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CONTENTS

1 Multiple Random Variable

1.1 Joint Discrete Distributions

In many applications there will be more than one random variable of interest, say $X_1, X_2, \ldots, X_k$. It is convenient mathematically to regard these variables as components of a $k-$dimensional vector, $X = (X_1, X_2, \ldots, X_k)$, which is capable of assuming values $x = (x_1, x_2, \ldots, x_k)$ in a $k-$ dimensional Euclidean space. Note, for example, that an observed value x may be the result of measuring k characteristics once each, or the result of measuring one characteristic k times.

Definition 1. The joint probability density function (joint pdf) of the $k-$dimensional discrete random variable $X = (X_1, X_2, \ldots, X_k)$ is defined to be

$$f(x_1, x_2, \ldots, x_k) = P[X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k)$$

for all possible values $x = (x_1, x_2, \ldots, x_k)$ of X.

thm1 **Theorem 1.** A function $f(x_1, x_2, \ldots, x_k)$ is the joint pdf for some vector-valued random variable

$$X = (x_1, x_2, \ldots, x_k)$$

if and only if the following properties are satisfied

1. $f(x_1, x_2, \ldots, x_k) > 0$ for all possible values $x_1, x_2, \ldots, x_k$
2. $\sum_{x_1} \sum_{x_2} \cdots \sum_{x_k} f(x_1, x_2, \ldots, x_k) = 1$

Definition 2.
If the $X = (x_1, x_2, \ldots, x_k)$ of discrete random variables has the joint pdf $f(x_1, x_2, \ldots, x_k)$, then the marginal pdf's of X_j is

$$f_j(x_j) = \sum \underset{\text{all} i \neq j}{\cdots} \sum f(x_1, \ldots, x_j, \ldots, x_k)$$

ex1

Example 1.

Let the joint pmf of X_1 and X_2 be defined by

$$p(x_1, x_2) = \frac{x_1 + x_2}{32}, \quad x_1 = 1, 2, x_2 = 1, 2, 3, 4.$$

(a) Display the joint probability distribution of X_1 and X_2 in a table.

Sol:

		x_1	
		1	2
x_2	1	$\frac{2}{32}$	$\frac{3}{32}$
	2	$\frac{3}{32}$	$\frac{4}{32}$
	3	$\frac{4}{32}$	$\frac{5}{32}$
	4	$\frac{5}{32}$	$\frac{6}{32}$

(b) Verify that the probability function satisfies Theorem thm1 1.

Sol:

(a) $p(x_1, x_2) \geq 0$ for all x_1 and x_2.

(b) $\sum_{x_1, x_2} p(x_1, x_2)$

$$= \frac{2}{32} + \frac{3}{32} + \frac{3}{32} + \frac{4}{32} + \frac{4}{32} + \frac{5}{32} + \frac{5}{32} + \frac{6}{32}$$

$$= 1$$

(c) Find $P(X_1 < X_2)$.

Sol:

$$P(X_1 < X_2)$$
$$= p(1,2) + p(1,3) + p(1,4) + p(2,3) + p(2,4)$$
$$= \frac{3}{32} + \frac{4}{32} + \frac{5}{32} + \frac{5}{32} + \frac{6}{32}$$
$$= \boxed{\frac{23}{32}}$$

(d) Find $P(X_1 + X_2 = 4)$.

Sol:

$$P(X_1 + X_2 = 4)$$
$$= p(1,3) + p(2,2)$$
$$= \frac{4}{32} + \frac{4}{32}$$
$$= \frac{8}{32}$$
$$= \boxed{\frac{1}{4}}$$

Definition 3. Joint CDF The joint cumulative distribution function of the k random variables $X_1, X_2, \ldots X_k$ is the function defined by

$$F(x_1, x_2, \ldots, x_k) = P[X_1 \leq x_1, \ldots, X_k \leq x_k]$$

Theorem 2. A function $F(x_1, x_2)$ is a bivariate CDF if and only if

- $\lim\limits_{x_1 \to -\infty} F(x_1, x_2) = F(-\infty, x_2) = 0 \;\forall\; x_2$
- $\lim\limits_{x_2 \to -\infty} F(x_1, x_2) = F(x_1, -\infty) = 0 \;\forall\; x_1$
- $\lim\limits_{x_1 \to \infty, x_2 \to \infty} F(x_1, x_2) = F(\infty, \infty) = 1 \;\forall\; x_1, x_2$
- $F(b, d) - F(b, c) - F(a, d) + F(a, c) \geq 0 \;\forall\; a < b, c < d$
- $\lim\limits_{h \to 0^+} F(x_1 + h, x_2) = \lim\limits_{h \to 0^+} F(x_1, x_2 + h) = F(x_1, x_2)$

Example 2. If X and Y are discrete random variables with joint pdf

$$f(x,y) = c\frac{2^{x+y}}{x!y!} \quad x = 0,1,2,\ldots; y = 0,1,2,\ldots$$

and zero otherwise.

(a) Find the constant c.

Sol:
$c\sum_{x=0}^{\infty}\sum_{y=0}^{\infty}\frac{2^x 2^y}{x!y!} = 1$
$c\sum_{x=0}^{\infty}\frac{2^x}{x!}\sum_{y=0}^{\infty}\frac{2^y}{y!} = 1$
$ce^2e^2 = 1$
$c = e^{-4}$

(b) Find the marginal pdf's of X and Y.

Sol:
$f(x) = \sum_{y=0}^{\infty}(e^{-4})\frac{2^{x+y}}{x!y!} = (e^{-4})\frac{2^x}{x!}\sum_{y=0}^{\infty}\frac{2^y}{y!}$
$= (e^{-4})\frac{2^x}{x!}e^2$
$= \frac{2^x e^{-2}}{x!}$
Similarly, $f(y) = \frac{2^y e^{-2}}{y!}$

1.2 Joint Continuous Distributions

Definition 4. A $k-$dimensional vector valued random variable $X = (X_1, X_2, \ldots, X_k)$ is said to be continuous if there is a function $f(x_1, x_2, \ldots, x_k)$, called the joint probability density function (joint pdf), of X, such that the joint CDF can be written as
$F(x_1, x_2, \ldots, x_k)$
$= \int_{-\infty}^{x_k} \cdots \int_{-\infty}^{x_1} f(t_1, t_2, \ldots, t_k) dt_1 \cdots dt_k$
$\forall\ x = (x_1, x_2, \ldots, x_k)$.

Theorem 3. Any function $f(x_1, x_2, \ldots, x_k)$ is a joint pdf of a $k-$dimensional random variable if and only if

1. $f(x_1, x_2, \ldots, x_k) \geq 0\ \forall\ x_1, x_2, , \ldots, x_k$
2. $\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, x_2, \ldots, x_k) dx_1 \cdots dx_k = 1$

Definition 5.
1f $X = (X_1, X_2, \ldots, X_k)$ is a $k-$dimensional random variable with joint CDF $F(x_1, x_2, \ldots, x_k)$, then the marginal CDF of X is

$$F_j(x_j) = \lim_{x_i \to \infty, \text{all } i \neq j} F(x_1, \ldots, x_j, \ldots, x_k)$$

Furthermore,the marginal pdf is

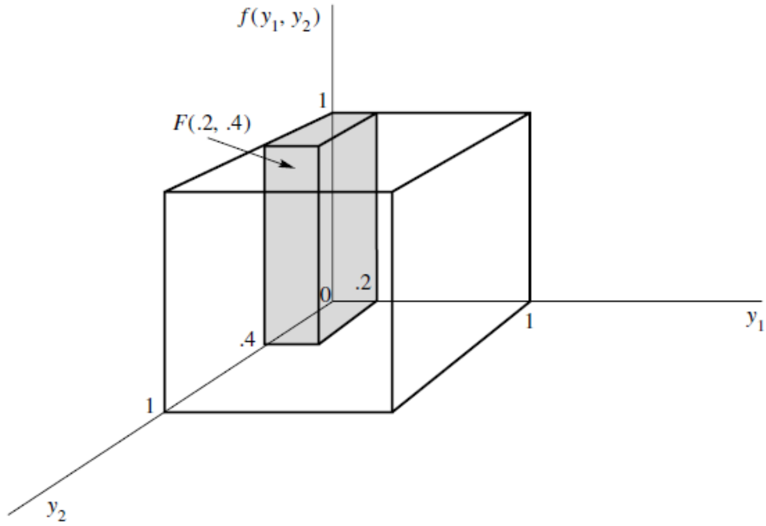
$$f_j(x_j) = \int \underset{\text{all } i \neq j}{\cdots} \int f(x_1, \ldots, x_j, \ldots, x_k) dx_1 \ldots dx_k$$

Example 3.

Suppose that a radioactive particle is randomly located in a square with sides of unit length. That is, if two regions within the unit square and of equal area are considered, the particle is equally likely to be in either region. Let X_1 and X_2 denote the coordinates of the particle's location. A reasonable model for the relative frequency histogram for X_1 and X_2 is the bivariate analogue of the univariate uniform density function:

$$f(x_1, x_2) = \begin{cases} 1, & 0 \le x_1 \le 1, 0 \le x_2 \le 1, \\ 0, & \text{otherwise} \end{cases}$$

(a) Sketch the probability density surface.



(b) Find $F(.2, .4)$.

Sol:

$F(.2, .4)$
$= P(X_1 \leq .2, X_2 \leq .4)$
$= (.2)(.4)$
$= \boxed{0.08}$

(c) Find $P(.1 \leq X_1 \leq .3, 0 \leq X_2 \leq .5)$

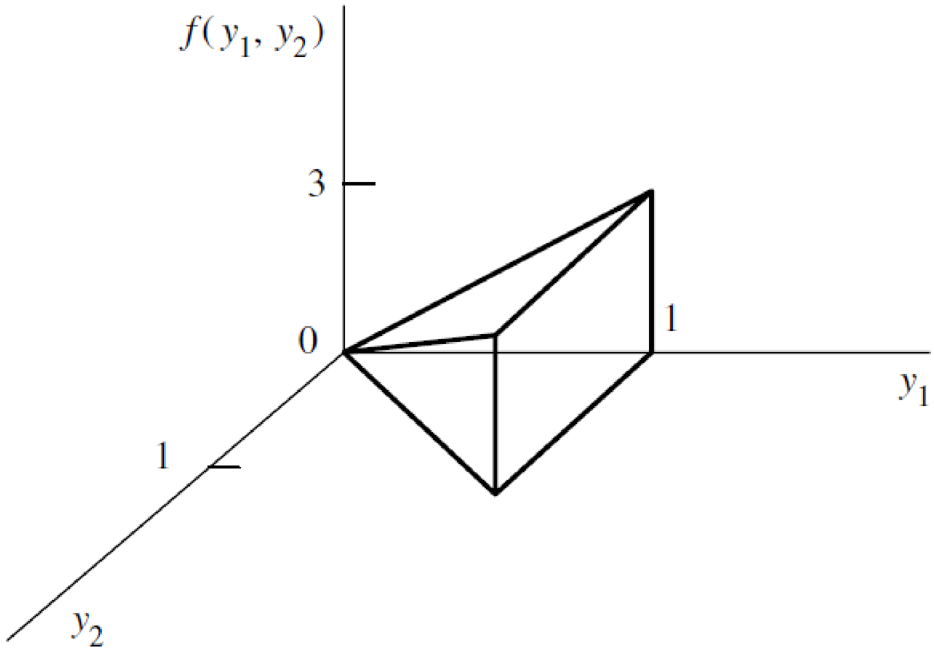
Sol:

$P(.1 \leq X_1 \leq .3, 0 \leq X_2 \leq .5)$
$= 0.2(0.5)$
$= \boxed{0.1}$

Example 4. The joint probability density function of X_1 and X_2 is

$$f(x_1, x_2) = \begin{cases} 3x_1, & 0 \leq x_2 \leq x_1 \leq 1, \\ 0, & \text{otherwise} \end{cases}$$

(a) Sketch the probability density surface.



(b) Find $P(0 \leq X_1 \leq .5, X_2 \geq 0.25)$.

Sol: $P(0 \leq X_1 \leq .5, X_2 \geq 0.25)$
$= \int_{.25}^{.5} \int_{.25}^{x_1} 3x_1 dx_2 dx_1$
$= \int_{.25}^{.5} 3x_1 [x_2]_{.25}^{x_1} dx_1$
$= \int_{.25}^{.5} 3x_1 (x_1 - .25) dx_1$
$= 3[\frac{x_1^3}{3} - \frac{.25x_1^2}{2}]_{.25}^{.5}$
$= \boxed{\frac{5}{128}}$

1.3 Conditional Distributions

Definition 6. Conditional pdf If X_1 and X_2 are discrete or continuous random variables with joint pdf $f(x_1, x_2)$, then the conditional probability density function (conditional pdf) of X_2 given $X_1 = x_1$ is defined to be

$$f(x_2|x_1) = \frac{f(x_1, x_2)}{f_1(x_1)}$$

for values x_1 such that $f_1(x_1) > 0$ and zero otherwise.

Similarly, the conditional pdf of X_1 given $X_2 = x_2$ is defined to be

$$f(x_1|x_2) = \frac{f(x_1, x_2)}{f_2(x_2)}$$

for values x_2 such that $f_2(x_2) > 0$ and zero otherwise.

Theorem 4. If X_1 and X_2 are random variables with joint pdf $f(x_1, x_2)$ and marginal pdf's $f_1(x_1)$ and $f_2(x_2)$, then

$$f(x_1, x_2) = f_1(x_1)f(x_2|x_1) = f_2(x_2)f(x_1|x_2)$$

and if X_1 and X_2 are independent, then

$$f(x_2|x_1) = f_2(x_2)$$

and

$$f(x_1|x_2) = f_1(x_1)$$

Example 5.

Let

$$f(x_1,x_2,x_3,x_4)=\begin{cases}\frac{3}{4}(x_1^2+x_2^2+x_3^2+x_4^2), & 0<x_i<1, i=1,2,3,4\\ 0, & \text{otherwise}\end{cases}$$

(a) Find the marginal pdf of (X_1, X_2).

(b) Find the conditional pdf of (X_3, X_4) given $X_1 = \frac{1}{3}$ and $X_2 = \frac{2}{3}$.

Sol:

$f(x_1, x_2)$
$= \int_0^\infty \int_0^\infty f(x_, x_2, x_3, x4) dx_3 dx_4$
$= \int_0^1 \int_0^1 \frac{3}{4}(x_1^2 + x_2^2 + x_3^2 + x_4^2) dx_3 dx_4$
$= \frac{3}{4}(x_1^2 + x_2^2) + \frac{1}{2}$
for $0 < x_1 < 1$ and $0 < x_2 < 1$.

$f(x_3, x_4|x_1, x_2)$
$= \frac{f(x_1,x_2,x_3,x_4)}{f(x_1,x_2)}$
$= \frac{\frac{3}{4}(x_1^2+x_2^2+x_3^2+x_4^2)}{\frac{3}{4}(x_1^2+x_2^2)+\frac{1}{2}}$
$= \frac{x_1^2+x_2^2+x_3^2+x_4^2}{x_1^2+x_2^2+\frac{2}{3}}$

$f(x_3, x_4|x_1 = \frac{1}{3}, x_2 = \frac{2}{3})$
$= \frac{(\frac{1}{3})^2+(\frac{2}{3})^2+x_3^2+x_4^2}{(\frac{1}{3})^2+(\frac{2}{3})^2+\frac{2}{3}}$
$= \frac{5}{11} + \frac{9}{11}x_3^2 + \frac{9}{11}x_4^2$

Example 6. The joint density function of X_1 and X_2 is given by

$$f(x_1,x_2) = \begin{cases} 30x_1x_2^2, & x_1 - 1 \le x_2 \le 1 - x_1, 0 \le x_1 \le 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Show that the marginal density of X_1 is a beta density with $a = 2$ and $b = 4$.

Sol:

$f_1(x_1)$
$= \int_{x_1-1}^{1-x_1} 30x_1x_2^2 dx_2$
$= 30x_1[\frac{x_2^3}{3}]_{x_1-1}^{1-x_1}$
$= 10x_1[(1-x_1)^3 + (1-x_1)^3]$
$= 20x_1(1-x_1)^3, 0 \le x_1 \le 1$

(b) Derive the marginal density of X_2.

Sol:

For $0 < x_2 < 1$
$f_2(x_2)$
$= \int_0^{1-x_2} 30x_1x_2^2 dx_1$
$= 30x_2^2[\frac{x_1^2}{2}]_0^{1-x_2}$

$= 15x_2^2(1-x_2)^2, 0 < x_2 < 1$

For $-1 < x_2 < 0$
$f_2(x_2)$
$= \int_0^{1+x_2} 30x_1x_2^2 dx_1$
$= \int_0^{1+x_2} 30x_1x_2^2 dx_1$
$= 30x_2^2[\frac{x_1^2}{2}]_0^{1+x_2}$
$= 15x_2^2(1+x_2)^2, -1 < x_2 < 0$

$$f(x_2) = \begin{cases} 15x_2^2(1-x_2)^2, & 0 < x_2 < 1 \\ 15x_2^2(1+x_2)^2, & -1 < x_2 < 0 \end{cases}$$

(c) Derive the conditional density of X_2 given $X_1 = x_1$.

Sol:

$f(x_2|x_1)$
$= \frac{f(x_1,x_2)}{f_1(x)}$
$= \frac{30x_1x_2^2}{20x_1(1-x_1)^3}$

$= \frac{3x_2^2}{2(1-x_1)^3}, x_1 - 1 \le x_2 \le 1 - x_1$

(d) Find $P(X_2 > 0|X_1 = .75)$.

Sol:

$f(x_2|x_1 = 0.75)$
$= \frac{3x_2^2}{2(1-.75)^3}, 0.75 - 1 \le x_2 \le 1 - 0.75$
$= 96x_2^2, -.25 \le x_2 \le .25$

$P(X_2 > 0|X_1 = .75)$
$= \int_0^{0.25} 96x_2^2 dx_2$
$= 96[\frac{x_2^3}{3}]_0^{0.25}$
$= 32[.25^3]$
$= \boxed{0.5}$

1.4 Conditional Expectation

Definition 7. If X and Y are jointly distributed random variables, then the conditional expectation of Y given $X = x$ is given by

$$E(Y|x) = \sum_y yf(y|x) \text{ if } X \text{ and } Y \text{ are discrete}$$

$$E(Y|x) = \int yf(y|x)dy \text{ if } X \text{ and } Y \text{ are continuous}$$

Example 7. Below is a table giving a joint probability function for discrete random variables X_1 and X_2.

	x_2			
x_1	3	4	5	6
4	.1	.05	.05	0
3	.05	0.2	0.2	0
2	0	0	.2	.05
1	0	0	0	.1

(a) Find the conditional mean of X_2 given $X_1 = 4$, $E[X_2|X_1 = 4]$.

Sol:

	x_2			
	3	4	5	6
$p(x_2|x_1 = 4)$	.5	.25	.25	0

$E[X_2|X_1 = 4] = 3(.5) + 4(.25) + 5(.25) + 6(0) = 3.75$

(b) Find the conditional variance of X_2 given $X_1 = 4$, $V[X_2|X_1 = 4]$.

Sol:

$E[X_2^2|X_1 = 4] = 3^2(.5) + 4^2(.25) + 5^2(.25) + 6^2(0) = 14.75$

$V[X_2|X_1 = 4] = 14.75 - 3.75^2 = 0.6875$

Example 8. Let X_1 and X_2 have the joint pdf

$$f(x_1, x_2) = \begin{cases} 1, & 0 < x_2 < 2x_1, 0 < x_1 < 1 \\ 0, & \text{otherwise} \end{cases}$$

Find $E(X_2|X_1 = x_1)$.

Sol:
$f(x_1|x_2) = k, 0 < x_2 < 2x_1,$
$k \int_0^{2x_1} dx_2 = 1$
$k(2x_1) = 1$
$k = \frac{1}{2x_1}$
$f(x_2|x_1) = \frac{1}{2x_1}, 0 < x_2 < 2x_1,$
$E(X_2|X_1 = x_1) = \int_0^{2x_1} x_2(\frac{1}{2x_1})dx_2 = \frac{x_2^2}{4x_1}|_0^{2x_1} = \frac{4x_1^2}{4x_1} = x_1$

Theorem 5. If X and Y are independent random variables, then $E(Y|x) = E(Y)$ and $E(X|y) = E(X)$.

Sol:

$$\begin{aligned} E(Y|x) &= \int yf(y|x)dy \\ &= \int y\frac{f(x,y)}{f(x)}dy \\ &= \int y\frac{f(x)f(y)}{f(x)}dy \text{ If } X \text{ and } Y \text{ are independent} \\ &= \int yf(y)dy \\ &=`E(Y) \end{aligned}$$

Theorem 6.

Let X and Y denote random variables. Then

$$\boxed{E(X) = E[E(X|Y)]}$$

where, on the right hand side, the inside expectation is with respect to the conditional distribution of X given Y, and the outside expectation is with respect to the distribution of Y.

Sol:

$$
\begin{aligned}
E[E(X|Y)] &= \int_{y=0}^{\infty} E(X|Y) f(y) dy \\
&= \int_{y=0}^{\infty} \left[\int_{x=0}^{\infty} x f(x|y) dx\right] f(y) dy \\
&= \int_{y=0}^{\infty} \left[\int_{x=0}^{\infty} x \left(\frac{f(x,y)}{f(y)}\right) dx\right] f(y) dy \\
&= \int_{y=0}^{\infty} \int_{x=0}^{\infty} x f(x,y) dx dy \\
&= E(X)
\end{aligned}
$$

Theorem 7.
Let X and Y denote random variables and $h(x,y)$ is a function. Then

$$E[h(X,Y)] = E_Y[E(h(X,Y)|Y)]$$

or

$$E[h(X,Y)] = E_X[E(h(X,Y)|X)]$$

Definition 8. The conditional variance of Y given $X = x$ is given by

$$V(Y|x) = E\{[Y - E(Y|x)]^2 2|x\}$$

An equivalent form,is

$$V(Y|x) = E(Y^2|x) - [E(Y|x)]^2$$

Theorem 8.
Let X and Y denote random variables. Then

$$\boxed{V(X) = E[V(X|Y)] + V[E(X|Y)]}$$

Sol:

$$\begin{aligned}
&E[V(X|Y)] + V[E(X|Y)] \\
&= E[E(X^2|Y) - E^2(X|Y)] \\
&\quad +(E(E^2(X|Y)) - E^2(E(X|Y)) \\
&= E(X^2) - E^2(X) \\
&= V(X)
\end{aligned}$$

Example 9. A quality control plan for an assembly line involves sampling $n = 10$ finished items per day and counting X, the number of defectives. If p denotes the probability of observing a defective, then X has a binomial distribution, assuming that a large number of items are produced by the line. But p varies from day to day and is assumed to have a uniform distribution on the interval from 0 to $\frac{1}{4}$. Find the expected value and variance of X.

Sol:
$X|p \sim Bin(10, p)$; $P \sim U(0, \frac{1}{4})$
$E(X) = E(E(X|p)) = E(10P) = 10E(P) = 10(\frac{1}{8}) = 1.25$

Example 10. If $X_2|X_1 = x_1 \sim POI(x_1)$, and $X_1 \sim EXP(1)$, find $E(X_2)$ and $V(X_2)$.

Sol:

$E(X_2) = E(E(X_2|X_1)) = E(X_1) = 1$

$V(X_2) = E[V(X_2|X_1)] + V[E(X_2|X_1)] = E(X_1) + V(X_1) = 1 + 1^1 = 2$

Example 11. Let X_1 be the number of customers arriving in a given minute at the drive-up window of a local bank, and let X_2 be the number who make the withdrawals. Assume X_1 is Poisson distributed with expected value $E(X_1) = 3$, and that the conditional expectation and variance X_2 given $X_1 = x_1$ are $E(X_2|x_1) = \frac{x_1}{2}$ and $V(X_2|x_1) = \frac{x_1+1}{3}$. Find

(a) $E(X_2)$

Sol:

$E(X_2) = E[E(X_2|X_1)] = E(\frac{X_1}{2}) = \frac{1}{2}(3) = 1.5$

(b) $V(X_2)$

Sol:

$V(X_2)$
$= E(V(X_2|X_1)) + V(E(X_2|X_1))$
$= E(\frac{X_1+1}{3}) + V(\frac{X-1}{2})$

$$= \frac{4}{3} + \frac{3}{4}$$
$$= \frac{25}{12}$$

(c) $E(X_1X_2)$

Sol:

$$E(X_1X_2) = E[E(X_1X_2|X_1)] = E[X_1(\frac{X_1}{2})]$$
$$= \frac{1}{2}E(X_1^2) = \frac{1}{2}(3+3^3) = 6$$

1.5 Multinomial Distribution

Suppose that there are $k+1$ mutually exclusive and exhaustive events, say $E_1, E_2, \ldots, E_k, E_{k+l}$, which can occur on any trial of an experiment, and let $p_i = P(E_i)$ for $i = 1, 2, \ldots, k+1$. On n independent trials of the experiment, we let X_i be the number of occurrences of the event E_i. The vector $X = (X_1, X_2, \ldots, X_k)$ is said to have the multinomial distribution which has a joint pdf of the form

$$f(x_1, x_2, \ldots, x_k) = \frac{n!}{x_1!x_2!\cdots x_k!} p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k}$$

Theorem 9. If $X = (X_1, X_2, \ldots, X_k)$ have a multinomial distribution with parameters n and $p_1, p_2, \ldots, p_k$, then

1. $E(X_i) = np_i$, $V(X_i) = np_iq_i$
2. $Cov(X_s, X_t) = -np_sp_t$, if $s \neq t$

Sol:
Let X be the number of trials falling into cell i. Let all other cells excluding cell i combined into a single cell. Then $X_i \sim Bin(n, p_i)$. Thus $E(X_i) = np_i$, $V(X_i) = np_iq_i$.
Define for $s \neq t$,

$$U_i = \begin{cases} 1, & \text{if trial } i \text{ resulting in class} s \\ 0, & \text{otherwise} \end{cases}$$

$$W_j = \begin{cases} 1, & \text{if trial } j \text{ resulting in class} t \\ 0, & \text{otherwise} \end{cases}$$

Then $X_s = \sum_{i=1}^{n} U_i$, $X_t = \sum_{j=1}^{n} W_j$
Notice that U_i and W_j cannot both equal 1. Thus $U_iW_j = 0$ and $E(U_iW_j) = 0$
$E(U_i) = p_s$ and $E(W_j) = p_t$.
$Cov(U_i, W_j) = 0$ if $i \neq j$ because the trials are

independent.

$Cov(U_i, W_j) = E(U_iW_j) - E(U_i)E(W_j) = -p_sp_t$

$Cov(X_s, X_t) = \sum_i \sum_j Cov(U_i, W_j) = \sum_{i=j=1}^{n} Cov(U$

$\sum \sum_{i \neq j} Cov(U_i, W_j) = \sum_{i=j=1}^{n} -p_sp_t - 0 = -np_sp_t$

Example 12. According to recent census figures, the proportions of adults (persons over 18 years of age) in the United States associated with five age categories are as given in the following table.

Age	Proportion
18-24	.18
25-34	.23
35-44	.16
45-64	.27
65 & above	.16

If these figures are accurate and five adults are randomly sampled, find the probability that the sample contains one person between the ages of 18 and 24, two between the ages of 25 and 34, and two between the ages of 45 and 64.

Sol:

$P(X_1 = 1, X_2 = 2, X_3 = 0, X_4 = 2, X_5 = 0)$
$= \frac{5!}{1!2!0!2!0!}(.18)(.23^2)(.27^2)$
$= \boxed{0.02082}$

Example 13. A large lot of manufactured items contains 10% with exactly one defect, 5% with more than one defect, and the remainder with no defects. Ten items are randomly selected from this lot for sale. If X_1 denotes the number of items with one defect and X_2, the number with more than one defect, the repair costs are $X_1 + 3X_2$. Find the mean and variance of the repair costs.

Sol:
$(X_1, X_2, X_3) \sim Multinomial(n = 10, p_1 = .1, p_2 = .05, p_3 = .85)$
$E(X_1) = 10(.1) = 1$; $E(X_2) = 10(.05) = 0.5$;
$V((X_1) = 10(.1)(.9) = 0.9$; $V((X_2) = 10(.05)(.95) = 0.475$
$Cov(X_1, X_2) = -10(.1)(.05) = -.05$
$E[X_1 + 3X_2] = E(X_1) + 3E(X_2) = 1 + 3(0.5) = 1.6$
$V[X_1+3X_2] = V(X_1)+9V(X_2)+2(1)(3)Cov(X_1, X_2)$
$0.9 + 9(.475) + 6(-0.05) = 4.875$

1.6 Bivariate Normal Distribution

A pair of continuous random variables X and Y is said to have a bivariate normal distribution if it has a joint pdf of the form

$$
\begin{aligned}
f(x, y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \\
&\times \exp\{-\frac{1}{2(1-\rho^2)}[(\frac{x-\mu_1}{\sigma_1})^2 + (\frac{y-\mu_2}{\sigma_2})^2 \\
&-2\rho(\frac{x-\mu_1}{\sigma_1})(\frac{y-\mu_2}{\sigma_2})]\}, x \in R, y \in R
\end{aligned}
$$

A special notation for this is

$$(X, Y) \sim BVN(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$$

which depends on five parameters, $\mu_1, \mu_2 \in R$, $\sigma_1^2 > 0, \sigma_2^2 > 0$ and $-1 < \rho < 1$.

Theorem 10. If $(X, Y) \sim BVN(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$, then $x_1 \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$.

Theorem 11. If $(X, Y) \sim BVN(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$, then

1. conditional on $X = x$,

$$Y|x \sim N(\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x - \mu_1), \sigma_2^2(1 - \rho^2))$$

2. conditional on $Y = y$,

$$X|y \sim N(\mu_1 + \rho\frac{\sigma_1}{\sigma_2}(y - \mu_2), \sigma_1^2(1 - \rho^2))$$

Example 14. Let X_1 and X_2 be independent normal random variables, $X_i \sim N(\mu_i, \sigma_i^2)$, and let $Y_1 = X_1$ and $Y_2 = X_1 + X_2$.

(a) What are the means, variances, and correlation coefficient of Y_1 and Y_2?

(b) Find the conditional distribution of Y_2 given $Y_1 = y_1$.

Sol:

$Y_1 \sim N(\mu_1, \sigma_1^2)$, $Y_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

$E(Y_1) = \mu_1$, $V(Y_1) = \sigma_1^2$

$E(Y_2) = \mu_1 + \mu_2$, $V(Y_2) = \sigma_1^2 + \sigma_2^2$

$Cov(Y_1, Y_2) = Cov(X_1, X_1 + X_2) = V(X_1) = \sigma_1^2$

$\rho = \frac{Cov(Y_1,Y_2)}{\sqrt{V(Y_1)V(Y_2)}} = \frac{\sigma_1^2}{\sigma_1\sqrt{\sigma_1^2+\sigma_2^2}} = \frac{\sigma_1}{\sqrt{\sigma_1^2+\sigma_2^2}}$

$E(Y_2|Y_1 = y_1) = \mu_1+\mu_2+\frac{\sigma_1}{\sqrt{\sigma_1^2+\sigma_2^2}}\left(\frac{\sqrt{\sigma_1^2+\sigma_2^2}}{\sigma_1}\right)(y_1-\mu_1) = \mu_2 + y_1$

$V(Y_2|Y_1 = y_1) = (\sigma_1^2+\sigma_2^2)(1-(\frac{\sigma_1}{\sqrt{\sigma_1^2+\sigma_2^2}})^2) = \sigma_2^2$

$$Y_2|Y_1 = y_1 \sim N(\mu_2 + y_1, \sigma_1^2)$$

1.7 Joint Moment Generating Function

The joint MGF of $X = (X_1, \ldots, X_k)$, if it exists, is defined to be

$$M_X(t) = E\left[\exp\left(\sum_{i=1}^{k} t_i X_i\right)\right]$$

Note that it also is possible to obtain the MGF of the marginal distributions from the joint MGF. For example,

$$M_X(t_1) = M_{X,Y}(t_1, 0)$$
$$M_Y(t_2) = M_{X,Y}(0, t_2)$$

Theorem 12. If $M_{XY}(t_1, t_2)$ exists, then the random variables X and Y are independent if and only if

$$M_{XY}(t_1, t_2) = M_X(t_1) M_Y(t_2)$$

Example 15. Suppose that X and Y are continuous with joint pdf $f(x, y) = 2e^{-x-y}$ if $0 < x < y < \infty$ and zero otherwise.

(a) Derive the joint MGF of X and Y.

(b) Derive the MGF of X and Y respectively.

Sol:

$M_{X,Y}(t_1, t_2)$

$= E[e^{t_1X+t_2Y}]$

$= \int_0^\infty \int_x^\infty e^{t_1X+t_2Y}(2e^{-x-y})dydx$

$= 2\int_0^\infty e^{-(1-t_1)x} \int_x^\infty e^{-(1-t_2)y}dydx$

$= 2\int_0^\infty e^{-(1-t_1)x}[\frac{e^{-(1-t_2)x}}{1-t_2}]dx$

$= \frac{2}{(1-t_2)(2-t_1-t_2)}$

$M_X(t_1) = M_{X,Y}(t_1, 0) = \frac{2}{2-t_1}$

$M_Y(t_2) = M_{X,Y}(0, t_2) = \frac{2}{(1-t_2)(2-t_2)}$