MEME15203 Statistical Inference

Assignment 1

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Faculty:	FES	Unit Code:	MEME15203
Course:	MAC	Unit Title:	Statistical Inference
Year:	1,2	Lecturer:	Dr Yong Chin Khian
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Q1. Let X_1, X_2 be two random variables with joint pdf $f(x_1, x_2) = \frac{1}{7!(70^9)} x_1^7 e^{-x_2/70}$, for $0 < x_1 < x_2 < \infty$, zero otherwise.

(a) Determine the joint mgf of X_1, X_2, $M_{X_1,X_2}(t_1, t_2)$.

(b) Determine the marginal distribution of X_1.

(c) Determine the marginal distribution of X_2.

(d) Are X_1 and X_2 independent?

(10 marks)

Q2. Suppose that the random variables X_1 and X_2 have joint probability density function $f(x_1, x_2)$ given by

$$f(x_1, x_2) = \begin{cases} \frac{30}{2} x_1^4 x_2, & 0 \le x_1 \le x_2, x_1 + x_2 \le 2 \\ 0, & \text{otherwise} \end{cases}.$$

(a) Show that the marginal density of X_1 is a beta density with $a = 5$ and $b = 2$.

(b) Derive the conditional density of X_2 given $X_1 = x_1$.

(c) Find $P(X_2 < 1.1 | X_1 = 0.6)$.

(d) Derive the marginal density of X_2.

(16 marks)

Q3. Show that if $X = (X_1, X_2, \ldots, X_k)$ have a multinomial distribution with parameters n and $p_1, p_2, \ldots, p_k$, then

(a) $E(X_i) = np_i$, $V(X_i) = np_iq_i$

(b) $Cov(X_s, X_t) = -np_sp_t$, if $s \neq t$

(10 marks)

Q4. Show that if $(X, Y) \sim BVN(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$, then conditional on $X = x$,

$$Y|x \sim N(\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x - \mu_1), \sigma_2^2(1 - \rho^2)).$$

(4 marks)

Q5. Suppose that X_1 and X_2 denote a random sample of sixe 2 from a gamma distribution $X_i \sim GAM(0.5, 5)$. Find the pdf of $\frac{X_1}{X_2}$.

(4 marks)

Q6. Suppose that $X_1, X_2, \ldots, X_{11}$ denote a random sample of size 11 from a gamma distribution $X_i \sim GAM(\alpha = \frac{1}{11}, \theta = 8)$. Find the pdf of $U = \sqrt[11]{X_1 + X_2 + \cdots + X_{11}}$ and state the name of the distribution of U.

(4 marks)

Q7. Let X_1 and X_2 be a random sample of size $n = 2$ from a continuous distribution with pdf of the form $f(x) = 3x^2$ if $0 < x < 1$ and zero otherwise.

(a) Find the joint pdf of $Y_1 = \min(X_1, X_2)$ and $Y_2 = \max(X_1, X_2)$.

(b) Find the pdf of the sample range $R = Y_2 - Y_1$.

(10 marks)

Q8. Let Y_9 denote the 9^{th} smallest item of a random sample of size n from a distribution of the continuous type that has cdf $F_X(x)$ and pdf $f_X(x) = F'_X(x)$. Find the limiting distribution of $W_n = nF_{Y_9}(y)$.

(4 marks)

Q9. Consider a random sample from a gamma distribution, $X_i \sim GAM(\alpha, \theta)$. Find the asymtotic normal distribution of $Y_n = \bar{X}_n^3$.

(4 marks)

Q10. Consider a random sample from a Gamma distribution with parameters α and θ. Let $W_i = e^{X_i}$ and $V_n = W_1 \times W_2 \times \cdots W_n$. $V_n^{\frac{1}{n}}$ converges in probability to a constant, identify the constant.

(4 marks)

Q11. Let $Y_n \sim GAM(7n, \theta)$. Find the limiting distribution of $Z_n = \frac{Y_n - 7n\theta}{\sqrt{7n}\theta}$ as $n \to \infty$, using moment generating function.

(4 marks)

Q12. Suppose that $X_i \sim N(\mu, \sigma^2), i = 1, \ldots, 14$, $Z_j \sim N(0, 1), j = 1, \ldots, 7$, and $W_k \sim \chi^2(10), k = 1, \ldots, 13$ and all random variables are independent. State the distribution of each of the following variables if it is a "named" distribution. [For example $X_1 + X_2 \sim N(2\mu, 2\sigma^2)$]

(a) $\frac{6\sum_{i=1}^{14}(X_i - \bar{X})^2}{13\sigma^2 \sum_{j=1}^{7}(Z_j - \bar{Z})^2}$.

(b) $\frac{6\sum_{k=1}^{7} W_k}{70\sum_{j=1}^{7}(Z_j - \bar{Z})^2}$

(c) $\frac{\sqrt{140}(\bar{X} - \mu)}{\sigma\sqrt{W_1}}$

(d) $\frac{W_1}{W_1+W_2+W_3+W_4}$

(e) $\frac{Z_i^2/W_1}{1+Z_1^2/W1}$

(f) $\frac{\frac{\sum_{k=1}^{7} W_k}{\sum_{j=1}^{7}(Z_j-\bar{Z})^2}}{1+\frac{\sum_{k=1}^{7} W_k}{\sum_{j=1}^{7}(Z_j-\bar{Z})^2}}$

(12 marks)

Q13. Suppose $Y \sim Beta(a = 4, b = 8)$, use the relationship between Beta distribution and F distribution, find $P[Y > 0.396]$.

(3 marks)

Q14. Suppose $Y \sim Beta(a = 6, b = 12)$, use the relationship between Beta distribution and F distribution, find 93^{th} percentile of Y.

(3 marks)

Q15. Recall that $Y \sim LOGN(\mu, \sigma^2)$ if $\ln Y \sim N(\mu, \sigma^2)$. Assume that $Y_i \sim LOGN(\mu_i, \sigma_i^2), i = 1, \ldots, n$ are independent.

(a) Find the distribution of $\prod_{i=1}^{n} Y_i$.

(b) Find the distribution of $\prod_{i=1}^{n} Y_i^a$.

(c) Find the dietribution of $\frac{Y_1}{Y_2}$.

(d) Find $E\left[\prod_{i=1}^{n} Y_i\right]$.

(8 marks)