MEME15203 Statistical Inference Marking Guide

Assignment 2

UNIVERSITI TUNKU ABDUL RAHMAN

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Q1. Consider a random sample from a Exponential distribution, $X_i \sim Exp(\theta)$. Find the asymtotic normal distribution of $Y_n = \bar{X}_n^4$.

(10 marks)

Ans.

$E(\bar{X}_n) = \theta$, $V(\bar{X}_n) = \frac{1}{n}V(X) = \frac{\theta^2}{n}$

By CLT, $\bar{X}_n \sim N\left(\theta, \frac{\theta^2}{n}\right)$

$g(\theta) = \theta^4$, $g'(\theta) = 4\theta^3$, $[g'(\theta)]^2 = 16\theta^6$, thus, by Theorem 11,

$\frac{c^2[g'(m)]^2}{n} = \frac{\theta^2(16\theta^6)}{n} = \frac{16\theta^8}{n}$

$Y_n \sim N\left(\theta^4, \frac{16\theta^8}{n}\right)$

Q2. Suppose that $X_i \sim N(\mu, \sigma^2), i = 1, \ldots, 13$, $Z_j \sim N(0,1), j = 1, \ldots, 5$, and $W_k \sim \chi^2(v), k = 1, \ldots, 12$ and all random variables are independent. State the distribution of each of the following variables if it is a "named" distribution. [For example $X_1 + X_2 \sim N(2\mu, 2\sigma^2)$]

(a) $\frac{Z_i^2/W_1}{1+Z_1^2/W1}$

(b) $\frac{\frac{\sum_{k=1}^{5} W_k}{\sum_{j=1}^{5}(Z_j-\bar{Z})^2}}{1+\frac{\sum_{k=1}^{5} W_k}{\sum_{j=1}^{5}(Z_j-\bar{Z})^2}}$

(c) $\frac{W_1}{W_1+W_2+W_3+W_4}$

(10 marks)

Ans.

(a) $Z_1^2 \sim \chi^2(1)$

$\frac{Z_1^2}{W_1/v} \sim F(1, v)$

$\frac{(\frac{1}{v})\frac{Z_1^2}{W_1/v}}{1+(\frac{1}{v})\frac{Z_1^2}{W_1/v}} = \frac{Z_i^2/W_1}{1+Z_1^2/W1} \sim Beta(1/2, v/2)$

(b) $\sum_{k=1}^{5} W_k \sim \chi^2(5v)$

$\sum_{j=1}^{5}(z_j - \bar{Z})^2 \sim \chi^2(4)$

$\frac{\sum_{k=1}^{5} W_k/5v}{\sum_{j=1}^{5}(z_j-\bar{Z})^2/4} \sim F(5v, 4)$

Thus, $\frac{4\sum_{k=1}^{5} W_k}{5v\sum_{j=1}^{5}(z_j-\bar{Z})^2} \sim F(5v, 4)$

$\frac{(\frac{5v}{4})\frac{4\sum_{k=1}^{5} W_k}{5v\sum_{j=1}^{5}(Z_j-\bar{Z})^2}}{1+(\frac{5v}{4})\frac{4\sum_{k=1}^{5} W_k}{5v\sum_{j=1}^{5}(Z_j-\bar{Z})^2}} = \frac{\frac{\sum_{k=1}^{5} W_k}{\sum_{j=1}^{5}(Z_j-\bar{Z})^2}}{1+\frac{\sum_{k=1}^{5} W_k}{\sum_{j=1}^{5}(Z_j-\bar{Z})^2}} \sim Beta(5v/2, 2)$

(c) $\frac{W_1}{W_1+W_2+W3+W_4} = \frac{W_1}{W_1+(W_2+W3+W_4)}$

$W_1 \sim \chi^2(v) \sim GAM(v/2, 2)$, $W_2 + W_3 + W_4 \sim \chi(3v) \sim GAM(3v/2, 2)$

Thus, $\frac{W_1}{W_1+W_2+W3+W_4} \sim Beta(v/2, 3v/2)$

Q3. Suppose that $X_i \sim N(\mu, \sigma^2), i = 1, \dots, 11$, $Z_j \sim N(0, 1), j = 1, \dots, 9$, and $W_k \sim \chi^2(v), k = 1, \dots, 10$ and all random variables are independent. State the distribution of each of the following variables if it is a "named" distribution. [For example $X_1 + X_2 \sim N(2\mu, 2\sigma^2)$]

(a) $\frac{8\sum_{i=1}^{11}(X_i-\bar{X})^2}{10\sigma^2\sum_{j=1}^{9}(Z_j-\bar{Z})^2}$.

(b) $\frac{W_1}{\sum_{k=1}^{9} W_k}$

(c) $\frac{\bar{X}}{\sigma^2} + \frac{\sum_{i=1}^{11} Z_i}{11}$

(10 marks)

Ans.

(a) $\frac{\sum_{i=1}^{11}(X_i-\bar{X})^2}{\sigma^2} \sim \chi^2(10)$

$\sum_{j=1}^{9}(Z_j - \bar{Z})^2 \sim \chi^2(8)$

$\frac{\frac{\sum_{i=1}^{11}(X_i-\bar{X})^2}{10\sigma^2}}{\sum_{j=1}^{9}(Z_j-\bar{Z})^2/8} \sim F(10, 8)$

Thus, $\boxed{\frac{8\sum_{i=1}^{11}(X_i-\bar{X})^2}{10\sigma^2\sum_{j=1}^{9}(Z_j-\bar{Z})^2} \sim F(10, 8)}$.

(b) Let $\frac{W_1}{\sum_{k=1}^{9} W_k} = \frac{W_1}{W_1+\sum_{k=2}^{9} W_k}$, then

$W_1 \sim GAM(\frac{v}{2}, 2)$ and $\sum_{k=2}^{9} W_k \sim GAM(\frac{8v}{2}, 2)$.

Thus, $\boxed{\frac{W_1}{\sum_{k=1}^{9} W_k} \sim BETA(\frac{v}{2}, \frac{8v}{2})}$.

(c) $\bar{X} \sim N(\mu, \frac{\sigma^2}{11})$, then $\frac{\bar{X}}{\sigma^2} \sim N(\frac{\mu}{\sigma^2}, \frac{1}{11\sigma^2})$

$\sum_{i=1}^{11} Z_i \sim N(0, 11)$, and $\frac{\sum_{i=1}^{11} Z_i}{11} \sim N(0, \frac{1}{11})$.

Thus $\boxed{\frac{\bar{X}}{\sigma^2} + \frac{\sum_{i=1}^{11} Z_i}{11} \sim N(\frac{\mu}{\sigma^2}, \frac{1}{11\sigma^2} + \frac{1}{11})}$.

Q4. Let $X_1, X_2, \ldots, X_n$ denote a random sample from the density function given by

$$f(x) = \begin{cases} \frac{5}{\theta} x^4 e^{-x^5/\theta}, & \theta > 0, x > 0, \\ 0, & \text{otherwise} \end{cases}$$

(a) find the MME of θ.

(b) Find the MLE of θ.

(c) Find the CRLB of θ.

(15 marks)

Ans.

(a)
$X \sim WEI(\tau = 5, \beta = \theta^{1/5})$
$E(X) = \theta^{1/5}\Gamma(1 + 1/5) = \theta^{1/5}\Gamma(6/5)$
$\tilde{\theta}^{1/5}\Gamma(6/5) = \bar{X}$
$\tilde{\theta} = \left(\frac{\bar{X}}{\Gamma(6/5)}\right)^5$

(b)
$\ln L = n \ln 5 - n \ln \theta + 4 \sum_{i=1}^n \ln x_i - \sum_{i=1}^n \frac{x_i^5}{\theta}$
$\frac{dL}{d\theta} = \frac{-n}{\theta} + \frac{\sum_{i=1}^n x_i^5}{\theta^2} = 0$
$\hat{\theta} = \frac{\sum_{i=1}^n x_i^5}{n}$
let $u = \sum_{i=1}^n x_i^5$, then
$\frac{dL^2}{d\theta^2} = \frac{n}{\theta^2} - \frac{2u}{\theta^3}$
$\frac{dL^2}{d\theta^2}\Big|_{\theta=\hat{\theta}} = \frac{n}{(u/n)^2} - \frac{2u}{(u/n)^3} = \frac{n^3}{u^2} - \frac{2n^3}{u^2} = -\frac{n^3}{u^2} < 0$
The the MLE of θ is $\hat{\theta} = \frac{\sum_{i=1}^n x_i^5}{n}$

(c)
Let $u = x^5$, $w(u) = x = u^{1/5}$, $w'(u) = \frac{1}{5}u^{1/5-1}$
$f_U(u) = \frac{1}{\theta}(5)(u^{1/5})^4 e^{-(u^{1/5})^5/\theta}(\frac{1}{5}u^{1/5-1}) = \frac{1}{\theta}u^{-u/\theta}$
$\Rightarrow U \sim EXP(\theta)$
$\tau(\theta) = \theta$
$\ln f(x;\theta) = -\ln \theta + \ln 5 + 4 \ln x - x^5/\theta$
$\frac{\partial \ln f(x;\theta)}{\partial \theta} = -\frac{1}{\theta} + \frac{x^5}{\theta^2}$
$\frac{\partial^2 \ln f(x;\theta)}{\partial \theta^2} = \frac{1}{\theta^2} - \frac{2x^5}{\theta^3}$
$E\left(\frac{\partial^2 \ln f(x;\theta)}{\partial \theta^2}\right) = \frac{1}{\theta^2} - \frac{2E(x^5)}{\theta^3} = \frac{1}{\theta^2} - \frac{2\theta}{\theta^3} = -\frac{1}{\theta^2}$

$CRLB = \frac{\tau'(\theta)}{-nE\left(\frac{\partial^2 \ln f(x;\theta)}{\partial \theta^2}\right)} = \frac{1}{-n(-1/\theta^2)} = \frac{\theta^2}{n}$

Q5. Let $X_1, X_2, \ldots, X_n$ denote a random sample from an exponentially distributed population with mean $\lambda = \frac{1}{\theta}$. Let $\Theta \sim \chi^2(2v)$.

(a) Find the Bayes estimator for $\lambda = \frac{1}{\theta}$ under square error loss.

(b) Show that it is a biased but consistent estimator for $\lambda = \frac{1}{\theta}$.

(20 marks)

Ans.

(a) $f(x_i|\theta) = \theta e^{-\theta x_i}$
$\Theta \sim \chi^2(2v) = GAM(\alpha = v, \beta = 2)$
$\pi(\theta) = \frac{1}{\Gamma(v)2^v}\theta^{v-1}e^{-\theta/2}, \theta > 0$
$\pi(\theta|\mathbf{x}) = k\theta^{v+n-1}e^{-\theta(\sum x_i+1/2)}, \theta > 0$
$\therefore \Theta|\mathbf{x} \sim GAM(v+n, (\sum x_i + 1/2)^{-1})$

$$\begin{aligned}\hat{\lambda} = E(\Theta^{-1}) &= \int_0^\infty \theta^{-1}\frac{(\sum x_i + 1/2)^{v+n}}{\Gamma(v+n)}\theta^{v+n-1}e^{-\theta(\sum x_i+1/2)}d\theta \\ &= \tfrac{(\sum x_i+1/2)^{v+n}}{\Gamma(v+n)}\int_0^\infty \theta^{v+n-2}e^{-\theta(\sum x_i+1/2)}d\theta \\ &= \tfrac{(\sum x_i+1/2)^{v+n}}{\Gamma(v+n)}\tfrac{\Gamma(v+n-1)}{(\sum x_i+1/2)^{v+n-1}} \\ &= \tfrac{\sum x_i+1/2}{v+n-1} \\ &= \tfrac{\sum x_i}{v+n-1} + \tfrac{1}{2(v+n-1)}\end{aligned}$$

(b) $E(\hat{\lambda}) = \frac{\sum E(X_i)}{v+n-1} + \frac{1}{2(v+n-1)} = \frac{n(1/\theta)}{v+n-1} + \frac{1}{2(v+n-1)} \neq 1/\theta$. thus $\hat{\lambda}$ is a biased estimator of $\lambda = \frac{1}{\theta}$.
$\lim_{n\to\infty} E(\hat{\lambda}) = 1/\theta$, Thus $\hat{\lambda}$ is asymtotically unbiased.
$V(\hat{\lambda}) = \frac{\sum V(X_i)}{(v+n-1)^2} = \frac{n(1/\theta^2)}{(v+n-1)^2}$
$\lim_{n\to\infty} V(\hat{\lambda}) = 0$. Thus $\hat{lambda}$ is MSE consistent and hence consistent.

Q6. A pmf on the integers with an integer parameter, θ , is

$$f(x|\theta) = \frac{1}{2}I[x = \theta - 5 \text{ or } x = \theta + 5].$$

For X_1 and X_2 iid from this distribution, compare MSE's for the two estimators of θ,

$$\hat{\theta} = \begin{cases} X_1 + 5, & \text{if } X_1 = X_2 \\ \bar{X}, & \text{if } X_1 \neq X_2 \end{cases} \text{ or } \tilde{\theta} = \hat{\theta} - \frac{5}{2}.$$

(15 marks)

Ans.

$$\begin{aligned}E(\hat{\theta}) &= (x_1+5)P(X_1 = X_2) + \bar{x}P(X_1 \neq X_2) \\ &= (\theta - 5 + 5)P[X_1 = \theta - 5, X_2 = \theta - 5] + (\theta + 5 + 5)P[X_1 = \theta + 5, X_2 = \theta + 5] \\ &+ (\tfrac{\theta-5+\theta+5}{2})P[X_1 = \theta - 5, X_2 = \theta + 5] + (\tfrac{\theta+5+\theta-5}{2})P[X_1 = \theta + 5, X_2 = \theta - 5] \\ &= \theta\left(\tfrac{1}{4}\right) + (\theta + 2(5)\left(\tfrac{1}{4}\right) + \theta\left(\tfrac{1}{4}\right) + \theta\left(\tfrac{1}{4}\right) \\ &= \theta + \tfrac{5}{2}\end{aligned}$$

$Bias(\hat{\theta}) = E(\hat{\theta}) - \theta = \frac{5}{2}$

$$
\begin{aligned}
E(\hat{\theta})^2 &= (x_1+5)^2P(X_1=X_2) + \bar{x}^2P(X_1 \neq X_2)\\
&= (\theta-5+5)^2P[X_1=\theta-5, X_2=\theta-5] + (\theta+5+5)^2P[X_1=\theta+5, X_2=\theta+5]\\
&+ (\tfrac{\theta-5+\theta+5}{2})^2P[X_1=\theta-5, X_2=\theta+5] + (\tfrac{\theta+5+\theta-5}{2})^2P[X_1=\theta+5, X_2=\theta-5]\\
&= \theta^2\left(\tfrac{1}{4}\right) + (\theta+2(5))^2\left(\tfrac{1}{4}\right) + \theta^2\left(\tfrac{1}{4}\right) + \theta^2\left(\tfrac{1}{4}\right)\\
&= \theta^2 + 5\theta + 25
\end{aligned}
$$

$V(\hat{\theta}) = E(\hat{\theta})^2 - E^2(\hat{\theta}) = \theta^2+5\theta+25-(\theta+\frac{5}{2})^2 = \theta^2+5\theta+25-(\theta^2+5\theta+\frac{25}{4})$
$= 18.75$
$MSE(\hat{\theta}) = V(\hat{\theta}) + Bias^2(\hat{\theta}) = \frac{3}{4}(25) + (\frac{5}{2})^2 = 25.0$
$E(\tilde{\theta}) = E(\hat{\theta}-\frac{5}{2}) = \theta$, thus, $\tilde{\theta}$ is an unbiased estimator of θ.
$V((\tilde{\theta}) = V(\hat{\theta}-\frac{5}{2}) = V(\hat{\theta}) = 18.75$
$MSE(\tilde{\theta}) = V((\tilde{\theta}) = 18.75$
Thus, $MSE(\tilde{\theta}) < MSE(\hat{\theta})$

Q7. Consider a random sample of size n from a distribution with discrete pdf $f(x : p) = p(1-p)^x; x = 0, 1, \ldots,$ zero otherwise.

(a) Find the MLE of p.

(b) Find the MLE of $\theta = \frac{1-p}{p}$.

(c) Find the CRLB for variance of unbiased estimators of θ.

(d) Is MLE of θ a UMVUE?

(e) Is MLE of θ MSE consistent?

(f) Find the asymptotic distribution of the MLE of θ.

(20 marks)

Ans.

(a) $L(p) = p^n(1-p)^{\sum x_i}$
$l(p) = n\ln(p) + \sum x_i \ln(1-p)$
$l'(p) = \frac{n}{p} - \frac{\sum x_i}{1-p} = 0$
$\frac{n}{\hat{p}} = \frac{\sum x_i}{1-\hat{p}}$
$n - n\hat{p} = \sum x_i\hat{p}$
$\hat{p} = \frac{n}{n+\sum x_i} = \frac{n}{n+n\bar{x}} = \frac{1}{1+\bar{x}}$

(b) By invariance property, $\hat{\theta} = \frac{1-\frac{1}{1+\bar{x}}}{\frac{1}{1+\bar{x}}} = \bar{x}$

(c) $\ln(f(x;p)) = \ln(p) + x\ln(1-p)$
$\frac{\partial \ln(f(x;p))}{\partial p} = \frac{1}{p} - \frac{x}{1-p}$
$\frac{\partial^2 \ln(f(x;p))}{\partial p^2} = -\frac{1}{p^2} + \frac{x}{(1-p)^2}$
Let $Y = X+1$, the $f_Y(y) = f_X(y-1) = p(1-P)^{y-1}, y = 1, 2, \ldots$

Thus $Y \sim Geo(p)$ and $E(Y) = \frac{1}{p}$ and $V(Y) = \frac{1-p}{p^2}$, so
$E(X) = E(Y) - 1 = \frac{1}{p} - 1 = \frac{1-p}{p}$ and $V(X) = V(Y) = \frac{1-p}{p^2}$
$E\left[\frac{\partial^2 \ln(f(x;p))}{\partial p^2}\right] = -\frac{1}{p^2} + \frac{E(X)}{(1-p)^2} = -\frac{1}{p^2} + \frac{\frac{1-p}{p}}{(1-p)^2} = -\frac{1}{p^2} + \frac{1}{p(1-p)} = \frac{-1}{p^2(1-p)}$
$\tau(p) = \frac{1-p}{p}$, $\tau'(p) = -\frac{1}{p^2}$.
The CRLB for $\theta = \frac{1-p}{p}$ is $\frac{[\tau'(p)]^2}{-nE\left[\frac{\partial^2 \ln(f(x;p))}{\partial p^2}\right]} = \frac{1/p^4}{-n\frac{-1}{p^2(1-p)}} = \frac{1-p}{np^2}$

(d) $V(\bar{X}) = \frac{V(X)}{n} = \frac{1-p}{np^2}$. Since $V(\bar{X})$ attined the CRLB for θ, thus $\hat{\theta} = \bar{X}$ is the UMVUE of θ.

(e) $\lim_{n\to\infty} V(\bar{X}) = \lim_{n\to\infty} \frac{1-p}{np^2} = 0$, Thus $\hat{\theta}$ is MSE consistent.

(f) $\bar{X} \sim N\left(\frac{1-p}{p}, \frac{1-p}{np^2}\right)$