CONTENTS

4 More General Annuities

4.1 Annuities with "Off Payments"

4.1.1 Payments Less Frequent Than the Interest Period

There are two general approaches for handling these types of annuities:

1. Use interest functions at the equivalent effective rate of interest for the **payment period**.
2. Use interest function at the effective rate of interest **given in the problem**.

Example 1.
Suppose the effective rate of interest is given as 5% per annum. An annuity with payments of 1 at the end of each 5-year period over 40 years (Payments are less frequent that the interest period of one year). Determine the PV of this annuity in terms of interest functions at the effective rate for a 5-year period. 3.1054

Example 2.

Determine the PV of the annuity with payments of 1 at the end of each 5-year period over 40 years at an annual effective rate of 5%, in terms of interest functions at 5%. $\boxed{3.1054}$

Example 3.

Find the PV of an annuity-immediate of 2,100 a year for 8 year at a nominal rate of interest of 8% compounded quarterly, in terms of interest functions at 2%.

Example 4 (T04Q01)**.**

Find the PV of an annuity with payments of 500 at the beginning of every 3 years for 18 years at 5% effective per annum, in terms of interest functions at 5%.

Example 5.

The present value of 1 payable at the end of years 7, 11, 15, 19, 23, 27 is

$(A)\dfrac{a_{\overline{25|}}-a_{\overline{4|}}}{a_{\overline{4|}}}$ $(B)\dfrac{a_{\overline{28|}}-a_{\overline{4|}}}{s_{\overline{4|}}}$ $(C)\dfrac{a_{\overline{28|}}-a_{\overline{4|}}}{s_{\overline{3|}}+d}$

$(D)\dfrac{a_{\overline{28|}}-a_{\overline{4|}}}{s_{\overline{3|}}-a_{\overline{1|}}}$ $(E)\dfrac{a_{\overline{28|}}-a_{\overline{4|}}}{s_{\overline{3|}}+a_{\overline{1|}}}$

Example 6.
You are given:

- the present value of a $6n-$year annuity-immediate of 1 at the end of every year is 9.996.
- the present value of a $6n-$year annuity-immediate of 1 at the end of every second year is 4.760.
- the present value of a $6n-$year annuity-immediate of 1 at the end of every third year is X.

Calculate X. $\boxed{3.02}$

4.1.2 Payments More Frequent Than the Interest Period

Let m be the number of payment periods in one interest conversion period, let n be the term of the annuity measured in interest conversion periods, and let i be the interest rate per interest conversion period. Assume that each interest conversion period contains an integral number of payment periods, i.e. m and n are both positive integers. The number of annuity payments made is mn.
For example, express the present value of an annuity-immediate with monthly payments of $\frac{1}{12}$ for 10 years at 5% effective per annum in terms of annuity functions at 5%.

$$PV = \frac{1}{12}(v^{\frac{1}{12}} + v^{\frac{2}{12}} + \cdots + v^{\frac{120}{12}})$$

The standard symbol for the PV is $a^{(12)}_{\overline{10}|}$ (1 per annum payable in monthly installments of $\frac{1}{12}$ each for 10 years). It can shown that by summing the geometric progression, the following formula can be derived:

$$\begin{aligned}
a_{\overline{10}|}^{(12)} &= \frac{1}{12}(v^{\frac{1}{12}} + v^{\frac{2}{12}} + \cdots + v^{\frac{120}{12}}) \\
&= \frac{1}{12}\left[\frac{v^{\frac{1}{12}}(1-v^{\frac{1}{12}(120)})}{1-v^{\frac{1}{12}}}\right] \\
&= \frac{1}{12}\left[\frac{1-v^{10}}{v^{-\frac{1}{12}}-1}\right] \\
&= \frac{1-v^{10}}{12\left[(1+i)^{\frac{1}{12}}-1\right]} \\
&= \frac{1-v^{10}}{i^{(12)}}
\end{aligned}$$

(Recall that $i^m = m[(1+i)^{\frac{1}{m}} - 1]$)

Since $a_{\overline{10}|} = \frac{1-v^{10}}{i}$, then

$$a_{\overline{10}|}^{(12)} = \frac{i}{i^{(12)}} a_{\overline{10}|}$$

This can be interpreted as an annuity with annual payments of $\frac{i}{i^{(12)}}$.

Consider the payments in the first year:

```
                  i%
      <-------------------------------->
          1/12     1/12    1/12       1/12
years |-------|-------|-------|----...---|
      0       1/12    2/12    3/12       1
     PV                                  R
```

"R" at time 1 represents the accumulated value of the twelve monthly payments of $\frac{1}{12}$.

$$\begin{aligned}
R &= \frac{1}{12}[1 + (1+i)^{\frac{1}{12}} + \cdots + (1+i)^{\frac{11}{12}}] \\
&= \frac{1}{12} \cdot \frac{1-[(1+i)^{\frac{1}{12}}]^{12}}{1-(1+i)^{\frac{1}{12}}} \\
&= \frac{i}{12[(1+i)^{\frac{1}{12}}-1]} \\
&= \frac{i}{i^{(12)}}
\end{aligned}$$

Since $\frac{i}{i^{(12)}}$ is an accumulated value, it can be represented by an "s" symbol. Since the accumulation is over one interest period $n = 1$ and the symbol is $s_{\overline{1}|}$. Since payments are monthly, and since the total payment in interest period is 1, thus $\frac{i}{i^{(12)}} = s_{\overline{1}|}^{(12)}$.

In general, the PV of an n-year annuity-immediate of $\frac{1}{m} \times m = 1$ per annum payable in m-thly installments is:

$$a_{\overline{n}|}^{(m)} = \frac{1-v^n}{i^{(m)}} = \frac{i}{i^{(m)}} a_{\overline{n}|} = s_{\overline{1}|}^{(m)} a_{\overline{n}|}$$

Similarly, the AV is:

$$s_{\overline{n}|}^{(m)} = \frac{(1+i)^n - 1}{i^{(m)}} = \frac{i}{i^{(m)}} s_{\overline{n}|} = s_{\overline{1}|}^{(m)} s_{\overline{n}|}$$

For annuity-due,

$$\ddot{a}_{\overline{n}|}^{(m)} = (1+i)^{1/m} a_{\overline{n}|}^{(m)} = \frac{1-v^n}{d^{(m)}} = \frac{i}{d^{(m)}} a_{\overline{n}|} = \ddot{s}_{\overline{1}|}^{(m)} a_{\overline{n}|}$$

and

$$\ddot{s}_{\overline{n}|}^{(m)} = (1+i)^{1/m} s_{\overline{n}|}^{(m)} = \frac{(1+i)^n - 1}{d^{(m)}} = \frac{i}{d^{(m)}} s_{\overline{n}|} = \ddot{s}_{\overline{1}|}^{(m)} s_{\overline{n}|}$$

If the m-thly payments continue forever, we have a **perpetuity**:

- $\lim_{n\to\infty} a_{\overline{n}|}^{(m)} = a_{\overline{\infty}|}^{(m)} = \lim_{n\to\infty} \left(\frac{1-v^n}{i^{(m)}}\right) = \frac{1}{i^{(m)}}$
- $\ddot{a}_{\overline{\infty}|}^{(m)} = \frac{1}{d^{(m)}}$
- $\ddot{a}_{\overline{\infty}|}^{(m)} - a_{\overline{\infty}|}^{(m)} = \frac{1}{d^{(m)}} - \frac{1}{i^{(m)}} = \frac{1}{m}$

Note: The coefficient of the $a_{\overline{n}|}^{(m)}$, $\ddot{a}_{\overline{n}|}^{(m)}$, $s_{\overline{n}|}^{(m)}$, and $\ddot{s}_{\overline{n}|}^{(m)}$ are the sum of payments in each interest payment.

Example 7.

An annuity will pay 200 at the end of every month for 16 years at an effective rate of interest of 6%. Express the present value of the payments in terms of an annuity symbol at 6%.

Example 8 (T04Q02)**.**

The proceeds of a 10,000 death benefit are left on deposit with an insurance company for seven years at an annual effective interest rate of 9%. The balance at the end of seven years is paid to the beneficiary in 180 equal monthly payments of X, with the first payment made immediately. During the payout period, interest is credited at an annual effective interest rate of 7%. Calculate X.

Example 9 (T04Q03).

You are given a perpetuity, with annual payments as follows:

- Payments of 1 at the end of first year and every three years thereafter.
- Payments of 2 at the end of second year and every three years thereafter.
- Payments of 3 at the end of third year and every three years thereafter.

The interest rate is 10% convertible semiannually. Calculate the present value of this perpetuity.

Example 10.

Which of the following statements are true?

(i) $s_{\overline{n}|} - a_{\overline{n}|} = i a_{\overline{n}|} s_{\overline{n}|}$

(ii) $\ddot{s}^{(m)}_{\overline{n}|} - s^{(m)}_{\overline{n}|} = \dfrac{i^{(m)}}{m} s^{(m)}_{\overline{n}|}$

(iii) ${}_{1/4|}\ddot{a}^{(2)}_{\overline{n}|} + a^{(2)}_{\overline{n}|} = a^{(4)}_{\overline{n}|}$

4.1.3 Continuous Annuities

We can evaluate a continuous annuity by taking the limit of $a_{\overline{n}|}^{(m)}$ or $\ddot{a}_{\overline{n}|}^{(m)}$ as $m \to \infty$.

$$\lim_{m\to\infty} a_{\overline{n}|}^{(m)} = \lim_{m\to\infty} \left(\frac{1-v^n}{i^{(m)}} \right)$$

As $i^{(m)} = \frac{(1+i)^{\frac{1}{m}}-1}{1/m}$

Using l'Hospital's rule,

$$\begin{aligned} \lim_{m\to\infty} i^{(m)} &= \lim_{m\to\infty} (1+i)^{\frac{1}{m}} \frac{-\frac{1}{m^2}\ln(1+i)}{-\frac{1}{m^2}} \\ &= \ln(1+i) \\ &= \delta \end{aligned}$$

Similarly,

$$\lim_{m\to\infty} d^{(m)} = -\ln(1-d) = ln(1+i) = \delta$$

Thus,

$$\lim_{m\to\infty} a_{\overline{n}|}^{(m)} = \frac{1-v^n}{\delta}$$

and

$$\lim_{m\to\infty} \ddot{a}_{\overline{n}|}^{(m)} = \lim_{m\to\infty} \left(\frac{1-v^n}{d^{(m)}} \right) = \frac{1-v^n}{\delta}$$

The limit could be written for short as $a_{\overline{n}|}^{(\infty)}$ but the standard symbol for it is $\bar{a}_{\overline{n}|}$ read as "a bar angle n."

$\bar{a}_{\overline{n}|}$ can also be expressed in terms of $a_{\overline{n}|}$.

$$\bar{a}_{\overline{n}|} = \frac{i}{\delta} a_{\overline{n}|}$$

Another way to evaluate $\bar{a}_{\overline{n}|}$ is to use integral:

$$\bar{a}_{\overline{n}|} = \int_0^n v^t dt$$

Because

$$\begin{aligned} \int_0^n v^t dt &= \int_0^n e^{t\ln(v)} dt \\ &= \int_0^n e^{-\delta t} dt \\ &= [\tfrac{-e^{-\delta t}}{\delta}]_0^n \\ &= \tfrac{1-v^n}{\delta} \\ &= \bar{a}_{\overline{n}|} \end{aligned}$$

Example 11.

Find the PV of a 14-year annuity with continuous payments at the rate of 700 a year at an effective interest rate of 6.93%.

Example 12.

You are given $\int_0^n \bar{a}_{\overline{t}|} dt = 100$. Calculate $\bar{a}_{\overline{n}|}$.

$\boxed{n - 100\delta}$

4.1.4 Double -dots and Upper m's Cancel

1. Double -dots cancel

$$\frac{\ddot{a}_{\overline{n}|}}{\ddot{a}_{\overline{p}|}} = \frac{\frac{1-v^n}{d}}{\frac{1-v^p}{d}} = \frac{1-v^n}{1-v^p} = \frac{a_{\overline{n}|}}{a_{\overline{p}|}}$$

2. Not only double-dots cancel, so do upper m's

$$\frac{\ddot{a}^{(m)}_{\overline{n}|}}{\ddot{a}^{(m)}_{\overline{p}|}} = \frac{\frac{1-v^n}{d^{(m)}}}{\frac{1-v^p}{d^{(m)}}} = \frac{1-v^n}{1-v^p} = \frac{a_{\overline{n}|}}{a_{\overline{p}|}} = \frac{\frac{1-v^n}{i^{(m)}}}{\frac{1-v^p}{i^{(m)}}} = \frac{a^{(m)}_{\overline{n}|}}{a^{(m)}_{\overline{p}|}}$$

Note: These relationships assume that all functions are at the **same interest rate**.

Example 13.

Given $a^{(12)}_{\overline{n}|} = 10$, $a^{(12)}_{\overline{2n}|} = 15$. Determine i.

$\frac{a^{(12)}_{\overline{2n}|}}{a^{(12)}_{\overline{n}|}} = \frac{a_{\overline{2n}|}}{a_{\overline{n}|}} = 1 + v^n = \frac{15}{10}$

$v^n = 0.5$

$a^{(12)}_{\overline{n}|} = \frac{1-v^n}{i^{(12)}} = \frac{1-0.5}{i^{(12)}} = 10$

$i^{(12)} = 0.05$

$1 + i = \left(1 + \frac{.05}{12}\right)^{12}$

$i = 0.051162$

4.1.5 The $s_{\bar{n}|}$ Trap When Interest Rate Varies

When the accumulation function is $a(t)$, $a(t)$ is the AV of a deposit made at time 0 and not at any other time. If we would like to find $s_{\overline{n}|}$, the AV's at various point are not accumulating from time 0. For example if we would like to find $s_{\overline{4}|}$, the AV at time 4 of 1 deposited at time 3 is $\frac{a(4)}{a(3)}$, not $a(1)$, the AV at time 4 of 1 deposited at time 2 is $\frac{a(4)}{a(2)}$, not $a(2)$, and the AV at time 4 of 1 deposited at time 1 is $\frac{a(4)}{a(1)}$, not $a(1)$. Therefore, the correct expression for $s_{\overline{4}|}$ is:

$$s_{\overline{4}|} = 1 + \frac{a(4)}{a(3)} + \frac{a(4)}{a(2)} + \frac{a(4)}{a(1)}$$

Example 14 (T04Q04)**.**

You ara given $\delta_t = 3/(71+t)$ for $0 \le t \le 5$. Calculate $s_{\overline{5}|}$.

4.2 Increasing and Decreasing Annuities

Consider a general annuity-immediate with a term of n periods in which payments begin at P and increase by Q per period thereafter. The interest rate is i per period.

```
        P    P+Q   P+2Q    P+(n-1)Q
  |-----|-----|-----|---...---|
  0     1     2     3         n
```

The present value of the annuity, PVI, is

$$PVI = Pv + (P+Q)v^2 + \cdots + [P+(n-1)Q]v^n \quad (1)$$
$$(1+i)PVI = P + (P+Q)v + (P+2Q)v^2 + \cdots + [P+(n-1)Q]v^{n-1} \quad (2)$$

Subtracting the first equation from the second equation

$$iPVI = P + Q[v + v^2 + \cdots + v^{n-1}] - [P+(n-1)Q]v^n$$

$$\begin{aligned} PVI &= \frac{P+Q[v+v^2+\cdots+v^{n-1}]-[P+(n-1)Q]v^n}{i} \\ &= \frac{P+Qa_{\overline{n-1}|}-[P+(n-1)Q]v^n}{i} \\ &= P\frac{1-v^n}{i} + Q\frac{a_{\overline{n-1}|}-nv^n+v^n}{i} \\ &= Pa_{\overline{n}|} + Q\frac{a_{\overline{n}|}-nv^n}{i} \end{aligned}$$

$$\boxed{PVI = Pa_{\overline{n}|} + Q\frac{a_{\overline{n}|} - nv^n}{i}}$$

PVI of the same payments on the date of the first payment (annuity due)(call it $P\ddot{V}I$) is

$$P\ddot{V}I = (1+i)PVI = P\ddot{a}_{\overline{n}|} + Q\frac{a_{\overline{n}|} - nv^n}{d}$$

To obtained the corresponding accumulated value, multiple PV by $(1+i)^n$:

$$AVI = (1+i)^n PVI = Ps_{\overline{n}|} + Q\frac{s_{\overline{n}|} - n}{i}$$

$$A\ddot{V}I = (1+i)^n P\ddot{V}I = P\ddot{s}_{\overline{n}|} + Q\frac{s_{\overline{n}|} - n}{d}$$

If Q is negative, we have decreasing payments, thus:

- $PVD = Pa_{\overline{n}|} - Q\frac{a_{\overline{n}|}-nv^n}{i}$
- $P\ddot{V}D = (1+i)PVD = P\ddot{a}_{\overline{n}|} - Q\frac{a_{\overline{n}|}-nv^n}{d}$
- $AVD = (1+i)^n PVD = Ps_{\overline{n}|} - Q\frac{s_{\overline{n}|}-n}{i}$
- $A\ddot{V}D = (1+i)^n P\ddot{V}D = P\ddot{s}_{\overline{n}|} - Q\frac{s_{\overline{n}|}-n}{d}$

Example 15.
Consider an annuity-immediate with the following payments: $5, 8, 11, \ldots, 32$ in years $1, 2, 3, \ldots, 10$ at an annual effective rate of 5% per annum. Determine the present value and accumulated value of this annuity.

Example 16.

Consider an annuity-immediate with the following payments: $900, 800, 700, 600, 500$ in years $1, 2, 3, 4, 5$ at an annual effective rate of 7% per annum. Determine the accumulated value at year 5 of this annuity.

Example 17 (T04Q05)**.**

John receives 13-year increasing annuity-immediate paying 900 the first year and increasing by 900 each year thereafter. Mavis receives a 13-year decreasing annuity-immediate paying Y the first year and decreasing by $\frac{Y}{13}$ each year thereafter. At an effective annual interest rate of 12%, both annuities have the same present value. Calculate Y.

Example 18 (T04Q06)**.**

Two annuities have equal present values. The first is an annuity-immediate with quarterly payments of X for 13 years. The second is an increasing-annuity with 13 annual payments. The first payment is 600 and subsequent payments increase by 60.0 per year. You may assume an annual effective interest rate of 8%. Determine X.

4.2.1 The Increasing Annuity

When $P = Q = 1$, we have **Increasing Annuity**. The PV and AV are denoted $(Ia)_{\overline{n}|}$ and $(Is)_{\overline{n}|}$.

$$\begin{aligned}(Ia)_{\overline{n}|} &= a_{\overline{n}|} + \frac{a_{\overline{n}|} - nv^n}{i} \\ &= \frac{ia_{\overline{n}|} + a_{\overline{n}|} - nv^n}{i} \\ &= \frac{(1+i)a_{\overline{n}|} - nv^n}{i} \\ &= \frac{\ddot{a}_{\overline{n}|} - nv^n}{i}\end{aligned}$$

$$\boxed{(Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i}}$$

The "due" form can be obtained by multiplying the "immediate" form by $(1 + i)$:

$$(I\ddot{a})_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{d}$$

$(Is)_{\overline{n}|}$ can be obtained by multiplying $(Ia)_{\overline{n}|}$ by $(1 + i)^n$:

$$(Is)_{\overline{n}|} = (1+i)^n (Ia)_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{i} = \frac{s_{\overline{n+1}|} - (n+1)}{i}$$

and the "due" form is:

$$(I\ddot{s})_{\overline{n}|} = (1+i)^n (I\ddot{a})_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{d} = \frac{s_{\overline{n+1}|} - (n+1)}{d}$$

Note that $(Is)_{\overline{n}|}$ can be expressed as $s_{\overline{1}|} + s_{\overline{2}|} + \cdots + s_{\overline{n}|}$ as shown in the figure below:

```
                                        1
                                   1    1
                              1    1    1
                              :    :    :
                              :    :    :
           1    1             1    1    1
           1    1             1    1    1
      1    1    1             1    1    1
|----|----|----|--...--|----|----|
0    1    2    3        n-2   n-1  n
```

So we have the following relationship:

$$(Is)_{\overline{n}|} = \sum_{t=1}^{n} s_{\overline{t}|} = \sum_{t=1}^{n} \frac{(1+i)^t - 1}{i} = \frac{\ddot{s}_{\overline{n}|} - n}{i}$$

Example 19.

Find the PV and AV of a 15-year increasing annuity immediate with payments of $1, 2, 3, \ldots, 15$ at 7.5% effective. $\boxed{58.94,\ 174.4}$

Example 20.

A perpetuity costs 144.5 and makes annual payments at the end of the year. The perpetuity pays 1 at the end of year 2, 2 at the end of year $3, \ldots, n$ at the end of year $(n+1)$. After year $(n+1)$, the payments remain constant at n. The annual effective interest rate is 7.1%. Calculate n.

4.2.2 The Decreasing Annuity

When $P = n$ and $Q = -1$, i.e., payments of $n, (n-1), \ldots, 1$ in year 1 to n, then we have **decreasing annuity** and its PV has symbol $(Da)_{\overline{n}|}$.

$$\begin{aligned}(Da)_{\overline{n}|} &= na_{\overline{n}|} - \frac{a_{\overline{n}|} - nv^n}{i} \\ &= \frac{ina_{\overline{n}|} - a_{\overline{n}|} + nv^n}{i} \\ &= \frac{n(1-v^n) - a_{\overline{n}|} + nv^n}{i} \\ &= \frac{n - a_{\overline{n}|}}{i}\end{aligned}$$

Similarly $(Da)_{\overline{n}|}$ can be expressed as the sum of $a_{\overline{t}|}$'s as shown in the figure below:

```
      1
      1     1
      1     1      1
      :     :      :
      1     1      1      1
      1     1      1      1     1
      1     1      1      1     1     1
|----|----|-----|-...-|----|----|
0     1     2      3     n-2   n-1    n
```

So,

$$(Da)_{\overline{n}|} = \sum_{t=1}^{n} a_{\overline{t}|} = \sum_{t=1}^{n} \left(\frac{1 - v^t}{i} \right) = \frac{n - a_{\overline{n}|}}{i}$$

Example 21.

Find the PV and AV of a 12-year decreasing annuity-immediate with payments of $12, 11, \ldots, 1$ at 4.25% effective.

Example 22 (T04Q07).

An-annuity-immediate pays 20 at the end of years 1 and 2, 19 at the ends of years 3 and 4, etc.., with payments decreasing by 1 every second year, until nothing is paid. The effective annual rate of interest is 7%. Calculate the present value of this annuity-immediate.

4.2.3 Combination of Increasing (or Decreasing) Annuity and Level annuity

The PV of any annuity in arithmetic progression $(P, P+Q, P+2Q, \ldots)$ can be express as a combination of the increasing annuity $(Ia)_{\overline{n|}}$ and a level annuity $a_{\overline{n|}}$. The steps are as follows:

1. The common difference in payment is Q, so immediately we write $Q(Ia)_{\overline{n|}}$.
2. The increasing annuity in step 1 is $Q, 2Q, 3Q, \ldots,$, but we need $P, P+Q, P+2Q, \ldots,$. So we need to add level payments of $P-Q$,, i.e., $(P-Q)a_{\overline{n|}}$.
3. So, $PV = Q(Ia)_{\overline{n|}} + (P-Q)a_{\overline{n|}}$
4. As $AV = (1+i)^n$, then
   $AV = Q(Is)_{\overline{n|}} + (P-Q)s_{\overline{n|}}$

Example 23.
Find the PV of the annuity-immediate with payments of $8, 13, 18, 23, 28$.

Example 24.
Find the AV of a 12-year annuity-immediate with annual payments of $1225, 2575, 3925, 5275, \ldots$

The PV of any decreasing annuity in arithmetic progression $(P, P-Q, P-2Q, \ldots, P-(n-1)Q)$ can be express as a combination of the decreasing annuity $(Da)_{\overline{n}|}$ and a level annuity $a_{\overline{n}|}$. The steps are as follows:

1. The common difference in payment is Q, so immediately we write $Q(Da)_{\overline{n}|}$.
2. The decreasing annuity in step 1 in reversing order is $nQ, (n-1)Q, (n-2)Q, \ldots, Q$, but we need $P, P-Q, P-2Q, \ldots,$. So we need to add level payments of $P - nQ$,, i.e., $(P - nQ)a_{\overline{n}|}$.
3. So, $PV = Q(Da)_{\overline{n}|} + (P - nQ)a_{\overline{n}|}$.

Example 25.
Find the PV of an annuity-due with 17 annual payments of $100, 96, 92, \ldots, 36$.

4.2.4 Increasing Perpetuity

To calculate varying perpetuities, we take the limit of the perpetuity-immediate:

$$\lim_{n\to\infty} PV = P\lim_{n\to\infty} a_{\overline{n}|} + Q\lim_{n\to\infty}\frac{a_{\overline{n}|} - nv^n}{i} = \frac{P}{i} + \frac{Q}{i^2}$$

since $\lim_{n\to\infty} a_{\overline{n}|} = \frac{1}{i}$ and $\lim_{n\to\infty} nv^n = 0$.
For Increasing perpetuity-immediate, $P = Q = 1$, we have

$$(Ia)_{\overline{\infty}|} = \frac{1}{i} + \frac{1}{i^2}$$

The PV of an increasing perpetuity-due is

$$(I\ddot{a})_{\overline{\infty}|} = \lim_{n\to\infty}(I\ddot{a})_{\overline{n}|} = \lim_{n\to\infty}\frac{\ddot{a}_{\overline{n}|} - nv^n}{d} = \frac{1}{d^2}$$

Example 26.
Find the present value of a perpetuity-immediate whose successive payments are 1,2,3,4,$\cdots$, at an effective rate of interest of 5%. $\boxed{420}$

Example 27.
The PV at effective rate i of a perpetuity-immediate with a first payment of 3 and with subsequent payments increase by 2 each year is 406.81. Determine i. $\boxed{.0739}$

Example 28.
Sandy purchases a perpetuity-immediate that makes annual payments. The first payment is 100, and each payment thereafter increases by 10. Danny purchases a perpetuity-due which makes annual payments of 180. Using the same effective interest rate, $i > 0$, the present value of both perpetuity are equal. calculate i. $\boxed{0.1017195}$

Example 29 (T04Q08)**.**

Bob purchases an increasing perpetuity with payments occuring at the end of every 2 years. The first payment is 1, the second one is 2, the third one is 3, etc. The price of the perpetuity is 200. Calculate the annual effective interest rate.

4.2.5 Increasing and Then Level Perpetuities

Consider a perpetuity-immediate with payments of $1, 2, 3, \ldots, n$ at the end of years 1 to n, and then with payments of n at the end of each year. One way to determine the PV of this perpetuity is as combination of an $n-$year increasing annuity followed by an $n-$-year deferred perpetuity-immediate with level payments of n:

$$PV = (Ia)_{\overline{n|}} + v^n\left(\frac{n}{i}\right) = \frac{\ddot{a}_{\overline{n|}} - nv^n + nv^n}{i} = \frac{\ddot{a}_{\overline{n|}}}{i}$$

The PV can be expressed as an $n-$year level payment annuity-due with payments of $\frac{1}{i}$.

Example 30.

Determine the PV of a perpetuity-immediate with payments 1, 2, 3, $\ldots, n$ for the first n years and then level payments of $(n+1)$ in years $(n+1)$ and subsequent.

Example 31.

You are given:

- Annuity 1: A 11-year decreasing annuity immediate, with annual payments of 11, 10, 9, ..., 1.
- Annual 2: A perpetuity immediate with annual payments. The perpetuity pays 1 in year 1, 2 in year 2, 3 in year 3, ..., and 12in year 12. After 12, the payments remains constant at 12.

At an annual effective interest rate of i, the present value of Annuity 2 is twice the present value of Annuity 1. Calculate the value of Annuity 1.

4.3 Payments in Geometric Progression

Consider an annuity-immediate with a term of n periods in which the first payments is 1 and successive payments increase in geometric progression with common ratio $1+k$.
The present value is

$$\begin{aligned} PV &= v + v^2(1+k) + \cdots + v^n(1+k)^{n-1} \\ &= v\frac{1-(\frac{1+k}{1+i})^n}{1-(\frac{1+k}{1+i})} \\ &= \frac{1-(\frac{1+k}{1+i})^n}{i-k} \end{aligned}$$

and the accumulated value is

$$\begin{aligned} AV &= (1+i)^{n-1} + (1+i)^{n-2}(1+k) + \cdots \\ &\quad +(1+i)(1+k)^{n-2} + (1+k)^{n-1} \\ &= \frac{(1+i)^n-(1+k)^n}{i-k} \end{aligned}$$

In the case of annuity-due, the present value is

$$\begin{aligned} PV &= 1 + v(1+k) + \cdots + v^{n-1}(1+k)^{n-1} \\ &= 1 + \frac{1+k}{1+i} + \cdots + (\frac{1+k}{1+i})^{n-1} \\ &= 1 + \frac{1}{1+i'} + \cdots + \frac{1}{(1+i')^{n-1}} \\ &= \ddot{a}_{\overline{n}|i'} \end{aligned}$$

where $i' = \frac{i-k}{1+k}$; the accumulated value is

$$AV = \ddot{a}_{\overline{n}|i'}(1+i)^n.$$

When $k < i$, the present value of perpetuity-immediate is

$$PV = \frac{1}{i-k},$$

the present value of perpetuity-due is

$$PV = \frac{1+i}{i-k}.$$

When $k > i$, the infinite geometric progression diverges and the present value of the perpetuity does not exist.

Example 32.

An annuity provides for 17 annual payments. The first payment is 190, paid at the end of first year, and each subsequent payment is 6% more that the one preceding it. Calculate the present value of this annuity if $i = 8\%$.

Example 33.

Chris makes annual deposits into a bank account at the beginning of each year for 20 years. Chris's initial deposit is equal to 100, with each subsequent deposit $k\%$ greater than the previous year's deposit. The bank credits interest at an annual effective rate of 5%. At the end of 20 years, the accumulated amount in Chris's account is equal to 7276.35. Given $k > 5$, calculate k.

Example 34.

You are given a perpetual annuity-immediate with annual payments increasing in geometric progression, with common ratio of 1.07. The annual effective interest rate is 12%. The first payment is 1. Calculate the present value of this annuity. $\boxed{20}$

Example 35 (T04Q09).

Chass deposits 360 per month beginning one month from now. The monthly deposits increases by 9% every two years. At a nominal interest rate of 12% convertible monthly, calculate the accumulated value of the deposits at the end of 26 years.

Example 36.

A perpetuity pays $2X$ one year from today. The annual payments increase by 5% per year thereafter. The effective annual interest rate on this perpetuity is 6%. The present value is 32,400. A second perpetuity pays Y one year from now, and the annual payments increase by X per year thereafter. The effective annual interest rate on this perpetuity is i and the present value is 24,000. A third perpetuity pays Y per year, with the first payment one year from now. The effective annual interest rate on this perpetuity is i and the present value is 4,000. What is Y.

$\boxed{360}$

Example 37.

Justin buys a perpetuity-immediate with varying annual payments. During the first 5 years, the payment is constant and equal 11. Beginning in year 6, then payment start to increase. For year 6 and all future years, the current years's payment is $k\%$ larger than the previous year's payment. At annual effective interest rate of 9.4%, the perpetuity has a present value of 177.54. Calculate k.

4.4 Continuous Varying Annuities

The last type of varying annuity is one in which payments are being made continuous at a varying rate. Such annuities are primarily of theoretical interest.

Consider an increasing annuity for n interest conversion periods in which payments are being made continuously at the rate of t per period at exact moment t. The present value of this annuity, $(\bar{I}\bar{a})_{\overline{n}|}$, is

$$(\bar{I}\bar{a})_{\overline{n}|} = \int_0^n tv^t dt$$

It can be simplified to by using integration by part

$$(\bar{I}\bar{a})_{\overline{n}|} = \left[\frac{tv^t}{\ln v}\right]_0^n - \left[\frac{v^t}{(\ln v)^2}\right]_0^n = \frac{\bar{a}_{\overline{n}|} - nv^n}{\delta}$$

$$\begin{aligned}(\bar{I}\bar{a})_{\overline{n}|} &= \lim_{m\to\infty}(I^{(m)}a)^{(m)}_{\overline{n}|} \\ &= \frac{\lim\limits_{m\to\infty} \ddot{a}^{(m)}_{\overline{n}|} - \lim\limits_{m\to\infty} nv^n}{\lim\limits_{m\to\infty} i^{(m)}} \\ &= \frac{\bar{a}_{\overline{n}|} - nv^n}{\delta}\end{aligned}$$

In general, if the amount of payment being made at exact moment t is $f(t)dt$ and the interest varies continuously by a force of interest δ_r, then an expression for the present value of an n-period continuous varying annuity would be

$$\int_0^n f(t)e^{-\int_0^t \delta_r dr}dt = \int_0^n f(t)a(t)dt.$$

Example 38 (T04Q10)**.**

You are given:

- The force of interest at time t is $0.8t^3$.
- R is the present value of of a 3 year continuously increasing annuity which has a rate of payment of $0.3t^3$ at time t.

Calculate R.

Example 39.

Payments are made to an account at a continuous rate of $(8k + tk)$, where $0 \leq t \leq 20$. Interest is credited at a force of interest $\delta_t = \frac{1}{8+t}$. After 20 years, the account is worth 95,609. Calculate k.