MEME15203 Statistical Inference Marking Guide

Assignment 4

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Q1. Let $X \sim NB(r, 0.49)$. Derive the most powerful test of size $\alpha = 0.134$ of $H_0 : r = 1$ against $H_1 : r = 3$ based on an observed value of X. Compute the power of this test for the alternative $r = 3$. (20 marks)

Ans.

$f(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}, x = r, r+1, \dots$

$\lambda = \frac{f(x;r=1)}{f(x;r=3)} = \frac{(0.49)(0.51)^{x-1}}{\binom{x-1}{2}(0.49)^3(0.51^{x-3})} = \frac{(0.51^2)}{\binom{x-1}{2}(0.49)^2} < k \Rightarrow x \geq k$. Thus the most powerful test of size $\alpha = 0.134$ of $H_0 : r = 1$ against $H_1 : r = 3$ is to reject H_0 if $x > k$ such that

$\gamma P(X = k|H_0) + P(X > k) = 0.134.$

Note that $P(X = k|H_0) = p(1-p)^{k-1}$, and

$P(X > k|H_0) = p(1-p)^k + p(1-p)^{k+1} = \dots = p(1-p)^k[1 + (1-p) + (1-p)^2 + \dots] = p(1-p)^k \left(\frac{1}{p}\right) = (1-p)^k$

Thus, $\gamma(0.49)(0.51)^{k-1} + (0.51)^k = 0.134$

$k = 3.0$, $\gamma = \frac{0.134 - 0.51^{3.0}}{0.49(0.51^{2.0})} = 0.0106$

Thus the most powerful test of size $\alpha = 0.134$ of $H_0 : r = 1$ against $H_1 : r = 3$ is

$$\phi(x) = \begin{cases} 0, & x < 3.0 \\ 0.0106, & x = 3.0 \\ 1, & x > 3.0 \end{cases}$$

$$\begin{aligned} \text{Power} &= 0.0106P(X = 3.0|r = 3) + P(X > 3.0|r = 3) \\ &= 0.0106P(X = 3.0|r = 3) + 1 - P(X < 3.0|r = 3) \\ &= 0.0106(2.0)(0.49^3(0.51)) + 1 - 0.49^3 \\ &= 0.883623020988 \end{aligned}$$

Q2. Consider a random sample of size n from a uniform distribution, $X_i \sim U(0, \theta)$. Find the UMP test of size α of $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$ by first deriving a most powerful test of simple hypotheses and then extending it to composite hypotheses. (20 marks)

Ans.

Let $\theta_1 > \theta_0$ and consider $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$.
The most powerful test is to reject H_0 if
$\lambda(x_1, \ldots, x_n; \theta_0, \theta_1) = \frac{\prod_{i=1}^n 1/\theta_0 I(0<x_i<\theta_0)}{\prod_{i=1}^n 1/\theta_1 I(0<x_i<\theta_1)} = \left(\frac{\theta_1}{\theta_0}\right)^n I(x_{(n)} < \theta_0) < k.$
The rejection rule depends on the data only through $x_{(n)}$. Since $\theta_1 > \theta_0$, we will reject H_0 if $x_{(n)}$ is large. We will consider the rejection region given by $x_{(n)} > k$, choosing k so that
$P(X_{(n)} \geq k|\theta = \theta_0) = \alpha$. Hence
$1 - [F_X(k)]^n = \alpha$
$1 - \left(\frac{k}{\theta_0}\right)^n = \alpha$
$k = \theta_0(1-\alpha)^{1/n}$. The most powerful critical region is thus $x_{(n)} > \theta_0(1-\alpha)^{1/n}$.
Since the choice of critical region depended only the fact that $\theta_0 < \theta_1$ and does not depend on θ_1, so it is UMP.
The power function is
$\pi(\theta) = P(X_{(n)} \geq \theta_0(1-\alpha)^{1/n}|\theta) = 1 - [P(X_1 \leq \theta_0(1-\alpha)^{1/n}|\theta)]^n = 1 - [\frac{\theta_0(1-\alpha)^{1/n}}{\theta}]^n = 1 - (1-\alpha)(\frac{\theta_0}{\theta})^n$.
Since $\pi(\theta) \leq \alpha \;\forall\; \theta < \theta_0$, the same test is UMP for $H_0 : \theta \leq \theta_0$.

Q3. Let $X_1, X_2, \ldots, X_n$ denote a random sample from a normal distribution with mean μ(unknown) and variance σ^2. For testing $H_0 : \sigma^2 = \sigma_0^2$ against $H_1 : \sigma^2 < \sigma_0^2$, show that the likelihood ratio test is equivalent to the χ^2 test. (20 marks)

Ans.

The null hypothesis specifies $\Omega_0 = \{\sigma^2 : \sigma^2 = \sigma_0^2\}$, while $\Omega = \Omega_0 \cup \Omega_1 = \{\sigma^2 : \sigma^2 \leq \sigma_0^2\}$.
In the restricted space Ω_0, the likelihood function is

$$L(\Omega_0) = \prod_{i=1}^{n} \frac{1}{(2\pi)^{1/2}\sigma_0} e^{-\frac{\sum(x_i-\mu)^2}{2\sigma_0^2}}$$

The MLE of μ is $\bar{x}$, so that

$$L(\hat{\Omega}_0) = \frac{1}{(2\pi)^{n/2}\sigma_0^n} e^{-\frac{\sum(x_i-\bar{x})^2}{2\sigma_0^2}}$$

In the unrestricted space

$$L(\Omega) = \prod_{i=1}^{n} \frac{1}{(2\pi)^{n/2}\sigma} e^{-\frac{\sum(x_i-\mu)^2}{2\sigma^2}}$$

The MLEs of μ is $\bar{x}$ and σ^2 is $\hat{\sigma}^2 = \max\left(\sigma_0^2, \hat{\sigma}^2 = \frac{\sum(x_i-\bar{x})^2}{n}\right)$, so

$$L(\hat{\Omega}) = \frac{1}{(2\pi)^{n/2}\hat{\sigma}^n} e^{-\frac{\sum(x_i-\bar{x})^2}{2\hat{\sigma}^2}}$$

$\lambda = \frac{L(\hat{\Omega}_0)}{L(\hat{\Omega})}$
$= \left(\frac{\hat{\sigma}^2}{\sigma_0^2}\right)^n e^{-\frac{\sum(x_i-\bar{x})^2}{2\sigma_0^2}+\frac{\sum(x_i-\bar{x})^2}{2\hat{\sigma}^2}}$
$= \begin{cases} 1, & \text{if } \hat{\sigma} = \sigma_0 \\ \left[\frac{\sum(x_i-\bar{x})^2}{n\sigma_0^2}\right]^{n/2} e^{-\frac{\sum(x_i-\bar{x})^2}{2\sigma_0^2}} e^{n/2}, & \text{if } \hat{\sigma} < \sigma_0 \end{cases}$

Hence the rejection region $\lambda \leq k$ is equivalent to

$$g(\chi^2) = (\chi^2)^{n/2} e^{-\chi^2/2} n^{-n/2} e^{n/2} \leq k$$

where $\chi^2 = \frac{(n-1)s^2}{\sigma_0^2} \sim \chi^2_{(n-1)}$
Further, if $\hat{\sigma} < \sigma_0$, $g(\chi^2)$ is monotonically increasing function of χ^2. Hence the region $\lambda \leq k$ is equivalent to $\chi^2 \leq c$ where c is determined such that $P(\chi^2_{(n-1)} \leq c) = \alpha$.

Q4. Let $X_1, \ldots, X_{20}$ denote a random sample from a Weibull distribution, $X_i \sim WEI(2, \theta)$. Show that a UMP size 0.03 test of $H_0 : \theta \geq 2$ versus $H_1 : \theta < 2$ using Theorem 3 is $\{\sum X_i^2 \leq k\}$, and then determine k. (20 marks)

Ans.

$f(\mathbf{x}; \theta) = [\frac{2}{\theta^2}]^n \prod x_i^{2-1} e^{-\sum(x_i/\theta)^2} = c(\theta)h(\mathbf{x})e^{q(\theta)t(\mathbf{x})}$
where $q(\theta) = -\frac{1}{\theta^2}$ is an increasing function of θ and $t(\mathbf{x}) = \sum x_i^2$. Thus, by Theorem 3, a UMP size α test of $H_0 : \theta \geq \theta_0$ versus $H_1 : \theta < \theta_0$ is to reject H_0 if $\sum x_i^2 \leq k$, where $P[\sum x_i^2 \leq k|\theta_0] = \alpha$.
Let $Y = X_i^2$,
$F_Y(y) = P[Y \leq y] = P[X^2 \leq y] = P[X \leq y^{1/2}] = 1 - e^{-(\frac{y^{1/2}}{\theta})^2} = 1 - e^{-y/\theta^2}$
$\Rightarrow X_i^2 \sim Exp(\theta^2)$ and $\sum X_i^2 \sim GAM(n, \theta^2)$
Thus an equivalent test is to reject H_0 if $\frac{2\sum X_i^2}{\theta_0^2} \leq \chi^2_{(1-\alpha)}(2n)$ or $\sum X_i^2 \leq \frac{\theta_0^2 \chi^2_{1-\alpha}(2n)}{2} = \frac{2^2 \chi^2_{0.97}(40)}{2} = \frac{1}{2}(2^2)qchisq(0.97, 40) = \frac{1}{2}(2^2)(58.43) = \boxed{116.86}$

Q5. Consider a random sample of size n from a Bernoulli distribution, $X_i \sim BIN(10, p)$. Derive a UMP test of $H_0 : p \geq p_0$ versus $H_1 : p < p_0$ using monotone likelihood ratio property. (10 marks)

Ans.

$f(\mathbf{x}; p) = \prod_{i=1}^n \binom{10}{x_i} p^{\sum x_i}(1-p)^{10n-\sum x_i}$
Let $p_1 < p_2$, so

$$\frac{f(\mathbf{x}; p_2)}{f(\mathbf{x}; p_1)} = \prod_{i=1}^n \binom{10}{x_i} \left(\frac{p_2}{p_1}\right)^{\sum x_i} \left(\frac{1-p_2}{1-p_1}\right)^{n-\sum x_i} = \prod_{i=1}^n \binom{10}{x_i} \left(\frac{p_2(1-p_1)}{p_1(1-p_2)}\right)^{\sum x_i} \left(\frac{1-p_2}{1-p_1}\right)^n$$

As $p_1 < p_2$, then $1 - p_1 > 1 - p_2$ and hence $\frac{p_2(1-p_1)}{p_1(1-p_2)} > 1$.
Thus $\frac{f(\mathbf{x};p_2)}{f(\mathbf{x};p_1)}$ is nondecreasing function of $t(\mathbf{x}) = \sum x_i$.
Hence, $f(\mathbf{x}; p)$ has the MLR propertyin the statitic $T = \sum X_i$.
By the theorem, a UMP test of size α for $H_0 : p \geq p_0$ versus $H_1 : p < p_0$ is to rejectH_0 if $\sum x_i \leq k$, where $P[\sum X_i \leq k|p_0] = \alpha$.

Q6. If $X_i|\lambda \sim POI(\lambda)$ and a Bayesian uses a prior for λ that is Gamma with parameters $\alpha = 7$ and $\theta = \frac{1}{100}$, suppose $x_1, x_2, \ldots, x_n$ have been observed, what is the Bayes test of $H_0 : \lambda \leq 5$ versus $H_1 : \lambda > 5$? (10 marks)

Ans. $X \sim POI(\lambda)$; $\Lambda \sim GAM(\alpha = 7, \theta = 100)$, thus $\Lambda|x \sim GAM(n\bar{x} + 7, \frac{1}{100+n})$
Then, the Bayes test is

$$\phi(x) = \begin{cases} 1, & P(\Lambda \leq 5) < 0.5 \\ 0, & \text{otherwise} \end{cases}$$

where $P(\Lambda \leq 5) = \int_0^5 \frac{(100+n)^{n\bar{x}+7}}{\Gamma(n\bar{x}+7)} \lambda^{x+6} e^{-(100+n)\lambda} d\lambda$