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## *CONTENTS*





# 3 Normal Theory Inference

## 3.1 Normal Distribution

**Definition 1.**

A random variable $Y$ with density function

$$f(y) = \frac{1}{\sqrt{2\pi}\sigma}\, e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}$$

is said to have a **normal** (*Gaussian*) **distribution** with

$$E(Y) = \mu \quad \text{and} \quad V(Y) = \sigma^2.$$

We will use the notation

$$Y \sim N(\mu, \sigma^2)$$

Suppose $Z$ has a normal distribution with $E(Z) = 0$ and $V(Z) = 1$, i.e.,

$$Z \sim N(0, 1),$$

then $Z$ is said to have a *standard normal* distribution.





**Definition 2.**

Suppose $\mathbf{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_m \end{bmatrix}$ is a random vector whose elements are independently distributed standard normal random variables. For any $m \times n$ matrix $A$, We say that

$$\mathbf{y} = \boldsymbol{\mu} + \mathbf{A}^T\mathbf{Z}$$

has a ***multivariate normal distribution*** with mean vector

$$\begin{aligned} E(\mathbf{y}) &= E(\boldsymbol{\mu} + \mathbf{A^T Z}) \\ &= \boldsymbol{\mu} + \mathbf{A^T} E(\mathbf{Z}) \\ &= \boldsymbol{\mu} + \mathbf{A^T 0} \\ &= \boldsymbol{\mu} \end{aligned}$$

and variance-covariance matrix

$$\begin{aligned} V(\mathbf{y}) &= \mathbf{A^T} V(\mathbf{Z})\mathbf{A} \\ &= \mathbf{A^T A} \equiv \boldsymbol{\Sigma} \end{aligned}$$





We will use the notation

$$\mathbf{y} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

When $\boldsymbol{\Sigma}$ is positive definite, the joint density function is

$$f(\mathbf{y}) = \frac{1}{(2\pi)^{n/2}|\boldsymbol{\Sigma}|^{1/2}}\, e^{-\frac{1}{2}(\mathbf{y}-\boldsymbol{\mu})^{\boldsymbol{T}}\boldsymbol{\Sigma}^{-1}(\mathbf{y}-\boldsymbol{\mu})}$$

The multivariate normal distribution has many useful properties:

**Result 1.** Normality is preserved under linear transformations: If

$$\mathbf{y} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}),$$

then

$$w = \mathbf{c^T y} \sim N(\mathbf{c}^T\boldsymbol{\mu}, \mathbf{c}^T\boldsymbol{\Sigma}\mathbf{c})$$

$$\mathbf{W} = \mathbf{c} + B\mathbf{y} \sim N(\mathbf{c} + B\boldsymbol{\mu}, B\boldsymbol{\Sigma}B^T)$$

for any non-random $\mathbf{c}$ and $B$.

**Result 2.**
Suppose

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix} \sim N\left( \begin{bmatrix} \boldsymbol{\mu_1} \\ \boldsymbol{\mu_2} \end{bmatrix}, \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{bmatrix} \right)$$

then

$$\mathbf{y}_1 \sim N(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_{11}).$$

**Note:** This result applies to any subset of the elements of $\mathbf{y}$ because you can move that subset to the top of the vector by multiplying $\mathbf{y}$ by an appropriate matrix of zeros and ones.



**Example 1.** Suppose

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \sim N\left( \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 & 1 & -1 \\ 1 & 3 & -3 \\ -1 & -3 & 9 \end{bmatrix} \right)$$

Find the distribution of

(a) $y_1$

(b) $y_2$

(c) $y_3$

(d) $\begin{bmatrix} y_1 \\ y_3 \end{bmatrix}$





If $w_1 = y_1 - 2y_2 + y_3$ and $w_2 = 3y_1 + y_2 - 2y_3$, then find the distribution of

(e) $w_1$

(f) $w_2$

(g) $\mathbf{W} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$





**Comment:**
If $\mathbf{y}_1 \sim N(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1)$ and $\mathbf{y}_2 \sim N(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_2)$, it is **not** always true that $\mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix}$ has a normal distribution.

**Result 3.**
If $\mathbf{y}_1$ and $\mathbf{y}_2$ are **independent** random vectors such that

$$\mathbf{y}_1 \sim N(\boldsymbol{\mu}_1, \boldsymbol{\Sigma_1}) \quad \text{and} \quad \mathbf{y}_2 \sim N(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_2)$$

then

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix} \sim N\left( \begin{bmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{bmatrix}, \begin{bmatrix} \boldsymbol{\Sigma}_1 & 0 \\ 0 & \boldsymbol{\Sigma}_2 \end{bmatrix} \right)$$

**Result 4.**

If $\mathbf{y}^T = [\mathbf{y}_1 \cdots \mathbf{y}_k]$ is a random vector with a multivariate normal distribution, then $\mathbf{y}_1$, $\mathbf{y}_2$, $\cdots$, $\mathbf{y}_k$ are **independent** if and only if $Cov(\mathbf{y}_i, \mathbf{y}_j) = 0$ for all $i \neq j$.

**Comments:**

(i) If $\mathbf{y}_i$ is independent of $\mathbf{y}_j$, then $Cov(\mathbf{y}_i, \mathbf{y}_j) = 0$.

(ii) When $\mathbf{y} = (y_1, \cdots, y_n)^T$ has a multivariate normal distribution, $y_i$ uncorrelated with $y_j$ implies $y_i$ is independent of $y_j$. This is usually not true for other distributions.



**Result 5.**

If

$$\begin{bmatrix} \mathbf{y} \\ \mathbf{x} \end{bmatrix} \sim N\left( \begin{bmatrix} \boldsymbol{\mu}_y \\ \boldsymbol{\mu}_x \end{bmatrix} \begin{bmatrix} \Sigma_{yy} & \Sigma_{yx} \\ \Sigma_{xy} & \Sigma_{xx} \end{bmatrix} \right)$$

with a positive definite covariance matrix, the **conditional distribution** of $\mathbf{y}$ given the value of $\mathbf{X}$ is a normal distribution with mean vector

$$E(\mathbf{y}|\mathbf{x}) = \boldsymbol{\mu}_y + \Sigma_{yx}\Sigma_{xx}^{-1}(\mathbf{x} - \boldsymbol{\mu}_x)$$

and positive definite covaraince matrix

$$V(\mathbf{y}|\mathbf{x}) = \Sigma_{yy} - \Sigma_{yx}\Sigma_{xx}^{-1}\Sigma_{xy}$$

note that this does not depend on the value of $\mathbf{x}$





## 3.2 Quadratic forms: $\mathbf{y}^{\mathbf{T}}\mathbf{A}\mathbf{y}$

Some useful information about the distribution of quadratic forms is summarized in the following results.

**Result 6.**

If $\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ is a random vector with

$$E(\mathbf{y}) = \boldsymbol{\mu}$$

and

$$V(\mathbf{y}) = \boldsymbol{\Sigma}$$

and $\mathbf{A}$ is an $n \times n$ non-random matrix, then

$$E(\mathbf{y}^{\mathbf{T}}\mathbf{A}\mathbf{y}) = \boldsymbol{\mu}^T\mathbf{A}\boldsymbol{\mu} + tr(\mathbf{A}\boldsymbol{\Sigma})$$





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**Example 2.**
Consider a Gauss-Markov model with

$$E(\mathbf{y}) = \mathbf{X}\boldsymbol{\beta} \text{ and } V(\mathbf{y}) = \sigma^2 I.$$

Show that $\hat{\sigma}^2 = \frac{SSE}{n - rank(\mathbf{X})}$ is an unbiased estimator of $\sigma^2$.



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## 3.3 Chi-square Distributions

**Definition 3.**

Let $\mathbf{Z} = \begin{bmatrix} Z_1 \\ \vdots \\ Z_n \end{bmatrix} \sim N(\mathbf{0}, I)$, i.e., the elements of $Z$ are $n$ independent standard normal random variables. The distribution of

$$W = \mathbf{Z}^T\mathbf{Z} = \sum_{i=1}^{n} Z_i^2$$

is called the **central chi-square distribution** with $n$ degrees of freedom.

We will use the notation

$$W \sim \boldsymbol{\chi}^2_{(n)}$$

The density function is

$$f(w) = \frac{1}{\Gamma(\frac{n}{2})2^{\frac{n}{2}}} w^{n/2-1} e^{-w/2}$$





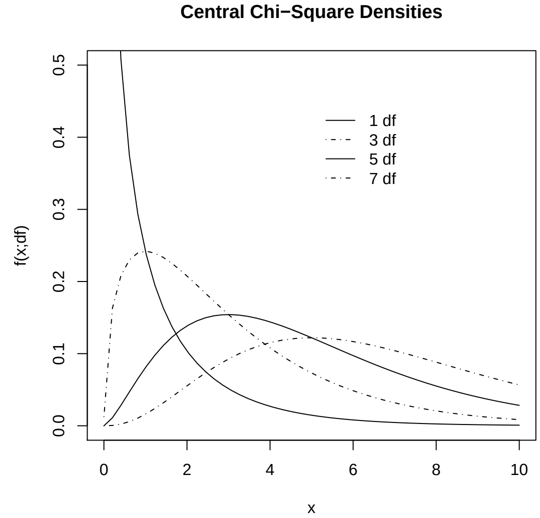
**Moments:**
If $W \sim \boldsymbol{\chi}^2_n$, then

(i) $E(W) = n$

(ii) $V(W) = 2n$

(iii) $M_W(t) = E(e^{tW}) = \frac{1}{(1-2t)^{n/2}}$

Note: The R-codes is store in the file: chidenR.txt.






**Definition 4.**
Let

$$\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \sim N(\boldsymbol{\mu}, I)$$

i.e., the elements of $\mathbf{y}$ are independent normal random variables with $y_i \sim N(\mu_i, 1)$. The distribution of the random variable

$$W = \mathbf{y}^T\mathbf{y} = \sum_{i=1}^{n} y_i^2$$

is called a **noncentral chi-square distribution** with $n$ degrees of freedom and noncentrality parameter

$$\lambda \ = \ \boldsymbol{\mu}^T\boldsymbol{\mu} \ = \ \sum_{i=1}^{n} \mu_i^2$$

We will use the notation

$$W \sim \boldsymbol{\chi}_n^2(\lambda)$$

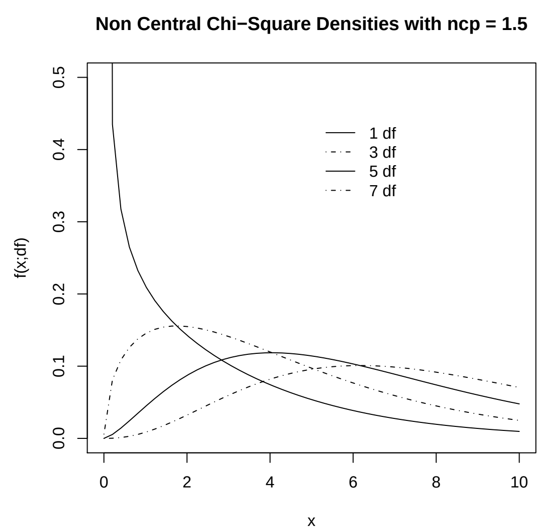




The density function is:

$$f(w) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \frac{w^{\frac{1}{2}n+k-1} e^{-w/2}}{2^{\frac{1}{2}n+k}\Gamma(\frac{1}{2}n+k)}$$

Note: The R-codes is store in the file: ncchidenR.txt.








## 3.4 $F$ Distribution

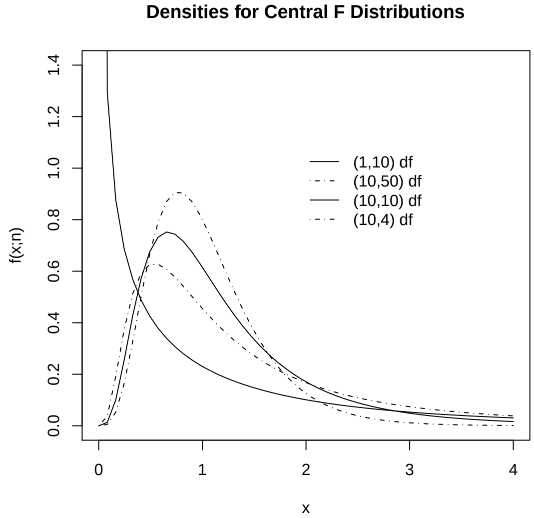
**Definition 5.**
If $W_1 \sim \boldsymbol{\chi}_{n_1}^2$ and $W_2 \sim \boldsymbol{\chi}_{n_2}^2$ and $W_1$ and $W_2$ are **independent**, then the distribution of

$$F = \frac{W_1/n_1}{W_2/n_2}$$

is called the **central F distribution** with $n_1$ and $n_2$ degrees of freedom.
We will use the notation

$$F \sim F_{n_1,n_2}$$

Note: The R-codes is store in the file: fdenR.txt.






**Definition 6.**
If $W_1 \sim \boldsymbol{\chi}^2_{n_1}(\lambda)$ and $W_2 \sim \boldsymbol{\chi}^2_{n_2}$ and $W_1$ and $W_2$ are **independent**, then the distribution of

$$F = \frac{W_1/n_1}{W_2/n_2}$$

is called a **noncentral F distribution** with $n_1$ and $n_2$ degrees of freedom and noncentrality parameter $\lambda$.

We will use the notation

$$F \sim F_{n_1,n_2}(\lambda)$$





## 3.5 Students's $t-$distribution

**Definition 7.**
If $Z \sim N(0,1)$ and $W \sim \chi^2_n$and $Z$ and$W$ are independent, then the distribution of

$$T = \frac{Z}{\sqrt{W/n}}$$

is called a student's $t-$distribution with $n$ degrees of freedom.

Its density function is

$$f(t) = \frac{\Gamma(\frac{1}{2}n + \frac{1}{2})}{\sqrt{n\pi}\Gamma(\frac{1}{2}n)} \left(1 + \frac{t^2}{n}\right)^{-\frac{1}{2}(n+1)}$$

We will use the notation

$$T \sim t_n$$

**Definition 8.**
If $y \sim N(\mu,1)$ and $W \sim \chi^2_n$ and $y$ and$W$ are independent, then the distribution of

$$T = \frac{Z}{W/n}$$





is called a noncentral student's $t-$distribution with $n$ degrees of freedom and non-central parameter $\mu$.

We will use the notation

$$T \sim t_n(\mu)$$

The density function is:

$$f(t) = \frac{n^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n)} \frac{e^{-\frac{1}{2}\mu^2}}{(n+t^2)^{\frac{1}{2}(n+1)}} \sum_{k=0}^{\infty} \frac{\Gamma(\frac{1}{2}n + \frac{1}{2}k + \frac{1}{2})\mu^k 2^{\frac{1}{2}k} t^k}{k!(n+t^2)^{\frac{1}{2}k}}$$

## 3.6 Sums of squares in ANOVA tables

Sums of squares in ANOVA tables are quadratic forms

$$\mathbf{y}^{\mathbf{T}}\mathbf{A}\mathbf{y}$$

where $\mathbf{A}$ is a non-negative definite symmetric matrix (**usually a projection matrix**).

To develop F-tests we need to identify conditions under which

- $\mathbf{y}^{\mathbf{T}}\mathbf{A}\mathbf{y}$ has a central (or noncentral) chi-square distribution
- $\mathbf{y}^{\mathbf{T}}\mathbf{A}_{\mathbf{i}}\mathbf{y}$ and $\mathbf{y}^{\mathbf{T}}\mathbf{A}_{\mathbf{j}}\mathbf{y}$ are independent



**Result 7.**
Let $\mathbf{A}$ be an $n \times n$ symmetric matrix with $\text{rank}(\mathbf{A}) = k$, and let

$$\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

where $\boldsymbol{\Sigma}$ is an $n \times n$ symmetric positive definite matrix. If

$$\mathbf{A}\boldsymbol{\Sigma} \text{ is idempotent}$$

then

$$\mathbf{y}^{\mathbf{T}}\mathbf{A}\mathbf{y} \sim \chi^2_k\left(\boldsymbol{\mu}^T\mathbf{A}\boldsymbol{\mu}\right)$$

In addition, if $\mathbf{A}\boldsymbol{\mu} = \mathbf{0}$ then

$$\mathbf{y}^{\mathbf{T}}\mathbf{A}\mathbf{y} \sim \chi^2_k$$





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**Example 3.**
For the Gauss-Markov model with

$$E(\mathbf{y}) = \mathbf{X}\boldsymbol{\beta} \quad \text{and} \quad V(\mathbf{y}) = \sigma^2\mathbf{I}$$

include the assumption that

$$\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \sim N(X\boldsymbol{\beta}, \sigma^2\mathbf{I}).$$

Show that $\frac{SSE}{\sigma^2} \sim \chi^2_{n-k}$.



**Example 4.**

Continuing Example 3, show that $\frac{1}{\sigma^2}\sum_{i=1}^{n}\hat{\mathbf{y}}_i^2 \sim \chi^2(\lambda)$, where $\lambda$ is the non-central parameter.





The next result addresses the independence of several quadratic forms

**Result 8.**
Let $\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$

and let $\mathbf{A_1}, \mathbf{A_2}, \ldots, \mathbf{A_p}$ be $n \times n$ symmetric matrices. If

$$\mathbf{A_i}\boldsymbol{\Sigma}\mathbf{A_j} = 0 \text{ for all } i \neq j$$

then

$$\mathbf{y^T A_1 y},\ \mathbf{y^T A_2 y},\ \ldots,\ \mathbf{y^T A_p y}$$

are independent random variables.





**Example 5.**
Continuing Example 3, show that the "uncorrected" model sum of squares

$$\sum_{i=1}^{n}\hat{y}_i^2 = \mathbf{y^T P_X y}$$

and the sum of squared residuals

$$\sum_{i=1}^{n}(y_i - \hat{y}_i)^2 = \mathbf{y^T(I - P_X)y}$$

are independently distributed for the "normal theory" Gauss-Markov model where

$$\mathbf{y} \sim N(X\boldsymbol{\beta}, \sigma^2\mathbf{I}).$$

**Example 6.**
If $\mathbf{y} = (y_1, y_2, \ldots, y_n)^T \sim N(\mu\mathbf{1}, \sigma^2\mathbf{I})$. Find the distribution of $\frac{(n-1)S^2}{\sigma^2} = \frac{\sum_{i=1}^{n}(y_i - \bar{y})^2}{\sigma^2}$.



**Example 7.**
Suppose that $\mathbf{y}$ is $N_3(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where

$$\boldsymbol{\mu} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}.$$

Let

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & -8 \\ -3 & 2 & -6 \\ -8 & -6 & 3 \end{bmatrix}$$

(a) Does $\mathbf{y^T A y}$ have a chi-square distribution?

(b) If $\boldsymbol{\Sigma} = \sigma^2 I$, does $\mathbf{y^T A y}/\sigma^2$ have a chi-square distribution?





**Example 8.** Suppose that $\mathbf{y}$ is $N_3(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and let

$$\boldsymbol{\mu} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}, \mathbf{A} = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}.$$

(a) What is the distribution of $\mathbf{y^T A y}/\sigma^2$?





(b) Are $\mathbf{y^T A y}$ and $\mathbf{By}$ independent?

(c) Are $\mathbf{y^T A y}$ and $y_1 + y_2 + y_3$ independent?

**Example 9.**

Consider the model

$$y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \epsilon_{ijk}$$

$$i = 1,2; j = 1,2,3; k = 1,2$$

where $\epsilon_{ijk} \sim NID(0, \sigma^2)$, $\alpha_i$ is associated with the $i$-th level of factor $A$, $\beta_j$ is associated with the $j$-th level of factor $B$, and $\gamma_{ij}$ is an interaction parameter.

(a) Define $SSE = \sum_{i1=}^{2}\sum_{j=1}^{3}\sum_{k=1}^{2}(y_{ijk} - \bar{y}_{ij\bullet})^2$, where $\bar{y}_{ij\bullet} = \frac{1}{2}(y_{ij1} + y_{ij2})$. Show that $\frac{SSE}{\sigma^2}$ has a chi-squares distribution. States the degrees of fredom.



(b) Consider the estimator

$$\hat{C} = \bar{y}_{\bullet 3 \bullet} - \bar{y}_{\bullet 1 \bullet},$$

where

$$\bar{y}_{\bullet j \bullet} = \frac{1}{4}\sum_{i=1}^{2}\sum_{k=1}^{2} y_{ijk}.$$

Show that

$$F = \frac{m(\hat{C})^2}{SSE}$$

has an F-distribution for some constant $m$. Report the value of $m$ and the degrees of freedom for the F-distribution.





## 3.7 Hypotesis Test for $E(\mathbf{y})$

In Example 3 we showed that

$$\frac{1}{\sigma^2}\sum_{i=1}^{n}\hat{y}_i^2 \sim \chi_k^2\left(\frac{\boldsymbol{\beta}^T\mathbf{X}^\mathbf{T}\mathbf{X}\boldsymbol{\beta}}{2\sigma^2}\right)$$

and

$$\frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2 \sim \chi_{n-k}^2$$

where $k = \text{rank}(\mathbf{X})$.

By Defn 6,

$$F = \frac{\frac{1}{k\sigma^2}\sum_{i=1}^{n}\hat{y}_i^2}{\frac{1}{(n-k)\sigma^2}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}$$

uncorrected model mean square

$$= \frac{\frac{1}{k}\sum_{i=1}^{n}\hat{y}_i^2}{\frac{1}{n-k}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}$$

Residual mean square





$$\sim F_{k,n-k}\left(\frac{1}{2\sigma^2}\boldsymbol{\beta}^T\mathbf{X}^\mathbf{T}\mathbf{X}\boldsymbol{\beta}\right)$$

This reduces to a central F distribution with $(k, n-k)$ d.f. when $\mathbf{X}\boldsymbol{\beta} = \mathbf{0}$

Use

$$F = \frac{\frac{1}{k}\sum_{i=1}^{n}\hat{y}_i^2}{\frac{1}{n-k}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}$$

to test the null hypothesis

$$H_0 : E(\mathbf{y}) = \mathbf{X}\boldsymbol{\beta} = \mathbf{0}$$

against the alternative

$$H_A : E(\mathbf{y}) = \mathbf{X}\boldsymbol{\beta} \neq \mathbf{0}$$

**Comments**

(i) The null hypothesis corresponds to the condition under which F has a central F distribution (**the non-centrality parameter is zero**).

$$\lambda = \frac{1}{2\sigma^2}(\mathbf{X}\boldsymbol{\beta})^T(\mathbf{X}\boldsymbol{\beta}) = 0$$

if and only if $\mathbf{X}\boldsymbol{\beta} = \mathbf{0}$.

(ii) If $k = \text{rank}(\mathbf{X})$ = number of columns in $\mathbf{X}$, then $H_0 : \mathbf{X}\boldsymbol{\beta} = \mathbf{0}$ is equivalent to $H_0 : \boldsymbol{\beta} = \mathbf{0}$.

(iii) If $k = \text{rank}(X)$ is less than the number of columns in $\mathbf{X}$, then $\mathbf{X}\boldsymbol{\beta} = \mathbf{0}$ for some $\boldsymbol{\beta} \neq \mathbf{0}$ and $H_0 : \mathbf{X}\boldsymbol{\beta} = 0$ is **not** equivalent to $H_0 : \boldsymbol{\beta} = \mathbf{0}$.

Example 4 is a simple illustration of a typical

$$\begin{aligned}
\sum_{i=1}^{n} y_i^2 &= \mathbf{y}^T\mathbf{y} \\
&= \mathbf{y}^T\left[(\mathbf{I} - \mathbf{P_X}) + \mathbf{P_X}\right]\mathbf{y} \\
&= \mathbf{y}^T\underbrace{(\mathbf{I} - \mathbf{P_X})}_{\text{call this } \mathbf{A_2}}\mathbf{y} + \mathbf{y^T}\underbrace{\mathbf{P_X}}_{\text{call this } \mathbf{A_1}}\mathbf{y} \\
&= \underbrace{\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}_{\text{d.f.} = \text{rank}(\mathbf{A_2})} + \underbrace{\sum_{i=1}^{n}\hat{y}_i^2}_{\text{d.f.} = \text{rank}(\mathbf{A_1})}
\end{aligned}$$



More generally an uncorrected total sum of squares can be partitioned as

$$\begin{aligned}
\sum_{i=1}^{n} y_i^2 &= \mathbf{y^T}\mathbf{y} \\
&= \mathbf{y^T}\mathbf{A_1}\mathbf{y} + \mathbf{y^T}\mathbf{A_2}\mathbf{y} + \\
&= \cdots + \mathbf{y^T}\mathbf{A_k}\mathbf{y}
\end{aligned}$$

using orthogonal projection matrices

$$\mathbf{A_1} + \mathbf{A_2} + \cdots + \mathbf{A_k} = \mathbf{I_{n \times n}}$$

where

$$\text{rank}(\mathbf{A_1}) + \text{rank}(\mathbf{A_2}) + \cdots + \text{rank}(\mathbf{A_k}) = n$$

and

$$\mathbf{A_i}\mathbf{A_j} = \mathbf{0} \quad \text{for any } i \neq j\,.$$

Since we are dealing with orthogonal projection matrices we also have

$$\mathbf{A_i^T} = \mathbf{A_i} \qquad \text{(symmetry)}$$

$$\mathbf{A_i}\mathbf{A_i} = \mathbf{A_i} \qquad \text{(idempodent matrices)}$$





**Result 9.**
Let $\mathbf{A_1}, \mathbf{A_2}, \cdots, \mathbf{A_k}$ be $n \times n$ symmetric matrices such that

$$\mathbf{A_1} + \mathbf{A_2} + \cdots + \mathbf{A_k} = \mathbf{I}\,.$$

Then the following statments are equivalent

(i) $\mathbf{A_i}\mathbf{A_j} = \mathbf{0}$ for any $i \neq j$

(ii) $\mathbf{A_i}\mathbf{A_i} = \mathbf{A_i}$ for all $i = 1, \ldots, k$

(iii) $\text{rank}(\mathbf{A_1}) + \cdots + \text{rank}(\mathbf{A_k}) = n$





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**Result 10.** (Cochran's Theorem)

Let $\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \sim N(\boldsymbol{\mu}, \sigma^2 I)$

and let $\mathbf{A_1}, \mathbf{A_2}, \cdots, \mathbf{A_k}$ be $n \times n$ symmetric matrices with

$$\mathbf{I} = \mathbf{A_1} + \mathbf{A_2} + \cdots + \mathbf{A_k}$$

and

$$n = r_1 + r_2 + \cdots + r_k$$

where $r_i = \text{rank}(\mathbf{A_i})$ . Then, for $i = 1, 2, \ldots, k$

$$\frac{1}{\sigma^2}\mathbf{y^T A_i y} \sim \chi^2_{r_i}\left(\frac{1}{\sigma^2}\boldsymbol{\mu}^{\mathbf{T}}\mathbf{A_i}\boldsymbol{\mu}\right)$$

and

$$\mathbf{y^T A_1 y},\ \mathbf{y^T A_2 y},\ \cdots,\ \mathbf{y^T A_k y}$$

are distributed independently.





**Example 10.**
Consider the model

$$y_i = \beta_1 X_{1i} + \beta_2 X_{2i} + \epsilon_i$$

where $\epsilon_i \sim N(0, \sigma^2)$ and the data as follow:

| $y$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ | $y_6$ |
|---|---|---|---|---|---|---|
| $X_1$ | 0 | 15 | 30 | 0 | 15 | 30 |
| $X_2$ | -1 | -1 | -1 | 1 | 1 | 1 |

(a) Let SSE denote the sum of squared residuals for this model, what is the distribution of SSE?





(b) Let **b** be a solution to the normal equations. What are the properties of **b**?

(c) Show that

$$F = \frac{2(y_4 + y_5 + y_6 - y_1 - y_2 - y_3)^2}{3SSE}$$

has an F-distribution. Report degrees of freedom.

(d) With respect to $\boldsymbol{\beta} = (\beta_1, \beta_2)^T$, describe the null hypothesis that can be tested with the $F$-test in Part (c). What is the alternative hypothesis?

(e) Does

$$F = \frac{3\left(\sum a_i y_i\right)^2}{(\sum a_i^2)SSE} = \frac{2(\mathbf{a^T y})^2}{(\mathbf{a^T a})SSE}$$

have an $F$-distribution for any vector of constants $\mathbf{a} = (a_1, a_2, a_3, a_4, a_5, a_6)^T$?



**Example 11.** Suppose $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$ and $\boldsymbol{\epsilon} \sim N(\mathbf{0}, \sigma^2\mathbf{I})$. Define

$$\mathbf{X} = \begin{bmatrix} 1 & X_1 & X_1^2 \\ 1 & X_2 & X_2^2 \\ \vdots & \vdots & \\ 1 & X_{40} & X_{40}^2 \end{bmatrix} \text{ and } \mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \text{ and } \mathbf{P_x} = \mathbf{X}(\mathbf{X^T X})^{-}\mathbf{X^T}$$

and $\mathbf{P_1} = \mathbf{1}(\mathbf{1^T 1})^{-}\mathbf{1^T}$. Find the distribution of $\frac{1}{\sigma^2}\mathbf{Y^T}(\mathbf{P_X} - \mathbf{P_1})\mathbf{Y}$ and $\frac{1}{\sigma^2}\mathbf{Y^T}(\mathbf{I} - \mathbf{P_X})\mathbf{Y}$. Then, derive the distribution of $V = \dfrac{c\mathbf{Y^T}(\mathbf{P_X} - \mathbf{P_1})\mathbf{Y}}{\mathbf{Y^T}(\mathbf{I} - \mathbf{P_X})\mathbf{Y}}$. Report $c$, degrees of freedom and a formula for the noncentrality parameter.