x) - \hat{g}_i(x)|^p w(x)\,dx = \int_0^1 |f(x) - \hat{g}(x)|^p w(x)\,dx$$

and we got a contradiction.

$\blacksquare$

(e) (i) Assume $f_i(x) \neq \hat{f}_i(x)$ for all $x \in I_i$. Thus, the term $1/(f_i(x)-\hat{f}_i(x))^2$ is well defind for all $x \in I_i$.
Define $w_{f_i,\hat{f}_i} : I_i \to \mathbb{R}$ by $w_{f_i,\hat{f}_i}(x) = |f_i(x)-\hat{f}_i(x)|^p \cdot \frac{1}{(f_i(x)-\hat{f}_i(x))^2}$ (and $w_{f_i,\hat{f}_i}$ is well defined and positive), and we get the desired.

(ii)

$$\mathcal{E}_i^p(f_i,\hat{f}_i) = \min_{\hat{f}_i} \int_{I_i} w(x)|f_i(x)-\hat{f}_i(x)|^p dx = \min_{\hat{f}_i} \int_{I_i} \underbrace{w(x)\cdot\frac{|f_i(x)-\hat{f}_i(x)|^p}{(f_i(x)-\hat{f}_i(x))^2}}_{:= w_{f_i,\hat{f}_i}} \cdot (f_i(x)-\hat{f}_i(x))^2 dx$$

(iii) Recall that $\hat{f}_i$ is constant on I_i. Than solving the optimization problem for $\hat{f}_i$ is like trying to find $y_i \in \mathbb{R}$ that minimize $\int_{I_i} w_{f,y_i}(x) \cdot |f(x)-y_i|^p dx$.
If w_{f,y_i} was independent from y_i we could derive the target function with respect to y_i and find the wanted value as we did before.
If, on the other hand, w_{f_i,y_i} is not independent of y_i we then we had to use derivation of a multiplication and then extract y_i from there, and that will be harder, as w_{f_i,y_i} is not necessarily simple.

(iv) In the case where f_i and $\hat{f}_i$ might be equal, the term $|f_i(x)-\hat{f}_i(x)|^p/(f_i(x)-\hat{f}_i(x))^2$ might not be defined in some part of the points in I_i. So, $w_{f_i,\hat{f}_i}$ can't have the discussed above form. Even so, we would believe that there is a sequence of functions $\{w^{(n)}_{f_i,\hat{f}_i}\}_{n=1}^{\infty}$ that "converge" to $w(x)\cdot\frac{|f_i(x)-\hat{f}_i(x)|^p}{(f_i(x)-\hat{f}_i(x))^2}$, where the last is defined to be ∞ in places where $f_i(x)=\hat{f}_i(x)$.

But then, solving for $\operatorname{argmin}_{\hat{f}} \int_{I_i} w_{f_i,\hat{f}_i}(x) |f(x) - \hat{f}(x)|^p dx$ might be impossible, or involve very high numbers.
By limiting the values w can get, the problem becomes feasible.

(v) <u>Input:</u> (1) f - the function to approximate.
(2) w - the weight function.
(3) ε - small fixed number.
(4) stoping criteria.

<u>The algorithm:</u>

<u>Init:</u> - $\hat{f}_i \leftarrow 1$ (some random initialization)
<u>loop:</u> until reaching some stoping criteria, do:

$$-w' \leftarrow \min\left\{\frac{1}{\varepsilon}, w(x) \cdot \frac{|f_i(x) - \hat{f}_i(x)|^p}{(f_i(x) - \hat{f}_i(x))^2}\right\}$$

$$-\hat{f}_i^{next} = \frac{1}{\int_{I_i} w'(x) dx} \cdot \int_{I_i} f_i(x) \cdot w'(x) dx$$

The solution to the L^2 problem (see part (b) of the question.

$$-\hat{f}_i \leftarrow \hat{f}_i^{next}$$

<u>Output:</u> $\hat{f}_i$

(f) To solve the L^p problem using L^2 optimization we will use the algorithm from part (e)(v), denoted here as $Alg^{(i)}$ for interval I_i:

<u>Input:</u> (1) f - the function to approximate.
(2) w - the weight function.

(3) ε - small fixed number.

(4) stoping criteria.

The algorithm:

For $i = 1, \dots, N$:

$\hat{f}_i = Alg^{(i)}(f, w, \varepsilon, \text{stoping criteria})$

Output: $f = \sum_{i=1}^{N} \hat{f}_i \cdot \mathbb{1}_{I_i}$

(9) IRLS - Iteratively Reweighted Least Square

2- Haar matrix and Walsh-Hadamard matrix

a) (i) We need to show that $\langle u_i, u_j \rangle = \delta_{ij}$ where $1 \le i,j \le 4$ and u_1, u_2, u_3, u_4 are the columns of H_4. So:

$$\langle u_1, u_2 \rangle = \left\langle \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \frac{1}{2}\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix} \right\rangle = 0 \qquad \langle u_2, u_3 \rangle = \frac{1}{4}\left\langle \begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}, \begin{bmatrix}\sqrt{2}\\-\sqrt{2}\\0\\0\end{bmatrix} \right\rangle = 0$$

$$\langle u_1, u_3 \rangle = \frac{1}{4}\left\langle \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}\sqrt{2}\\-\sqrt{2}\\0\\0\end{bmatrix} \right\rangle = 0 \qquad \langle u_2, u_4 \rangle = \frac{1}{4}\left\langle \begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}, \begin{bmatrix}0\\0\\\sqrt{2}\\-\sqrt{2}\end{bmatrix} \right\rangle = 0$$

$$\langle u_1, u_4 \rangle = \frac{1}{4}\left\langle \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\\sqrt{2}\\-\sqrt{2}\end{bmatrix} \right\rangle = 0 \qquad \langle u_3, u_4 \rangle = \frac{1}{4}\left\langle \begin{bmatrix}\sqrt{2}\\-\sqrt{2}\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\\sqrt{2}\\-\sqrt{2}\end{bmatrix} \right\rangle = 0$$

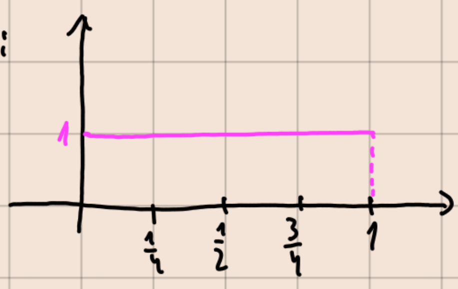
$$\langle u_1, u_1 \rangle = \frac{1}{4}\left\langle \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \right\rangle = 1 \qquad \langle u_3, u_3 \rangle = \frac{1}{4}\left\langle \begin{bmatrix}\sqrt{2}\\-\sqrt{2}\\0\\0\end{bmatrix}, \begin{bmatrix}\sqrt{2}\\-\sqrt{2}\\0\\0\end{bmatrix} \right\rangle = 1$$

$$\langle u_2, u_2 \rangle = \frac{1}{4}\left\langle \begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}, \begin{bmatrix}1\\1\\-1\\-1\end{bmatrix} \right\rangle = 1 \qquad \langle u_4, u_4 \rangle = \frac{1}{4}\left\langle \begin{bmatrix}0\\0\\\sqrt{2}\\-\sqrt{2}\end{bmatrix}, \begin{bmatrix}0\\0\\\sqrt{2}\\-\sqrt{2}\end{bmatrix} \right\rangle = 1$$

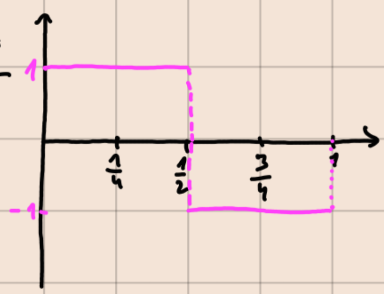
(ii)
$$\begin{bmatrix}\psi_1^H\\\psi_2^H\\\psi_3^H\\\psi_4^H\end{bmatrix} = H_4^T \cdot \begin{bmatrix}\psi_1^S\\\psi_2^S\\\psi_3^S\\\psi_4^S\end{bmatrix} = \cancel{\frac{1}{2}}\begin{bmatrix}1 & 1 & 1 & 1\\1 & 1 & -1 & -1\\\sqrt{2} & -\sqrt{2} & 0 & 0\\0 & 0 & \sqrt{2} & -\sqrt{2}\end{bmatrix}\begin{bmatrix}\cancel{\sqrt{4}} \cdot 1_{[0,\frac{1}{4})}\\\cancel{\sqrt{4}} \cdot 1_{[\frac{1}{4},\frac{1}{2})}\\\cancel{\sqrt{4}} \cdot 1_{[\frac{1}{2},\frac{3}{4})}\\\cancel{\sqrt{4}} \cdot 1_{[\frac{3}{4},1]}\end{bmatrix} = \begin{bmatrix}1_{[0,1]}\\1_{[0,\frac{1}{2})} - 1_{[\frac{1}{2},1]}\\\sqrt{2}\cdot 1_{[0,\frac{1}{4}]} - \sqrt{2}\cdot 1_{[\frac{1}{4},\frac{1}{2})}\\\sqrt{2}\cdot 1_{[\frac{1}{2},\frac{3}{4})} - \sqrt{2}\cdot 1_{[\frac{3}{4},1]}\end{bmatrix}$$

graphs

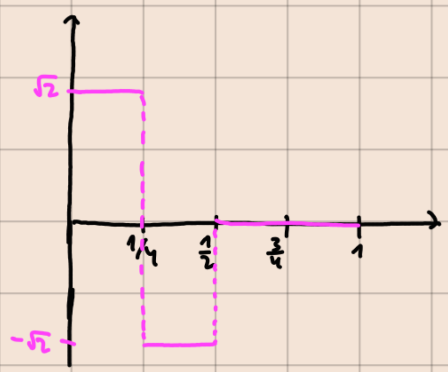
ψ_1^H:



ψ_2^H:



ψ_3^H:



ψ_4^H:

(iii) The approximation of ϕ using this Haar basis is given by $\hat{\Phi}_H(t) = \sum_{i=1}^{4} \langle \Phi, \psi_i^H \rangle \cdot \psi_i^H(t)$.
So, let us start by calculating $\langle \Phi, \psi_i^H \rangle$ for $i=1,2,3,4$.
To do so, we will start by calculating $\langle \Phi, \mathbb{1}_{[\frac{j}{4}, \frac{j+1}{4})} \rangle$ for $j=0,1,2,3$, and then use additivity and linearity of the integral.

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$$\langle \Phi, \mathbb{1}_{[\frac{j}{4}, \frac{j+1}{4})} \rangle = \int_0^1 \Phi(t) \cdot \mathbb{1}_{[\frac{j}{4}, \frac{j+1}{4})}(t)\,dt = \int_{j/4}^{\frac{j+1}{4}} (a + b\cos(2\pi t) + c \cdot \cos^2(\pi t))\,dt$$

$$= \int_{\frac{j}{4}}^{\frac{j+1}{4}} \left(a + b\cos(2\pi t) + \frac{c}{2}\cos(2\pi t) + \frac{c}{2}\right) dt$$

$$= \frac{a + \frac{c}{2}}{4} + \frac{b + \frac{c}{2}}{2\pi} \cdot \sin(2\pi t) \Big|_{\frac{j}{4}}^{\frac{j+1}{4}}$$

$\underline{j=0}$: $\langle \Phi, \mathbb{1}_{[0, \frac{1}{4})} \rangle = \frac{a}{4} + \frac{b}{2\pi} + \frac{c}{8} + \frac{c}{4\pi}$

$\underline{j=1}$: $\langle \Phi, \mathbb{1}_{[\frac{1}{4}, \frac{1}{2})} \rangle = \frac{a}{4} - \frac{b}{2\pi} + \frac{c}{8} - \frac{c}{4\pi}$

$\underline{j=2}$: $\langle \Phi, \mathbb{1}_{[\frac{1}{2}, \frac{3}{4})} \rangle = \frac{a}{4} - \frac{b}{2\pi} + \frac{c}{8} - \frac{c}{4\pi}$

$\underline{j=3}$: $\langle \Phi, \mathbb{1}_{[\frac{3}{4}, 1)} \rangle = \frac{a}{4} + \frac{b}{2\pi} + \frac{c}{8} + \frac{c}{4\pi}$

<u>כמובן!</u>

$$\langle \Phi, \psi_1^H \rangle = \sum_{j=0}^{3} \langle \Phi, \mathbb{1}_{[\frac{j}{4}, \frac{j+1}{4})} \rangle = a + \frac{c}{2}$$

$$\langle \Phi, \psi_2^H \rangle = \langle \Phi, \mathbb{1}_{[0, \frac{1}{4})} \rangle + \langle \Phi, \mathbb{1}_{[\frac{1}{4}, \frac{1}{2})} \rangle - \langle \Phi, \mathbb{1}_{[\frac{1}{2}, \frac{3}{4})} \rangle - \langle \Phi, \mathbb{1}_{[\frac{3}{4}, 1)} \rangle = 0$$

$$\langle \Phi, \psi_3^H \rangle = \frac{\sqrt{2}\,b}{\pi} + \frac{\sqrt{2}\,c}{2\pi}$$

$$\langle \Phi, \psi_4^H \rangle = -\frac{\sqrt{2}\,b}{\pi} - \frac{\sqrt{2}\,c}{2\pi}$$

$$\Rightarrow \boxed{\hat{\Phi}_H^{(4)}(t) = \left(a + \frac{c}{2}\right) \cdot \mathbb{1}_{[0, \frac{1}{4})} + \left(\frac{\sqrt{2}\,b}{\pi} + \frac{\sqrt{2}\,c}{2\pi}\right)\left(\mathbb{1}_{[\frac{1}{2}, \frac{3}{4})} - \mathbb{1}_{[\frac{3}{4}, 1]}\right)}$$

The associated MSE is given by:

$$\Psi^H_{MSE}(\Phi(t)-\hat{\Phi}^{(4)}_H(t)) = \int_0^1 \phi^2(t)\,dt - \sum_{i=1}^{4} \langle \phi, \psi_i^H \rangle^2 .$$

<u>So:</u>

$$\int_0^1 \phi^2(t)\,dt = \int_0^1 (a + b\cos(2\pi t) + c\cdot\cos^2(\pi t))^2\,dt$$

$$= \int_0^1 (a^2 + 2ab\underset{0}{\cancel{\cos(2\pi t)}} + 2ac\cdot\cos^2(\pi t) + b^2\cos^2(2\pi t) + c^2\cos^4(\pi t) + 2bc\cos(2\pi t)\cdot\cos^2(\pi t)\,dt$$

$$= a^2 + \int_0^1 ac\cdot(\underset{0}{\cancel{\cos 2\pi t}} + 1) + \frac{b^2}{2}(\underset{0}{\cancel{\cos(4\pi t)}} + 1) + \frac{c^2}{4}((\cos(2\pi t)+1)^2) + 2bc\cos(2\pi t)\cdot\cos^2(\pi t)\,dt$$

$$= a^2 + ac + \frac{b^2}{2} + \int_0^1 \frac{c^2}{4}\cdot(\cos^2(2\pi t) + 2\underset{0}{\cancel{\cos(2\pi t)}} + 1) + bc\cos(2\pi t)\cdot(\cos(2\pi t)+1)\;dt$$

$$= a^2 + ac + \frac{b^2}{2} + \int_0^1 \frac{c^2}{4}\cdot\left(\underset{0}{\cancel{\frac{\cos 4\pi t}{2}}} + \frac{1}{2} + 1\right) + bc(\cos^2(2\pi t) + \underset{0}{\cancel{\cos 2\pi t}})\,dt$$

$$= a^2 + ac + \frac{b^2}{2} + \frac{3c^2}{8} + \int_0^1 bc\left(\underset{0}{\cancel{\frac{\cos(4\pi t)}{2}}} + \frac{1}{2}\right)dt$$

$$= a^2 + ac + \frac{b^2}{2} + \frac{3c^2}{8} + \frac{bc}{2}$$

$$\boxed{\begin{aligned} \Rightarrow \Psi^H_{MSE}(\Phi(t)-\Phi^{(4)}_H(t)) &= a^2 + ac + \frac{b^2}{2} + \frac{3c^2}{8} + \frac{bc}{2} - \left(a + \frac{c}{2}\right)^2 - 2\left(\frac{\sqrt{2}b}{\pi} + \frac{\sqrt{2}c}{2\pi}\right)^2 \\ &= \cancel{a^2} + \cancel{ac} + \frac{b^2}{2} + \frac{3c^2}{8} + \frac{bc}{2} - \cancel{a^2} - \cancel{ac} - \frac{c^2}{4} - \frac{4}{\pi^2}\left(b^2 + bc + \frac{c^2}{4}\right) \\ &= \frac{b^2}{2} + \frac{c^2}{8} + \frac{bc}{2} - \frac{4b^2}{\pi^2} + \frac{4bc}{\pi^2} - \frac{c^2}{\pi^2} \\ &= \left(\frac{1}{2} - \frac{4}{\pi^2}\right)b^2 + \left(\frac{1}{2} + \frac{4}{\pi^2}\right)bc + \left(\frac{1}{8} - \frac{1}{\pi^2}\right)c^2 \end{aligned}}$$

(iv) Assume $a \geq b \geq 0$ and $c \geq 0$.

From the expression of the MSE we can see that we need to sort $\{|\langle \phi, \psi_i^H \rangle|\}_{i=1}^{4}$ in a non-increasing order:

$1 > \frac{\sqrt{2}}{\pi}$ and $\begin{cases} a \geq b \geq 0 \\ c \geq 0 \end{cases}$ so $a + \frac{c}{2} > \frac{\sqrt{2}}{\pi} b + \frac{1}{2}\cdot\frac{\sqrt{2}}{\pi} c$.

Then:

<u>1-term approx.</u>: $\hat{\Phi}^{(1)}_H(t) = \left(a + \frac{c}{2}\right)\cdot \mathbb{1}_{[0,1]}$

<u>2-term approx.</u>: $\hat{\Phi}^{(2)}_H(t) = \left(a + \frac{c}{2}\right)\cdot \mathbb{1}_{[0,1]} + \left(\frac{2}{\pi} b + \frac{2}{2\pi} c\right)\cdot\left(\mathbb{1}_{[0,\frac{1}{4})} - \mathbb{1}_{[\frac{1}{4},\frac{1}{2})}\right)$

3-term approx.:

$$\hat{\Phi}^{(3)}_H(t) = \left(a + \frac{c}{2}\right)\cdot \mathbb{1}_{[0,1]} + \left(\frac{2}{\pi} b + \frac{2}{2\pi} c\right)\cdot\left(\mathbb{1}_{[0,\frac14)} - \mathbb{1}_{[\frac14,\frac12)} - \mathbb{1}_{[\frac12,\frac34)} + \mathbb{1}_{[\frac34,1]}\right)$$

4-term approx.: the same as the 3-term approx., as $\langle \Phi, \Psi^2_H\rangle = 0$.

(v) Now assume that $a = \frac{1}{\pi}$, $b = 1$, $c = \frac{3}{2}$. We can compute $\langle \Phi, \Psi_i^H\rangle$ for $i = 1, \dots, 4$ exactly:

$|\langle\Phi,\Psi_1^H\rangle| = \left|\frac{1}{\pi} + \frac{1}{2}\cdot\frac{3}{2}\right| = \frac{1}{\pi} + \frac{3}{4}$

$|\langle\Phi,\Psi_2^H\rangle| = 0$

$|\langle\Phi,\Psi_3^H\rangle| = |\langle\Phi,\Psi_4^H\rangle| = \frac{\sqrt2}{\pi}\cdot 1 + \frac{\sqrt2}{2\pi}\cdot\frac{3}{2} = \frac{\sqrt2}{\pi}\left(1 + \frac{3}{4}\right)$

Comparing $|\langle\Phi,\Psi_1^H\rangle|$ and $|\langle\Phi,\Psi_3^H\rangle|$:

$$\left(\frac{1}{\pi} + \frac{3}{4}\right) - \left(\frac{\sqrt2}{\pi} + \frac{3\sqrt2}{4\pi}\right) = \frac{4 + 3\pi - 4\sqrt2 - 3\sqrt2}{4\pi} = \frac{4 + 3\pi - 7\sqrt2}{4\pi} = (*)$$

and $7\sqrt2 < 7\cdot 1.5 = 11.5$, $4 + 3\pi > 4 + 3\cdot 3 = 13$

$\Rightarrow (*) < 0$

$\Rightarrow |\langle\Phi,\Psi_3^H\rangle| > |\langle\Phi,\Psi_1^H\rangle|$.

So:

1-term approx.: $\hat{\Phi}^{(1)}_H(t) = \left(\frac{2b}{\pi} + \frac{2c}{2\pi}\right)\cdot\left(\mathbb{1}_{[0,\frac14)} - \mathbb{1}_{[\frac14,\frac12)}\right)$

2-term approx.: $\hat{\Phi}^{(1)}_H(t) = \left(\frac{2b}{\pi} + \frac{2c}{2\pi}\right)\cdot\left(\mathbb{1}_{[0,\frac14)} - \mathbb{1}_{[\frac14,\frac12)} - \mathbb{1}_{[\frac12,\frac34)} + \mathbb{1}_{[\frac34,1]}\right)$

3-term approx.:

$$\hat{\Phi}^{(3)}_H(t) = \left(a + \frac{c}{2}\right)\cdot \mathbb{1}_{[0,1]} + \left(\frac{2}{\pi} b + \frac{2}{2\pi} c\right)\cdot\left(\mathbb{1}_{[0,\frac14)} - \mathbb{1}_{[\frac14,\frac12)} - \mathbb{1}_{[\frac12,\frac34)} + \mathbb{1}_{[\frac34,1]}\right)$$

4-term approx.: the same as the 3-term approx., as $\langle \Phi, \Psi^2_H\rangle = 0$.

(b) (i) $\langle \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \frac{1}{2}\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}\rangle = \frac{1}{4}\cdot(1+1-1-1) = 0$, $\langle \frac{1}{2}\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}, \frac{1}{2}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}\rangle = 0$

$\langle \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \frac{1}{2}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}\rangle = 0$, $\langle \frac{1}{2}\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}, \frac{1}{2}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}\rangle = 0$

$\langle \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \frac{1}{2}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}\rangle = 0$, $\langle \frac{1}{2}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}, \frac{1}{2}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}\rangle = 0$

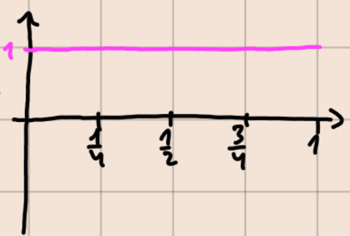
$\langle \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\rangle = \frac{1}{4}\cdot 4 = 1$, $\langle \frac{1}{2}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}, \frac{1}{2}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}\rangle = 1$

$\langle \frac{1}{2}\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}, \frac{1}{2}\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}\rangle = \frac{1}{4}\cdot 4 = 1$, $\langle \frac{1}{2}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}, \frac{1}{2}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}\rangle = 1$

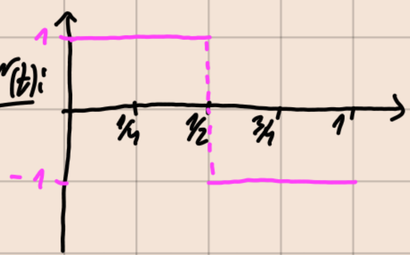
$\Rightarrow \langle col_i(W_4), col_j(W_4)\rangle = \delta_{ij}$ for $(i,j) \in \{1,2,3,4\}^2$, and therfore W_4 is unitary.

(ii) $$\begin{bmatrix}\psi_1^w(t)\\\psi_2^w(t)\\\psi_3^w(t)\\\psi_4^w(t)\end{bmatrix} = W_4 \cdot \begin{bmatrix}\psi_1^S(t)\\\psi_2^S(t)\\\psi_3^S(t)\\\psi_4^S(t)\end{bmatrix} = \cancel{\frac{1}{2}}\begin{bmatrix}1&1&1&1\\1&1&-1&-1\\1&-1&-1&1\\1&-1&1&-1\end{bmatrix} \cdot \begin{bmatrix}\cancel{\sqrt{4}}\cdot 1_{[0,\frac{1}{4})}\\\cancel{\sqrt{4}}\cdot 1_{[\frac{1}{4},\frac{1}{2})}\\\cancel{\sqrt{4}}\cdot 1_{[\frac{1}{2},\frac{3}{4})}\\\cancel{\sqrt{4}}\cdot 1_{[\frac{3}{4},1]}\end{bmatrix} = \begin{bmatrix}1_{[0,1]}\\1_{[0,\frac{1}{2})} - 1_{[\frac{1}{2},1]}\\1_{[0,\frac{1}{4})} - 1_{[\frac{1}{4},\frac{3}{4})} + 1_{[\frac{3}{4},1]}\\1_{[0,\frac{1}{4})} - 1_{[\frac{1}{4},\frac{1}{2})} + 1_{[\frac{1}{2},\frac{3}{4})} - 1_{[\frac{3}{4},1]}\end{bmatrix}$$

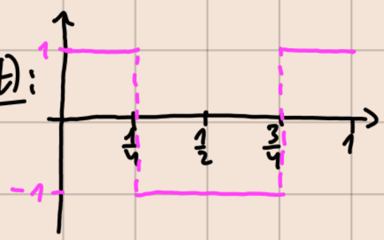
$\psi_1^w(t)$:



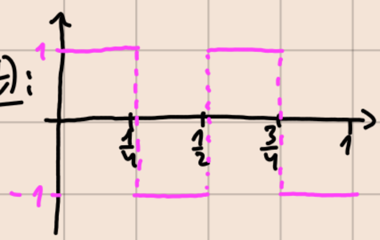
$\psi_2^w(t)$:



$\psi_1^w(t)$:



$\psi_1^w(t)$:



(iii) From (a)(iii) we already know $\langle \phi, 1_{[\frac{j}{4}, \frac{j+1}{4})}\rangle$ for $j = 0,1,2,3$. We will now use this to compute $\langle \phi, \psi_i^w\rangle$ for $i = 1,2,3,4$.

$\langle \phi, \psi_1^w\rangle = \langle \phi, \psi_1^H\rangle = a + \frac{c}{2}$

$$\langle \phi, \psi_2^W \rangle = \langle \phi, \psi_2^H \rangle = 0$$

$$\langle \phi, \psi_3^W \rangle = \langle \phi, \mathbb{1}_{[0,\frac{1}{4})} \rangle - \langle \phi, \mathbb{1}_{[\frac{1}{4},\frac{1}{2})} \rangle - \langle \phi, \mathbb{1}_{[\frac{1}{2},\frac{3}{4})} \rangle + \langle \phi, \mathbb{1}_{[\frac{3}{4},1]} \rangle$$

$$= \frac{a}{4} + \frac{b}{2\pi} + \frac{c}{8} + \frac{c}{4\pi} - \frac{a}{4} + \frac{b}{2\pi} - \frac{c}{8} + \frac{c}{4\pi} - \frac{a}{4} + \frac{b}{2\pi} - \frac{c}{8} + \frac{c}{4\pi} + \frac{a}{4} + \frac{b}{2\pi} + \frac{c}{8} + \frac{c}{4\pi}$$

$$= \frac{2b + c}{\pi}$$

$$\langle \phi, \psi_4^W \rangle = \langle \phi, \mathbb{1}_{[0,\frac{1}{4})} \rangle - \langle \phi, \mathbb{1}_{[\frac{1}{4},\frac{1}{2})} \rangle + \langle \phi, \mathbb{1}_{[\frac{1}{2},\frac{3}{4})} \rangle - \langle \phi, \mathbb{1}_{[\frac{3}{4},1]} \rangle$$

$$\frac{a}{4} + \frac{b}{2\pi} + \frac{c}{8} + \frac{c}{4\pi} - \frac{a}{4} + \frac{b}{2\pi} - \frac{c}{8} + \frac{c}{4\pi} + \frac{a}{4} - \frac{b}{2\pi} + \frac{c}{8} - \frac{c}{4\pi} - \frac{a}{4} - \frac{b}{2\pi} - \frac{c}{8} - \frac{c}{4\pi}$$

$$= 0$$

$$\boxed{\Rightarrow \hat{\Phi}_W^{(4)} = \sum_{i=1}^{4} \langle \phi, \psi_i^W \rangle \cdot \psi_i^W(t) = \left(a + \frac{c}{2}\right) \cdot \mathbb{1}_{[0,1]} + \frac{2b+c}{\pi} \left(\mathbb{1}_{[0,\frac{1}{4})} - \mathbb{1}_{[\frac{1}{4},\frac{3}{4})} + \mathbb{1}_{[\frac{3}{4},1]} \right)}$$

$$\boxed{\begin{aligned}
\psi_{MSE}^W\left(\Phi(t) - \Phi_W^{(4)}(t)\right) &= \int_0^1 \Phi^2(t)\,dt - \sum_{i=1}^{4} \langle \phi, \psi_i^W \rangle^2 \\
&= a^2 + ac + \frac{b^2}{2} + \frac{3c^2}{8} + \frac{bc}{2} - \left(a + \frac{c}{2}\right)^2 - \left(\frac{2b+c}{\pi}\right)^2 \\
&= \cancel{a^2} + \cancel{ac} + \frac{b^2}{2} + \frac{3c^2}{8} + \frac{bc}{2} - \cancel{a^2} - \frac{c^2}{4} - \cancel{ac} - \frac{4b^2}{\pi^2} - \frac{c^2}{\pi^2} - \frac{4bc}{\pi^2} \\
&= \left(\frac{1}{2} - \frac{4}{\pi^2}\right) b^2 + \left(\frac{1}{8} - \frac{1}{\pi^2}\right) c^2 + \left(\frac{1}{2} - \frac{4}{\pi^2}\right) bc
\end{aligned}}$$

(iv) Assume $a \geq b \geq 0$, $c \geq 0$. As in (a)(iv), we will sort $\{\langle \phi, \psi_i^W \rangle\}_{i=1}^{4}$ in a non-increasing order:

$a \geq b \geq 0$ and $1 > \frac{2}{\pi}$ so $a > \frac{2b}{\pi}$. Also, $\frac{1}{2} > \frac{1}{\pi}$ and $c \geq 0$, so $\frac{c}{2} > \frac{c}{\pi}$, so $\langle \phi, \psi_1^W \rangle = a + \frac{c}{2} > \frac{2b+c}{\pi} = \langle \phi, \psi_3^W \rangle$.

<u>Then:</u>

<u>1-term approx.:</u> $\hat{\Phi}_W^{(1)}(t) = \left(a + \frac{c}{2}\right) \cdot \mathbb{1}_{[0,1]}$

<u>2-term approx.:</u> $\hat{\Phi}_W^{(2)}(t) = \left(a + \frac{c}{2}\right) \cdot \mathbb{1}_{[0,1]} + \frac{2b+c}{\pi} \cdot \left(\mathbb{1}_{[0,\frac{1}{4})} - \mathbb{1}_{[\frac{1}{4},\frac{3}{4})} + \mathbb{1}_{[\frac{3}{4},1]} \right)$

3-term and 4-term approx. are the same, as
$\langle \phi, \psi_2^w \rangle = \langle \phi, \psi_4^w \rangle = 0$.

(iv) Now $a = \frac{1}{\pi}$, $b = 1$, $c = \frac{3}{2}$.

$\Rightarrow \langle \phi, \psi_1^w \rangle = a + \frac{c}{2} = \frac{1}{\pi} + \frac{3}{4}$

$\langle \phi, \psi_4^w \rangle = \frac{2b+c}{\pi} = \frac{2 + \frac{3}{2}}{\pi} = \frac{2}{\pi} + \frac{3}{2\pi}$

$\langle \phi, \psi_4^w \rangle - \langle \phi, \psi_1^w \rangle = \frac{2}{\pi} + \frac{3}{2\pi} - \frac{1}{\pi} - \frac{3}{4} = \frac{5}{2\pi} - \frac{3}{4} = \frac{10 - 3\pi}{4\pi} > 0$

$\Rightarrow \langle \phi, \psi_4^w \rangle > \langle \phi, \psi_1^w \rangle$

Then:

1-term approx.: $\left(\frac{2}{\pi} + \frac{3}{2\pi}\right) \cdot \left(\mathbb{1}_{[0,\frac{1}{4})} - \mathbb{1}_{[\frac{1}{4},\frac{3}{4})} + \mathbb{1}_{[\frac{3}{4},1]}\right)$

2-term approx.: $\hat{\phi}_w^{(2)}(t) = \left(a + \frac{c}{2}\right) \cdot \mathbb{1}_{[0,1]} + \frac{2b+c}{\pi} \cdot \left(\mathbb{1}_{[0,\frac{1}{4})} - \mathbb{1}_{[\frac{1}{4},\frac{3}{4})} + \mathbb{1}_{[\frac{3}{4},1]}\right)$

3-term and 4-term approx. are the same, as
$\langle \phi, \psi_2^w \rangle = \langle \phi, \psi_4^w \rangle = 0$.