MEME15203 Statistical Inference Marking Guide

Test 2

UNIVERSITI TUNKU ABDUL RAHMAN

Faculty:	FES	Unit Code:	MEME15203
Course:	MAC	Unit Title:	Statistical Inference
Year:	1,2	Lecturer:	Dr Yong Chin Khian
Session:	January 2022		

Show your workings. If no workings are shown, ZERO is awarded.

Q1. Consider a random sample of size n from a uniform distribution, $X_i \sim U(0, \theta)$. Find the UMVUE of the p^{th} percentile.

(20 marks)

Ans.

$$f(x_i) = \frac{1}{\theta}, 0 \le x_i \le \theta$$
$$f(x_1, x_2, \dots, x_n) = \theta^{-n}, x_{(n)} < \theta$$
$$= g(s, \theta)h(x_1, \dots, x_n)$$

where $g(s,\theta) = \theta^{-n} I(X_{(n)}) < \theta)$ and $h(x_1, \dots, x_n) = 1$,
By factorization Theorem, $S = X_{(n)}$ is a sufficient statistic for θ.

$$f_S(s) = n f_X(s)[F_X(s)]^{(n-1)} = n\frac{1}{\theta}\left(\frac{s}{\theta}\right)^{(n-1)} = \frac{n}{\theta^n}s^{n-1}, 0 \le s \le \theta$$

$$E[u(S)] = \int_0^\theta u(s)\frac{n}{\theta^n}s^{n-1}ds = 0 \ \forall\theta$$
$$\Rightarrow \int_0^\theta u(s)s^{n-1}ds = 0 \ \forall\theta$$
$$\frac{d}{d\theta}\int_0^\theta u(s)s^{n-1}ds = u(\theta)\theta^{n-1} = 0 \ \forall\theta$$

This implies $u(\theta) = 0$ for all θ, so $S = X_{(n)}$ is a complete sufficient statistic.

$$E(S) = \int_0^\theta s\frac{n}{\theta^n}s^{n-1}ds = \frac{n}{\theta^n}\left[\frac{s^{n+1}}{n+1}\right]_0^\theta = \frac{n}{n+1}\theta$$

$$F_X(\pi_p) = p$$
$$\frac{\pi_p}{\theta} = p$$
$$\pi_p = p\theta$$

Let $S_1 = \frac{(n+1)p}{n}S$, then
$E(S_1) = \frac{(n+1)p}{n}E(S) = \frac{(n+1)p}{n}\left[\frac{n}{n+1}\theta\right] = p\theta$
Since $S_1 = \frac{(n+1)p}{n}S$ is a function of css for θ and unbisaed for π_p. Then, $S_1 = \frac{(n+1)p}{n}S$ is the UMVUE of the p^{th} percentile.

Q2. Suppose that $X_1, \ldots, X_{30}$ is a random sample from a Gamma distribution, $X_i \sim$ GAM$(\alpha = 6, \theta)$,

(a) Show that the p.d.f. of X belongs to the regular exponential family.

(b) Find a complete and sufficient statistic for θ.

(c) Find the UMVUE for $\frac{1}{1-5\theta}$.

(15 marks)

Ans.

(a) $f(x) = \frac{1}{\Gamma(6)\theta^6} x^{6-1} e^{-x/\theta} = c(\theta)h(x)e^{q(\theta)t(x)}$
where $c(\theta) = \theta^{-6}$, $h(x) = 1/\Gamma(6)x^{6-1}$, $q(\theta) = 1/\theta$, and $t(x) = x$.
Thus the p.d.f. of X belongs to the regular exponential family.

(b) Since the p.d.f. of X belongs to the regular exponential family, thus by the theorem, $S = \sum_{i=1}^{30} X_i$ is a c.s.s of θ

(c) $S \sim GAM(180, \theta)$
$E(e^{5S}) = (1 - 5\theta)^{-180}$
$E(e^{5S(\frac{1}{180})}) = [(1 - 5\theta)^{-180}]^{\frac{1}{180}}$
Thus $e^{\frac{5S}{180}}$ is an UE of $\frac{1}{1-5\theta}$.
Since $e^{\frac{5S}{180}}$ is a function of the c.s.s. of θ
which is an UE of $\frac{1}{1-5\theta}$,
thus $e^{\frac{5S}{180}}$ is the UMVUE of $\frac{1}{1-5\theta}$.

Q3. Suppose that $X_1, \ldots, X_{48}$ is a random sample from a Poisson distribution, $X_i \sim$ POI(θ). Find the UMVUE of $e^{-10\theta}$ using Rao-Blackwell theorem.

(15 marks)

Ans.

$f(x) = \frac{1}{\theta} e^{-x/\theta} = c(\theta)h(x)e^{t(x)q(\theta)}$ which is in a member of REC. Hence $S = \sum X_i$ is a CSS of θ.
Let

$$T = \begin{cases} 1, & X_1 + \cdots + X_{10} = 0 \\ 0, & \text{otherwise} \end{cases}.$$

$E(T) = P(X_1 + \cdots + X_{10} = 0) = e^{-10\theta}$. Thus T is and unbiased estimator of $e^{-10\theta}$. Since S is CSS of θ. Hence by Rao-Blackwell therem, $T^* = E(T|S)$ is an UMVUE of $e^{-10\theta}$.

$E\left[T|\sum_{i=0}^{n} X_i = s\right]$

$= 1 \cdot P[X_1 + \cdots + X_{10} = 0|X_1 + X_2 + \cdots X_n = s]$

$= \dfrac{P(X_1 + \cdots + X_{10} = 0, X_{11} + \cdots + X_n = s)}{P(X_1 + \cdots + X_n = s)}$

$= \dfrac{P(X_1 + \cdots + X_{10} = 0) \times P(X_{11} + \cdots + X_n = s)}{P(X_1 + \cdots + X_n = s)}$ Since $X_1, X_2, \ldots, X_n$ are independent.

$= \dfrac{e^{-10\theta}[(n-10)\theta]^s e^{-(n-10)\theta}}{(n\theta)^s s! e^{-n\theta}/s!}$

$= \left(\frac{n-10}{n}\right)^s$

Q4. Let $X_1, X_2, \ldots, X_n$ denote a random sample from an exponentially distributed population with density $f(x|\theta) = \theta e^{-\theta x}, 0 < x$. Let $\Theta \sim GAM(\alpha = 4, \beta = \frac{1}{900})$. Find the Bayes estimator for θ^{-r} under square error loss.

(15 marks)

Ans.

$\pi(\theta) = \frac{1}{\Gamma(4)900^4}\theta^{4-1}e^{-900\theta}, \theta > 0$

$\pi(\theta|\mathbf{x}) = k\theta^{4+n-1}e^{-\theta(\sum x_i + 900)}, \theta > 0$

$\therefore \Theta|\mathbf{x} \sim GAM(4+n, (n\bar{x}+900)^{-1})$

$$\hat{\mu} = E(\Theta^{-r}) = \int_0^\infty \theta^{-r}\frac{(n\bar{x}+900)^{4+n}}{\Gamma(4+n)}\theta^{4+n-1}e^{-\theta(n\bar{x}+900)}d\theta$$
$$= \frac{(n\bar{x}+900)^{4+n}}{\Gamma(4+n)}\int_0^\infty \theta^{4+n-r-1}e^{-\theta(n\bar{x}+900)}d\theta$$
$$= \frac{(n\bar{x}+900)^{4+n}}{\Gamma(4+n)}\frac{\Gamma(4+n-r)}{(n\bar{x}+900)^{4+n-r}}$$
$$= \frac{\Gamma(4+n-r)(n\bar{x}+900)^r}{\Gamma(4+n)}$$

Q5. A pmf on the integers with an integer parameter, θ , is

$$f(x|\theta) = \frac{1}{2}I[x = \theta - 7 \text{ or } x = \theta + 7].$$

For X_1 and X_2 iid from this distribution, compare MSE's for the two estimators of θ,

$$\hat{\theta} = \begin{cases} X_1 + 7, & \text{if } X_1 = X_2 \\ \bar{X}, & \text{if } X_1 \neq X_2 \end{cases} \quad \text{or } \tilde{\theta} = \hat{\theta} - \frac{7}{2}.$$

(20 marks)

Ans.

$$\begin{aligned} E(\hat{\theta}) &= (x_1 + 7)P(X_1 = X_2) + \bar{x}P(X_1 \neq X_2) \\ &= (\theta - 7 + 7)P[X_1 = \theta - 7, X_2 = \theta - 7] + (\theta + 7 + 7)P[X_1 = \theta + 7, X_2 = \theta + 7] \\ &+ (\tfrac{\theta-7+\theta+7}{2})P[X_1 = \theta - 7, X_2 = \theta + 7] + (\tfrac{\theta+7+\theta-7}{2})P[X_1 = \theta + 7, X_2 = \theta - 7] \\ &= \theta\left(\tfrac{1}{4}\right) + (\theta + 2(7)\left(\tfrac{1}{4}\right) + \theta\left(\tfrac{1}{4}\right) + \theta\left(\tfrac{1}{4}\right) \\ &= \theta + \tfrac{7}{2} \end{aligned}$$

$$Bias(\hat{\theta}) = E(\hat{\theta}) - \theta = \tfrac{7}{2}$$

$$\begin{aligned} E(\hat{\theta})^2 &= (x_1 + 7)^2P(X_1 = X_2) + \bar{x}^2P(X_1 \neq X_2) \\ &= (\theta - 7 + 7)^2P[X_1 = \theta - 7, X_2 = \theta - 7] + (\theta + 7 + 7)^2P[X_1 = \theta + 7, X_2 = \theta + 7] \\ &+ (\tfrac{\theta-7+\theta+7}{2})^2P[X_1 = \theta - 7, X_2 = \theta + 7] + (\tfrac{\theta+7+\theta-7}{2})^2P[X_1 = \theta + 7, X_2 = \theta - 7] \\ &= \theta^2\left(\tfrac{1}{4}\right) + (\theta + 2(7))^2\left(\tfrac{1}{4}\right) + \theta^2\left(\tfrac{1}{4}\right) + \theta^2\left(\tfrac{1}{4}\right) \\ &= \theta^2 + 7\theta + 49 \end{aligned}$$

$V(\hat{\theta}) = E(\hat{\theta})^2 - E^2(\hat{\theta}) = \theta^2 + 7\theta + 49 - (\theta + \frac{7}{2})^2 = \theta^2 + 7\theta + 49 - (\theta^2 + 7\theta + \frac{49}{4}) = 36.75$
$MSE(\hat{\theta}) = V(\hat{\theta}) + Bias^2(\hat{\theta}) = \frac{3}{4}(49) + (\frac{7}{2})^2 = 49.0$
$E(\tilde{\theta}) = E(\hat{\theta} - \frac{7}{2}) = \theta$, thus, $\tilde{\theta}$ is an unbiased estimator of θ.
$V((\tilde{\theta}) = V(\hat{\theta} - \frac{7}{2}) = V(\hat{\theta}) = 36.75$
$MSE(\tilde{\theta}) = V((\tilde{\theta}) = 36.75$
Thus, $MSE(\tilde{\theta}) < MSE(\hat{\theta})$

Q6. Let $X_1, X_2, \ldots, X_n$ denote a random sample of size n from a population whose density is given by

$$f(x) = \begin{cases} 6\theta^6x^{-7}, & x \geq \theta \\ 0, & \text{otherwise} \end{cases}$$

where $\beta > 0$ is unknown.

(a) Find the MME of θ.

(b) Find the MLE of θ.

(c) Find the CRLB of θ.

(15 marks)

Ans.

(a) $X \sim SP(\alpha = 6, \theta)$
$E(X) = \frac{6\theta}{5}$
$\frac{6\theta}{5} = \bar{X}$
$\hat{\theta} = \frac{5\bar{X}}{6}$

(b) $L(\theta) = 6^n \theta^{6n} \prod_{i=1}^{n} x_i^{-7}, x_{(1)} < \theta$
Since $L(\theta)$ is an increasing function of θ, thus, the MLE of θ is $\hat{\theta} = X_{(1)}$.

(c) $\ln(f(x)) = \ln(6) + 6\ln(\theta) - 7\ln(x)$
$\frac{\partial \ln(f(x))}{\partial \theta} = \frac{6}{\theta}$
$\frac{\partial^2 \ln(f(x))}{\partial \theta^2} = \frac{-6}{\theta^2}$
$-E\left[\frac{\partial^2 \ln(f(x))}{\partial \theta^2}\right] = \frac{6}{\theta^2}$
CRLB of $\theta = \frac{1}{-nE\left[\frac{\partial^2 \ln(f(x))}{\partial \theta^2}\right]} = \frac{\theta^2}{6n}$