CONTENTS

6 Bonds

One of the major applications of the theory of interest is the determination of prices and values for bonds and other securities, such as preferred stock and common stock. There are three main questions related to securities:

1. Given the desired yield rate of an investor, what price should be paid for a given security?
2. Given the purchase price of a security, what is the resulting yield rate to an investor?
3. What is the value of a security on a given date after it has been purchased?

6.1 Price of a Bond

A bond is basically a loan to a governmental entity or corporation on which we typically receive interest payments called "coupons", and then receive the redemption value on the redemption date.

Formal notations:

- P = the *price* of a bond;
- F = the *par value*, or *face amount*. It is printed on the front of the bond and is often the amount payable at the maturity date. It is customary to quote bond prices in terms of a par value of 100;
- C = the *redemption value* of a bond. It is the amount of money paid at a redemption date to the holder of the bond. Often $C = F$, however there are exceptions.

- r = the *coupon rate*, rate per coupon payment period. The most common frequency for bond coupons in the US is semiannual. E.g. an 8% bond with semiannual coupons has $r = 0.04$. In the international financial markets, there are other coupon frequencies.
- Fr = the amount of the coupon;
- $g = \frac{Fr}{C}$ is the *modified rate* of the bond. We have $Cg = Fr$. (If a bond is redeemable at par, $C = F$ and $g = r$.)
- i = the *yield rate* of the bond, i.e. the rate realized by the investor, or the *internal rate of return*, or the *yield to maturity*.
- n = the number of coupon payment periods;
- $K = Cv^n$ = the present value of the redemption;
- $G = \frac{Fr}{i}$, the base amount of a bond;
- $v = \frac{1}{1+i}$.

6.1.1 Basic Formulas:

The price of a bond to yield an effective rate i is the PV of bond payments at that rate. The PV of the payments is the PV of the coupons plus the PV of the redemption value:

$$P = Fra_{\overline{n}|} + Cv^n$$

or

$$P = Cga_{\overline{n}|} + Cv^n$$

6.1.2 The Premium/Discount Formula

Since $a_{\overline{n}|} = \frac{1-v^n}{i}$, and $v^n = 1 - ia_{\overline{n}|}$:

$$\begin{aligned} P &= Fra_{\overline{n}|} + Cv^n \\ &= Fra_{\overline{n}|} + C(1 - ia_{\overline{n}|}) \\ &= C + (Fr - Ci)a_{\overline{n}|} \\ &= C + (Cg - Ci)a_{\overline{n}|} \end{aligned}$$

Example 1.

A 20-year 3,000 par value bond with 10% semi-annual coupons is purchansed to earn a yield of 7% convertible semiannually. What is the price of the bond?

Example 2.

Find the price to yield 4% effective for a 100 bond with 5.5% annual coupons redeemable at 110 in 10 years, using the basic formula and premium/discount formula. 118.92

Example 3.

John buys a bond that is due to mature at par in 3 year. It has a 200 par value and coupons at 3% convertible semiannually. John pays 197.63 to obtain a yield rate i convertible semiannually, $i > 0$. Calculate i.

Example 4.

John purchases a 1000 par value 15-year bond with coupons at 10% convertible semiannually which will be redeemed for C. The purchase price is 585.0 and the present value of the redemption value is 71.0. Calculate C.

Example 5.

A 1,000 bond with with annual coupons is redeemable at par at the end of 9 years. At a purchase price of 850, the yield rate is i. The coupon rate $i - 0.03$. Calculate i.

Example 6.

A 1000 par value 24-year bond with annual coupons and redeemable at maturity at 1050 is purchased for P to yield an annual effective rate of 8.86%. The first coupon is 90. Each subsequent coupon is 5% greater than the preceding coupon. Determine P.

Example 7.

Two bonds are purchased for the same price to yield 5%. Bond X has 4% annual coupons and matures for its face value of 100. Bond Y has annual coupons of 3 and matures for 180. Both bonds mature at the end of n years. Calculate n.

33

Example 8.
A 1,000 par value 3-year bond with annual coupons of 50 for the first year, 70 for the second year and 90 for the third year is bought to yield a force of interest

$$\delta_t = \frac{2t-1}{2(t^2-t+1)} \text{ for } t \geq 0.$$

Calculate the price of this fund. 502.4

6.2 Premium and Discount

When a bond is purchased for more than its redemption value, the excess of the price over the redemption value is called the "premium". When a bond is purchased for less than its redemption value, the excess of the price over the redemption value is called the "discount".

The premium/discount formula is ideal for quickly determining whether a bond is sold at a premium or discount:

$$P = C + (Cg - Ci)a_{\overline{n}|}$$

Thus

1. If $g > i$, then $P > C$ and the bond is purchases at a premium equal to $P - C = (Cg - Ci)a_{\overline{n}|}$.
2. If $g < i$, then $P < C$ and the bond is purchases at a discount equal to $C - P = (Cg - Ci)a_{\overline{n}|}$.

Example 9.
A 1,000 bond with 8% semiannual coupons redeemable at 1,050 in 1.5 years, purchased to yield a nominal annual rate of 6% compounded semiannually.

(a) Determine whether the bond is purchased at a premium or discount.

(b) What is the amount of premium/discount?

Example 10.
You are given:

(i) A 10-year 8% semiannual coupon bond is purchased at a discount of X.

(ii) A 10-year 9% semiannual coupon bond is purchased at a premium of Y.

(iii) A 10-year 10% semiannual coupon bond is purchased at a premium of $2X$.

(iv) All bonds were purchased at the same yield rate and have par values of 1000.

Calculate Y. 0.5X

Example 11.

Two 16-year bonds with 100 redemption values are each purchased to yield an effective annual interest rate of 5%. The first bond bears annual $g\%$ coupons and is purchased at a premium of 14.13. The second bond bears annual $(g+2)\%$ coupons. What is the purchased price of the second bond?

6.3 Bond Amortization

When a bond is bought at a premium ($g > i$), the book value will gradually be adjusted downward. This process is called amortization of premium or "write down." When a bond is bought at a discount ($g < i$), the book value will gradually be adjusted upward. This process is called accumulation of discount or "write ups." The concept of the bond amortization is the same as the loan amortization, however, bond terminology is different:

(i) "Book value" is used instead of "outstanding loan balance."

(ii) Instead of "principal repaid," the periodic reduction in the book value is called the "amount for amortization of premium." if the bond is purchased at premium and the periodic increase in the book value is called the 'amount for accumulation of discount".

(iii) Instead of "payment amount," coupon is used.

6.3.1 Bond Amortization When a Bond is Purchased At a Premium

Consider a 1,000 bond with 8% semiannual coupons redeemable at 1,050 in 1.5 years, purchased to yield a nominal annual rate of 6% compounded semiannually. We construct the bond a bond amortization schedule:
$B_0 = P = Fra_{\overline{n}|i} + Cv^n = 1000(.04)\left[\frac{1-1.03^{-3}}{.03}\right] + 1050(1.03^{-3}) = 1074.04$

		Interest Earned	Amount for Amortization of Premium	Book Value
Period t	Coupon	$I_t = iB_{t-1}$	P_t = Coupon $- I_t$	$B_t = B_{t-1} - P_t$
0				1,074.04
1	40	32.22	7.78	1,066.26
2	40	31.99	8.01	1,058.25
3	40	31.75	8.25	1,050.00
Totals	120	95.96	24.04	

Notes:

1. Book Value, $B_t = B_{t-1} - P_t$ or
   Prospectively: $B_t = Fra_{\overline{n-t}|} + Cv^{n-t}$
   retrospectively: $B_t = B_0(1+i)^t - Frs_{\overline{t}|}$

2. $I_t = \boxed{iB_{t-1}}$
   $= i[Fra_{\overline{n-t+1}|} + Cv^{n-t+1}]$
   $= Fr(1 - v^{n-t+1}) + Civ^{n-t+1}$
   $= \boxed{Fr - (Fr - Ci)v^{n-t+1}}$

3. The amounts for amortization of premium (write down),

   $P_t = Fr - I_t$
   $= (Fr - Ci)v^{n-t+1}$
   $= (Cg - Ci)v^{n-t+1}$

4. P_t are also in geometric progression with common ratio $(1+i)$.

   $$P_t = (1+i)^{t-1}P_1$$

5. P_t in this case is also called "write down" because the asset value of a bond is "written down" by this amount each period.

6. Total Interest Earned
   = Total payments received - Amount invested

$$\sum_{t=1}^{n} I_t = (nFr + C) - P$$

   For this bond:
   Total payments received
   = 3 coupons + redemption value
   $= 3 \times 40 + 1,050 = 1,170$
   The amount invested = 1,074.04, so,
   Total Interest Earned $= 1,170 - 1074.04 =$ 95.56

7. The total of the amounts for amortization of premium = Premium.

$$\sum_{t=1}^{n} P_t = P - C$$

   Premium $= P - C = (cg - ci)a_{\overline{n}|}$
   $= [40 - 1050(0.03)]a_{\overline{3}|}$
   $= 24.04$

Example 12.

A 1000 par value 15-year bond with 8% semiannual coupons was bought to yield 7.5% convertible semiannually. Determine the amount of premium amortized(write down) in the 8th coupon payment.

Example 13.

An actuary finds a 18-year bond that was purchased at a premium has determined the following:

- The bonds pays semiannual interest.
- The amount for amortization of the premium in the 2nd coupon payment was 983.46.
- The amount for amortization of the premium in the 4th coupon payment was 1129.23.

What is the value of the premium?

Example 14.

A 18-year bond with par value of 1000 is purchased to yield 9% convertible semiannually. Par value equals redemption value. The interest paid portion of the first semiannually coupon is 43.73. At what nominal rate of interest(express in %) convertible semiannually are the coupons paid?

Example 15.

Bryan buys a $2n-$year 1000 par value bond with 8.1% annual coupons at a price P. The price assumes an annual effective yield of 13%. At the end of n years, the book value of the bond, X, is 48.92 greater than the purchase price, P. Assume $v_{13\%}^{n} < 0.5$. Calculate X.

Example 16.

An $n-$ year 1000 par value bond with 8% annual coupons has an annual effective yield of $i, i > 0$. The book value of the bond at the end of the year 3 is 1099.84 and the book value at the end of year 5 is 1082.27. Calculate the purchase price of the bond. 1122.38

6.3.2 Bond Amortization When a Bond is Purchased At a Discount

When a bond is bought at a discount ($g < i$), the book value will gradually be adjusted upward. This process is called accumulation of discount or "write ups."

Consider a 1,000 bond with 8% semiannual coupons redeemable at 1,050 in 1.5 years, purchased to yield a nominal annual rate of 10% compounded semiannually. The price of the bond is:

$$\begin{aligned} P &= 1,050 + [40 - (1,050)(0.05)]a_{\overline{3}|} \\ &= 1,050 - 12.5a_{\overline{3}|} \\ &= 1,015.96 \end{aligned}$$

We construct the bond a bond amortization schedule, to avoid using negative numbers, we will write the entries in the principal repaid column as positive amounts and remember to add each of them to the previous book value.

		Interest Earned	Amount for Accumulation of Discount	Book Value
Period t	Coupon	$I_t = iB_{t-1}$	$P_t =$ Coupon $- I_t$	$B_t = B_{t-1} + P_t$
0				1,015.96
1	40	50.80	10.80	1,026.76
2	40	51.34	11.34	1,038.10
3	40	51.90	11.90	1,050.00
Totals	120	154.04	34.04	

Notes:

- Total Interest Earned
  = Total payments received - Amount invested

$$\sum_{t=1}^{n} I_t = (nFr + C) - P$$

In this case, $\sum_{t=1}^{n} I_t = 3 \times 40 + 1,050 - 1015.96 = 154.04$

- The total of the "amount for accumulation of discount" is equal to the discount.

$$\sum P_t = C - P$$

- Another term used for "P_t = amount for accumulation of discount" is "write up" since the asset value of a bond is "written up" by this amount each period.

$$P_t = (Ci - Cg)v^{n-t+1} = (Ci - Fr)v^{n-t-1}$$

- The amount for accumulation of discount are in geometric progression.

$$P_t = P_1(1+i)^{t-1}$$

Note that in the case of a bond purchased at a discount, we have defined the amount for accumulation of discount P_t as the negative of the normal principal repaid, to avoid negative signs.

Example 17.

A 24-year bond with semiannual coupons has a redemption value of 100. It is purchased at a discount to yield 12% compounded semiannually. If the amount for accumulation of discount in the 41th coupon payment is 3.03. Determine the total amount of discount in the original purchase price?

Example 18.

A 35-year 10,000 bond that pays 3% annual cuopons matures at par. It is purchased to yield 5% for the first 17 year and 4% thereafter. Calculate the amount for accumulation of discount for year 9.

Example 19.

Laura buys two bonds at time 0. Bond X is a 5,000 par value 13-year bond with 12% annual coupons. It is bought at a price to yield an annual effective rate of 10%. Bond Y is a 13-year par value bond with 8.1% annual coupons and a face amount of F. Laura pays P for Bond Y to yield an annual effective rate of 10%. During year 5, the write-down in premium (principal adjustment) on bond X is equal to the write-up in discount (principal adjustment) on bond Y. Calculate P.

Example 20.
A 1,000 par value bond bearing 4% annual coupons is purchased at a discount to yield an effective annual rate of 5%, the write-up in value during the first year is 4.36. Determine the purchased price.
887.2

6.4 Determination of Yield Rates

Up to now, we have usually assumed that the yield rate is known and that the price has to be computed. In this section, the yield rate is unknown, and most of the times the equation of values has to be solve numerically.
In the case of a bond, using the Basic formula, we are looking for the solution of the equation for i:

$$P = Fra_{\overline{n}|} + Cv^n$$

where P, Fr, C and n are known. We have to use the Table in TI30 calculator to search the solution.

Example 21.

A 100 par value 14-year bond with 10% semiannual coupons is selling for 87. Find the yield rate convertible semiannually.

Example 22.

A 1,000 19-year 8% bond with semiannual coupons is purchased for 1018. The redemption value is 1,000. The coupons are reinvested at a nominal annual rate of 9%, compounded semiannually. Determined the purchaser's annual effective yield rate over the 19 year period.