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3 Annuities

Annuity is a series of payments made at equal intervals of time. If the payments are certain to be made, term **annuity-certain** is used. When payments are not certain, the series is called **contingent annuity**.

- A pension is paid so long as the person survives. Pension is a contingent annuity.
- House rents, mortgage payments and installment payments on automobiles are all examples of annuity-certain.

3.1 Annuity-Immediate

An annuity having the first payment a period from now is called **annuity-immediate**.

3.1.1 Present Value of an Annuity-Immediate

Let's consider a simple annuity - a payment of 1 at the end of each year for n years.

$$\begin{array}{cccccccccc} & & 1 & & 1 & & 1 & & 1 \\ | & - - - & | & - - - & | & - \cdots - & | & - - - & | \\ 0 & & 1 & & 2 & & n-1 & & n \\ PV & & & & & & & & . \end{array}$$

The present value of this annuity is:

$$PV = v + v^2 + \ldots + v^{n-1} + v^n$$

Recall the geometric series:

$$a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r}$$

Thus,

$$PV = \frac{v(1-v^n)}{1-v} = \frac{v(1-v^n)}{iv} = \frac{1-v^n}{i}$$

This PV is denoted by $a_{\overline{n}|i}$. This is read as "a angle n at i." (If the rate of interest is understood, we can just write $a_{\overline{n}|}$.)

$$\boxed{a_{\overline{n}|i} = \frac{1-v^n}{i}}$$

Example 1.

Find the PV of a 19-year annuity-immediate with annual payments of 1,700 at an effective interest rate of 5% per annum.

Ans:

```
  5%
<---->
   1700 1700       1700
|----|----|-- ... --|
0    1    2          19
PV
```

$$PV = 1700a_{\overline{19}|5\%} = 1700\left[\frac{1-1.05^{-19}}{0.05}\right] = \boxed{20{,}545.0455}$$

Example 2.

Find the PV of an annuity-immediate with payments of \$90 every 6 months for 10 years at a nominal rate of interest of 5% compounded semi-annually.

Ans:

```
             2.5%
            <---->
                9090     90
1/2 years   |----|----|-- ... --|
            0    1    2         20
           PV
```

$$PV = 90a_{\overline{20}|2.5\%} = 90\left[\frac{1-1.025^{-20}}{0.025}\right] = \boxed{1{,}403.0246}$$

Example 3.

A 12-year annuity-immediate is purchased for 100,000 at 5.5% effective. What is the level annual payment provided by this annuity? $\boxed{11{,}602.92}$

Ans:

A 12-year annuity-immediate is purchased for 144,000 at 7.5% effective. What is the level annual payment provided by this annuity?

Ans:

```
 7.5%
<---->
     R      R       R
|----|------|--...--|
0    1      2      12
PV = 144000
```

$144000 = Ra_{\overline{12}|7.5\%}$

$144000 = R\left[\frac{1-1.075^{-12}}{0.075}\right]$

$R = \boxed{18{,}616.0077}$

Example 4.

A loan of 2,000 is to be repaid by equal quarterly installments of X at the end of each quarter over a 5-year period at a nominal rate of interest of 12% compounded quarterly. Determine X.

Ans:

```
              3.0%
           <---->
           X    X      X
1/4 years  |----|------|--...--|
           0    1      2       20
          PV = 1,000
```

$2,000 = Xa_{\overline{20}|3.0\%}$

$2,000 = X\left[\frac{1-1.03^{-20}}{0.03}\right]$

$X = \boxed{134.4314}$

Example 5.

At an annual effective interest rate of 7.4%, an annuity-immediate with $4n$ level annual payments of 1000 has present value of 12,736. Determine the fraction of the total present value represented by the first set of N payments and third set of N payments combined.

Ans:

$1000a_{\overline{4n}|7.4\%} = 12736.0$

$1000\left[\frac{1-1.074^{-4n}}{0.074}\right] = 12736.0$

$n = 10.0$

The required fraction is

$\frac{1000a_{\overline{4n}|7.4\%}+1000a_{\overline{n}|7.4\%}v^n}{12736.0}$

$= \frac{1000\left[\frac{1-1.074^{-10.0}}{0.074}\right]+1000\left[\frac{1-1.074^{-10.0}}{0.074}\right]1.074^{-2(10.0)}}{12736.0}$

$= \frac{1000\left[\frac{1-1.074^{-10.0}}{0.074}\right](1+1.074^{-2(10.0)})}{12736.0}$

$= \boxed{0.6713}$

3.1.2 Accumulated Value of an Annuity-Immediate

The AV of an annuity-immediate means the AV as of the date of last payment or deposit. The AV has the special symbol $s_{\overline{n}|}$.

$$\begin{array}{ccccccccc} & & 1 & & 1 & & 1 & & 1 \\ | & --- & | & --- & | & -\cdots- & | & --- & | \\ 0 & & 1 & & 2 & & n-1 & & n \\ & & & & & & & & s_{\overline{n}|} \end{array}$$

Accumulating the payments to time n, starting with the last payment, we have:

$s_{\overline{n}|} = 1 + (1+i) + \ldots + (1+i)^{n-2} + (1+i)^{n-1}$

$= \frac{[1-(1+i)^n]}{1-(1+i)}$

$= \frac{(1+i)^n-1}{i}$

As a matter of fact, $a_{\overline{n}|}$ and $s_{\overline{n}|}$ represent the value of the same payments on two different dates which are n years apart. Thus

$$s_{\overline{n}|} = (1+i)^n a_{\overline{n}|} = (1+i)^n\left(\frac{1-v^n}{i}\right) = \frac{(1+i)^n-1}{i}$$

Example 6.

John deposit 400 at the end of each month for 10 years in an account that credits interests at a nominal rate of 6% per annum compounded monthly. How much is in his account on the date of the last deposit?

Ans:

```
                  0.5%
                  <---->
                      400  400       400
1/12 years        |----|------|--.......--|
                  0    1      2           120
```

Let $j = \frac{i^{(12)}}{12} = \frac{0.06}{12} = 0.005$

$s_{\overline{n}|} = \frac{(1+j)^{12n}-1}{j} = \frac{(1.005)^{120}-1}{0.005}$

$AV = 400 s_{\overline{n}|} = 400(163.8793) = \boxed{65{,}552}$

Example 7.

1,200 is deposited in an account on January 1 of each year from 2015 to 2028, inclusive, at 5 January 1, 2028?

Ans:

```
          5%
      <---->
         1200  1200      1200
      |----|------|--.......--|
      0    1      2          14
   1/1/2015               1/1/2028
```

$s_{\overline{n}|} = \frac{(1+i)^{n}-1}{i} = \frac{(1.05)^{14}-1}{0.05}$

$AV = 1200 s_{\overline{n}|} = 1200(19.5986) = \boxed{23{,}518}$

Example 8.

What quarterly deposit for 8 years will accumulate to 29,225 on the date of last deposit at a nominal rate of interest of 6% compounded quarterly?

Ans:

```
                        1.5%
                        <---->
                             X        X              X
1/4 years              |----|------|--.......--|
                       0    1      2             32
```

Let $j = \frac{i^{(4)}}{4} = \frac{0.06}{4} = 0.015$

$s_{\overline{n}|} = \frac{(1+j)^{4n}-1}{j} = \frac{(1.015)^{32}-1}{0.015} = 40.6883$

$29,225 = X s_{\overline{n}|}$

$X = \frac{29,225}{s_{\overline{n}|}} = \frac{29,225}{40.6883} = \boxed{718}$

3.2 Annuity-Due

When the first payment is today, an annuity is called an **annuity-due**.

3.2.1 Present Value of an Annuity-Due

Suppose we want the PV of an annuity with n payments where the first payment is today, rather than a year from now.

```
 1      1                 1          .
|---|---|---...---|---|
0      1      2      n-1       n
PV                                   .
```

$$PV = 1 + v + \cdots + v^{n-1} = \frac{1-v^n}{1-v} = \frac{1-v^n}{d}$$

This PV has the symbol $\ddot{a}_{\overline{n}|}$ which is read as "a double-dot angle n".

Notes:

- Annuity-$i$mmediate has i in the denominator:

$$\boxed{a_{\overline{n}|} = \frac{1-v^n}{i}}$$

- Annuity-$d$ue has d in the denominator:

$$\boxed{\ddot{a}_{\overline{n}|} = \frac{1-v^n}{d}}$$

- Since $\ddot{a}_{\overline{n}|}$ is equal to the value of $a_{\overline{n}|}$ a year later.

$$\boxed{\ddot{a}_{\overline{n}|} = (1+i)a_{\overline{n}|}}$$

- If we cover up the payment at time 0. What is left is an annuity-immediate with $(n-1)$ payments.

$$\boxed{\ddot{a}_{\overline{n}|} = 1 + a_{\overline{n-1}|}}$$

Example 9.

Find the PV of an annuity-due with payments of 1,800 every 6 months for 8 years at a nominal rate of interest of 5
semiannually?

Ans:

```
                2.5%
              <---->
                  1800     1800         1800
1/2 years |-------|-------|-- ... ----|
           0      1    2               16
         PV
```

Let $j = \frac{i^{(2)}}{2} = \frac{0.05}{2} = 2.5$

$d = \frac{j}{1+j} = \frac{2.5}{1.025} = 0.02439$

$PV = 1800a_{\overline{16}|2.5\%} = 1800\left[\frac{1-1.025^{-16}}{0.02439}\right] = \boxed{24{,}086.72}$

Example 10.

Mary deposits 15,000 today in a bank crediting interest at a nominal rate of 5% compounded monthly. This sufficient to permit her to make monthly withdrawals of X for 6 years, first withdrawals today. Determine X.

Ans:

```
                 0.41667%
                 <---->
                 X    X               X
1/12 years       |----|------|--......|-----|
                 0    1      2       71    72
```

Let $j = \frac{i^{(12)}}{12} = \frac{0.05}{12} = 0.00417$
$d = \frac{j}{1+j} = \frac{0.00417}{1.00417} = 0.00415$
$\ddot{a}_{\overline{n}|} = \frac{1-(1+j)^{-12n}}{d} = \frac{1-1.00417^{-72}}{0.00415} = 62.3848$
$15000 = X\ddot{a}_{\overline{72}|}$
$X = \frac{15000}{\ddot{a}_{\overline{72}|}} = \frac{15000}{62.3848} = \boxed{240.44}$

3.2.2 Accumulated Value of an Annuity-Due

By the AV of an annuity-due, we mean the AV one period **after** the last payment, the symbol for this AV is $\ddot{s}_{\overline{n}|}$.

$$
\begin{array}{cccccccccc}
 & 1 & & 1 & & 1 & & 1 & & . \\
| & - - - & | & - - - & | & - \cdots - & | & - - - & | \\
0 & & 1 & & 2 & & n-1 & & n \\
 & & & & & & & & \ddot{s}_{\overline{n}|}
\end{array}
$$

$$
\begin{aligned}
\ddot{s}_{\overline{n}|} &= (1+i) + (1+i)^2 + \cdots + (1+i)^{n-1} + (1+i)^n \\
&= \frac{(1+i)[1-(1+i)^n]}{1-(1+i)} \\
&= \frac{(1+i)^n - 1}{iv} \\
&= \frac{(1+i)^n - 1}{d}
\end{aligned}
$$

Thus

$$\boxed{\ddot{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{d}}$$

Similarly, Since $\ddot{s}_{\overline{n}|}$ is equal to the value of $s_{\overline{n}|}$ a year later. So

$$\boxed{\ddot{s}_{\overline{n}|} = (1+i)s_{\overline{n}|}}$$

$$\begin{array}{ccccccccc}
1 & & 1 & & 1 & & 1 & & (1) \\
| & - - - & | & - - - & | & - \cdots - & | & - - - & | \\
0 & & 1 & & 2 & & n-1 & & n \\
 & & & & & & & & \ddot{(1)}
\end{array}$$

If we place a fictitious deposit at time n above the time line but to immediately withdrawal it at time n below the time line. Thus

$$\boxed{\ddot{s}_{\overline{n}|} = s_{\overline{n+1}|} - 1.}$$

Example 11.

Determine the AV of 13 annual deposits of 1429 one year after the last deposit, at 2.10% effective.

$\boxed{21551.03}$

Example 12.
Andrea makes deposits of 100 on the first day of each month in calendar years 2010 through 2015, inclusive, at a nominal rate of 7% per annum convertible monthly. How much is in her account on January 1, 2016? $\boxed{8968.1}$

Example 13.
Ed makes 15 annual deposits of X, which are just sufficient to allow him to make 8 annual withdrawals of 1,000, first withdrawal one year after last deposit. Interest is at 5% effective. Determine X. $\boxed{299.52}$

Example 14.

Deposits of 100 are made every month for 5 years into an account crediting interest at a nominal rate of 9% convertible monthly. Starting one month after the last deposit, monthly withdrawals of X are made for 10 years, exhausting the account. Determine X. $\boxed{95.54}$

Example 15.

(a) 1 is deposited at the end of each year for 5 years at i effective. How much is in the account on the date of the last deposit?

Ans:

$s_{\overline{5}|i}$

(b) 1 is deposited at the end of the 6th through 10th years. How much is in the account on the date of the last deposit?

Ans:

$s_{\overline{5}|}$

(c) 1 is deposited at the beginning of each year for 5 years at i effective. How much is in the account on the date of the last deposit?

Ans:

$s_{\overline{5}|}$

3.3 Annuity Values on any Date

Consider the following series of payments:

```
              1     1     1     1     1     1     1
|---|---|---|---|---|---|---|---|---|---|
1   2       3     4     5     6     7     8     9     10    11
```

Determine the value of these payments as of times:

1. $t = 2$

   Ans: $a_{\overline{7}|}$

2. $t = 3$

   Ans: $\ddot{a}_{\overline{7}|}$

3. $t = 9$

   Ans: $s_{\overline{7}|}$

4. $t = 10$

   Ans: $\ddot{s}_{\overline{7}|}$

As of $t = 0$, we have a number of choices.

$$PV = v^2 a_{\overline{7}|} = v^3 \ddot{a}_{\overline{7}|} = v^9 s_{\overline{7}|} = v^{10} \ddot{s}_{\overline{7}|}$$

Present values more than one period before the first payment due date is called **deferred annuity** and has a special symbol ${}_{m|}a_{\overline{n}|}$. In the above example, it is ${}_{2|}a_{\overline{7}|}$.

We can interpret this symbol as follows: "Go to time 2, pay what the symbol to the right of the vertical line says - in this case, an annuity-immediate for 7 years."

We can also write this PV as a deferred annuity-due: ${}_{3|}\ddot{a}_{\overline{7}|}$("Go to time 3, pay what the symbol to the right of the vertical line says - in this case, an annuity-due for 7 years.")

We could also determine the present value by placing *two'* fictional payments on the diagram and immediately withdraw them:

```
(1)(1)     1    1    1    1    1    1    1
|---|---|---|---|---|---|---|---|---|---|
1   2   3   4   5   6   7   8   9   10  11
(1)(1)
```

The PV of the payments above the time line (including the two fictitious ones) is $a_{\overline{9}|}$, but this includes the PV of the fictitious payments, which is $a_{\overline{2}|}$. Thus the PV of this deferred annuity is $a_{\overline{9}|} - a_{\overline{2}|}$.

The two most common ways to evaluate ${}_{2|}a_{\overline{7}|}$ are:

$$ {}_{2|}a_{\overline{7}|} = v^2 a_{\overline{7}|} = a_{\overline{9}|} - a_{\overline{2}|} $$

In general, consider an m-year deferred n-payment annuity-immediate (Note that the first payment is at time $m+1$). This annuity could also described as an $(m+1)$-year deferred n-payment annuity-due. We have:

$$\boxed{\begin{aligned} {}_{m|}a_{\overline{n}|} &= {}_{m+1|}\ddot{a}_{\overline{n}|} \\ &= v^m a_{\overline{n}|} \\ &= v^{m+1}\ddot{a}_{\overline{n}|} \\ &= a_{\overline{m+n}|} - a_{\overline{m}|} \end{aligned}}$$

```
                    1    1         1
|---|--...--|---|---|--...--|
0   1       m  m+1 m+2     m+n
```

Example 16. Determine an expression for $\frac{a_{\overline{5}|}}{a_{\overline{6}|}}$.

(A) $\frac{a_{\overline{2}|}+a_{\overline{3}|}}{2a_{\overline{3}|}}$

(B) $\frac{a_{\overline{2}|}+s_{\overline{3}|}}{1+a_{\overline{3}|}+s_{\overline{2}|}}$

(C) $\frac{a_{\overline{2}|}+s_{\overline{3}|}}{a_{\overline{3}|}+s_{\overline{3}|}}$

(D) $\frac{1+a_{\overline{2}|}+s_{\overline{3}|}}{a_{\overline{3}|}+s_{\overline{3}|}}$

Example 17.
Consider an annuity that pays 1 at the beginning of each year for $k+m$ years. Which of the following expression does not give the value of this annuity at the end of year k:

(A) $a_{\overline{k+m}|}(1+i)^{k+1}$

(B) $s_{\overline{k+m}|}v^m$

(C) $s_{\overline{k+1}|}+a_{\overline{m-1}|}$

(D) $\ddot{s}_{\overline{k}|}+\ddot{a}_{\overline{m}|}$

(E) $1+\ddot{s}_{\overline{k}|}+a_{\overline{m-1}|}$

Example 18.

A loan of amount $a_{\overline{10}|}$, made at time $t = 0$, is to be repaid by 10 annual payments of 1, beginning at time $t = 1$ and ending at time $t = 10$. At time $t = 4$, the borrower has financial troubles and can only pay $(1 - v^7)$. If he then returns to his original payment schedule of 1 at times $t = 5$ through $t = 9$, how much will his payment at time $t = 10$ need to be in order to pay the loan off in full? $\boxed{1 + v}$

Ans:

Let X be the payment at time $t = 10$ need to be in order to pay the loan off in full.

Note that the borrower underpaid by v^7 at time 4. He must pay v^7 at time 10, in addition to the regular payment of 1. Thus

$X = v^7(1 + i)^6 + 1 = v + 1$ or

Let R be the amount that need to cover for v^7, thus $a_{\overline{10}|} = a_{\overline{10}|} - v^7 \times v^4 + Rv^{10}$

$V^{11} = RV^{10}$

$R = v$

Thus $X = 1 + v$

3.4 Annuities with Block Payments

Annuities with block payments means that payments are level for a period of year, then change to another level for another period of years, etc.

3.4.1 Present Value of Annuities with Block Payments

Consider the following annuity with block payments: 5 for the first 8 years, 12 for the next 7 years, 10 for the next 7 years and 15 for the next 6 years:

The rules of calculating the PV are:

1. start with the payment furthest from the comparison date;
2. make adjustments (+ or −) as you move in closer to the comparison date.

Refer to the above example:

1. We start with the furthest payment (15 at time 28) and immediately write $15a_{\overline{28}|}$.

2. We move in closer from time 28 toward 0 until there is a change, which is time 22, when payments decrease by 5 (from 15 to 10), so we write $-5a_{\overline{22}|}$
3. We move in closer, see another change at $t = 15$, when payments increase by 2 (from 10 to 12), so we write $+2a_{\overline{15}|}$.
4. Finally, the last change is at time 8, a decrease of 7, so we write $-7a_{\overline{8}|}$.

Pulling all this together, we have:

$$PV = 15a_{\overline{28}|} - 5a_{\overline{22}|} + 2a_{\overline{15}|} - 7a_{\overline{8}|}$$

Example 19.

Write down directly in simplest annuity form of PV of the following payments:

Time	Payment
1 to 10	5
11 to 18	8
19 to 23	12
24 to 30	20

Ans:

```
   5        5  8        8  12       12  20       20
|--|--...--|--|--...--|--|--...--|--|--...--|
0  1       10 11       18 19       23 24       30
```

$PV == 20a_{\overline{30}|} - 8a_{\overline{23}|} - 4a_{\overline{18}|} - 3a_{\overline{10}|}$

Example 20.

You are given:

- the present value of an annuity-due that pays 300 every 6 months during the first 15 years and 200 every 6 months during the second 15 years is 6000;
- the present value of a 15-year deferred annuity-due that pays 350 every 6 months for 15 years is 4000; and
- the present value of an annuity-due that pays 100 every 6 months during the first 15 years and 200 every 6 months during the second 15 years is X;

Determine X. $\boxed{3523.81}$

Ans:

Example 21.

Annuity X and Y provide the following payments:

End of Year	Annuity X	Annuity Y
1-10	1	K
11-20	2	0
21-30	1	K

Annuities X and Y have equal present values at an effective annual interest i such that $v^{10} = 1/2$. Determine K. $\boxed{9/5}$

Ans:

$PV_X = a_{\overline{30|}} + a_{\overline{20|}} - a_{\overline{10|}}$

$PV_Y = K(a_{\overline{30|}} - a_{\overline{20|}} + a_{\overline{10|}})$

$a_{\overline{30|}} + a_{\overline{20|}} - a_{\overline{10|}} = K(a_{\overline{30|}} - a_{\overline{20|}} + a_{\overline{10|}})$

$\frac{1-v^{30}}{i} + \frac{1-v^{20}}{i} - \frac{1-v^{10}}{i} = K(\frac{1-v^{30}}{i} - \frac{1-v^{20}}{i} + \frac{1-v^{10}}{i})$

$1 - v^{30} - v^{20} + v^{10} = K(1 - v^{30} + v^{20} - v^{10})$

$1 - .5^3 - .5^2 + .5 = K(1 - .5^3 + .5^2 - .5)$

$1.125 = K(0.625)$

$K = \frac{1.125}{0.625} = \boxed{1.8}$

3.4.2 Accumulated Value of Annuities with Block Payments

The AV of annuities with block payments is obtained similar to the PV. Consider the annuities from Example 19, the comparison date is 30 if we want the AV on the date of last payment:

- We start with the furthest payment from time 30, which is 5 at time 1, we immediately write $5s_{\overline{30}|}$.
- As we moved toward the comparison date of time 30, we see that the first change is an increase of 3 (from 5 to 8) at time 11, so we write $+3s_{\overline{20}|}$.
- The next change is an increase of 4 at time 19, so we write $+4s_{\overline{12}|}$
- Finally, payments increase by 8 at time 24, so the adjustment term is $+s_{\overline{7}|}$.

Putting them together, we have:

$$AV = 5s_{\overline{30}|} + 3s_{\overline{20}|} + 4s_{\overline{12}|} + 8s_{\overline{7}|}$$

Example 22.
Write the present value at time 0 and the accumulated value on the date of the last payment for the following annuities:

Annuity #1		Annuity #2	
time	Payment	Time	Payment
1 to 5	5	1 to 10	8
6 to 12	10	11 to 15	0
13 to 18	5	16 to 22	10
19 to 27	10	23 to 25	12

□

Example 23.

Chuck needs to purchase an item in 10 years. The item costs 200 today, but its price inflates at 4% per year. To finance the purchase, Chuck deposits 20 into an account at the beginning of each year for 6 years. He deposits and additional X at the beginning of years 4,5 and 6 to meet his goal. The annual effective into rate is 10%. Calculate X. $\boxed{8.92}$

Ans:

$200(1.04^{10}) = (20s_{\overline{6}|} + Xs_{\overline{3}|})(1.10^5)$

$X = \dfrac{200(1.04^{10})(1.10^{-5}) - 20s_{\overline{6}|}}{s_{\overline{3}|}}$

$X = \dfrac{200(1.04^{10})(1.10^{-5}) - 20(7.7156)}{3.31}$

$X = 8.92$

Example 24.

Ellyn plans to accumulate 100,000 at the end of 42 years. She makes the following deposits:

- X at the beginning of years 1-14;
- No deposits at the beginning of years 15-32, and
- Y at the beginning of years 33-42. The annuity effective interest rate is 7%

$$X - Y = 100$$

Calculate Y. $\boxed{479}$

Ans:

$X - Y = 100$

$X = 100 + Y$ $100,000 = X\ddot{s}_{\overline{42}|} - Xs_{\overline{28}|} + Ys_{\overline{10}|}$

$100,000 = (100 + Y)(s_{\overline{42}|} - s_{\overline{28}|}) + Ys_{\overline{10}|}$

$100,000 = (100 + Y)(160.43) + Y(14.78)$

$Y = 479$

3.5 Perpetuities

A **perpetuity** is an annuity whose payments continue forever, i.e. the term of the annuity is not finite.

3.5.1 Perpetuity-immediate

A **perpetuity-immediate** is an annuity-immediate with annual payments of 1 where the payments continue forever.

$$a_{\overline{\infty}|} = v + v^2 + \cdots = \frac{v}{1-v} = \frac{v}{d} = \frac{v}{iv} = \frac{1}{i}.$$

3.5.2 Perpetuity-due

A **perpetuity-due** is an annuity-due with annual payments of 1 where the payments continue forever.

$$\ddot{a}_{\overline{\infty}|} = 1 + v + v^2 + \cdots = \frac{1}{1-v} = \frac{1}{d}$$

Another approach is to see that a perpetuity-due provides 1 today plus exactly the same payments as a perpetuity-immediate. we have

$$\ddot{a}_{\overline{\infty}|} = 1 + a_{\overline{\infty}|} = 1 = \frac{1}{i} = \frac{1+i}{i} = \frac{1}{d}$$

Since $d = \frac{i}{1+i}$.

Since $\ddot{a}_{\overline{\infty}|}$ exceed $a_{\overline{\infty}|}$ by 1, we have:

$$\ddot{a}_{\overline{\infty}|} - a_{\overline{\infty}|} = 1$$
$$\frac{1}{d} - \frac{1}{i} = 1$$

Example 25.

Deposits of 1000 are placed into a fund at the beginning of each year for 30 years. At the end of 40th years, annual payments commence and continue forever. Interest is at an effective annual rate of 5%. Calculate the annual payment. $\boxed{5{,}411}$

Ans: $1000s_{\overline{30|}}(1.05^{10}) = X(\frac{1}{0.05}$

$X = 1000s_{\overline{30|}}(1.05^{10})(0.05) = 5411$

Example 26.

The present value of a series payments of 2 at the end of every 8 years, forever, is equal to 5. Calculate the effective rate of interest. $\boxed{.043}$

Ans: $\frac{2}{j} = 5$

$j = \frac{2}{5} = 0.4$

$1 + 0.4 = (1 + I)^{1/8}$

$I = 0.043$

Example 27.

Mark receives 500,000 at his retirement. He invests 500,000-X in an annual payment 10-year annuity-immediate and X in an annual payment perpetuity-immediate. His total annual payments received during the first 10 years are twice as large as those received thereafter. the annual effective rate of interest is 6%. Calculate X. $\boxed{346{,}836}$

Ans: $500,000 - X = Pa_{\overline{10}|6\%}$
$X = Q(\frac{1}{0.06})$
$P + Q = 2Q$
$Q = P$
$500,000 = Q[a_{\overline{10}|6\%} + \frac{1}{0.06}]$
$Q = \frac{500{,}000}{7.36+\frac{1}{0.06}} = 20,810.21$
$X = \frac{20810.21}{0.06} = 346,836$

3.6 The $a_{\overline{2n}|}/a_{\overline{n}|}$ Trick and Variations

From a purely algebraic approach:

$$\begin{aligned}\frac{a_{\overline{2n}|}}{a_{\overline{n}|}} &= \frac{\frac{1-v^{2n}}{i}}{\frac{1-v^n}{i}} \\ &= \frac{1-v^{2n}}{1-v^n} \\ &= \frac{(1-v^n)(1+v^n)}{1-v^n} \\ &= 1 + v^n\end{aligned}$$

Another way to derive this result is by a general reasoning approach. Think of $a_{\overline{2n}|}$ as consisting of two annuities:

1. an annuity-immediate for n years, followed by
2. a deferred annuity for another n years.

The relationship is:

$$a_{\overline{2n}|} = a_{\overline{n}|} + v^n a_{\overline{n}|} = a_{\overline{n}|}(1 + v^n)$$

Hence $\frac{a_{\overline{2n}|}}{a_{\overline{n}|}} = 1 + v^n$

Notes:

1. "Double dots cancel": The quotient $\frac{\ddot{a}_{\overline{2n}|}}{\ddot{a}_{\overline{n}|}}$ is the same as the quotient $\frac{a_{\overline{2n}|}}{a_{\overline{n}|}}$:

   $\frac{\ddot{a}_{\overline{2n}|}}{\ddot{a}_{\overline{n}|}} = \frac{a_{\overline{2n}|}}{a_{\overline{n}|}} = 1 + v^n$

2. Formulas can also be derived for other annuities:

$$\begin{aligned} a_{\overline{3n}|} &= a_{\overline{n}|} + v^n a_{\overline{n}|} + v^{2n} a_{\overline{n}|} \\ &= a_{\overline{n}|}(1 + v^n + v^{2n}) \end{aligned}$$

   So, $\frac{a_{\overline{3n}|}}{a_{\overline{n}|}} = 1 + v^n + v^{2n}$

Example 28.

Becky receives payments of X at the end of each year for n years. The present value of her annuity is 493.
Sam receives payments of $3X$ at the end of each year for $2n$ years. The present value of his annuity is 2748.
Both present values are calculated at the same annual effective interest rate.
Determine v^n. $\boxed{.86}$

Ans:
$Xa_{\overline{n}|} = 493$
$3Xa_{\overline{2n}|} = 2748$
$\frac{3Xa_{\overline{2n}|}}{Xa_{\overline{n}|}} = \frac{2748}{493}$
$3(1 + v^n) = 5.574$
$v^n = 0.86$

Example 29.

You are given:

- $a_{\overline{n}|} = 10.00$; and
- $a_{\overline{3n}|} = 24.40$.

Determine $a_{\overline{4n}|}$.

Ans:

$\frac{a_{\overline{3n}|}}{a_{\overline{n}|}} = 1 + v^n + v^{2n} = \frac{24.40}{10} = 2.44$

$v^{2n} + v^n - 1.44 = 0$

$v^n = \frac{-1+\sqrt{1-4(-1.44)}}{2} = 0.8$

$a_{\overline{4n}|} = a_{\overline{n}|} + v^n a_{\overline{3n}|} = 10 + (0.8)(24.4) = 29.52$

3.7 Unknown Time

Suppose a loan of 10,000 is to be repaid at 5% effective by payments of 1,000 at the end of each year until the loan is repaid. How many payments are required?

It is very unlikely that an integral number of payments of 1,000 would just happen to have a PV of 10,000 at 5% effective.

Let's solve the following equation:

$$10,000 = 1,000a_{\overline{n}|5\%}$$

Using calculator, we found $n = 14.21$.

Since $14 < n < 15$, we know that 14 payments of 1,000 would not repay the loan but 15 payments of 1,000 would overpay the loan. So in order to exactly repay the loan, we have to do something special between time 14 and 15 that doesn't follow the regular schedule of 1,000 payments. There are three alternatives:

1. A final payment larger than 1,000 could be made at time 14. This is called a **balloon payment**. In this case, we first determine $14 < n < 15$. Thus, there are 14 regular payments of 1,000 and an additional amount (call it X_1) at time 14 that will repay the loan. The equation of value is:

$$10,000 = 1,000a_{\overline{14}|} + X_1 v^{14}$$
$$10,000 = 9898.64 + X_1(1.05)^{-14}$$
$$X_1 = 200.69$$

2. A final payment smaller than 1,000 could be made at time 15. This is called a **drop payment**. In this case, let the drop payment X_2 at time 15, the equation of value is:

$$10,000 = 1,000a_{\overline{14}|} + X_2 v^{15}$$
$$10,000 = 9898.64 + X_2(1.05)^{-15}$$
$$x_2 = 210.72$$

3. A final payment (X_3), smaller than 1,000 could be made at time n that is the solution to the equations of value, i.e. at $n = 14.21$.

$$10,000 = 1,000a_{\overline{14}|} + v^{14.21}X_3$$
$$10,000 = 9898.64 + (1.05^{-14.21}X_3$$
$$X_3 = 202.75$$

Example 30.

Bob purchased Amy's engagement ring on January 1, 2012 with a 10,000 loan. His loan carries an interest rate of 21% per yaer convertible monthly. He pays 600 per month starting February 1, 2012, plus an additional 1,600 on June 1, 2012. His last payment will be a partial payment. Determine when he will make the last full payment of 600.

Ans:

$$10,000 = 600a_{\overline{n}|1.75\%} + 1600v^5$$
$$a_{\overline{n}|1.75\%} = 10,000 - 1,600(1.0175^{-5}) = 142215$$
$$\frac{1-v^n}{0.0175} = 14.2215$$
$$v^n = 0.7511$$
$$n = 16.5$$

Thus, the last full payment of 600 will be make at the end of 16 months. 16 months after 1.1.92 is 5.1.93 = $\boxed{\text{May 1, 2013}}$

Example 31.

A 74.93 loan is to be repaid in two 50 installments. The effective annual rate of interest on the loan is 3%. The first interest is due in six years. When is the second installment due? $\boxed{14}$

Ans:

$$74.95 = 50v^6 + 50v^n$$
$$v^n = 0.65615$$
$$n = \frac{\ln(0.6615)}{\ln(1.03^{-1}} = 14$$

Example 32.

You are given an n-year annuity-due of one per year plus a final payment at time $n+k-1 (0 < k < 1)$. The present value of the payments can be simplified to:

$$\frac{1-v^{n+k}}{d}$$

Calculate the final payment. $\boxed{\frac{[(1+i)^k-1]}{i}}$

Ans:

$$\ddot{a}_{\overline{n}|} + Xv^{n+k-1} = \frac{1-v^{n+k}}{d}$$

$$\frac{1-v^n}{d} = +Xv^{n+k-1} = \frac{1-v^{n+k}}{d}$$

$$1-v^n + Xdv^{n+k-1} = 1-v^{n+k}$$

$$X = \frac{[v^n - v^{n+k}](1+i)^{n+k-1}}{d}$$

$$X = \frac{v[(1+i)^k-1]}{d}$$

$$X = \frac{[(1+i)^k-1]}{i}$$

3.8 Unknown Rate of Interest

This section consider the case when the rate of interest, i, is unknown. We can use TI-30XB calculators to compute the unknown interest rate. For example, if we wish to find the nominal annual rate of interest compounded semiannually of an 15,400 annuity-immediate with semiannual payment of 1,300 for 9 years:

- By using TI-30XB calculators: Let $j = \frac{i^{(2)}}{2}$

  The equation of value is: $1,300a_{\overline{18}|j} = 15,400$

  $$1,300\left[\frac{1-(1+j)^{-18}}{j}\right] = 15,400$$

  $$y = 1,300\left[\frac{1-(1+x)^{-18}}{x}\right] - 15,400$$

  In the calculators, hit $\boxed{\text{table}}$, then key in :

  $y = 1,300\left[\frac{1-(1+x)^{-18}}{x}\right] - 15,400$ and hit $\boxed{\text{enter}}$

  Since $0 < i < 1$, then we start with $\boxed{\text{Start} = 0.05}$ and use $\boxed{\text{Step} = 0.001}$ and choose $\boxed{\text{Auto}}$, the answer will be between the values that change y from $+$ to $-$ or from $-$ to $+$, here we find

$$x = 0.048 \text{or} x = 0.049$$

If we need more decimal places, the decrease the step size, let change Step from 0.001 t0 $\boxed{\text{Step} = 0.000001}$, then we get

$$x = 0.04830$$

i.e. $j = 0.0483$ and hence $i^{(2)} = 0.0483 \times 2 = 0.0966$

Example 33.

A renter with 1140 has one year lease. the landlord is willing to accept two payment options:

(i) 1104 now; or

(ii) 100 paid at the beginning of each month for twelve months.

What monthly interest rate would be required for two options to be equivalent? $\boxed{.0155}$

Ans:

$100\ddot{a}_{\overline{12}|j} = 1104$

To avoid using [BGN] mode on the BA II Plus calculators, we can substitute $1 + a_{\overline{11}|} = \ddot{a}_{\overline{12}|}$,

$$100a_{\overline{11}|} = 1004$$

Enter: 11 $\boxed{\text{N}}$ 1004 $\boxed{\text{PV}}$ 100 $\boxed{+-}$ $\boxed{\text{PMT}}$ $\boxed{\text{CPT}}$ $\boxed{\text{YY}}$

The answer is 1.55%

3.9 Varying Interest

When the effective rate of interest varies over the term of an investment, we have to be careful about how a question describes the rates.

Example 34.
Annual deposits of 100 are made at the beginning of each year for 10 years. Find the accumulated value at the end of 10 years if the effective rate of interest is 8% for the first 6 years and 7% for the last 4 years. $\boxed{1513.59}$

Ans:

$$\begin{aligned} AV &= 100\ddot{s}_{\overline{6}|8\%}(1.07^4) + 100\ddot{s}_{\overline{4}|7\%} \\ &= 100(s_{\overline{7}|8\%} - 1)(1.07^4) + 100(s_{\overline{5}|7\%} - 1) \\ &= 100(8.9228 - 1)(1.07^4) + 100(5.75074 - 1) \\ &= 1513.59 \end{aligned}$$

Example 35.
Annual deposits of 100 are made at the beginning of each year for 10 years. Find the accumulated value at the end of 10 years if the first 6 deposits are invested at 8% effective and the last 4 deposits are invested at 7% effective. $\boxed{1552.96}$

Ans:

$$\begin{aligned} AV &= 100\ddot{s}_{\overline{6}|8\%}(1.08^4) + 100\ddot{s}_{\overline{4}|7\%} \\ &= 100(s_{\overline{7}|8\%} - 1)(1.08^4) + 100(s_{\overline{5}|7\%} - 1) \\ &= 100(8.9228 - 1)(1.08^4) + 100(5.75074 - 1) \\ &= 1552.96 \end{aligned}$$

Example 36.

Smith deposits 100 at the beginning of each year in Saving Account A and 100 at the beginning of each year in Saving account B. At the end of N years the balance in each account is 2,350.

- The effective annual rate of interest on Saving Account A is 3.5%.
- The effective annual rate of interest on Saving Account B is $i\%$ for the first $N-6$ years and 5% for the last 6 years.

Calculate i. $\boxed{1.726}$

Ans:

$100s_{\overline{N}|3.5\%} = 2350$
$100(s_{\overline{N+1}|3.5\%} - 1) = 2350$
$100(s_{\overline{N+1}|3.5\%}) = 2450$
$N + 1 = 18$
$N = 17$

$100s_{\overline{N-5}|i\%}(1.05^6) + 100\ddot{s}_{\overline{5}|5\%} = 2350$
$134s_{\overline{12}|i\%} + 100(s_{\overline{5}|5\%} - 1) = 2350$ $134s_{\overline{12}|i\%} = 1769.81$
$i = 1.726\%$

Example 37.

You are given $\delta_t = \frac{2}{10+t}, t \geq 0$. Calculate $a_{\overline{4}|}$. $\boxed{2.62}$

Ans:

$$\begin{aligned}
a(t) &= e^{\int_0^t \frac{2}{10+u} du} \\
&= e^{2[\ln(10+u)]_0^2} \\
&= e^{2[\ln(\frac{10+t}{10}]} \\
&= \left(\frac{10+t}{10}\right)^2
\end{aligned}$$

$$\begin{aligned}
a_{\overline{4}|} &= \frac{1}{a(1)} + \frac{1}{a(2)} + \frac{1}{a(3)} + \frac{1}{a(4)} \\
&= \left(\frac{10}{11}\right)^2 + \left(\frac{10}{12}\right)^2 + \left(\frac{10}{13}\right)^2 + \left(\frac{10}{14}\right)^2 \\
&= 2.62
\end{aligned}$$