CONTENTS

3 Greatest Accuracy Theory

3.1 Introduction

A model-based approach to the solution of the credibility problem is referred to as greatest accuracy credibility theory.

For a particular policyholder, we have observed n exposure units of past claims $\mathbf{X} = (X_1, X_2, \ldots, X_n)^T$. Let us assume that the risk level of each policyholder in the rating class may be characterized by risk parameter θ, but the value of θ varies by policy holder. Thus, there is a probability distribution with pf $\pi(\theta)$ of these values across the rating class. We assume $\pi(\theta)$ is known.

3.2 The Bayesian Approach and the Bayesian Premium

3.2.1 The Bayesian methodology

Let X_i have conditional pf

$$f_{X_i|\Theta}(x_i|\theta), i = 1, \ldots, n, n+1$$

Note that if the X_i are iid (conditional on $\Theta = \theta$), then $f_{X_i|\Theta}(x_i|\theta)$ does not depend on i. Ideally, we are interested in the conditional distribution of X_{n+1} given $\Theta = \theta$ in order to predict the claims experience X_{n+1} of the same policyholder. If we knew θ, we could use $f_{X_i|\Theta}(x_i|\theta)$.

However, $f_{X_i|\Theta}(x_i|\theta)$ is unknown, but we do know $\mathbf{X}$ for the same policyholder. Consequently, we calculate the conditional distribution of $\mathbf{X_{n+1}}$ given $\mathbf{X} = \mathbf{x}$ called the predictive distribution.

Because the $X_i's$ are independent conditional on $\Theta = \theta$, we have

$$f(x_1,\ldots,x_n|\theta)\pi(\theta) = \left[\prod_{i=1}^{n} f_{X_i|\theta}(x_i|\theta)\right]\pi(\theta)$$

The joint distribution of $\mathbf{X}$ is thus the marginal distribution obtained by integrating/summing θ out, i.e.

$$f_{\mathbf{X}}(\mathbf{x}) = \int \left[\prod_{i=1}^{n} f_{X_i|\theta}(x_i|\theta)\right]\pi(\theta)d\theta$$

if the prior is continuous

$$f_{\mathbf{X}}(\mathbf{x}) = \sum_{\theta\in\Theta} \left[\prod_{i=1}^{n} f_{X_i|\theta}(x_i|\theta)\right]\pi(\theta)$$

if the prior is discrete

Similarly, the joint distribution of $X_1,\ldots,X_{n+1}$ is the right hand side of equation above with n replace by $n+1$ in the product/summation.

The condition density of X_{n+1} given $\mathbf{X}=\mathbf{x}$ is

$$f_{X_{n+1|\mathbf{X}}}(x_{n+1|\mathbf{x}}) = \frac{\int \left[\prod_{i=1}^{n+1} f_{X_i|\theta}(x_i|\theta)\right]\pi(\theta)d\theta}{f_{\mathbf{X}}(\mathbf{x})}$$

if the prior is continuous

$$f_{X_{n+1|\mathbf{X}}}(x_{n+1|\mathbf{x}}) = \frac{\sum_{\theta\in\Theta} \left[\prod_{i=1}^{n+1} f_{X_i|\theta}(x_i|\theta)\right]\pi(\theta)}{f_{\mathbf{X}}(\mathbf{x})}$$

if the prior is discrete

The **posterior density** of Θ given $\mathbf{X}$ is

$$\begin{aligned}&\pi_{\Theta|\mathbf{X}}(\theta|\mathbf{x})\\ &= \frac{f_{\mathbf{X},\Theta}(\mathbf{x},\theta)}{f_{\mathbf{X}}(\mathbf{x})}\\ &= \frac{\left[\prod_{i=1}^{n} f_{X_i|\theta}(x_i|\theta)\right]\pi(\theta)}{f_{\mathbf{X}}(\mathbf{x})}\end{aligned}$$

In other words,

$$\left[\prod_{i=1}^{n} f_{X_i|\theta}(x_i|\theta)\right]\pi(\theta) = \pi_{\Theta|\mathbf{X}}(\theta|\mathbf{x}) f_{\mathbf{X}}(\mathbf{x})$$

Thus,

$f_{X_{n+1|\mathbf{X}}}(x_{n+1}|\mathbf{x})$
$= \int_{\theta\in\Theta} f_{X_{n+1|\theta}}(x_{n+1}|\theta)\pi_{\Theta|\mathbf{X}}(\theta|\mathbf{x})d\theta$
if the prior is continuous

and

$f_{X_{n+1|\mathbf{X}}}(x_{n+1}|\mathbf{x})$
$= \sum_{\theta\in\Theta} f_{X_{n+1|\theta}}(x_{n+1}|\theta)\pi_{\Theta|\mathbf{X}}(\theta|\mathbf{x})$
if the prior is discrete

and

$= E(X_{n+1}|\mathbf{X})$
$= \int x_{n+1} f_{X_{n+1|\mathbf{X}}}(x_{n+1}|\mathbf{x})dx_{n+1}$
$= \int x_{n+1} \int_{\theta\in\Theta} f_{X_{n+1|\theta}}(x_{n+1}|\theta)\pi_{\Theta|\mathbf{X}}(\theta|\mathbf{x})d\theta dx_{n+1}$
$= \int_{\theta\in\Theta}[\int x_{n+1} f_{X_{n+1|\theta}}(x_{n+1}|\theta)dx_{n+1}]\ \pi_{\Theta|\mathbf{X}}(\theta|\mathbf{x})d\theta$
$= \int_{\theta\in\Theta} E(X_{n+1}|\theta)\pi_{\Theta|\mathbf{X}}(\theta|\mathbf{x})d\theta$
if the prior is continuous

Similarly,
$E(X_{n+1}|\mathbf{x}) = \sum_{\theta\in\Theta} E(X_{n+1}|\theta)\pi_{\Theta|\mathbf{X}}(\theta|\mathbf{x})$
if the prior is discrete

Notes:
$f_{X_{n+1|\mathbf{X}}}(x_{n+1}|\mathbf{x})$is called the predictive probability and $E(X_{n+1}|\mathbf{X})$ is called the Bayesian premium.

3.2.2 Bayesian Estimation and Credibility - Discrete Prior

The application of Bayesian analysis to credibility estimation can be summarized as follows:

1. The random variable X (usually loss frequency or severity) had pf/pdf f that depends on a parameter Θ, where Θ is a random variable.
2. The assume distribution of Θ (can be discrete or continuous) is refer to the prior distribution, and has pf/pdf $\pi(\theta)$.
3. The distribution of $X|\theta$ is called model distribution.
4. An observation (several observations) is (are) made from the distribution of X, and then an updated form from the distribution of Θ is found, this is called the posterior distribution.
5. It is then is possible to extend this analysis to find the predictive distribution of the next occurrence of X.

The Bayesian statistician assumes that the universe follow a parametric model, with unknown parameters. The distribution of the model given the value of the parameters is called the model distribution. Unlike frequentist, who estimates the parameters from the data, the Bayesian assigns a prior probability distribution to the parameters. After observing data, a new distribution, called the posterior distribution is developed for the parameters.

Recall for discrete prior distribution,

The condition density of X_{n+1} given $X = x$ is

$$f_{X_{n+1|\mathbf{X}}}(x_{n+1}|\mathbf{x}) = \sum_{\theta \in \Theta} f_{X_{n+1|\theta}}(x_{n+1}|\theta)\pi_{\Theta|\mathbf{X}}(\theta|\mathbf{x})$$

and

$$E(X_{n+1}|\mathbf{x}) = \sum_{\theta \in \Theta} E(X_{n+1}|\theta)\pi_{\Theta|\mathbf{X}}(\theta|\mathbf{x})$$

The calculations can be summarized in the following table:

	$\Theta=\theta_1$	$\Theta=\theta_2$	$\cdots$	$\Theta=\theta_k$	Sum
Prior prob.	$\pi(\Theta=\theta_1)$ $=a_1$	$\pi(\Theta=\theta_2)$ $=a_2$	$\cdots$ $\cdots$	$\pi(\Theta=\theta_k)$ $=a_k$	1
Model prob.	$\prod f(x_j\|\theta_1)$ $=b_1$	$\prod f(x_j\|\theta_2)$ $=b_2$	$\cdots$ $\cdots$	$\prod f(x_j\|\theta_k)$ $=b_k$	
Joint prob.	a_1b_1	a_2b_2	$\cdots$	a_kb_k	$\sum_{i=1}^k a_ib_i$
Posterior prob.	$\frac{a_1b_1}{\sum a_ib_i}$ $=c_1$	$\frac{a_2b_2}{\sum a_ib_i}$ $=c_2$	$\cdots$ $\cdots$	$\frac{a_kb_k}{\sum a_ib_i}$ $=c_k$	1
Predictive Prob.	$\sum f(x_{n+1}\|\theta_i)c_i$				
Hypothetical mean	$E[X\|\theta_1]$	$E[X\|\theta_2]$	$\cdots$	$E[X\|\theta_k]$	
Bayesian premium	$\sum E[X\|\theta_i]c_i$				

Example 1.

There are two types of driver. Good drivers make up 75% of the population and in one year have zero claims with probability 0.7, one claim with probability 0.2, and two claims with probability 0.1 Bad drivers make up the other 25% of the population and have zero, one, or two claims with probabilities 0.5, 0.3, 0.2, respectively. For a particular policyholder suppose we have observed $x_1 = 0$ and $x_2 = 1$. Determine the posterior distribution of $\Theta|x_1 = 0, x_2 = 1$.

Example 2.

Using the information from previous example, determine the predictive distribution of $X_3|x_1 = 0, x_2 = 1$.

Example 3 (T3Q1).

An automobile liability coverage is sold in three territories, A, B, and C. 30% of the business is sold in A, 20% in B, and 50% in C. Claim frequencies on this coverage are given in the following table:

Territory	Number of Claims		
	0	1	2
A	0.61	0.22	0.17
B	0.67	0.26	0.07
C	0.23	0.66	0.11

An insured selected at random had no claims in one period. Determine the probability of one claim from this insured in the next period. $\boxed{0.3495}$

Example 4 (T3Q02).

An automobile liability coverage is sold in three cities, J, K, and L. 31% of the business is sold in J, 22% in K, and 47% in L. Claim frequencies on this coverage are given in the following table:

City	Number of Claims		
	0	1	2
J	0.6	0.23	0.17
K	0.7	0.28	0.02
L	0.23	0.65	0.12

An insured selected at random had no claims in the first period and two claims in the second period. Determine the expected number of claims from this insured in the next period. $\boxed{0.6538}$

Example 5 (T3Q03).

You are given:

- The annual number of claims on a given policy has a poisson distribution with parameter λ.
- 27% of the policies have $\lambda = 1.4$, 46% of the policies have $\lambda = 2.1$, and the remaining 27% have $\lambda = 3.2$.

A randomly selected policy had 2 claims in Year 1. Calculate the Bayesian expected number of claims for the selected policy in Year2.

Example 6 (T3Q04).

Two eight-sided dice, A and B, are used to determine the number of claims for an insured. The faces of each die are marked with either 0 or 1, representing the number of claims for that insured for the year.

Die	$P(\text{claims}=0)$	$P(\text{claims}=1)$
A	$\frac{6}{8}$	$\frac{2}{8}$
B	$\frac{4}{8}$	$\frac{4}{8}$

Two spinners, X and Y, are used to determined claim cost. Spinner X has two areas marked 13 and c. Spinner Y has only one area marked 13.

Spinner	$P(\text{cost}=13)$	$P(\text{cost}=c)$
X	$\frac{1}{2}$	$\frac{1}{2}$
Y	1	0

To determine the losses for the year, a die is randomly selected from A and B and rolled. If a claim occurs, a spinner is randomly selected from X and Y. For subsequent years, the same die and spinner are used to determine losses. Losses for the first year are 13. Based upon the results

of the first year, you determine that the expected losses for the second year are 9.0. Calculate c.

Example 7 (T3Q5).

You are given the following information about six coins:

Coin	Probability of heads
$1-4$	0.51
5	0.49
6	0.00

A coin is selected at random and then flipped repeatedly. X_i denotes the outcome of the i-th flip, where "1" indicates heads and "0" indicates tail. The following sequence is obtained:

$$S = (X_1, X_2, X_3, X_4) = (1, 0, 0, 0)$$

Determine $E(X_5|S)$ using Bayesian analysis.

3.2.3 Bayesian Estimation and Credibility - Continuous Prior

Definition 1.
The prior density, $\pi(\theta)$, is the initial density function for the parameter which describes the model. It is permissible to use an improper prior density, one which is nonnegative but whose integral is infinite.

For example, $\pi(\theta) = 1/\theta$, an improper prior, is a good prior for a scale parameter θ for which you have no idea what its distribution is.

Definition 2.
The **model density**, $f(x|\theta)$ is the density function describing the item we're modeling - number of claims or claim size - conditional on the parameter θ. It does not vary as we get more data.

Definition 3.
$f(x)$ is the **unconditional density function** for the item we're modeling if we pick an insured at random from the group. It is given by

$$f(x) = \int f(x|\theta)\pi(\theta)d\theta$$

Definition 4.
The **posterior density**, $\pi(\theta|x_1, \ldots, x_n)$ is the revised density function for the parameter based on data $x_1, \ldots, x_n$.

$\pi(\theta|x_1, \ldots, x_n)$
$= \frac{f(x_1|\theta)f(x_2|\theta)\cdots f(x_n|\theta)\pi(\theta)}{\int f(x_1|\theta)f(x_2|\theta)\cdots f(x_n|\theta)\pi(\theta)d\theta}$
$= \frac{f(x_1,x_2,\ldots,x_n,\theta)}{f(x_1,x_2,\ldots,x_n)}$

Definition 5.
The **predictive density**, $f(x_{n+1}|x_1, \ldots, x_n)$ is the revised unconditional $f(x)$ based on the observations $x_1, \ldots, x_n$. It is given by re-weighting the model with the posterior instead of the prior:

$$f(x_{n+1}|x_1, \ldots, x_n) = \int f(x|\theta)\pi(\theta|x_1, \ldots, x_n)d\theta$$

Definition 6.
The predictive expectation,

$= E[x_{n+1}|x_1, \ldots, x_n]$
$= \int E(X|\theta)\pi(\theta|x_1, \ldots, x_n)d\theta$
$= E_{\Theta}[E(X|\theta)]$

where Θ here refer to the posterior distribution.

Example 8.

The amount of claim has the exponential distribution with mean $\frac{1}{\Theta}$. Among the class of insureds and potential insureds, the parameter Θ varies according to gamma distribution with $\alpha = 4$ and scale parameter $\beta = 0.001$. Suppose a person had claims of 100, 950, and 450. Determine the posterior distribution of Θ. $\boxed{\frac{1}{\Gamma(7)2500^{-7}}\theta^6 e^{-2500\theta}}$

Example 9.

Using the information from previous example, determine the predictive distribution of the fourth claim. $\boxed{\text{Pareto}(\alpha = 7, \theta = 2500)}$

Example 10.

A random sample of $n = 8$ values for the distribution of X is given:

$$3, 4, 8, 10, 12, 18, 22, 35$$

The distribution of X is assumed to be exponential with parameter θ. The prior distribution of Θ has pdf $\pi(\theta) = 1/\theta, \theta > 0$. Find the posterior density of Θ. $\boxed{\frac{112^8}{\Gamma(8)}\theta^{-9}e^{-112/\theta}}$

Example 11.

Claim size follows an uniform distribution on [0, ω]. Θ has a single Pareto distribution with parameters $\alpha = 3$ and $\theta = 550$. An insured selected at random submits 2 claims of sizes 420 and 650. Calculate

(i) The expected size of the next claim. $\boxed{406.25}$

(ii) The posterior probability Θ greater than 760.

$\boxed{0.4576}$

(iii) Find $P[X_3 = x | x_1 = 420, x_2 = 650]$.

(iv) The probability that the next claim will be more than 620. $\boxed{0.2051}$.

(v) The probability the next claim will be less than 770. $\boxed{0.9286}$

Example 12 (T3Q06)**.**

Claim size follows a single-parameter Pareto distribution with parameters $\alpha = 4$ and θ. Over all insureds, Θ has a uniform distribution on $[2, 18]$. An insured is selected at random submits 3 claims of sizes 8, 10, and 13. Determine

(i) the posterior density.

(ii) the posterior mean. $\boxed{7.4286}$

(iii) the expected size of next claim. $\boxed{9.9048}$

(iv) the predictive distribution.

(v) the probability that the next claim will be less than 7.9. $\boxed{0.1998}$

(vi) the probability that the next claim will be greater than 11.0. $\boxed{0.21394}$

Example 13 (T3Q07).

The conditional distribution of a frequency model X, given the risk parameter Θ is

$$P(X=0|\Theta=\theta)=2\theta, P(X=1|\Theta=\theta)=2\theta,$$
$$P(X=2|\Theta=\theta)=1-4\theta$$

The parameter Θ is assumed to be uniformly distributed on the interval $[0,1/4]$. Determine $P(X_2=0|X_1=0)$.

Example 14 (T3Q08).

Losses are uniformly distributed on $[0,\ \theta]$. Θ varies by insured uniformly over $[5,\ 10]$. For a randomly selected insured, one observation of loss size is less that 7.5, Calculate the probability that the next observation of loss size from the same insured is less than 7.5.

3.3 Conjugate Prior

3.3.1 Poisson-Gamma

Bayesian analysis is easy when the posterior hypothesis comes from the same family of distributions as the prior hypothesis. If a prior hypothesis has this property for a given model, it is called the conjugate prior of the model.

An important example is the Poisson model, for which the gamma distribution is a conjugate prior.

The Poisson model is commonly used for claim frequency.

Model distribution: $X|\lambda \sim Poisson(\lambda)$
Prior distribution: $\Lambda \sim gamma(\alpha, \theta)$.

Suppose there are k exposures - This could be k years for one insured, or k insureds for one year.

Suppose there are $\sum_{j=1}^{k} x_j$ claims. Then,

the **posterior distribution** of λ given $\mathbf{x}$:

$$\Lambda|\mathbf{x} \sim gamma(\alpha^* = \alpha + \sum x_i, \theta^* = 1/(\theta^{-1} + k))$$

The **predictive distribution** of X_{k+1} given $X = x$:

$$(X_{k+1}|X = x) \sim NB(r^* = \alpha + x, \theta^* = 1/(\theta^{-1} + k))$$

The **Bayesian premium** (estimate) per individual is the expected value of the predictive distribution , i.e.

$$E(X_{k+1}|X = x) = \alpha^*\theta^* = (\alpha + x)/(\theta^{-1} + k)$$

For multiple exposure units, the total premium for $m_{(k+1)}$ individuals in year $k + 1$ is

$$m_{(k+1)}\alpha^*\theta^*$$

Example 15.

The random variable X denotes the number of claims in a year arising from a risk. The distribution of X is to be modeled using a Poisson(λ) random variable, where λ is unknown. Regarded as a random variable, λ has a prior distribution which is gamma, with parameters α and θ.The mean number of claims in the past k years is known, and is denoted $\bar{x}$.

(a) Derive the posterior distribution of λ.

(b) Show that the Bayesian premium can be written in the form of a credibility estimate. □

(c) Find the predictive distribution of X_{k+1} and state which family of distributions it belongs to. □

Example 16.

Assume an individual insured is selected at random from a population of insureds. The number of claims experienced in a given year by each insured follows a Poisson distribution. The mean value θ of the Poisson distribution is distributed across the population according to the following gamma distribution:

$$f(\theta) = \frac{7^3}{\Gamma(3)}\theta^2 e^{-7\theta}, \theta > 0$$

Given that a particular insured experienced a total of 6 claims in the previous 2 years, what is the posterior estimate of the future expected annual claim frequency, given the experience of this particular insured?

Example 17 (T3Q09).

You are given:

- The number of claims per auto insured follows a Poisson distribution with mean λ
- The prior distribution for Λ has the following probability density function:

$$\pi(\lambda) = [780\lambda^{80}e^{-780\lambda}/\lambda\Gamma(80)].$$

- A company observes the following claims experience:

	Year 1	Year 2
Number of claims	90	260
Number of autos insured	740	970

The company expects to insure 1180 autos in Year 3. Determine the expected number of claims in Year 3.

Example 18.

An insured's number of claims per year follows a Poisson distribution with mean λ. λ varies in accordance with a gamma distribution with $\alpha = 30$ and $\theta = 0.035$. You have the following information on the number of claims made by an insured in the past 6 years:

$$0, 2, 1, 1, 1, 0$$

Calculate the predictive variance of the number of claims per year for this insured.

3.3.2 Normal/Normal

The normal distribution is the conjugate prior of a model having the normal distribution with a fixed variance.

Model distribution: $X|\Theta = \theta \sim N(\theta, v)$
Prior distribution: $\Theta \sim N(\mu, a)$

Posterior distribution:

$$\Theta|\mathbf{x} \sim N\left(\mu^* = \frac{v\mu + na\bar{x}}{v + na}, a^* = \frac{va}{v + na}\right)$$

Predictive Distribution of X_{n+1} Given $X = x$:

$$X_{n+1}|\mathbf{x} \sim N\left(\mu^* = \frac{v\mu + na\bar{x}}{v + na}, a^* + v = \frac{va}{v + na} + v\right)$$

The **Bayesian Premium** (estimate) is

$$\mu^* = \frac{v\mu + na\bar{x}}{v + na}$$

Example 19. The random variable X denotes the annual aggregate claim amounts from a risk. The distribution of X is to be modeled using a $N(\theta, v)$ where v is known but θ is unknown and is to be estimated. Regarded as a random variable, Θ has a $N(\mu, a)$ distribution where both parameters are known.
The mean aggregate claim amount from the risk over the past n years is known, and is denoted $\bar{x}$. Derive the posterior distribution of Θ given the data and show that the Bayesian premium can be written in the form of a credibility estimate. □

Example 20.

The annual aggregate claim amounts for the past 5 years for a risk is

$$256, \quad 240, \quad 283, \quad 181, \quad 253.$$

Using normal/normal model, with parameter values $\mu = 230$, $v = 390$, $a = 200$, estimate the pure premium for the coming year for the risk.

Example 21 (T3Q10)**.**

Claim sizes are normally distributed with mean θ and variance 100,000. θ varies by risk, and is normally distributed with mean 1,600 and variance 1,000,000. For a certain risk, 13 claims averaging 2000 are observed. Determine the posterior probability that θ is less than 2049.0.

3.3.3 Binomial-Beta

We observe k outcomes of binomial $(X|\Theta = \theta)$, $X_i = x_i, i = 1, ..., k$, with a total of $\sum_{i=1}^{k} x_i$ losses.

Model Distribution:

$$X_i|\Theta = \theta \sim Binomial(m, \theta), i = 1, 2, \ldots, k$$

Prior Distribution:

$$\Theta \sim Beta(a, b)$$

Posterior Distribution of Θ Given $X = x$:

$$\Theta|X = x \sim Beta(a^* = a + \sum x_i, b^* = b + mk - \sum x_i)$$

Bayesian Premium per exposure:

The Bayesian premium is the expected value of the predictive distribution
$E(X_{k+1}|X = x) = \frac{ma^*}{(a^*+b^*)} = \frac{m(a+\sum x_i)}{(a+b+mk)}$

Example 22.
The random variable X denotes the number of claims in a year arising from a risk. The distribution of X is to be modeled using a Binomial(m, q) random variable, where q is unknown. Regarded as a random variable, Q has a prior distribution which is beta, with parameters a and b.The total number of claims in the past k years is known, and is denoted $\sum x_i$.

(a) Derive the posterior distribution of Q.

(b) Derive the Bayesian premium and show that the Bayesian premium can be written in the form of a credibility estimate.

Example 23 (T3Q11).

The number of claims per year on an insurance coverage has a binomial distribution with parameter $m = 3$ and Q. Q varies by insured and is distributed according to the following density function:

$$f(q) = cq(1-q)^7, 0 \leq q \leq 1,$$

where c is a constant.

An insured submits 1 claims in 7 years. Calculate the posterior probability that for this insured, q is less than 0.043000000000000003.

Example 24.

You are given:

- Claim number random variables per exposure per year are i.i.d. Bernoulli with parameter q.
- q has a beta distribution with parameters $a = 5$ and $b = 19$.
- For a given risk you observe:

Year	Exposures	Claim Numbers
2017	100	11
2018	130	14
2019	120	16

- Claim size is fixed at 1 if there is a claim.

Determine the total Bayesian premium for a group of 210 exposures from the same class for the next year. $\boxed{25.8289}$

Example 25.

The number of claims X, for an individual risk in one year follows the binomial distribution:

$$f(x|\theta) = \binom{6}{x}\theta^x(1-\theta)^{6-x}, x = 0, 1, \ldots, 6$$

The parameter Θ has a prior distribution in the form of a beta:

$$f(\theta) = 60\theta^3(1-\theta)^2, 0 \le \theta \le 1$$

Two claims occurred in the first year and three claims occurred in the second year. Determine the Bayesian estimate for the expected number of claims in the third year. $\boxed{54/19}$

3.3.4 Exponential-Inverse Gamma

This is a loss severity model X in which the conditional distribution $X|\Theta$ is Exponential. The size of losses per year $(X|\Theta = \theta)$ is conditionally Exponential and Θ is inverse gamma with parameters α and β. It is required that $\alpha > 1$. We observe n outcomes of $(X|\Theta = \theta)$ denoted $X_1 = x_1, X_2 = x_2, \ldots X_n = x_n$ with a total of $\sum_{i=1}^{n} x_i$ losses.

Model Distribution:

$$(X|\Theta = \theta) \sim Exponential(\theta)$$

Prior Distribution:

$$\Theta \sim InverseGamma(\alpha, \beta)$$

Marginal Distribution of X:

$$X \sim Pareto(\alpha, \beta)$$

Posterior Distribution of Θ Given $X = x$:

$$(\Theta|X = x) \sim InverseGamma(\alpha^* = \alpha + n, \beta^* = \beta + \sum x_i)$$

Predictive Distribution of X_{n+1} Given $X = x$:

$$(X_{n+1}|X = x) \sim Pareto(\alpha^* = \alpha + n, \beta^* = \beta + \sum x_i)$$

Bayesian Premium:
The Bayesian premium is the expected value of the predictive distribution, that is,

$$E(X_{n+1}|X = x) = \frac{\beta^*}{\alpha^* - 1} = \frac{\beta + \sum x_i}{\alpha + n - 1}$$

Note that the Bayesian premium is the same as the posterior mean.

Example 26. The random variable X denotes the loss severity from a risk. The distribution of X is to be modeled using an $Exp(\theta)$ where θ is unknown and is to be estimated. Regarded as a random variable, Θ has a $InverseGamma(\alpha, \beta)$ distribution where both parameters are known. The mean calim amount from the risk over the past n years is known, and is denoted $\bar{x}$.

(a) Derive the posterior distribution of Θ given the data.

(b) Show that the Bayesian Premium can be written in the form of a credibility estimate.

(c) Find the predictive distribution of X_{n+1} and state which family of distributions it belongs to.

Example 27 (T3Q12)**.**

We assume that the amount of an individual claim, Y , follows an exponential distribution with mean δ. The mean claim amount, δ, follows an inverse gamma distribution with density function

$$\pi(\delta) = \frac{6^3 e^{-6/\delta}}{\Gamma(3)\delta^4}, \delta > 0$$

Suppose 26 claims are observed with total aggregate claim amount of 8. Find

$$P\left(Y_{27} > 1 | \sum_{i=1}^{26} Y_i = 8\right)$$

Example 28 (T3Q13)**.**

For an insurance portfolio with 1197 exposures, you are given:

- The number of claims for each exposure follows a Poisson distribution.
- The mean claim count varies by exposure. the distribution of mean claim counts is a gamma distribution with parameters $\alpha_1 = 0.5$, $\theta_1 = 4$.
- The size of claims for each exposure follows an exponential distribution.
- The mean claim size varies by exposure. The distribution of mean claim sizes is an inverse gamma distribution with parameters $\alpha_2 = 4$, $\theta_2 = 4$.
- the standard for full credibility of aggregate claims is that aggregate claims must be within 8% of expected 90% of the time.

Determine the credibility assigned to this portfolio. $\boxed{0.8994}$

3.4 Inference and Prediction

3.4.1 Point Estimation

After obtaining the posterior distribution, you would want a point estimate of the parameter, just as you would with non-Bayesian (e.g. MME and MLE) estimation. To do this, a loss function $l(\hat{\theta}, \theta)$ is defined. $\hat{\theta}$ will be your estimate, and the loss function defines the penalty for $\hat{\theta}$ being different from θ. The Bayes estimate is the $\hat{\theta}$ which minimizes the expected loss, using the posterior distribution to calculate the expectation. i.e.
$\min_{\theta\in\Theta} E[l(\hat{\theta}, \theta)] = \min_{\theta\in\Theta} \int l(\hat{\theta}, \theta)\pi(\theta|x)d\theta$
Three loss function that are commonly use by Bayesian are:

1. Square error loss function, $l(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2$
2. Absolute error loss function, $l(\hat{\theta}, \theta) = |\hat{\theta} - \theta|$
3. Almost constant loss function,

$$l(\hat{\theta}, \theta) = \begin{cases} c, \hat{\theta} \neq \theta, \\ 0, \text{otherwise} \end{cases}$$

Theorem 1.
Let $\pi(\theta|x)$ denote the posterior density function of θ, given the vector of observed data $\mathbf{x}$. If the loss function is $l(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2$, then the Bayesian point estimator, $\hat{\theta}$, which minimizes the expected value of the loss function is the **mean** of the posterior distribution.

Theorem 2.
If the loss function is $l(\hat{\theta}, \theta) = |\hat{\theta} - \theta|$, then the Bayesian point estimator, $\hat{\theta}$, which minimizes the expected value of the loss function is the **median** of the posterior distribution.

Theorem 3.
If the loss function is $l(\hat{\theta}, \theta) = \begin{cases} c, \hat{\theta} \neq \theta \\ 0, \text{otherwise} \end{cases}$,then the Bayesian point estimator, $\hat{\theta}$, which minimizes the expected value of the loss function is the mode of the posterior distribution.

Example 29. Losses follow a gamma distribution with $\alpha = 3$ and θ unknown. The prior distribution of θ has density function $\pi(\theta) = \frac{1}{\theta}$, Five losses are observed:

$$100, 200, 140, 120, 200.$$

Determine the Bayesian estimate of the loss under square error loss. $\boxed{54.2875}$

Example 30 (T3Q14)**.**

Losses follow a distribution with desity function

$$f(x) = \delta x^{\delta - 1}, 0 \leq x \leq 1$$

δ varies by insured according to a gamma distribution with $\alpha = 3$, $\theta = 8$. A loss size of 0.69 is observed. Determine the posterior estimate of δ using zero-one loss fuction.

3.4.2 Credible Interval

A credible interval (or in general, a credible set) is the Bayesian analogue of a confidence interval. A $100(1-\alpha)\%$ credible set C is a subset of Θ such that

$$\int_C \pi(\theta|\mathbf{x}) d\theta = 1 - \alpha$$

If the parameter space Θ is discrete, a sum replaces the integral.

Definition 7.

If a is the $\frac{\alpha}{2}$ posterior quantile for θ, and b is the $1 - \frac{\alpha}{2}$ posterior quantile for θ, then (a, b) is a $100(1-\alpha)\%$ **equal probability credible interval** for θ.

Note:

$P(\theta < a|\mathbf{x}) = \frac{\alpha}{2}$ and $P(\theta > b|\mathbf{x}) = \frac{\alpha}{2}$

$\Rightarrow P(\theta \in (a, b)|\mathbf{x})$

$= 1 - P(\theta \notin (a, b)|\mathbf{x})$

$= 1 - (P(\theta < a|\mathbf{x}) + P(\theta > b|\mathbf{x})) = 1 - \alpha$

Example 31. The following amounts were paid on a hospital liability policy:

125	132	141	107	133	319	126	104	145	223

The amount of a single payment has the single-parameter Pareto distribution with $\theta = 100$ and α unknown. The prior distribution has the gamma distribution with $\alpha = 2$ and $\theta = 1$. Determine the 95% equal probability credible interval for α.

$\boxed{\text{a}=1.2915,\ \text{b} = 4.0995}$

The equal-tail credible interval approach is ideal when the posterior distribution is symmetric. If $\pi(\theta|\mathbf{x})$ is skewed, a better approach is to create an interval of $\theta-$values having the Highest Posterior Density (HPD).

Definition 8.
A $100(1-\alpha)\%$ HPD region for θ is a subset $C \in \Theta$ defined by

$$C = \{\theta : \pi(\theta|\mathbf{x}) \geq k\}$$

where k is the largest number such that

$$\int_{\theta:\pi(\theta|\mathbf{x})\geq k} \pi(\theta|\mathbf{x})d\theta$$

The value k can be thought of as a horizontal line placed over the posterior density whose intersection(s) with the posterior define regions with probability $1-\alpha$.

Theorem 4.

If the posterior random variable $\theta|\mathbf{x}$ is continuous and unimodal, then the $100(1-\alpha)\%$ HPD credible interval is the unique solution to

$$\int_a^b \pi(\theta|\mathbf{x})d\theta = 1-\alpha$$

$$\pi(a|\mathbf{x}) = \pi(b|\mathbf{x})$$

Example 32 (T3Q15)**.**

The annual aggregate claim amounts (X) for the past 5 years for a risk is

$$203, \quad 204, \quad 212, \quad 183, \quad 223.$$

Suppose $X|\theta \sim N(\theta, v = 390)$ and $\Theta \sim N(\mu = 210, a = 190)$, determine the lower bound of the 90% HPD credibility interval for θ.

Example 33.

The following amounts were paid on a hospital liability policy:

125	132	141	107	133
319	126	104	145	223

The amount of a single payment has the single-parameter Pareto distribution with $\theta = 100$ and α unknown. The prior distribution has the gamma distribution with $\alpha = 2$ and $\theta = 1$. Determine the 95% HPD credible interval for α.

a=1.1832, b = 3.9384

Example 34 (T3Q16)**.**

You are given the following:

- Claim sizes for a given policyholder follow a distribution with density function
$$f(x|\theta) = \frac{5x^4}{\theta^5}, 0 < x < \theta.$$
- The prior distribution of Θ has density function
$$\pi(\theta) = \frac{5}{\theta^6}, \theta > 1.$$

The policyholder experiences three claim sizes of 200, 800, 900. Find a 94% "HPD" credible set for θ.

3.5 The Bühlmann Credibility

The Bayesian method is hard to apply in practice, since it requires not only a hypothesis for the loss distribution of specific risk class, but also a hypothesis for the distribution of that hypothesis over the risk classes.

The Buhlmann method is a least squares approximation of Bayesian result.

Recall

Moments of X$|\Theta$:

1. Expectation of Conditional Means:
$$E[E(X|\theta)] = E(X)$$
2. Variance of Conditional Means:
$$V[E(X|\theta)]$$
3. Expectation of Conditional Variance:
$$E[V(X|\theta)]$$

4. Components of Variance of X:
$$V(X) = E[V(X|\theta)] + V[E(X|\theta)]$$

S In credibility and estimation problems, X is going to be the client's loss severity, loss frequency, or aggregate loss per period. Θ is going to be the distribution of the client's parameter.

The conditional mean is referred to also as the hypothetical mean and is denoted by $\mu(\theta) = E(X|\Theta = \theta)$. The other component is the conditional variance, usually referred to as the process variance. It is denoted by $v(\theta) = V(X|\Theta = \theta)$. Let $\mu = E[\mu(\theta)]$, $v = E[v(\theta)]$, and $a = V[\mu(\theta)]$. The theorem on conditional moments says that

$$E(X) = \mu;$$
$$V(X) = a + v;$$

and

$$Cov(X_i, X_j) = a (i \neq j)$$

The Bühlmann credibility estimate is determined as follows:

1. Model: $f(x|\theta), \pi(\theta)$
2. Parameters: $\mu = E(X)$, $k = \frac{E[V(X|\theta)]}{V[E(X|\theta)]} = \frac{v}{a}$.
3. Observations: $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$
4. Credibility: $Z = \frac{n}{(n+k)}$
5. Estimate: $$E(\widehat{X_{n+1}|}\mathbf{X}) = Z\bar{x} + (1-Z)\mu = \mu + Z(\bar{x} - \mu)$$

Example 35. You are given the following:

- X is a random variable with mean m and variance ν.
- M is a random variable with mean 2 and variance 4.
- ν is a random variable with mean 8 and variance 32.

Determine the value of the Bühlmann credibility factor Z, after three observations of X. $\boxed{3/5}$

Example 36 (T3Q17).

- Suppose the losses $X_1, X_2, \ldots, X_n$ have $E(X_j) = 205$, $V(X_j) = 195$, and $Cor(X_i, X_j) = 0.72$ for $i \neq j$.
- You are given $X_1 = 260$, $X_2 = 190$, $X_3 = 175$.
- The credibility premium for the 5^{th} observation is 260 based on the first 4 observations.

Determine the credibility premium for the 6^{th} observation if $X_5 = 340$.

Example 37 (T3Q18).

You are given the following:

- Two risks have the following severity distribution.

	Probability of	Claim Amount
Amount of Claim	Risk 1	Risk 2
430	0.38	0.44
4,580	0.31	0.33
23,300	0.31	0.23

- Risk 2 is three times as likely as Risk 1 of being observed.
- A claim of 430 is observed, but the observed risk is unknown.

Determine the Buhlmann estimate of the expected value of a second claim amount from the same risk.

Example 38 (T3Q19).

You are given the following:

- The conditional distribution $f_{X|\Theta}(x|\theta)$ is a member of the linear exponential family.
- The prior distribution $\pi(\theta)$ is a conjugate prior for $f_{X|\Theta}(x|\theta)$.
- $E(X) = 3.33$.
- $E(X_2|X_1 = 100) = 27.50$, where X_1 is the value of single observation.
- The expected value of the process variance is 16.67.

Determine the variance of the hypothetical means.

Example 39 (T3Q20).

You are given:

- There are two groups of insureds, A and B. Each group is equally large.
- The number of claims for each member of either group follows a Poisson distribution.
- You are given the following information on mean number of claims for members of each group.

Group	Average Hypothetical Mean	Variance of Hypothetical Mean
A	0.2	0.04
B	0.4	0.25

Calculate the Buhlmann credibility to assign to one of a member. $\boxed{0.3407}$

Example 40 (T3Q21).

You sell individual health coverage. Aggregate claim costs vary for each insured, based on the insured's diet and exercise habits. The following table lists the mean and variance of annual aggregate claim costs per insured.

Exercise Habit	Annual aggregate claim costs			
	Bad Diet		Good Diet	
	Expected Claims	Claim Variance	Expected Claims	Claim Variance
Sedentary	11	21	7	15
Active	9	11	5	8
Total	10.0	17.0	6.0	12.5

80% of insureds have a bad diet and 20% have a good diet. Calculate the Bühlmann credibility factor for one year of experience. $\boxed{0.1908}$

Example 41 (T3Q22).

- Claims sizes for each insured have mean θ and variance v.
- θ varies by insured according to a gamma distribution with $\alpha = 5$, $\beta = 860$.
- For one insured, 2 claim sizes of 1120 and 1840, plus a third unknown claim size, are observed.
- Using Buhlmann credibility with the known information, the expected claim size is 1800.
- The third claim size turns out to be 1510.

Using Buhlmann credibility methods and using all three claim sizes, determine the revised value of the expected claim size.

Example 42 (T3Q23)**.**

You are given:

- An insured's loss size follows a single-parameter Pareto distribution with parameters $\alpha = 4$ and θ.
- The parameter θ varies by insured uniformly on [570,950].
- An insured submits claims of 760, 940, 1170

Using Buhlmann credibility methods, estimate the expected size of the next claims.

Example 43 (T3Q24)**.**

You are given the following:

- The number of claims follow a distribution with mean λ and variance $e^{0.07200000000000001\lambda}$.
- Claim sizes follow a distribution with mean θ and variance $e^{0.25\theta}$.
- the number of claims and claim sizes are independent.
- Λ and Θ have a prior probability distribution with joint density function
$$f(\lambda, \theta) = A\lambda^5\theta^3 e^{-(0.1\lambda+0.2\theta)}, \lambda, \theta > 0$$
where A is a constant.
- During the first year we observed 2 claims and the claim amounts are 550, and 410.
- During the second year we observed 3 claims and the claim amounts are 200, 280 and 270.

Determine the Buhlmann estimate of the expected aggregate loss for the third year. $\boxed{1008.08}$

Example 44 (T3Q25)**.**

You are given;

- The number of claims follows a binomial distribution with parameters 3 and λ.
- Claim sizes follow a distribution with mean σ and variance $2\sigma^2$.
- The number of claims and claim sizes are independent.
- λ and σ have a prior probability distribution with joint density function

  $$f(\lambda, \sigma) = k\lambda^2(8-\sigma)^1, 0 < \lambda < 1, 0 < \sigma < 8.$$

- During the first year we observe 3 claims and the claims are 1, 2, and 3.
- During the second year we observe 2 claims and the claims are 3, and 4.

Determine the Buhlmann estimate of the expected aggregate loss for the third year. $\boxed{6.225}$

Example 45. You are given;

- The number of claims follows a distribution with mean λ and variance 2λ
- Claim sizes follow a distribution with mean σ and variance $2\sigma^2$.
- The number of claims and claim sizes are independent.
- λ and σ have a prior probability distribution with joint density function

  $$f(\lambda,\sigma) = \frac{5}{112}\lambda^2(8-\sigma), 0 < \lambda < 2, \lambda < \sigma < 2\lambda.$$
- During the first year we observe 2 claims and the claims are 1, and 3.
- During the second year we observe 2 claims and the claims are 0, and 2.

Determine the Buhlmann estimate of the expected aggregate loss for the third year. $\boxed{3.7144}$

3.6 Bühlmann-Straub credibility

The Bühlmann model is the simplest of the credibility models because it effectively requires that the past claims experience of a policyholder comprise of i.i.d. components with respect to each past year. An important practical difficulty with this assumption is that it does not allow for variations in the exposure or size.

For example, what if the first year's claims experience of a policyholder reflected only a portion of a year due to unusual policyholder anniversary? For group insurance, what if the size of group changed over time?

The Bühlmann–Straub model is a generalization of the Bühlmann model. There are two generalizations of Bühlmann Credibility:

1. Varying Exposure
2. Generalized Variance of Observations

3.6.1 Varying Exposure

Suppose there are m_j exposure units in year j, $j = 1, \ldots, n$. Here the loss random variables $X_1, X_2, \ldots, X_n$ for n risks are independent, conditionally on Θ, with common mean

$$\mu(\theta) = E(X_j|\Theta = \theta)$$

but with conditional variances

$$V(X_j|\Theta = \theta) = \frac{v(\theta)}{m_j}$$

This model would be appropriate if each X_j were the average m_j independent (conditional on Θ) random variables each with mean $\mu(\theta)$ and variance $v(\theta)$.

Bühlmann-Straub Model

- m_j is a known constant, the exposure for risk j.
- $X_1, X_2, \ldots, X_n$ for n risks are independent (X_j is the average of m_j independent random variables.)
- $E(X_j|\Theta = \theta) = \mu(\theta)$
- $V(X_j|\Theta = \theta) = v(\theta)/m_j$
- $\mu = E[\mu(|\Theta)]$
- $v = E[v(\Theta)]$
- $a = V[\mu(\Theta)]$
- $k = \frac{v}{a}$
- $\begin{aligned} V(X_j) &= E[V(X_j|\Theta)] + V[E(X_j|\Theta)] \\ &= E\left[\frac{v(\Theta)}{m_j}\right] + V[\mu(\Theta)] \\ &= \frac{v}{m_j} + a \end{aligned}$
- $m = m_1 + m_2 + \ldots + m_n$
- $Z = \frac{m}{(m+k)}$

- $\bar{X} = \sum_{j=1}^{n} \frac{m_j}{m} X_j$
- The credibility premium is

$$E(\widehat{X_{1,n+1}}|\mathbf{X}) = Z\bar{X} + (1-Z)\mu$$

- The credibility premium to be charged to the group in year $n+1$) would thus be

$$m_{n+1}[Z\bar{X} + (1-Z)\mu]$$

for m_{n+1} members in the next year.

Example 46 (T3Q26).

Number of claims for each member of a group follows a Poisson distribution with mean λ. λ varies by insured according to a uniform distribution on (0, 0.5).
You are given three years of experience for the group:

Year	Number of members	Number of claims
1	130	3
2	160	6
3	180	6

The group will have 200 members in year 4. Calculate the Bühlmann credibility premium for the group in year 4.
$\boxed{7.46}$

Example 47 (T3Q27).

For each exposure in a group, the hypothetical mean of aggregate losses is Θ and the process variance is $e^{0.41000000000000003\theta}$. Θ varies by group. Its distribution is gamma with $\alpha = 2$ and $\beta = 1.95$. For three years experience from a group, you have the following data:

Year	Exposures	Aggregate Losses
1	24	80
2	26	97
3	35	110

There will be 38 exposures in the group next year. Calculate the Buhlmann-Straub credibility premium for the group.

Example 48 (T3Q28).

You are given the following:

- A portfolio of risks consists of three classes, A, B and C.
- The number of claims per year per risk is the same for each member in a class. The distribution for each class is:

class	Number of Claims			
	0	1	2	3
A	0.51	0.26	0.13	0.1
B	0.34	0.28	0.21	0.17
C	0.41	0.33	0.16	0.1

- The ratio of the number of insureds in Class A, B and C is 6:7:8.
- Customers insure risks, all of which must belong to the same class.

A randomly selected customer has the following experience:

- In year 1 the customer insurers 16 risks and has 3 claims.

- In year 2 the customer insurers 17 risks and has 7 claims.

In year 3 the customer seeks to insure 20 risks. Determine the Buhlmann-Straub estimate of the number of claims for this customer for year 3.

Example 49.

An insurance company writes a book of business that contains several classes of policyholders. You are given:

- The average claim frequency for a policyholder over the entire book is 0.425.
- The variance of the hypothetical means is 0.370.
- The expected value of the process variance is 1.793.

One class of policyholders is selected at random from the book. Nine policyholders are selected at random from this class and are observed to have produced a total of seven claims. Five additional policyholders are selected at random for the same class. Determine the Bühlmann credibility estimate for the total number of claims for these five policyholders. $\boxed{3.2715}$

Example 50 (T3Q29).

For a group dental coverage, you have the following three years of experience from a covered group:

Year	Number of members in group	Number of claims	Aggregate claims
2017	110	150	36,000
2018	140	175	46,000
2019	110	160	46,000

There will be 160 members in the next year. The number of claims per member in any year follows a binomial distribution with parameters $m = 3$ and q. q is the same for all members in the group, but varies over groups, and is distributed uniformly over (0.63, 0.73). Claim size follows a gamma distribution with parameters $\alpha = 12$, $\theta = 34$. Claim sizes and claim counts are independent.

Calculate the Buhlmann-Straub estimate of aggregate claims in the next year.

Example 51 (T3Q30).

You are given five classes of insureds, each of whom may have zero or one claim, with the following probabilities:

	Number of Claims	
Class	0	1
I	0.91	0.09
II	0.69	0.31
III	0.52	0.48
IV	0.34	0.66
V	0.29	0.71

A class is selected at random (with probability 1/5), and 6 insureds are selected at random from the class. The total number of claims is 3. If 13 insureds are selected at random from the same class, estimate the total number of claims using Buhlmann-Straub credibility.

Example 52. For a portfolio of insurance risks, claim frequency per year has a Poisson distribution with mean λ. λ varies by insured, and its distribution has density function $\pi(\lambda) = (\frac{1}{8})(4-\lambda)$, $0 \le \lambda \le 4$. A given risk has the following experience:

Exposure Period	Number of Claims
6 months	0
12 months	2
3 months	0

Determine the Bühlmann-Straub estimate of the number of claims for this risk in the next 9 months. $\boxed{12/13}$

3.6.2 Generalized Variance of Observations

- m_j is a known constant, the exposure for risk j.
- $X_1, X_2, \ldots, X_n$ for n risks are independent (X_j is the average of m_j independent random variables.)
- $$E(X_j|\Theta=\theta)=\mu(\theta)$$ $$V(X_j|\Theta=\theta)=w(\theta)+v(\theta)/m_j$$
- $$\mu=E[\mu(\theta)]$$ $$v=E[v(\theta)]$$ $$a=V[\mu(\theta)]$$
  as before.
- $$m^*=\sum_{j=1}^{n}\frac{m_j}{(v+wm_j)}=\sum_{j=1}^{n}\left(\frac{1}{w+\frac{v}{m_j}}\right)$$

- $$\bar{X}=\frac{\sum_{j=1}^{n}\frac{m_jX_j}{m_jw+v}}{m^*}$$
- $$Z=am^*/(1+am^*)$$
- The credibility premium is $$E(\widehat{X_{1,n+1}}|\mathbf{X})=Z\bar{X}+(1-Z)\mu$$
- The credibility premium to be charged to the group in year $n+1$) would thus be $$m_{n+1}[Z\bar{X}+(1-Z)\mu]$$
  for m_{n+1} members in the next year.
- $\bar{X}=\dfrac{\sum_{j=1}^{n}\left(\frac{\alpha_j}{w+v/m_j}\right)}{m^*}=\dfrac{\sum_{j=1}^{n}\left(\frac{m_jX_j}{m_jw+v}\right)}{m^*}$

Example 53. You are given the following information for a policyholder regarding actual aggregate losses and the conditional variance of aggregate losses in each of 3 years:

	Average losses	Variance
Year 1	10	5
Year 2	9	4
Year 3	11	2

The variance of the hypothetical means of the losses is 1.

Calculate the Buhlmann-Straub credibility factor Z for this experience. $\boxed{19/39}$

Example 54.

An insurance portfolio has two type of risk, A and B, each comprising half the portfolio. The mean claim count per year for a risk of Type A is 0.1. The mean claim count for a risk of Type B is 0.3. For either type, the variance of the claim count is $0.1 + \frac{1}{m_j}$, where m_j is the number of exposures. A group in the portfolio has 40 members with 5 claims in the first year, and 10 members with 0 claims in the second year. Determine the Bühlmann-Straub credibility estimate for the number of claims per member in the third year. $\boxed{0.1858}$

Example 55 (T3Q31).

For a portfolio of insurance risks, average aggregate losses per exposure have mean θ and variance $9000 + \frac{50000}{m_j}$, where m_j is the number of exposures in year j. θ varies by risk, and follows a log normal distribution with parameters $\mu = 5$, $\sigma = 0.7$. The following is the experience for this risk over 5 years:

Year	Number of Exposures	Average Losses Per Expo
1	20	1,800
2	35	2,200
3	60	3,200
4	65	3,400
5	90	3,700

Determine the Buhlmann-Straub estimate of average aggregate losses per exposure in the next year for this risk.

3.7 Bühlmann As Least squares estimate of Bayes

Bühlmann has shown that the credibility estimate

$$E(\widehat{X_{n+1}|\mathbf{X}}) = Z\bar{X} + (1-Z)\mu$$

is the best linear approximation to Bayesian estimate of pure premium

$$[E(X_{n+1}|\mathbf{x})].$$

If we want to estimate Y_i by $\hat{Y}_i$ which is a linear function of X_i,

$$\hat{Y}_i = aX_i + b,$$

in such a way as to minimizes the weighted least square difference, where p_i is the weight of observation i, then, treating the p_i as probabilities, the formula are

$$a = \frac{cov(X,Y)}{V(X)} = \frac{\sum p_i X_i Y_i - \bar{X}\bar{Y}}{\sum p_i X_i^2 - \bar{X}^2}$$

and

$$b = E(Y) - aE(X)$$

where

- X_i are the observations
- Y_i are the Bayesian predictions
- $\hat{Y}_i$ are the Bühlmann predictions.

So, $a = Z$, the credibility factor. Also, the overall mean doesn't change and the mean of Bühlmann prediction is the true mean, so

$$E(Y) = E(X) = E(\hat{Y}_i).$$

This implies that

$$b = (1 - Z)E(X).$$

Example 56. You are given the following information about the Bayesian estimates of an event:

Outcome	Initial Probability of Outcome $P(X_1)$	Bayesian Estimate $E(X_2\|X_1)$
0	0.5	1
2	0.25	2
8	0.25	-

Determine the Bühlmann estimates of $E(X_2|X_1)$ for each possible outcome. □

Example 57. You are given the following information about the Bayesian estimates of an event:

Outcome(R_i)	Initial Probability of Outcome $P(R_1)$	Bayesian Estimate $E(E_i\|R_i)$
0	2/3	7/4
2	2/9	55/24
14	1/9	35/12

The Bühlmann credibility factor after one experiment is $\frac{1}{12}$. Determine the values for the parameters a and b that minimize the expression $\sum_{i=1}^{3}(P_i(aR_i + b - E_i)^2$: $\boxed{1/12,\ 11/6}$

Example 58 (T3Q32)**.**

You are given the following information about a credibility model:

Observed Losses	Probability	Bayesian Estimate
10	2/10	28.61
12	1/10	29.21
24	1/10	32.81
40	2/10	37.61
50	4/10	y_5

Determine the Buhmann credibility estimate of the second observation, given that the first observation is 24.