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CONTENTS

4 Binomial Interest Rates Trees

In this chapter, we will discuss models for interest rates, or for bond prices. The basic objective is to model derivatives on bonds, thus we need a model for bond prices.

Let $P(T, T+s)$ be the time-T price of an s-year zero-coupon bond that matures at time $T+s$.

```
                P(T,T+s)      $1
 -|-----------|-----------|---> time


  0           T           T+s
```

Let $r(T, T+s)$ be the time-T yield to maturity of an s-year zero-coupon bond, then

$$P(T, T+s) = [1 + r(T, T+s)]^{-s}$$

and hence

$$r(T, T+s) = [P(T, T+s)]^{-\frac{1}{s}} - 1$$

for non-continuous compounding.

$$P(T, T+s) = e^{-r(T,T+s)s}$$

and hence

$$r(T, T+s) = -\frac{1}{s}\ln P(T, T+s)$$

for continuous compounding.

Suppose that today is time 0 and we observed zero-coupon bond prices P(0,s) for $s = 1, 2, 3, 4$, we can compute the following:

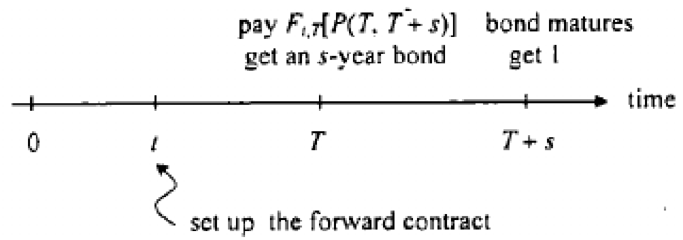
S	$P(0, s)$	$r(0, s)$ (non-continuous)	$r(0, s)$ continuous
1	0.95	5.263%	5.129%
2	0.90	5.409%	5.268%
3	0.85	5.557%	5.417%
4	0.80	5.737%	5.579%

A plot of $r(0, s)$ versus s is called the zero-coupon yield curve.

4.1 Forward Bond Price

If we enter into a forward contract at time t for the time-T delivery of an s-year zero-coupon bond.

1. At time t, we agree to pay $F_{t,T}[P(T, T+s)]$ at time T. Nothing is paid at time t.
2. At time T, we pay $F_{t,T}[P(T, T+s)]$ and get s-year zero-coupon bond.
3. At time $T+s$, the bond matures and we get 1.



The following replicating portfolio can be used to illustrate the formula of forward price:

Transaction set up at time t	Time-t cost	Payoff	
		Time T	Time $T+s$
Short $F_{t,T}[P(T,T+s)]$ units of zero-coupon bonds maturing at $T+s$	$-F_{t,T}[P(T,T+s)] \times P(t,T)$	$-F_{t,T}[P(T,T+s)]$	0
Buy a zero-coupon bond maturing at time $T+s$	$P(t,T+s)$	0	1

The cash flow at time T and $T+s$ match those from the forward contract, and therefore we have replicated the forward contract. The time-t cost of the contract is zero. This gives

$$P(t,T+s) - F_{t,T}[P(T,T+s)]P(t,T) = 0$$

and hence

$$F_{t,T}[P(T,T+s)] = \frac{P(t,T+s)}{P(t,T)}$$

Example 1.

Suppose that today is time 0 and we observe the following zero-coupon bond prices:

s	1	2	3	4
$P(0,s)$	0.95	0.90	0.85	0.80

Calculate the time-0 forward bond price for

(a) a time-1 delivery of a 2-year zero-coupon bond. $\boxed{0.89474}$

(b) a time-2 delivery of a zero-coupon bond that matures at time 3. $\boxed{0.9444}$

Example 2.

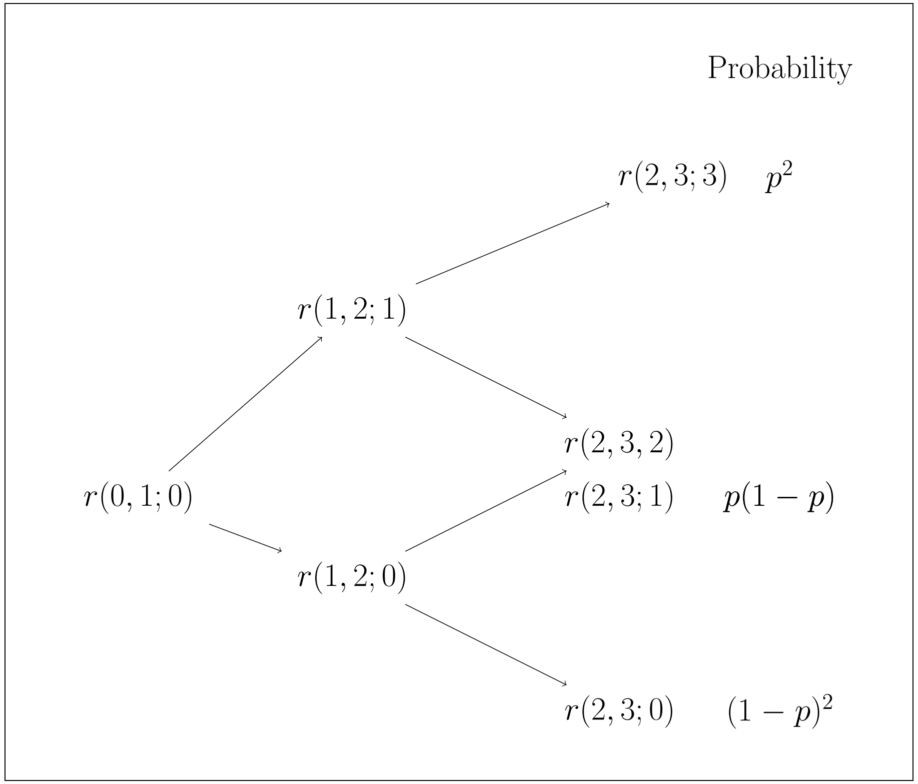
You are given that continuously compounded interest rates for a zero-cuopon bonds are

- 5% for a 2-year bond.
- 6% for a 5-year bond.

Calculate the time-0 forward bond price for a time-2 delivery of a 3-year zero-cuopon bond. $\boxed{0.8187}$

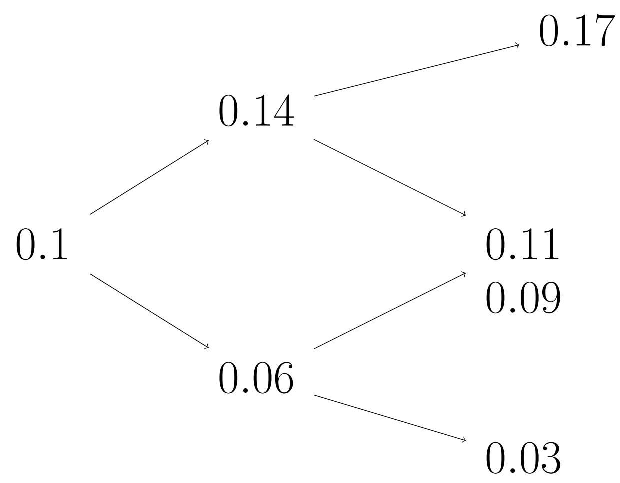
4.2 Binomial Trees

One way to model interest rates is to construct a not-necessarily recombining binomial tree, with each period of the tree being the period for which you are stating interest rate. For example, if we want to use one year interest rates, each period is one year.



The rates $r(t_1, t_2; j)$ are interest rates for a zero-coupon bond issued at time t_1 maturing at time t_2, j is the node counter, counting from the bottom node to the top node. At each node, there is a p risk neutral probability that rate will increase and $1-p$ risk neutral probability that the rate will decrease. Unlike the binomial trees, the risk neutral probabilities are given. They are not calculated from the interest rates.

Example 3.
You are given the continuously compounded interest tree below:



Suppose the risk-neutral probability of an up move is $p = 0.45$.

(a) Find the price of T-year zero-coupon bonds for $T = 1, 2, 3$.
$\boxed{0.904837,\ 0.822662,\ 0.751708}$

(b) Calculate the continuously compounded annualized yields to maturity for the zero-coupon bonds.
$\boxed{10\%,\ 9.7605\%,\ 9.5136\%}$

(c) Find the price of a 0.88-strike two-year put on a 1-year zero-coupon bond. $\boxed{0.00573}$

Example 4 (T4Q1).

Consider the following 3-period binomial interest rate tree where the initial interest (continuously compounded) rate is 12% and rates can move up or down by 2.4% at the end of each year. The risk-neutral probability of an up move is 0.54.

$$
\begin{array}{ccccc}
 & & & & 16.8\% \\
 & & 14.4\% & \nearrow & \\
 & \nearrow & & \searrow & 12.0\% \\
12\% & & & & \\
 & \searrow & & \nearrow & 12.0\% \\
 & & 9.6\% & \searrow & \\
 & & & & 7.2\%
\end{array}
$$

Find the price of a two-year 274.0-strike call on a 1-year zero-coupon bond of face value 300.

Example 5.

The initial one-year continuously compounded spot interest rate is 6%. To model future interest rate, we use a binomial tree with risk-neutral probabilities of 0.5 of an increase and 0.5 of a decrease. Increase and decrease in the continuously compounded interest rates are 2% in each period. Calculate the 1-year forward price of a 2-year bond. $\boxed{0.883}$

Example 6 (T4Q2).

Consider the following 3-period binomial interest rate tree modelling the effective annual yields. The risk-neutral probability of an up move is 0.35.

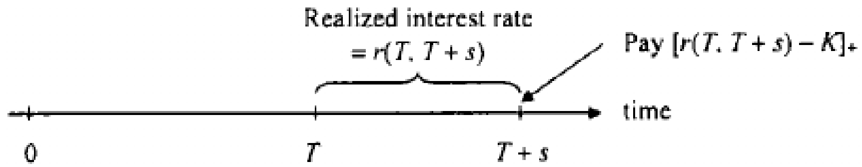
$$\begin{array}{ccccc}
 & & & & 8.64\% \\
 & & 7.2\% & \nearrow & \\
 & \nearrow & & \searrow & 6.0\% \\
6\% & & & & \\
 & \searrow & & \nearrow & 6.0\% \\
 & & 5.0\% & & \\
 & & & \searrow & 4.1667\%
\end{array}$$

Find the yield rate of a three-year 13% annual-coupon bond of face value 100.

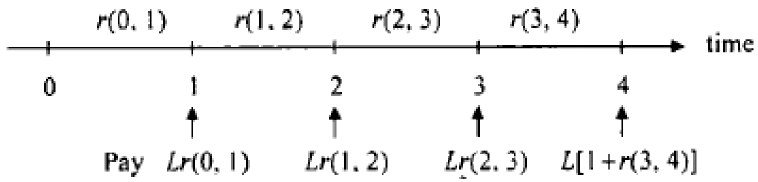
4.3 Interest Rate Caplets and Caps

Apart from ordinary bond calls and puts, there are derivatives that use the interest rate r as the underlying "asset".

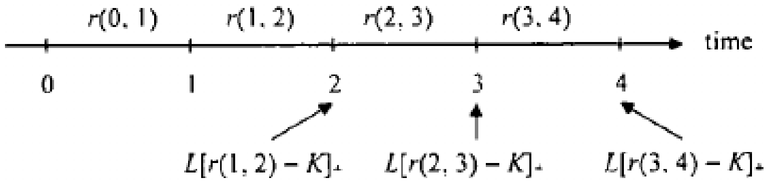
An interest **caplet** pays $[r(T, T+s) - K]_+$ **at time** $T+s$ (not T). In the discussion of caplets and caps, rates are not continuously compounded and no-annualized.



Interest rate caplets arise naturally from **floating rate bond**.



A borrower will be hurt if $r(1,2)$, $r(2,3)$ and $r(3,4)$ are too high. Only the first interest payment is non-random since $r(0,1)$ is already know at time 0. To hedge the interest rate risk, the borrower can purchase three interest rate caplets:



The three caplets above constitute a 4-year interest rate cap with principal L and strike K.

Notes:

- A n year cap pays whenever the prevailing rate is higher than the cap rate in n year period (except at issue). To value a cap, we must include potential payments in all periods.
- A caplet for year k pays only if the prevailing rate in year k is higher than the cap rate. To value a caplet, we only include potential payments for year k.

Example 7 (T4Q3)**.**

You are given the following binomial interest rate tree modeling the annual effective interest rate:

$$r_0 = 8.9\%,\ r_u = 11.163\%,\ r_d = 7.992\%,$$
$$r_{uu} = 13.931\%,\ r_{ud} = r_{du} = 10.687\%,$$
$$r_{dd} = 8.232\%$$

The risk-neutral probability that the annual effective interest rate moves up or down is 0.5. Find the price of a caplet with a guaranteed rate of 10% for a loan of 100 for year 3.

Example 8 (T4Q4).

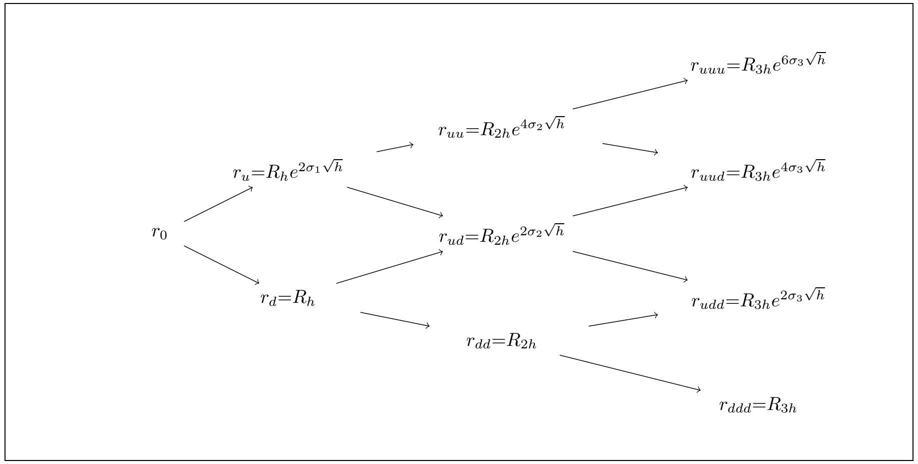
Consider the following 3-period binomial interest rate tree for effective annual rates. The risk-neutral probability of an up move is 0.6.

		13.9%
	12.0%	
9.6%		10.0% / 10.1%
	8.1%	
		5.8%

Find the price of a 9.6% interest rate cap on a 900 three-year loan with annual interest payments.

4.4 The Black-Derman-Toy Model

The Black-Derman-Toy (BDT) model is a recombining binomial tree. It models the evolution of **effective annual yields**, which are always non-continuously compounded! The structure of the tree of with time step h is:



Notes:

- The BDT interest rate tree is completely specified by R_{ih}'s and σ_i's.
- R_{ih} is the rate level parameter at time ih. R_h is the effective annual interest rate for the cur-

rent time at the lowest node.

- σ_i is the volatility parameter for rates in $[ih, i(h+1)]$. It is also called the **lognormal yield volatility** for one year bonds issued at the beginning of a period.

$$\sigma_i = \frac{\ln\left(\frac{R_{ih,1}}{R_{ih,0}}\right)}{2\sqrt{h}}$$

For example, $\sigma_2 = \frac{\ln(r_{uu}/r_{ud})}{2\sqrt{h}} = \frac{\ln(r_{ud}/r_{dd})}{2\sqrt{h}}$

- The ratios of the effective annual yields in adjacent nodes in the i^{th}-period are always $e^{2\sigma_i\sqrt{h}}$ for $t \geq 2$.
- The risk-neutral probabilities for up-moves are always 0.5.
- The most important feature about a BDT tree is that the nodes for a particular time period always form a geometric sequence. For example:

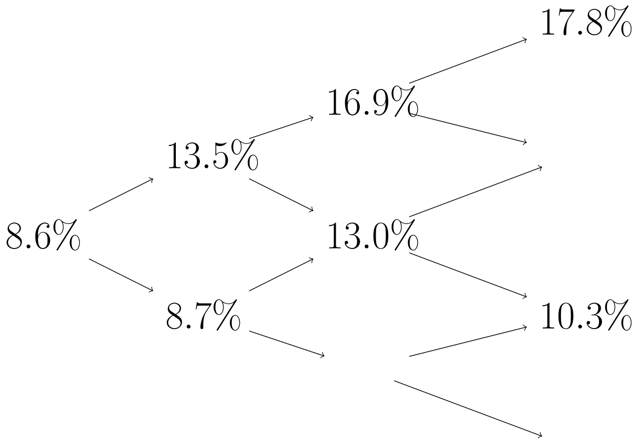
$$r_{ddd} \times e^{2\sigma_i\sqrt{h}} = r_{udd}$$

$$r_{udd} \times e^{2\sigma_i\sqrt{h}} = r_{uud}$$
$$r_{uud} \times e^{2\sigma_i\sqrt{h}} = r_{uuu}$$

- There is no relation between nodes in different vertical columns. It can happen that r_u is greater than r_{uu} and r_d is less than r_{dd}.

Example 9 (T4Q5).

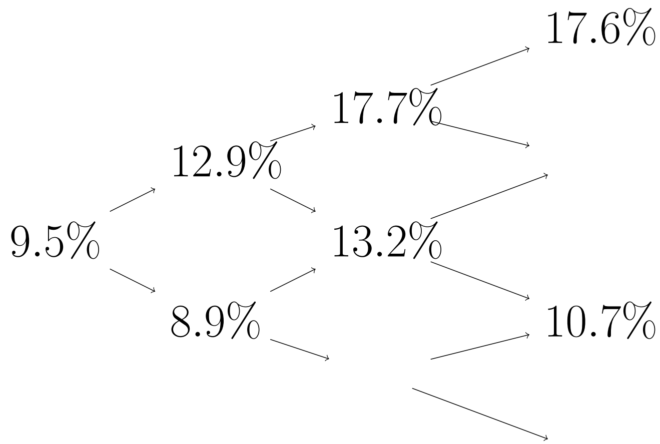
Consider the following incomplete BDT tree model for the effective annual interest rates:



Calculate the price of a 87.4-strike 2-year put on a 2-year 5% annual coupon bond with face value 100, maturing at time 4.

Example 10 (T4Q6).

Consider the following incomplete BDT tree model for the effective annual interest rates:



Calculate the price of a 3-year caplet with a cap rate of 10.2% for the notational amount of 100.

Example 11 (T4Q7).

In a Black-Derman-Toy tree with period of one year:

- The lognormal yield volatility of 3-year zero-coupon bonds after two years is 0.14.
- The lognormal yield volatility of 2-year zero-coupon bonds after one years is 0.1.
- The effective annual yield on 1-year zero-coupon bonds issued at the end of two years is 0.025 at the lowest node.
- The effective annual yield on 1-year zero-coupon bonds issued at the end of one year is 0.047 at the lowest node.

Determine the lognormal yield volatility of 3-year zero-coupon bonds after one year.

4.5 Put Call Parity for Bond Options

The prices of bond call and put options written on the same bond are related by put-call parity, provided that they have the same strike and maturity. If the underlying assets is an s-year zero-cuopon bond with face value F maturing at $T+s$, then

$$\boxed{c(F,K,T) - p(F,K,T) = FP(0,T+s) - KP(0,T)}$$

More generally, If the underlying assets is an s-year coupon bond with face falue F maturing at $T+s$, then

$$\boxed{c(F,K,T) - p(F,K,T) = \text{PV(Cash flow)} - KP(0,T)}$$

Example 12.

You are given the following information:

s	1	2	3
$P(0,s)$	0.904873	0.822662	0.751708

Suppose that the price of a 0.88-strike 2-year put on a 1-year zero-cuopon bond is 0.005788. Calculate the price of a 0.88-strike 2-year call on the same zero-cuopon bond. $\boxed{0.033553}$

Example 13.

You are given the follwing infomation:

Maturity (years)	Bond Price
1	0.9174
2	0.8271
2	0.7279
3	0.6479

Calculate the difference between the price of a 85-strike 2-year put on a 2-year 5.0% annual coupon bond with face value 100 maturing at time 4 and the price of a 85-strike 2-year call on the same underlying bond.

4.6 The Black Formula

Recall that the prepaid forward version of Black-Scholes formula are:

$$c(S(0), K, T) = F^P_{0,T}(S)N(d_1) - F^P_{0,T}(K)N(d_2)$$

$$p(S(0), K, T) = F^P_{0,T}(K)N(-d_2) - F^P_{0,T}(S)N(-d_1)$$

where $d_1 = \frac{\ln \frac{F^P_{0,T}(S)}{F^P_{0,T}(K)} + \frac{\sigma^2 T}{2}}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$.

The Black version of the Black-Scholes formula uses forward prices instead. If interest rate is constant, then

$$F^P_{0,T}(S) = e^{-rT}F_{0,T}(S) = P(0,T)F_{0,T}(S)$$

and

$$F^P_{0,T}(K) = e^{-rT}F_{0,T}(K) = KP(0,T)$$

So,

$$c(S(0), K, T) = P(0,T)[F_{0,T}(S)N(d_1) - KN(d_2)]$$

$$p(S(0), K, T) = P(0,T)[KN(-d_2) - F_{0,T}(S)N(-d_1)]$$

where $d_1 = \frac{\ln \frac{F_{0,T}(S)}{K} + \frac{\sigma^2}{2}}{\sigma\sqrt{T}}$ and $d_1 = d_2 - \sigma\sqrt{T}$.

The Black formula can also be used to price options on stocks, futures and currencies.

If we would like to compute the time-0 price of a T-year K-strike call on an s-year zero-cuopon bond, by subsituting $F_{0,T}(S)$ by $F_{0,T}[P(T, T+s)]$ into the Black formula, we get

$$c(K, T) = P(0,T)[FN(d_1) - KN(d_2)]$$

$$p(K, T) = P(0,T)[KN(-d_2) - FN(-d_1)]$$

where
$F = F_{0,T}[P(T, T+s)]$,
$d_1 = \frac{\ln(F/K) + 0.5\sigma^2 T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$.

Here, σ is the s-year forward bond price volatility:

$$\begin{aligned}\sigma^2 &= \tfrac{1}{t}V[\ln F_{t,T}(T, T+s)] \\ &= \tfrac{1}{t}V\left[\ln \tfrac{P(t,T+s)}{P(t,T)}\right], 0 < t \le T.\end{aligned}$$

If we put $t = T$, then

$$\sigma^2 = \frac{1}{T}V[\ln P(T, T+s)]$$

The value of σ depends on both T and s. For each option maturity and each underlying bond, there is a particular σ.

Example 14. [T4Q8]
You are given then following information:

Bond maturity (years)	1	2	3
Zero-coupon bond price	0.9605	0.9182	0.875

A 1-year European call option gives you the right to purchase a zero-coupon bond that matures at time 3 for 0.97. The bond forward price is lognormally distributed with volatility 0.16. Using the Black formula, calculate the price of the call option. $\boxed{0.0336}$

Example 15. [T4Q9]
You are given the following information for a 1-year zero-coupon bonds:

t	1	2	3	4	5
$(t-1)$-year forward price for 1-year bond	.87	.86	.85	.84	.83
Volatility of t-year prepaid forward price for 3-year bond	.04	.06	.08	.1	.12

Using Black's formula, calculate the price of a 2-year European call option with strike price 0.69 on a 3-year bond. $\boxed{0.0008}$

Example 16. [T4Q10]
Let $P(0,T)$ be the time-0 price of a zero-coupon bond that pays 1 at time T. You are given:

T	$P(0,T)$	$Var[\ln P(T,T+0.5)]/T]$
0.5	0.9207	0.0625
1	0.8693	0.076176
1.5	0.823	0.093636
2	0.7823	0.1089

Calculate the price of 1.5-year 0.98-strike put on a 6-month zero-coupon bond of face value 1 using Black formula.

4.7 Interest Rate Caplet

In order to price interest caplet, we discount the payoff to time T. The discounted payoff at time T is

$$\begin{aligned}&P(T,T+s)[r(T,T+s)-K]_+\\&=P(T,T+s)(1+K)\left[\tfrac{1+r(T,T+s)}{1+K}-1\right]_+\\&=(1+K)\left[\tfrac{1}{1+K}-P(T,T+s)\right]_+\end{aligned}$$

In other words, the caplet is equivalent to $(1+K)$ units of T-year put on $P(T,T+s)$, with strike $\frac{1}{1+K}$.

Consider a caplet that pays $[r(T,T+s)-K]_+$ at time $T+s$, then the time-0 price is

$$\text{caplet}(K,T,T+s)=(1+K)P(0,T)\left[\frac{N(-d_2)}{1+K}-FN(-d_1)\right]$$

where $F=\frac{P(0,T+s)}{P(0,T)}$, $d_1=\frac{\ln[F(1+K)]+0.5\sigma^2T}{\sigma\sqrt{T}}$ and $d_2=d_1-\sigma\sqrt{T}$.

Example 17. [T4Q11]
You are given the following information for zero-coupon bonds:

Bond Maturity (T in years)	1	2	3	4
$P(0,T)$	0.9529	0.8495	0.7722	0.7020
1-year Forward Price Volatility for a Bond Maturing at T	(N/A)	0.1000	0.1050	0.1100

Calculate the price of an interest rate caplet that provides an 11% (effective annual rate) cap on 1-year borrowing of 100 3 years from now. $\boxed{5.526}$

4.8 One Factor Interest rate Models

We let $r(t)$ be the instantaneous risk-free interest rate at time t. That is, for a 1 dollar at time t, the interest earned in an infinitesimally short interval $(t, t+dt)$ is $r(t)dt$. $r(t)$ is called force of interest or short rate at time t. If $r(s)$ is not random for any s, then time-t value of 1 dollar payable at time T is

$$P(t,T) = e^{-\int_t^T r(s)ds}.$$

If the short rate is constant, i.e., $r(s) = r$ for any s, then $P(t,T) = e^{-r(t-t)}$.

For $r(t)$ random, the dynamic of $r(t)$ is governed by the SDE

$$dr(t) = a(r)dt + \sigma(r)dZ(t)$$

Consider a nondividend-paying interest rate derivative which matures at time T. Let the time-t price of the derivative be $V(r,t,T)$. Then

$dV(r,t,T)$
$= V_t dt + V_r dr(t) + \frac{1}{2}V_{rr} dr(t)^2$
$= V_t dt + V_r[a(r)dt + \sigma(r)dZ(t) + \frac{1}{2}V_{rr}\sigma(r)^2 dt$
$= [V_t + V_r a(r) + \frac{1}{2}V_{rr}(\sigma(r)^2]dt + V_r\sigma(r)dZ(t)$
$= \alpha(r,t,T)V(r,t,T)dt - q(r,t,T)V(r,t,T)dZ(t)$

where
$\alpha(r,t,T) = \frac{1}{V(r,t,T)}\left(V_t + V_r a(r) + \frac{1}{2}V_{rr}(\sigma(r)^2\right)$
$q(r,t,T) = -\frac{1}{V(r,t,T)}V_r\sigma(r)$
Note that $\sigma(r)$ is positive, and V_r is negative since bond prices decrease as r increase. This is why a negative sign is attached to $q(r,t,T)$ in $dV(r,t,T)$.
Suppose that the dynamic of $V(r,t,T)$ follows

$$\frac{dV(r,t,T)}{V(r,t,T)} = \alpha(r,t,T)dt - q(r,t,T)dZ(t),$$

then the sharpe ratio of an Interest Rate Derivative is

$$\phi(r,t) = \frac{\alpha(r,t,T) - r(t)}{q(r,t,T)}.$$

Note that the Sharpe ratio ϕ only has two arguments, r and t. It cannot vary with T.

Example 18. [T4Q12]
Let $P(r,t,T)$ be the time-t price of a zero-coupon bond that matures at time T, when the time-t short rate is r. You are given:

(i) The stochastic process of P is given by

$$dP(r,t,T) = \alpha(r,t,T)P(r,t,T)dt - q(r,t,T)P(r,t,T)dZ(t),$$

where $Z(t)$ is a standard Brownian motion.

(ii) The Sharpe ratio of the interest rate risk is of the form $\phi(r,t) = kr$, where k is a constant.

(iii) $\alpha(0.1,2,5) = 0.56$

(iv) $q(0.1,2,5) = 2.25$ and $q(0.2,3,7) = 3.92$

Determine $\alpha(0.2,3,7)$. $\boxed{1.80281}$

4.9 The Term Structure Equation

Analogous to Black-Scholes equation, we have the term structure equation for interest rate derivative.

$$V_t + [a(r) + \sigma(r)\phi(r,t)]V_r + \frac{1}{2}\sigma(r)^2 V_{rr} = rV$$

The pricing formula for any derivative must satisfy the term structure equation.

Notes:

- The expression for q gives the volatility of the interest rate derivative.
- Since the bond price decreases with r, we have $V_r < 0$. The negative sign in the SDE of V and the Sharpe ratio makes q and ϕ non-negative.

Example 19. [T4Q13]
You are given:

(i) The stochastic process of the short rate is

$$dr(t) = a(r)dt + \sigma(r)dZ(t),$$

where $\sigma(r) > 0$ and $Z(t)$ is a standard Brownian motion.

(ii) Let $P(r, t, T)$ be the time-t price of a zero-cuopon bond that matures at time T, when the time-t short rate is r. Then $P(r, t, T)$ staisfies the equation

$$\frac{\partial V}{\partial t} + (0.42 - 0.2r)\frac{\partial V}{\partial r} + 0.08r\frac{\partial^2 V}{\partial r^2} = rV,$$

(iii) The Sharpe ratio $\phi(r, t) = 0.1\sqrt{r}$.

Determine $a(0.05)$.

4.10 Risk Neutral Valuation

In the one-factor interest rate model, it can be shown that the risk-neutral world Q is defined by

$$\tilde{Z}(t) = Z(t) - \int_0^t \phi(r, s)ds$$

or

$$d\tilde{Z}(t) = dZ(t) - \phi(r, t)dt$$

Here, an integral is needed because the Sharpe ratio can vary with time and the short rate. In this risk-neutral world, the interest rate process is as follows.

$$dr(t) = [a(r) + \sigma(r)\phi(r, t)]dt + \sigma(r)d\tilde{Z}(t)$$

The price-t of the derivative is

$$V(r, t) = E^*\left(e^{-\int_t^T r(s)ds}V(r, T)|r\right)$$

The following table summarizes the analogy between the Black-Scholes model and the bond factor interest rate model.

True dynamics	BS:$dS(t) = (\alpha - \delta)S(t)dt + \sigma S(t)dZ(t)$ IR:$dr = a(r)dt + \sigma(r)dZ(t)$
Equation	BS:$V_t + (r-\delta)SV_S + \frac{1}{2}\sigma^2 S^2 V_{SS} = rV$ IR:$V_t + (a + \sigma\phi)V_r + \frac{1}{2}\sigma^2 V_{rr} = rV$
Risk-neutral dynamics	BS:$dS(t) = (r-\delta)S(t)dt + \sigma S(t)d\tilde{Z}(t)$ IR:$dr = [a(r) + \sigma(r)\phi(r,t)]dt + \sigma(r)d\tilde{Z}(t)$
Relationship between P and Q	BS:$d\tilde{Z}(t) = dZ(t) + \phi dt$ IR:$d\tilde{Z}(t) = dZ(t) - \phi(r(t))dt$ defines a standard BM in Q
Time-t price	BS: $e^{rT}E^*[V(S(T),T)\|S(0)]$ $E^*[e^{-\int_0^t r(s)ds}V(r(T),T)\|r(0)]$

Example 20.

You are given:

- The true stochastic process of the short-rate is given by
$$dr(t) = [0.09 - 0.5r(t)]dt + 0.3dZ(t),$$
where $\{Z(t)\}$ is a standard Brownian motion under the true probability measure.
- The risk-neutral process of the short-rate is given by
$$dr(t) = [0.15 - 0.5r(t)]dt + \sigma(r(t))d\tilde{Z}(t),$$
where $\{\tilde{Z}(t)\}$ is a standard Brownian motion under the risk-neutral probability measure.
- $g(r,t)$ denotes the price of an interest-rate derivative at time t, if the short rate at that time is r. The interest-rate derivative does not pay any dividend or interest.
- $g(r(t),t)$ satisfies
$$dg(r(t),t) = \mu(r(t), g(r(t),t))dt - 0.4g(r(t),t)dZ(t).$$

Determine $\mu(r,g)$. $\boxed{(r+0.08)g}$

4.11 Greek for Interest Rate Derivatives

The Greek letters for interest rate derivatives are:

$$\Delta = \frac{\partial V(r,t,T)}{\partial r} = V_r$$

and

$$\Gamma = \frac{\partial^2 V(r,t,T)}{\partial r^2}$$

The Delta-Gamma Approximation is

$$V(r+\epsilon,t) \approx V(r,t) + \Delta\epsilon + \frac{1}{2}\Gamma\epsilon^2$$

If bond prices follow an "affine structure," the two Greek letters can be computed easily. Suppose that $V(r,t,T) = P(r,t,T)$ is the time-t price of a zero-coupon bond that matures at time T. We say that P has an affine structure if

$$P(r,t,T) = A(t,T)e^{-rB(t,T)}$$

To find the Greeks under an affine structure,

$$\begin{aligned}\Delta &= \tfrac{\partial P(r,t,T)}{\partial r}\\ &= -B(t,T)A(t,T)e^{-rB(t,T)}\\ &= -B(t,T)P(r,t,T)\end{aligned}$$

and

$$\Gamma = [B(t,T)]^2 P(r,t,T).$$

Moreover, the volatility of the bond is

$$q(r,t,T) = -\frac{\sigma(r)P_r(r,t,T)}{P(r,t,T)} = \sigma(r)B(t,T).$$

Example 21 (T4Q14)**.**

For $t \leq T$, let $P(r,t,T)$ be the price at time t of a zero-coupon bond that pays \$1 at time T, if the short-rate at time t is r. You are given:

- $P(r,t,T) = A(t,T)e^{-B(t,T)r}$ for some functions $A(t,T)$ and $B(t,T)$.
- $B(0,8)$= 3.1069.

Based on $P(0.048,0,8)$, you use the delta-gamma approximation to estimate $P(r,0,8)$, and denote the value as $P_{Est}(r,0,8)$. If $1000\left(\frac{P_{Est}(r,0,8)}{P(0.048,0,8)} - 1\right) = -15.1319$ and $r < 0.1$. Calculate r.

4.12 Delta Hedging

If we own an interest rate derivative V_1, whose dynamics is

$$\frac{dV_1(r,t,T)}{V_1(r,t,T)} = \alpha_1(r,t,T)dt - q_1(r,t,T)dZ(t),$$

and if there is another interest rate derivative V_2, whose dynamics is

$$\frac{dV_2(r,t,T)}{V_2(r,t,T)} = \alpha_2(r,t,T)dt - q_2(r,t,T)dZ(t),$$

then we can **delta hedge** the interest risk by trading

$$N = -\frac{q_1(r,t,T)V_1(r,t,T)}{q_2(r,t,T)V_2(r,t,T)} = \frac{V_{1r}(r,t,T)}{V_{2r}(r,tT)}$$

units of V_2.

Example 22.

Let $P(r,t,T)$ be the time-t price of a zero-coupon bond that matures at time T, when the time-t short rate is r. You are given:

(i) The stochastic process of P is given by

$$dP(r,t,T) = \alpha(r,t,T)P(r,t,T)dt - q(r,t,T)P(r,t,T)dZ(t),$$

where $Z(t)$ is a standard Brownian motion.

(ii) $\alpha(0.03,0,5) = 0.1165$ and $\alpha(0.03,0,10) = 0.5208$

At time-0, you sold 100 units of 5-year zero-coupon bond when the short rate is 3%. You decide to hedge interest rate risk by purchasing 10-year zero-coupon bonds. If $P(0.03,0,5) = 0.6740$, find the cost of purchasing 10-year bonds. $\boxed{11.8788}$

4.13 The Vasicek Model

In the Vasicek model, $a(r) = a(b-r)dt$, $\sigma(r) = \sigma$:

$$dr = a(b-r)dt + \sigma dZ(t)$$

As a result, $r(t)$ follows an Ornstein-Uhlenback process. The solution of the SDE for $r(t)$ is

$$r(t) = b + [r(0) - b]e^{-at} + \sigma \int_0^t e^{-a(t-u)} dZ(u).$$

This model is mean reversion, a controls the speed of mean-reversion, b is the level to which r reverts. When $r > b$, the drift is negative and r is expected to decrease, and vice versa. The disadvantages of this model are r can be negative and the volatility is constant.

A Vasicek model is specified using four parameters: a, b, σ and ϕ. In a Vasicek model, the Sharpe ratio ϕ does not vary with r or t, i.e. ϕ is constant.

The bond price under Vasicek Model is

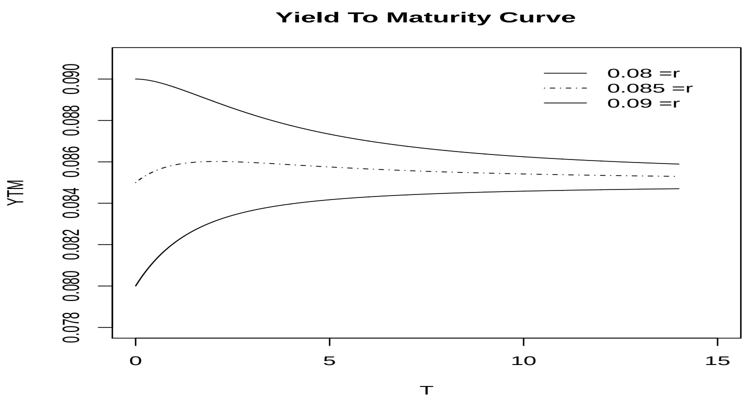
$$P(r,t,T) = A(T-t)e^{-rB(T-t)}$$

where $A(T-t) = e^{\bar{r}(B(T-t)+t-T)-\frac{B^2(T-t)\sigma^2}{4a}}$, $\boxed{B(T-t) = \bar{a}_{\overline{T-t}|a}}$ and $\boxed{\bar{r} = b + \frac{\sigma\phi}{a} - \frac{\sigma^2}{2a^2}}$.

Notes:

- It is very unlikely that we need to use formula for $A(T-t)$.
- P is an affine structure
$$P(r,t,T) = A(t,T)e^{-rB(t,T)}.$$
Both A and B depend only on the time to expiration $T-t$. This means that
$$P(0.01,1,3) = P(0.01,8,10).$$
- The formula of $B(T-t)$ is nice. It is a continuous annuity evaluated at rate a.
- The volatility coefficient $q(r,t,T)$ is
$$q(r,t,T) = \sigma B(T-t) = \frac{\sigma(1-e^{-a(T-t)})}{a}.$$

- The risk-neutral process of $r(t)$ is
$$dr = [a(b-r)+\sigma\phi]dt+\sigma d\tilde{Z} = (a(b'-r)dt+\sigma d\tilde{Z}$$
where $b' = b + \frac{\sigma\phi}{a}$. Thus, in Q measure, r still follows an Ornstein-Uhlenback process.
- We can calculate the annualized and continuously compounded yield to maturity in (t,T) from $P(r,t,T)$ using
$$P(r,t,T) = e^{-y(r,t,T)(T-t)}$$
or equivalently,
$$\begin{aligned} y(r,t,T) &= -\tfrac{1}{T-t}\ln P(r,t,T) \\ &= \tfrac{\ln A(T-t)-rB(T-t)}{T-t} \end{aligned}$$
- Figure below shown a plot of the time-0 yield to maturity for Vasicek model model as a function of T. The assume parameters are $a = 0.6$, $b = 0.04$, $\sigma = 0.06$ and $\phi = 0.5$. Three value of $r(0)$ are 8%, 8.5% and 9% are considered.



We observe that:

- The yield curve is not flat, it can be upward sloping, slightly humped or downward sloping.
- When $r(0)$ changes, the yield curve does not shift in parallel manner.

- When $T \to \infty$, $y(r,t,T) \to \bar{r} = b + \frac{\sigma\phi}{a} - \frac{\sigma^2}{2a^2}$.
- $\bar{r}$ is the yield to maturity on an infinite lived bond.
- Mean and variance of $r(t)$
  In order to find mean and variance of $r(t)$,

we need the following properties of Stochastic Integrals:

- The Ito integral has zero expectation.

$$E\left[\int_0^t f(u,w)dZ(u)\right] = 0$$

- Ito Isometry.

$$E\left[\left(\int_0^t f(u)dZ(u)\right)^2\right] = \int_0^t f^2(u)du$$

Example 23 (T4Q15)**.**
Let $r(t)$ be the time-t short rate. You re given that

- The stochastic process of $r(t)$ is given by

$$dr(t) = 2\sigma[0.15 - r(t)]dt + \sigma dZ(t)$$

  where σ is a positive constant and $Z(t)$ is a standard Brownian motion.

- The Sharpe ratio of interest risk is 0.1.

- Let $P(r, t, T)$ be the time-t price of a zero-coupon bond that matures at time T, when the time-t short rate is r.

Find $\lim_{T\to\infty} \frac{\ln P(0.02,0,T)}{T}$.

Example 24. [T4Q16]
Let $P(r, t, T)$ denote the price at time t of \$1 to be paid with certainty at time T, $t \le T$, if the short rate at time t is equal to r.
For a Vasicek model you are given:

$$P(0.04, 0, 2) = 0.9445 \quad P(0.05, 1, 3) = 0.9321$$
$$P(r^*, 2, 4) = 0.8960$$

Calculate r^*. $\boxed{0.07988}$

Example 25.
You are using the Vasicek one-factor interest-rate model with the short-rate process calibrated as

$$dr(t) = 0.6[b - r(t)]dt + \sigma dZ(t).$$

For $t \geq T$, let $P(r, t, T)$ be the price at time t of a zero-coupon bond that pays \$1 at time T, if the short-rate at time t is r. The price of each zero-coupon bond in the Vasicek model follows an Ito process,

$$\frac{dP[r(t), t, T)}{P[r(t), t, T]} = \alpha[r(t), t, T]dt - q[r(t), t, T]dZ(t).$$

You are given that $\alpha(0.04, 0, 2) = 0.04139761$. Find $\alpha(0.05, 1, 4)$. $\boxed{0.0516694}$

Example 26. [T4Q17]
You are given that $r(t)$, the short term interest rate at time t, satisfies the following equation:

$$r(t) = r(0)e^{-0.033t} + 0.06(1 - e^{-0.033t}) + 0.08\int_0^t e^{0.033(s-t)}dZ(s)$$

where $Z(s)$ is a Brownian motion.
The price of a zero-coupon bond issued at time t and expiring at time T when the short term interest rate is r is denoted by $P(r, t, T)$. Calculate $\frac{P_r(0.04,0,5)}{P(0.04,0,5)}$. $\boxed{-4.6093}$

4.14 The Cox-Ingersoll Ross (CIR) Model

In the CIR model

$$dr = a(b - r)dt + \sigma\sqrt{r}dZ$$

There is no explicit solution for $r(t)$.
The model has the following characteristics:

- Similar to the Vasicek model. $r(t)$ exhibits mean-reversion: a controls the speed of mean-reversion, b is the level to which r reverts.
- The instantaneous variance of the change in $r(t)$, $\sigma^2 r(t)$, is proportional to $r(t)$, instead of being constant as in the Vasicek model.
- r is always non-negative.
- A cox-ingersoll model is specified by four parameters a, b, σ and $\bar{\phi}$. The sharpe ratio is related to the parameters $\bar{\phi}$ as

$$\phi(r,t) = \frac{\bar{\phi}}{\sigma}\sqrt{r} = c\sqrt{r}$$

The Bond price under CIR Model is

$$P(r,t,T) = A(T-t)e^{-rB(T-t)}$$

where

$$A(T-t) = \left[\frac{2\gamma e^{(a-\bar{\phi}+\gamma)(T-t)/2}}{(a-\bar{\phi}+\gamma)(e^{\gamma(T-t)}-1)+2\gamma}\right]^{2ab/\sigma^2},$$

$$B(T-t) = \frac{2(e^{\gamma(T-t)}-1)}{(a-\bar{\phi}+\gamma)(e^{\gamma(T-t)-1}+2\gamma)}$$

and

$$\boxed{\gamma = \sqrt{(a-\bar{\phi})^2 + 2\sigma^2}}$$

Notes:

- It is very unlikely that we need to use formula for $A(T-t)$ and $B(T-t)$. However, we need to use the formula for γ.
- P is an affine structure

$$P(r,t,T) = A(t,T)e^{-rB(t,T)}.$$

  Both A and B depend only on the time to expiration $T-t$. This means that

$$P(0.01,1,3) = P(0.01,8,10).$$

- The formula of $B(T-t)$ is not nice. We will be given a table of A and B.
- We can use the given value of A and B to compute

$$q(r,t,T) = \sigma\sqrt{r}B(T-t)$$

with $q(r,t,T)$, we can calculate the hedge ratio easily.
- The risk-neutral process of $r(t)$ is

$$\begin{aligned} dr &= [a(b-r) + \frac{\bar{\phi}\sqrt{r}}{\sigma}\sigma\sqrt{r}]dt + \sigma\sqrt{r}d\tilde{Z} \\ &= (a'(b'-r)dt + \sigma d\tilde{Z} \end{aligned}$$

where $a' = a - \bar{\phi}$ and $b' = \frac{ab}{a-\bar{\phi}}$. Thus, the process of $r(t)$ in the Q measure has the same form as that in the P measure.
- The yield curve implied by the CIR model also can be upward sloping, slightly humped or downward sloping.
- The limit of the yield to maturity $\bar{r}$ is $\frac{2ab}{a-\bar{\phi}+\gamma}$

Example 27 (T4Q18)**.**
The short-rate process $\{r(t)\}$ in a Cox-Ingersoll-Ross model follows

$$dr(t) = [0.011 - 0.1r(t)]dt + 0.09\sqrt{r(t)}dZ(t),$$

where $\{Z(t)\}$ is a standard Brownian motion under the true probability measure. For $t \leq T$, let $P(r,t,T)$ denote the price at time t of a zero-coupon bond that pays 1 at time T, if the short-rate at time t is r. You are given:

- The Sharpe ratio takes the form $\phi(r,t) = c\sqrt{r}$.
- $\lim_{T\to\infty}\frac{1}{T}\ln[P(r,0,T)] = -0.09513$ for each $r > 0$.

Find the constant c.

Example 28. [T4Q19]
In a Cox-Ingersoll-Ross model for the short rate, $q(0.033, 1, 5) = 0.917$. Determine $q(0.058, 3, 7)$. $\boxed{1.2157}$

Example 29. Let $r(t)$ be the time-t short rate. You are given that

- The stochastic process of $r(t)$ is given by the Cox-Ingersoll-Ross model
$$dr(t) = [0.005 - 0.1r(t)]dt + 0.3\sqrt{r(t)}dZ(t),$$
where $Z(t)$ is a standard Brownian motion.
- The Sharpe ratio of interest rate risk when $t = 3$ and $r(3) = 0.04$ is $\phi(0.04, 3) = 0.05$.

Compute the drift of the stochastic differential equation of $r(t)$ in the risk-neutral measure when $t = 2$ and $r(2) = 0.07$. $\boxed{0.00325}$

Example 30 (T4Q20).

Let $r(t)$ be the short rate at time t. You are given:

- The stochastic process of $r(t)$ is
$$dr(t) = [0.12 - 0.2r(t)]dt + 0.1\sqrt{r(t)}dZ(t),$$
where $Z(t)$ is a standard Brownian motion under the true probability measure.
- The Sharpe ratio of Z is of the form $\phi(r,t) = c\sqrt{r}$.
- $\phi(0.04, 0) = 0.3$.
- Let $P(r,t,T)$ be the price at time t of a zero coupon bond paying 1 at time T, when the short rate at time t is r. Then
$$P(0.04, 0, 4) = 0.7942 \text{ and } P(0.1, 0, 4) = 0.6422.$$
- The stochastic process for the bond price process is
$$\frac{dP(r(t),t,T)}{P(r(t),t,T)} = \alpha(r(t),t,T)dt - q(r(t),t,T)dZ(t).$$

Find $\alpha(0.06, 0, 4)$.