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CONTENTS

3 Exotic Options

3.1 Asian Options

A T-year Asian option has a payoff that is based on the **average price** of the underlying stock over the time interval $[0,T]$. There are two average prices:

- Arithmetic average: $A(S) = \frac{1}{n}\sum_{t=1}^{n} S_t$
- Geometric average: $G(S) = \left(\prod_{t=1}^{n} S_t\right)^{1/n}$

The payoff from an ordinary option depends on $S(T)$ and K. An Asian option is formed by replacing either $S(T)$ or K by an average.

- If we replace $S(T)$ by and average, then we have **average price** Asian option.
- If we replace K by and average, then we have **average strike** Asian option

	Payoff	
	Call	Put
Arithmetic Average Price	$[A(S) - K]_+$	$[K - A(S)]_+$
Arithmetic Average Strike	$[S(T) - A(S)]_+$	$[A(S) - S(T)]_+$
Geometric Average Price	$[G(S) - K]_+$	$[K - G(S)]_+$
Geometric Average Strike	$[S(T) - G(S)]_+$	$[G(S) - S(T)]_+$

When pricing Asian Options with binomial trees, the payoff depends on the path of stock prices, thus the binomial tree will not recombined.

Example 1 (T03Q1)**.**

For a stock:

$$S = 48; r = 0.046; \delta = 0$$

An Asian arithmetic average price put with strike price 47 on the stock pays based on 3 monthly stock prices. It is valued using a binomial tree with $u = 1.1$, $d = 0.9$. Calculate the option value. $\boxed{1.8118}$

Example 2 (T03Q2)**.**

For a stock:

$$S = 40; r = 0.05; \delta = 0$$

An Asian geometric average strike call on the stock pays based on 3 monthly stock prices. It is valued using a binomial tree with $u = 1.1$, $d = 0.9$. Calculate the option value. $\boxed{1}.4587$

Example 3 (T03Q3).

For a dollar-denominatedAsian call option on 100 yen:

- The continuously compounded risk-free rate for dollars is 0.059.
- The continuously compounded risk-free rate for yen is 0.014.
- The current exchange rate is ¥100 = \$1.05.
- The option will pay, at the end of three years, the excess of the arithmetic average dollar value of 100 yen at the ends of each of the three years over \$1.12.
- An otherwise similar Asian put option costs \$0.237.

Determine the value of the call option.

Example 4 (T03Q4).

An Asian arithmetic average price call option has a payoff based on the average of stock prices at the ends of each of three months from the date of issue. You are given:

- The underlying stock follows the Black-Scholes framework.
- The stock's volatility is 0.3.
- The continuously compounded risk-free interest is 0.043.
- The stock pays no dividends.
- The initial price of the stock is 74.
- The price of the stock one month after the date of issue is 70.0.
- The price of the stock two months after the date of issue is 72.0.
- The strike price is 69.

Calculate the value of this option at the end of two months.

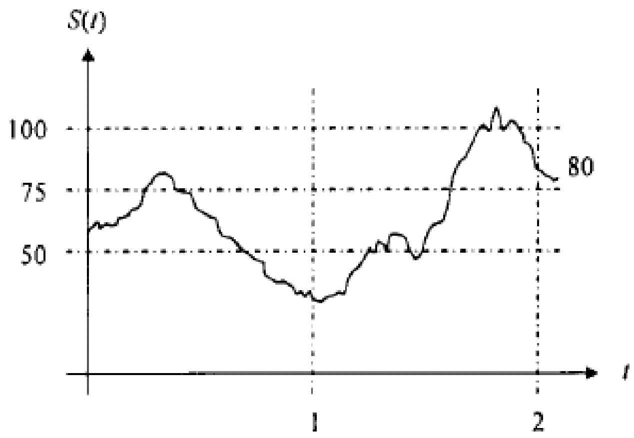
3.2 Barrier Options

A barrier option has a payoff that depends on whether, over the life of the option, $S(t)$ reaches a pre-specified level. It is a path-dependent option. There are three basic kinds of barrier options:

1. **Knock-out options:** Go out of existence if the asset price reach the barrier.
   (a) **down-and-out (DO):** If the price of the underlying asset has to fall to reach the barrier.
   (b) **up-and-out (UO):** If the price of the underlying asset has to rise to reach the barrier.
2. **Knock-in options:** Come into existence if the asset price reach the barrier.
   (a) **Down-and-in (DI):** If the underlying asset price falls to reach the barrier
   (b) **Up-and-in: (UI):** If the underlying asset price rises to reach the barrier

3. **Rebate options:** Make a fixed payment if the asset price reaches the barrier.
   (a) **Down rebates:** If the asset price falls to reach the barrier.
   (b) **Up rebates:** If the asset price rises to reach the barrier

Example 5.



On the basic of the diagram from above, which of the following option(s) would give a zero payoff?

(I) A 2-year 75-strike down-and-in call with barrier 50

(II) A 1-year 75-strike down-and-in call with barrier 50

(III) A 2-year 75-strike up-and-out put with barrier 100

(IV) A 1-year 75-strike up-and-out put with barrier 100

(V) A 1-year 75-strike up-and-in put with barrier 75

Example 6.

On the basic of the diagram from Example 5, calculate the payoff from the following options.

(a) A 2-year 75-strike up-and-in call with barrier 75

(b) A 2-year 100-strike up-and-in call with barrier 75

(c) A 2-year 100-strike down-and-in call with barrier 50

(d) A 2-year 75-strike down-and-in put with barrier 50

3.2.1 Parity Relations for Barrier Options

The important relation for barrier options is

$$\boxed{\text{“Knock-in” option + “knock-out” option = Ordinary option}}$$

When the payoffs of knock-in and knock-out options are added with the same barrier, the effect of the barrier canceled out, and the ramain is a payoff from ordinary option. For example,

$$\boxed{\text{Down-and-in call + Down-and-out call = ordinary call}}$$

Notes:

1. Barrier option price $\leq$ Ordinary option price.
2. When the price of a “knock-out” option is given, the price of a “knock-in” option can be determined.

Example 7 (T03Q5)**.**

You are given that:

- The current stock price is 68.
- The stock pays dividends continuously at a rate that is proportional to its price, the dividend yield is 0.029.
- The stock's volatility is 0.33.
- The continuously compounded risk-free interest rate is 0.078.
- A 1-year 57.12-strike up-and-out call with and up-and-out barrier of 78 has a Black-Scholes price of 13.2224.

Compute the Black-Scholes price of a 1-year 57.12-strike up-and-in call with an up-and-in barrier of 78.

Example 8.

Suppose that S=100, $r = 8\%$, $\sigma = 30\%$, $\delta = 0$, and $T = 1$. By constructing a 2-period binomial forward tree, calculate the price of a 1-year 110-strike down-and-in put barrier option with a down-and-in barrier of 90. $\boxed{11.42}$

Example 9.

Suppose that S=100, $r = 8\%$, $\sigma = 30\%$, $\delta = 0$, and $T = 1$. By constructing a 2-period binomial forward tree, calculate the price of

(a) a 1-year deferred up-rebate option with a barrier of 105 and a rebate of 1;

(b) a 1-year immediate up-rebate option with a barrier of 105 and a rebate of 1

3.3 Compound Options

A **compound option** is an option whose underlying asset is another option which expires later. Suppose the second option has expiration T and strike price K. Then a compound option sold at time t_0 would have an expiration $t_1 < T$ and a strike price x, which means you could buy (call) or sell (put) the second option at time t_1 for price x. Depending on whether the first option is a put or a call and the second option is a put or a call. There are 4 main types of compound options:

1. A **Put-on-Call(POC)** allows selling a call option at time t_1.
2. A **Put-on-Put(POP)** allows selling a put option at time t_1.
3. A **Call-on-Call(COC)** allows buying a call option at time t_1.
4. A **Call-on-Put(COP)** allows buying a put option at time t_1.

We will assume that the option is European style; the option to purchase an option can only be execised at time t_1 and no earlier, and the option purchased at time t_1 can only be exercised at time T and no earlier.
To price a compound option, a formula is developed based on the assumption of lognormal distribution on the underlying stock prices. This requires using a bivariate normal distribution due to two exercises times. Thus we will use the compound option parity only.

For compound options, we will use the notation $(S(0), K, T, x, t_1)$, where

- K is the strike price of the underlying option (the amount paid to purchase the stock)
- x is the strike price of the compound option (the amount paid to purchase the underlying option)
- t_1 is the expiry of the compound option
- T is the expiry of the underlying option

3.3.1 Compound Option Parity

Recall that the put-call parity for option is

Call on an asset -Put on an asset = Prepaid Forward Price of Asset - PV of strike

If the underlying asset is a **call**, then
$COC(S(0), K, T, x, t_1) - POC(S(0), K, T, x, t_1)$
$. \quad . \quad = \mathbf{C}(S(0), K, T)$ - xe^{-rt_1}

If the underlying asset is a **put**, then
$COP(S(0), K, T, x, t_1) - POP(S(0), K, T, x, t_1)$
$. \quad . \quad = \mathbf{P}(S(0), K, T)$ - xe^{-rt_1}

Example 10 (T03Q6)**.**
Assume the Black-Scholes framework for a future index. You are given that:

- The current index value is 1186.
- The index volatility is 45%.
- The continuously compounded risk free rate is 10%.

A 1-year 234-strike put option on a 1233.0-strike put option that matures 1.5 years from now has a price of 59.33. Compute the price of a 1-year 234-strike call option on a 1233.0-strike put option that matures 1.5 years from now.

Example 11.

A 6-month European put-on-call option is modeled using a 1-year 2-period binomial tree for the underlying stock with $u = 1.3$, $d = 0.7$. You are given:

- The price of the underlying stock is 45.
- The continuous annual dividend rate is 0.03.
- The strike price of the underlying call is 50.
- The underlying call expires at time 1.
- The strike price of the put-on-call is 0.55.
- The continuously compounded risk-free interest rate is 0.06.

Determine the premium of the put-on-call option. $\boxed{0.25345}$

Example 12.

For a European call-on-call option:

- The price of the underlying stock is 42.
- $\sigma = 0.2$
- The continuous annual dividend rate is 2%.
- The continuously compounded risk-free interest rate is 6%.
- For the call-on-call option, the premium is 0.85, time to expiry is 3 months, and the strike price is 4.
- For the underlying call option, time to expiry is 6 months and the strike price is 40.
- Options are priced using Black-Scholes formula.

Determine the premium for a European put-on-call with the same underlying asset and strike price. $\boxed{0.6023}$

Example 13 (T03Q7)**.**
Let $x(t)$ be the value of €1 in terms of US dollars at time t. You are given that:

- The continuously compounded risk-free rate in US is 5.6%.
- Under the risk-neutral measure, the stochastic differential equation of x is

  $$dx(t) = 0.023x(t)dt + 0.11d\tilde{Z}(t), \quad x(0) = 0.9$$

  where $\tilde{Z}(t)$ is a standard Brownian motion under the risk-neutral measure.
- A call option that gives the option holder the right to pay \$0.02 six months from today to buy a call option that gives the the right to buy €1 using \$0.95 one year from now is costs \$0.0233.

Calculate the price of a put option that gives the option holder the right to sell at \$0.02 six months from today a call that gives the right to buy €1 using \$0.95 one year from now.

3.4 Exchange Options

An exchange option is an option with a payoff that is dependent on two risky assets.

3.4.1 Payoff Characteristic

Consider a European exchange option that lets the purchaser choose to receive $S(T)$ in return or exchange for K units $Q(T)$ at time T. The payoff from the option is

$$[S(T) - KQ(T)]_+ = \max[0, S(T) - KQ(T)]$$

The corresponding price of the option is denoted by

$$c[S(t), KQ(t), t, T]$$

In other words, this is a call option that exchange K units of $Q(T)$ for one unit of $S(T)$. We call S the underlying asset and Q the strike asset.

For a European exchange option that lets the purchaser choose to give $S(T)$ in return/exchange for K units $Q(T)$ at time T. The payoff from the option is

$$[KQ(T) - S(T)]_+ = \max[0, KQ(T) - S(T)]$$

The corresponding price of the option is denoted by

$$p[S(t), KQ(t), t, T]$$

In other words, this is a put option that exchange one unit of $S(T)$ for K units of $Q(T)$. We call S the underlying asset and Q the strike asset.

If $Q(t) = 1$ for any t, then c and p are just prices of ordinary call and put options.

3.4.2 The generalized put-call-parity for exchange options

Let $x = S(T) - KQ(T)$ in the identity

$$x_+ - (-x_+) = x,$$

we have

$$[S(T) - KQ(T)]_+ - [KQ(T) - S(T)]_+ \\ = S(T) - KQ(T)$$

Thus

$$c[S(t), Q(t), t; T] - p[S(t), Q(t), t; T] \\ = F^P_{t,T}(S) - KF^P_{t,T}(Q).$$

Example 14 (T03Q8)**.**

For two stocks S and Q, you are given:

- The current price of S is 300.0.
- The current price pf Q is 400.
- Both stocks pay dividends continuously at a rate proportional to its price. The dividend yields for S and Q are 3.4% and 5.7% respectively.
- The current price of an option to exchange 5 units of Q for 6 units of S at time 1 is 345.12.

Compute the price of an option to exchange 24 units of S for 20 units of Q at time 1.

Example 15 (T03Q8).

A British company will receive \$1,000,000 at the end of 6 month. To hedge its currency risk, it buys an option allowing to exchange dollars for pounds at a rate of £0.64/\$. You are given:

- The spot exchange rate is £0.65/\$.
- The continuously compounded risk-free interest rate for dollars is 0.06.
- The continuously compounded risk-free interest rate for pounds is 0.04.
- The volatility of the exchange rate between the two currencies is 0.1.
- The Black-Scholes framework is assumed to apply to the currency rate.

Calculate the cost in pounds of the hedge.

3.4.3 Pricing

Let the volatility of S and Q be σ_S and σ_Q, respectively, and ρ be the constant correlation between the continuously compounded returns on S and Q. Then the price of European option to obtain one unit of S in exchange for K units of Q is

$$F_{t,T}^P(S)N(d_1) - KF_{t,T}^P(Q)N(d_2)$$

where

$d_1 = \frac{\ln[F_{t,T}^P(S)/KF_{t,T}^P(Q)]+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$

$d_2 = d_1 - \sigma\sqrt{T-t}$

$\sigma^2 = \sigma_S^2 + \sigma_Q^2 - 2\rho\sigma_S\sigma_Q$

Example 16.

Suppose that for stock S, $S(0) = 21$, $\sigma_S = 0.3$, $\delta_S = 0$, for stock Q, $Q(0) = 42$, $\sigma_Q = 0.53$, $\delta_Q = 0.037$. The correlation between the continuously compounded returns on S and Q is $\rho = 0.53$ and the continuously compounded risk-free interest rate is $r = 0.08$. What is the price of an exchange option with payoff $[5S(1)-4Q(1)]_+$?
$\boxed{5.166}$

Example 17 (T03Q11).

Let $S(t)$ be the time-t price of stock S and $Q(t)$ be the time-t price of stock Q. These prices satistfy the following stochastic differential equation in the risk-neutral measure:

$$\frac{dS(t)}{S(t)} = 0.034dt + 0.16dZ(t)$$

$$\frac{dQ(t)}{Q(t)} = 0.023dt + 0.29dZ'(t)$$

$Z(t)$ and $Z'(t)$ are standard Brownian motions in the risk-neutral measure that satisfy:

$$Z(t) = W_1(t)$$

$$Z'(t) = 0.7W_1t + 0.7W_2t$$

where $W_1(t)$ and $W_2(t)$ are independent standard Brownian motions. You are given:

- $S(0) = 60$ and $Q(0) = 240$
- The continuously compounded risk-free interest rate is 0.06.

A European exchange option allows the purchaser to exchange 4 shares of S for one share of Q at the end of one year. Calculate the value of this option.

Example 18.
Suppose that for stock S, $S(0) = 20$, $\sigma_S = 30\%$, $\delta_S = 0\%$, for stock Q, $Q(0) = 35$, $\sigma_Q = 50\%$, $\delta_Q = 4\%$. The correlation between the continuously compounded returns on S and Q is $\rho = 0.5$ and the continuously compounded risk-free interest rate is $r = 8\%$. What is the price of an exchange option with payoff $[3S(1) - 2Q(1)]_+$?
$\boxed{7.8142}$

3.4.4 Labeling Options: Currency Options

A call option can be viewed as a put option and vice versa. The idea that calls can be relabeled as put is used frequently by currency traders. Let x_t be the value in dollars of one foreign currency (f) at time t: That is, 1f = \$$x_t$. A $K-$strike dollar-denominated call option on a foreign currency that expires at time T gives the owner the right to receive one (f) in exchange of K dollars. This option has a payoff at expiration, in dollars, given by

$$\max(0, x_T - K)$$

In the foreign currency, this payoff at the expiration time T would be

$$\begin{aligned}\frac{1}{x_T}\max(0, x_T - K) &= \max\left(0, 1 - \frac{K}{x_T}\right) \\ &= K\max\left(0, \frac{1}{K}, \frac{1}{x_T}\right)\end{aligned}$$

The payoff at expiration of a 1 $K-$strike foreign-denominated put on dollars is in that foreign currency is

$$\max\left(0, \frac{1}{K}, \frac{1}{x_T}\right)$$

Hence, we have put-call duality,
K(premium in foreign currency of 1 K-strike foreign-denominated put on dollars)
= (premium in foreign currency of $K-$strike dollar-denominated call on foreign currency)
= (premium in dollars of $K-$strike dollar-denominated call on foreign currency)/x_0.

In symbols,

$$\frac{c_{\$}(x_0, K, T)}{x_0} = Kp_f\left(\frac{1}{x_0}, \frac{1}{K}, T\right)$$

or

$$c_{\$}(x_0, K, T) = x_0Kp_f\left(\frac{1}{x_0}, \frac{1}{K}, T\right)$$

similarly,

$$p_{\$}(x_0, K, T) = x_0Kc_f\left(\frac{1}{x_0}, \frac{1}{K}, T\right)$$

Example 19 (T03Q9)**.**

A 1-year European euro-denominated put option for \$100 with strike price of €0.8/\$1 has a premium of €1.5. You are given:

- The continuously compounded risk-free interest rate for dollars is 0.051.
- The continuously compounded risk-free interest rate for euros is 0.025.
- The current exchange rate €0.82/\$1.

Calculate the price of a dollar-denominated put option allowing the sale of €100 for \$125.0.

Example 20 (T03Q10)**.**

Let \$ denote the Australian dollars. Suppose the (spot) exchange rate is £0.4/\$, the pound-denominated continuously compounded interest rate is 0.073,the dollar-denominated continuously compounded interest rate is 0.067, and the price of 1-year £0.38-strike pound-denominated European put on the dollar is £0.0144. What is the price of a dollar-denominated pounds put?□

3.4.5 Application of Maximum and Minimum

We can use the formula for pricing exchange options to price options that depends on the maximum or minimum of $S(T)$ and $Q(T)$. Note that

- $\max(a,b) = a + (b-a)_+ = b + (a-b)_+$
- $\min(a,b) = a - (a-b)_+ = b - (b-a)_+$

Thus, for maximum,

$$\begin{aligned}\max[S(T), KQ(T)] &= KQ(T) + [S(T) - KQ(T)]_+ \\ &= S(T) + [KQ(T) - S(T)]_+\end{aligned}$$

and hence

$$\begin{aligned}&V(S(t), Q(t), t) \\ &= KF^P_{t,T}(Q) + c[S(t), Q(t), t : K, T] \\ &= F^P_{t,T}(S) + p[S(t), Q(t), t : K, T]\end{aligned}$$

To price an option with a payoff of $\max[S(T), K]$, we can set $Q(t) = 1$ for all t (and hence $F^P_{t,T}(Q) = e^{-r(T-t)}$), so that

$$\begin{aligned}V(S(t), 1, t) &= Ke^{-r(T-t)} + c(S(t), t, K, T) \\ &= F^P_{t,T}(S) + p(S(t), t; , K, T)\end{aligned}$$

For minimum,

$$\begin{aligned}\min[S(T), KQ(T)] &= S(T) - [S(T) - KQ(T)]_+ \\ &= KQ(T) - [KQ(T) - S(T)]_+\end{aligned}$$

and hence

$$\begin{aligned}&V(S(t), Q(t), t) \\ &= F^P_{t,T}(S) - c[S(t), Q(t), t : K, T] \\ &= KF^P_{t,T}(Q) - p[S(t), Q(t), t : K, T]\end{aligned}$$

To price an option with a payoff of $\min[S(T), K]$, we can set $Q(t) = 1$ for all t, so that

$$\begin{aligned}V(S(t), 1, t) &= F^P_{t,T}(S) - c(S(t), t; , K, T) \\ &= Ke^{-r(T-t)} - p(S(t), t : K, t)\end{aligned}$$

There is a parity relation between the two options. Since

$$\max(a,b) + \min(a,b) = a + b,$$

we have

time-t price of $\max[S(T), KQ(T)]+$ time-t price of $\min[S(T), KQ(T)]$

$= F^P_{t,T}(S) + KF^P_{t,T}(Q)$

Example 21 (T03Q12).

Consider a model with two stocks. Each stock pays dividends continuously at a rate proportional to its price. $S_j(t)$ denotes the price of one share of stock j at time t. Consider a claim maturing at time 3. The payoff of the claim is $\max[S_1(3), S_2(3)]$. You are given:

- $S_1(0) = 198$
- $S_2(0) = 396$
- Stock 1 pays dividends of amount $0.045S_1(t)dt$ between time t and time $t + dt$.
- Stock 2 pays dividends of amount $0.09S_2(t)dt$ between time t and time $t + dt$.
- The price of a European option to exchange Stock 2 for Stock 1 at time 3 is 17.

Calculate the price of the claim. $\boxed{319.2983}$

Example 22 (T03Q13).

Assume the Black-Scholes framework for a stock whose time-t price is $S(t)$. You are given:

- $S(0) = 75$
- S pays dividends of amount $0.031S_1(t)dt$ between time-t and time $t + dt$.
- $V[\ln S(t)] = 0.1156t$
- The continuously compounded risk-free interest rate is 0.083.

Compute the price of $\min(S(0.2), 80)$ that mature at time 0.2.

Example 23 (T03Q14)**.**

You will have to pay max(£ 10, €15) to someone in one year. You are given

- Both pounds and euros follow the Blank-Scholes framework.
- The volatility of the pounds/euros exchange rate is 0.28.
- The continuously compounded risk-free interest rate for pounds is 0.042.
- The continuously compounded risk-free interest rate for euros is 0.11.
- The current exchange rate is 1.5€/£.

Your domestic currency is pounds. You delta-hedge the payment you will have to make by buying euros. Determine the number of euros you should buy.

3.5 Gap Option

For Gap options, we have two Ks, the payoff characteristics are:

- $$\text{call} = \begin{cases} 0 & \text{when} S(T) \leq K_2 \\ S(T) - K_1 & \text{when} S(T) > K_2 \end{cases}$$
- $$\text{put} = \begin{cases} K_1 - S(T) & \text{when} S(T) \leq K_2 \\ 0 & \text{when} S(T) > K_2 \end{cases}$$

In the above, K_1 is the strike price of the option and K_2 is the payment trigger because it specifies the region where the option would be forced to exercise.

3.5.1 Pricing

Under the Black-Scholes framework, gap option can be priced using binary options. For example, for a gap call,the payoff is

$V(S(T),T)$
$= [S(T) - K_1]|S(T) > K_2)$
$= S(T)I(S(T) > K_2) - K_1 I(S(T) > K_2)$,

which shows that a gap call is simply

Asset-or-nothing call(strike K_2)
- $K_1\times$ Cash-or-nothing call (strike K_2).

As a result, the time-t price of gap call is

$$\boxed{V_c(S(t),t) = S(t)e^{-\delta(T-t)}N(d_1) - K_1 e^{-r(T-t)}N(d_2)},$$

where $d_1 = \frac{\ln[S(t)/K_2]+(r-\delta+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$ and $d_2 = d_1 - \sigma\sqrt{T-t}$.
It is important to know that the two ds are computed using K_2.

Similarly, the time-t price of gap put is

$$\boxed{V_p(S(t),t) = K_1 e^{-r(T-t)}N(-d_2) - S(t)e^{-\delta(T-t)}N(-d_1)}$$

Example 24 (T03Q15)**.**

Let $S(t)$ denote the price at time t of a stock. Consider a 9-month European gap option. If the stock price after 9-month is less than 28, the payoff is $28.5 - S\left(\frac{9}{12}\right)$; otherwise, the payoff is zero. You are given:n

- $S(0) = 30$.
- The stock will pay a dividend of amount 4 after 4-months. This is the only dividend that will be paid before the gap option expires.
- The prepaid forward price of the stock follows a geometric Brownian motion with a volatility of 33%.
- The continuously compounded risk-free rate of interest is 11%.

Calculate the price of the gap option.

3.5.2 Gap Put-Call Parity

For gap calls and puts with same strike, payment trigger and time to expiration, the put-call-parity relation for gap option is:

Time t price of a gap call
- Time-t price of gap put
$= F^P_{t,T}(S) - K_1 e^{-r(T-t)}$.

3.5.3 Relation Between Gap Options and Ordinary Options

For a gap call

$$\begin{aligned}&V_c(S(T),T)\\&= [S(T) - K_2 + (K_2 - K_1)]I(S(T) > K_2)\\&= (S(T) - K_2]I(S(T) > K_2) + (K_2 - K_1)I(S(T) > K_2)\end{aligned}$$

which shows that a gap call is

Ordinary call(Strike-K_2)
$-(K_2 - K_1)\times$ Cash-or-nothing call (Strike K_2)

As a result, the time-t price of the gap call is

$$\boxed{V_c(S(t),t) = c(S(t);K_2) + (K_2 - K_1)e^{-r(T-t)}N(d_2)}$$

For a gap put

$$\begin{aligned}&V_p(S(T),T)\\&= [K_2 - S(T) - (K_2 - K_1)]I(S(T) \le K_2)\\&= (K_2 - S(T)]I(S(T) \le K_2) - (K_2 - K_1)I(S(T) \le K_2)\end{aligned}$$

which shows that a gap call is

Ordinary put(Strike-K_2)
$-(K_2 - K_1)\times$ Cash-or-nothing put (Strike K_2)

As a result, the time-t price of the gap put is

$$\boxed{V_p(S(t),t) = p(S(t);K_2) - (K_2 - K_1)e^{-r(T-t)}N(-d_2)}$$

Example 25 (T03Q16).

Assume the Black-Scholes framework. Let $S(t)$ denote the price at time t of a nondividend-paying stock. You are given:

- $S(0) = 50$.
- $\sigma = 0.22$.
- $r = 0.08$

A market-maker sells a 1-year European gap put option with trigger 44.0 and strike price 56.0. Calculate the number of shares of stock to buy to delta-hedge this option. $\boxed{\text{-0.43047}}$