MEME15203 Statistical Inference Marking Guide

Assignment 2

UNIVERSITI TUNKU ABDUL RAHMAN

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Q1. A random sample of size n is taken from a distribution with probability density funtion (pdf)

$$f(x) = \frac{4x^3}{\theta^4}, 0 < x < \theta, \text{zero otherwise.}$$

(a) Find the Maximum Likelihood Estimator(MLE) of θ. Call it $\hat{\theta}$.

Ans.
$L(\theta) = 4^n \prod_{i=1}^{n} x_i^3 \theta^{-4n}, x_{(n)} < \theta$
Since $L(\theta)$ is a decreasing function of θ, hence, the MLE of θ is $\boxed{\hat{\theta} = X_{(n)}}$

(b) Find the MLE of the median of the distribution.

Ans.
Let m be the median,
$\int_0^m \frac{4x^3}{\theta^4} dx = \frac{1}{2}$
$\frac{1}{\theta^4}[x^4]_0^m = \frac{1}{2}$
$\frac{m^4}{\theta^4} = \frac{1}{2}$
$m = \frac{\theta}{2^{\frac{1}{4}}}$
So, the MLE of m is $\hat{m} = \frac{\hat{\theta}}{2^{\frac{1}{4}}} = \boxed{\frac{X_{(n)}}{2^{\frac{1}{4}}}}$

(c) Find the Method of Moment Estimator(MME) of θ. Call it $\tilde{\theta}$.

Ans.
$E(X) = \int_0^\theta \frac{4x^4}{\theta^4} dx = \frac{4}{5\theta^4}[x^5]_0^\theta = \frac{4\theta}{5}$
$\frac{4\tilde{\theta}}{5} = \bar{X}$
$\boxed{\tilde{\theta} = \frac{5\bar{X}}{4}}$

(d) Find the constant c so that $c\hat{\theta}$ becomes an unbiased estimator of θ.

Ans.

$F_X(x) = \int_0^x \frac{4u^3}{\theta^4} du = \frac{1}{\theta^4}[u^4]_0^x = \left[\frac{x}{\theta}\right]^4$

$g_n(x) = n[F_X(x)]^{n-1} f_X(x) = n(\frac{x}{\theta})^{4n-4}(\frac{4x^3}{\theta^4}) = \frac{4n}{\theta^{4n}} x^{4n-1}$

$E(\hat{\Theta}) = E(X_{(n)}) = \int_0^\theta x g_n(x) dx = \int_0^\theta \frac{4n}{\theta^{4n}} x^{4n} dx = \frac{4n}{\theta^{4n}}[\frac{x^{4n+1}}{4n+1}]_0^\theta = \frac{4n}{4n+1}\theta$

$E(c\hat{\Theta}) = \theta$

$cE(\hat{\Theta}) = \theta$

$c(\frac{4n}{4n+1}\theta) = \theta$

$c = \frac{4n+1}{4n}$

(e) Find the Mean Square Error(MSE) of $\hat{\theta}$.

Ans.

$E(\hat{\Theta}^2) = E(X_{(n)}^2) = \int_0^\theta x^4 g_n(x) dx = \int_0^\theta \frac{4n}{\theta^{4n}} x^{4n+1} dx = \frac{4n}{\theta^{4n}}[\frac{x^{4n+2}}{4n+2}]_0^\theta = \frac{4n}{4n+2}\theta^2$

$MSE(\hat{\Theta}) = E(\hat{\Theta} - \theta)^2 = E(\hat{\Theta}^2 - 2\hat{\Theta}\theta + \theta^2) = \frac{4n}{4n+2}\theta^2 - 2(\frac{4n}{4n+1}\theta)(\theta) + \theta^2 = \left[\frac{4n}{4n+2} - 2(\frac{4n}{4n+1}) + 1)\right]\theta^2 = \left[\frac{4n(4n+1) - 8n(4n+2) + (4n+1)(4n+2)}{(4n+1)(4n+2)})\right]\theta^2 = \left[\frac{16n^2+4n-32n^2-16n+16n^2+3(4)n+2}{(4n+1)(4n+2)})\right]\theta^2 = \boxed{\frac{2\theta^2}{(4n+1)(4n+2)}}$

(f) Find the MSE of $\tilde{\theta}$.

Ans.

$E(X^2) = \int_0^\theta \frac{4x^5}{\theta^4} dx = \frac{4}{6\theta^4}[x^6]_0^\theta = \frac{4\theta^2}{6}$

$V(X) = \frac{4\theta^2}{6} - \frac{16\theta^2}{25} = \frac{4\theta^2}{150} = (2/75)\theta^2$

$E(\tilde{\Theta}) = E(\frac{5\bar{X}}{4}) = \frac{5}{4}E(\bar{X}) = \frac{5}{4}(\frac{4}{5}\theta) = \theta$

Thus, $MSE(\tilde{\Theta}) = V(\tilde{\Theta}) = \frac{25}{16}V(\bar{X}) = \frac{25}{16}\left(\frac{4\theta^2}{150n}\right) = \boxed{(1/24)\theta^2}$

(30 marks)

Q2. Consider a random sample of size n from a distribution with discrete pdf

$$f(x; p) = p(1-p)^x; x = 0, 1, \ldots, \text{ zero otherwise.}$$

(a) Find the MLE of p.

(b) Find the MLE of $\theta = \frac{1-p}{p}$.

(c) Find the CRLB for variance of unbiased estimators of θ.

(d) Is MLE of θ a UMVUE?

(e) Is MLE of θ MSE consistent?

(f) Find the asymptotic distribution of the MLE of θ.

(30 marks)

Ans.

(a) $L(p) = p^n(1-p)^{\sum x_i}$
$l(p) = n\ln(p) + \sum x_i \ln(1-p)$
$l'(p) = \frac{n}{p} - \frac{\sum x_i}{1-p} = 0$
$\frac{n}{\hat{p}} = \frac{\sum x_i}{1-\hat{p}}$
$n - n\hat{p} = \sum x_i \hat{p}$
$\hat{p} = \frac{n}{n+\sum x_i} = \frac{n}{n+n\bar{x}} = \frac{1}{1+\bar{x}}$

(b) By invariance property, $\hat{\theta} = \frac{1-\frac{1}{1+\bar{x}}}{\frac{1}{1+\bar{x}}} = \bar{x}$

(c) $\ln(f(x;p)) = \ln(p) + x\ln(1-p)$
$\frac{\partial \ln(f(x;p))}{\partial p} = \frac{1}{p} - \frac{x}{1-p}$
$\frac{\partial^2 \ln(f(x;p))}{\partial p^2} = -\frac{1}{p^2} + \frac{x}{(1-p)^2}$
Let $Y = X+1$, the $f_Y(y) = f_X(y-1) = p(1-P)^{y-1}, y = 1, 2, \ldots$
Thus $Y \sim Geo(p)$ and $E(Y) = \frac{1}{p}$ and $V(Y) = \frac{1-p}{p^2}$, so
$E(X) = E(Y) - 1 = \frac{1}{p} - 1 = \frac{1-p}{p}$ and $V(X) = V(Y) = \frac{1-p}{p^2}$
$E\left[\frac{\partial^2 \ln(f(x;p))}{\partial p^2}\right] = -\frac{1}{p^2} + \frac{E(X)}{(1-p)^2} = -\frac{1}{p^2} + \frac{\frac{1-p}{p}}{(1-p)^2} = -\frac{1}{p^2} + \frac{1}{p(1-p)} = \frac{-1}{p^2(1-p)}$
$\tau(p) = \frac{1-p}{p}$, $\tau'(p) = -\frac{1}{p^2}$.
The CRLB for $\theta = \frac{1-p}{p}$ is $\frac{[\tau'(p)]^2}{-nE\left[\frac{\partial^2 \ln(f(x;p))}{\partial p^2}\right]} = \frac{1/p^4}{-n\frac{-1}{p^2(1-p)}} = \frac{1-p}{np^2}$

(d) $V(\bar{X}) = \frac{V(X)}{n} = \frac{1-p}{np^2}$. Since $V(\bar{X})$ attined the CRLB for θ, thus $\hat{\theta} = \bar{X}$ is the UMVUE of θ.

(e) $\lim_{n\to\infty} V(\bar{X}) = \lim_{n\to\infty} \frac{1-p}{np^2} = 0$, Thus $\hat{\theta}$ is MSE consistent.

(f) $\bar{X} \sim N\left(\frac{1-p}{p}, \frac{1-p}{np^2}\right)$

Q3. Let $X_1, \ldots, X_n$ be a random sample from the uniform distribution on the interval $(0, \theta)$, where $\theta > 0$ is unknown. Let the prior of θ be the log-normal distribution with parameter (μ, σ), where $\mu \in R$ and $\sigma > 0$ are known constants. Find the posterior density of $\ln(\theta)$.

(15 marks)

Ans.

$X_i\theta \sim U(0,\theta)$, $f(x_i|\theta) = \frac{1}{\theta}, 0 < x_i < \theta$
$\Theta \sim LN(\mu, \sigma)$
Let $V = \ln(\theta)$, then $V \sim N(\mu, \sigma^2)$, $\pi(v) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(v-\mu)^2/2\sigma^2}, v \in R$

$$\begin{aligned}\pi(v|\mathbf{x}) &= k\theta^{-n} e^{-(v^2-2v\mu)/2\sigma^2} \\ &= ke^{-nv} e^{-(v^2-2v\mu)/2\sigma^2} \\ &= ke^{-[v^2-2(\mu-n\sigma^2)v]/2\sigma^2}\end{aligned}$$

$$= ke^{-[v-(\mu-n\sigma^2)]^2/2\sigma^2}, v > \ln(x_{(n)}), \text{ where } X_{(n)} = \max(X_i)$$

$\int_{\ln(x_{(n)})}^{\infty} ke^{-[v-(\mu-n\sigma^2)]^2/2\sigma^2} dv = 1$
$k\sqrt{2\pi}\sigma \int_{\ln(x_{(n)})}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-[v-(\mu-n\sigma^2)]^2/2\sigma^2} dv = 1$
$k\sqrt{2\pi}\sigma \left[1 - \Phi\left(\frac{\ln(x_{(n)})-(\mu-n\sigma^2)}{\sigma}\right)\right] = 1$
$k = \frac{1}{\sqrt{2\pi}\sigma\left[\Phi\left(\frac{(\mu-n\sigma^2-\ln(x_{(n)})}{\sigma}\right)\right]}$
$\pi(v|\mathbf{x}) = \frac{1}{\sqrt{2\pi}\sigma\left[\Phi\left(\frac{(\mu-n\sigma^2-\ln(x_{(n)})}{\sigma}\right)\right]} e^{-[v-(\mu-n\sigma^2)]^2/2\sigma^2}, v > \ln(x_{(n)})$

Q4. Let $X_1, X_2, \ldots, X_n$ denote a random sample from the density function given by

$$f(x) = \begin{cases} \frac{3}{\theta}x^2 e^{-x^3/\theta}, & \theta > 0, x > 0, \\ 0, & \text{otherwise} \end{cases}$$

(a) find the MME of θ.

(b) Find the MLE of θ.

(c) Find the CRLB of θ.

(15 marks)

Ans.

(a) $X \sim WEI(\tau = 3, \beta = \theta^{1/3})$
$E(X) = \theta^{1/3}\Gamma(1 + 1/3) = \theta^{1/3}\Gamma(4/3)$
$\tilde{\theta}^{1/3}\Gamma(4/3) = \bar{X}$
$\tilde{\theta} = \left(\frac{\bar{X}}{\Gamma(4/3)}\right)^3$

(b) $\ln L = n\ln 3 - n\ln\theta + (3-1)\sum_{i=1}^{n} \ln x_i - \sum_{i=1}^{n} \frac{x_i^3}{\theta}$
$\frac{dL}{d\theta} = \frac{-n}{\theta} + \frac{\sum_{i=1}^{n} x_i^3}{\theta^2} = 0$
$\hat{\theta} = \frac{\sum_{i=1}^{n} x_i^3}{n}$
let $u = \sum_{i=1}^{n} x_i^3$, then
$\frac{dL^2}{d\theta^2} = \frac{n}{\theta^2} - \frac{2u}{\theta^3}$
$\frac{dL^2}{d\theta^2}\Big|_{\theta=\hat{\theta}} = \frac{n}{(u/n)^2} - \frac{2u}{(u/n)^3} = \frac{n^3}{u^2} - \frac{2n^3}{u^2} = -\frac{n^3}{u^2} < 0$
The the MLE of θ is $\hat{\theta} = \frac{\sum_{i=1}^{n} x_i^3}{n}$

(c) Let $u = x^3$, $w(u) = x = u^{1/3}$, $w'(u) = \frac{1}{3}u^{1/3-1}$
$f_U(u) = \frac{1}{\theta}(3)(u^{1/3})^2 e^{-(u^{1/3})^3/\theta}(\frac{1}{3}u^{1/3-1}) = \frac{1}{\theta}u^{-u/\theta}$
$\Rightarrow U \sim EXP(\theta)$
$\tau(\theta) = \theta$
$\ln f(x;\theta) = -\ln\theta + \ln 3 + 2\ln x - x^3/\theta$
$\frac{\partial \ln f(x;\theta)}{\partial\theta} = -\frac{1}{\theta} + \frac{x^3}{\theta^2}$

$\frac{\partial^2 \ln f(x;\theta)}{\partial\theta^2} = \frac{1}{\theta^2} - \frac{2x^3}{\theta^3}$
$E\left(\frac{\partial^2 \ln f(x;\theta)}{\partial\theta^2}\right) = \frac{1}{\theta^2} - \frac{2E(x^3)}{\theta^3} = \frac{1}{\theta^2} - \frac{2\theta}{\theta^3} = -\frac{1}{\theta^2}$

$CRLB = \frac{\tau'(\theta)}{-nE\left(\frac{\partial^2 \ln f(x;\theta)}{\partial\theta^2}\right)} = \frac{1}{-n(-1/\theta^2)} = \frac{\theta^2}{n}$

Q5. Suppose $X|\theta \sim U(\theta - \frac{1}{2}, \theta + \frac{1}{2})$ and that a prior distribution of θ is $N(0,1)$. Find the Bayes estimator of θ under squared error loss.

(10 marks)

Ans.

$f(x|\theta) = 1, \theta - \frac{1}{2} < x < \theta + \frac{1}{2}$
$\pi(\theta) = \frac{1}{\sqrt{2\pi}} e^{-\theta^2/2}, \theta \in \mathbb{R}$
$\pi(\theta|x) = ke^{-\theta^2/2}, x - \frac{1}{2} < \theta < x + \frac{1}{2}$
$\int_{x-\frac{1}{2}}^{x+\frac{1}{2}} ke^{-\theta^2/2} d\theta = 1$
$k\sqrt{2\pi}[\Phi(x+\frac{1}{2}) - \Phi(x-\frac{1}{2})] = 1$
$k = \frac{1}{\sqrt{2\pi}[\Phi(x+\frac{1}{2}) - \Phi(x-\frac{1}{2})]}$, thus
$\pi(\theta|x) = \frac{e^{-\theta^2/2}}{\sqrt{2\pi}[\Phi(x+\frac{1}{2}) - \Phi(x-\frac{1}{2})]}, x - \frac{1}{2} < \theta < x + \frac{1}{2}$
Under the square error loss, the Bayes estimator of θ is the posterior mean.
$E(\Theta) = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \frac{\theta e^{-\theta^2/2}}{\sqrt{2\pi}[\Phi(x+\frac{1}{2}) - \Phi(x-\frac{1}{2})]} d\theta = \frac{1}{\sqrt{2\pi}[\Phi(x+\frac{1}{2}) - \Phi(x-\frac{1}{2})]}\left[-e^{-\theta^2/2}\right]_{x-\frac{1}{2}}^{x+\frac{1}{2}} =$
$\frac{e^{-\frac{1}{2}(x-\frac{1}{2})^2} - e^{-\frac{1}{2}(x+\frac{1}{2})^2}}{\sqrt{2\pi}[\Phi(x+\frac{1}{2}) - \Phi(x-\frac{1}{2})]}$ $\boxed{1}$