MEME15203 Statistical Inference Marking Guide

Assignment 2

UNIVERSITI TUNKU ABDUL RAHMAN

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Q1. Consider $f(x|\theta) = \begin{cases} p, & x = 0 \\ (1-p)\frac{(\ln\theta)^x}{\theta x!}, & x = 1, 2, 3, \dots. \\ 0, & \text{otherwise} \end{cases}$

Suppose parameters are $p \in [0,1]$ and $\theta \geq 0$. Then, for $X_1, X_2, \dots, X_n$ iid with this, find a method of moments estimator for the parameter vector (p, θ) based on the first two sample moments.

(15 marks)

Ans.

Let $\lambda = \ln\theta$, then $\theta = e^{\lambda}$

$$f(x|\theta) = \begin{cases} p, & x = 0 \\ (1-p)\frac{(\lambda)^x e^{-\lambda}}{x!}, & x = 1, 2, 3, \dots. \\ 0, & \text{otherwise} \end{cases}$$

$E(X) = \sum_1^{\infty}(1-p)\frac{x\lambda^x e^{-\lambda}}{x!} = (1-p)\lambda = \mu_1$ .... (1)

$E(X^2) = \sum_1^{\infty}(1-p)\frac{x^2\lambda^x e^{-\lambda}}{x!} = (1-p)(\lambda+\lambda^2) = \mu_2$ .... (2)

$\frac{(2)}{(1)}, \frac{\lambda+\lambda^2}{\lambda} = \frac{\mu_2}{\mu_1}, \Rightarrow, \lambda = \frac{\mu_2}{\mu_1} - 1 = \frac{\mu_2-\mu_1}{\mu_1}$, and

$p = 1 - \frac{\mu_1}{\lambda} = 1 - \frac{\mu_1^2}{\mu_2-\mu_1} = \frac{\mu_2-\mu_1-\mu_1^2}{\mu_2-\mu_1}$

The MME of μ_1 is $\tilde{\mu}_1 = \bar{x}$ and $\mu_2 = \frac{1}{n}\sum_1^n x_i^2$.

Thus the MME of (λ, p) are

$\tilde{\lambda} = \frac{\frac{1}{n}\sum_1^n x_i^2 - \bar{x}}{\bar{x}} \Rightarrow \tilde{\theta} = e^{\frac{\frac{1}{n}\sum_1^n x_i^2-\bar{x}}{\bar{x}}}$ and $\tilde{p} = \frac{\frac{1}{n}\sum_1^n x_i^2 - \bar{x} - (\bar{x})^2}{\frac{1}{n}\sum_1^n x_i^2 - \bar{x}}$

Q2. Let $X_1, X_2, \dots, X_n$ be a random sample from the probability density function:

$$f(x_i) = \begin{cases} 4\theta x_i^{4\theta-1}, & 0 < x_i < 1, \theta > 0 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the MLE of θ.

(b) Find c such that $c\hat{\theta}$ is an unbiased estimator of θ, where $\hat{\theta}$ is the MLE of θ.

(15 marks)

Ans.

$L(\theta) \propto \theta^n \prod_{i=1}^n x_i^{4\theta}$
$l(\theta) \propto n\ln(\theta) + 4\theta \sum_{i=1}^n \ln x_i$
$\frac{dl(\theta)}{d\theta} = \frac{n}{\theta} + 4\sum_{i=1}^n \ln x_i = 0$
$\hat{\theta} = -\frac{n}{4\sum_{i=1}^n \ln x_i}$

Let $v_i = -ln(x_i)$.Thus $0 < v_i < \infty$. This correspond to a 1-1 transformation of $x_i = e^{-v_i}$
$h^{-1}(v_i) = e^{-v_i}$
$f_V(v_i) = f_X(h^{-1}(v_i))\frac{dh^{-1}(v_i)}{dv_i} = 4\theta e^{-(4\theta-1)v_i}e^{-v_i} = 4\theta e^{-4\theta}$
$\Rightarrow V_i \sim EXP(1/4\theta)$ and
$U = -\sum_{i=1}^n \ln x_i = \sum_{i=1}^n V_i \sim gamma(\alpha = n, \beta = \frac{1}{4\theta})$

$$
\begin{aligned}
E(\hat{\theta}) &= E(\tfrac{n}{U}) \\
&= nE(U^{-1}) \\
&= n\int_0^\infty u^{-1}\frac{(4\theta)^n}{\Gamma(n)}u^{n-1}e^{-4\theta u}du \\
&= \frac{n(4\theta)^n}{\Gamma(n)}\int_0^\infty u^{n-2}e^{-4\theta u}du \\
&= \frac{n(4\theta)^n}{\Gamma(n)}\left[\frac{\Gamma(n-1)}{(4\theta)^{n-1}}\right] \\
&= \frac{n(4\theta)}{n-1}
\end{aligned}
$$

$E\left(c\hat{\theta}\right) = \theta$
$c\left[\frac{n(4\theta)}{n-1}\right] = \theta$
$c = \frac{n-1}{4n}$

Q3. Let $X \sim POI(\mu)$. Suppose $\theta = e^{-\mu}$, $\hat{\theta} = e^{-X}$ and $\tilde{\theta} = u(x) = \begin{cases} 1, \text{for } x = 0 \\ 0, \text{for } x = 1, 2, \ldots \end{cases}$.

Compare the MSEs of $\hat{\theta}$ and $\tilde{\theta}$ for estimating θ when $\mu = 5$

(20 marks)

Ans.

$E(\hat{\theta}) = E(e^{-X}) = M_X(-1) = e^{\mu(e^{-1}-1)}$.
$Bias(\hat{\theta}) = E(\hat{\theta}) - \theta = e^{\mu(e^{-1}-1)} - e^{-\mu}$
$Bias(\hat{\theta})|_{\mu=5} = e^{5(e^{-1}-1)} - e^{-5} = 0.0357$
$E(\hat{\theta})|_{\mu=5} = e^{5(e^{-1}-1)} = 0.0424$
$E(\hat{\theta})^2 = E(e^{-X})^2 = M_X(-2) = e^{\mu(e^{-2}-1)}$
$E(\hat{\theta})^2|_{\mu=5} = e^{5(e^{-2}-1)} = 0.0133$
$V(\hat{\theta}) = E(\hat{\theta})^2 - E^2(\hat{\theta}) = 0.0133 - 0.0424^2 = 0.0115$
$MSE(\hat{\theta}) = Bias(\hat{\theta}) + V(\hat{\theta}) = 0.0357^2 + 0.0115 = 0.0128$

$E(\tilde{\theta}) = E(U(X) = 1 \times P(X_0) + 0 \times \sum_{x=1}^{\infty} xP(X = x) = e^{-\mu}$.
$E(\tilde{\theta})^2 = E(U(X))^2 = 1^2 \times P(X_0) + 0 \times \sum_{x=1}^{\infty} x^2 P(X = x) = e^{-\mu}$.
$V(\tilde{\theta}) = e^{-\mu} - [e^{-\mu}]^2 = e^{-5} - [e^{-5}]^2 = 0.0067$
$MSE(\tilde{\theta}) = V(\tilde{\theta}) = 0.0067$

Thus, $MSE(\tilde{\theta}) < MSE(\hat{\theta})$ when $\mu = 5$.

Q4. Let $X_1, X_2, \ldots, X_n$ denote a random sample from the density function given by

$$f(x) = \begin{cases} \frac{4}{\theta} x^3 e^{-x^4/\theta}, & \theta > 0, x > 0, \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the MLE of θ.

(b) Find the CRLB of θ.

(c) Find the UMVUE for θ.

(15 marks)

Ans.

(a) $\ln L = n \ln 4 - n \ln \theta + (4-1) \sum_{i=1}^{n} \ln x_i - \sum_{i=1}^{n} \frac{x_i^4}{\theta}$
$\frac{dL}{d\theta} = \frac{-n}{\theta} + \frac{\sum_{i=1}^{n} x_i^4}{\theta} = 0$
$\hat{\theta} = \frac{\sum_{i=1}^{n} x_i^4}{n}$

(b) Let $u = x^4$, $w(u) = x = u^{1/4}$, $w'(u) = \frac{1}{4} u^{1/4-1}$
$f_U(u) = \frac{1}{\theta}(4)(u^{1/4})^3 e^{-(u^{1/4})^4/\theta} (\frac{1}{4} u^{1/4-1}) = \frac{1}{\theta} u^{-u/\theta}$
$\Rightarrow U \sim EXP(\theta)$
$\tau(\theta) = \theta$
$\ln f(x;\theta) = -\ln \theta + \ln 4 + 3 \ln x - x^4/\theta$
$\frac{\partial \ln f(x;\theta)}{\partial \theta} = -\frac{1}{\theta} + \frac{x^4}{\theta^2}$
$\frac{\partial^2 \ln f(x;\theta)}{\partial \theta^2} = \frac{1}{\theta^2} - \frac{2x^4}{\theta^3}$
$E\left(\frac{\partial^2 \ln f(x;\theta)}{\partial \theta^2}\right) = \frac{1}{\theta^2} - \frac{2E(x^4)}{\theta^3} = \frac{1}{\theta^2} - \frac{2\theta}{\theta^3} = -\frac{1}{\theta^2}$

$CRLB = \frac{\tau'(\theta)}{-nE\left(\frac{\partial^2 \ln f(x;\theta)}{\partial \theta^2}\right)} = \frac{1}{-n(-1/\theta^2)} = \frac{n}{\theta^2}$

(c) $\hat{\theta} = \bar{U}$
$Var(\bar{U}) = \frac{Var(U)}{n} = \frac{\theta^2}{n}$
Since $Var(\hat{\theta})$ attained the CRLB, thus $\hat{\theta} = \frac{\sum_{i=1}^{n} x_i^4}{n}$ is the CRLB for θ

Q5. Let $X_1, X_2, \ldots, X_n$ denote a random sample from an exponentially distributed population with mean $\lambda = \frac{1}{\theta}$. Let $\Theta \sim \chi^2(2v)$.

(a) Find the Bayes estimator for $\lambda = \frac{1}{\theta}$ under square error loss.

(b) Show that it is a biased but consistent estimator for $\lambda = \frac{1}{\theta}$.

(20 marks)

Ans.

(a) $f(x_i|\theta) = \theta e^{-\theta x_i}$

$\Theta \sim \chi^2(v) = GAM(\alpha = v, \beta = 2)$

$\pi(\theta) = \frac{1}{\Gamma(v)2^v}\theta^{v-1}e^{-\theta/2}, \theta > 0$

$\pi(\theta|\mathbf{x}) = k\theta^{v+n-1}e^{-\theta(\sum x_i + 1/2)}, \theta > 0$

$\therefore \Theta|\mathbf{x} \sim GAM(v+n, (\sum x_i + 1/2)^{-1})$

$$\begin{aligned}\hat{\lambda} = E(\Theta^{-1}) &= \int_0^\infty \theta^{-1}\frac{(\sum x_i + 1/2)^{v+n}}{\Gamma(v+n)}\theta^{v+n-1}e^{-\theta(\sum x_i+1/2)}d\theta \\ &= \tfrac{(\sum x_i+1/2)^{v+n}}{\Gamma(v+n)}\int_0^\infty \theta^{v+n-2}e^{-\theta(\sum x_i+1/2)}d\theta \\ &= \tfrac{(\sum x_i+1/2)^{v+n}}{\Gamma(v+n)}\tfrac{\Gamma(v+n-1)}{(\sum x_i+1/2)^{v+n-1}} \\ &= \tfrac{\sum x_i+1/2}{v+n-1} \\ &= \tfrac{\sum x_i}{v+n-1} + \tfrac{1}{2(v+n-1)}\end{aligned}$$

(b) $E(\hat{\lambda}) = \frac{\sum E(X_i)}{v+n-1} + \frac{1}{2(v+n-1)} = \frac{n(1/\theta)}{v+n-1} + \frac{1}{2(v+n-1)} \neq 1/\theta$. thus $\hat{\lambda}$ is a biased estimator of $\lambda = \frac{1}{\theta}$.

$\lim_{n\to\infty} E(\hat{\lambda}) = 1/\theta$, Thus $\hat{\lambda}$ is asymtotically unbiased.

$V(\hat{\lambda}) = \frac{\sum V(X_i)}{(v+n-1)^2} = \frac{n(1/\theta^2)}{(v+n-1)^2}$

$\lim_{n\to\infty} V(\hat{\lambda}) = 0$. Thus $\hat{lambda}$ is MSE consistent and hence consistent.

Q6. Suppose $X|\theta \sim U(\theta - \frac{1}{6}, \theta + \frac{5}{6})$ and that a prior distribution of θ is $N(\mu, 1)$. Find the Bayes estimator of θ under squared error loss.

(15 marks)

Ans.

$f(x|\theta) = 1, \theta - \frac{1}{6} < x < \theta + \frac{5}{6}$

$\pi(\theta) = \frac{1}{\sqrt{2\pi}}e^{-(\theta-\mu)^2/2}, \theta \in \mathbb{R}$

$\pi(\theta|x) = ke^{-(\theta-\mu)^2/2}, x - \frac{5}{6} < \theta < x + \frac{1}{6}$

$\int_{x-\frac{5}{6}}^{x+\frac{1}{6}} ke^{-(\theta-\mu)^2/2}d\theta = 1$

Let $z = \theta - \mu$, $dz = d\theta$

$\int_{x-\frac{5}{6}-\mu}^{x+\frac{1}{6}-\mu} ke^{-z^2/2}dz = 1$

$k\sqrt{2\pi}[\Phi(x + \frac{1}{6} - \mu) - \Phi(x - \frac{5}{6} - \mu)] = 1$

$k = \frac{1}{\sqrt{2\pi}[\Phi(x+\frac{1}{6}-\mu)-\Phi(x-\frac{5}{6}-\mu)]}$, thus

$\pi(\theta|x) = \frac{e^{-(\theta-\mu)^2/2}}{\sqrt{2\pi}[\Phi(x+\frac{1}{6}-\mu)-\Phi(x-\frac{5}{6}-\mu)]}, x - \frac{5}{6} < \theta < x + \frac{1}{6}$

Under the square error loss, the Bayes estimator of θ is the posterior mean.

$$
\begin{aligned}
E(\Theta) &= \int_{x-\frac{5}{6}}^{x+\frac{1}{6}} \frac{\theta e^{-(\theta-\mu)^2/2}}{\sqrt{2\pi}[\Phi(x+\frac{1}{6}-\mu)-\Phi(x-\frac{5}{6}-\mu)]} d\theta \\
&= \frac{1}{\sqrt{2\pi}[\Phi(x+\frac{1}{6}-\mu)-\Phi(x-\frac{5}{6}-\mu)]} \int_{x-\frac{5}{6}}^{x+\frac{1}{6}} \theta e^{-(\theta-\mu)^2/2} d\theta \\
&\text{Let } z = \theta - \mu,\ dz = d\theta \\
&= \frac{1}{\sqrt{2\pi}[\Phi(x+\frac{1}{6}-\mu)-\Phi(x-\frac{5}{6}-\mu)]} \int_{x-\frac{5}{6}-\mu}^{x+\frac{1}{6}-\mu} (z+\mu) e^{-z^2/2} dz \\
&= \frac{1}{\sqrt{2\pi}[\Phi(x+\frac{1}{6}-\mu)-\Phi(x-\frac{5}{6}-\mu)]} \left[\int_{x-\frac{5}{6}-\mu}^{x+\frac{1}{6}-\mu} z e^{-z^2/2} dz + \mu \int_{x-\frac{5}{6}-\mu}^{x+\frac{1}{6}-\mu} e^{-z^2/2} dz\right] \\
&= \frac{1}{\sqrt{2\pi}[\Phi(x+\frac{1}{6}-\mu)-\Phi(x-\frac{5}{6}-\mu)]} \left[[-e^{-z^2/2}]_{x-\frac{5}{6}-\mu}^{x+\frac{1}{6}-\mu} + \mu\sqrt{2\pi}[\Phi(x+\tfrac{1}{6}-\mu) - \Phi(x-\tfrac{5}{6}-\mu)]\right] \\
&= \frac{e^{-\frac{1}{2}(x+\frac{1}{6}-\mu)^2} - e^{-\frac{1}{2}(x+\frac{1}{6}-\mu)^2}}{\sqrt{2\pi}[\Phi(x+\frac{1}{6}-\mu)-\Phi(x-\frac{5}{6}-\mu)]} + \mu
\end{aligned}
$$