Random Variables and Their Distributions-Review 2

For each of the following distributions,

1. Bernoulli
2. Binomial
3. Hypergeometric
4. Geometeric
5. Negative Binomial
6. Poisson
7. Uniform(Discrete)
8. Uniform(Continuous)
9. Gamma
10. Exponential
11. Weibull
12. Pareto
13. Beta
14. Normal
15. Lognormal
16. Cauchy
17. Two-Parameter Exponential
18. Double Exponential
19. Single Parameter Pareto

(a) State the probability density function.

(b) Give a practical example of its application.

(c) Derive the cumulative distribution function if the close form exists.

(d) Prove the pdf sums (integrates) to one.

(e) Derive the moment generating function.

(f) Derive the mean and variance using moment generating function if it exists, otherwise use the definition. .

1. $\mathbf{X} \sim Bernoulli(p)$

   (a) State the probability density function.

Ans.
$f(x) = p^x q^{1-x}$

(b) Give a practical example of its application.

Ans.
To observe whether a student is aware of a certain political issue or not.

(c) Derive the cumulative distribution function.

Ans.
NA

(d) Prove the pdf sums (integrates) to one.

Ans.
$\sum_x f(x) = p^0 q^1 + p^1 q^0 = p + q = 1$

(e) Derive the moment generating function.

Ans.
$M_X(t) = E(e^{tX}) = e^{t(0)}q + e^t p = pe^t + q$

(f) Derive the mean and variance.

Ans.
$M'_X(t) = pe^t$, $E(X) = M'_X(0) = p$
$M''_X(t) = pe^t$, $E(X^2) = M''_X(0) = p$
$V(X) = E(X^2) - E^2(X) = p - p^2 = p(1-p) = pq$

$E(X) = 0(p^0)(q) + 1(p)(q^0) = p$
$E(X^2) = 0^2(p^0)(q) + 1^2(p)(q^0) = p$
$V(X) = E(X^2) - E^2(X) = p - p^2 = p(1-p) = pq$

2. $\mathbf{X} \sim Binomial(n, p)$

   (a) State the probability density function.

   Ans.
   $f(x) = \binom{n}{x} p^x q^{n-x}, x = 0, 1, \dots, n$

   (b) Give a practical example of its application.

   Ans.
   100 voters were ask to response to their prefrence to democractic or republican.

   (c) Derive the cumulative distribution function.

   Ans.
   NA

   (d) Prove the pdf sums (integrates) to one.

Ans.

$\sum_{x=0}^{n} f(x) = \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x} = (p+q)^n = 1^n = 1$ by using Binomial Theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

(e) Derive the moment generating function.

Ans.

let $Y_i \sim Bernoulli(p)$, Then $X = \sum_{i=1}^{n} Y_i \sim BIN(n.p)$
$M_Y(t) = [M_X(t)]^n = (pe^t + q)^n$

(f) Derive the mean and variance.

Ans.

$E(X) = \sum_{x=0}^{n} x\binom{n}{x} p^x q^{n-x} = \sum_{x=0}^{n} n\binom{n-1}{x-1} p^x q^{n-x} = np\sum_{y=0}^{n-1} \binom{n-1}{y} p^y q^{n-y-1} = np$

$E(X^2) = \sum_{x=0}^{n} x^2\binom{n}{x} p^x q^{n-x} = \sum_{x=0}^{n} xn\binom{n-1}{x-1} p^x q^{n-x} = n\sum_{x=1}^{n}(n-1)\binom{n-2}{x-2} p^x q^{n-x} = n(n-1)p^2 \sum_{y=0}^{n} \binom{n-2}{y} p^y q^{n-y-2} = n(n-1)p$

$Var(X) = n(n-1)p^2 - n^2p^2 = np[(n-1)p - np) = np(1-p) = npq$
$M'_Y(t) = npe^t(pe^t+q)^{n-1}$
$M''_Y(t) = npe^t(pe^t+q)^{n-1} + n(n1-)(pe^t)^2(pe^t+q)^{n-2}$

$E(X^2) = M''_Y(0) = np + n(n-1)p^2 = np(1-p) + (np)^2$
$Var(X) = E(X^2) - (E(X)^2 = np(1-p) + (np)^2 - (np)^2 = np(1-p) = npq$

3. $\mathbf{X} \sim HYP(n, M, N)$

(a) State the probability mass/density function.

Ans.

$$f(x) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}},$$

$$x = 0, 1, \ldots, \min(n, M), n - x \leq N - M.$$

(b) Give a practical example of its application.

Ans.

5 coponents have been selected without replacement and the number of defectives were observed.

(c) Derive the cumulative distribution function.

Ans.

NA

(d) Prove the pdf sums (integrates) to one.

Ans.

$\sum_{x=1}^{\min(n,M)} \binom{M}{x}\binom{N-M}{n-x} = \binom{M}{0}\binom{N-M}{n} + \binom{M}{1}\binom{N-M}{n-1} + \cdots + \binom{M}{\min(n,M)}\binom{N-M}{n-\min(n,M)} = \binom{N}{}$
$\sum_{x=1}^{\min(n,M)} \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$
$=\sum_{x=1}^{\min(n,M)} \frac{\binom{N}{n}}{\binom{N}{n}}$
$= 1$

(e) Derive the moment generating function.

Ans.

The MGF does not exist.

(f) Derive the mean and variance.

Ans.

$E(X)$
$= \sum_{x=1}^{\min(n,M)} x\frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$
$= \sum_{x=1}^{\min(n,M)} \frac{M\binom{M-1}{x-1}\binom{N-M}{n-x}}{\frac{N}{n}\binom{N-1}{n-1}}$
$= \frac{nM}{N}$

$E(X^2)$
$= \sum_{x=1}^{\min(n,M)} x^2\frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$
$= \sum_{x=1}^{\min(n,M)} \frac{xM\binom{M-1}{x-1}\binom{N-M}{n-x}}{\frac{N}{n}\binom{N-1}{n-1}}$
$= \frac{nM}{N}\sum_{y=0}^{\min(n,M)-1} \frac{(y+1)\binom{M-1}{y}\binom{N-M}{n-y-1}}{\binom{N-1}{n-1}}$
$= \frac{nM}{N}\sum_{y=0}^{\min(n,M)-1} \frac{(y)\binom{M-1}{y}\binom{N-M}{n-y-1}+\binom{M-1}{y}\binom{N-M}{n-y-1}}{\binom{N-1}{n-1}}$
$= \frac{nM}{N}\left[\frac{(n-1)(M-1)}{N-1} + 1\right]$

4. $\mathbf{X} \sim GEO(p)$

(a) State the probability mass/density function.

Ans.

$$f(x) = pq^{x-1} \qquad x = 1, 2, 3, \ldots$$

(b) Give a practical example of its application.

Ans.

A test is run until the first success is achieved.

(c) Derive the cumulative distribution function.

Ans.

$$F(x) == p + pq = pq^2 + \cdots + pq^{x-1} = 1 - q^x \qquad x = 1, 2, 3, \ldots$$

(d) Prove the pdf sums (integrates) to one.

Ans.

$\sum_x pq^{x-1} = p + pq + pq^2 + \cdots = \frac{p}{1-q} = \frac{p}{p} = 1$

(e) Derive the moment generating function.

Ans.

$M_X(t) = \sum_x e^{tx} pq^{x-1} = p/q \sum_x (qe^t)^x = \frac{pe^t}{1-qe^t}$

(f) Derive the mean and variance.

Ans.

$\mu = \sum_{x=1}^{\infty} xpq^{x-1} = p \sum_{x=0}^{\infty} \frac{d}{dq} q^x = p \frac{d}{dq} \sum_{x=0}^{\infty} q^x = p \frac{d}{dq} \frac{1}{1-q} = p(1-p)^{-2} = \frac{1}{p},$
$E(X^2) = \sum_{x=1}^{\infty} x^2 pq^{x-1}\ \sigma^2 = \frac{q}{p^2}$

5. $\mathbf{X} \sim NegativeBinomial(r, p)$

(a) State the probability mass/density function.

Ans.

$$f(x) = \binom{x-1}{r-1} p^r q^x, x = r, r+1, \ldots$$

(b) Give a practical example of its application.

Ans.

A test is run until the r successes are achieved.

(c) Derive the cumulative distribution function.

Ans. NA

(d) Prove the pdf sums (integrates) to one.

Ans.

$\sum_{x=r}^{\infty} \binom{x-1}{r-1} p^r q^{x-r} = p^r \sum_{i=0}^{\infty} \binom{i+r-1}{r-1} q^i = p^r (1-q)^{-r} = 1$

(e) Derive the moment generating function.

Ans.

$M_X(t) = \sum_{x=r}^{\infty} e^{tx} \binom{x-1}{r-1} p^r q^x = p^r \sum_{i=0}^{\infty} \binom{i+r-1}{r-1} (qe^t)^i = \left(\frac{pe^t}{1-qe^t}\right)^r$

(f) Derive the mean and variance.

Ans.

$M_x'(t) = r\left(\frac{pe^t}{1-qe^t}\right)^{r-1} \frac{(1-qe^t)(pe^t)-pe^t(-qe^t)}{(1-qe^t)^2} = \frac{r(pe^t)^r}{(1-qe^t)^{r+1}}$
$E(X) = M_X'(0) = \frac{r(pe^0)^r}{(1-qe^0)^{r+1}} = \frac{rp^r}{p^{r+1}} = \frac{r}{p}$
$M_x''(t) = \frac{r^2(pe^t)^{r-1}(pe^t)(1-qe^t)^{r+1} - r(pe^t)^r(r+1)(1-qe^t)^r(-qe^t)}{(1-qe^t)^{2(r+1)}} = \frac{r(pe^t)^r(1-qe^t)^r[r(1-qe^t)+(r+1)(qe^t)}{(1-qe^t)^{2(r+1)}}$
$E(X^2) = M_x''(0) = \frac{r(p)^r(1-q)^r[r(1-q)+(r+1)(q)}{(1-q)^{2(r+1)}} = \frac{r(r+q)}{p^2}$
$V(X) = \frac{r(r+q)}{p^2} - \frac{r^2}{p^2} = \frac{rq}{p^2}$

$E(X)$
$= \sum_{x=r}^{\infty} x \binom{x-r}{r-1} p^r q^{x-r}$
Let $x = i + r$

$= \sum_{i=0}^{\infty} (i+r) \binom{i+r-1}{r-1} p^r q^i$
$= \sum_{i=0}^{\infty} \frac{(i+r)(i+r-1)!}{(r-1)!(i!)} p^r q^i$
$= \sum_{i=0}^{\infty} \frac{r(i+r)!}{(r)!(i!)} p^r q^i$
$= \sum_{i=0}^{\infty} r \binom{i+r}{i} p^r q^i$
$\mu = \frac{r}{p}, \sigma^2 = \frac{rq}{p^2}$

6. $\mathbf{X} \sim POI(\mu)$

(a) State the probability mass/density function.

Ans.

$$f(x) = \frac{e^{-\mu}\mu^x}{x!} \qquad x = 0, 1, 2, \ldots$$

(b) Give a practical example of its application.

Ans.

Number claims of motor vehicle insurance is follows a Poisson distribution.

(c) Derive the cumulative distribution function.

Ans.

NA

(d) Prove the pdf sums (integrates) to one.

Ans. $\sum_x \frac{e^{-\mu}\mu^x}{x!} = e^{-\mu} \sum_x \frac{\mu^x}{x!} = e^{-\mu} e^{\mu} = 1$

(e) Derive the moment generating function.

Ans.

$M_X(t) = E[e^{tX}] = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\mu}\mu^x}{x!} = \sum_{x=0}^{\infty} \frac{e^{-\mu}(\mu e^t)^x}{x!} = e^{-\mu} e^{\mu e^t} = e^{\mu(e^t-1)}$

(f) Derive the mean and variance.

Ans.

$E(X) = \sum_{x=0}^{\infty} x \frac{e^{-\mu}\mu^x}{x!} = \mu \sum_{x=1}^{\infty} \frac{e^{-\mu}\mu^{x-1}}{(x-1)!} = \mu$
$E[X(X-1)] = \sum_{x=0}^{\infty} x(x-1) \frac{e^{-\mu}\mu^x}{x!} = \mu^2 \sum_{x=2}^{\infty} \frac{e^{-\mu}\mu^{x-2}}{(x-2)!} = \mu^2$
$E(X^2) = E[X(X-1)] + E(X) = \mu^2 + \mu$
$V(X) = E(X^2) - E^2(X) = \mu^2 + \mu - \mu^2 = \mu$

7. $\mathbf{X} \sim DU(N)$

(a) State the probability mass/density function.

Ans.

$$f(x) = \frac{1}{N}, X = 1, 2, \ldots, N$$

(b) Give a practical example of its application.

Ans.

Rolling an unbiased dice.

(c) Derive the moment generating function.

Ans.
$M_X(t) = \sum_{x=1}^{N} e^{tx} \frac{1}{N} = \frac{1}{N} \frac{e^t - e^{(N+1)t}}{1-e^t}$

(d) Derive the cumulative distribution function.

Ans.
$F(x) = \sum_{i}^{x} \frac{1}{N} = \frac{x(1+x)}{2N}$

(e) Prove the pdf sums (integrates) to one.

Ans.
$\sum_{x=1}^{N} \frac{1}{N} = \frac{N}{N} = 1$

(f) Derive the mean and variance.

Ans.
$\mu = \sum_{x=1}^{N} \frac{x}{N} = \frac{N(N+1)}{2N} = \frac{N+1}{2}$,
$E(X^2) = \sum_{x=1}^{N} \frac{x^2}{N} = \frac{N(N+1)(2N)+1}{6N} = \frac{(N+1)(2N+1)}{6}$
$\sigma^2 = \frac{(N+1)(2N+1)}{6} - \frac{(N+1)^2}{4} = \frac{(N+1)(4N+2-3N-3}{12} = \frac{(N+1)(N-1)}{12} = \frac{N^2-1}{12}$

8. $\mathbf{X} \sim U(a, b)$

(a) State the probability mass/density function.

Ans.

$$f(x) = \frac{1}{b-a}, a < x < b$$

and zero otherwise

(b) Give a practical example of its application.

Ans.
The hardness of a certain alloy is unifromly distributed between 50 to 75.

(c) Derive the cumulative distribution function.

Ans.

$$F(x) = \begin{cases} 0, & x \leq a \\ \frac{x-a}{b-a}, & a < x < b \\ 1, & b \leq x \end{cases}$$

(d) Prove the pdf sums (integrates) to one.

Ans.
$\int_a^b \frac{1}{b-a} = \frac{b-a}{b-a} = 1$

(e) Derive the moment generating function.

Ans.
$M_X(t) = \int_a^b e^{tx} \frac{1}{b-1} = \frac{1}{b-1} [e^{tx}]_a^b = \frac{e^{tb} - e^{ta}}{b-a}$

(f) Derive the mean and variance.

Ans.
$\mu = \sum_a^b \frac{x}{b-a} = \frac{1}{b-a} [x^2/2]_a^b = \frac{b^2-a^2}{2(b-a)} = \frac{a+b}{2}$,
$E(X^2) = \sum_a^b \frac{x^2}{b-a} = \frac{1}{b-a} [x^3/3]_a^b = \frac{b^3-a^3}{2(b-a)} = \frac{b^2+ab+a^2}{2}$,
$\sigma^2 = \frac{b^2-ab-a^2}{3} - \frac{(a+b)^2}{4} = \frac{4b^2+4ab+4a^2-3b^2-6ab-3a^2}{12} = \frac{b^2-2ab+a^2}{12} = \frac{(b-a)^2}{12}$

9. $\mathbf{X} \sim Gamma(\alpha, \theta)$

(a) State the probability mass/density function.

Ans.

$$f(x) = \frac{1}{\theta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\theta}, x > 0$$

(b) Give a practical example of its application.

Ans.
The amount of claim for fire insurance follow a gamma distribution.

(c) Derive the cumulative distribution function for for integral values of α.

Ans.

$$F(x) = P(S_\alpha \leq x = P(N \geq \alpha = 1 - \sum_{i=0}^{\alpha-1} \frac{(x/\theta)^i}{i!} e^{x/\theta}$$

where $S_\alpha \sim Gamma(\alpha, \theta)$ and $N \sim POI(x\theta)$

(d) Prove the pdf sums (integrates) to one.

Ans.
$\int_0^\infty \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-x/\theta}$
Let $t = \frac{x}{\theta}$, $dt = \frac{1}{\theta} dx$
$\int_0^\infty \frac{1}{\Gamma(\alpha)\theta^\alpha} (\theta t)^{\alpha-1} e^{-t} \theta dt$
$\frac{1}{\Gamma(\alpha)\theta^\alpha} \theta^\alpha \int_0^\infty t^{\alpha-1} e^{-t} dt = \frac{1}{\Gamma(\alpha)\theta^\alpha} \theta^\alpha \Gamma(\alpha) = 1$

(e) Derive the moment generating function.

Ans.

$$\begin{aligned} M_X(t) &= \int_0^\infty \frac{1}{\Gamma(\alpha)\theta^\alpha} e^{tx} x^{\alpha-1} e^{-x/\theta} dx \\ &= \int_0^\infty \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-x(1/\theta - t)} dx \\ &= \frac{1}{\Gamma(\alpha)\theta^\alpha} \Gamma(\alpha) (\frac{\theta}{1-\theta t})^\alpha \\ &= (\frac{1}{1-\theta t})^\alpha \end{aligned}$$

(f) Derive the mean and variance.

Ans.
$E(X) = \int_0^\infty x \frac{1}{\theta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\theta} dx = \frac{1}{\theta^\alpha \Gamma(\alpha)} \int_0^\infty x^\alpha e^{-x/\theta} = \frac{1}{\theta^\alpha \Gamma(\alpha)} \Gamma(\alpha+1) \theta^{\alpha+1} = \alpha\theta$
$E(X^2) = \int_0^\infty x^2 \frac{1}{\theta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\theta} dx = \frac{1}{\theta^\alpha \Gamma(\alpha)} \int_0^\infty x^{\alpha+1} e^{-x/\theta} = \frac{1}{\theta^\alpha \Gamma(\alpha)} \Gamma(\alpha+2) \theta^{\alpha+2} = \alpha(\alpha+1)\theta^2$
$V(X) = \alpha(\alpha+1)\theta^2 - \alpha^2\theta^2 = \alpha\theta^2$

10. $\mathbf{X} \sim EXP(\theta)$

(a) State the probability mass/density function.

Ans.

$$f(x) = \frac{1}{\theta} e^{-x/\theta}, x > 0$$

and zero otherwise.

(b) Give a practical example of its application.

Ans.

The life times of a light bulb.

(c) Derive the cumulative distribution function.

Ans.

$$F(x) = \int_0^x \frac{1}{\theta} e^{-t/\theta} dt = [-e^{-t/\theta}]_0^x = 1 - e^{-x/\theta}, x > 0$$

$$F(x) = 1 - e^{-x/\theta}, x > 0$$

(d) Prove the pdf sums (integrates) to one.

Ans.

$\int_0^x \frac{1}{\theta} e^{-t/\theta} dt = [-e^{-t/\theta}]_0^\infty = 1$

(e) Derive the moment generating function.

Ans.

$M_X(t) = \int_0^\infty \frac{1}{\theta} e^{tx} e^{-x/\theta} dx = \int_0^\infty \frac{1}{\theta} e^{-x(1/\theta - t} dx = \frac{1}{\theta}\left(\frac{\theta}{1-\theta t}\right) = \left(\frac{1}{1-\theta t}\right)$

(f) Derive the mean and variance.

Ans.

$\mu = \int_0^\infty \frac{1}{\theta} x e^{-x/\theta} = \frac{1}{\theta}\Gamma(2)\theta^2 = \theta,$

$E(X^2) = \int_0^\infty \frac{1}{\theta} x^2 e^{-x/\theta} dx = \frac{1}{\theta}\Gamma(3)\theta^3 = 2\theta^2,$

$\sigma^2 = 2\theta^2 - \theta^2 = \theta^2$

11. $\mathbf{X} \sim WEI(\tau, \theta)$

(a) State the probability mass/density function.

Ans.

$$f(x) = \frac{\tau}{\theta^\tau} x^{\tau-1} e^{-(x/\theta)^\tau}, x > 0$$

and zero otherwise.

(b) Give a practical example of its application.

Ans.

(c) Derive the cumulative distribution function.

Ans.

$F(x)$

$= \int_0^x \frac{\tau}{\theta^\tau} t^{\tau-1} e^{-(t/\theta)^\tau} dt$

Let $u = t^\tau$, $du = \tau t^{\tau-1} dt$

$= \int_0^{x^\tau} \frac{\tau}{\theta^\tau} u^{1/\tau-1} e^{-u/\theta^\tau} du u^{1-1/\tau}/(\tau)$

$= \int_0^{x^\tau} \frac{1}{\theta^\tau} e^{-u/\theta^\tau} du$

$= 1 - e^{-(x/\theta)^\tau}$

(d) Prove the pdf sums (integrates) to one.

Ans.

$\int_0^\infty \frac{\tau}{\theta^\tau} t^{\tau-1} e^{-(t/\theta)^\tau} dt$

Let $u = t^\tau$, $du = \tau t^{\tau-1} dt$

$= \int_0^\infty \frac{\tau}{\theta^\tau} u^{1/\tau-1} e^{-u/\theta^\tau} du u^{1-1/\tau}/(\tau)$

$= \int_0^\infty \frac{1}{\theta^\tau} e^{-u/\theta^\tau} du$

$= 1$

(e) Derive the moment generating function.

Ans.

The MGF does not exist

(f) Derive the mean and variance.

Ans.

$\mu = \int_0^\infty \frac{\tau}{\theta^\tau} x^\tau e^{-(x/\theta)^\tau} dx$

Let $u = x^\tau$, $du = \tau x^{\tau-1} dx$

$= \int_0^\infty \frac{\tau}{\theta^\tau} u e^{-u/\theta^\tau} du u^{1+1/\tau}/(\tau)$

$= \int_0^\infty \frac{1}{\theta^\tau} u^{2+1/\tau} e^{-u/\theta^\tau} du$

$= \frac{1}{\theta^\tau}\Gamma(1 + \frac{1}{\tau})\theta^{\tau(1+1/\tau)}$

$= \theta\Gamma\left(1 + \frac{1}{\tau}\right),$

$E(X^2) = \int_0^\infty \frac{\tau}{\theta^\tau} x^{\tau+1} e^{-(x/\theta)^\tau} dx$

Let $u = x^\tau$, $du = \tau x^{\tau-1} dx$

$= \int_0^\infty \frac{\tau}{\theta^\tau} u^{1+1/\tau} e^{-u/\theta^\tau} du u^{1+1/\tau}/(\tau)$

$= \int_0^\infty \frac{1}{\theta^\tau} u^{2+2/\tau} e^{-u/\theta^\tau} du$

$= \frac{1}{\theta^\tau}\Gamma(1 + \frac{2}{\tau})\theta^{\tau(1+2/\tau)}$

$= \theta^2\Gamma\left(1 + \frac{2}{\tau}\right),$

$\sigma^2 = \theta^2\Gamma(1 + 2/\tau) - \theta^2[\Gamma(1 + 1/\tau)]^2$

$= \theta^2\left[\Gamma\left(1 + \frac{2}{\tau}\right) - \Gamma^2\left(1 + \frac{1}{\tau}\right)\right]$

12. $\mathbf{X} \sim PAR(\alpha, \theta)$

(a) State the probability mass/density function.

Ans.

$$f(x) = \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}}, x > 0$$

(b) Give a practical example of its application.

Ans.

(c) Derive the cumulative distribution function.

Ans.

$F(x)$
$= \int_0^x \frac{\alpha\theta^\alpha}{(t+\theta)^{\alpha+1}} dt$
Let $u = (t+\theta)$, $du = dt$
$= \int_\theta^{x+\theta} \frac{\alpha\theta^\alpha}{u^{\alpha+1}} du$
$= \alpha\theta^\alpha [\frac{u^{-\alpha}}{-\alpha)}]_\theta^{x+\theta}$
$= \alpha\theta^\alpha [\frac{\theta^{-\alpha}}{\alpha} - \frac{(x+\theta)^{-\alpha}}{\alpha}]$
$= 1 - (\frac{\theta}{x+\theta})^\alpha$

(d) Prove the pdf sums (integrates) to one.

Ans.

$\int_0^\infty \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}} dx$
Let $u = (x+\theta)$, $du = dx$
$= \int_\theta^\infty \frac{\alpha\theta^\alpha}{u^{\alpha+1}} du$
$= \alpha\theta^\alpha [\frac{u^{-\alpha}}{-\alpha)}]_\theta^\infty$
$= \alpha\theta^\alpha [\frac{\theta^{-\alpha}}{\alpha} - 0]$
$= 1$

(e) Derive the moment generating function.

Ans. The MGF does not exist.

(f) Derive the mean and variance.

Ans.

μ
$= \int_0^\infty x \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}} dx$
Let $u = (x+\theta)$, $du = dx$
$= \int_\theta^\infty (u-\theta) \frac{\alpha\theta^\alpha}{u^{\alpha+1}} du$
$= \int_\theta^\infty [u \frac{\alpha\theta^\alpha}{u^{\alpha+1}} - \theta \frac{\alpha\theta^\alpha}{u^{\alpha+1}}] du$
$= \alpha\theta^\alpha [\frac{u^{-\alpha+1}}{-\alpha+1} - \frac{\theta u^{-\alpha}}{-\alpha}]_\theta^\infty$
$= \alpha\theta^\alpha [\frac{\theta^{-\alpha+1}}{\alpha-1} - \frac{\theta(\theta^{-\alpha})}{\alpha}]$
$= \frac{\alpha\theta}{\alpha-1} - \theta$
$= \frac{\theta}{\alpha-1}$,

$E(X^2) = \int_0^\infty x^2 \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}} dx$
Let $u = (x+\theta)$, $du = dx$
$= \int_\theta^\infty (u-\theta)^2 \frac{\alpha\theta^\alpha}{u^{\alpha+1}} du$
$= \int_\theta^\infty [u^2 \frac{\alpha\theta^\alpha}{u^{\alpha+1}} - 2u \frac{\alpha\theta^\alpha}{u^{\alpha+1}} + \theta^2 \frac{\alpha\theta^\alpha}{u^{\alpha+1}}] du$
$= \alpha\theta^\alpha [\frac{u^{-\alpha+2}}{-\alpha+2} - \frac{2u^{-\alpha+1}}{-\alpha+1} + \frac{\theta^2 u^{-\alpha}}{-\alpha}]_\theta^\infty$
$= \alpha\theta^\alpha [\frac{\theta^{-\alpha+2}}{\alpha-2} - \frac{2\theta^{-\alpha+1}}{\alpha-1} + \frac{\theta^2(\theta^{-\alpha})}{\alpha}]$
$= \frac{2\theta^2}{(\alpha-1)(\alpha-2)}$,
$\sigma^2 = \frac{\theta^2}{(\alpha-1)^2(\alpha-2)}$

13. $\mathbf{X} \sim Beta(a, b)$

(a) State the probability mass/density function.

Ans.

$$f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1},$$

for $0 < x < 1$

(b) Give a practical example of its application.

Ans.

The proportion of defective items in a shipment

(c) Derive the cumulative distribution function.

Ans.

(d) Prove the pdf sums (integrates) to one.

Ans.

$\int_0^1 \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1} dx$
$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 x^{a-1}(1-x)^{b-1} dv$
$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} B(a, b)$
$= 1$ since $B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$

(e) Derive the moment generating function.

Ans. The MGF does not exist.

(f) Derive the mean and variance.

Ans.

$E(X^k)$
$= \int_0^1 x^k \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1} dx$
$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 x^{a+k-1}(1-x)^{b-1} dx$
$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(a+k)\Gamma(b)}{\Gamma(a+b+k)}$
$= \frac{\Gamma(a+b)\Gamma(a+k)}{\Gamma(a)\Gamma(a+b+k)}$

$E(X)$
$= \frac{\Gamma(a+b)\Gamma(a+1)}{\Gamma(a)\Gamma(a+b+1)}$
$= \frac{\Gamma(a+b)a\Gamma(a)}{\Gamma(a)(a+b)\Gamma(a+b)}$
$= \frac{a}{a+b}$

$E(X^2)$
$= \frac{\Gamma(a+b)\Gamma(a+2)}{\Gamma(a)\Gamma(a+b+2)}$
$= \frac{\Gamma(a+b)a(a+1)\Gamma(a)}{\Gamma(a)(a+b)(a+b+1)\Gamma(a+b+2)}$
$= \frac{a(a+1)}{(a+b)(a+b+1)}$

$V(X)$
$= \frac{a(a+1)}{(a+b)(a+b+1)} - (\frac{a}{a+b})^2$
$= \frac{a(a+1)(a+b)-a^2(a+b+1)}{(a+b)^2(a+b+1)}$
$= \frac{ab}{(a+b)^2(a+b+1)}$

14. $\mathbf{X} \sim N(\mu, \sigma^2)$

(a) State the probability mass/density function.

Ans.

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2},$$

for $x \in R$, $\mu \in R$ and $\sigma > 0$.

(b) Give a practical example of its application.

Ans.

The score of a subject in a country.

(c) Derive the cumulative distribution function.

Ans.

$F(x)$
$= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} e^{-(u-\mu)^2/2\sigma^2} du$
Let $z = \frac{u-\mu}{\sigma}$, $dz = \frac{1}{\sigma} dx$
$= \int_{-\infty}^{\frac{x-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$
$= \Phi(\frac{x-\mu}{\sigma})$

(d) Prove the pdf sums (integrates) to one.

Ans.

$I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{(x-\mu)^2/2\sigma^2} dx$
Let $z = \frac{x-\mu}{\sigma}$, $dz = \frac{1}{\sigma} dx$
$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-z^2/2} \sigma dz$
$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$
$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$
$= 2\int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$

Let $w = z^2/2$, $z = \sqrt{2w}$, $dt = w^{-1/2}/\sqrt{2} dw$

$I = \int_{0}^{\infty} \frac{w^{-1/2}}{\sqrt{\pi}} e^{-} w dw = \frac{\Gamma(1/2)}{\sqrt{pi}} = 1$

(e) Derive the moment generating function.

Ans.

$M_X(t)$
$= E(e^{tX}$
$= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} dx$
$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{tx - \frac{1}{2\sigma^2}(x^2 - 2\mu x + \mu^2)} dx$
$= e^{-\frac{\mu^2}{2\sigma^2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x^2 - 2(\mu+\sigma^2 t)x)} dx$
$= e^{-\frac{\mu^2}{2\sigma^2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-(\mu+\sigma^2 t))^2 - (\mu+\sigma^2 t)^2)} dx$
$= e^{\frac{\mu^2 + (\mu+\sigma^2 t)^2}{2\sigma^2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-(\mu+\sigma^2 t))^2} dx$
$= e^{\mu t + \sigma^2 t^2/2}$

(f) Derive the mean and variance.

Ans.

$M'_X(t)$
$= (\mu + \sigma^2 t) e^{\mu t + \sigma^2 t^2/2}$
$E(X)$
$= M'_X(0)$
$= \mu$

$M''_X(t)$
$= \sigma^2 e^{\mu t + \sigma^2 t^2/2} + (\mu + \sigma^2 t)^2 e^{\mu t + \sigma^2 t^2/2}$
$E(X^2)$
$= M''_X(0)$
$= \sigma^2 + \mu^2$

$V(X) = \sigma^2 + \mu^2 - \mu^2 = \sigma^2$

15. $\mathbf{X} \sim LN(\mu, \sigma)$

(a) State the probability mass/density function.

Ans.

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-(\ln x - \mu)^2/2\sigma^2},$$

for $x > 0$, $\mu \in R$ and $\sigma > 0$

(b) Give a practical example of its application.

Ans.

Log normal distribution is used to model insurance claim amount.

(c) Derive the cumulative distribution function.

Ans.

$F(x) = \int_{0}^{x} \frac{1}{u\sigma\sqrt{2\pi}} e^{-(\ln x - \mu)^2/2\sigma^2} du$
Let $z = \frac{\ln u - \mu}{\sigma}$, $dz = \frac{1}{u\sigma}$
$F(x) = \int_{-\infty}^{\frac{\ln x - \mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$
$= \Phi\left(\frac{\ln x - \mu}{\sigma}\right)$

(d) Prove the pdf sums (integrates) to one.

Ans.

$\int_{0}^{\infty} \frac{1}{u\sigma\sqrt{2\pi}} e^{-(\ln x - \mu)^2/2\sigma^2} du$
Let $z = \frac{\ln u - \mu}{\sigma}$, $dz = \frac{1}{u\sigma}$
$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$
$= 1$

(e) Derive the moment generating function.

Ans.

(f) Derive the mean and variance.

Ans.

Let $X \sim N(\mu, \sigma^2)$, the $Y = e^X \sim LN(\mu, \sigma)$

$E(Y^k) = E(X^{kX}) = e^{k\mu + \frac{k^2\sigma^2}{2}}$

$E(Y) = e^{\mu + \frac{\sigma^2}{2}}$

$E(Y^2) = e^{2\mu + 2\sigma^2}$

$V(Y) = e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2} = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)$

16. $\mathbf{X} \sim CAU($
$theta, \eta)$

(a) State the probability mass/density function.

Ans.

$f(x) = \frac{1}{\theta\pi\left[1+\left(\frac{x-\eta}{\theta}\right)^2\right]}$

(b) Give a practical example of its application.

Ans.

It is used to model the points of impact of a fixed straight line of particles emitted from a point source.

(c) Derive the cumulative distribution function.

Ans.

$F(x) = \int_{-\infty}^{x} \frac{1}{\theta\pi\left[1+\left(\frac{u-\eta}{\theta}\right)^2\right]} du$

Let $z = \frac{u-\eta}{\theta}$, $dz = \frac{1}{\theta} du$

$= \int_{-\infty}^{\frac{x-\eta}{\theta}} \frac{1}{\theta\pi[1+z^2]} \theta dz$

$= \frac{1}{\pi}[tan^{-1}(z)]_{-\infty}^{\frac{x-\eta}{\theta}}$

$= \frac{1}{\pi}[tan^{-1}(\frac{x-\eta}{\theta}) - (-\pi/2)]$

$= \frac{1}{2} + \frac{1}{\pi} tan^{-1}(\frac{x-\eta}{\theta})$

(d) Prove the pdf sums (integrates) to one.

Ans.

$\int_{-\infty}^{\infty} \frac{1}{\theta\pi\left[1+\left(\frac{x-\eta}{\theta}\right)^2\right]} dx$ Let $z = \frac{x-\eta}{\theta}$, $dz = \frac{1}{\theta} dx$

$= \int_{-\infty}^{\infty} \frac{1}{\theta\pi[1+z^2]} \theta dz$

$= \frac{1}{\pi}[tan^{-1}(z)]_{-\infty}^{\infty}$

$= \frac{1}{\pi}[\pi/2 + \pi/2]$

$= 1$

(e) Derive the mean and variance.

Ans.

Mean and variance do not exists

(f) Derive the moment generating function.

Ans.

MGF does not exists.

17. $\mathbf{X} \sim EXP(\eta, \theta)$

(a) State the probability mass/density function.

Ans.

$$f(x) = \frac{1}{\theta} e^{-\frac{x-\eta}{\theta}} \qquad x > \eta$$

(b) Give a practical example of its application.

Ans.

Two parameter exponential distribution can be used in reliability.

(c) Derive the cumulative distribution function.

Ans.

$F(x)$

$= \int_{\eta}^{x} \frac{1}{\theta} e^{-\frac{u-\eta}{\theta}} du$

$= [-e^{-\frac{u-\eta}{\theta}}]_{\eta}^{x}$

$= 1 - e^{-\frac{x-\eta}{\theta}}$

(d) Prove the pdf sums (integrates) to one.

Ans.

$\int_{\eta}^{\infty} \frac{1}{\theta} e^{-\frac{x-\eta}{\theta}} dx$

$= [-e^{-\frac{x-\eta}{\theta}}]_{\eta}^{\infty}$

$= 1$

(e) Derive the moment generating function.

Ans.

$M_X(t)$

$= E(e^{tX}$

$= \int_{\eta}^{\infty} e^{tx} \frac{1}{\theta} e^{-\frac{x-\eta}{\theta}} dx$

$= \frac{e^{\eta}}{\theta} \int_{\eta}^{\infty} e^{-\frac{(1-\theta t)x}{\theta}} dx$

$= \frac{e^{\eta}}{\theta} \left[\frac{\theta}{1-\theta t} e^{-\frac{(1-\theta t)x}{\theta}}\right]_{\eta}^{\infty}$

$= \frac{e^{\eta}}{\theta} \left[\frac{\theta}{1-\theta t} e^{-\frac{(1-\theta t)\eta}{\theta}}\right]$

$= \frac{e^{\eta t}}{1-\theta t}$

(f) Derive the mean and variance.

Ans.

$M'_X(t) = \frac{(1-\theta t)(\eta e^{\eta t}) - e^{\eta t}(-\theta)}{(1-\theta t)^2} = \frac{e^{\eta t}[(1-\theta t)\eta + \theta]}{(1-\theta t)^2}$

$E(X) = M'_X(0) = \eta + \theta$

$M''_X(t) = \frac{(1-\theta t)^2[\eta e^{\eta t}((1-\theta t)\eta+\theta) + e^{\eta t}(-\theta\eta)] - e^{\eta t}[(1-\theta t)\eta+\theta](2)(1-\theta t)(-\theta))}{(1-\theta t)^4}$

$E(X^2) = M''_X(0) = \eta^2 + 2\eta\theta + 2\theta^2$

$V(X) = \eta^2 + 2\eta\theta + 2\theta^2 - (\eta + \theta)^2 = \theta^2$

18. $\mathbf{X} \sim DE(\eta, \theta)$

(a) State the probability mass/density function.

Ans.

$$f(x) = \frac{1}{2\theta} e^{-|x-\eta|/\theta} \qquad -\infty < x < \infty$$

and zero otherwise.

(b) Give a practical example of its application.

Ans.

Double exponential distribution is used to model exotic options auch as compound option and asian option.

(c) Derive the cumulative distribution function.

Ans.

$F_X(x)$
$= \int_{-\infty}^{x} \frac{1}{2\theta} e^{-|u-\eta|/\theta} du$
$= \begin{cases} \frac{1}{2\theta} \int_{-\infty}^{x} e^{(u+\eta)/\theta} du, & u \leq \eta \\ \frac{1}{2\theta} \int_{\eta}^{x} e^{-(u-\eta)/\theta} du, & u > \eta \end{cases}$
$= \begin{cases} \frac{1}{2\theta} [\theta e^{(u+\eta)/\theta}]_{-\infty}^{x}, & u \leq \eta \\ \frac{1}{2\theta} [-\theta e^{-(u-\eta)/\theta}]_{\eta}^{x}, & u > \eta \end{cases}$
$= \begin{cases} \frac{1}{2} e^{(x+\eta)/\theta}, & u \leq \eta \\ \frac{1}{2} [1 - e^{-(x+\eta)/\theta}], & u > \eta \end{cases}$

(d) Prove the pdf sums (integrates) to one.

Ans.

$\int_{-\infty}^{\infty} \frac{1}{2\theta} e^{-|x-\eta|/\theta} dx$
$= \int_{-\infty}^{\eta} \frac{1}{2\theta} e^{x+\eta/\theta} dx + \int_{\eta}^{\infty} \frac{1}{2\theta} e^{-(x-\eta)/\theta} dx$
$= \frac{1}{2\theta} [\theta e^{(x+\eta)/\theta}]_{-\infty}^{\eta} + \frac{1}{2\theta} [-\theta e^{-(x-\eta)/\theta}]_{\eta}^{\infty}$
$= \frac{1}{2} + \frac{1}{2}$
$=1$

(e) Derive the moment generating function.

Ans.

$M_X(t)$
$= E(e^{tX})$
$= \int_{-\infty}^{\infty} e^{tx} \frac{1}{2\theta} e^{-|x-\eta|/\theta} dx$
$= \frac{1}{2\theta} \int_{-\infty}^{\eta} e^{tx} e^{(-x-\eta)/\theta} dx + \frac{1}{2\theta} \int_{\eta}^{\infty} e^{tx} e^{-(x-\eta)/\theta} dx$
$= \frac{e^{-\eta/\theta}}{2\theta} \int_{-\infty}^{\eta} e^{(1+\theta t)x/\theta} dx + \frac{e^{\eta/\theta}}{2\theta} \int_{\eta}^{\infty} e^{-(1-\theta t)x/\theta} dx$
$= \frac{1}{2\theta} \left[e^{-\eta/\theta} \frac{\theta e^{(1+\theta t)x/\theta}}{1+\theta t} |_{-\infty}^{\eta} - e^{\eta/\theta} \frac{\theta e^{-(1-\theta t)x/\theta}}{1-\theta t} |_{\eta}^{\infty} \right]$
$= \frac{1}{2\theta} [e^{-\eta/\theta} \frac{\theta e^{(1+\theta t)\eta/\theta}}{1+\theta t} + e^{\eta/\theta} \frac{\theta e^{-(1-\theta t)\eta/\theta}}{1-\theta t}]$
$= \frac{1}{2\theta} [\frac{\theta e^{\eta t}}{1+\theta t} + \frac{\theta e^{\eta t}}{1-\theta t}]$
$= \frac{1}{2\theta} [\frac{\theta e^{\eta t} (1-\theta t+1+\theta t}{1-\theta^2 t^2}]$
$= \frac{e^{\eta t}}{1-\theta^2 t^2}$

(f) Derive the mean and variance.

Ans.

$M'_X(t) = \frac{(1-\theta^2t^2)\eta e^{\eta t} - e^{\eta t}(-2\theta^2 t)}{(1-\theta^2t^2)^2} = \frac{e^{\eta t}(\eta - \eta\theta^2t^2 + 2\theta^2 t)}{(1-\theta^2t^2)^2}$
$E(X) = M'_X(0) = \frac{e^{\eta(0)}(\eta - \eta\theta^2(0)^2 + 2\theta^2(0))}{(1-\theta^2(0)^2)^2} = \eta$
$M''_X(t) = \frac{(1-\theta^2t^2)^2[\eta e^{\eta t}(\eta-\eta\theta^2t^2+2\theta^2t)+e^{\eta t}(2\eta\theta^2t+2\theta^2)]+e^{\eta t}(\eta-\eta\theta^2t^2+2\theta^2t)(2(1-\theta^2t^2)(2\theta^2t^2))}{(1-\theta^2t^2)^4}$
$E(X^2) = M''_X(0) = \eta^2 + 2\theta^2$
$V(X) = \eta^2 + 2\theta^2 - \eta^2 = 2\theta^2$

19. $\mathbf{X} \sim$ Single Parameter Pareto (α, θ)

(a) State the probability mass/density function.

Ans.

$$f(x) = \frac{\alpha\theta^\alpha}{x^{\alpha+1}}, x > \theta$$

and zero otherwise.

(b) Give a practical example of its application.

Ans.

(c) Derive the cumulative distribution function.

Ans.

$F_X(x)$
$= \int_{\theta}^{x} \frac{\alpha\theta^\alpha}{u^{\alpha+1}} du$
$= \alpha\theta^\alpha [\frac{u^{-\alpha}}{-\alpha}]_{\theta}^{x}$
$= \alpha\theta^\alpha [\frac{x^{-\alpha} - \theta^{-\alpha}}{-\alpha}]$
$= 1 - [\frac{\theta}{x}]^\alpha$

(d) Prove the pdf sums (integrates) to one.

Ans.

$\int_{\theta}^{\infty} \frac{\alpha\theta^\alpha}{x^{\alpha+1}} dx$
$= \alpha\theta^\alpha [\frac{x^{-\alpha}}{-\alpha}]_{\theta}^{\infty}$
$= \alpha\theta^\alpha [0 + \frac{\theta^{-\alpha}}{-\alpha}]$
$=1$

(e) Derive the moment generating function.

Ans.

NA

(f) Derive the mean and variance.

Ans.

$E(X) = \int_{\theta}^{\infty} \frac{\alpha\theta^\alpha}{x^{\alpha+2}} dx = \alpha\theta^\alpha [\frac{x^{-\alpha+1}}{-\alpha+1}]_{\theta}^{\infty} = \alpha\theta^\alpha [0 + \frac{\theta^{-\alpha+1}}{\alpha-1}] = \frac{\alpha\theta}{\alpha-1}$
$E(X^2) = \int_{\theta}^{\infty} \frac{\alpha\theta^\alpha}{x^{\alpha+3}} dx = \alpha\theta^\alpha [\frac{x^{-\alpha++2}}{-\alpha+2}]_{\theta}^{\infty} = \alpha\theta^\alpha [0 + \frac{\theta^{-\alpha+2}}{\alpha-2}] = \frac{\alpha\theta^2}{\alpha-2}$
$V(X) = \frac{\alpha\theta^2}{\alpha-2} - (\frac{\alpha\theta}{\alpha-1})^2$