MEME15203 Statistical Inference Marking Guide

Assignment 3

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Q1. Suppose that $X_1, \dots, X_n$ is a random sample from a Poisson distribution, $X_i \sim$ POI(θ),

(a)Find a complete and sufficient statistic for θ.

(b)Find the UMVUE for θ

(c)Find the UMVUE of $e^{-7\theta}$ using Lehmann Scheffe Theorem.

(d)Find the UMVUE of $e^{-7\theta}$ using Rao Blackwell Theorem.

(20 marks)

Ans.

(a)$f(x) = e^{-\theta}(1/x!)e^{x\ln\theta} = c(\theta)h(x)e^{q(\theta)t(x)}$
where $c(\theta) = e^{-\theta}$, $h(x) = 1/(x!)$, $q(\theta) = \ln\theta$, and $t(x) = x$.
Thus the p.d.f. of X belongs to the regular exponential family. Since the p.d.f. of X belongs to the regular exponential family, thus by the theorem, $S = \sum_{i=1}^{n} X_i$ is a c.s.s of θ

(b)Since $E(\sum_{i=1}^{n} X_i/n) = E(\bar{X}) = \theta$, thus, $\bar{X}$ is the UMVUE of θ

(c)Let $E(a^S) = e^{-7\theta}$
$E(a^S)$
$= E(e^{S\ln a})$
$= e^{n\theta(e^{\ln a}-1)}$
$= e^{n\theta(a-1)}$
$\Longrightarrow n(a-1) = -7$
$a = \frac{n-7}{n}$
Thus $E(\frac{n-7}{n})^S = e^{-7\theta}$
Since $(\frac{n-7}{n})^S$ is a function of the c.s.s. of θ which is an UE of $e^{-7\theta}$, thus $(\frac{n-7}{n})^S$ is the UMVUE of $e^{-7\theta}$.

(d)Let

$$T = \begin{cases} 1, & X_1 + \cdots + X_7 = 0 \\ 0, & \text{otherwise} \end{cases}.$$

$E(T) = P(X_1 + \cdots + X_7 = 0) = e^{-7\theta}$. Thus T is and unbiased estimator of $e^{-7\theta}$. Since S is CSS of θ. Hence by Rao-Blackwell theorem, $T^* = E(T|S)$

is an UMVUE of $e^{-7\theta}$.

$E\left[T|\sum_{i=0}^{n} X_i = s\right]$

$= 1 \cdot P[X_1 + \cdots + X_7 = 0 | X_1 + X_2 + \cdots X_n = s]$

$= \dfrac{P(X_1 + \cdots + X_7 = 0, X_8 + \cdots + X_n = s)}{P(X_1 + \cdots + X_n = s)}$

$= \dfrac{P(X_1 + \cdots + X_7 = 0) \times P(X_8 + \cdots + X_n = s)}{P(X_1 + \cdots + X_n = s)}$ Since $X_1, X_2, \ldots, X_n$ are independent.

$= \dfrac{e^{-7\theta}[(n-7)\theta]^s e^{-(n-7)\theta}}{(n\theta)^s s! e^{-n\theta}/s!}$

$= \left(\frac{n-7}{n}\right)^s$

$= \left(\frac{n-7}{n}\right)^s$

Q2. Let $X_1, X_2, \ldots, X_n$ be random sample of size n from an Exponential distribution with unknown mean θ. Find the UMVUE of $\gamma = e^{-t/\theta}$ using Rao-Blackwell theorem.

(20 marks)

Ans.

Let

$$T = \begin{cases} 1, & X_1 > t \\ 0, & \text{otherwise} \end{cases}.$$

Then, $E(T) = P(X > t) = e^{-t/\theta}$. Thus T is and unbiased estimator of γ.
$f(x) = \frac{1}{\theta} e^{-x/\theta} = c(\theta)h(x)e^{q(\theta)t(x)}$, where $c(\theta) = \frac{1}{\theta}$, $h(x) = 1$, $q(\theta) = \frac{1}{\theta}$, and $t(x) = x$, hence $f(x)$ is a member of $REC(\theta)$ and $S = \sum_{i=1}^{n} X_i$ is a complete sufficient statistics for θ.
Thus, by Rao-Blackwell therem, $T^* = E(T|S)$ is an UMVUE of γ.

$f_{X_1,S}(x_1, s) = f_{X_1,S_1}(x_1, s - x_1)$ where $S_1 = X_2 + \cdots, X_n \sim gamma(n-1, \theta)$

$= f_{X_1}(x_1) f_{S_1}(s - x_1)$

$= \frac{1}{\theta} e^{-x_1/\theta} \frac{1}{\Gamma(n-1)\theta^{n-1}} (s - x_1)^{n-2} e^{-(s-x_1)/\theta}$

$= \frac{1}{\Gamma(n-1)\theta^{n-1}} (s - x_1)^{n-2} e^{-s/\theta}$

$f_{X_1|S}(x_1) = k(s - x_1)^{n-2}, 0 < x_1 < s$

$\int_0^s k(s - x_1)^{n-2} dx_1 = 1$

$k\left[\frac{(s-x_1)^{n-1}}{-(n-1)}\right]_0^s = 1$

$k\left[\frac{s^{n-1}}{n-1}\right] = 1$

$k = \frac{n-1}{s^{n-1}}$

$\therefore f_{X_1|S}(x_1) = \frac{(n-1)(s-x_1)^{n-2}}{s^{n-1}}, 0 < x_1 < s$

$$\begin{aligned} E[T|S] &= P[X_1 > t|s] \\ &= \int_t^s \frac{(n-1)(s-x_1)^{n-2}}{s^{n-1}} dx_1 \\ &= \frac{n-1}{s^{n-1}} \left[\frac{(s-x_1)^{n-1}}{-(n-1)} \right]_t^s \\ &= \left(\frac{n-1}{s^{n-1}} \right) \left(\frac{(s-t)^{n-1}}{n-1} \right) \\ &= \left(\frac{s-t}{s} \right)^{n-1} \end{aligned}$$

Q3. Let $X_1, X_2, \dots, X_n$ be a random sample from a distribution with pdf

$$f(x;\theta) = 8\theta x^{8\theta-1} I_{(0,1)}(x).$$

Find the UMVUE of θ,

(20 marks)

Ans.

$f(x;\theta) = 8\theta x^{8\theta-1} = 8\theta e^{(8\theta-1)\ln x} = c(\theta)h(x)e^{q(\theta t(x)}$ which is a member of REC(θ) with $q(\theta) = 8\theta - 1$ and $t(x) = \ln x$. Thus, $T = \sum \ln X_i$ is a complete sufficient statistic for θ.

Let $v_i = -ln(x_i)$.Thus $0 < v_i < \infty$. This correspond to a 1-1 transformation of $x_i = e^{-v_i}$

$h^{-1}(v_i) = e^{-v_i}$

$f_V(v_i) = f_X(h^{-1}(v_i))\left|\frac{dh^{-1}(v_i)}{dv_i}\right| = 8\theta e^{-(8\theta-1)v_i} e^{-v_i} = 8\theta e^{-8\theta v_i}$

$\Rightarrow V_i \sim EXP(1/8\theta)$ and

$U = -\sum_{i=1}^n \ln x_i = \sum_{i=1}^n V_i \sim gamma(\alpha = n, \beta = \frac{1}{8\theta})$

$$\begin{aligned} E(U^{-1}) &= \int_0^\infty u^{-1} \frac{(8\theta)^n}{\Gamma(n)} u^{n-1} e^{-8\theta u} du \\ &= \frac{(8\theta)^n}{\Gamma(n)} \int_0^\infty u^{n-2} e^{-8\theta u} du \\ &= \frac{(8\theta)^n}{\Gamma(n)} \left[\frac{\Gamma(n-1)}{(8\theta)^{n-1}} \right] \\ &= \frac{8\theta}{n-1} \end{aligned}$$

$E\left(\frac{n-1}{8} U^{-1}\right) = \theta$

Thus by Lehmann Scheffie Theorem, $\frac{n-1}{8U}$ is an UMVUE of θ

Q4. Let $X_1, X_2, \dots, X_n$ be random sample of size n from $f(x|\theta) = \binom{k}{x}\theta^x(1-\theta)^{k-x}, x = 0, 1, \dots, k$. Find the uniformly minimum variance unbiased estimator (UMVUE) of $g(\theta) = \binom{k}{2}\theta^2(1-\theta)^{k-x}$.

(20 marks)

Ans.

$f(x|\theta) = \binom{k}{x}\theta^x(1-\theta)^{k-x} = \binom{k}{x}(1-\theta)^k e^{x\ln[\theta(1-\theta)]}$ which is $REC(q_1)$ with $q_1(\theta) = \ln[\theta(1-\theta)]$ and $t_1(x) = x$. By the theorem, $S = \sum_{i=1}^n X_i$ is a complete sufficient statistic for θ.

Let

$$T = \begin{cases} 1, & X_1 = 2 \\ 0, & \text{otherwise} \end{cases}.$$

$E(T) = P(X = 2) = \binom{k}{2}\theta^2(1-\theta)^{k-x} = g(\theta)$. Thus T is and unbiased estimator of θ. Since S is CSS of θ. Hence by Rao-Blackwell therem, $T^* = E(T|S)$ is an UMVUE of $g(\theta)$.

$E[T|\sum_{i=0}^n X_i = s]$
$= 1 \cdot P[X_1 = 2|X_1 + X_2 + \cdots X_n = s]$
$= \frac{P(X_1=2, X_2+\cdots+X_n=s-2)}{P(X_1+\cdots+X_n=s)}$
$= \frac{P(X_1=2)\times P(X_2+\cdots+X_n=s-2)}{P(X_1+\cdots+X_n=s)}$ Since $X_1, X_2, \ldots, X_n$ are independent and $S \sim Bin(kn, \theta)$, $X_2 + \cdots + X_n \sim Bin(k(n-1), \theta)$

$$= \frac{\binom{k}{2}\theta^2(1-\theta)^{k-x} \times \binom{k(n-1)}{s-2}\theta^{s-2}(1-\theta)^{k(n-1)-(s-2)}}{\binom{kn}{s}\theta^s(1-\theta)^{kn-s}}$$

$$= \frac{\binom{k}{2} \times \binom{k(n-1)}{s-2}}{\binom{kn}{s}}$$

Q5. Let $X_1, \ldots, X_{30}$ be a random sample from a distribution with probability density function(p.d.f.)

$$f(x) = \frac{\theta^6}{\Gamma(6)}x^5 e^{-\theta x} I(0, \infty), \theta > 0.$$

(a)Show that the p.d.f. of X belongs to the regular exponential family.

(b)Find a complete and sufficient statistic for θ.

(c)Find the UMVUE for $V(X_1)$.

(d)Find the UMVUE for θ.

(20 marks)

Ans.

(a)$f(x) = \frac{\theta^6}{\Gamma(6)}x^5 e^{-\theta x} I(0, \infty), \theta > 0$
$= c(\theta)h(x)e^{q(\theta)t(x)}$
where $c(\theta) = \theta^6$, $h(x) = 1/\Gamma(6)x^5$, $q(\theta) = -\theta$, and $t(x) = x$.
Thus the p.d.f. of X belongs to the regular exponential family.

(b) Since the p.d.f. of X belongs to the regular exponential family, thus by the theorem, $S = \sum_{i=1}^{30} X_i$ is a c.s.s of θ

(c) $X \sim GAM(6, \frac{1}{\theta})$ and $S \sim GAM(180, \frac{1}{\theta})$
Let $g(\theta) = V(X_1) = \frac{6}{\theta^2}$
$E(S^2) = V(S) + E^2(S) = \frac{180}{\theta^2} + \frac{180^2}{\theta^2} = \frac{32580}{\theta^2}$
Thus $E\left[\frac{S^2}{5430}\right] = \frac{32580}{5430\theta^2} = \frac{6}{\theta^2}$
Since $E\left[\frac{S^2}{5430}\right] = \frac{6}{\theta^2}$, thus, by Lehmann Scheffe Theorem, $\frac{S^2}{5430}$ is the UMVUE of $V(X_1)$.

(d) $E(S^{-1}) = \int_0^\infty s^{-1} \frac{\theta^{180}}{\Gamma(180)} s^{179} e^{-\theta s} ds = \int_0^\infty \frac{\theta^{180}}{\Gamma(180)} s^{178} e^{-\theta s} ds = \frac{\theta^{180}}{\Gamma(180)} \frac{\Gamma(179)}{\theta^{179}} = \frac{\theta}{179}$
Thus $E(179S^{-1}) = \theta$.
Since $E(179S^{-1}) = \theta$, thus, by Lehmann Scheffe Theorem, $179S^{-1}$ is the UMVUE of θ.