**MEME16203Linear Models Marking Guide**

**Assignment 1**

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| Faculty: | FES | Unit Code: | MEME16203 |
| Course: | MAC | Unit Title: | Linear Models |
| Year: | 1,2 | Lecturer: | Dr Yong Chin Khian |
| Session: | May 2022 | | |
| Due by: | 18/6/2022 | | |

Q1. Let $\mathbf{A}$ be an $n \times n$ symmetric matrix with rank $(\mathbf{A}) = r$. Here $r$ may be smaller than $n$. Let

$$\mathbf{A} = \mathbf{L}\begin{bmatrix} \mathbf{\Delta}_r & 0 \\ 0 & 0 \end{bmatrix}\mathbf{L^T}$$

represent the spectral decomposition of $A$. Then, $\mathbf{\Delta}_r$ is an $r \times r$ diagonal matrix containing the positive eigenvalues of $\mathbf{A}$, and $\mathbf{L}$ is an $n \times n$ orthogonal matrix where the columns are eignenvectors of $\mathbf{A}$. Show that

$$\mathbf{G} = \mathbf{L}\begin{bmatrix} \mathbf{\Delta}_r^{-1} & 0 \\ 0 & 0 \end{bmatrix}\mathbf{L^T}$$

satisfies the definition of the Moore-Penrose inverse of $\mathbf{A}$. (20 marks)

*Ans.*

Since $\mathbf{A}$ is an $n \times n$ symmetric matrix with $rank(\mathbf{A}) = r$, we can use the spectral decomposition to write $\mathbf{A}$ as

$$\underset{n\times n}{\mathbf{A}} = \underset{n\times n}{\mathbf{L}}\begin{bmatrix} \Delta_r & \underset{r\times(n-r)}{\mathbf{0}} \\ \underset{(n-r)\times r}{\mathbf{0}} & \underset{(n-r)\times(n-r)}{\mathbf{0}} \end{bmatrix}\underset{n\times n}{\mathbf{L^T}}$$

$$= \begin{bmatrix} \underset{n\times r}{\mathbf{L_1}} & \underset{p\times(n-r)}{\mathbf{L_2}} \end{bmatrix}\begin{bmatrix} \Delta_r & \underset{r\times(n-r)}{\mathbf{0}} \\ \underset{(n-r)\times r}{\mathbf{0}} & \underset{(n-r)\times(n-r)}{\mathbf{0}} \end{bmatrix}\begin{bmatrix} \underset{r\times n}{\mathbf{L_1^T}} \\ \underset{(n-r)\times n}{\mathbf{L_2^T}} \end{bmatrix}$$

$$= \underset{n\times r}{\mathbf{L_1}}\Delta_r\underset{r\times n}{\mathbf{L_1^T}}$$

Note that

$$\underset{n\times n}{\mathbf{G}} = \underset{n\times n}{\mathbf{L}}\begin{bmatrix} \Delta_r^{-1} & \underset{r\times(n-r)}{\mathbf{0}} \\ \underset{(n-r)\times r}{\mathbf{0}} & \underset{(n-r)\times(n-r)}{\mathbf{0}} \end{bmatrix}\underset{n\times n}{\mathbf{L^T}}$$

$$= \begin{bmatrix} \underset{n\times r}{\mathbf{L_1}} & \underset{p\times(n-r)}{\mathbf{L_2}} \end{bmatrix}\begin{bmatrix} \Delta_r^{-1} & \underset{r\times(n-r)}{\mathbf{0}} \\ \underset{(n-r)\times r}{\mathbf{0}} & \underset{(n-r)\times(n-r)}{\mathbf{0}} \end{bmatrix}\begin{bmatrix} \underset{n\times r}{\mathbf{L_1}} \\ \underset{n\times(n-r)}{\mathbf{L_2}} \end{bmatrix}$$

$$= \underset{n\times r}{\mathbf{L_1}}\Delta_r^{-1}\underset{r\times n}{\mathbf{L_1^T}}$$

Now show that the four properties of the Moore-Penrose inverse are satisfied.

- $\mathbf{AGA} = \mathbf{A}$

$$
\begin{aligned}
&\mathbf{AGA}\\
&= \underset{n\times r}{\mathbf{L_1}} \Delta_r \underset{r\times n}{\mathbf{L_1^T}} \underset{n\times r}{\mathbf{L_1}} \Delta_r^{-1} \underset{r\times n}{\mathbf{L_1^T}} \underset{n\times r}{\mathbf{L_1}} \Delta_r \underset{r\times n}{\mathbf{L_1^T}}\\
&= \underset{n\times r}{\mathbf{L_1}} \Delta_r \mathbf{I_r} \Delta_r^{-1} \mathbf{I_r} \Delta_r \underset{r\times n}{\mathbf{L_1^T}} \text{ since } \underset{r\times n}{\mathbf{L_1^T}} \underset{n\times r}{\mathbf{L_1}} = \mathbf{I_r}\\
&= \underset{n\times r}{\mathbf{L_1}} \mathbf{I_r} \Delta_r \underset{r\times n}{\mathbf{L_1^T}} \text{ since } \Delta_r \Delta_r^{-1} = \mathbf{I_r}\\
&= \underset{n\times r}{\mathbf{L_1}} \Delta_r \underset{r\times n}{\mathbf{L_1^T}}\\
&= \mathbf{A}
\end{aligned}
$$

- $\mathbf{GAG} = \mathbf{G}$

$$
\begin{aligned}
&\mathbf{GAG}\\
&= \underset{n\times r}{\mathbf{L_1}} \Delta_r^{-1} \underset{r\times n}{\mathbf{L_1^T}} \underset{n\times r}{\mathbf{L_1}} \Delta_r \underset{r\times n}{\mathbf{L_1^T}} \underset{n\times r}{\mathbf{L_1}} \Delta_r^{-1} \underset{r\times n}{\mathbf{L_1^T}}\\
&= \underset{n\times r}{\mathbf{L_1}} \Delta_r^{-1} \mathbf{I_r} \Delta_r \mathbf{I_r} \Delta_r^{-1} \underset{r\times n}{\mathbf{L_1^T}} \text{ since } \underset{r\times n}{\mathbf{L_1^T}} \underset{n\times r}{\mathbf{L_1}} = \mathbf{I_r}\\
&= \underset{n\times r}{\mathbf{L_1}} \mathbf{I_r} \Delta_r^{-1} \underset{r\times n}{\mathbf{L_1^T}} \text{ since } \Delta_r^{-1} \Delta_r = \mathbf{I_r}\\
&= \underset{n\times r}{\mathbf{L_1}} \Delta_r^{-1} \underset{r\times n}{\mathbf{L_1^T}}\\
&= \mathbf{G}
\end{aligned}
$$

- $\mathbf{AG}$ is symmetric, i.e. $(\mathbf{AG})^T = \mathbf{AG}$

$$
\begin{aligned}
\mathbf{AG} &= \underset{n\times r}{\mathbf{L_1}} \Delta_r \underset{r\times n}{\mathbf{L_1^T}} \underset{n\times r}{\mathbf{L_1}} \Delta_r^{-1} \underset{r\times n}{\mathbf{L_1^T}}\\
&= \underset{n\times r}{\mathbf{L_1}} \Delta_r \mathbf{I_r} \Delta_r^{-1} \underset{r\times n}{\mathbf{L_1^T}} \text{ since } \underset{r\times n}{\mathbf{L_1^T}} \underset{n\times r}{\mathbf{L_1}} = \mathbf{I_r}\\
&= \underset{n\times r}{\mathbf{L_1}} \mathbf{I_r} \underset{r\times n}{\mathbf{L_1^T}} \text{ since } \Delta_r \Delta_r^{-1} = \mathbf{I_r}\\
&= \underset{n\times r}{\mathbf{L_1}} \underset{r\times n}{\mathbf{L_1^T}}
\end{aligned}
$$

$$
(\mathbf{AG})^T = (\underset{n\times r}{\mathbf{L_1}} \underset{r\times n}{\mathbf{L_1^T}})^T = \underset{n\times r}{\mathbf{L_1}} \underset{r\times n}{\mathbf{L_1^T}} = \mathbf{AG}
$$

- $\mathbf{GA}$ is symmetric, i.e. $(\mathbf{GA})^T = \mathbf{GA}$

$$
\begin{aligned}
\mathbf{GA} &= \underset{n\times r}{\mathbf{L_1}} \Delta_r^{-1} \underset{r\times n}{\mathbf{L_1^T}} \underset{n\times r}{\mathbf{L_1}} \Delta_r \underset{r\times n}{\mathbf{L_1^T}}\\
&= \underset{n\times r}{\mathbf{L_1}} \Delta_r^{-1} \mathbf{I_r} \Delta_r \underset{r\times n}{\mathbf{L_1^T}} \text{ since } \underset{r\times n}{\mathbf{L_1^T}} \underset{n\times r}{\mathbf{L_1}} = \mathbf{I_r}\\
&= \underset{n\times r}{\mathbf{L_1}} \mathbf{I_r} \underset{r\times n}{\mathbf{L_1^T}} \text{ since } \Delta_r^{-1} \Delta_r = \mathbf{I_r}\\
&= \underset{n\times r}{\mathbf{L_1}} \underset{r\times n}{\mathbf{L_1^T}}
\end{aligned}
$$

$$
(\mathbf{GA})^T = (\underset{n\times r}{\mathbf{L_1}} \underset{r\times n}{\mathbf{L_1^T}})^T = \underset{n\times r}{\mathbf{L_1}} \underset{r\times n}{\mathbf{L_1^T}} = \mathbf{GA}
$$

Q2. Suppose $\mathbf{X}$ and $\mathbf{W}$ are any two matrices with $n$ rows for which $C(\mathbf{X}) = C(\mathbf{W})$. Show that $\mathbf{P_X} = \mathbf{P_W}$, where $\mathbf{P_X} = \mathbf{X}(\mathbf{X^TX})^{-}\mathbf{X^T}$ and $\mathbf{P_W} = \mathbf{W}(\mathbf{W^TW})^{-}\mathbf{W^T}$.

(20 marks)

*Ans.*

$C(\mathbf{X}) = C(\mathbf{W})$
$\Rightarrow \mathbf{W} = \mathbf{XF}$ for some $\mathbf{F}$ and $\mathbf{X} = \mathbf{WG}$ for some $\mathbf{G}$. Thus,

$$\begin{aligned}\mathbf{P_X P_W} &= \mathbf{P_X W(W^T W)^- W^T}\\ &= \mathbf{P_X XF(W^T W)^- W^T}\\ &= \mathbf{XF(W^T W)^- W^T}\\ &= \mathbf{W(W^T W)^- W^T}\\ &= \mathbf{P_W}\end{aligned}$$

Likewise

$$\begin{aligned}\mathbf{P_W P_X} &= \mathbf{P_W X(X^T X)^- X^T}\\ &= \mathbf{P_W WG(X^T X)^- X^T}\\ &= \mathbf{WG(X^T X)^- X^T}\\ &= \mathbf{X(X^T X)^- X^T}\\ &= \mathbf{P_X}\end{aligned}$$

Now

$$\begin{aligned}(\mathbf{P_X - P_W})^\mathbf{T}(\mathbf{P_X - P_W}) &= \mathbf{P_X P_X - P_X P_W - P_W P_X + P_W P_W}\\ &= \mathbf{P_X - P_W - P_X + P_W}\\ &= \mathbf{0}\end{aligned}$$

Therefore, $\mathbf{P_X - P_W = 0}$ which implies $\mathbf{P_X = P_W}$

Q3. Suppose $\mathbf{X}$ is an $45 \times 8$ matrix. Prove that $\mathcal{C}(\mathbf{X}) = \mathcal{C}(\mathbf{X(X^T X)^- X^T})$. (20 marks)

*Ans.*

Key:
1. $\mathbf{a} \in \mathcal{C}(\mathbf{X}) \iff \mathbf{a} = \mathbf{Xb}$ for some $\mathbf{b}$.
2. $\mathbf{X(X^T X)^- X^T X = X}$ by property of projection matrix

Prove that $\mathcal{C}(\mathbf{X}) = \mathcal{C}(\mathbf{X(X^T X)^- X^T})$:

$$\begin{aligned}\mathbf{a} \in \mathcal{C}(\mathbf{X}) &\iff \mathbf{a} = \mathbf{Xb} \text{ for some } \mathbf{b} \text{ by key 1}\\ &\iff \mathbf{a} = \underbrace{\mathbf{X(X^T X)^- X^T X}}_{\mathbf{X}}\mathbf{b} \text{ for some } \mathbf{b} \text{ by key 2}\\ &\iff \mathbf{a} = \mathbf{X(X^T X)^- X^T} \underbrace{\mathbf{Xb}}_{45\times 1} \text{ treat as } \mathbf{X(X^T X)^- X^T} \text{ product a } 45\times 1 \text{ vector}\\ &\iff \mathbf{a} = \mathbf{X(X^T X)^- X^T k} \text{ for some } \mathbf{k} = \mathbf{Xb}\\ &\implies \mathbf{a} \in \mathcal{C}(\mathbf{X(X^T X)^- X^T}) \text{ by key 1}\end{aligned}$$

So, $\mathcal{C}(\mathbf{X}) \subseteq \mathcal{C}(\mathbf{X(X^T X)^- X^T})$
Then similarly,

$$
\begin{aligned}
\mathbf{g} \in \mathcal{C}(\mathbf{X}(\mathbf{X^TX})^{-}\mathbf{X^T}) &\iff \mathbf{g} = \mathbf{X}(\mathbf{X^TX})^{-}\mathbf{X^T}\mathbf{h} \text{ for some } \mathbf{h} \text{ by key 1} \\
&\iff \mathbf{g} = \underbrace{\mathbf{X}(\mathbf{X^X})^{-}\mathbf{XT}}_{\mathbf{X}(\mathbf{X^TX})^{-}\mathbf{X^T}}\mathbf{h} \text{ for some } \mathbf{h} \\
&\iff \mathbf{g} = \mathbf{X}\underbrace{\mathbf{X}(\mathbf{X^TX})^{-}\mathbf{X^TX}}_{8\times 1} \text{ treat as } \mathbf{X} \text{ product a } 8\times 1 \text{ vector} \\
&\iff \mathbf{g} = \mathbf{Xq} \text{ for some } \mathbf{q} = (\mathbf{X^TX})^{-}\mathbf{Xh} \\
&\implies \mathbf{q} \in \mathcal{C}(\mathbf{X}) \text{ by key 1}
\end{aligned}
$$

So, $\mathcal{C}(\mathbf{X}(\mathbf{X^TX})^{-}\mathbf{X^T}) \subseteq \mathcal{C}(\mathbf{X})$
And hence,
$\mathcal{C}(\mathbf{X}) = \mathcal{C}(\mathbf{X}(\mathbf{X^TX})^{-}\mathbf{X^T})$

Q4. Suppose $\mathbf{Z} = \underset{5\times 1}{\mathbf{1}}$, $\mathbf{G} = 36$, $\mathbf{R} = 49\underset{5\times 5}{\mathbf{I}}$. If $\mathbf{\Sigma} = \mathbf{ZGZ^T} + \mathbf{R}$, find $\mathbf{\Sigma^{-1}}$. (10 marks)

*Ans.*

$$\mathbf{\Sigma} = \mathbf{ZGZ^T} + \mathbf{R} = \underset{5\times 1}{\mathbf{1}}(36)\underset{1\times 5}{\mathbf{1^T}} + 49\underset{5\times 5}{\mathbf{I}} = 36\underset{5\times 1}{\mathbf{1}}\,\underset{1\times 5}{\mathbf{1^T}} + 49\underset{5\times 5}{\mathbf{I}} = 36\underset{5\times 5}{\mathbf{J}} + 49\underset{5\times 5}{\mathbf{I}}$$

$$\mathbf{\Sigma^{-1}} = [49\underset{5\times 5}{\mathbf{I}} + 36\underset{5\times 5}{\mathbf{J}}]^{-1} = \frac{1}{49}\left[\underset{5\times 5}{\mathbf{I}} - \frac{36}{49+5\times 36}\underset{5\times 5}{\mathbf{J}}\right] = \frac{1}{49}\begin{bmatrix} \frac{193}{229} & \frac{-36}{229} & \cdots & \frac{-36}{229} \\ \frac{-36}{229} & \frac{193}{229} & \cdots & \frac{-36}{229} \\ \vdots & \vdots & \ddots & \\ \frac{-36}{229} & \frac{-36}{229} & \cdots & \frac{193}{229} \end{bmatrix}$$

Q5. Show that the matrix $\underset{n\times n}{\mathbf{A}} = \mathbf{I_n} - \frac{1}{n}\mathbf{J_n}$ is singular. (10 marks)

*Ans.*

$$
\begin{aligned}
&\underset{n\times n}{\mathbf{A}}^2 \\
&= [\mathbf{I_n} - \tfrac{1}{n}\mathbf{J_n}][\mathbf{I_n} - \tfrac{1}{n}\mathbf{J_n}] \\
&= \mathbf{I_n}^2 - 2\tfrac{1}{n}\mathbf{I_nJ_n} + \tfrac{1}{n^2}\mathbf{J_n}^2 \\
&= \mathbf{I_n} - \tfrac{2}{n}\mathbf{J_n} + \tfrac{1}{n^2}\mathbf{1_n1_n^T1_n1_n^T} \\
&= \mathbf{I_n} - \tfrac{2}{n}\mathbf{J_n} + \tfrac{1}{n}\mathbf{J_n} \\
&= \mathbf{I_n} - \tfrac{1}{n}\mathbf{J_n} \\
&= \underset{n\times n}{\mathbf{A}}^2.
\end{aligned}
$$

Thus $\underset{n\times n}{\mathbf{A}}$ is idempotent.
Rank($\underset{n\times n}{\mathbf{A}}$) = tr($\underset{n\times n}{\mathbf{A}}$) = tr($\mathbf{I_n}$) - tr($\frac{1}{n}\mathbf{J_n}$) $= n - \frac{1}{n}(n) = n - 1 < n$. Thus $\underset{n\times n}{\mathbf{A}}$ is singular.

Q6. A useful result from linear algebra (that you may use it without proof) is as follows:

$$\text{rank}(\mathbf{UV}) \leq \min[\text{rank}(\mathbf{U}), \text{rank}(\mathbf{V})]$$

for any two matrices $\mathbf{U}$ and $\mathbf{V}$ with dimensions that allow multiplication (number of columns of $\mathbf{U}$ equals the number of rows of $\mathbf{V}$). In words, this result says that the rank of a product of matrices is no greater than the rank of any matrix in the product. Show that for any matrix $\mathbf{X}$, $\text{rank}(\mathbf{X}) = \text{rank}(\mathbf{P_X})$, where $\mathbf{P_X} = \mathbf{X}(\mathbf{X^TX})^{-1}\mathbf{X^T}$. (20 marks)

*Ans.*

We are given that for any two matrices $\mathbf{U}$ and $\mathbf{V}$ that allow for the product matrix $\mathbf{UV}$ ,

$$\text{rank}(\mathbf{UV}) \leq \min[\text{rank}(\mathbf{U}), \text{rank}(\mathbf{V})]$$

This says rank(UV ) is no larger than the smaller of the two quantities rank($\mathbf{U}$) and rank($\mathbf{V}$),which implies

$$\text{rank}(\mathbf{UV}) \leq \text{rank}(\mathbf{U}) \text{ and } \text{rank}(\mathbf{UV}) \leq \text{rank}(\mathbf{V}).$$

Let $\mathbf{X}$ be any matrix, then

$$\begin{aligned}
\text{rank}(\mathbf{X}) &= \text{rank}(\mathbf{P_X X})\text{since } \mathbf{P_X X} = \mathbf{X} \\
&\leq \min[\text{rank}(\mathbf{P_X}), \text{rank}(\mathbf{X})] \\
&\leq \text{rank}(\mathbf{P_X}) \\
&= \text{rank}(\mathbf{X}(\mathbf{X^T X})^{-}\mathbf{X^T}) \\
&\leq \min[\text{rank}(\mathbf{X}), \text{rank}((\mathbf{X^T X})^{-}\mathbf{X^T})] \\
&\leq \text{rank}(\mathbf{X}).
\end{aligned}$$

Inequality in both directions implies equality; therefore,

$$\text{rank}(\mathbf{X}) = \text{rank}(\mathbf{P_X}).$$