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7 Interval Estimation

7.1 Confidence Intervals

Definition 1. Confidence Interval

An interval

$$(l(x_1, \dots, x_n), u(x_1, \dots, x_n))$$

is called a $100\gamma\%$ confidence interval for θ if

$$P[l(x_1, \dots, x_n) < \theta < u(x_1, \dots, x_n)] = \gamma$$

where $0 < \gamma < 1$.

The observed values $l(x_1, \dots, x_n)$ and $u(x_1, \dots, x_n)$ are called lower and upper confidence limits, respectively.

Definition 2. One-Sided Confidence Limits If

$$P[l(x_1, \dots, x_n) < \theta] = \gamma$$

then $l(x_1, \dots, x_n)$ is called a one-sided lower $100\gamma\%$ confidence limit for θ .

If

$$P[u(x_1, \dots, x_n) > \theta] = \gamma$$

then $u(x_1, \dots, x_n)$ is called a one-sided upper $100\gamma\%$ confidence limit for θ .

In general, if (θ_L, θ_U) is a $100\gamma\%$ confidence interval for a parameter θ , and if $\tau(\theta)$ is a monotonic increasing function of $\theta \in \Omega$, The $(\tau(\theta_L), \tau(\theta_U))$ is a $100\gamma\%$ confidence interval for $\tau(\theta)$.

Example 1. Consider a random sample of size n from an exponential distribution, $X_i \sim \text{Exp}(\theta)$.

- (a) Construct a one-sided lower $100\gamma\%$ confidence limit for θ .

- (b) Construct a one-sided upper $100\gamma\%$ confidence limit for θ .

(c) Construct a $100\gamma\%$ confidence interval for θ .

- (d) Find a one-sided lower $100\gamma\%$ confidence limit for $P(X > t) = e^{-t/\theta}$.

Example 2.

Consider independent random samples from two gamma distributions, $X \sim \text{gamma}(4, \beta_1)$ and $Y_j \sim \text{gamma}(10, \beta_2); i = 1, \dots, n_1, j = 1, \dots, n_2$.

- (a) Find the distribution of $\left(\frac{\beta_2}{\beta_1}\right) \left(\frac{5\bar{X}}{2\bar{Y}}\right)$.
- (b) Derive a $100(1 - \alpha)\%$ confidence for $\frac{\beta_2}{\beta_1}$.

Example 3.

Consider a random sample of size 40 from a uniform distribution, $X_i \sim U(0, \theta)$, $\theta > 0$, and let $X_{n:n}$ be the largest order statistic. Find the constant c such that $(x_{n:n}, cx_{n:n})$ is a 92% confidence interval for θ .

7.2 Pivotal Quantity Method

Definition 3. Pivotal Quantity

If $Q = q(X_1, \dots, X_n; \theta)$ is a random variable that is a function only of (X_1, \dots, X_n) and θ , then Q is called a pivotal quantity if its distribution does not depend on θ or any other unknown parameters. That is, if $X \sim F(\mathbf{x}|\theta)$, then Q has the same distribution for all values of θ .

Example 4. (Gamma pivot)

Suppose that X_1, \dots, X_n are iid $Exp(\theta)$, find the pivotal quantity based on the sufficient statistics $T = \sum X_i$.

Example 5.

Consider a random sample from a normal distribution, $X \sim N(\mu, \sigma^2)$, where both μ and σ^2 are unknown. If $\hat{\mu}$ and $\hat{\sigma}$ are the MLEs of μ and σ ,

(a) show that $\frac{\hat{\mu} - \mu}{\hat{\sigma}}$ and $\hat{\sigma}/\sigma$ are pivotal quantities;

(b) find a $100(1 - \alpha)\%$ confidence interval for μ .

-
- (c) find an equal tail $100(1 - \alpha)\%$ confidence interval for σ^2 .

Example 6.

Let X_1, X_2, \dots, X_n be a random sample from a Weibull distribution, $X \sim WEI(\theta, 4)$.

- (a) Show that $Q = 2 \sum_{i=1}^n X_i^4 / \theta^4 \sim \chi^2(2n)$.
- (b) Use Q to derive an equal tailed $100\gamma\%$ confidence interval for θ .

Example 7.

Let X_1, \dots, X_n , be a random sample from a gamma distribution with parameters $\alpha = 5$ and unknown θ .

- (a) Find a pivotal quantity for the parameter θ based on the sufficient statistic.
- (b) Derive an equal tail 92% confidence interval for θ based on the pivotal quantity from part (a).

It may not always be possible to find a pivotal quantity, but for a sample from a continuous distribution with a single unknown parameter, at least one pivotal quantity can always be derived by use of the probability integral transform.

If

$$X \sim f(x; \theta)$$

and if

$$F(x; \theta)$$

is the CDF of X , then

$$F(X; \theta) \sim U(0, 1)$$

and consequently

$$Y_i \sim -\ln F(X_i, \theta) \sim EXP(1).$$

For a random sample X_1, \dots, X_n , it follows that

$$-2 \sum_{i=1}^n \ln F(X_i; \theta) \sim \chi^2(2n)$$

so that

$$P[\chi_{\alpha/2}^2(2n) < -2 \ln F(X_i; \theta) < \chi_{1-\alpha/1}^2(2n)] = 1-\alpha$$

and inverting this statement will provide a confidence region for θ .

If the CDF is not in closed form or if it is too complicated, then the inversion may have to be done numerically.

If $F(x; \theta)$ is a monotonic increasing (or decreasing) function of θ , then the resulting confidence region will be an interval.

Notice also that $1 - F(X_i; \theta) \sim U(0, 1)$ and

$$-2 \sum_{i=1}^n \ln[1 - F(X_i; \theta)] \sim \chi^2(2n)$$

Example 8.

Consider a random sample from a Pareto distribution, $X \sim PAR(\alpha, \theta = 300)$, find a $100(1 - \alpha)\%$ confidence interval for α .

7.3 Approximate Confidence Intervals

For discrete distributions, and for some multiparameter problems, a pivotal quantity may not exist. However, an approximate pivotal quantity often can be obtained based on asymptotic results. Let X_1, \dots, X_n be a random sample from a distribution with pdf $f(x; \theta)$. As noted in previous chapter, MLEs are asymptotically normal under certain condition.

Example 9.

Consider a random sample from a Bernoulli distribution, $X \sim BIN(1, p)$. Find an approximate confidence limits for p .

7.4 Credible Interval

A credible interval (or in general, a credible set) is the Bayesian analogue of a confidence interval. A $100(1 - \alpha)\%$ credible set C is a subset of Θ such that

$$\int_C \pi(\theta|\mathbf{x})d\theta = 1 - \alpha$$

If the parameter space Θ is discrete, a sum replaces the integral.

Definition 4.

If a is the $\frac{\alpha}{2}$ posterior quantile for θ , and b is the $1 - \frac{\alpha}{2}$ posterior quantile for θ , then (a, b) is a $100(1 - \alpha)\%$ **equal probability credible interval** for θ .

Example 10.

The following amounts were paid on a hospital liability policy:

125, 138, 142, 103, 137, 311, 127, 102, 144, 231.

The amount of a single payment has the single-parameter Pareto distribution with $\theta = 102$ and α unknown. The prior distribution has the gamma distribution with $\alpha = 3$ and $\theta = 1$. Determine the 96% equal probability credible interval for α .

The equal-tail credible interval approach is ideal when the posterior distribution is symmetric. If $\pi(\theta|\mathbf{x})$ is skewed, a better approach is to create an interval of θ -values having the Highest Posterior Density (HPD).

Definition 5.

A $100(1 - \alpha)\%$ HPD region for θ is a subset $C \in \Theta$ defined by

$$C = \{\theta : \pi(\theta|\mathbf{x}) \geq k\}$$

where k is the largest number such that

$$\int_{\theta: \pi(\theta|\mathbf{x}) \geq k} \pi(\theta|\mathbf{x}) d\theta = 1 - \alpha$$

The value k can be thought of as a horizontal line placed over the posterior density whose intersection(s) with the posterior define regions with probability $1 - \alpha$.

Theorem 1.

If the posterior random variable $\theta|\mathbf{x}$ is continuous and unimodal, then the $100(1 - \alpha)\%$ HPD credible interval is the unique solution to

$$\int_a^b \pi(\theta|\mathbf{x})d\theta = 1 - \alpha$$

$$\pi(a|\mathbf{x}) = \pi(b|\mathbf{x})$$

Example 11.

You are given the following:

$$f(x|\theta) = \frac{5x^4}{\theta^5}, 0 < x < \theta.$$

$$\pi(\theta) = \frac{6}{\theta^7}, \theta > 1.$$

Three observations were observed: 500, 600, 1300.
Find a 95% "HPD" credible set for θ .

Example 12.

The following amounts were paid on a hospital liability policy:

| | | | | |
|-----|-----|-----|-----|-----|
| 125 | 132 | 141 | 107 | 133 |
| 319 | 126 | 104 | 145 | 223 |

The amount of a single payment has the single-parameter Pareto distribution with $\theta = 100$ and α unknown. The prior distribution has the gamma distribution with $\alpha = 2$ and $\theta = 1$. Determine the 95% HPD credible interval for α .

a=1.1832, b = 3.9384

```
f = function(x){  
  y = numeric(2)  
  y[1] = pgamma(x[2],12,4.801121) - pgamma(x[1],12,4.801121) - 0.95  
  y[2] = dgamma(x[1],12,4.801121) - dgamma(x[2],12,4.801121)  
  y  
}  
library(nleqslv)  
xstart = c(1,3)  
nleqslv(xstart, f, control=list(btol=.01),  
method="Newton")
```

Example 13.

Let X_1, \dots, X_n , be a random sample from $N(\theta, 1)$. Assume that the prior distribution of Θ is $N(\mu, \sigma^2)$ with known μ and σ^2 .

- (a) Derive the posterior distribution of Θ .
- (b) Find a Bayesian interval of θ with confidence coefficient $1 - \alpha$.
- (c) Find the corresponding non-Bayesian confidence interval of θ using pivotal quantity method.