MEME16203Linear Models Marking Guide

Assignment 4

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Faculty:	FES	Unit Code:	MEME16203
Course:	MAC	Unit Title:	Linear Models
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Q1. Consider the model $Y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \epsilon ijk$, $i = 1,2,3; j = 1,2; k = 1$, where $\epsilon_{ijk} \overset{iid}{\sim} N(0, \sigma^2)$. Determine which of the following hypotheses are testable. Justify your answer.

(a) $H_0 : \gamma_{11} = 0$. (4 marks)

(b) $H_0 : \mu + \alpha_1 + \frac{1}{2}(\beta_1 + \beta_2 + \gamma_{11} + \gamma_{12}) = 0$. (3 marks)

(c) $H_0 : (\alpha_1 - \alpha_2) + \gamma_{11} - \gamma_{21}$. (3 marks)

Ans.

(a) Let $\mathbf{c}^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, then $\mathbf{c}^T \boldsymbol{\beta} = \gamma_{11}$

$$X = \begin{bmatrix} 1&1&0&0&1&0&1&0&0&0&0&0 \\ 1&1&0&0&0&1&0&1&0&0&0&0 \\ 1&0&1&0&1&0&0&0&1&0&0&0 \\ 1&0&1&0&0&1&0&0&0&1&0&0 \\ 1&0&0&1&1&0&0&0&0&0&1&0 \\ 1&0&0&1&0&1&0&0&0&0&0&1 \end{bmatrix}$$

Let $\mathbf{d}^T = [1\ 0\ 0\ 0\ 0\ 0\ -1\ -1\ -1\ -1\ -1\ -1]$, then $X\mathbf{d} = \mathbf{0}$, but

$$\mathbf{c}^T\mathbf{d} = [0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0] \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix} = -1 \neq 0$$

Thus, $H_0 : \gamma_{11}$ is not estimable and hence not testable.

(b) $E(\bar{Y}_{1..}) = \frac{1}{2}E(Y_{111} + Y_{121}) = \frac{1}{2}(\mu + \alpha_1 + \beta_1 + \gamma_{11} + \mu + \alpha_1 + \beta_2 + \gamma_{12} = \mu + \alpha_1 + \frac{1}{2}(\beta_1 + \beta_2 + \gamma_{11} + \gamma_{12})$, hence $\mu + \alpha_1 + \frac{1}{2}(\beta_1 + \beta_2 + \gamma_{11} + \gamma_{12})$ is estimable.

$H_0 : \mathbf{C}^T\boldsymbol{\beta} = [1\ 1\ 0\ 0\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ 0\ 0\ 0\ 0]\boldsymbol{\beta}$
$Rank(C) = 1 =$ number of row of C. Thus $\mu+\alpha_1+\frac{1}{2}(\beta_1+\beta_2+\gamma_{11}+\gamma_{12})$ is testable.

(c) $E(Y_{111} - Y_{211} = (\mu + \alpha_1 + \beta_1 + \gamma_{11}) - (\mu + \alpha_2 + \beta_1 - \gamma_{21}) = \alpha_1 - \alpha_2 + \gamma_{11}) - \gamma_{21})$, hence $\alpha_1 - \alpha_2 + \gamma_{11}) - \gamma_{21})$ is estimable.
$H_0 : \mathbf{C}^T\boldsymbol{\beta} = [1\ 1\ -1\ 0\ 0\ 0\ 1\ 0\ -1\ 0\ 0\ 0]\boldsymbol{\beta}$
$Rank(C) = 1 =$ number of row of C. Thus $\alpha_1 - \alpha_2 + \gamma_{11}) - \gamma_{21})$ is testable.

Q2. An researcher recorded moisture content for three types of cheese made by two different methods. Three pieces of cheese were measure for each type and each method. The data are shown below.

Treatment	Moisture Content Measurements		
Type A made with Method 1	$y_{11} = 38.02$	$y_{12} = 39.79$	$y_{13} = 37.79$
Type B made with Method 1	$y_{21} = 36.74$	$y_{22} = 33.41$	$y_{23} = 38.41$
Type C made with Method 1	$y_{31} = 38.02$	$y_{32} = 35.00$	$y_{33} = 34.00$
Type A made with Method 2	$y_{41} = 39.96$	$y_{42} = 39.06$	$y_{43} = 38.01$
Type B made with Method 2	$y_{51} = 34.58$	$y_{52} = 36.52$	$y_{53} = 35.52$
Type C made with Method 2	$y_{61} = 34.60$	$y_{62} = 36.05$	$y_{63} = 38.0$

Consider the model $y_{ij} = \mu + \alpha_i + \epsilon_{ij}$, where $\epsilon_{ij} \sim NID(0, \sigma^2)$, $i = 1, 2, 3, 4, 5, 6$, and $j = 1, 2, 3$. This model can be expressed in matrix form as $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$.

Express the each of the following hypotheses in the form $H_0 : \mathbf{C}\boldsymbol{\beta} = \mathbf{0}$. Use R to compute the value of the corresponding SSH_0, SSE and F-statistic.

(a) After averaging across the two methods of making cheese, the average moisture content is the same for all three types of cheese. (5 marks)

Ans.
The hypothesis to be tested is

$$H_0 : (\alpha_1 + \alpha_4)/2 = (\alpha_2 + \alpha_5)/2 = (\alpha_3 + \alpha_6)/2$$

or

$$H_0 : \mathbf{C}\boldsymbol{\beta} = \begin{bmatrix} 0 & 1 & -1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & -1 \end{bmatrix} \boldsymbol{\beta} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$F = \frac{MSH_0}{MSE} = \frac{32.91023/2}{33.72387/12} = 5.8552$ with $df = (2, 12)$

(b) For each type of cheese, the average moisture content is not affected by the method for making cheese. (This hypothesis allows the average moisture content to vary across types of cheese). (5 marks)

Ans. The hypothesis to be tested is

$$H_0 : \alpha_1 = \alpha_4; \alpha_2 = \alpha_5; \alpha_3 = \alpha_6$$

or

$$H_0 : \mathbf{C}\boldsymbol{\beta} = \begin{bmatrix} 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix} \boldsymbol{\beta} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$F = \frac{MSH_0}{MSE} = \frac{1.4109/3}{33.72387/12} = 0.16735$ with $df = (3, 12)$

R-Codes:

```
y = c( 38.02, 39.79, 37.79, 36.74, 33.41,38.41, 38.02, 35, 34,39.96, 39.06, 38
34.58, 36.52, 35.52, 34.6, 36.05, 38)
n = length(y)
X = matrix( 0, ncol=7, nrow=n )
X[,1] = 1
X[1:3, 2] = 1
X[4:6, 3] = 1
X[7:9,4] = 1
X[10:12,5] = 1
X[13:15,6] = 1
X[16:18,7] = 1
X

library(MASS)
betahat = ginv(t(X) \%*\% X) \%*\% t(X) \%*\% y
Px = X \%*\% ginv(t(X) \%*\% X) \%*\% t(X)
In = diag( rep(1,n) )

SSE = t(y) \%*\% ( In- Px) \%*\% y
SSE

df.sse = n - qr(X)$rank
df.sse
```

```
c1 = matrix( c( 0, 1, -1, 0, 1, -1, 0,0, 1, 0, -1, 1, 0, -1 ),
ncol=7, byrow=T )
cbhat1 = c1 \%*\% betahat

SS.H0.c1 = t(cbhat1) \%*\%
solve(c1 \%*\% ginv(t(X) \%*\% X)\%*\% t(c1))\%*\% cbhat1
SS.H0.c1

F1 = (SS.H0.c1/2)/(SSE/(df.sse))
F1

pvalue = 1 - pf( q=F1, df1=2, df2=df.sse )
pvalue

c2 = matrix( c( 0, 1, 0, 0, -1, 0, 0,0, 0, 1, 0, 0, -1, 0,0,
0, 0, 1, 0, 0, -1 ), ncol=7, byrow=T )
c2
cbhat2 = c2 \%*\% betahat

SS.H0.c2 = t(cbhat2) \%*\%
solve(c2 \%*\% ginv(t(X) \%*\% X) \%*\% t(c2))\%*\% cbhat2
SS.H0.c2

F2 = (SS.H0.c2/3)/(SSE/(df.sse))
F2
pvalue = 1 - pf( q=F2, df1=3, df2=df.sse )
pvalue
```

Q3. An researcher recorded moisture content for three types of cheese made by two different methods. A 3×2 factorial experiment with types of cheese made by two different methods was conducted. The data had unequal replications among the six treatment combinations of the two factors, Cheese and Method. The collected data are given below.

	Cheese		
Method	1	2	3
1	38.49 39.99	39.47 39.88 38.49	39.0 38.79 39.01 39.13
2	38.83 38.83 39.18	39.75	39.4 39.24 39.31

Consider the model $y_{ijk} = \mu_{ij} + \epsilon_{ijk}$, where $\epsilon_{ijk} \sim NID(0, \sigma^2)$, $i = 1, 2,$ and

$j = 1, 2, 3$ and $k = 1, \ldots, n_{ij}$. This model can be expressed in matrix form as $\mathbf{Y} = \mathbf{D}\boldsymbol{\beta} + \boldsymbol{\epsilon}$. Examine type III sums of squares for these data.

(a) Specify the $\mathbf{C}$ matrix needed to write the null hypothesis associated with the F-test for Method effects in the form $H_0 : \mathbf{C_1}\boldsymbol{\beta} = \mathbf{0}$. (5 marks)

Ans.

$\mathbf{C_1} = [\mathbf{I}_{a-1}| - \mathbf{1}_{a-1}] \otimes \mathbf{1_b^T} = [\mathbf{I}_1| - \mathbf{1}_1] \otimes \mathbf{1_3^T} = \begin{bmatrix}1 & -1\end{bmatrix} \otimes \begin{bmatrix}1 & 1 & 1\end{bmatrix} = \begin{bmatrix}1 & 1 & 1 & -1 & -1 & -1\end{bmatrix}$

(b) Present a formula for $SS_{H_{0,1}}$, corresponding to the null hypothesis in part (a), and state it's distribution when the null hypothesis is true. (5 marks)

Ans.

$$
\begin{aligned}
SS_{H_{0,1}} &= (\mathbf{C_1 b} - \mathbf{0})^\mathbf{T}[\mathbf{C_1}(\mathbf{D^T D})^{-1}\mathbf{C_1^T}]^{-1}(\mathbf{C_1 b} - \mathbf{0}) \\
&= \mathbf{y^T D}(\mathbf{D^T D})^{-1}\mathbf{C_1^T}[\mathbf{C_1}(\mathbf{D^T D})^{-1}\mathbf{C_1^T}]^{-1}\mathbf{C_1}(\mathbf{D^T D})^{-1}\mathbf{D^T y}
\end{aligned}
$$

Under H_0, $SSH_{0,1} \sim \frac{1}{\sigma^2}\chi_1^2$

(c) Compute $SS_{H_{0,1}}$. (5 marks)

Ans.

$\mathbf{b} = \begin{bmatrix}\bar{Y}_{11.} & \bar{Y}_{12.} & \bar{Y}_{13.} & \bar{Y}_{21.} & \bar{Y}_{22.} & \bar{Y}_{23.}\end{bmatrix}^T = \begin{bmatrix}39.24 & 39.28 & 38.982 & 38.947 & 39.75 & 39.317\end{bmatrix}^T$

$$
\mathbf{C_1 b} = \begin{bmatrix}1 & 1 & 1 & -1 & -1 & -1\end{bmatrix}\begin{bmatrix}39.24\\39.28\\38.982\\38.947\\39.75\\39.317\end{bmatrix} = \begin{bmatrix}-0.512\end{bmatrix}
$$

$\mathbf{D^T D^{-1}} = diag(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{3}, 1, \frac{1}{3})$

$$
\begin{aligned}
C_1(D^T D)^{-1}C^T] &= \begin{bmatrix}1 & 1 & 1 & -1 & -1 & -1\end{bmatrix}\begin{bmatrix}\frac{1}{2} & 0 & 0 & 0 & 0 & 0\\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0\\ 0 & 0 & \frac{1}{4} & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{3} & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3}\end{bmatrix}\begin{bmatrix}1\\1\\1\\-1\\-1\end{bmatrix} \\
&= \begin{bmatrix}2.75\end{bmatrix}
\end{aligned}
$$

$SSH_{01} = (\mathbf{C_1 b} - \mathbf{0})^\mathbf{T}[\mathbf{C_1}(\mathbf{D^T D})^{-1}\mathbf{C_1^T}]^{-1}(\mathbf{C_1 b} - \mathbf{0}) = \begin{bmatrix}-0.512\end{bmatrix}\begin{bmatrix}0.3636\end{bmatrix}\begin{bmatrix}-0.512\end{bmatrix} = 0.0953$

(d) Specify the $\mathbf{C_2}$ matrix needed to write the null hypothesis associated with the F-test for Cheese effects in the form $H_0 : \mathbf{C_2}\boldsymbol{\beta} = \mathbf{0}$. (5 marks)

Ans.

$$\mathbf{C_2} = \mathbf{1_a^T} \otimes [\mathbf{I}_{b-1}| - \mathbf{1}_{b-1}] = \mathbf{1_2^T} \otimes [\mathbf{I}_{3-1}| - \mathbf{1}_{3-1}] = \begin{bmatrix}1\\1\end{bmatrix} \otimes \begin{bmatrix}1 & 0| -1\\0 & 1| -1\end{bmatrix} = \begin{bmatrix}1 & 0 & -1 & 1 & 0 & -1\\0 & 1 & -1 & 0 & 1 & -1\end{bmatrix}$$

(e) Present a formula for $SS_{H_{0,2}}$, corresponding to the null hypothesis in part (d), and state it's distribution when the null hypothesis is true. (5 marks)

Ans.

$$\begin{aligned} SS_{H_{0,2}} &= (\mathbf{C_2b-0})^{\mathbf{T}}[\mathbf{C_2}(\mathbf{D^TD})^{-1}\mathbf{C_2^T}]^{-1}(\mathbf{C_2b-0}) \\ &= \mathbf{y^TD}(\mathbf{D^TD})^{-1}\mathbf{C_2^T}[\mathbf{C_2}(\mathbf{D^TD})^{-1}\mathbf{C_2^T}]^{-1}\mathbf{C_2}(\mathbf{D^TD})^{-1}\mathbf{D^Ty} \end{aligned}$$

Under H_0, $SSH_{0,2} \sim \frac{1}{\sigma^2}\chi^2_2$

(f) Compute $SS_{H_{0,2}}$. (5 marks)

Ans.

$$\mathbf{b} = \begin{bmatrix}\bar{Y}_{11.} & \bar{Y}_{12.} & \bar{Y}_{13.} & \bar{Y}_{21.} & \bar{Y}_{22.} & \bar{Y}_{23.}\end{bmatrix}^T = \begin{bmatrix}39.24 & 39.28 & 38.982 & 38.947 & 39.75 & 39.317\end{bmatrix}^T$$

$$\mathbf{C_2b} = \begin{bmatrix}1 & 0 & -1 & 1 & 0 & -1\\0 & 1 & -1 & 0 & 1 & -1\end{bmatrix}\begin{bmatrix}39.24\\39.28\\38.982\\38.947\\39.75\\39.317\end{bmatrix} = \begin{bmatrix}-0.112\\0.731\end{bmatrix}$$

$$\mathbf{D^TD^{-1}} = diag(\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \tfrac{1}{3}, 1, \tfrac{1}{3})$$

$$\begin{aligned} C_2(D^TD)^{-1}C_2^T] &= \begin{bmatrix}1 & 0 & -1 & 1 & 0 & -1\\0 & 1 & -1 & 0 & 1 & -1\end{bmatrix}\begin{bmatrix}\frac{1}{2} & 0 & 0 & 0 & 0 & 0\\0 & \frac{1}{3} & 0 & 0 & 0 & 0\\0 & 0 & \frac{1}{4} & 0 & 0 & 0\\0 & 0 & 0 & \frac{1}{3} & 0 & 0\\0 & 0 & 0 & 0 & 1 & 0\\0 & 0 & 0 & 0 & 0 & \frac{1}{3}\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\\-1 & -1\\1 & 0\\0 & 1\\-1 & -1\end{bmatrix} \\ &= \begin{bmatrix}1.4167 & 0.5833\\0.5833 & 1.9167\end{bmatrix} \end{aligned}$$

$$SSH_{0,2} = (\mathbf{C_2b-0})^{\mathbf{T}}[\mathbf{C_2}(\mathbf{D^TD})^{-1}\mathbf{C_2^T}]^{-1}(\mathbf{C_2b-0}) = \begin{bmatrix}-0.112 & 0.731\end{bmatrix}\begin{bmatrix}0.807 & -0.2456\\-0.2456 & 0.5965\end{bmatrix}\begin{bmatrix}-0.112\\0.731\end{bmatrix} = 0.3691$$

(g) Verify that the hypothesis in part (d) is testable. (5 marks)

Ans. $E\begin{bmatrix}\bar{y}_{1..} & -\bar{y}_{3..}\\ \bar{y}_{2..} & -\bar{y}_{3..}\end{bmatrix} = \begin{bmatrix}\mu_{1.} - \mu 3.\\ \mu_{2.} - \mu_{3.}\end{bmatrix} = \mathbf{C_1}\boldsymbol{\beta}$, thus, $\mathbf{C_1}\boldsymbol{\beta}$ is estimable.
Rank $\mathbf{C} = 2 =$ number of rows of $\mathbf{C}$.
Hence, $\mathbf{C_1}\boldsymbol{\beta}$ is testable.

(h) Suppose $SSE = 1.097608$, compute the test statistic for testing the null hypothesis associated with the F-test for Cheese effects. (5 marks)

Ans.
$MSH_{0,2} = \frac{SSH_{0,2}}{2} = \frac{0.3691}{2} = 0.18455$
$MSE = -\frac{SSE}{16-6} = \frac{1.097608}{10} = 0.1097608$
$F = \frac{MSH_{0,2}}{MSE} = \frac{0.18455}{0.1097608} = 1.6814$

Q4. An researcher recorded moisture content for four types of cheese made by two different methods. Three pieces of cheese were measure for each type and each method. The data are shown below.

Treatment	Moisture Content Measurements		
Type A made with Method 1	$y_{111} = 38.02$	$y_{112} = 39.79$	
Type A made with Method 2	$y_{121} = 39.96$	$y_{122} = 39.06$	$y_{123} = 38.01$
Type B made with Method 1	$y_{211} = 36.74$	$y_{212} = 33.41$	
Type B made with Method 2	$y_{221} = 34.58$	$y_{222} = 36.52$	$y_{223} = 35.52$
Type C made with Method 1	$y_{311} = 38.02$	$y_{312} = 35.00$	$y_{313} = 34.00$
Type C made with Method 2	$y_{321} = 34.60$		
Type D made with Method 1	$y_{411} = 48.02$	$y_{412} = 45.00$	$y_{413} = 42.00$
Type D made with Method 2	$y_{421} = 44.60$	$y_{422} = 46.05$	

Consider the model $y_{ijk} = \mu_{ij} + \epsilon_{ijk}$, where $\epsilon_{ijk} \sim NID(0, \sigma^2)$, $i = 1, 2, 3, 4$, and $j = 1, 2$ and $k = 1, ..., n_i$. This model can be expressed in matrix form as $\mathbf{Y} = \mathbf{D}\boldsymbol{\beta} + \boldsymbol{\epsilon}$. Examine type III sums of squares for these data.

(a) Use R t0 complete the ANOVA table below.

Source	Type III SS	DF	Mean Square	F Value	P-Value
Type					
Method					
Type×Method					

(10 marks)

Ans.

Source	Type III SS	DF	Mean Square	F Value	P-Value
	Df	Sum Sq	Mean Sq	F value	Pr(¿F)
Type	3	292.7142	97.5714	27.6406	0.0000202 *
Method	1	0.0089	0.0089	0.00253	0.9608
Type:Method	3	1.2427	0.4142	0.1173	0.9480
Residuals	11	38.8300	3.53		

```
> Y = c(38.02,39.79,39.96,39.06,38.01,36.74,
33.41,34.58,36.52,35.52,38.02,35,34,34.6,48.02,
45,42,44.6,46.05)
> d1 = c(rep(1,2), rep(0,17))
> d2 = c(0,0,rep(1,3),rep(0,14))
> d3 = c(rep(0,5),rep(1,2), rep(0,12))
> d4 = c(rep(0,7),rep(1,3),rep(0,9))
> d5 = c(rep(0,10), rep(1,3), rep(0,6))
> d6 = c(rep(0,13), 1, rep(0,5))
> d7 = c(rep(0,14), rep(1,3), rep(0,2))
> d8 = c(rep(0,17), rep(1,2))
> D = cbind(d1,d2,d3,d4,d5,d6,d7,d8)
> a = 4
> b = 2
> beta = solve(t(D)\%*\%D)\%*\%t(D)\%*\%Y
> Yhat = D\%*\%beta
> SSE = crossprod(Y-Yhat)
> df2 = NROW(Y) - a*b
> am1 = a-1
> bm1 = b-1
> Iam1 = diag(rep(1,am1))
> Ibm1 = diag(rep(1,bm1))
> Onea = c(rep(1,a))
> Oneam1 = c(rep(1,am1))
> Oneb = c(rep(1,b))
> Onebm1 = c(rep(1,bm1))
> C1 = kronecker(cbind(Iam1, -Oneam1),t(Oneb))
> C1b = C1\%*\%beta
> SSH0a = t(C1b)\%*\%solve(C1\%*\%solve(crossprod(D))\%*\%t(C1))\%*\%C1b
> df1 = a-1
> MSH0a = SSH0a/df1
> F = (SSH0a/df1)/(SSE/df2)
```

```
> p = 1-pf(F, df1,df2)
> C1
     [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8]
[1,]    1    1    0    0    0    0   -1   -1
[2,]    0    0    1    1    0    0   -1   -1
[3,]    0    0    0    0    1    1   -1   -1
> data.frame(SSH0=SSH0a, dfSSH0 = df1, MSH0a = MSH0a,  F.Stat = F, p.val
      SSH0 dfSSH0    MSH0a   F.Stat      p.value
1 292.7142      3 97.57141 27.64057 2.019936e-05
> C2 = kronecker(t(Onea), cbind(Ibm1, -Onebm1))
> C2b = C2\%*\%beta
> SSH0b = t(C2b)\%*\%solve(C2\%*\%solve(crossprod(D))\%*\%t(C2))\%*\%C2b
> df1 = b-1
> MSH0b = SSH0b/df1
> F = (SSH0b/df1)/(SSE/df2)
> p = 1-pf(F, df1,df2)
> C2
     [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8]
[1,]    1   -1    1   -1    1   -1    1   -1
> data.frame(SS=SSH0b, dfSSH0 = df1, MSH0b = MSH0b, F.Stat = F, p.value
           SS dfSSH0       MSH0b      F.Stat   p.value
1 0.008928261      1 0.008928261 0.002529247 0.9607916
> C3 = kronecker(cbind(Iam1, -Oneam1), cbind(Ibm1, -Onebm1))
> C3b = C3\%*\%beta
> SSH0ab = t(C3b)\%*\%solve(C3\%*\%solve(crossprod(D))\%*\%t(C3))\%*\%C3
> df1 = (a-1)*(b-1)
> MSH0ab = SSH0ab/df1
> F = (SSH0ab/df1)/(SSE/df2)
> p = 1-pf(F, df1,df2)
> C3
     [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8]
[1,]    1   -1    0    0    0    0   -1    1
[2,]    0    0    1   -1    0    0   -1    1
[3,]    0    0    0    0    1   -1   -1    1
> data.frame(SS=SSH0ab, ddfSSH0ab = df1, MSH0ab = MSH0ab,F.Stat = F, p.v
        SS ddfSSH0ab    MSH0ab    F.Stat   p.value
1 1.242677         3 0.4142256 0.1173441 0.9480444
> MSE = SSE/df2
> data.frame(SS = SSE, df = df2, MSE = MSE)
        SS df      MSE
1 38.83008 11 3.530008
>
```

(b) Specify the $\mathbf{C}$ matrix needed to write the null hypothesis associated with the F-test for type effects in the form $H_0 : \mathbf{C}\boldsymbol{\beta} = \mathbf{0}$. What can you conclude from the results of this test? (10 marks)

Ans.

$$\begin{aligned}
\mathbf{C} &= [\mathbf{I}_{a-1}|-\mathbf{1}_{a-1}] \otimes [\mathbf{1}_b^T] \\
&= [\mathbf{I}_3|-\mathbf{1}_3] \otimes [\mathbf{1}_2^T] \\
&= \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \end{bmatrix}
\end{aligned}$$

From the ANOVA table, the test statistic is 27.64057 with degrees of freedom (3, 11) and p-value 0.0000202. This indicates the mean moisture content, averaging with equal weights across the Methods, are different for all three Types (i.e., we reject the null hypothesis).

(c) Specify the $\mathbf{C}$ matrix needed to write the null hypothesis associated with the F-test for method effects in the form $H_0 : \mathbf{C}\boldsymbol{\beta} = \mathbf{0}$. What can you conclude from the results of this test? (10 marks)

Ans.

$$\begin{aligned}
\mathbf{C} &= \mathbf{1}_a^T \otimes [1 - b - 1| - 1_{b-1}] \\
&= \mathbf{1}_4^T \otimes [1| - 1] \\
&= \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix}
\end{aligned}$$

From the ANOVA table, the test statistic is .002529247 with degrees of freedom (1, 11) and p-value 0.9607916. This indicates the mean moisture content, averaging with equal weights across the Types, are nearly the same for all two Methods (i.e., we cannot reject the null hypothesis).

(d) Specify the $\mathbf{C}$ matrix needed to write the null hypothesis associated with the F-test for interaction effects in the form $H_0 : \mathbf{C}\boldsymbol{\beta} = \mathbf{0}$. What can you conclude from the results of this test? (10 marks)

Ans.

$$
\begin{aligned}
\mathbf{C} &= [\mathbf{I}_{a-1}|-\mathbf{1}_{a-1}] \otimes [\mathbf{I}_{b-1}|-\mathbf{1}_{b-1}] \\
&= [\mathbf{I}_3|-\mathbf{1}_3] \otimes [\mathbf{I}_1|-\mathbf{1}_1] \\
&= \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix}
\end{aligned}
$$

From the ANOVA table, the test statistic is 0.1173441 with degrees of freedom (3, 11) and p-value 0.9480. This indicates the interaction effects are not significance. (10 marks)