MEME15203 Statistical Inference Marking Guide

Assignment 1

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Year:	1,2	Lecturer:	Dr Yong Chin Khian
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Q1. Suppose the joint probability function of X_1 and X_2 is given by

$$p(x_1, x_2) = k, \quad \textbf{for} \quad x_1 = 1, 2, \ldots, 10; x_2 = 1, 2, \ldots, x_1$$

(a) Find k

Ans.

	x_2			
x_1	1	2	$\cdots$	10
1	k			
2	k	k		
$\vdots$	$\vdots$	$\vdots$	$\ddots$	
10	k	k	$\cdots$	k

$[\frac{10(9)}{2} + 10](k) = 1$
$55.0k = 1$
$k = \frac{1}{55.0}$

(b) Find $P(X_1 = 10) + P(X_2 = 7)$.

Ans.
$P(X_1 = 10) + P(X_2 = 7) = k[10 + 4] = \frac{14}{55.0}$

(c) Find the conditional mean of X_2 given $X_1 = 7$, i.e. find $E(X_2|X_1 = 7)$.

Ans.
$E(X_2|X_1 = 7) = \frac{k}{7k} + \frac{2k}{7k} + \cdots + 1 = \frac{1}{7} + \frac{2}{7} + \cdots + 1 = \boxed{4.0}$

(4 marks)

Q2. The joint density function of X_1 and X_2 is given by

$$f(x_1, x_2) = \begin{cases} cx_1^5 x_2^6, & x_1 - 1 \le x_2 \le 1 - x_1, 0 \le x_1 \le 1 \\ 0, & \text{otherwise} \end{cases},$$

(a)Find c.

Ans.
$\int_0^1 \int_{x_1-1}^{1-x_1} f(x_1, x_2)dx_2dx_1 = 1$
$\int_0^1 \int_{x_1-1}^{1-x_1} cx_1^5x_2^6dx_2dx_1 = 1$
$c\int_0^1 x_1^5 \left[\frac{x_2^7}{7}\right]_{x_1-1}^{1-x_1} dx_1 = 1$
$\frac{2c}{7}\int_0^1 x_1^5(1-x_1)^7 = 1$
$\left[\frac{2c}{7}\right]\left[\frac{\Gamma(6)\Gamma(8)}{\Gamma(6+8)}\right] = 1$
$c = 36036.0$

(b)Show that the marginal density of X_1 is a beta density with $a = 6$ and $b = 8$.

Ans.
$f_1(x_1)$
$= \int_{x_1-1}^{1-x_1} 36036x_1^5x_2^6dx_2$
$= 36036x_1^5[\frac{x_2^{6+1}}{6+1}]_{x_1-1}^{1-x_1}$
$= \frac{36036}{7}x_1^5[(1-x_1)^7 + (1-x_1)^7]$
$= \frac{36036}{7}x_1^5[2(1-x_1)^7]$
$= 10,296x_1^5(1-x_1)^7, 0 \leq x_1 \leq 1$
$\Rightarrow X_1 \sim Beta(a = 6, b = 8)$

(c)Derive the conditional density of X_2 given $X_1 = x_1$.

Ans.
$f(x_2|x_1) = kx_2^6, x_1 - 1 \leq x_2 \leq 1 - x_1$
$k\int_{x_1-1}^{1-x_1} x_2^6dx_2 = 1$
$k\left[\frac{x_2^7}{7}\right]_{x_1-1}^{1-x_1} = 1$
$k\left[\frac{2(1-x_1)^7}{7}\right] = 1$
$k = \frac{7}{2(1-x_1)^7}$
$\therefore f(x_2|x_1) = \frac{7x_2^6}{2(1-x_1)^7}, x_1 - 1 \leq x_2 \leq 1 - x_1$

(d)Find $P(X_2 > 0|X_1 = 0.53)$.

Ans.
$f(x_2|x_1 = 0.53)$
$= \frac{7x_2^6}{2(1-0.53)^7}, 0.53 - 1 \leq x_2 \leq 1 - 0.53$
$= 690.85x_2^6, -0.47 \leq x_2 \leq 0.47$

$P(X_2 > -0.17|X_1 = 0.53)$
$= \int_{-0.17}^{0.47} 691x_2^6dx_2$

$= 690.85[\frac{x_2^7}{7}]_{-0.17}^{0.47}$
$= \frac{690.85}{7}(0.47^7 - (-0.17)^7)$
$= 0.5004$

(e)Derive the marginal density of X_2.

Ans.

For $0 < x_2 < 1$
$f_2(x_2)$
$= \int_0^{1-x_2} 36036 x_1^5 x_2^6 dx_1$
$= 36036 x_2^6 [\frac{x_1^6}{6}]_0^{1-x_2}$
$= 6006.0 x_2^6 (1-x_2)^6, -1 < x_2 < 1$

For $-1 < x_2 < 0$
$f_2(x_2)$
$= \int_0^{1+x_2} 36036 x_1^5 x_2^6 dx_1$
$= 36036 x_2^6 [\frac{x_1^6}{6}]_0^{1+x_2}$
$= 6006.0 x_2^6 (1+x_2)^6, 0 < x_2 < 1$

$$f(x_2) = \begin{cases} 6006.0 x_2^6 (1-x_2)^6, & 0 < x_2 < 1 \\ 6006.0 x_2^6 (1+x_2)^6, & -1 < x_2 < 0 \end{cases}$$

(10 marks)

Q3. Given that the nonnegative function $g(x)$ has the property that

$$\int_0^\infty g(x)dx = 1,$$

show that

$$f(x_1, x_2) = \frac{2g(\sqrt{x_1^2 + x_2^2})}{\pi\sqrt{x_1^2 + x_2^2}}, 0 < x_1 < \infty, 0 < x_2 < \infty,$$

zero elsewhere, satisfies the conditions for a pdf of two continuous-type random variables X_1 and X_2. *Hint:* Use polar coordinates

(3 marks)

Ans.
$$\int_0^\infty \int_0^\infty \frac{2g(\sqrt{x_1^2 + x_2^2})}{\pi\sqrt{x_1^2 + x_2^2}} dx_1 dx_2$$
Let $r^2 = x_1^2 + X_2^2$, $x_1 = r sin\theta$, $x_2 = r cos\theta$, then $dx_1 dx_2 = r d\theta dr$

$$= \int_0^\infty \int_0^{\pi/2} \frac{2g(r)}{\pi r} r d\theta dr$$
$$= \int_0^\infty g(r) dr$$
$$= 1$$

Q4. Suppose X and Y are continuous random variables with joint pdf $f(x, y) = cx^3y^3$ if $x > 0, y > 0$, and $x + y < 1$, and zero otherwise, where c is a constant.

(a) Find c.

Ans.

$$\int_0^1 \int_0^{1-y} cx^3y^3 dxdy = 1$$
$$c \int_0^1 y^3 [\tfrac{x^4}{4}]_0^{1-y} dy = 1$$
$$\tfrac{c}{4} \int_0^1 y^3(1-y)^4 dy = 1$$
$$\tfrac{c}{4} \left[\frac{\Gamma(4)\Gamma(5)}{\Gamma(9)}\right] = 1$$
$$c = \frac{4\Gamma(9)}{\Gamma(4)\Gamma(5)}$$

(b) find $V(5X + 8Y)$.

Ans.

$$E(X^rY^s)$$
$$= \int_0^1 \int_0^{1-y} \frac{4\Gamma(9)}{\Gamma(4)\Gamma(5)} x^{r+3}y^{s+3} dxdy$$
$$= \frac{4\Gamma(9)}{\Gamma(4)\Gamma(5)} \int_0^1 y^{s+3} \int_0^{1-y} x^{r+3} dxdy$$
$$= \frac{4\Gamma(9)}{\Gamma(4)\Gamma(5)} \int_0^1 y^{s+3} \left[\frac{x^{r+4}}{r+4}\right]_0^{1-y} dy$$
$$= \frac{4\Gamma(9)}{\Gamma(4)\Gamma(5)(r+4)} \int_0^1 y^{s+3}(1-y)^{r+4} dy$$
$$= \frac{4\Gamma(9)}{\Gamma(4)\Gamma(5)(r+4)} \left[\frac{\Gamma(s+4)\Gamma(r+5)}{\Gamma(r+s+9)}\right]$$
$$E(X) = \frac{4\Gamma(9)}{\Gamma(4)\Gamma(5)(5)} \left[\frac{\Gamma(4)\Gamma(6)}{\Gamma(1+9)}\right] = \frac{4}{9}$$
$$E(Y) = \frac{4\Gamma(9)}{\Gamma(4)\Gamma(5)(4)} \left[\frac{\Gamma(5)\Gamma(5)}{\Gamma(1+9)}\right] = \frac{4}{9}$$
$$E(X^2) = \frac{4\Gamma(9)}{\Gamma(4)\Gamma(5)(6)} \left[\frac{\Gamma(4)\Gamma(7)}{\Gamma(2+9)}\right] = \frac{20}{90}$$
$$E(Y^2) = \frac{4\Gamma(9)}{\Gamma(4)\Gamma(5)(4)} \left[\frac{\Gamma(6)\Gamma(5)}{\Gamma(11)}\right] = \frac{20}{90}$$
$$E(XY) = \frac{4\Gamma(9)}{\Gamma(4)\Gamma(5)(5)} \left[\frac{\Gamma(5)\Gamma(6)}{\Gamma(11)}\right] = \frac{16}{90}$$
$$V(X) = V(Y) = E(X^2) - E^2(X) = \tfrac{20}{90} - \left[\tfrac{4}{9}\right]^2 = 0.0246914$$
$$Cov(X, Y) = E(XY) - E(X)E(Y) = \tfrac{16}{90} - \left[\tfrac{4}{9}\right]^2 = -0.0197531$$
$$V(5X + 8Y)$$
$$= 5^2V(X) + 8^2V(Y) + 2(5)(8)Cov(X, Y)$$
$$= 5^2(0.0246914) + 8^2(0.0246914) + 2(5)(8)(-0.0197531)$$
$$= \boxed{0.61729}$$

(6 marks)

Q5. Let X_1, X_2 be two random variables with joint pdf $f(x_1,x_2) = \frac{1}{8!(50^{10})}x_1^8 e^{-x_2/50}$, for $0 < x_1 < x_2 < \infty$, zero otherwise.

(a) Determine the joint mgf of X_1, X_2, $M_{X_1,X_2}(t_1,t_2)$.

Ans.

$M_{X_1,X_2}(t_1,t_2)$
$= E(e^{t_1X_1+t_2X_2})$
$= \int_0^\infty \int_{x_1}^\infty e^{t_1x_1+t_2x_2}(\frac{1}{8!(50^{10})})x_1^8 e^{-x_2/50}dx_2dx_1$
$= \int_0^\infty (\frac{1}{8!(50^{10})})x_1^8 e^{t_1x_1}\int_{x_1}^\infty e^{-x_2(1/50-t_2)}dx_2dx_1$
$= \int_0^\infty (\frac{1}{8!(50^{10})})x_1^8 e^{t_1x_1}\frac{50e^{-x_1\left(\frac{1-50t_2}{50}\right)}}{1-50t_2}dx_1$
$= (\frac{50}{8!(50^{10})})(\frac{1}{1-50t_2})\int_0^\infty x_1^8 e^{-x_1\left(\frac{1-50t_1-50t_2}{50}\right)}dx_1$
$= (\frac{50}{8!(50^{10})})(\frac{1}{1-50t_2})\frac{8!(50^9)}{(1-50t_1-50t_2)^9}$
$= \frac{1}{(1-50t_2)(1-50t_1-50t_2)^9}$
provided that $50t_1 + 50t_2 < 1$ and $50t_2 < 1$.

(b) Determine the marginal distribution of X_1.

Ans.

$M_{X_1}(t_1,0) = \frac{1}{(1-50(0))(1-50t_1-50(0))^9} = \frac{1}{(1-50t_1)^9}$
$\Rightarrow X_1 \sim GAM(\alpha = 9, \theta = 50)$

(c) Determine the marginal distribution of X_2.

Ans.

$M_{X_2}(0,t_2) = \frac{1}{(1-50t_2)(1-50(0)-50t_2)^9} = \frac{1}{(1-50t_2)^{10}}$
$\Rightarrow X_2 \sim GAM(\alpha = 10, \theta = 50)$

(7 marks)

Q6. Suppose that $X \sim \chi^2(25)$, $S = X + Y \sim \chi^2(60)$, and X and Y are independent. Use MGFs to find the distrbution of $S - X$.

(4 marks)

Ans.

$S - X = X + Y - X = Y$
$M_X(t) = (1-2t)^{-25/2}$,

$M_S(t) = (1-2t)^{-60/2}$
$M_S(t) = M_X(t)M_Y(t)$
$(1-2t)^{-60/2} = (1-2t)^{-25/2}M_Y(t)$
$M_Y(t) = (1-2t)^{-35/2}$
$\Rightarrow Y = S - X \sim \chi^2(35)$

Q7. Consider a random sample of size n from an exponential distribution, $X_i \sim EXP(1)$. Derive the pdf of the sample range, $R = Y_n - Y_1$, where $Y_1 = \min(X_1, \dots, X_n)$ and $Y_n = \max(X_1, \dots, X_n)$.

(8 marks)

Ans.

$f(x) = e^{-x}, x > 0$
$F(x) = 1 - e^{-x}, x > 0$
$f_{Y_1,Y_n}(y_1, y_n)$
$= \frac{n!}{(n-2)!} f(y_1)[F(y_n) - F(y_1)]^{n-2} f(y_n)$
$= \frac{n!}{(n-2)!} e^{-y_1}[e^{-y_1} - e^{-y_n}]^{n-2} e^{-y_n}, y_1 > 0, y_n > 0$
Making the tranformation $R = Y_n - Y_1$, $S = Y_1$, yields the inverse tranformation $y_1 = s$, $y_n = r + s$, and $|J| = 1$. Thus the joint pdf of R and S is
$f_{R,S}(r, s)$
$= f_{Y_1,Y_n}(s, s+t)|J|$
$= \frac{n!}{(n-2)!} e^{-s}[e^{-s} - e^{-(r+s)}]^{n-2} e^{-(r+s)}$
$= \frac{n!}{(n-2)!} e^{-r} e^{-2s}[e^{-s}(1 - e^{-r})]^{n-2}$
$= \frac{n!}{(n-2)!} e^{-r}[1 - e^{-r})]^{n-2} e^{-ns}, r > 0, s > 0$

$$\begin{aligned} f_R(r) &= \tfrac{n!}{(n-2)!} e^{-r}[1 - e^{-r})]^{n-2} \int_0^\infty e^{-ns} ds \\ &= \tfrac{n!}{(n-2)!} e^{-r}[1 - e^{-r})]^{n-2} \tfrac{1}{n} \\ &= (n-1)e^{-r}[1 - e^{-r})]^{n-2} \end{aligned}$$

Q8. Let X_1 and X_2 be a random sample of size 2 from a distribution $N(\theta, 2^2)$, and let

$$U = X_1 + X_2 \quad \text{and} \quad W = X_1 - X_2.$$

(a)Find the joint pdf of U and W.

(b)Find the marginal pdf of U.

(c)Find the marginal pdf of W.

(d)Show that U and W are independent.

(10 marks)

Ans.

(a) This transformation corresponds to $u = x_1 + x_2$ and $w = x_1 - x_2$ which has unique solution $x_1 = \frac{u+w}{2}$ and $x_2 = \frac{u-w}{2}$. The support set for uw-plaine is determined by the inequalities: $-\infty < x_1 < \infty \Rightarrow -\infty < \frac{u+w}{2} < \infty$ and $-\infty < x_2 < \infty \Rightarrow -\infty < \frac{u-w}{2} < \infty$

Thus, $B = [(u,w)| -\infty < u < \infty, -\infty < w < \infty]$

$$J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

$$\begin{aligned} f_{X_1,X_2}(x_1,x_2) &= \frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{2(2^2)}(x_1-\theta)^2} \frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{2(2^2)}(x_2-\theta)^2} \\ &= \frac{1}{2\pi(2^2)} e^{-\frac{1}{2(2^2)}[(x_1-\theta)^2+(x_2-\theta)^2]} \end{aligned}$$

Note that

$$\begin{aligned} (x_1-\theta)^2 + (x_2-\theta)^2 &= [(x_1-\theta)-(x_2-\theta)]^2 + 2(x_1-\theta)(x_2-\theta) \\ &= (x_1-x_2)^2 + 2[x_1x_2 - (x_1+x_2)\theta + \theta^2] \end{aligned}$$

In terms of u, w, $x_1x_2 = \frac{u+w}{2}\frac{u-w}{2} = \frac{1}{4}(u^2-w^2)$

$f_{U,W}(u,w)$

$$= \begin{cases} f_{X_1,X_2}(\frac{u+w}{2}, \frac{u-w}{2})|J|, & (u,w) \in B \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{1}{2\pi(2^2)} e^{-\frac{1}{2(2^2)}[w^2+\frac{1}{2}(u^2-w^2)-2u\theta+2\theta^2]}(\frac{1}{2}), & (u,w) \in B \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{1}{4\pi(2^2)} e^{-\frac{1}{2(2^2)}[\frac{w^2}{2}+\frac{1}{2}(u^2-4u\theta+4\theta^2)]}, & (u,w) \in B \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{1}{4\pi(2^2)} e^{-\frac{1}{2(2^2)}[\frac{w^2}{2}+\frac{1}{2}(u-2\theta)^2]}, & (u,w) \in B \\ 0, & \text{otherwise} \end{cases}$$

(b)
$$\begin{aligned} f_U(u) &= \int_{-\infty}^{\infty} f_{U,W}(u,w)dw \\ &= \int_{-\infty}^{\infty} \frac{1}{4\pi(2^2)} e^{-\frac{1}{2(2^2)}[\frac{w^2}{2}+\frac{1}{2}(u-2\theta)^2]} dw \\ &= \frac{1}{4\pi(2^2)} e^{-\frac{1}{2(2^2)}[\frac{1}{2}(u-2\theta)^2]} \int_{-\infty}^{\infty} e^{-\frac{1}{2(2^2)}[\frac{w^2}{2}]} dw \\ &= \frac{1}{4\pi(2^2)} e^{-\frac{1}{2(2^2)}[\frac{1}{2}(u-2\theta)^2]} \left[\sqrt{2\pi(2(2^2))}\right] \\ &= \frac{\sqrt{2\pi(2(2^2))}\sqrt{2\pi(2(2^2))}}{4\pi(2^2)\sqrt{2\pi(2(2^2))}} e^{-\frac{1}{2[2(2^2)]}[(u-2\theta)^2]} \\ &= \frac{1}{\sqrt{2\pi(2(2^2))}} e^{-\frac{1}{2[2(2^2)]}[(u-2\theta)^2]} \end{aligned}$$

$\Rightarrow U \sim N(2\theta, 2(2^2))$

(c)
$$\begin{aligned} f_W(w) &= \int_{-\infty}^{\infty} f_{U,W}(u,w)du \\ &= \int_{-\infty}^{\infty} \frac{1}{4\pi(2^2)} e^{-\frac{1}{2(2^2)}[\frac{w^2}{2}+\frac{1}{2}(u-2\theta)^2]} du \\ &= \frac{1}{4\pi(2^2)} e^{-\frac{1}{2(2^2)}[\frac{w^2}{2}]} \int_{-\infty}^{\infty} e^{-\frac{1}{2[2(2^2)]}[(u-2\theta)^2]} du \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4\pi(2^2)} e^{-\frac{1}{2(2^2)}\left[\frac{w^2}{2}\right]} \left[\sqrt{2\pi(2(2^2))}\right] \\
&= \frac{\sqrt{2\pi(2(2^2))}\sqrt{2\pi(2(2^2))}}{4\pi(2^2)\sqrt{2\pi(2(2^2))}} e^{-\frac{1}{2[2(2^2)]}w^2]} \\
&= \frac{1}{\sqrt{2\pi(2(2^2))}} e^{-\frac{1}{2[2(2^2)]}[w^2]}
\end{aligned}
$$

$\Rightarrow W \sim N(0, 2(2^2))$

(d) $$f_{U,W}(u,w) = \begin{cases} \frac{1}{\sqrt{2\pi(2(2^2))}} e^{-\frac{1}{2[2(2^2)]}[(u-2\theta)^2]} \frac{1}{\sqrt{2\pi(2(2^2))}} e^{-\frac{1}{2[2(2^2)]}[w^2]}, & (u,w) \in B \\ 0, & \text{otherwise} \end{cases}$$

Since $f_{U,W}(u,w)$ can be factor into $f_U(u)$ and $f_W(w)$ and the support B is a cartisian product, thus U and W are independent.

Q9. Let $X_1, \ldots, X_4$ be a random sample of size 4 from a distribution $N(270, 50^2)$. Let $U = \max(X_1, X_2, \ldots, X_4)$, find the value of the p.d.f. of U at $u = 363.75$.

(3 marks)

Ans.

$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}(x_\mu)^2}, x \in \mathbb{R}$

$$
\begin{aligned}
f_U(u) &= n[F_X(u)]^{n-1} f_X(u) \\
&= n\left[\Phi\left(\frac{u-\mu}{\sigma}\right)\right]^{n-1} \phi\left(\frac{u-\mu}{\sigma}\right)
\end{aligned}
$$

$$
\begin{aligned}
f_U(363.75) &= 4\left[\Phi\left(\frac{363.75-270}{50}\right)\right]^3 \phi\left(\frac{363.75-270}{50}\right) \\
&= 4\left[\Phi(1.88)\right]^3 \phi(1.88) \\
&= 4\left[0.9699\right]^3 \frac{1}{\sqrt{2\pi}} e^{-1.88^2/2} \\
&= 4\left[0.9699\right]^3 (0.0681) \\
&= \boxed{0.2485}
\end{aligned}
$$

Q10. Consider a random sample from a Poisson distribution, $X_i \sim POI(\mu)$. Show that $\bar{X}_n e^{-\bar{X}_n}$ converges in probability to a constant, identify the constant.

(3 marks)

Ans.

$E(\bar{X}_n) = \mu$, $V(\bar{X}_n) = \frac{1}{n}V(X) = \frac{\mu}{n}$

$P\left[|\overline{X_n} - \mu| \geq \epsilon\sqrt{\frac{\mu}{n}}\sqrt{\frac{n}{\mu}}\right] < \frac{\mu}{n\epsilon^2} \to 0$

$\therefore \bar{X}_n \xrightarrow{P} \mu$

By Theorem **??**, $\bar{X}_n e^{\bar{X}_n} \xrightarrow{P} \mu e^{\mu}$

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Q11. Let $X_1, \ldots, X_n$, be a random sample from a uniform distribution, $X \sim U(0, \theta)$, and let $Y_n = X_{n:n}$ the largest order statistic. Find the limiting distribution of $Z_n = n(\theta - Y_n)$.

(3 marks)

Ans. $F_X(x) = \frac{x}{\theta}$
$F_n(y) = P[Y_n \leq y] = [F_X(y)]^n = \left[\frac{y}{\theta}\right]^n$
$F_n(z) = P[Z_n \leq z] = P[Y_n > \theta - z/n] = 1 - \left[\frac{\theta - z/n}{\theta}\right]^n = 1 - \left[1 - \frac{z/\theta}{n}\right]^n$
$\lim_{n\to\infty} F_n(z) = 1 - \lim_{n\to\infty} \left[1 - \frac{z/\theta}{n}\right]^n = 1 - e^{-y/\theta}, y > 0$
$\Rightarrow F(z) \sim EXP(\theta)$

Q12. Consider a random sample from a Exponential distribution, $X_i \sim Exp(\theta)$. Find the asymtotic normal distribution of $Y_n = [\ln(\bar{X}_n)]^4$.

(3 marks)

Ans.

$E(\bar{X}_n) = \theta$, $V(\bar{X}_n) = \frac{1}{n}V(X) = \frac{\theta^2}{n}$
By CLT, $\bar{X}_n \sim N\left(\theta, \frac{\theta^2}{n}\right)$
$g(\theta) = (\ln\theta)^4$, $g'(\theta) = \frac{4}{\theta}(\ln\theta)^3$, $[g'(\theta)]^2 = \frac{16}{\theta^2}(\ln\theta)^6$, thus, by Theorem 11,
$\frac{c^2[g'(m)]^2}{n} = \frac{16\theta^2}{n\theta^2}(\ln\theta)^6 = \frac{16}{n}(\ln\theta)^6$
$Y_n \sim N\left([\ln(\theta)]^4, \frac{16}{n}(\ln\theta)^6\right)$

Q13. Suppose that $W_1, W_2, \ldots$ are iid $Lognormal(\mu, \sigma)$. Let $V_n = W_1 \times W_2 \times \cdots \times W_n$. Both $(V_n)^{1/n}$ and $(V_n)^{1/n^2}$ converge in probability to constants. Identify those constants. .

(3 marks)

Ans.

Suppose $X_1, X_2, \ldots \overset{iid}{\sim} N(\mu, \sigma^2)$, then $W_i = e^{X_i} \overset{iid}{\sim} Lognormal(\mu, \sigma)$.
By weak law of large number, $\bar{X}_n \xrightarrow{P} \mu$ and $\frac{1}{n}\bar{X}_n \xrightarrow{P} 0$.
$(V_n)^{1/n} = (W_1 \times W_2 \times \cdots \times W_n)^{1/n} = e^{\bar{X}_n}$
Thus, $(V_n)^{1/n} \xrightarrow{P} e^{\mu}$ and
$(V_n)^{1/n^2} = (W_1 \times W_2 \times \cdots \times W_n)^{1/n^2} = e^{\frac{1}{n}\bar{X}_n}$
and hence, $(V_n)^{1/n} \xrightarrow{P} e^0 = 1$

Q14. Let the random variable Y_n have a distribution that is $Bin(n, p)$. Prove that $\left(\frac{Y_n}{n}\right)\left(1 - \frac{Y_n}{n}\right)$ converges in probability to a constant, identify the constant.

(3 marks)

Ans.

$E(Y_n/n) = \frac{1}{n}E(Y_n) = p$, $V(Y_n/n) = \frac{1}{n^2}V(Y) = \frac{1}{n^2}np(1-p) = \frac{p(1-p)}{n}$

For all $\epsilon > 0$, $P\left[|Y_n/n - p| \geq \epsilon\sqrt{\frac{n}{p(1-p)}}\sqrt{\frac{p(1-p)}{n}}\right] < \frac{p(1-p)}{n\epsilon^2} \to 0$

$\therefore Y_n/n \xrightarrow{P} p.$

Since $\frac{Y_n}{n} \xrightarrow{P} p$, then by Theorem 5, $\left(\frac{Y_n}{n}\right)\left(1-\frac{Y_n}{n}\right) \xrightarrow{P} p(1-p)$.

Q15. Let $\bar{X}_n$ denote the mean of a random sample of size n from a Poisson distribution with parameter μ. Determine the limiting distribution of $Y_n = \frac{\sqrt{n}(\bar{X}_n-\mu)}{\sqrt{\mu}}$ uusing moment generating function.

(3 marks)

Ans.

$X_i \sim POI(\mu)$, $M_{X_i}(t) = e^{\mu(e^t-1)}$

$M_{\bar{X}_i}(t) = M_{\frac{X_1+\cdots+X_n}{n}}(t) = [M_{X_i}(t/n)]^n = e^{n\mu(e^{t/n}-1)}$

$M_{Y_n}(t) = \exp\left(-\frac{\sqrt{n\mu}}{\sqrt{\mu}}t\right) M_{\bar{X}_n}\left(\frac{\sqrt{n}}{\sqrt{\mu}}t\right)$

$= \exp\left(-\sqrt{n\mu}t\right)\exp\left(n\mu(\exp\left(\frac{\sqrt{n\mu}}{n\mu}t\right)-1\right)$

$= \exp\left(-\sqrt{n\mu}t\right)\exp\left(n\mu\left(\frac{\sqrt{n\mu}}{n\mu}t + (\frac{\sqrt{n\mu}}{n\mu}t)^2/2 + (\frac{\sqrt{n\mu}}{n\mu}t)^3/6 + \cdots\right)\right)$

$= \exp\left(\frac{t^2}{2} + \frac{\mu t^3}{6\sqrt{n}} + \cdots\right)$

$$\begin{aligned}\lim_{n\to\infty} M_{Y_n}(t) &= \lim_{n\to\infty} \exp\left(\frac{t^2}{2} + \frac{\mu t^3}{6\sqrt{n}} + \cdots\right)\\ &= e^{t^2/2}\end{aligned}$$

Q16. Let $Y_n \sim \chi^2(n)$. Find the limiting distribution of $\frac{Y_n-n}{\sqrt{2n}}$ as $n \to \infty$, using moment generating function.

(3 marks)

Ans.

$M_{Y_n}(t) = (1-2t)^{-\frac{n}{2}}$

$$\begin{aligned}M_{\frac{Y_n-n}{\sqrt{2n}}}(t) &= e^{-\frac{n}{\sqrt{2n}}t}M_{Y_n}\left(\frac{1}{\sqrt{2n}}t\right)\\ &= e^{-\frac{n}{\sqrt{2n}}t}\left(1-\frac{2}{\sqrt{2n}}t\right)^{-\frac{n}{2}}\\ &= e^{-\frac{\sqrt{2n}}{2}t}\left(1-\frac{\sqrt{2n}}{n}t\right)^{-\frac{n}{2}}\end{aligned}$$

$$= e^{-\frac{\sqrt{2n}}{2}t} e^{-\frac{n}{2}\ln(1-\frac{\sqrt{2n}}{n}t)}$$
$$= e^{-\frac{\sqrt{2n}}{2}t} e^{-\frac{n}{2}\left[-\frac{\sqrt{2n}}{n}t-\frac{2n}{2n^2}t^2-\frac{(2n)^{3/2}}{3n^3}t^3-\cdots\right]}$$
$$= e^{-\frac{\sqrt{2n}}{2}t+\frac{\sqrt{2n}}{2}t+\frac{t^2}{2}+\frac{(2)^{3/2}}{3n^{1/2}}t^3-\cdots}$$
$$= e^{\frac{t^2}{2}+\frac{(2)^{3/2}}{3n^{1/2}}t^3-\cdots}$$

$$\lim_{n\to\infty} M_{\frac{Y_n-n}{\sqrt{2n}}}(t) = \lim_{n\to\infty} e^{\frac{t^2}{2}+\frac{(2)^{3/2}}{3n^{1/2}}t^3-\cdots} = e^{\frac{t^2}{2}}$$
$$\Rightarrow \frac{Y_n-n}{\sqrt{2n}} \xrightarrow{d} N(0,1)$$

Q17. Suppose that $X_i \sim N(\mu,\sigma^2), i = 1,\ldots,21$ and $Z_i \sim N(0,1), i = 1,\ldots,28$, $W_i \sim \chi^2(11), i = 1,\ldots,11$, $Y_i \sim EXP(130), i = 1,\ldots,7$, and all variables are independent. State the distribution of each of the following variables if it is a "named" distribution or otherwise state "unknown."

(a) $\frac{3X_1+5X_2-8\mu}{\sigma S_Z\sqrt{34}}$

(b) $\frac{11Z_1^2}{W_1}$

(c) $\frac{\sqrt{588}(\bar{X}-\mu)}{\sigma\sqrt{\sum_{i=1}^{28} Z_i^2}}$

(d) $\frac{\sum_{i=1}^{21}(X_i-\mu)^2}{\sigma^2} + \sum_{i=1}^{28}(Z_i-\bar{Z})^2 + \sum_{i=1}^{11} W_i$

(e) $\frac{(27)\sum_{i=1}^{21}(X_i-\bar{X})^2}{(20)\sigma^2\sum_{i=1}^{28}(Z_i-\bar{Z})^2}$

(f) $\frac{2\sigma^2(20)\sum_{i=1}^{7} Y_i}{130\sum_{i=1}^{21}(X_i-\bar{X})^2}$

(12 marks)

Ans.

(a) $3X_1 + 5X \sim N(8, 34\sigma^2)$, $\frac{3X_1+5X_2-8}{\sqrt{34}\sigma} \sim N(0,1)$
$(28-1)S_Z^2 \sim \chi^2(27)$
$\frac{3X_1+5X_2-8}{\sqrt{34}\sigma\sqrt{27S_Z^2/27}} = \frac{3X_1+5X_2-8\mu}{\sigma S_Z\sqrt{34}} \sim T(27)$

(b) $Z_1^2 \sim \chi^2(1)$, $W_1 \sim \chi^2(11)$
$\frac{Z_1^2}{W_1/11} = \frac{11Z_1^2}{W_1} \sim F(1,11)$

(c) $\frac{\sqrt{21}(\bar{X}-\mu)}{\sigma} \sim N(0,1)$
$\sum_{i=1}^{28} Z_i^2 \sim \chi^2(28)$
$\frac{\sqrt{21}(\bar{X}-\mu)}{\sigma\sqrt{\sum_{i=1}^{28} Z_i^2/28}} = \frac{\sqrt{588}(\bar{X}-\mu)}{\sigma\sqrt{\sum_{i=1}^{28} Z_i^2}} \sim T(28)$

(d) $\frac{\sum_{i=1}^{21}(X_i-\mu)^2}{\sigma^2} \sim \chi^2(21)$
$\sum_{i=1}^{28}(Z_i-\bar{Z})^2 \sim \chi^2(27)$

$\sum_{i=1}^{11} W_i \sim \chi^2(11 \times 11)$

$\frac{\sum_{i=1}^{21}(X_i-\mu)^2}{\sigma^2} + \sum_{i=1}^{28}(Z_i - \bar{Z})^2 + \sum_{i=1}^{11} W_i \sim \chi^2(169)$

(e) $\frac{\sum_{i=1}^{21}(X_i-\bar{X})^2}{\sigma^2} \sim \chi^2(20)$

$\sum_{i=1}^{28}(Z_i - \bar{Z})^2 \sim \chi^2(27)$

$\frac{27\sum_{i=1}^{21}(X_i-\bar{X})^2}{20\sigma^2 \sum_{i=1}^{28}(Z_i-\bar{Z})^2} \sim F(20, 27)$

(f) $\frac{2\sum_{i=1}^{7} Y_i}{130} \sim \chi^2(14)$

$\frac{2\sigma^2(20)\sum_{i=1}^{7} Y_i}{130\sum_{i=1}^{21}(X_i-\bar{X})^2} \sim F(14, 20)$

Q18. Suppose $Y \sim Beta(a = 8, b = 6)$, use the relationship between Beta distribution and F distribution, find $P[Y > 0.388]$.

(3 marks)

Ans.

Let $X \sim F_{2(8),2(6)}$ and $c = \frac{8}{6}$, then $Y = \frac{cX}{1+cX} \sim Beta(a = 8, b = 6)$

$$
\begin{aligned}
P[Y > 0.388] &= P\left[\frac{cX}{1+cX} > 0.388\right] \\
&= P[cX > 0.388 + cX(0.388)] \\
&= P[cX(1 - 0.388) > 0.388] \\
&= P[X > \frac{0.388}{c(1-0.388)}] \\
&= P[X > 0.4755] \\
&= 1 - pf(0.4755, 16, 12) \\
&= 1 - 0.0828 \\
&= \boxed{0.9172}
\end{aligned}
$$

Q19. Suppose $Y \sim Beta(a = 4, b = 6)$, use the relationship between Beta distribution and F distribution, find 90^{th} percentile of Y.

(3 marks)

Ans.

Let $X \sim F_{2(4),2(6)}$ and $c = \frac{4}{6}$, then $Y = \frac{cX}{1+cX} \sim Beta(a = 4, b = 6)$

$$
\begin{aligned}
P[Y \leq \pi_{0.9}] &= P\left[\frac{cX}{1+cX} \leq \pi_{0.9}\right] \\
&= P[cX \leq \pi_{0.9} + cX(\pi_{0.9})] \\
&= P[cX(1 - \pi_{0.9}) \leq \pi_{0.9}] \\
&= P\left[X \leq \frac{\pi_{0.9}}{c(1-\pi_{0.9})}\right]
\end{aligned}
$$

Thus,

$\frac{\pi_{0.9}}{c(1-\pi_{0.9})} = F_{8,12,0.9}$

$\pi_{0.9} = \frac{cF_{8,12,0.9}}{1+cF_{8,12,0.9}} = \frac{\frac{4}{6}(2.2446)}{1+\frac{4}{6}(2.2446)} = \boxed{0.5994}$
where $F_{8,12,0.9} = qf(0.9, 8, 12) = 2.2446$

Q20. Suppose that $X_i \sim N(\mu, \sigma^2), i = 1, \ldots, 17$, $Z_j \sim N(0,1), j = 1, \ldots, 28$, and $W_k \sim \chi^2(v), k = 1, \ldots, 16$ and all random variables are independent. State the distribution of each of the following variables if it is a "named" distribution. [For example $X_1 + X_2 \sim N(2\mu, 2\sigma^2)$]

(a) $\frac{27\sum_{i=1}^{17}(X_i-\bar{X})^2}{16\sigma^2\sum_{j=1}^{28}(Z_j-\bar{Z})^2}$.

(b) $\frac{W_1}{\sum_{k=1}^{28} W_k}$

(c) $\frac{\bar{X}}{\sigma^2} + \frac{\sum_{j=1}^{28} Z_j}{28}$

(6 marks)

Ans.

(a) $\frac{\sum_{i=1}^{17}(X_i-\bar{X})^2}{\sigma^2} \sim \chi^2(16)$
$\sum_{j=1}^{28}(Z_j-\bar{Z})^2 \sim \chi^2(27)$
$\frac{\frac{\sum_{i=1}^{17}(X_i-\bar{X})^2}{16\sigma^2}}{\sum_{j=1}^{28}(Z_j-\bar{Z})^2/27} \sim F(16, 27)$
Thus, $\boxed{\frac{27\sum_{i=1}^{17}(X_i-\bar{X})^2}{16\sigma^2\sum_{j=1}^{28}(Z_j-\bar{Z})^2} \sim F(16, 27)}$.

(b) Let $\frac{W_1}{\sum_{k=1}^{28} W_k} = \frac{W_1}{W_1+\sum_{k=2}^{28} W_k}$, then
$W_1 \sim GAM(\frac{v}{2}, 2)$ and $\sum_{k=2}^{28} W_k \sim GAM(\frac{27v}{2}, 2)$.
Thus, $\boxed{\frac{W_1}{\sum_{k=1}^{28} W_k} \sim BETA(\frac{v}{2}, \frac{27v}{2})}$.

(c) $\bar{X} \sim N(\mu, \frac{\sigma^2}{17})$, then $\frac{\bar{X}}{\sigma^2} \sim N(\frac{\mu}{\sigma^2}, \frac{1}{17\sigma^2})$
$\sum_{i=1}^{28} Z_j \sim N(0, 28)$, and $\frac{\sum_{i=1}^{28} Z_i}{28} \sim N(0, \frac{1}{28})$.
Thus $\boxed{\frac{\bar{X}}{\sigma^2} + \frac{\sum_{j=1}^{28} Z_j}{28} \sim N(\frac{\mu}{\sigma^2}, \frac{1}{17\sigma^2} + \frac{1}{28})}$.