MEME15203 Statistical Inference Marking Guide

Assignment 1

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Q1. Let X_1, X_2 be two random variables with joint pdf $f(x_1, x_2) = \frac{1}{7!(70^9)} x_1^7 e^{-x_2/70}$, for $0 < x_1 < x_2 < \infty$, zero otherwise.

(a) Determine the joint mgf of X_1, X_2, $M_{X_1,X_2}(t_1, t_2)$.

Ans.

$$
\begin{aligned}
&M_{X_1,X_2}(t_1,t_2)\\
&= E(e^{t_1X_1+t_2X_2})\\
&= \int_0^\infty \int_{x_1}^\infty e^{t_1x_1+t_2x_2}\left(\tfrac{1}{7!(70^9)}\right)x_1^7 e^{-x_2/70}dx_2dx_1\\
&= \int_0^\infty \left(\tfrac{1}{7!(70^9)}\right)x_1^7 e^{t_1x_1}\int_{x_1}^\infty e^{-x_2(1/70-t_2)}dx_2dx_1\\
&= \int_0^\infty \left(\tfrac{1}{7!(70^9)}\right)x_1^7 e^{t_1x_1}\frac{70e^{-x_1\left(\frac{1-70t_2}{70}\right)}}{1-70t_2}dx_1\\
&= \left(\tfrac{70}{7!(70^9)}\right)\left(\tfrac{1}{1-70t_2}\right)\int_0^\infty x_1^7 e^{-x_1\left(\frac{1-70t_1-70t_2}{70}\right)}dx_1\\
&= \left(\tfrac{70}{7!(70^9)}\right)\left(\tfrac{1}{1-70t_2}\right)\frac{7!(70^8)}{(1-70t_1-70t_2)^8}\\
&= \frac{1}{(1-70t_2)(1-70t_1-70t_2)^8}
\end{aligned}
$$

provided that $70t_1 + 70t_2 < 1$ and $70t_2 < 1$.

(b) Determine the marginal distribution of X_1.

Ans.

$M_{X_1}(t_1, 0) = \frac{1}{(1-70(0))(1-70t_1-70(0))^8} = \frac{1}{(1-70t_1)^8}$

$\Rightarrow X_1 \sim GAM(\alpha = 8, \theta = 70)$

(c) Determine the marginal distribution of X_2.

Ans.

$M_{X_2}(0, t_2) = \frac{1}{(1-70t_2)(1-70(0)-70t_2)^8} = \frac{1}{(1-70t_2)^9}$

$\Rightarrow X_2 \sim GAM(\alpha = 9, \theta = 70)$

(d) Are X_1 and X_2 independent?

Ans.
$M(t_1, t_2) \neq M(t_1, 0)M(0, t_2)$, thus X_1 and X_2 are not independent.

(10 marks)

Q2. Suppose that the random variables X_1 and X_2 have joint probability density function $f(x_1, x_2)$ given by

$$f(x_1, x_2) = \begin{cases} \frac{30}{2}x_1^4 x_2, & 0 \leq x_1 \leq x_2, x_1 + x_2 \leq 2 \\ 0, & \text{otherwise} \end{cases}.$$

(a) Show that the marginal density of X_1 is a beta density with $a = 5$ and $b = 2$.

Ans.
$f_1(x_1)$
$= \int_{x_1}^{2-x_1} \frac{30}{2}x_1^4 x_2 dx_2$
$= \frac{30}{2}x_1^4 \int_{x_1}^{2-x_1} x_2 dx_2$
$= \frac{30}{2}x_1^4 [\frac{x_2^2}{2}]_{x_1}^{2-x_1}]$
$= \frac{30}{4}x_1^4 [(2 - x_1)^2 - x_1^2]$
$= \frac{30}{4}x_1^4 (4 - 4x_1)$
$= 30x_1^4(1 - x_1), 0 \leq x_1 \leq 1$
$\Rightarrow X_1 \sim Beta(a = 5, b = 2)$

(b) Derive the conditional density of X_2 given $X_1 = x_1$.

Ans.
$f(x_2|x_1) = kx_2, x_1 \leq x_2 \leq 2 - x_1$
$k \int_{x_1}^{2-x_1} x_2 dx_2 = 1$
$k[\frac{x_2^2}{2}]_{x_1}^{2-x_1} = 1$
$\frac{k}{2}[(2 - x_1)^2 - x_1^2] = 1$
$\frac{k}{2}(4 - 4x_1) = 1$
$k = \frac{1}{2(1-x_1)}$
$\therefore f(x_2|x_1) = \frac{x_2}{2(1-x_1)}, x_1 \leq x_2 \leq 2 - x_1$

(c) Find $P(X_2 < 1.1|X_1 = 0.6)$.

Ans.
$f(x_2|x_1 = 0.6) = \frac{x_2}{2(1-0.6)} = \frac{x_2}{0.8}, 0.6 \leq x_2 \leq 1.4$
$P(X_2 < 1.1|X_1 = 0.6)$

$= \int_{0.6}^{1.1} \frac{x_2}{0.8} dx_2$
$= \frac{1}{2(0.8)}[x_2^2]0.6^{1.1}$
$= \frac{1}{2(0.8)}[1.1^2 - 0.6^2]$
$= 0.53125$

(d) Derive the marginal density of X_2.

Ans.

For $0 \le x_2 < 1$,
$f_2(y_2)$
$= \int_0^{x_2} \frac{30}{2} x_1^4 x_2 dx_1$
$= \frac{30}{2} x_2 [\frac{x_1^5}{5}]_0^{x_2}$
$= \frac{30}{2} x_2 [\frac{x_2^5}{5}]$
$= 3.0x_2^6$

For $1 \le x_2 < 2$,
$f_2(y_2)$
$= \int_0^{2-x_2} \frac{30}{2} x_1^4 x_2 dx_1$
$= \frac{30}{2} x_2 [\frac{x_1^5}{5}]_0^{2-x_2}$
$= \frac{30}{2} x_2 [\frac{(2-x_2)^5}{5}]$
$= 3.0x_2(2 - x_2)^5$

(16 marks)

Q3. Show that if $X = (X_1, X_2, \ldots, X_k)$ have a multinomial distribution with parameters n and $p_1, p_2, \ldots, p_k$, then

(a) $E(X_i) = np_i$, $V(X_i) = np_i q_i$

(b) $Cov(X_s, X_t) = -np_s p_t$, if $s \ne t$

(10 marks)

Ans.

(a) Let X be the number of trials falling into cell i. Let all other cells excluding cell i combined into a single cell. Then $X_i \sim Bin(n, p_i)$. Thus $E(X_i) = np_i$, $V(X_i) = np_i q_i$.

(b) Define for $s \ne t$,

$$U_i = \begin{cases} 1, & \text{if trial } i \text{ resulting in class} s \\ 0, & \text{otherwise} \end{cases}$$

$$W_j = \begin{cases} 1, & \text{if trial } j \text{ resulting in class} t \\ 0, & \text{otherwise} \end{cases}$$

Then $X_s = \sum_{i=1}^n U_i$, $X_t = \sum_{j=1}^n W_j$

Notice that U_i and W_j cannot both equal 1. Thus $U_iW_j = 0$ and $E(U_iW_j) = 0$

$E(U_i) = p_s$ and $E(W_j) = p_t$.

$Cov(U_i, W_j) = 0$ if $i \neq j$ because the trials are independent.

$Cov(U_i, W_j) = E(U_iW_j) - E(U_i)E(W_j) = -p_sp_t$

$Cov(X_s, X_t) = \sum_i \sum_j Cov(U_i, W_j) = \sum_{i=j=1}^n Cov(U_i, W_j) + \sum\sum_{i\neq j} Cov(U_i, W_j) = \sum_{i=j=1}^n -p_sp_t - 0 = -np_sp_t$

Q4. Show that if $(X, Y) \sim BVN(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$, then conditional on $X = x$,

$$Y|x \sim N(\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x - \mu_1), \sigma_2^2(1 - \rho^2)).$$

(4 marks)

Ans.

$f(x, y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}[(\frac{x-\mu_1}{\sigma_1})^2 + (\frac{y-\mu_2}{\sigma_2})^2 - 2\rho(\frac{x-\mu_1}{\sigma_1})(\frac{y-\mu_2}{\sigma_2})]\right], x \in R, y \in R$

$$\begin{aligned} f(y|x) &\propto \exp\left[-\frac{1}{2(1-\rho^2)}[(\frac{y-\mu_2}{\sigma_2})^2 - 2\rho(\frac{x-\mu_1}{\sigma_1})(\frac{y-\mu_2}{\sigma_2})]\right] \\ &= c\exp\left[-\frac{1}{2\sigma_2^2(1-\rho^2)}[(y-\mu_2)^2 - 2\rho\frac{\sigma_2(x-\mu_1)(y-\mu_2)}{\sigma_1}]\right] \\ &= c_1\exp\left[-\frac{1}{2\sigma_2^2(1-\rho^2)}[(y-\mu_2) - \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)]^2\right] \\ &= c_1\exp\left[-\frac{1}{2\sigma_2^2(1-\rho^2)}[y - (\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1))]^2\right] \end{aligned}$$

$\Rightarrow Y|x \sim N(\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x - \mu_1), \sigma_2^2(1 - \rho^2))$

Q5. Suppose that X_1 and X_2 denote a random sample of sixe 2 from a gamma distribution $X_i \sim GAM(0.5, 5)$. Find the pdf of $\frac{X_1}{X_2}$.

(4 marks)

Ans.

$X_i \sim GAM(0.5, 5)$

$f(x_i) = \frac{1}{\Gamma(0.5)5^{0.5}} x_i^{-0.5}e^{-x_i/5}$

$f(x_i, x_2) = \frac{1}{(\Gamma(0.5))^2 5^{2(0.5)}} x_1^{-0.5}x_2^{-0.5}e^{-(x_1+x_2)/5} = \frac{1}{5\pi}x_1^{-0.5}x_2^{-0.5}e^{-(x_1+x_2)/5}$

let $V = X_2$ and $W = \frac{X_1}{X_2}$. Then this corresponds to the transformation $X_1 = WV$ and $X_2 = V$ which have unique solutions $h_1(v,w) = x_1 = wv$ and $h_2(v,w) = x_2 = v$,
$J = \begin{vmatrix} \frac{\partial h_1}{\partial w} & \frac{\partial h_1}{\partial v} \\ \frac{\partial h_2}{\partial w} & \frac{\partial h_2}{\partial v} \end{vmatrix} = \begin{vmatrix} v & w \\ 0 & 1 \end{vmatrix} = v$
$f_{V,W}(v,w) = f_{V,W}(v,wv)|J| = \frac{1}{5\pi}(wv)^{-0.5}v^{-0.5}e^{-(wv+v)/5}|v| = \frac{1}{5\pi}(w)^{-0.5}e^{-v(w+1)/5}$, $v > 0, w > 0$
$f_W(w) = \frac{1}{5\pi}(w)^{-0.5}\int_0^\infty e^{-v(w+1)/5}dv = \frac{1}{5\pi}w^{-0.5}\frac{5}{w+1} = \frac{w^{-0.5}(1+w)^{-1}}{\pi}, w > 0$, 0 otherwise.

Q6. Suppose that $X_1, X_2, \ldots, X_{11}$ denote a random sample of size 11 from a gamma distribution $X_i \sim GAM(\alpha = \frac{1}{11}, \theta = 8)$. Find the pdf of $U = \sqrt[11]{X_1 + X_2 + \cdots + X_{11}}$ and state the name of the distribution of U.

(4 marks)

Ans.

Let $S = X_1 + X_2 + \cdots + X_{11}$, then $S \sim GAM(1,8) \sim Exp(8)$
$f_S(s) = \frac{1}{8}e^{-s/8}, s > 0$ and 0 otherwise
$U = S^{\frac{1}{11}}$. This corresponds to the tranformation of $u = s^{\frac{1}{11}}$ which has unique solution $s = w(u) = u^{11}$ and $\frac{ds}{du} = 11u^{10}$. So
$f_U(u) = f_S(u^{11})|11u^{10}| = \frac{1}{8}e^{-u^{11}/8}(11u^{10}) = \frac{11}{8}u^{10}e^{-u^{11}/8}, u > 0$ and 0 otherwise.
$\Rightarrow U \sim Weibull(\tau = 11, \theta = \sqrt[11]{8})$

Q7. Let X_1 and X_2 be a random sample of size $n = 2$ from a continuous distribution with pdf of the form $f(x) = 3x^2$ if $0 < x < 1$ and zero otherwise.

(a) Find the joint pdf of $Y_1 = \min(X_1, X_2)$ and $Y_2 = \max(X_1, X_2)$.

(b) Find the pdf of the sample range $R = Y_2 - Y_1$.

(10 marks)

Ans.

$f_{Y_1,Y_2}(y_1,y_2) = 2!f(y_1)f(y_2) = 2!(3y_1^2)(3y_2^2) = 18y_1^2y_2^2, 0 < y_1 < y_2 < 1$
Making the tranformation $R = Y_2 - Y_1$, $S = Y_1$, yields the inverse tranformation $y_1 = s$, $y_2 = r + s$, and $|J| = 1$. Thus the joint pdf of R and S is
$f_{R,S}(r,s) = f_{Y_1,Y_2}(s, s+t)|J| = 18s^2(r+s)^2, 0 < s < 1 - r, 0 < r < 1$

$$\begin{aligned} f_R(r) &= \int_0^{1-r} 18s^2(r+s)^2 ds \\ &= \int_0^{1-r} 18s^2(r^2 + 2rs + s^2)ds \end{aligned}$$

$$= 18[\frac{r^2s^3}{3} + \frac{rs^4}{2} + \frac{s^5}{5}]_0^{1-r}$$
$$= 18[\frac{r^2(1-r)^3}{3} + \frac{r(1-r)^4}{2} + \frac{(1-r)^5}{5}]$$

Q8. Let Y_9 denote the 9^{th} smallest item of a random sample of size n from a distribution of the continuous type that has cdf $F_X(x)$ and pdf $f_X(x) = F'_X(x)$. Find the limiting distribution of $W_n = nF_{Y_9}(y)$.

(4 marks)

Ans.

$g_9(y) = \frac{n(n-1)(n-2)\cdots(n-8)}{(8)!} F_X^8(y)[1 - F_X(y)]^{n-9} f_X(y), -\infty < y < \infty$
$w = nF_{Y_9}(y)$, $\frac{dy}{dw} = \frac{1}{nf_{Y_9}(y)}$
$h(w) = \frac{(n-1)(n-2)\cdots(n-8)}{(8)!n^8} w^8(1 - w/n)^{n-9}, 0 < w < n$
$\lim_{n\to\infty} H_n(w) = \lim_{n\to\infty} \int_0^w \frac{(n-1)(n-2)\cdots(n-8)}{(8)!n^8} u^8(1 - u/n)^{n-9} du = \int_0^w \frac{1}{8!} u^8 e^{-u} du$
which is cdf of $Gamma(\alpha = 9, \theta = 1)$ distribution.

Q9. Consider a random sample from a gamma distribution, $X_i \sim GAM(\alpha, \theta)$. Find the asymtotic normal distribution of $Y_n = \bar{X}_n^3$.

(4 marks)

Ans.

$E(\bar{X}_n) = \alpha\theta$, $V(\bar{X}_n) = \frac{1}{n}V(X) = \frac{\alpha\theta^2}{n}$
By CLT, $\bar{X}_n \sim N\left(m = \alpha\theta, \frac{c^2}{n} = \frac{\alpha\theta^2}{n}\right)$
$g(\alpha\theta) = (\alpha\theta)^3$, $g'(\theta) = 3(\alpha\theta)^2$, $[g'(\alpha\theta)]^2 = 9(\alpha\theta)^4$, thus, by Theorem 13,
$\frac{c^2[g'(m)]^2}{n} = \frac{\alpha\theta^2(9(\alpha\theta)^4)}{n} = \frac{9\alpha^5\theta^6}{n}$
$Y_n \sim N\left((\alpha\theta)^3, \frac{9\alpha^5\theta^6}{n}\right)$

Q10. Consider a random sample from a Gamma distribution with parameters α and θ. Let $W_i = e^{X_i}$ and $V_n = W_1 \times W_2 \times \cdots W_n$. $V_n^{\frac{1}{n}}$ converges in probability to a constant, identify the constant.

(4 marks)

Ans.

$E(\bar{X}_n) = \alpha\theta$, $V(\bar{X}_n) = \frac{1}{n}V(X) = \frac{\alpha\theta^2}{n}$
$P\left[|\overline{X}_n - \alpha\theta| \geq \epsilon\right] < \frac{\alpha\theta^2}{n\epsilon^2} \to 0$

$\therefore \bar{X}_n \xrightarrow{P} \alpha\theta$
$(V_n)^{1/n} = (W_1 \times W_2 \times \cdots \times W_n)^{1/n} = e^{\bar{X}_n}$
Thus, $(V_n)^{1/n} \xrightarrow{P} e^{\alpha\theta}$

Q11. Let $Y_n \sim GAM(7n, \theta)$. Find the limiting distribution of $Z_n = \frac{Y_n - 7n\theta}{\sqrt{7n}\theta}$ as $n \to \infty$, using moment generating function.

(4 marks)

Ans.

$M_{Y_n}(t) = (1 - \theta t)^{-7n}$

$$\begin{aligned} M_{Z_n}(t) &= M_{\frac{Y_n - 7n\theta}{\sqrt{7n}\theta}}(t) \\ &= e^{-\frac{7n}{\sqrt{7n}}t} M_{Y_n}\left(\frac{1}{\sqrt{7n}\theta}t\right) \\ &= e^{-\frac{7n}{\sqrt{7n}}t}\left(1 - \frac{1}{\sqrt{7n}}t\right)^{-7n} \\ &= e^{-\frac{7n}{\sqrt{7n}}t} e^{-7n\ln\left(1 - \frac{1}{\sqrt{7n}}t\right)} \\ &= e^{-\frac{7n}{\sqrt{7n}}t} e^{-7n\left[-\frac{1}{\sqrt{7n}}t - \frac{1}{27n}t^2 - \frac{1}{37n^{3/2}}t^3 - \cdots\right]} \\ &= e^{-\frac{\sqrt{27n}}{2}t + \frac{\sqrt{27n}}{2}t + \frac{t^2}{2} + \frac{t^3}{3\sqrt{7n}} - \cdots} \\ &= e^{\frac{t^2}{2} + \frac{t^3}{3\sqrt{7n}} - \cdots} \end{aligned}$$

$$\lim_{7n\to\infty} M_{\frac{Y_n - 7n}{\sqrt{27n}}}(t) = \lim_{n\to\infty} e^{\frac{t^2}{2} + \frac{t^3}{3\sqrt{7n}} - \cdots} = e^{\frac{t^2}{2}}$$
$$\Rightarrow Z_n = \frac{Y_n - 7n\theta}{\sqrt{7n}\theta} \xrightarrow{d} N(0,1)$$

Q12. Suppose that $X_i \sim N(\mu, \sigma^2), i = 1, \ldots, 14,\ Z_j \sim N(0,1), j = 1, \ldots, 7$, and $W_k \sim \chi^2(10), k = 1, \ldots, 13$ and all random variables are independent. State the distribution of each of the following variables if it is a "named" distribution. [For example $X_1 + X_2 \sim N(2\mu, 2\sigma^2)$]

(a) $\frac{6\sum_{i=1}^{14}(X_i - \bar{X})^2}{13\sigma^2\sum_{j=1}^{7}(Z_j - \bar{Z})^2}$.

(b) $\frac{6\sum_{k=1}^{7} W_k}{70\sum_{j=1}^{7}(Z_j - \bar{Z})^2}$

(c) $\frac{\sqrt{140}(\bar{X} - \mu)}{\sigma\sqrt{W_1}}$

(d) $\frac{W_1}{W_1 + W_2 + W_3 + W_4}$

(e) $\frac{Z_i^2/W_1}{1 + Z_1^2/W1}$

(f) $\frac{\frac{\sum_{k=1}^{7} W_k}{\sum_{j=1}^{7}(Z_j - \bar{Z})^2}}{1 + \frac{\sum_{k=1}^{7} W_k}{\sum_{j=1}^{7}(Z_j - \bar{Z})^2}}$

(12 marks)

Ans.

(a) $\frac{\sum_{i=1}^{14}(X_i-\bar{X})^2}{\sigma^2} \sim \chi^2(13)$

$\sum_{j=1}^{7}(Z_j-\bar{Z})^2 \sim \chi^2(6)$

$\frac{\frac{\sum_{i=1}^{14}(X_i-\bar{X})^2}{13\sigma^2}}{\sum_{j=1}^{7}(Z_j-\bar{Z})^2/6} \sim F(13,6)$

Thus, $\boxed{\frac{6\sum_{i=1}^{14}(X_i-\bar{X})^2}{13\sigma^2\sum_{j=1}^{7}(Z_j-\bar{Z})^2} \sim F(13,6)}$.

(b) $\sum_{k=1}^{7} W_k \sim \chi^2(70)$

$\sum_{j=1}^{7}(z_j-\bar{Z})^2 \sim \chi^2(6)$

$\frac{\sum_{k=1}^{7} W_k/70}{\sum_{j=1}^{7}(z_j-\bar{Z})^2/6} \sim F(70,6)$

Thus, $\boxed{\frac{6\sum_{k=1}^{7} W_k}{70\sum_{j=1}^{7}(z_j-\bar{Z})^2} \sim F(70,6)}$

(c) $\frac{\sqrt{14}(\bar{X}-\mu)}{\sigma} \sim N(0,1)$

$\frac{\frac{\sqrt{14}(\bar{X}-\mu)}{\sigma}}{\sqrt{W_1/10}} \sim t(10)$

Thus, $\boxed{\frac{\sqrt{140}(\bar{X}-\mu)}{\sigma\sqrt{W_1}} \sim t(10)}$

(d) $\frac{W_1}{W_1+W_2+W3+W_4} = \frac{W_1}{W_1+(W_2+W3+W_4)}$

$W_1 \sim \chi^2(10) \sim GAM(5.0,2)$, $W_2+W_3+W_4 \sim \chi(30) \sim GAM(15.0,2)$

Thus, $\frac{W_1}{W_1+W_2+W3+W_4} \sim Beta(5.0,15.0)$

(e) $Z_1^2 \sim \chi^2(1)$

$\frac{Z_1^2}{W_1/10} \sim F(1,10)$

$\frac{(\frac{1}{10})\frac{Z_1^2}{W_1/10}}{1+(\frac{1}{10})\frac{Z_1^2}{W_1/10}} = \frac{Z_i^2/W_1}{1+Z_1^2/W1} \sim Beta(1/2,5.0)$

(f) $\sum_{k=1}^{7} W_k \sim (70)$

$\sum_{j=1}^{7}(z_j-\bar{Z})^2 \sim \chi^2(6)$

$\frac{\sum_{k=1}^{7} W_k/70}{\sum_{j=1}^{7}(z_j-\bar{Z})^2/6} \sim F(70,6)$

Thus, $\frac{6\sum_{k=1}^{7} W_k}{70\sum_{j=1}^{7}(z_j-\bar{Z})^2} \sim F(70,6)$

$\frac{(\frac{70}{6})\frac{6\sum_{k=1}^{7} W_k}{70\sum_{j=1}^{7}(Z_j-\bar{Z})^2}}{1+(\frac{70}{6})\frac{6\sum_{k=1}^{7} W_k}{70\sum_{j=1}^{7}(Z_j-\bar{Z})^2}} = \frac{\frac{\sum_{k=1}^{7} W_k}{\sum_{j=1}^{7}(Z_j-\bar{Z})^2}}{1+\frac{\sum_{k=1}^{7} W_k}{\sum_{j=1}^{7}(Z_j-\bar{Z})^2}} \sim Beta(35.0,3.0)$

Q13. Suppose $Y \sim Beta(a=4, b=8)$, use the relationship between Beta distribution and F distribution, find $P[Y > 0.396]$.

(3 marks)

Ans.

Let $X \sim F_{2(4),2(8)}$ and $c = \frac{4}{8}$, then $Y = \frac{cX}{1+cX} \sim Beta(a = 4, b = 8)$

$$\begin{aligned} P[Y > 0.396] &= P\left[\tfrac{cX}{1+cX} > 0.396\right] \\ &= P[cX > 0.396 + cX(0.396)] \\ &= P[cX(1-0.396) > 0.396] \\ &= P[X > \tfrac{0.396}{c(1-0.396)}] \\ &= P[X > 1.3113] \\ &= 1 - pf(1.3113, 8, 16) \\ &= 1 - 0.6942 \\ &= \boxed{0.3058} \end{aligned}$$

Q14. Suppose $Y \sim Beta(a = 6, b = 12)$, use the relationship between Beta distribution and F distribution, find 93^{th} percentile of Y.

(3 marks)

Ans.

Let $X \sim F_{2(6),2(12)}$ and $c = \frac{6}{12}$, then $Y = \frac{cX}{1+cX} \sim Beta(a = 6, b = 12)$

$$\begin{aligned} P[Y \leq \pi_{0.93}] &= P\left[\tfrac{cX}{1+cX} \leq \pi_{0.93}\right] \\ &= P[cX \leq \pi_{0.93} + cX(\pi_{0.93})] \\ &= P[cX(1-\pi_{0.93}) \leq \pi_{0.93}] \\ &= P\left[X \leq \tfrac{\pi_{0.93}}{c(1-\pi_{0.93})}\right] \end{aligned}$$

Thus,

$\frac{\pi_{0.93}}{c(1-\pi_{0.93})} = F_{12,24,0.93}$

$\pi_{0.93} = \frac{cF_{12,24,0.93}}{1+cF_{12,24,0.93}} = \frac{\frac{6}{12}(2.0124)}{1+\frac{6}{12}(2.0124)} = \boxed{0.5015}$

where $F_{12,24,0.93} = qf(0.93, 12, 24) = 2.0124$

Q15. Recall that $Y \sim LOGN(\mu, \sigma^2)$ if $\ln Y \sim N(\mu, \sigma^2)$. Assume that $Y_i \sim LOGN(\mu_i, \sigma_i^2), i = 1, \ldots, n$ are independent.

(a) Find the distribution of $\prod_{i=1}^{n} Y_i$.

(b) Find the distribution of $\prod_{i=1}^{n} Y_i^a$.

(c) Find the dietribution of $\frac{Y_1}{Y_2}$.

(d) Find $E\left[\prod_{i=1}^{n} Y_i\right]$.

(8 marks)

Ans.

(a) $\ln Y_i \sim N(\mu_i, \sigma_i^2)$

$$\ln \prod_{i=1}^{n} Y_i = \sum_{i=1}^{n} \ln Y_i \sim N\left(\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2\right)$$

$$\prod_{i=1}^{n} Y_i = \exp\left(\ln \prod_{i=1}^{n} Y_i\right) \sim LOGN\left(\sum_{i=1}^{n} \mu_i, \sqrt{\sum_{i=1}^{n} \sigma_i^2}\right)$$

(b) $\ln Y_i \sim N(\mu_i, \sigma_i^2)$

$$\ln \prod_{i=1}^{n} Y_i^a = \sum_{i=1}^{n} a \ln Y_i \sim N\left(\sum_{i=1}^{n} a\mu_i, \sum_{i=1}^{n} a^2\sigma_i^2\right)$$

$$\prod_{i=1}^{n} Y_i^a = \exp\left(\ln \prod_{i=1}^{n} Y_i^a\right) \sim LOGN\left(\sum_{i=1}^{n} a\mu_i, \sqrt{\sum_{i=1}^{n} a^2\sigma_i^2}\right)$$

(c) $\ln Y_1 \sim N(\mu_1, \sigma_1^2)$, $\ln Y_2 \sim N(\mu_2, \sigma_2^2)$

$$\ln\left(\frac{Y_1}{Y_2}\right) = \ln Y_1 - \ln Y_2 \sim N(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2)$$

$$\frac{Y_1}{Y_2} = \exp\left(\ln\left(\frac{Y_1}{Y_2}\right)\right) \sim LOGN(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2)$$

(d) $$E\left[\prod_{i=1}^{n} Y_i\right] = \prod_{i=1}^{n} E(Y_i) = \prod_{i=1}^{n} e^{\mu_i t + \frac{1}{2}t^2\sigma_i^2} = e^{t\sum_{i=1}^{n}\mu_i + \frac{1}{2}t^2\sum_{i=1}^{n}\sigma_i^2}$$