MEME16203Linear Models Marking Guide

Assignment 2

UNIVERSITI TUNKU ABDUL RAHMAN

Faculty:	FES	Unit Code:	MEME16203
Course:	MAC	Unit Title:	Linear Models
Year:	1,2	Lecturer:	Dr Yong Chin Khian
Session:	May 2022		
Due by:			

Q1. Suppose that we are interested in the coefficients $\boldsymbol{\beta}$ of a linear model $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta}+\boldsymbol{\epsilon}$, where $\mathbf{Y}$ is $n \times 1$, $\mathbf{X}$ is nonsingular with dimension $n \times p$ and $\boldsymbol{\epsilon} \sim N(\mathbf{0}, \sigma^2\mathbf{I})$. Furthermore, suppose that it is of interest to partition that model in the form $\mathbf{X} = \begin{bmatrix}\mathbf{X}_1 & \mathbf{X}_2 & \mathbf{X}_3\end{bmatrix}$, for $n \times p_i$ matrices $\mathbf{X}_i, i = 1, 2, 3$. Finally, suppose that an investigator creates a partially orthogonal design, in which $\mathbf{X} = \begin{bmatrix}\mathbf{X}_1 & \mathbf{X}_2 & \mathbf{X}_3\end{bmatrix}$ has the property that $\mathbf{X_i^TX_j} = 0$ for $i \neq j$. We have been showing that the least squares estimate of $\boldsymbol{\beta}$ takes the form $\hat{\boldsymbol{\beta}} = \begin{bmatrix}\hat{\beta}_1\\ \hat{\beta}_2\\ \hat{\beta}_3\end{bmatrix}$, where

- $\hat{\boldsymbol{\beta}}_\mathbf{1} = (\mathbf{X_1^TX_1})^{-1}\mathbf{X_1^TY}$
- $\hat{\boldsymbol{\beta}}_\mathbf{2} = (\mathbf{X_2^TX_2})^{-1}\mathbf{X_2^TY}$
- $\hat{\boldsymbol{\beta}}_\mathbf{3} = (\mathbf{X_3^TX_3})^{-1}\mathbf{X_3^TY}$

Show that the estimates $\hat{\boldsymbol{\beta}}_\mathbf{1}$, $\hat{\boldsymbol{\beta}}_\mathbf{2}$ and $\hat{\boldsymbol{\beta}}_\mathbf{3}$ are uncorrelated. That is, show that the covariance matrix $V(\hat{\boldsymbol{\beta}})$ has a block-diagonal form

$$\begin{bmatrix}\mathbf{A}_1 & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{A}_2 & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{A}_3\end{bmatrix}$$

for some matrices $\mathbf{A}_1$,$\mathbf{A}_2$ and $\mathbf{A}_3$,

Ans.

$$V(\hat{\boldsymbol{\beta}}) = V\begin{bmatrix}(\mathbf{X_1^TX_1})^{-1}\mathbf{X_1^T}\\ (\mathbf{X_2^TX_2})^{-1}\mathbf{X_2^T}\\ (\mathbf{X_3^TX_3})^{-1}\mathbf{X_3^T}\end{bmatrix}\mathbf{Y}$$

$$= \begin{bmatrix}(\mathbf{X_1^TX_1})^{-1}\mathbf{X_1^T}\\ (\mathbf{X_2^TX_2})^{-1}\mathbf{X_2^T}\\ (\mathbf{X_3^TX_3})^{-1}\mathbf{X_3^T}\end{bmatrix} V(\mathbf{Y}) \begin{bmatrix}\mathbf{X_1}(\mathbf{X_1^TX_1})^{-1} & \mathbf{X_2}(\mathbf{X_2^TX_2})^{-1} & (\mathbf{X_3^TX_3})^{-1}\mathbf{X_3^T}\end{bmatrix}$$

$$= \begin{bmatrix}(\mathbf{X_1^TX_1})^{-1}\mathbf{X_1^T}\\ (\mathbf{X_2^TX_2})^{-1}\mathbf{X_2^T}\\ (\mathbf{X_3^TX_3})^{-1}\mathbf{X_3^T}\end{bmatrix} \sigma^2(\mathbf{I}) \begin{bmatrix}\mathbf{X_1}(\mathbf{X_1^TX_1})^{-1} & \mathbf{X_2}(\mathbf{X_2^TX_2})^{-1} & (\mathbf{X_3^TX_3})^{-1}\mathbf{X_3^T}\end{bmatrix}$$

$$= \begin{bmatrix}(\mathbf{X_1^TX_1})^{-1}\mathbf{X_1^TX_1}(\mathbf{X_1^TX_1})^{-1} & (\mathbf{X_1^TX_1})^{-1}\mathbf{X_1^TX_2}(\mathbf{X_2^TX_2})^{-1} & (\mathbf{X_1^TX_1})^{-1}\mathbf{X_1^TX_3}(\mathbf{X_3^TX_3})^{-1}\\ (\mathbf{X_2^TX_2})^{-1}\mathbf{X_2^TX_1}(\mathbf{X_1^TX_1})^{-1} & (\mathbf{X_2^TX_2})^{-1}\mathbf{X_2^TX_2}(\mathbf{X_2^TX_2})^{-1} & (\mathbf{X_2^TX_2})^{-1}\mathbf{X_2^TX_3}(\mathbf{X_3^TX_3})^{-1}\\ (\mathbf{X_3^TX_3})^{-1}\mathbf{X_3^TX_1}(\mathbf{X_3^TX_3})^{-1} & (\mathbf{X_3^TX_3})^{-1}\mathbf{X_3^TX_2}(\mathbf{X_3^TX_3})^{-1} & (\mathbf{X_3^TX_3})^{-1}\mathbf{X_3^TX_3}(\mathbf{X_3^TX_3})^{-1}\end{bmatrix}$$

$$= \begin{bmatrix} (\mathbf{X_1^T X_1})^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (\mathbf{X_2^T X_2})^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & (\mathbf{X_3^T X_3})^{-1} \end{bmatrix}$$

Q2. You are given:

- Model (1): $Y_{ij} = \mu + \alpha_i + \epsilon_{ij}$
- Model (2): $Y_{ij} = \gamma_0 + \gamma_1(X_i - 10) + \gamma_2(X_i - 10)^2 + \epsilon_{ij}$

where $i = 1, 2, 3,\ j = 1, 2,\ X_1 = 5, X_2 = 10, X_3 = 15$ and μ, α_1, α_2, α_3, γ_0, γ_1, γ_2 and γ_3 are unknown parameters.

(a) For model (2), write down a formula for the best linear unbiased estimator (BLUE) for $\boldsymbol{\gamma} = \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{bmatrix}$.

Ans.

$$\mathbf{Y} = \begin{bmatrix} Y_{11} \\ Y_{12} \\ Y_{21} \\ Y_{22} \\ Y_{31} \\ Y_{32} \end{bmatrix},$$

$$\mathbf{X} = \begin{bmatrix} 1 & -5 & 25 \\ 1 & -5 & 25 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 5 & 25 \\ 1 & 5 & 25 \end{bmatrix}, \ \mathbf{X^T X} = \begin{bmatrix} 6 & 0 & 100 \\ 0 & 100 & 0 \\ 100 & 0 & 2500 \end{bmatrix} \text{ and } \mathbf{X^T Y} = \begin{bmatrix} Y_{..} \\ 5(Y_{3.} - Y_{1.}) \\ 25(Y_{1.} + Y_{3.}) \end{bmatrix}$$

$$|\mathbf{X^T X}| = 6(100)(2500) - 100^3 = 500000$$

$$(\mathbf{X^T X})^{-1}| = \frac{1}{500000} \begin{bmatrix} \begin{vmatrix} 100 & 0 \\ 0 & 2500 \end{vmatrix} & \begin{vmatrix} 0 & 0 \\ 100 & 2500 \end{vmatrix} & \begin{vmatrix} 0 & 100 \\ 100 & 0 \end{vmatrix} \\ \begin{vmatrix} 0 & 100 \\ 0 & 2500 \end{vmatrix} & \begin{vmatrix} 6 & 100 \\ 100 & 2500 \end{vmatrix} & \begin{vmatrix} 6 & 0 \\ 100 & 0 \end{vmatrix} \\ \begin{vmatrix} 0 & 100 \\ 100 & 0 \end{vmatrix} & \begin{vmatrix} 6 & 100 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 6 & 0 \\ 0 & 100 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 0.5 & 0 & -0.02 \\ 0 & 0.01 & 0 \\ -0.02 & 0 & 0.0012 \end{bmatrix}$$

$$\hat{\boldsymbol{\gamma}} = (\mathbf{X^T X})^{1} \mathbf{X^T Y} = \begin{bmatrix} 0.5 & 0 & -0.02 \\ 0 & 0.01 & 0 \\ -0.02 & 0 & 0.0012 \end{bmatrix} \begin{bmatrix} Y_{..} \\ 5(Y_{3.} - Y_{1.}) \\ 25(Y_{1.} + Y_{3.}) \end{bmatrix} = \begin{bmatrix} \bar{Y}_{2.} \\ \frac{\bar{Y}_{3.} - \bar{Y}_{1.}}{10} \\ \frac{\bar{Y}_{1.} - 2\bar{Y}_{2.} + \bar{Y}_{3.}}{50} \end{bmatrix}$$

(b) For model (1), verify that $\tau = 3\mu + 3\alpha_1 - 6\alpha_2 + 6\alpha_3$ is an estimable function and writhe down a formula for $\hat{\tau}$, the BLUE for τ.

Ans.

$$\tau = 3\mu + 3\alpha_1 - 6\alpha_2 + 6\alpha_3$$

$$= 3(\mu+\alpha_1) - 6(\mu+\alpha_2) + 6(\mu+\alpha_3)$$
$$= E(3\bar{Y}_{1.} - 6\bar{Y}_{2.1} + 6\bar{Y}_{3.})$$

Hence, τ is estimable.
The BLUE for τ is $3\bar{Y}_{1.} - 6\bar{Y}_{2.1} + 6\bar{Y}_{3.}$.

(c) Formulate what is meant by the statement that model (2) is a reparameterization of model (1), and verify that this statement is correct.

Ans.

The model $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$ is a reparameterization of the the model $\mathbf{Y} = \mathbf{W}\boldsymbol{\gamma} + \boldsymbol{\epsilon}$ if there is a matrix $\mathbf{F}$ such that $\mathbf{W} = \mathbf{XF}$ and a matrix $\mathbf{G}$ such that $\mathbf{X} = \mathbf{WG}$. The the space spanned by the columns of $\mathbf{X}$ is a basis for space spanned by the columns of $\mathbf{W}$ and vice versa.

In this case we can write model (1) as

$$\mathbf{Y} = \mathbf{X}\begin{bmatrix}\mu\\ \alpha_1\\ \alpha_2\\ \alpha_3\end{bmatrix} + \boldsymbol{\epsilon},$$

where

$$\mathbf{X} = \begin{bmatrix}1&1&0&0\\1&1&0&0\\1&0&1&0\\1&0&1&0\\1&0&0&1\\1&0&0&1\end{bmatrix},$$

and we can write model (2) as

$$\mathbf{Y} = \mathbf{W}\begin{bmatrix}\gamma_0\\ \gamma_1\\ \gamma_2\end{bmatrix} + \boldsymbol{\epsilon},$$

where

$$\mathbf{W} = \begin{bmatrix}1&-5&25\\1&-5&25\\1&0&0\\1&0&0\\1&5&25\\1&-5&25\end{bmatrix}$$

Then,

$$\mathbf{W} = \mathbf{XF}, \text{ where } \mathbf{F} = \begin{bmatrix}1&0&0\\0&-5&25\\0&0&0\\0&5&25\end{bmatrix}, \text{ and}$$

$$\mathbf{X} = \mathbf{WG} \text{ where } \mathbf{G} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & -\frac{1}{10} & 0 & \frac{1}{10} \\ 0 & \frac{1}{50} & -\frac{1}{25} & \frac{1}{50} \end{bmatrix}.$$

Q3. Suppose that y_{11} and y_{12} are independent $N(\mu_1, 9\sigma^2)$ variables independent of y_{21} and y_{22} that are independent $N(\mu_2, 16\sigma^2)$ and $N(\mu_2, 16\sigma^2)$ variables respectively. What is the BLUE of $4\mu_1 + 2\mu_2$? Explain carefully.

Ans.

$$\begin{bmatrix} y_{11} \\ y_{12} \\ y_{21} \\ y_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} + \boldsymbol{\epsilon}$$
$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$$

where
$\boldsymbol{\epsilon} \sim N(\mathbf{0}, \sigma^2 diag(9, 9, 16, 16))$

$$\mathbf{X} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{Y} = \begin{bmatrix} y_{11} \\ y_{12} \\ y_{21} \\ y_{22} \end{bmatrix}$$

Let $\mathbf{V^{-1/2}} = diag(\frac{1}{\sqrt{9}}, \frac{1}{\sqrt{9}}, \frac{1}{\sqrt{16}}, \frac{1}{\sqrt{16}})$ and

$$\mathbf{Z} = \mathbf{V^{-1/2}Y} = \begin{bmatrix} \frac{1}{\sqrt{9}} y_{11} \\ \frac{1}{\sqrt{9}} y_{12} \\ \frac{1}{\sqrt{16}} y_{21} \\ \frac{1}{\sqrt{16}} y_{22} \end{bmatrix}$$

$$\begin{aligned} E(\mathbf{Z}) &= \mathbf{V^{-1/2}} E(\mathbf{Y}) \\ &= \mathbf{V^{-1/2}X}\boldsymbol{\beta} \\ &= \mathbf{W}\boldsymbol{\beta} \end{aligned}$$

where

$$\mathbf{W} = \begin{bmatrix} \frac{1}{\sqrt{9}} & 0 \\ \frac{1}{\sqrt{9}} & 0 \\ 0 & \frac{1}{\sqrt{16}} \\ 0 & \frac{1}{\sqrt{16}} \end{bmatrix}$$

$$\begin{aligned} Var(\mathbf{Z}) &= \mathbf{V^{-1/2}}^T Var(\mathbf{Y}) \mathbf{V^{-1/2}} \\ &= \begin{bmatrix} \frac{1}{\sqrt{9}} & & & \\ & \frac{1}{\sqrt{9}} & & \\ & & \frac{1}{\sqrt{16}} & \\ & & & \frac{1}{\sqrt{16}} \end{bmatrix} \begin{bmatrix} 9\sigma^2 & & & \\ & 9\sigma^2 & & \\ & & 16\sigma^2 & \\ & & & 16\sigma^2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{9}} & & & \\ & \frac{1}{\sqrt{9}} & & \\ & & \frac{1}{\sqrt{16}} & \\ & & & \frac{1}{\sqrt{16}} \end{bmatrix} \end{aligned}$$

$$= \sigma^2 \mathbf{I}$$

Thus, $\mathbf{Z}$ follows a Gauss-Markov model with model matrix $\mathbf{W}$.

$$(\mathbf{W^T W}) = \begin{bmatrix} \frac{1}{\sqrt{9}} & \frac{1}{\sqrt{9}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{16}} & \frac{1}{\sqrt{16}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{9}} & 0 \\ \frac{1}{\sqrt{9}} & 0 \\ 0 & \frac{1}{\sqrt{16}} \\ 0 & \frac{1}{\sqrt{16}} \end{bmatrix} = \begin{bmatrix} 0.2222222222222222 & 0 \\ 0 & 0.125 \end{bmatrix}$$

$$(\mathbf{W^T W})^{-1} = \begin{bmatrix} 4.5 & 0 \\ 0 & 8.0 \end{bmatrix}$$

$$\mathbf{W^T Z} = \begin{bmatrix} \frac{1}{\sqrt{9}} & \frac{1}{\sqrt{9}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{16}} & \frac{1}{\sqrt{16}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{9}} y_{11} \\ \frac{1}{\sqrt{9}} y_{12} \\ \frac{1}{\sqrt{16}} y_{21} \\ \frac{1}{\sqrt{16}} y_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{9} y_{11} + \frac{1}{9} y_{12} \\ \frac{1}{16} y_{21} + \frac{1}{16} y_{22} \end{bmatrix}$$

So, the BLUE of $4\mu_1 - 2\mu_2$ is
$[4 \quad -2][(\mathbf{W^T W})^{-1} \mathbf{W^T Z}]$
$= [4 \quad -2] \begin{bmatrix} 4.5 & 0 \\ 0 & 8.0 \end{bmatrix} \begin{bmatrix} \frac{1}{9} y_{11} + \frac{1}{9} y_{12} \\ \frac{1}{16} y_{21} + \frac{1}{16} y_{22} \end{bmatrix}$
$= 18.0[\frac{1}{9} y_{11} + \frac{1}{9} y_{12}] - 16.0[\frac{1}{16} y_{21} + \frac{1}{16} y_{22}]$

Q4. Consider a problem of quadratic regression in one variable, $\mathbf{X}$. In particular, suppose that $n = 5$ values of a response $\mathbf{y}$ are related to values $x = 0, 1, 2, 3, 4$ by a linear model $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$ for

$$\mathbf{y} = \begin{bmatrix} 2 \\ 6 \\ 6 \\ 11 \\ 12 \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix}, \quad \boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix}, \text{ and } \quad \boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \end{bmatrix}$$

Define

$$\mathbf{W} = \begin{bmatrix} 1 & -2 & 2 \\ 1 & -1 & -1 \\ 1 & 0 & -2 \\ 1 & 1 & -1 \\ 1 & 2 & 2 \end{bmatrix}$$

Show that $\mathbf{y} = \mathbf{W}\boldsymbol{\gamma} + \boldsymbol{\epsilon}$ is reparameterization of $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$, where $\boldsymbol{\gamma}^{\boldsymbol{T}} = [\gamma_1, \gamma_2, \gamma_3]$. Find the OLS estimate of $\boldsymbol{\gamma}$ in the model $\mathbf{y} = \mathbf{W}\boldsymbol{\gamma} + \boldsymbol{\epsilon}$ and then OLS estimate of $\boldsymbol{\beta}$ in the original model. (Find numerical values.)

Ans.

$\mathbf{W} = \mathbf{XF}$ where

$$\mathbf{F} = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$\mathbf{X} = \mathbf{WG}$ where

$$\mathbf{G} = \begin{bmatrix} 1 & 2 & 6 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$\mathbf{W^T W} = diag(15, 10, 14)$
$(\mathbf{W^T W})^{-1} = diag(\frac{1}{5}, \frac{1}{10}, \frac{1}{14})$

Then, $\hat{\gamma}_{OLS} = (\mathbf{W^T W})^{-1}\mathbf{W^T y} = diag(\frac{1}{5}, \frac{1}{10}, \frac{1}{14}) \begin{bmatrix} 37 \\ 25 \\ -1 \end{bmatrix} = \begin{bmatrix} 2.4667 \\ 2.5 \\ -0.0714 \end{bmatrix}$

Since $\mathbf{W}\hat{\gamma}_{OLS} = \mathbf{XF}\gamma_{OLS}$, we must have $\hat{\boldsymbol{\beta}}_{OLS} = \mathbf{F}\gamma_{OLS}$.

$$\boldsymbol{\beta}_{OLS} = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2.4667 \\ 2.5 \\ -0.0714 \end{bmatrix} = \begin{bmatrix} -2.6761 \\ 2.7856 \\ -0.0714 \end{bmatrix}$$

Q5. Suppose $y_i = \beta_0 + \beta_1 x_i + \epsilon_i$, where for $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2, \cdots, \epsilon_n)'$, $E(\boldsymbol{\epsilon}) = \mathbf{0}$. A particular experiment produces $n = 5$ data points as per

x_i	20	39	28	37	85
y	305	363	124	140	89

Suppose that $V(\boldsymbol{\epsilon}) = \sigma^2 diag\left(\frac{1}{4}, \frac{1}{16}, \frac{1}{16}, \frac{1}{49}, \frac{1}{100}\right)$. Evaluate an appropriate BLUE of $\boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$ under the model assumptions.

Ans.

Let $\mathbf{V}^{-\frac{1}{2}} = diag(2, 4, 4, 7, 10)$.

Then $\mathbf{Z} = \mathbf{V}^{-\frac{1}{2}}\mathbf{y} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 10 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 2y_1 \\ 4y_2 \\ 4y_3 \\ 7y_4 \\ 10y_5 \end{bmatrix} = \begin{bmatrix} 610 \\ 1452 \\ 496 \\ 980 \\ 890 \end{bmatrix}$

$$\begin{aligned} E(\mathbf{Z}) &= \mathbf{V}^{-1/2}E(\mathbf{y}) \\ &= \mathbf{V}^{-1/2}\mathbf{X}\boldsymbol{\beta} \\ &= \mathbf{W}\boldsymbol{\beta} \end{aligned}$$

where

$$\mathbf{W} = \mathbf{V^{-1/2}X} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 10 \end{bmatrix} \begin{bmatrix} 1 & 20 \\ 1 & 39 \\ 1 & 28 \\ 1 & 37 \\ 1 & 85 \end{bmatrix} = \begin{bmatrix} 2 & 40 \\ 4 & 156 \\ 4 & 112 \\ 7 & 259 \\ 10 & 850 \end{bmatrix}$$

$$\begin{aligned} V(\mathbf{Z}) &= \mathbf{V^{-1/2}}^T Var(\mathbf{y})\mathbf{V^{-1/2}} \\ &= \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 10 \end{bmatrix} \sigma^2 \begin{bmatrix} \frac{1}{4} & & & & \\ & \frac{1}{16} & & & \\ & & \frac{1}{16} & & \\ & & & \frac{1}{49} & \\ & & & & \frac{1}{100} \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 10 \end{bmatrix} \\ &= \sigma^2\mathbf{I} \end{aligned}$$

Thus, $\mathbf{Z}$ follows a Gauss-Markov model with model matrix $\mathbf{W}$. So, the BLUE of $\boldsymbol{\beta}$ is

$$\hat{\boldsymbol{\beta}} = [(\mathbf{W^TW})^{-1}W^T\mathbf{Z}]$$

$$\left[\begin{bmatrix} 2 & 4 & 4 & 7 & 10 \\ 40 & 156 & 112 & 259 & 850 \end{bmatrix} \begin{bmatrix} 2 & 40 \\ 4 & 156 \\ 4 & 112 \\ 7 & 259 \\ 10 & 850 \end{bmatrix} \right]^{-1} \begin{bmatrix} 2 & 4 & 4 & 7 & 10 \\ 40 & 156 & 112 & 259 & 850 \end{bmatrix} \begin{bmatrix} 610 \\ 1452 \\ 496 \\ 980 \\ 890 \end{bmatrix}$$

$$= \begin{bmatrix} [[249.05879919]] \\ [[-1.85816641]] \end{bmatrix}$$