CONTENTS

1 Models for Claim Severities

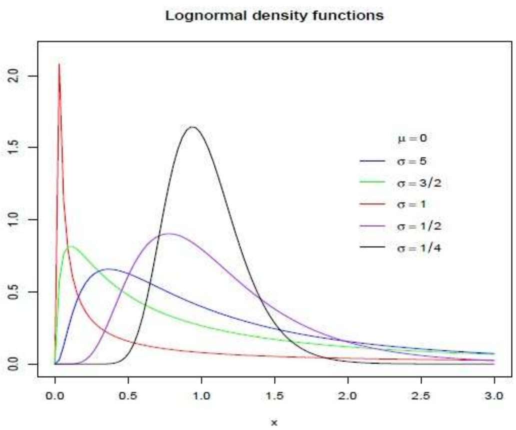
1.1 Introduction

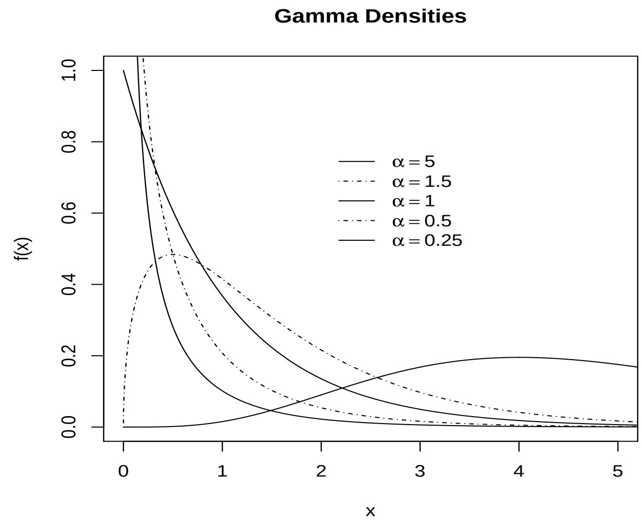
- Given that a claim occurs, the (individual) claim size X is typically referred to as claim severity.
- While typically this may be of continuous random variables, sometimes claim sizes can be considered discrete.
- When modeling claims severity, insurers are usually concerned with the tails of the distribution. There are certain types of insurance contracts with what are called long tails.

1.2 Some Parametric Claim Size Distributions

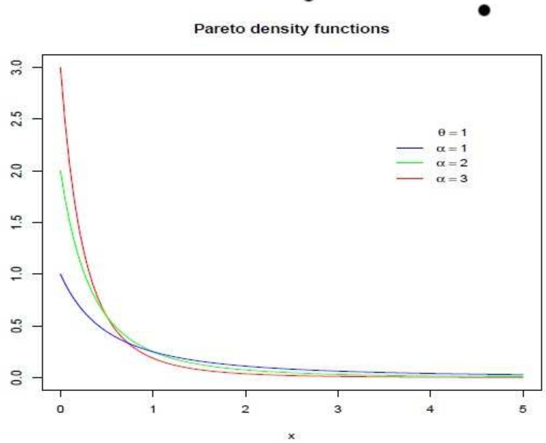
- Normal - easy to work with, but careful with getting negative claims. Insurance claims usually are never negative.
- Gamma/Exponential - use this if the tail of distribution is considered "light", applicable for example with damage to automobiles.
- Lognormal - somewhat heavier tails, applicable for example with fire insurance.
- Burr/Pareto - used for heavy-tailed business, such as liability insurance.
- Inverse Gaussian - not very popular because complicated mathematically.

Lognormal densities for various σ's

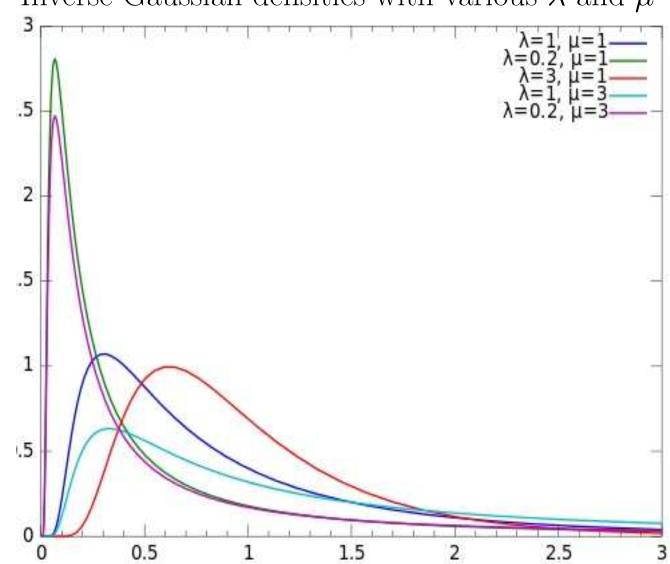




Pareto densities for various α's



Inverse Gaussian densities with various λ and μ



1.3 Basic Distributional Quantiles

1.3.1 Moments:

Definition 1. k^{th} **raw moment**
The k^{th} raw moment of X is define as

$$\mu'_k = E(X^k) = \int_{-\infty}^{\infty} x^k f(x)dx$$

if X is continuous

$$\mu'_k = E(X^k) = \sum_j x_j^k p(x_j)$$

if X is discrete.

Definition 2. k^{th} **central moment**
The k^{th} central moment of X is define as

$$\mu_k = E[(X-\mu)^k] = \int_{-\infty}^{\infty} (x-\mu)^k f(x)dx$$

if X is continuous

$$\mu_k = E[(X-\mu)^k)] = \sum_j (x_j-\mu)^k p(x_j)$$

if X is discrete.

Expectation is linear, so the central moments can be calculated from the raw moments by binomial expansion. In binomial expansion, the last 2 terms always merge, so we have

- $\mu_2 = \mu'_2 - \mu^2$ instead of $\mu'_2 - 2\mu'_1\mu + \mu^2$
- $\mu_3 = \mu'_3 - 3\mu'_2\mu + 2\mu^3$ instead of $\mu'_3 - 3\mu'_2\mu + 3\mu'_1\mu^2 - \mu^3$
- $\mu_4 = \mu'_4 - 4\mu'_3\mu + 6\mu'_2\mu^2 - 3\mu^4$ instead of $\mu'_4 - 4\mu'_3\mu + 6\mu'_2\mu^2 - 4\mu_1\mu^3 + \mu^4$

1.3.2 Special Functions of Moments

- **Variance:** $V(X) = \mu_2$ and is denoted by σ^2.
- **Standard Deviation:** $\sigma = \sqrt{\sigma^2}$.
- **Coefficient of Variation:** $CV = \frac{\sigma}{\mu}$.
  The coefficient of variation expresses the standard deviation as a percentage of the sample mean. This is useful when interest is in the size of variation relative to the size of the observation, and it has the advantage that the coefficient of variation is INDEPENDENT OF the UNITS of observation. For example, the value of the standard deviation of a set of weights will be different depending on whether they are measured in kilograms or pounds. The coefficient of variation, however, will be the same in both cases as it does not depend on the unit of measurement.

- **Skewness:** $\gamma_1 = \frac{\mu_3}{\sigma^3}$
  Skewness is a measure of symmetry, or more precisely, the lack of symmetry. A distribution, or data set, is symmetric if it looks the same to the left and right of the center point. Negative values for the skewness indicate data that are skewed left and positive values for the skewness indicate data that are skewed right. By skewed left, we mean that the left tail is long relative to the right tail. Similarly, skewed right means that the right tail is long relative to the left tail. The skewness for a normal distribution is zero, and any symmetric data should have a skewness near zero.

- **Kurtosis:** $\gamma_2 = \frac{\mu_4}{\sigma^4}$.
  Kurtosis is a measure of whether the data are peaked or flat relative to a normal distribution. That is, data sets with high kurtosis tend to have a distinct peak near the mean, decline rather rapidly, and have heavy tails. Data sets with low kurtosis tend to have a flat top near the mean rather than a sharp peak. A uniform distribution would be the extreme case. The kurtosis for a standard normal distribution is three.
- **Mode**
  A mode is x such that $f(x)$ is maximized (or $p(x)$ for discrete distribution).
- **Moment Generating Function**
  The moment generating function (MGF) is defined by
  $$M_X(t) = E(e^{tX})$$

Notes:

- $M^{(n)}(0) = E(X^n)$
- $M_{X_1+\ldots X_n}(t) = [M_X(t)]^n$ when $X_i's$ are identically and independently distributed.

- **Probability Generating Function**
  The probability generating function (PGF) is defined by
  $$P_X(z) = E(z^X)$$
  It is important to realize that we cannot have intuition about PGFs because they do not correspond to anything which is directly observable.

Notes:

- PGFs make calculations of expectations and of some probabilities very easy.
  - $P'(1) = E(X)$
  - $P''(1) = E[X(X-1)]$
  - $P^{(3)}(1) = E[X(X-1)(X-2)]$

- PGFs make sums of independent random variables easy to handle. i.e.,
  $$P_{X_1+\ldots+X_n}(z) = [P_X(z)]^n$$
  when $X_i's$ are identically and independently distributed.

- **Cumulant Generating Function**
  The cumulant-generating function $K(t)$, is the natural logarithm of the moment-generating function:
  $$K(t) = \ln E(e^{tX}) = \ln M_X(t)$$
  The cumulants κ_n are obtained from a power series expansion of the cumulant generating function:
  $$K(t) = \sum_{n=1}^{\infty} \kappa_n \frac{t^n}{n!}.$$
  This expansion is a Maclaurin series, so that n^{th} cumulant can be obtained by differentiating the above expansion n times and evaluat-

ing the result at zero:

$$\kappa_n = K^{(n)}(0).$$

The first cumulant is the expected value; the second and third cumulants are respectively the second and third central moments (the second central moment is the variance); but the higher cumulants are neither moments nor central moments, but rather more complicated polynomial functions of the moments.

Example 1.
A random variable X has a gamma distribution with parameters $\alpha = 5$ and $\beta = 0.1$, calculate the mode, CV, skewness and the kurtosis. $\boxed{\text{mode} = 0.4, \text{CV} = 0.4472}$
$\boxed{\text{Skewness} = 0.8945, \text{Kurtosis} = 4.2}$

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Example 2.
Claim severity has the following distribution:

Claim Size	100	200	300	400	500
Probability	0.05	0.20	0.50	0.20	0.05

Determine the distribution's skewness and kurtosis. $\boxed{0,3.125}$

Example 3 (T1Q1)**.**
A random variable has a mean of 8 and coefficient of variation of 11. The third raw moment is 1790. Determine the skewness.

Example 4 (T1Q2).

Claim severity has the following distribution:

Claim Size	200.0	210.0	220.0	230.0	240.0
Probability	0.37	0.22	0.20	0.12	0.09

Determine the distribution's coefficient of variation, skewness and Kurtosis.

1.3.3 Percentiles

Definition 3.
The $100p^{th}$ percentile of a random variable is any value π_p such that $F(\pi_p-) \leq p \leq F(\pi_p)$.

- If the distribution function has a value of p for one and only one x value, then the percentile is uniquely defined.
- If the distribution function jumps from a value below p to a value above p, then the percentile is at the location of the jump.
- If the distribution function is constant at a value of p over a range of values, then any value in that range can be used as the percentile.

Example 5.
Suppose

$$F(x) = \begin{cases} 0, & x < 0 \\ 0.01x, & 0 \leq x < 100 \\ 1, & x \geq 100. \end{cases}$$

Determine the 50^{th} and 80^{th} percentiles. $\boxed{50,\ 80}$

Example 6.
Suppose

$$F(x) = \begin{cases} 0, & x < 0, \\ 0.5, & 0 \leq x < 1, \\ 0.75, & 1 \leq x < 2, \\ 0.87, & 2 \leq x < 3, \\ 0.95, & 3 \leq x < 4, \\ 1, & x \geq 4 \end{cases}$$

Determine the 50^{th} and 80^{th} percentiles. $\boxed{[0,1],\ 2}$

Example 7.
A random variable X has the following distribution:

x	$P(X=x)$
1	0.20
3	0.25
7	0.45
8	0.10

Calculate the 50^{th} and 90^{th} percentiles of X. $\boxed{7, [7,8]}$

1.4 Classifying and Creating Distribution

1.4.1 Scaling

Definition 4. A parametric distribution is a set of distribution functions, each member of which is determined by specifying one or more values called parameters. The number of parameters is fixed and finite.

Definition 5. A parametric distribution is a scale distribution if, when a random variable from that set of distribution is multiplied by a positive constant, the resulting random variable is also in that set of distributions.

Note: All of the continuous distributions in Appendix A (except lognormal and inverse Gaussian) are scale families.

1.4.2 Multiplication by a Constant

This transformation is equivalent to applying inflation uniformly across all loss levels and is known as a change of scale. For example, if this year's losses are given by a random variable X, the uniform inflation of 5% indicates that next year's losses can be modeled with the random variable $Y = 1.05X$.

Theorem 1. Let X be a continuous random variable with pdf $f_X(x)$ and cdf $F_X(x)$. Let $Y = cX$ with $c > 0$. Then

$$F_Y(y) = F_X\left(\frac{y}{c}\right), f_Y(y) = \frac{1}{c} f_X\left(\frac{y}{c}\right)$$

Notes:

- If X has scale parameter θ and other parameters, then cX has scale parameter $c\theta$ and the same other parameters.
- If $X \sim lognormal(\mu, \sigma)$, then $cX \sim lognormal(\mu + lnc, \sigma)$.

Example 8.
X is a claim size in 2011, and has a Pareto distribution with parameters $\alpha = 5$ and $\theta = 20$. There is an 8% inflation in 2012. Determine the distribution of the inflated variable.

Example 9 (T1Q3)**.**

Claim sizes expressed in Ringgit Malaysia(RM) follow a pareto distribution with parameters $\alpha = 5$ and $\theta = 2,950$. A euro is worth 4.6 RM. Calculate the probability that a claim will be worth 818.0 euros or more.

1.4.3 Raising to a Power

Theorem 2. Let X be a continuous random variable with pdf $f_X(x)$ and cdf $F_X(x)$ with $F_X(0) = 0$. Let $Y = X^{\frac{1}{\tau}}$. Then if $\tau > 0$,

$$F_Y(y) = F_X(y^\tau), f_Y(y) = \tau y^{\tau-1} f_X(y^\tau), y > 0$$

while if $\tau < 0$,

$$F_Y(y) = 1 - F_X(y^\tau), f_Y(y) = -\tau y^{\tau-1} f_X(y^\tau).$$

Definition 6.

When raising a distribution to a power, if $\tau > 0$, the resulting distribution is called transformed; if $\tau = -1$, it is called inverse, and if if $\tau < 0$ (but not -1), it is called inverse transformed.

Example 10.

Suppose $X \sim exp(1)$. Determine the cdf of the inverse, transformed, and inverse transformed exponential distribution.

1.4.4 Exponentiation

Theorem 3.

Let X be a continuous random variable with pdf $f_X(x)$ and cdf $F_X(x)$ with $f_X(0) > 0$ for all real x. Let $Y = exp(X)$. Then, for $y > 0$,

$$F_Y(y) = F_X(lny), f_Y(y) = \frac{1}{y} f_X(lny)$$

Example 11.
Let X have the normal distribution with mean μ and variance σ^2. Determine the cdf of $Y = e^X$.

$\square$

1.4.5 Continuous Mixing

Theorem 4.
Let X have pdf $f_{X|\Lambda}(x|\lambda)$ and cdf $F_{X|\Lambda}(x|\lambda)$, where λ is a parameter of X, while X may have other parameters, there are not relevant. Let λ be a realization of the random variable Λ with pdf $\pi_{|\Lambda}(\lambda)$. Then the unconditional pdf of X is

$$f_X(x) = \int f_{X|\Lambda}(x|\lambda)\pi_\Lambda(\lambda)d\lambda$$

where the integral is taken over all values of λ with positive probability.

Two distributions that are derived from continuous mixing are:

1. Pareto distribution:
   If
   $$X|\beta \sim \text{Exp}(\beta)$$
   and
   $$\beta \sim \text{Inverse Gamma}(\alpha, \theta)$$
   then
   $$X \sim \text{Pareto}(\alpha, \theta)$$
2. Negative Binomial: If
   $$X|\lambda \sim \text{Poisson}(\lambda)$$
   and
   $$\Lambda \sim \text{gamma}(\alpha, \beta)$$
   then
   $$X \sim \text{Negative Binomial}(r = \alpha, \beta)$$

Example 12. You are given the following:

- The amount of an individual claim has an exponential distribution given by:
  $$f(x|\lambda) = \frac{1}{\lambda}e^{-x/\lambda} \quad x > 0, \lambda > 0$$
- The parameter λ has a probability density function given by:
  $$\pi(\lambda) = \frac{400}{\lambda^3}e^{-20/\lambda} \quad \lambda > 0$$

Determine the mean of the claim severity distribution. $\boxed{20}$

Example 13.
The claim count N has a Poisson distribution with mean λ. Λ has a gamma distribution with parameters $\alpha = 2$ and $\beta = 0.5$. Calculate the probability that $N = 1$. $\boxed{0.2963}$

1.4.6 Splicing

Definition 7. A k-component **spliced distribution** has a density function that can be expressed as follows:

$$f_X(x) = \begin{cases} a_1 f_1(x), & c_0 < x < c_1, \\ a_2 f_2(x), & c_1 < x < c_2, \\ \vdots \\ a_k f_k(x), & c_{k-1} < x < c_k. \end{cases}$$

For $j = 1, \ldots, k$, each $a_j > 0$ and each $f_j(x)$ must be a legitimate density function with all probability on the interval $c_{j-1}, c_j)$. Also $a_1 + \cdots a_k = 1$.
Note that the splicing does not ensure that the resulting density function will be continuous. Such a restriction could be added to the specification.

Example 14 (T1Q4)**.**
An insurance loss is being modeled as a continuous two-spliced distribution as follows:

$$f_X(x) = \begin{cases} c_1 e^{-x/200}, & 0 < x < 200 \\ c_2 e^{-x/800}, & x \geq 200 \end{cases}$$

Calculate the average loss.

1.5 Severity with Coverage Modifications

1.5.1 The Limited Loss Variable

Definition 8. The limited loss variable (Payment per loss with claims limit)

- The random variable for the amount not paid due to the deductible is the minimum of X and d. This random variable is

$$X \wedge d = \begin{cases} X, & X < d, \\ d, & X \geq d. \end{cases}$$

It expected value is called the limited expected value.

$$E(X \wedge d) = \int_{-\infty}^{d} x f(x) dx + dS(d) = \int_0^d S(x) dx$$

if the variable is continuous, and

$$E(X \wedge d) = \sum_{x \leq d} x_j P(x_j) + dS(d)$$

if the variable is discrete.

This variable is also called the right censored variable. The expected value is the expected amount not paid for loss below d, the integral, plus the expected amount not paid for above d, which is $dS(d)$. An insurance phenomena that related to this variable is the existence of a policy limit that sets a maximum on the benefit to be paid. Note that $(X - d)_+ + X \wedge d = X$. That is, buying one policy with limit of d and another with a deductible of d is equivalent to buying full coverage.

- The k^{th} moment of $X \wedge d$ is

$$E(X \wedge d)^k = \int_{-\infty}^{d} x^k f(x)dx + d^k S(d) = \int_0^d kx^{k-1} S(x)dx$$

if the variable is continuous, and

$$E(X \wedge d)^k = \sum_{x \leq d} x_j^k P(x_j) + d^k S(d)$$

if the variable is discrete.

Remark:
If $x \geq \theta$, then

$$\begin{aligned} E(X \wedge d)^k &= \int_\theta^d x^k f(x)dx + d^k S(d) \\ &= \theta^k S(\theta) + \int_\theta^d kx^{k-1} S(x)dx \end{aligned}$$

Example 15.
The claim size (X) distribution for an insurance coverage is modeled as a Pareto distribution with parameters $\alpha = 3$, $\theta = 1000$. Calculate $E(X \wedge 3000)$ and $E(X \wedge 3000)^2$. $\boxed{468.75,\ 562500}$

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Example 16.
Let X be a random variable with discrete loss distribution given by

x	100	200	300	400	500
$P(X=x)$	0.55	0.20	0.10	0.08	0.07

Calculate $V(X \wedge 300)$. $\boxed{7100}$

Example 17.
Claim severity follows a single-parameter Pareto distribution with $\alpha = 1$ and $\theta = 1000$. An insurance coverage has a claims limit of 10,000. Determine the mean and variance of the claim severity. $\boxed{3302.585,\ 8{,}092{,}932}$

Example 18 (T1Q5)**.**
For insurance coverage, you are given that claim size, X, follows a gamma distribution with parameters $\alpha = 3$, $\theta = 900$. Determine $V(X \wedge \wedge 2,000)$.

Example 19 (T1Q6)**.**
X is a random variable representing loss size. You are given that

$$E[X \wedge d] = 424.5 - \frac{283^3}{2d^2}$$

Loss sizes are affected by 15% inflation. Determine the average payment per loss under a policy with 325 ordinary deductible after inflation.

1.5.2 Deductibles

An ordinary deductible d means that the first d of each claim is not paid. An ordinary deductible modifies a random variable into either the left censored and shifted or excess loss variable.

- The per-loss variable is

$$Y^L = (X-d)_+ = \begin{cases} 0, & X \le d, \\ X-d, & X > d, \end{cases}$$

- For a given value of d with $P(X > d) > 0$, the per-payment variable is

$$Y^P = \begin{cases} Undefined, & X \le d, \\ X-d, & X > d, \end{cases}$$

- The corresponding densities are:

$$f_{Y^P} = \frac{f_X(y+d)}{S_X(d)}, y > 0$$

$$f_{Y^L}(y) = \begin{cases} F_X(d), & y = 0 \\ f_X(y+d), & y > 0 \end{cases}$$

- $$\begin{aligned} F_{Y^L}(x) &= P(Y^L \le x) \\ &= P(X-d \le x) \\ &= P(X \le x+d) \\ &= F_X(x+d) \end{aligned}$$

- $$\begin{aligned} F_{Y^P}(x) &= P(Y^P \le x) \\ &= P(X-d \le x | X > d) \\ &= P(X \le x+d | X > d) \\ &= \frac{P(d<X<x+d))}{P(X>d)} \\ &= \frac{F_X(x+d)-F_X(d)}{1-F_X(d)} \end{aligned}$$

  Notice the need to subtract $F_X(d)$ in the numerator, because of the joint condition $X > d$ and $X \le x_d$.

- $S_{Y^P}(x) = \frac{S_X(x+d)}{S_X(d)}$. Thus, working with survival functions is often easier.

- The expected value of $(X-d)_+$ can be calculated from

$$E(X-d)_+ = \int_d^\infty (x-d)f(x)dx = \int_d^\infty S(x)dx$$

  if the variable is continuous, and

$$E(X-d)_+ = \sum_{x_j > d} (x_j - d)p(x_j)$$

  if the variable is discrete.

- The k^{th} moment of $(X-d)_+$ is define as

$$E[(X-d)_+^k] = \int_d^\infty (x-d)^k f(x)dx$$

  if the variable is continuous, and

$$E[(X-d)_+^k] = \sum_{x_j > d} (x_j - d)^k p(x_j)$$

  if the variable is discrete.

- Expected cost per payment(the mean excess loss)

$$\begin{aligned} E(Y^P) &= E(X-d|X>d) = e_X(d) \\ &= \frac{E(X-d)_+}{1-F(d)} \\ &= \frac{\int_d^\infty (x-d)f(x)dx}{S(d)} \end{aligned}$$

- The k^{th} moment of the excess loss variable is

$$\begin{aligned} e_X^k(d) &= \frac{[E(X-d)_+^k]}{1-F(d)} \\ &= \frac{\int_d^\infty (x-d)^k f(x)dx}{S(d)} \end{aligned}$$

- Special cases

Distribution	$e_X(d)$
Exp(θ)	θ
Pareto(α, θ)	$\frac{\theta+d}{\alpha-1}$
Single Pareto(α, θ)	$\frac{d}{\alpha-1}, d \ge \theta$

Theorem 5. For an ordinary deductible,

$$E(Y^L) = E(X-d)_+ = E(X) - E(X \wedge d)$$

and

$$E(Y^P) = \frac{E(Y^L)}{1-F(d)} = \frac{E(X) - E(X \wedge d)}{1-F(d)}$$

Example 20.
A random sample of auto glass claims has yielded the following observed claim amounts:

$$1000 \quad 1,250 \quad 2,000 \quad 2,500 \quad 3,000$$

Calculate $E[(X-1,500)_+]$, $E[(X-1,500)_+^3]$ and $E[X \wedge 1,500]$. $\boxed{600,\ 9\times 10^8,\ 1{,}350}$

Example 21.
Claim severity has the following distribution:

Claim Size	50	150	500	1000	2000	5000	10000
Probability	0.305	0.225	0.220	0.155	0.055	0.030	0.010

Determine $E[(X-120)_+]$ and $V[(X-120)_+]$.
$\boxed{575.35,\ 1{,}705{,}942}$

Example 22 (T1Q7)**.**
You are given the following:

- Losses follow a Weibull distribution with parameters $\theta = 29$ and $\tau = 3$.
- The insurance coverage has an ordinary deductible of 10.

If the insurer makes a payment, what is the probability that an insurer's payment is less than or equal to 29.

Example 23 (T1Q8)**.**
The distribution of X is specified by it's hazard rate function

$$h(x) = \frac{xe^{-0.5x}}{\int_x^\infty se^{-0.5s}ds}, x > 0$$

Calculate $E(X-3)_+$.

Example 24 (T1Q9)**.**
Suppose $X \sim N(\mu = 150, \sigma^2 = 900)$, calculate $E[(X-90)_+]$.

Example 25.
For an insurance, losses, X, has the following distribution:

Claim Size	50	150	500	1000	2000	5000	10000
Probability	0.305	0.225	0.220	0.155	0.055	0.030	0.010

The insurance has an ordinary deductible of 800 per loss, Y^P is the payment per payment random variable, calculate $V(Y^P)$. $\boxed{4{,}256{,}400}$

Example 26.
Calculate the payment per loss for an insurance coverage with deductible of 5 if the loss distribution is

(i) exponential with mean 10. $\boxed{6.0653}$

(ii) Pareto with parameters $\alpha = 3$, $\theta = 20$. $\boxed{6.4}$

(iii) Single-parameter Pareto with parameters $\alpha = 2$, $\theta = 1$ $\boxed{.2}$

Example 27.

You are given:

x	$E[X \wedge x]$	$F(x)$
120	70	0.3
600	210	0.7
1,200	420	0.9
12000	1,120	1.0

Determine the mean excess loss for a deductible of 600.

Example 28 (T1Q10)**.**

A loss, X, follows a Pareto distribution with $\alpha = 7$ and unspecified parameter θ. You are given:

$E[X-791.00|X > 791.00] = 1.8E[X-115|X > 115]$.

Calculate $E[X - 2,440|X > 2,440]$.

Example 29 (T1Q11)**.**

You are given:

- The coverage limit is 12,900.
- The expected value of the loss before considering the coverage limit is 9,360.
- The probability of a claim for 12,900 or more is 0.16.
- The mean excess loss at 12,900 is 20,330.

Determine the average claim paid less than 12,900.

Example 30 (T1Q12)**.**

You are given the following information:

- The amount of an individual claim has an exponential distribution with mean λ.
- The parameter λ has a probability density function given by:

  $$\pi(\lambda) = k\lambda^{-5}e^{-31/\lambda}, \lambda > 0.$$

  where k is a constant.

Determine the expected claim size greater than 31.

Example 31 (T1Q13).

The probability density function of loss amounts is given by

$$f(x) = \frac{8(430-x)^7}{430^8}, 0 < x \leq 430$$

An insurance coverage for these losses has an ordinary deductible of 100. Calculate the mean excees loss at 100.

1.5.3 Franchise Deductible

A franchise deductible modifies the ordinary deductible by adding the deductible when there is a positive amount paid.

- The per-loss variable is

$$Y^L = \begin{cases} 0, & X \leq d, \\ X, & X > d, \end{cases}$$

- The per-payment variable is

$$Y^P = \begin{cases} Undefined, & X \leq d, \\ X, & X > d, \end{cases}$$

- The corresponding densities are:

$$f_{Y^P} = \frac{f_X(y)}{S_X(d)}, y > d$$

$$f_{Y^L}(y) = \begin{cases} F_X(d), & y = 0 \\ f_X(y), & y > d \end{cases}$$

- The expected value of Y^L for franchise deductible, d, can be calculated from

$$E(Y^L) = \int_d^\infty xf(x)dx$$

if the variable is continuous, and

$$E(Y^L) = \sum_{x_j > d} (x_j)p(x_j)$$

if the variable is discrete.

- The k^{th} moment of Y^L for franchise deductible, d is define as

$$E[(Y^L)^k] = \int_d^\infty x^k f(x)dx$$

if the variable is continuous, and

$$E[(Y^L)^k] = \sum_{x_j > d} x_j^k p(x_j)$$

if the variable is discrete.

- Notes:

1. $$\begin{aligned} E(Y^L) &= \textstyle\int_{-\infty}^\infty xf(x)dx - \int_0^d xf(x)dx \\ &= E(X) - [E(X \wedge d) - dS(d)] \\ &= E(X) - E(X \wedge d) + dS(d) \\ &= E(X-d)_+ + dS(d) \\ &= e(d)S(d) + dS(d) \\ &= [e(d) + d]S(d) \end{aligned}$$

2. $$\begin{aligned} E(Y^P) &= \tfrac{E(Y^L)}{S(d)} \\ &= \tfrac{E(X) - E(X \wedge d) + dS(d)}{S(d)} \\ &= \tfrac{E(X) - E(X \wedge d)}{S(d)} + d \\ &= e(d) + d \end{aligned}$$

Example 32.

Losses follow a Pareto distribution with $\alpha = 3.5$, $\theta = 5000$. A policy covers losses subjects to a 500 franchise deductible. Determine the average payment per loss and average payment per payment. $\boxed{1{,}934,\ 2{,}700}$

1.5.4 Loss Elimination Ratio (LER)

Definition 9. The ***loss elimination ratio*** is the ratio of the decrease in the expected payment with an ordinary deductible to the expected payment without the ordinary deductible. Therefore

$$LER = \frac{E(X) - [E(X) - E(X \wedge d)]}{E(X)} = \frac{E(X \wedge d)}{E(X)}$$

provided $E(X)$ exists.

LER can be meaningful in evaluating the impact of a deductible.

Note:

$LER = \frac{E(X) - E(X-d)_+}{E(X)} = 1 - \frac{e(d)S(d)}{E(X)}$

Example 33.

Determine the loss elimination ratio for a Pareto distribution with $\alpha = 3$, $\theta = 2000$ with an ordinary deductible of 500. $\boxed{0.36}$

Example 34 (T1Q14)**.**

Let X be a discrete random variable with probability generating function

$P_X(z) = 0.49z^{220} + 0.20z^{660} + 0.14z^{1100} + 0.11z^{1540} + 0.06z^{1980}$

Calculate LER(1,200).

1.5.5 The Effect of Inflation on Ordinary Deductible

Inflation increase costs, when there is a deductible, the effect of inflation is magnified.

Theorem 6. For an ordinary deductible of d after uniform inflation of $1+r$,

$$E[(1+r)X \wedge d] = (1+r)E\left[X \wedge \frac{d}{1+r}\right]$$

and

$$E[(1+r)Y^L] = (1+r)E(X) - (1+r)E\left(X \wedge \frac{d}{1+r}\right)$$

and, if $S\left(\frac{d}{1+r}\right) < 1$, then

$$E\left[(1+r)Y^P\right] = \frac{(1+r)E(X) - (1+r)E\left(X \wedge \frac{d}{1+r}\right)}{1 - F\left(\frac{d}{1+r}\right)}$$

Example 35.
The underlying loss distribution function for a certain line of business in 2008 is:

$$F(x) = 1 - x^{-6}, x > 1.$$

From 2008 to 2009, 10% inflation impacts all claims uniformly. Determine the expected amount not pay due deductible of 2.2 in year 2009. $\boxed{1.313125}$

1.5.6 The Effect of Inflation on Policy Limit

For a policy limit of u, after uniform inflation of $1+r$, the expected cost is

$$E\{[(1+r)X] \wedge u\} = (1+r)E\left[X \wedge \frac{u}{1+r}\right]$$

Example 36.
Impose a limit of 3,000 on a Pareto distribution with $\alpha = 3$, $\theta = 2000$. Determine the expected cost per loss with the limit and after 10% uniform inflation is applied. $\boxed{903.2}$

1.5.7 Coinsurance, Deductible, and Limits

Coinsurance of α means that a portion, α, of each loss is reimbursed by insurance. For example, 80% coinsurance means that insurance will pay 80% of the loss.

When ordinary deductible, limit coinsurance, and inflation are present, the per loss random variable is:

$$Y^L = \begin{cases} 0, & x < \frac{d}{1+r}, \\ \alpha[(1+r)X - d], & \frac{d}{1+r} \le X < \frac{u}{1+r}, \\ \alpha(u-d), & X \ge \frac{u}{1+r}. \end{cases}$$

Note that u is the maximum cover loss and $\alpha(u-d)$ is policy limit.
The quantities in this definition are applied in a particular order with the coinsurance applied last.

Theorem 7.
$$E(Y^L) = \alpha(1+r)\left[E\left(X \wedge \frac{u}{1+r}\right) - E\left(X \wedge \frac{d}{1+r}\right)\right]$$
and
$$E(Y^P) = \frac{E(Y^L)}{1 - F_X\left(\frac{d}{1+r}\right)}$$

Proof: From the definition of Y^L,
let $d^* = \frac{d}{1+r}$ and $u^* = \frac{u}{1+r}$

$$Y^L = \alpha(1+r)[X \wedge u^* - X \wedge d^*]$$

Thus,

$$E(Y^L) = \alpha(1+r)[E(X \wedge u^*) - E(X \wedge d^*)]$$

Theorem 8.
$$E[(Y^L)^2] = \alpha^2(1+r)^2\{E\left[(X \wedge \tfrac{u}{1+r})^2\right] \\ -E\left[(X \wedge \tfrac{r}{1+r})^2\right] - 2\left(\tfrac{d}{1+r}\right)E\left(X \wedge \tfrac{u}{1+r}\right) \\ +2\left(\tfrac{d}{1+r}\right)E\left(X \wedge \tfrac{d}{1+r}\right)\}$$

Proof:
$$\frac{(Y^L)^2}{\alpha^2(1+r)^2}$$
$$= [X \wedge u^* - X \wedge d^*]^2$$
$$= (X \wedge u^*)^2 + (X \wedge d^*)^2 - 2(X \wedge u^*)(X \wedge d^*)$$
$$= (X \wedge u^*)^2 - (X \wedge d^*)^2 - 2(X \wedge d^*)[(X \wedge u^*) - (X \wedge d^*]$$
$$= (X \wedge u^*)^2 - (X \wedge d^*)^2 - 2d^*[(X \wedge u^*) - (X \wedge d^*)]$$

Thus
$$E(Y^L)^2 = \alpha^2(1+r)^2[E(X \wedge u^*)^2 - E(X \wedge d^*)^2 - 2d^*[E(X \wedge u^*) - E(X \wedge d^*)]]$$

Note:
$$2(X \wedge d^*)[(X \wedge u^*) - (X \wedge d^*)] = 2d^*[(x \wedge u^*) - (X \wedge d^*)]$$

To see this, when $X < d^*$, both sides equal zero; when $d^* \le x \le u^*$, both sides equal $2d^*(X - d^*)$; and when $x \ge u^*$, both sides equal $2d^*(u^* - d^*)$.

Example 37 (T1Q15)**.**
An individual losses has the Pareto distribution with parameters $\alpha = 3$ and $\theta = 240$ with deductible of 57.2, coinsurance of 81% and a loss limit of 114.4 (before application of the deductible and coinsurance) are applied to each individual loss. Loss sizes are affected by 10% inflation. Determine the variance of the loss payment on the per payment basic.

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1.5.8 Bonus

An agent receives a bonus if his loss ratio is below certain amount, or a hospital receives a bonus if it doesn't submit too many claims.

$$\begin{aligned}
\text{Bonus} &= \max[0, c(rP - X)] \\
&= \begin{cases} 0, & c(rP - X) < 0 \\ c(rP - X), & c(rP - X) > 0 \end{cases} \\
&= \begin{cases} 0, & X > rP \\ c(rP - X), & X < rP \end{cases} \\
&= crP - c\min(rP, X) \\
&= crP - c(X \wedge rP)
\end{aligned}$$

Note: $\max(0, a - b) = a - \min(a, b)$

Example 38.
An insurance agent will receive a bonus if his loss ratio is less than 70%. you are given:

- His loss ratio is calculated as incurred losses divided by earned premium on his block of business.
- The agent will receive a percentage of earned premium equal tp 1/3 of the difference between 70% and his loss ratio.
- The agent receive no bonus if his loss ratio is greater than 70%.
- His earned premium is 500,000.
- His incurred losses are distributed according to the Pareto distribution

$$F(x) = 1 - \left(\frac{600,000}{x + 600,000}\right)^3$$

Calculate the expected value of his bonus. $\boxed{56,555.86}$

Example 39 (T1Q16).

In a major college football program, the revenue from ticket sales for a home game is being modeled as a Pareto distribution with $\alpha = 5$ and $\theta = 1{,}400{,}000$. For each home game, the coach receives a bonus only if revenue exceeds 700,000. The amount of bonus is 7% of the revenue in excess of 700,000. If there are 10 home games in each football season, calculate the expected bonus the football coach receives each football season.

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1.6 Tails of distributions

The tail of a distribution is the portion of the distribution corresponding to large values of the random variable. Random variables that tend to assign higher probabilities to larger values are said to be heavier-tailed.

1.6.1 Concept of Tail Weight

1. Relative: Model A has a heavier tail the Model B.
2. Absolute: Distributions with certain property are classified a heavy-tail.

When using a parametric distribution to model loss size, it is important to provide for the possibility of very high claims, which may occur rarely. The bigger the tail weight of the distribution, the more provision for high claims.

1.6.2 Classification of Tail weight

1. Classification based on **moments**: The more positive raw moment exist, the less tail weight.
2. Comparison based on **limiting tail behavior**: if
$$\lim_{x\to\infty} \frac{S_1(x)}{S_2(x)} = \lim_{x\to\infty} \frac{f_1(x)}{f_2(x)} = \infty$$
then X_1 has heavier tail.
3. Classification based on the **hazard rate function**: An increasing hazard rate function (w.r.t x) have light tail.
4. Classification based on **mean excess loss function**: If the mean excess loss function ($e_X(d)$) is increasing in d, the distribution is considered to have a heavy tail. If the mean excess loss function ($e_X(d)$) is decreasing in d, the distribution is considered to have a light tail.

Example 40.
Random variable X_1 with distribution function F_1 and probability density function f_1 has a heavier tail than random variable X_2 with distribution function F_2 and probability density function f_2. Which of the following statements is true?

1. X_1 will tend to have fewer positive moments than X_2.
2. The limiting ratio of the density functions, $\frac{f_1}{f_2}$, will go to infinity.
3. The hazard rate of X_1 will increase more rapidly than hazard rate of X_2.
4. The mean residual life of X_1 will increase more rapidly than the mean residual life of X_2.

Example 41.
Which of the following are true based on the existence of moments test?

1. The Loglogistic Distribution has a heavier tail than the Gamma Distribution.
2. The Paralogistic Distribution has a heavier tail than the Lognormal Distribution.
3. The Inverse Exponential has a heavier tail than the Exponential Distribution.

Example 42.
You are given:

- X has an density $f(x)$, where $f(x) = \frac{500{,}000}{x^3}$, for $x > 500$.
- Y has density $g(y) = ye^{-y/500}/500,000$.

Which of the following are true?

1. X has an increasing mean residual life function.
2. Y has an increasing hazard rate.
3. X has a heavier tail than Y based on the hazard rate test.

1.7 Measure of Risk

A risk measure is a mapping from the random variable representing the loss associated with the risks to the real line. A risk measure gives a single number that is intended to quantify the risk exposure. Risk measures are denoted by the function $\rho(X)$.

1.7.1 Value-at-Risk

Definition 10. Let X denote a loss random variable. The value-at-Risk of X at the $100p\%$ level, denoted $VaR_p(X)$ or π_p, is the $100p$ percentile (or quantile) of the distribution of X.

For continuous distributions, we can simply write $VaR_p(X)$ for the random variable X as the value of π_p satisfying $P(X > \pi_p) = 1 - p$.

For a discrete distribution of loss, the percentile may not be unique. We define the VaR as the lowest percentile.

$$\pi_p = min(\pi | P(X \le \pi) \ge p)$$

Note:

- VaR does not satisfy subadditivity requirement.
- **VaR for normal and lognormal distribution**
  Let z_p be the 100pth percentile of a standard normal distribution. Then
  if X is normal, $\text{VaR}_P(X) = \mu + z_p\sigma$,
  if X is lognormal, then $\text{VaR}_P(X) = e^{\mu + z_p\sigma}$
- **VaR for exponential distribution**
  $e^{-\pi_p/\theta} = 1 - p$
  $\pi_p = -\theta \ln(1 - p)$
- **VaR for Pareto distribution**
  $\left(\frac{\theta}{\theta+\pi_p}\right)^\alpha = 1 - p$
  $\frac{\theta}{\theta+\pi_p} = (1 - p)^{1/\alpha}$
  $\pi_p = \frac{\theta[1-(1-p)^{1/\alpha}]}{(1-p)^{1/\alpha}}$

Example 43.

Losses follow an Exponential distribution with θ = 1,240. Determine the VaR at 93.00%.

Example 44.

Losses have a lognormal distribution with mean 10 and variance 300. Calculate the VaR at 95% and 99%. $\boxed{34.68, 77.33}$

Example 45 (T1Q17)**.**

Annual losses follow a Pareto distribution with α = 3.70 and θ = 1,310. Calculate $\text{VaR}_{0.940}$.

Example 46.
Losses have the following distribution:

Loss size	Probability
0	0.5
5	0.3
10	0.15
20	0.05

Calculate the VaR at 95% and 99%. $\boxed{10,20}$

1.7.2 Tail-Value-at-Risk

Definition 11. Let X denote a loss random variable. The Tail-Value-at-Risk of X at the $100p\%$ level, denoted $TVaR_p(X)$ or Conditional Tail Expectation (CTE_p), is the expected loss given that the loss exceeds the $100p$ percentile of the distribution.

$$TVaR_p(X) = E(X|X > \pi_p)$$

For continuous distribution,

$$\begin{aligned} TVaR_p(X) &= \frac{\int_{\pi_p}^{\infty} xf(x)dx}{S(\pi_p)} \\ &= \frac{\int_{\pi_p}^{\infty} xf(x)dx}{1-p} \end{aligned}$$

Note that

$$\begin{aligned} \int_{\pi_p}^{\infty} xf(x)dx &= \int_0^{\infty} xf(x)dx - \int_0^{\pi_p} xf(x)dx \\ &= E(X) - [E(X \wedge \pi_p) - \pi_p S(\pi_p)] \\ &= E(X) - E(X \wedge \pi_p) + (1-p)\pi_p \end{aligned}$$

$TVaR_p$ can also be written as

$$\begin{aligned} TVaR_p &= E(X|X > \pi_p) \\ &= \pi_p + \frac{\int_{\pi_p}^{\infty}(x-\pi_p)f(x)dx}{1-p} \\ &= \pi_p + e(\pi_p) \end{aligned}$$

This formula should be use for Pareto and exponential distributions which have simple formulas for $e(\pi_p)$.

Notes:

- **TVaR for exponential distributions**

$$\begin{aligned} \text{TVaR} &= \pi_p + e(\pi_p) \\ &= -\theta \ln(1-p) + \theta \\ &= \theta[1 - \ln(1-p)] \end{aligned}$$

- **TVaR for pareto distributions**
  $\text{TVaR}(X) = \pi_p + e(\pi_p) = \pi_p + \frac{\theta+\pi_p}{\alpha-1} = \frac{\alpha\pi_p+\theta}{\alpha-1}$

- **TVaR for lognormal distributions**

$$\begin{aligned} \text{TVaR}(X) &= E(X)\left(\frac{(1-\Phi(\ln \pi_p - \mu - \sigma^2)/\sigma)}{1-p}\right) \\ &= E(X)\left(\frac{(1-\Phi((\mu + z_p\sigma) - \mu - \sigma^2)/\sigma)}{1-p}\right) \\ &= E(X)\left(\frac{\Phi(\sigma - z_p)}{1-p}\right) \end{aligned}$$

- **TVaR for normal distributions**

$$\begin{aligned} (1-p)\text{TVaR}_p(Z) &= \frac{1}{\sqrt{2\pi}} \int_{\pi_p(z)}^{\infty} ye^{-y^2/2}dx \\ &= \frac{1}{\sqrt{2\pi}}\left[-e^{y^2/2}\right]\Big|_{\pi_p(Z)}^{\infty} \\ &= \frac{e^{-z_p^2/2}}{\sqrt{2\pi}} \end{aligned}$$

So, $\text{TVaR}_p(Z) = \frac{e^{-z_p^2/2}}{(1-p)\sqrt{2\pi}} = \frac{\phi(z_p)}{1-p}$
Note that $\pi_p(Z) = \Phi^{-1}(p) = z_p$.
Thus,

$$E[X|X > x] = E\left[\mu + \sigma Z|Z > \frac{x-\mu}{\sigma}\right]$$
$$= \mu + \sigma E\left[Z|Z > \frac{x-\mu}{\sigma}\right]$$

$$\text{TVaR}(X) = \mu + \sigma TVaR(Z) = \mu + \sigma\frac{\phi(z_p)}{1-p}$$

- TVaR is coherent
- $TVaR_0(X) = E(X)$
- $TVaR_p(X) \geq VaR_p(X)$, with equality holding only if $VaR_p(X) = max(X)$

Example 47.
Losses have a lognormal distribution with mean 10 and variance 300. Calculate $\text{TVaR}_{0.95}$. $\boxed{63.84}$

Example 48. Losses have an exponential distribution with mean 10. Calculate $TVaR_{0.95}$. $\boxed{39.96}$

Example 49 (T1Q18)**.**
Annual losses follow a Pareto distribution with parameters $\alpha = 4$ and $\theta = 1000$. $\text{TVaR}_p = 1,537$, Determine p.

Example 50 (T1Q19).

The losses experienced by an insurance company have the following probability distribution:

Loss size	Probability
0	0.60
200	0.25
300	0.10
1,400	0.05

Calculate the $CTE_{0.66}$(or $TVaR_{0.66}$).

Example 51 (T1Q20).

Annual losses follow a Pareto distribution with α = 3.70 and θ = 1,150. Calculate the difference between $TVaR_{0.94}$ and $VaR_{0.94}$.