MEME16203Linear Models Marking Guide

Assignment 5

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Due by:			

Q1. A study was conducted on human subjects to measure the effects of 4 foods on serum glucose levels. Each of the 4 foods was randomly assigned to 5 subjects. The serum glucose was measured for each of the subjects at 6 different time points starting at 15 minutes and every 15 minutes after food was ingested. Consider the model

$$Y_{ijk} = \mu + \alpha_i + S_{ij} + \tau_k + \gamma_{ik} + e_{ijk}$$

where y_{ijk} is the serum glucose levels at the k^{th} time point for the j^{th} subject with the i^{th} food, α_i is the fixed diet effect, τ_k is the fixed time effect and γ_{ik} is the fixed diet $\times$ time effect, $S_{ij} \sim NID(0, \sigma_S^2)$ and is independent of $e_{ijk} \sim NID(0, \sigma_e^2)$.

(a) Find $V(\mathbf{Y_{ij}})$, for this model? (10 marks)

Ans.

$$\mathbf{Y_{ij}} = \begin{bmatrix} Y_{ij1} \\ Y_{ij2} \\ \vdots \\ Y_{ij6} \end{bmatrix}$$

$$\begin{aligned} V(Y_{ijk}) &= V(\mu + \alpha_i + S_{ij} + \tau_k + e_{ijk}, \mu + \alpha_i + S_{ij} + \tau_l + e_{ijl}) \\ &= V(S_{ij} + e_{ijk}) \\ &= \sigma_S^2 + \sigma_e^2 \end{aligned}$$

$$\begin{aligned} Cov(Y_{ijk}, Y_{ijl}) &= Cov(\mu + \alpha_i + S_{ij} + \tau_k + e_{ijk}, \mu + \alpha_i + S_{ij} + \tau_l + e_{ijl}) \\ &= Cov(S_{ij}, S_{ij}) \\ &= \sigma_S^2 \text{ for } k \neq l \end{aligned}$$

$$\begin{aligned} V(\mathbf{Y_{ij}}) &= \begin{bmatrix} V(Y_{ij1}) & Cov(Y_{ij1}, Y_{ij2}) & \cdots & Cov(Y_{ij1}, Y_{ij6}) \\ Cov(Y_{ij2}, Y_{ij1}) & V(Y_{ij2}) & \cdots & Cov(Y_{ij2}, Y_{ij6}) \\ \vdots & \ddots & & \vdots \\ Cov(Y_{ij6}, Y_{ij1}) & Cov(Y_{ij6}, Y_{ij2}) & \cdots & V(Y_{ij6}) \end{bmatrix} \\ &= \begin{bmatrix} \sigma_S^2 + \sigma_e^2 & \sigma_S^2 & \cdots & \sigma_S^2 \\ \sigma_S^2 & \sigma_S^2 + \sigma_e^2 & \cdots & \sigma_S^2 \\ \vdots & \ddots & & \vdots \\ \sigma_S^2 & \sigma_S^2 & \cdots & \sigma_S^2 + \sigma_e^2 \end{bmatrix} \\ &= \sigma_e^2 \mathbf{I_6} + \sigma_S^2 \mathbf{J_6} \end{aligned}$$

(b) Provide the formulas for the estimator of σ_e^2 and σ_S^2. (10 marks)

Ans.

$\hat{\sigma}_e^2 = MSE$
$\hat{\sigma}_S^2 = \frac{MS(Subject)-MSE}{6}$

(c) What is the correlation between observations taken on the same subject? (10 marks)

Ans.

$\rho = \frac{Cov(Y_{ijk},Y_{ijl})}{\sqrt{V(Y_{ijk})}\sqrt{V(Y_{ijl})}} = \frac{\sigma_S^2}{\sigma_S^2+\sigma_e^2}$

(d) Find the estimator of $V(\bar{Y}_{ij.})$ and provide it's degrees of freedom.(10 marks)

Ans.

$$\begin{aligned} V(\bar{Y}_{ij.}) &= V\left(\frac{1}{6}\sum_{k=1}^{6} Y_{ijk}\right) \\ &= \frac{1}{6^2}V(Y_{ij1}+Y_{ij2}+\cdots+Y_{ij6}) \\ &= \frac{1}{6^2}[\sum_{k=1}^{6} V(Y_{ijk}) + 2\sum\sum_{k\neq l} Cov((Y_{ijk},Y_{ijl})] \\ &= \frac{1}{6^2}[\sum_{k=1}^{6}(\sigma_S^2+\sigma_e^2) + 2\binom{6}{2}(\sigma_S^2) \\ &= \frac{1}{6^2}[6(\sigma_S^2+\sigma_e^2) + 2\binom{6}{2}(\sigma_S^2) \\ &= \frac{1}{6^2}[6^2\sigma_S^2 + 6\sigma_e^2] \\ &= \sigma_S^2 + \frac{1}{6}\sigma_e^2 \end{aligned}$$

$S^2_{\bar{Y}_{ij.}} = \frac{MS(Subject)-MSE}{6} + \frac{MSE}{6} = \frac{MS(Subject)}{6}$
$DF = 6-1 = 5$

(e) Find the estimator of $V(\bar{Y}_{i.k})$ and provide it's Satterthwaith degrees of freedom. (10 marks)

Ans.

$$\begin{aligned} V(\bar{Y}_{i.k}) &= V\left(\frac{1}{5}\sum_{j=1}^{5} Y_{ijk}\right) \\ &= \frac{1}{5^2}\sum_{j=1}^{5} V(Y_{ijk}) \\ &= \frac{1}{5}(\sigma_S^2+\sigma_e^2) \end{aligned}$$

$S^2_{\bar{Y}_{i.k}} = \sqrt{\frac{1}{5}\left[\frac{5MSE}{6} + \frac{MS(Subject)}{6}\right]}$
$D.F = \frac{\frac{1}{5}\left[\frac{5MSE}{6}+\frac{MS(Subject)}{6}\right]}{\frac{(\frac{5MSE}{6})^2}{5}+\frac{(\frac{MS(subject)}{6})^2}{6}}$

Q2. Researchers investigated the effects of 24 different drugs on the level of a protein in the blood of mice. On each of 6 days, 24 mice were randomly assigned to the 24 drugs with one mouse for each drug. Each mouse was injected with its assigned drug, and then blood samples were taken from each mouse at 4 time points: 1, 2, 3, and 4 hours after injection. The same process was repeated each day with 24 different mice, so a total of 144 mice were used in the experiment. The level of the protein of interest was measured in each of the 576 blood samples. For $i = 1, \ldots, 6$, $j = 1, \ldots, 24$, and $k = 1, \ldots, 4$, let y_{ijk} be the protein level measurement on day i for chemical drug j at time k. For $i = 1, \ldots, 6$, $j = 1, \ldots, 24$, and $k = 1, \ldots, 4$, consider the model

$$y_{ijk} = \mu_{jk} + d_i + e_{ijk},$$

where μ_{jk} terms are unknown fixed parameters and the other terms are random effects defined as follows. Let $\mathbf{d} = [d_1, \ldots, d_6]^T$. For $i = 1, \ldots, 6$ and $j = 1, \ldots, 24$, let $\mathbf{e}_{ij} = [e_{ij1}, \ldots, e_{ij4}]^T$.
Suppose

$$\mathbf{d} \sim N(\mathbf{0}, \sigma_d^2 \mathbf{I}_{6\times 6}),$$

and

$$\mathbf{e}_{ij} \sim N(\mathbf{0}, \mathbf{\Sigma}_e) \text{ for } i = 1, \ldots, 6 \text{ and } j = 1, \ldots, 24,$$

where σ_d^2 is an unknown positive variance parameter and

$$\mathbf{\Sigma}_e = \begin{bmatrix} \sigma_1^2 & \rho^4\sigma_1\sigma_2 & \rho^9\sigma_1\sigma_3 & \rho^{19}\sigma_1\sigma_4 \\ \rho^4\sigma_1\sigma_2 & \sigma_2^2 & \rho^5\sigma_2\sigma_3 & \rho^{15}\sigma_2\sigma_4 \\ \rho^9\sigma_1\sigma_3 & \rho^5\sigma_2\sigma_3 & \sigma_3^2 & \rho^{10}\sigma_3\sigma_4 \\ \rho^{19}\sigma_1\sigma_4 & \rho^{15}\sigma_2\sigma_4 & \rho^{10}\sigma_3\sigma_4 & \sigma_4^2 \end{bmatrix}$$

for some unknown standard deviation parameter $\sigma_i > 0, i = 1, 2, 3, 4$ and some unknown correlation parameter ρ. Finally, suppose that $\mathbf{d}$ and $\mathbf{e}_{11}, \ldots, \mathbf{e}_{6,24}$ are all independent. In terms of model parameters, give a simplified expression for the variance of the generalized least squares estimator of each of the following:

(a) μ_{13} (10 marks)

Ans.

$V(\bar{Y}_{.13}) = V(\bar{d}_. + \bar{e}_{.13}) = \boxed{\frac{\sigma_d^2}{6} + \frac{\sigma_1^2}{6}}$

(b) $\bar{\mu}_{1.}$ (10 marks)

Ans.

$V(\bar{Y}_{.1.}) = V(\bar{d}_. + \bar{e}_{.1.}) = \frac{\sigma_d^2}{6} + V(\frac{1}{24}\sum_{i=1}^{6}\sum_{k=1}^{4} e_{i1k}) = \frac{\sigma_d^2}{6} + \frac{1}{24^2}(6)\mathbf{1}^T\mathbf{\Sigma_e 1}$
$= \frac{\sigma_d^2}{6} + \frac{6[\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2+2(\rho^4\sigma_1\sigma_2+\rho^9\sigma_1\sigma_3+\rho^{19}\sigma_1\sigma_4+\rho^5\sigma_2\sigma_3+\rho^{15}\sigma_2\sigma_4\rho^{10}\sigma_3\sigma_4)]}{576}$

(c) $\mu_{11} - \mu_{21}$ (10 marks)

Ans.

$V(\bar{Y}_{.11} - \bar{Y}_{.21}) = V(\bar{e}_{.11} - \bar{e}_{.21}) = V(\bar{e}_{.11} + \bar{e}_{.21}) = \frac{\sigma_1^2}{6} + \frac{\sigma_1^2}{6} = \frac{2\sigma_e^2}{6}$

(d) $\mu_{11} - \mu_{14}$ (10 marks)

Ans.

$V(\bar{Y}_{.11} - \bar{Y}_{.14}) = V(\bar{e}_{.11} - \bar{e}_{.14}) = V\left[\frac{1}{6}\sum_{i=1}^{6}(e_{i11} - e_{i14})\right] = \frac{\sigma_1^2 + \sigma_4^2 - 2cov(e_{i11}, e_{i14})}{6}$
$= \frac{\sigma_1^2 + \sigma_4^2 - 2\rho^{19}\sigma_1\sigma_4}{6}$

(e) $\bar{\mu}_{.3}$ (10 marks)

Ans.

$V(\bar{Y}_{..3}) = V(\bar{d}_{.} + \bar{e}_{..3}) = \boxed{\frac{\sigma_d^2}{6} + \frac{\sigma_3^2}{144}}$