MEME16203Linear Models Marking Guide

Assignment 2

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Due by:			

Q1. Show that any two matrices $\mathbf{W}$ and $\mathbf{X}$ have the same column space if there exist matrices $\mathbf{F}$ and $\mathbf{G}$ such that $\mathbf{WG} = \mathbf{X}$ and $\mathbf{XF} = \mathbf{W}$.

Ans.

$$\begin{aligned}
\mathbf{a} \in \mathcal{C}(\mathbf{X}) &\implies \mathbf{a} = \mathbf{Xb} \text{ for some } \mathbf{b}\\
&\implies \mathbf{a} = \mathbf{WGb} \text{ for some } \mathbf{b} \text{ (Because } \mathbf{X} = \mathbf{WG})\\
&\implies \mathbf{a} = \mathbf{Wc} \text{ for some vector } \mathbf{c}\\
&\implies \mathbf{a} \in \mathcal{C}(\mathbf{W})
\end{aligned}$$

So, $\mathcal{C}(\mathbf{X}) \subseteq \mathcal{C}(\mathbf{W})$
Then similarly,

$$\begin{aligned}
\mathbf{g} \in \mathcal{C}(\mathbf{W}) &\implies \mathbf{g} = \mathbf{Wh} \text{ for some } \mathbf{h}\\
&\implies \mathbf{g} = \mathbf{XFh} \text{ for some } \mathbf{h} \text{ (Because } \mathbf{W} = \mathbf{XF})\\
&\implies \mathbf{g} = \mathbf{Xk} \text{ for some } \mathbf{k}\\
&\implies \mathbf{g} \in \mathcal{C}(\mathbf{X})
\end{aligned}$$

So, $\mathcal{C}(\mathbf{W}) \subseteq \mathcal{C}(\mathbf{X})$
And hence,
$\mathcal{C}(\mathbf{X}) = \mathcal{C}(\mathbf{W})$

Q2. Suppose

$$Y_{ij} = \mu_i + \epsilon_{ij}, i = 1,2; j = 1,2,3.$$

$$\text{Let } \boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_{11} \\ \epsilon_{12} \\ \epsilon_{13} \\ \epsilon_{21} \\ \epsilon_{22} \\ \epsilon_{23}) \end{bmatrix} \sim N\left(\begin{bmatrix} 0\\0\\0\\0\\0\\0 \end{bmatrix}, \sigma^2 \begin{bmatrix} 9 & 0 & 0 & 0 & 0 & 0\\ 0 & 36 & 0 & 0 & 0 & 0\\ 0 & 0 & 64 & 0 & 0 & 0\\ 0 & 0 & 0 & 9 & 0 & 0\\ 0 & 0 & 0 & 0 & 36 & 0\\ 0 & 0 & 0 & 0 & 0 & 64 \end{bmatrix}\right).$$

What is the BLUE of $3\mu_1 + 5\mu_2$? Explain carefully.

Ans.

$$\begin{bmatrix} Y_{11} \\ Y_{12} \\ Y_{13} \\ Y_{21} \\ Y_{22} \\ Y_{23} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} + \boldsymbol{\epsilon}$$

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$$

where

$$\mathbf{X} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{Y} = \begin{bmatrix} Y_{11} \\ Y_{12} \\ Y_{13} \\ Y_{21} \\ Y_{22} \\ Y_{23} \end{bmatrix}$$

Let $\mathbf{V^{-1/2}} = diag(\frac{1}{\sqrt{9}}, \frac{1}{\sqrt{36}}, \frac{1}{\sqrt{64}}, \frac{1}{\sqrt{9}}, \frac{1}{\sqrt{36}}, \frac{1}{\sqrt{64}})$ and

$$\mathbf{Z} = \mathbf{V^{-1/2}Y} = \begin{bmatrix} \frac{1}{\sqrt{9}}y_{11} \\ \frac{1}{\sqrt{36}}y_{12} \\ \frac{1}{\sqrt{64}}y_{13} \\ \frac{1}{\sqrt{9}}y_{21} \\ \frac{1}{\sqrt{36}}y_{22} \\ \frac{1}{\sqrt{64}}y_{23} \end{bmatrix}$$

$$\begin{aligned} E(\mathbf{Z}) &= \mathbf{V^{-1/2}}E(\mathbf{Y}) \\ &= \mathbf{V^{-1/2}X}\boldsymbol{\beta} \\ &= \mathbf{W}\boldsymbol{\beta} \end{aligned}$$

where

$$\mathbf{W} = \begin{bmatrix} \frac{1}{\sqrt{9}} & 0 \\ \frac{1}{\sqrt{36}} & 0 \\ \frac{1}{\sqrt{64}} & 0 \\ 0 & \frac{1}{\sqrt{9}} \\ 0 & \frac{1}{\sqrt{36}} \\ 0 & \frac{1}{\sqrt{64}} \end{bmatrix}$$

$$\begin{aligned} Var(\mathbf{Z}) &= \mathbf{V^{-1/2}}^T Var(\mathbf{Y})\mathbf{V^{-1/2}} \\ &= diag\left(\frac{1}{\sqrt{9}}, \frac{1}{\sqrt{36}}, \frac{1}{\sqrt{64}}, \frac{1}{\sqrt{9}}, \frac{1}{\sqrt{36}}, \frac{1}{\sqrt{64}}\right) \times diag\left(9\sigma^2, 36\sigma^2, 8\sigma^2, 3\sigma^2, 6\sigma^2, 3\sigma^2\right) \end{aligned}$$

$$\times diag\left(\frac{1}{\sqrt{9}}, \frac{1}{\sqrt{36}}, \frac{1}{\sqrt{64}}, \frac{1}{\sqrt{9}}, \frac{1}{\sqrt{36}}, \frac{1}{\sqrt{64}}\right)$$
$$= \sigma^2 \mathbf{I}$$

Thus, $\mathbf{Z}$ follows a Gauss-Markov model with model matrix $\mathbf{W}$.

$$(\mathbf{W^TW}) = \begin{bmatrix} \frac{1}{\sqrt{9}} & \frac{1}{\sqrt{36}} & \frac{1}{\sqrt{64}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{9}} & \frac{1}{\sqrt{36}} & \frac{1}{\sqrt{64}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{9}} & 0 \\ \frac{1}{\sqrt{36}} & 0 \\ \frac{1}{\sqrt{64}} & 0 \\ 0 & \frac{1}{\sqrt{9}} \\ 0 & \frac{1}{\sqrt{36}} \\ 0 & \frac{1}{\sqrt{64}} \end{bmatrix} = \begin{bmatrix} 0.1545 & 0 \\ 0 & 0.1545 \end{bmatrix}$$

$$(\mathbf{W^TW})^{-1} = \begin{bmatrix} 6.4725 & 0 \\ 0 & 6.4725 \end{bmatrix}$$

$$\mathbf{W^TZ} = \begin{bmatrix} \frac{1}{\sqrt{9}} & \frac{1}{\sqrt{36}} & \frac{1}{\sqrt{64}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{9}} & \frac{1}{\sqrt{36}} & \frac{1}{\sqrt{64}} \end{bmatrix} \begin{bmatrix} \frac{1}{9}y_{11} + \frac{1}{36}y_{12} + \frac{1}{64}y_{13} \\ \frac{1}{9}y_{21} + \frac{1}{36}y_{22} + \frac{1}{64}y_{23} \end{bmatrix}$$

So, the BLUE of $3\mu_1 + 5\mu_2$ is

$$[3 \quad 5][(\mathbf{W^TW})^{-1}\mathbf{W^TZ}]$$
$$= [3 \quad 5] \begin{bmatrix} 6.4725 & 0 \\ 0 & 6.4725 \end{bmatrix} \begin{bmatrix} \frac{1}{9}y_{11} + \frac{1}{36}y_{12} + \frac{1}{64}y_{13} \\ \frac{1}{9}y_{21} + \frac{1}{36}y_{22} + \frac{1}{64}y_{23} \end{bmatrix}$$
$$= 19.4175[\tfrac{1}{9}y_{11} + \tfrac{1}{36}y_{12} + \tfrac{1}{64}y_{13}] = 32.3625[\tfrac{1}{9}y_{21} + \tfrac{1}{36}y_{22} + \tfrac{1}{64}y_{23}]$$

Q3. Consider a problem of quartic regression in one variable, X. In particular, suppose that $n = 6$ values of a response y are related to values $x = 0, 1, 2, 3, 4, 5$ by a linear model $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \epsilon$ for

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \\ 1 & 5 & 25 & 125 \end{bmatrix}, \quad \boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}, \text{ and } \quad \boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{bmatrix}$$

Define

$$\mathbf{W} = \begin{bmatrix} 1 & -5 & 5 & -5 \\ 1 & -3 & -1 & 7 \\ 1 & -1 & -4 & 4 \\ 1 & 1 & -4 & -4 \\ 1 & 3 & -1 & -7 \\ 1 & 5 & 5 & 5 \end{bmatrix}$$

(a) Show that $\mathbf{y} = \mathbf{W}\boldsymbol{\gamma} + \boldsymbol{\epsilon}$ is reparameterization of $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$, where $\boldsymbol{\gamma}^{\boldsymbol{T}} = [\gamma_1, \gamma_2, \gamma_3, \gamma_4]$.

Ans.

$\mathbf{W} = \mathbf{X}\mathbf{F}$ where

$$\mathbf{F} = \begin{bmatrix} 1 & -5 & 5 & -5 \\ 0 & 2 & -7.5 & 137/6 \\ 0 & 0 & 1.5 & -12.5 \\ 0 & 0 & 0 & 5/3 \end{bmatrix}$$

and

$\mathbf{X} = \mathbf{W}\mathbf{G}$ where

$$\mathbf{G} = \begin{bmatrix} 1 & 2.5 & 55/6 & 37.5 \\ 0 & 0.5 & 2.5 & 11.9 \\ 0 & 0 & 2/3 & 5 \\ 0 & 0 & 0 & 0.6 \end{bmatrix}$$

(b) Notice that $\mathbf{W^T W}$ is diagonal. Suppose that $\mathbf{y^T} = (-2, 0, 4, 2, 2, 1)$. Find the OLS estimate of $\boldsymbol{\gamma}$ in the model $\mathbf{y} = \mathbf{W}\boldsymbol{\gamma} + \boldsymbol{\epsilon}$ and then OLS estimate of $\boldsymbol{\beta}$ in the original model. (Find numerical values.)

Ans.

$\mathbf{W^T W} = diag(6, 70, 84, 182)$

$(\mathbf{W^T W})^{-1} = diag(\frac{1}{6}, \frac{1}{70}, \frac{1}{84}, \frac{1}{182})$

Then, $\hat{\gamma}_{OLS} = (\mathbf{W^T W})^{-1}\mathbf{W^T y} = diag(\frac{1}{6}, \frac{1}{70}, \frac{1}{84}, \frac{1}{182}) \begin{bmatrix} 7 \\ 16 \\ -31 \\ 9 \end{bmatrix} = \begin{bmatrix} \frac{7}{6} \\ \frac{8}{35} \\ \frac{-31}{84} \\ \frac{9}{182} \end{bmatrix}$

Since $\mathbf{W}\hat{\boldsymbol{\gamma}}_{OLS} = \mathbf{X}\mathbf{F}\boldsymbol{\gamma}_{OLS}$, we must have $\hat{\boldsymbol{\beta}}_{OLS} = \mathbf{F}\boldsymbol{\gamma}_{OLS}$.

$$\boldsymbol{\beta}_{OLS} = \begin{bmatrix} 1 & -5 & 5 & -5 \\ 0 & 2 & -7.5 & 137/6 \\ 0 & 0 & 1.5 & -12.5 \\ 0 & 0 & 0 & 5/3 \end{bmatrix} \begin{bmatrix} \frac{7}{6} \\ \frac{8}{35} \\ \frac{-31}{84} \\ \frac{9}{182} \end{bmatrix} = \begin{bmatrix} -2.06868132 \\ 4.35412088 \\ -1.17170330 \\ 0.08241758 \end{bmatrix}$$

Q4. Two varieties of corn (variety A and variety B) were compared in a field trail. In addition to the varieties, three levels of nitrogen were used (0, 30, and 60 pounds per acre (lb/a). Six different fields were used, and the six combinations of varieties and nitrogen levels were randomly assigned to the fields. Let Y_{ij} denote the yield (in bushels per acre) of the i^{th} variety of corn when the j^{th} level of nitrogen is applied. Throughout this question, ϵ_{ij}, $i = 1, 2, j = 1, 2, 3$, denote independent $N(0, \sigma^2)$ random variables where σ^2 is an unknown variance. The following two models were proposed:

$$\text{Model 1:}\quad \begin{bmatrix} Y_{11}\\ Y_{12}\\ Y_{13}\\ Y_{21}\\ Y_{22}\\ Y_{23} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -30 & 900\\ 1 & 0 & 0 & 0\\ 1 & 0 & 30 & 900\\ 0 & 1 & -30 & 900\\ 0 & 1 & 0 & 0\\ 0 & 1 & 30 & 900 \end{bmatrix} \begin{bmatrix} \gamma_1\\ \gamma_2\\ \delta_1\\ \delta_2 \end{bmatrix} + \begin{bmatrix} \epsilon_{11}\\ \epsilon_{12}\\ \epsilon_{13}\\ \epsilon_{21}\\ \epsilon_{22}\\ \epsilon_{23} \end{bmatrix}$$

$$\text{Model 2:}\quad \begin{bmatrix} Y_{11}\\ Y_{12}\\ Y_{13}\\ Y_{21}\\ Y_{22}\\ Y_{23} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & -1 & 1\\ 1 & 1 & 0 & 0 & -2\\ 1 & 1 & 0 & 1 & 1\\ 1 & 0 & 1 & -1 & 1\\ 1 & 0 & 1 & 0 & -2\\ 1 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \mu\\ \alpha_1\\ \alpha_2\\ \beta_1\\ \beta_2 \end{bmatrix} + \begin{bmatrix} \epsilon_{11}\\ \epsilon_{12}\\ \epsilon_{13}\\ \epsilon_{21}\\ \epsilon_{22}\\ \epsilon_{23} \end{bmatrix}$$

(a) With respect to the effects of varieties and nitrogen levels on corn yields, interpret the parameters γ_1 and δ_1 in Model 1.

> *Ans.*
> $E(Y_{12}) = \gamma_1$, thus γ_1 is the mean corn yield for variety A grown with nitrogen applied at 30lb/a.
> $E[Y_{13} - Y_{11}] = 60\delta_1$, thus $60\delta_1$ is the difference between mean yields for nitrogen applied at 60lb/a and 0lb/a for either variety A or variety B.

(b) For Model 1, indicate which of the following quantities are estimable

$$\gamma_1 - \gamma_2; \qquad \gamma_1 - 10\delta_1 + 100\delta_2$$

Give a brief explanation, to support your conclusions.

> *Ans.*
> Since the model matrix has full column rank, any linear combination of the parameters is estimable. Hence, both quantities above are estimable.

(c) For Model 2, determine if $\mu + \alpha_1$ is estimable? Give a brief explanation to support your conclusion.

> *Ans.*
> $\mu + \alpha_1$ is estimable because $E[(Y_{11} + Y_{12} + Y_{13})/3)] = \mu + \alpha_1$

(d) Expressing Model 2 as $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$, a solution to the normal equations is $\mathbf{b} = (\mathbf{X}^\mathbf{T}\mathbf{X})^{-}\mathbf{X}^\mathbf{T}\mathbf{Y}$. Explain how a generalized inverse $(\mathbf{X}^\mathbf{T}\mathbf{X})^{-}$ can be computed. (You are not expected to obtain a numerical value for $(\mathbf{X}^\mathbf{T}\mathbf{X})^{-}$, just briefly outline a procedure for how it can be computed.)

Ans. Delete the third row and the third column of $(\mathbf{X^T X})^-$. Invert the remaining 4×4 matrix. Fill in zeros for the third row and third column of the generalized inverse. Alternatively you could use the spectral or singular value decomposition of $(\mathbf{X^T X})^-$ to obtain a generalized inverse.

(e) Using $\mathbf{b} = (\mathbf{X^T X})^-\mathbf{X^T Y}$ from Part (d), define the estimator

$$\hat{\alpha}_1 - \hat{\alpha}_2 = \begin{bmatrix} 0 & 1 & -1 & 0 & 0 \end{bmatrix} \mathbf{b}$$

What are the properties of this estimator?

Ans.
$E(Y_{21} - Y_{11}) = \alpha_1 - \alpha_2$, hence $\alpha_1 - \alpha_2$ is estimable. Then by Gauss Markov theorem, $\begin{bmatrix} 0 & 1 & -1 & 0 & 0 \end{bmatrix} \mathbf{b}$ is the unique best linear unbiased estimator(BLUE) for $\begin{bmatrix} 0 & 1 & -1 & 0 & 0 \end{bmatrix} \boldsymbol{\beta} = \alpha_1 - \alpha_2$.

(f) Would the residual sum of squares from fitting models (1) and (2) be the same?

Ans.
Yes, because the columns of the model matrices span the same linear space. Let $\mathbf{W} = [\mathbf{w_1}|\mathbf{w_2}|\mathbf{w_3}|\mathbf{w_4}]$ denote the model matrix for Model 1, and let $\mathbf{X} = [\mathbf{x_1}|\mathbf{x_2}|\mathbf{x_3}|\mathbf{x_4}|\mathbf{x_5}]$ denote the model matrix for Model 2. Then, $\mathbf{x_2} = \mathbf{w_1}$, $\mathbf{x_3} = \mathbf{w_2}$, $\mathbf{x_1} = \mathbf{w_1} + \mathbf{w_2}$, $\mathbf{x_4} = \mathbf{w_3}/30$, and $\mathbf{x_5} = (3\mathbf{w_4}/900) - 2\mathbf{w_1} - 2\mathbf{w_2}$. Also, $\mathbf{w_1} = \mathbf{x_2}$, $\mathbf{w_2} = \mathbf{x_3}$, $\mathbf{w_3} = 30\mathbf{x_4}$, and $\mathbf{w_4} = 900(\mathbf{x_5} + 2\mathbf{x_2} + 2\mathbf{x_3})/3$. Or

$$\mathbf{W} = \mathbf{XF}, \text{ where } F = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1800/3 \\ 0 & 1 & 0 & 1800/3 \\ 0 & 0 & 30 & 0 \\ 0 & 0 & 0 & 900/3 \end{bmatrix} \text{ and}$$

$$\mathbf{X} = \mathbf{WG}, \text{ where } G = \begin{bmatrix} 1 & 1 & 0 & 0 & -2 \\ 1 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1/30 & 0 \\ 0 & 0 & 0 & 0 & 3/900 \end{bmatrix}$$

Q5. Two varieties of corn (variety A and variety B) were compared in a field trail. In addition to the varieties, three levels of nitrogen were used (10, 20 and 30 pounds per acre (lb/a). Six different fields were used, and the six combinations of varieties and nitrogen levels were randomly assigned to the fields. Suppose the data are as follows.

Field	i	j	Amount of Nitrogen (x_{ij})	Bushels per acre (y_{ij})
1	1	1	10	80
2	1	2	20	120
3	1	3	30	140
4	2	1	10	60
5	2	2	20	150
6	2	3	30	170

Consider a Gauss-Markov model

$$y_{ij} = \mu + \alpha_i + \gamma_i X_{ij} + \gamma_3 X_{ij}^2 + \epsilon_{ij}$$

where

- y_{ij} is the yield (in bushels per acre) of the i^{th} variety of corn when the j^{th} level of nitrogen is applied.
- X_{ij} denote the level of nitrogen administered to the corn,
- $\mu, \alpha_1, \alpha_2, \gamma_1, \gamma_2, \gamma_3$ are unknown parameters, and
- ϵ_{ij} denotes a random error with $\epsilon_{ij} \sim NID(0, \sigma^2)$ where $\sigma^2 > 0$.

Let $\boldsymbol{\beta} = (\mu, \alpha_1, \alpha_2, \gamma_1, \gamma_2, \gamma_3)^T$, $\mathbf{y} = [y_{11}, y_{12}, y_{13}, y_{21}, y_{22}, y_{23}]^T$, and $\boldsymbol{\epsilon} = [\epsilon_{11}, \epsilon_{12}, \epsilon_{13}, \epsilon_{21}, \epsilon_{22}, \epsilon_{23}]^T$.

(a) Determine the design matrix $\mathbf{X}$ so that $E(\mathbf{y}) = \mathbf{X}\boldsymbol{\beta}$.

Ans.

$$\mathbf{X} = \begin{bmatrix} 1 & 1 & 0 & 10 & 0 & 100 \\ 1 & 1 & 0 & 20 & 0 & 400 \\ 1 & 1 & 0 & 30 & 0 & 900 \\ 1 & 0 & 1 & 0 & 10 & 100 \\ 1 & 0 & 1 & 0 & 20 & 400 \\ 1 & 0 & 1 & 0 & 30 & 900 \end{bmatrix}$$

(b) Determine whether α_2 is estimable. Prove that your answer is correct.

Ans.

let $\mathbf{d^T} = \begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 \end{bmatrix}$, then $\mathbf{Xd} = \mathbf{0}$.
Let $\mathbf{c} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$, then $\mathbf{c^T}\boldsymbol{\beta} = \alpha_2$, but $\mathbf{c^T d} = -1 \neq 0$, i.e., there exists a vector $\mathbf{d}$ such that $\mathbf{Xd} = \mathbf{0}$ but $\mathbf{c_1^T}\boldsymbol{\beta} \neq 0$. Thus, α_2 is not estimable.

(c) Show that $\mathcal{C}(\mathbf{X}) = \mathcal{C}(\mathbf{W})$, where

$$\mathbf{W} = \begin{bmatrix} 1 & 1 & -1 & 0 & 1 \\ 1 & 1 & 0 & 0 & -2 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & -1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 0 & -2 \\ 1 & -1 & 0 & 1 & 1 \end{bmatrix}.$$

Ans.

$\mathcal{C}(\mathbf{X}) = \mathcal{C}(\mathbf{W})$ means that the space spanned by the columns of $\mathbf{X}$ is a basis for space spanned by the columns of $\mathbf{W}$ and vice versa. Thus there is a matrix $\mathbf{F}$ such that $\mathbf{W} = \mathbf{XF}$ and a matrix $\mathbf{G}$ such that $\mathbf{X} = \mathbf{WG}$.

$$\mathbf{W} = \begin{bmatrix} 1 & 1 & -1 & 0 & 1 \\ 1 & 1 & 0 & 0 & -2 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & -1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 0 & -2 \\ 1 & -1 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 10 & 0 & 100 \\ 1 & 1 & 0 & 20 & 0 & 400 \\ 1 & 1 & 0 & 30 & 0 & 900 \\ 1 & 0 & 1 & 0 & 10 & 100 \\ 1 & 0 & 1 & 0 & 20 & 400 \\ 1 & 0 & 1 & 0 & 30 & 900 \end{bmatrix} \begin{bmatrix} 1 & 0.5 & -1 & -1 & 12.36 \\ 0 & 0.5 & -1 & 1 & 12.36 \\ 0 & -1.5 & 1 & -1 & 12.36 \\ 0 & 0.0 & 1/10 & 0 & -2.43 \\ 0 & 0.0 & 0 & 1/10 & -2.43 \\ 0 & 0.0 & 0 & 0 & 0.05 \end{bmatrix} = \mathbf{XF}$$

$$\mathbf{X} = \begin{bmatrix} 1 & 1 & 0 & 10 & 0 & 100 \\ 1 & 1 & 0 & 20 & 0 & 400 \\ 1 & 1 & 0 & 30 & 0 & 900 \\ 1 & 0 & 1 & 0 & 10 & 100 \\ 1 & 0 & 1 & 0 & 20 & 400 \\ 1 & 0 & 1 & 0 & 30 & 900 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & 0 & 1 \\ 1 & 1 & 0 & 0 & -2 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & -1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 0 & -2 \\ 1 & -1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0.5 & 0.5 & 10 & 10 & 436.67 \\ 0 & 0.5 & -0.5 & 10 & -10 & 0 \\ 0 & 0.0 & 0.0 & 10 & 0 & 445.00 \\ 0 & 0.0 & 0.0 & 0 & 10 & 445.00 \\ 0 & 0.0 & 0.0 & 0 & 0 & 18.33 \end{bmatrix} = \mathbf{WG}$$

Thus, $\mathcal{C}(\mathbf{X}) = \mathcal{C}(\mathbf{W})$

(d) Verify that $\tau = \mu + \alpha_1$ is estimable, then obtained the unique BLUE of τ.

Ans.

$E(3Y_{11} - 3Y_{12} + Y_{13})$
$= 3(\mu + \alpha_1 + \gamma_1 + \gamma_3) - 3(\mu + \alpha_1 + 2\gamma_1 + 4\gamma_3) + (\mu + \alpha_1 + 3\gamma_1 + 9\gamma_3)$
$= (3 - 3 + 1)(\mu + \alpha_1) + (3 - 6 + 3)\gamma_1 + (3 - 12 + 9)\gamma_3$
$= \mu + \alpha_1.$

Thus, $\mu + \alpha_1$ is estimable.

Since $\mathbf{c}^{\mathbf{T}}\boldsymbol{\beta} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix} = \mu + \alpha_1$ is estimable, thus, by Gauss

Markov theorem, $\mathbf{c^T}\hat{\boldsymbol{\beta}}$ is the unique BLUE of $\mathbf{c^T}\boldsymbol{\beta}$.
Since $\mathcal{C}(\mathbf{X}) = \mathcal{C}(\mathbf{W})$, then $\mathbf{c^T}\hat{\boldsymbol{\beta}} = \mathbf{c^T F}\hat{\boldsymbol{\gamma}}$, where $\hat{\boldsymbol{\gamma}} = (\mathbf{W^T W})^{-1}\mathbf{W^T y}$

$(\mathbf{W^T W}) = \begin{bmatrix} 6 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 12 \end{bmatrix}$; $(\mathbf{W^T W})^{-1} = \begin{bmatrix} \frac{1}{6} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{6} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{12} \end{bmatrix}$ and $(\mathbf{W^T y}) = \begin{bmatrix} 720 \\ -40 \\ 60 \\ 110 \\ -90 \end{bmatrix}$

$$\hat{\boldsymbol{\gamma}} = \begin{bmatrix} 120 \\ -6.67 \\ 30 \\ 55 \\ -7.5 \end{bmatrix}$$

$$\mathbf{c^T}\hat{\boldsymbol{\beta}} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0.5 & -1 & -1 & 12.36 \\ 0 & 0.5 & -1 & 1 & 12.36 \\ 0 & -1.5 & 1 & -1 & 12.36 \\ 0 & 0.0 & 1/10 & 0 & -2.43 \\ 0 & 0.0 & 0 & 1/10 & -2.43 \\ 0 & 0.0 & 0 & 0 & 0.05 \end{bmatrix} \begin{bmatrix} 120 \\ -6.67 \\ 30 \\ 55 \\ -7.5 \end{bmatrix} =$$