MEME15203 Statistical Inference Marking Guide

Assignment 3

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Due by:			

Q1. Consider a random sample of size n from a single-parameter pareto distribution, $X_i \sim SP(\alpha = 2, \theta)$. Find the UMVUE of the p^{th} percentile.

(10 marks)

Ans.

$f(x_i) = \frac{2\theta^2}{x_i^3}, x_i > \theta$

$f(x,x_2,\ldots,x_n) = 2^n\theta^{2n}\prod_{i=1}^n x_i^{-3}, x_{(1)} > \theta$

$\quad = g(s,\theta)h(x_1,\ldots,x_n)$

where $g(s,\theta) = \theta^{2n}I(X_{(1)}) > \theta)$ and $h(x_1,\ldots,x_n) = 2^n\prod_{i=1}^n x_i^{-3}$,
By factorization Theorem, $S = X_{(1)}$ is a sufficient statistic for θ.

$f_S(s) = nf_X(s)[1-F_X(s)]^{(n-1)} = \frac{2n\theta^2}{s^3}\left(\frac{\theta}{s}\right)^{2(n-1)} = \frac{2n\theta^{2n}}{s^{2n+1}}, s > \theta$

$\Rightarrow S \sim SP(\alpha = 2n, \theta)$

$E[u(S)] = \int_\theta^\infty u(s)\frac{2n\theta^{2n}}{s^{2n+1}}ds = 0\ \forall\theta$

$\Rightarrow \int_\theta^\infty u(s)s^{-2n}ds = 0\ \forall\theta$

$\frac{d}{d\theta}\int_\theta^\infty u(s)s^{-2n}ds = u(\theta)\theta^{-2n} = 0\ \forall\theta$

This implies $u(\theta) = 0$ for all θ, so $S = X_{(1)}$ is complete.

$E(S) = \frac{2n\theta}{2n-1}$

$P(X > \pi_p) = 1-p$

$\left(\frac{\theta}{\pi_p}\right)^2 = 1-p$

$\pi_p = \frac{\theta}{\sqrt{1-p}}$

Let $S_1 = \frac{(2n-1)S}{2n\sqrt{1-p}}$, then

$E(S_1) = \frac{(2n-1)E(S)}{2n\sqrt{1-p}} = \frac{\theta}{\sqrt{1-p}}$

Since $S_1 = \frac{(2n-1)S}{2n\sqrt{1-p}}$ is a function of css for θ and unbisaed for π_p. Then, $S_1 = \frac{(2n-1)S}{2n\sqrt{1-p}}$ is the UMVUE of the p^{th} percentile.

Q2. Suppose that $X_1, \ldots, X_n$ is a random sample from a Negative Binomial distribution, $X_i \sim \text{NB}(\text{r} = 6,\, \theta)$,

(a) Show that the p.d.f. of X belongs to the regular exponential family.

(b) Based on the answer in 1, find a complete and sufficient statistic for θ.

(c) Find the UMVUE of $\left[\frac{4\theta}{1-4(1-\theta)}\right]^{6n}$.

(15 marks)

Ans.

(a) $f(x) = \binom{x-1}{r-1}\theta^6(1-\theta)^{x-6} = \binom{x-1}{r-1}\left(\frac{\theta}{1-\theta}\right)^6 e^{x\ln(1-\theta)} = c(\theta)h(x)e^{q(\theta)t(x)}$
where $c(\theta) = \left(\frac{\theta}{1-\theta}\right)^6$, $h(x) = \binom{x-1}{r-1}$, $q(\theta) = \ln(1-\theta)$, and $t(x) = x$.
Thus the p.d.f. of X belongs to the regular exponential family.

(b) Since the p.d.f. of X belongs to the regular exponential family, thus by the theorem, $S = \sum_{i=1}^{n} X_i$ is a c.s.s of θ

(c) Let $E(t^S) = \left[\frac{4\theta}{1-4(1-\theta)}\right]^{6n}$
As $S \sim NB(6n, \theta)$,
$E(t^S) = E(e^{S\ln t}) = \left(\frac{t\theta}{1-t(1-\theta)}\right)^{6n}$
$\Rightarrow t = 4$
Since 4^S is a function of the c.s.s. of θ
which is an UE of $\left(\frac{4\theta}{1-4(1-\theta)}\right)^{6n}$,
thus 4^S is the UMVUE of $\left(\frac{4\theta}{1-4(1-\theta)}\right)^{6n}$.

Q3. Consider a random sample of size n from a gamma distribution $X_i \sim GAM(\alpha, \theta)$ and let $\bar{X} = (1/n)\sum X_i$ and $\tilde{X} = (\prod X_i)^{1/n}$ be the sample mean and geometric mean respectively.

(a) Show that $\bar{X}$ and $\tilde{X}$ are jointly complete and sufficient for θ and α.

(b) Find the UMVUE of $\mu = \alpha\theta$.

(c) Find the UMVUE of μ^n..

(d) Show that $\bar{X}$ and T are stochastically independent random variables.

(20 marks)

Ans.

(a) $$\begin{aligned} f(x_i) &= \frac{1}{\Gamma(\alpha)\theta^\alpha} x_i^{\alpha-1} e^{-x_i/\theta} \\ &= \frac{1}{\Gamma(\alpha)\theta^\alpha} x_i^{\alpha} e^{\alpha\ln x_i} e^{-x_i/\theta} \end{aligned}$$

$= \frac{1}{\Gamma(\alpha)\theta^\alpha} x_i^{-1} e^{\alpha \ln x_i - x_i/\theta}$

Thus $X_i \sim GAM(\alpha, \theta)$ belongs to REC familly where $c(\theta) = \frac{1}{\Gamma(\alpha)\theta^\alpha}$, $h(x) = x_i^{-1}$, $t_1(X_i) = \ln X_1$, $t_2(X_i) = X_1$, $q_1(\theta) = \alpha$ and $q_2(\theta) = \theta$.

Thus, $\sum \ln X_i$ and $\sum X_i$ are jointly complete and sufficient of α and θ. Since $\bar{X}$ is a function of $\sum X_i$ and $\tilde{X}$ is a function of $\sum \ln X_i$, thus $\bar{X}$ and $\tilde{X}$ are jointly complete and sufficient for θ and α.

(b) $\sum X_i \sim GAM(n\alpha, \theta)$
$2\sum X_i/\theta \sim \chi^2(2n\alpha)$
$E(2n\bar{X}/\theta) = 2n\alpha$
$E(\bar{X}) = \alpha\theta$
Since $\bar{X}$ is a CSS of θ and α, thus, $\bar{X}$ is the UMVUE of $\mu = \alpha\theta$.

(c) $E(\prod X_i) = (EX_i)^n = \mu^n$
Thus, $\prod X_i$ is an UE of μ.
Since $\prod X_i = \tilde{X}^n$, then $\prod X_i$ is the UMVUE of μ^n.

(d)
$$
\begin{aligned}
T &= u(X_1, X_2, \ldots, X_n) \\
&= u(cX_1, cX_2, \ldots, cX_n) \\
&= \frac{(cX_1 + cX_2 + \cdots, + cX_n)/n}{(cX_1 \cdot cX_2 \cdots cX_n)^{1/n}} \\
&= \frac{c(X_1 + X_2 + \cdots, + X_n)/n}{c(X_1 \cdot X_2 \cdots X_n)^{1/n}} \\
&= \frac{\bar{X}}{\tilde{X}}
\end{aligned}
$$
Since $\bar{X}$ is a CSS of θ and the distribution of T does not depend on θ. thus by BASU theorem, $\bar{X}$ and T are stochastically independent random variables.

Q4. %%%%% Suppose that $X_1, \ldots, X_n$ is a random sample from a Poisson distribution, $X_i \sim \mathrm{POI}(\lambda)$. Find the UMVUE of $P(X_1 + X_2 = 0 \text{ or } 1) = (1 + 2\lambda)e^{-2\lambda}$ using Rao-Blackwell theorem.

(10 marks)

Ans.

$f(x) = \frac{1}{lambda} e^{-x/\lambda} = c(\lambda)h(x)e^{t(x)q(\lambda)}$ which is in a member of REC. Hence $S = \sum X_i$ is a CSS of λ.
Let
$$
T = \begin{cases} 1, & X_1 + X_2 = 0 \text{ or } 1 \\ 0, & \text{otherwise} \end{cases}.
$$

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$E(T) = P(X_1 + X_2 = 0 \text{ or } 1) = e^{-2\lambda} + 2\lambda e^{-2\lambda} = (1+2\lambda)e^{-2\lambda}$. Thus T is and unbiased estimator of $(1+2\lambda)e^{-2\lambda}$. Since S is CSS of λ. Hence by Rao-Blackwell theorem, $T^* = E(T|S)$ is an UMVUE of $(1+2\lambda)e^{-2\lambda}$.

$E\left[T|\sum_{i=0}^{n} X_i = s\right]$

$= 1 \cdot P[X_1 + X_2 = 0 \text{ or } 1|X_1 + X_2 + \cdots X_n = s]$

$= \frac{P(X_1+X_2=0, X_3+\cdots+X_n=s)}{P(X_1+\cdots+X_n=s)} + \frac{P(X_1+X_2=1, X_3+\cdots+X_n=s-1)}{P(X_1+\cdots+X_n=s)}$

$= \frac{P(X_1+X_2=0)\times P(X_3+\cdots+X_n=s)}{P(X_1+\cdots+X_n=s)} + \frac{P(X_1+X_2=1)\times P(X_3+\cdots+X_n=s-1)}{P(X_1+\cdots+X_n=s)}$ Since $X_1, X_2, \ldots, X_n$ are independent.

$= \frac{e^{-2\lambda}[(n-2)\lambda]^s e^{-(n-2)\lambda}}{(n\lambda)^s s! e^{-n\lambda}/s!} + \frac{2\lambda e^{-2\lambda}[(n-2)\lambda]^{s-1}e^{-(n-2)\lambda}}{(n\lambda)^s (s-1)! e^{-n\lambda}/s!}$

$= \left(\frac{n-2}{n}\right)^s + \left(\frac{n-2}{n}\right)^s \left(\frac{2s}{n-2}\right)$

Q5. Consider a random sample of size n from a distribution with pdf

$$f(x;\theta) = \frac{(\ln\theta)^x}{\theta x!}, x = 1, 1, \ldots; \theta > 1, \text{ zero, otherwise.}$$

(a)Find a complete sufficient statistic for θ.

(b)Find the UMVUE of $\ln\theta$.

(c)Find the UMVUE of $(\ln\theta)^2$.

(15 marks)

Ans.

(a)Let $\mu = \ln\theta$

$f(x;\theta) = \frac{\mu^x}{e^\mu x!} = e^{-\mu}\frac{1}{x!}e^{x\ln(\mu)} = c(\mu)h(x)e^{q(\mu)t(x)}$

where $c(\mu) = e^{-\mu}$, $h(x) = \frac{1}{x!}$, $q(\mu) = \ln(\mu)$; $t(x) = x$. Thus, $f(x;\theta)$ belong to $REC(\theta)$ and hence $S = \sum X_i$ is a complete sufficient statistic for $\mu = \ln\theta$.

(b)$X \sim POI(\mu = \ln\theta)$, $E(X) = \ln\theta$ and $E\left(\frac{S}{n}\right) = E(\bar{X}) = \ln\theta$. Since $\bar{X}$ is a function of CSS and unbiased for $\ln\theta$, therefore, by Lehmann Scheffe theorem, $\bar{X}$ is the UMVUE of $\ln\theta$.

(c)$E\left[\bar{X}^2 - \frac{\bar{X}}{n}\right] = \frac{\mu}{n} + \mu^2 - \frac{\mu}{n} = \mu^2$. Since $\bar{X}^2 - \frac{\bar{X}}{n}$ is a function of CSS and unbiased for $\mu^2 = (\ln\theta)^2$, therefore, by Lehmann Scheffe theorem, $\bar{X}^2 - \frac{\bar{X}}{n}$ is the UMVUE of $(\ln\theta)^2$.

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Q6. Show that $X \sim N(0,\theta)$ is not a complete family.

(10 marks)

Ans.

$X \sim N(0,\theta)$
Let $U(X) = X$, Then $E[U(S)] = E(X) = 0$, but $X \neq 0$. Thus, $X \sim N(0,\theta)$ is not a complete family.

Q7. Let $X_1, X_2, \ldots, X_n$ be random sample of size n from a Gamma distribution with probability density function

$$\frac{1}{\theta^2} x e^{-x/\theta}, x > 0$$

zero otherwise. Find the UMVUE of $\gamma = P(X > t)$ using Rao-Blackwell theorem.

(20 marks)

Ans.

Let

$$T = \begin{cases} 1, & X_1 > t \\ 0, & \text{otherwise} \end{cases}.$$

Then, $E(T) = P(X > t)$. Thus T is and unbiased estimator of γ.
$f(x) = \frac{1}{\theta^2} e^{-x/\theta} = c(\theta)h(x)e^{q(\theta)t(x)}$, where $c(\theta) = \frac{1}{\theta^2}$, $h(x) = 1$, $q(\theta) = \frac{1}{\theta}$, and $t(x) = x$, hence $f(x)$ is a member of $REC(\theta)$ and $S = \sum_{i=1}^{n} X_i$ is a complete sufficient statistics for θ.
Thus, by Rao-Blackwell therem, $T^* = E(T|S)$ is an UMVUE of γ.

$$\begin{aligned} f_{X_1,S}(x_1, s) &= f_{X_1,S_1}(x_1, s - x_1) \text{ where } S_1 = X_2 + \cdots, X_n \sim gamma(2n-2,\theta) \\ &= f_{X_1}(x_1) f_{S_1}(s - x_1) \\ &= \tfrac{1}{\theta^2} x_1 e^{-x_1/\theta} \tfrac{1}{\Gamma(2n-2)\theta^{2n-2}} (s - x_1)^{2n-3} e^{-(s-x_1)/\theta} \\ &= \tfrac{1}{\Gamma(2n-2)\theta^{2n}} x_1 (s - x_1)^{2n-3} e^{-s/\theta} \end{aligned}$$

$f_{X_1|S}(x_1) = kx_1(s - x_1)^{2n-3}, 0 < x_1 < s$
Let $z = s - x_1$, $dz = -dx$
$\int_s^0 k(s - z)z^{2n-3} dx_1 = 1$
$\int_s^0 k(sz^{2n-3} - z^{2n-2}) dz = 1$
$k\left[\frac{sz^{2n-2}}{2n-2} - \frac{z^{2n-1}}{2n-1}\right]_0^s = 1$
$k\left[\frac{s^{2n-1}}{2n-2} - \frac{s^{2n-1}}{2n-1}\right] = 1$
$k\left[\frac{(2n-1)s^{2n-1} - (2n-2)s^{2n-1}}{(2n-1)(2n-2)}\right] = 1$
$k\left[\frac{s^{2n-1}}{(2n-1)(2n-2)}\right] = 1$
$k = (2n-2)(2n-1)s^{1-2n}$
$\therefore f_{X_1|S}(x_1) = (2n-2)(2n-1)s^{1-2n} x_1 (s - x_1)^{2n-3}, 0 < x_1 < s$

$$
\begin{aligned}
E[T|S] &= P[X_1 > t|s] \\
&= \int_t^s (2n-2)(2n-1)s^{1-2n} x_1 (s-x_1)^{2n-3} dx_1
\end{aligned}
$$

Let $z = s - x_1$, $dz = -dx$

$$
\begin{aligned}
&= (2n-2)(2n-1)s^{1-2n} \int_{s-t} t^0 (s-z) z^{2n-3}(-dz) \\
&= (2n-2)(2n-1)s^{1-2n} \left[\frac{sz^{2n-2}}{2n-2} - \frac{z^{2n-1}}{2n-1}\right]_0^{s-t} \\
&= (2n-2)(2n-1)s^{1-2n} \left[\frac{s(s-t)^{2n-2}}{2n-2} - \frac{(s-t)^{2n-1}}{2n-1}\right] \\
&= (2n-2)(2n-1) \left[\frac{\left(\frac{s-t}{s}\right)^{2n-2}}{2n-2} - \frac{\left(\frac{s-t}{s}\right)^{2n-1}}{2n-1}\right] \\
&= (2n-2)(2n-1) \left[\frac{(2n-1)\left(\frac{s-t}{s}\right)^{2n-2} - (2n-2)\left(\frac{s-t}{s}\right)^{2n-1}}{(2n-2)(2n-1)}\right] \\
&= (2n-1)\left(\frac{s-t}{s}\right)^{2n-2} - (2n-2)\left(\frac{s-t}{s}\right)^{2n-1}
\end{aligned}
$$