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5 Data Reduction

5.1 Sufficient Statistics

Definition 1. Jointly Sufficient Statistics

Let $\mathbf{X} = (X_1, \dots, X_n)$ have joint pdf $f(\mathbf{x}, \boldsymbol{\theta})$, and let $S = (S_1, \dots, S_k)$ be a k -dimensional statistic. Then S_1, \dots, S_k is a set of **jointly sufficient statistics** for $\boldsymbol{\theta}$ if for any other vector of statistics, \mathbf{T} , the conditional pdf of \mathbf{T} given $\mathbf{S} = \mathbf{s}$, denoted by $f_{\mathbf{T}|\mathbf{S}}(t)$, does not depend on $\boldsymbol{\theta}$. In the one-dimensional case, we simply say that S is a **sufficient statistic** for θ .

Example 1.

Let X_1, \dots, X_n be a random sample from a Bernoulli distribution. Show that $S = \sum X_i$ is a sufficient statistic for θ by definition.

Example 2.

Consider a random sample from an exponential distribution, $X_i \sim EXP(\theta)$. Show that $S = \sum X_i$ is a sufficient statistic for θ by definition.

5.2 Factorization Theorem

Theorem 1. Factorization Criterion

If X_1, \dots, X_n have joint pdf $f(x_1, \dots, x_n; \theta)$, and if $S = (S_1, \dots, S_k)$, then S_1, \dots, S_k are jointly sufficient for θ if and only if

$$f(x_1, \dots, x_k; \boldsymbol{\theta}) = g(\mathbf{s}; \boldsymbol{\theta})h(x_1, \dots, x_n)$$

where $g(\mathbf{s}; \boldsymbol{\theta})$ does not depend on x_1, \dots, x_n , except through s , and $h(x_1, \dots, x_n)$ does not involve θ .

Example 3.

Consider a random sample from a Gamma distribution, $X_i \sim \text{Gamma}(\alpha = 3, \theta)$. Show that $S = \sum X_i$ is a sufficient statistic for θ by factorization theorem.

Definition 2. If A is a set, then the indicator function of A , denoted by I_A is defined as

$$I_A = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$$

Example 4. Consider a random sample from a uniform distribution, $X_i \sim U(0, \theta)$, where θ is unknown. Find a sufficient statistic for θ .

Example 5. Consider a random sample from a uniform distribution, $X_i \sim U(\theta, \theta + 1)$. Notice that the length of the interval is one unit, but the endpoints are assumed to be unknown. Show that $S_1 = X_{1:n}$ and $S_2 = X_{n:n}$ are jointly sufficient for θ .

Example 6.

Consider a random sample from a normal distribution, $X_i \sim N(\mu, \sigma^2)$, where both μ and σ^2 are unknown. Show that $S_1 = \sum X_i$ and $S_2 = \sum X_i^2$ are jointly sufficient for $\boldsymbol{\theta} = (\mu, \sigma^2)$.

5.3 Rao-Blackwell

Theorem 2. Rao-Blackwell Let X_1, \dots, X_n , have joint pdf $f(x_1, \dots, x_n; \boldsymbol{\theta})$, and let $S = (S_1, \dots, S_k)$ be a vector of jointly sufficient statistics for $\boldsymbol{\theta}$. If T is any unbiased estimator of $\tau(\boldsymbol{\theta})$, and if $T^* = E(T|S)$, then

1. T^* is an unbiased estimator of $\tau(\boldsymbol{\theta})$
2. T^* is a function of S , and
3. $V(T^*) \leq V(T)$ for every $\boldsymbol{\theta}$, and $V(T^*) < V(T)$ for some $\boldsymbol{\theta}$ unless $T^* = T$ with probability 1.

Example 7.

Let X_1, X_2, \dots, X_n be random sample of size n from a Poisson distribution with unknown mean λ . Let

$$T = \begin{cases} 1, & X_1 = x \\ 0, & \text{otherwise} \end{cases}.$$

$$\text{Find } E \left[T \mid \sum_{i=1}^n X_i = s \right].$$

Example 8.

Let X_1, X_2, \dots, X_n be random sample of size n from an Exponential distribution with unknown mean θ . Find the UMVUE of $\gamma = 1 - e^{-t/\theta}$ using Rao-Blackwell theorem.

5.4 Completeness

Definition 3. Completeness A family of density functions $\{f_{\mathbf{T}}(t; \boldsymbol{\theta}); \boldsymbol{\theta} \in \Omega\}$ is called complete if $E[u(T)] = 0$ for all $\boldsymbol{\theta} \in \Omega$ implies $u(T) = 0$ with probability 1 for all $\boldsymbol{\theta} \in \Omega$.

A sufficient statistic the density of which is a member of a complete family of density functions will be referred to as a **complete sufficient statistic**.

Example 9.

Let X_1, \dots, X_n denote a random sample from a Geometric distribution, $X_i \sim \text{Geometric}(p)$. Show that $S = \sum X_i$ is the complete sufficient statistic for p .

Example 10.

Let X_1, \dots, X_n be iid $N(\theta, a\theta^2)$, where a is known constant and $\theta > 0$. Show that the family of distribution is not complete.

Example 11.

Consider a random sample of size n from a uniform distribution $X \sim U(0, \theta)$. Show that $S = X_{n:n}$ is a complete sufficient statistic for θ .

5.5 Exponential Class

Definition 4. Exponential Class A density function is said to be a member of the regular exponential class if it can be expressed in the form

$$f(\mathbf{x}; \boldsymbol{\theta}) = c(\boldsymbol{\theta})h(\mathbf{x})e^{-\sum_{j=1}^n q_j(\boldsymbol{\theta})t_j(\mathbf{x})}, \mathbf{x} \in A$$

and zero otherwise, where $\boldsymbol{\theta} = (\theta_1, \dots, \theta_k)$ is a vector of k unknown parameters, if the parameter space has the form

$$\Omega = \{\boldsymbol{\theta} | a_i \leq \theta_i \leq b_i, i = 1, \dots, k\}$$

(note that $a_i = -\infty$ and $b_i = \infty$ are permissible values), and if it satisfies regularity conditions 1, 2, and 3(a) or 3(b) given by:

1. The set $A = \{x : f(x; \boldsymbol{\theta}) > 0\}$ does not depend on $\boldsymbol{\theta}$.
2. The functions $q_j(\boldsymbol{\theta})$ are nontrivial, functionally independent, continuous functions of the $\boldsymbol{\theta}$.
- 3.(a) For a continuous random variable, the derivatives $t_j(x)$ are linearly independent contin-

uous functions of x over A .

- (b) For a discrete random variable, the $t_j(x)$ are nontrivial functions of x on A , and none is a linear function of the others.

For convenience, we will write that $f(x; \boldsymbol{\theta})$ is a member of $REC(q_1, \dots, q_k)$ or simply REC .

Theorem 3. X_1, \dots, X_n , is a random sample from a member of the regular exponential class $REC(q_1, \dots, q_k)$, then the statistics

$$S_1 = \sum_{i=1}^n t_1(X_i), \dots, S_k = \sum_{i=1}^n t_k(X_i)$$

are a minimal set of complete sufficient statistics for $\theta_1, \dots, \theta_k$.

Example 12.

Show that $X \sim \text{Gamma}(\alpha, \theta)$ belong to the regular exponential class, and use this information to find complete sufficient statistics based on a random sample X_1, \dots, X_n .

5.6 Lehmann-Scheffe

Theorem 4. Lehmann-Scheffe Let X_1, \dots, X_n have joint pdf $f(x_1, \dots, x_n; \boldsymbol{\theta})$, and let S be a vector of jointly complete sufficient statistics for $\boldsymbol{\theta}$. If $T^* = t^*(S)$ is a statistic that is unbiased for $\tau(\boldsymbol{\theta})$ and a function of S , then T^* is a UMVUE of $\tau(\boldsymbol{\theta})$.

Example 13.

Let X_1, \dots, X_n be a random sample from a Bernoulli distribution, $X_i \sim \text{Bin}(1, p)$; $0 < p < 1$. Find the UMVUE of p^2 .

Example 14.

Let X_1, \dots, X_{30} be a random sample from a distribution with probability density function (p.d.f.)

$$f(x) = \frac{\theta^6}{\Gamma(6)} x^5 e^{-\theta x} I(0, \infty), \theta > 0.$$

- (a) Show that the p.d.f. of X belongs to the regular exponential family.
- (b) Find a complete and sufficient statistic for θ .
- (c) Find the UMVUE for $V(X_1)$.
- (d) Find the UMVUE for θ .

Example 15.

Let X_1, X_2, \dots, X_n be random sample of size n from $f(x|\theta) = \binom{18}{x} \theta^x (1-\theta)^{18-x}$, $x = 0, 1, \dots, 18$. Find the uniformly minimum variance unbiased estimator(UMVUE) of $g(\theta) = \binom{18}{2} \theta^2 (1-\theta)^{16}$.

Example 16.

Let X_1, X_2, \dots, X_n be a random sample from a distribution with pdf

$$f(x; \theta) = 4\theta x^{4\theta-1}.$$

Find the UMVUE of θ ,

5.7 Basu Theorem

Theorem 5. Basu Let X_1, \dots, X_n have joint pdf $f(x_1, \dots, x_n; \theta)$; $\theta \in \Omega$. Suppose that $S = (S_1, \dots, S_k)$ where S_1, \dots, S_k are jointly complete sufficient statistics for θ , and suppose that T is any other statistic. If the distribution of T does not involve θ , then S and T are stochastically independent. In this case, T is called an ancillary statistic.

Example 17. Consider a random sample of size n from a normal distribution $X \sim N(\mu, \sigma^2)$. Use Basu Theorem to show that \bar{X} and S^2 are independent.

Example 18.

Consider a random sample of size n from a two-parameter exponential distribution, $X_i \sim EXP(1, \eta)$.

- (a) Show that $T(X) = X_{1:n}$ is complete and sufficient for η .
- (b) Find the UMVUE of η .
- (c) Find the UMVUE of η^2 .
- (d) Use Basu's Theorem to show that $X_{1:n}$ and $S^2 = \frac{\sum (X_i - \bar{X})^2}{n-1}$ are independent.