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1 Measurement of Interest

1.1 Accumulation and Amount Functions

Consider an investment of an amount of money at interest.

- **Principal**: amount initially invested
- **Accumulated value, AV**: amount after a period of time
- **Amount of interest**: difference between the accumulated value and the principal
- **Measurement period**: a unit of time, in most cases a year, sometimes a month, a day, etc

It would be convenient to have functions representing the amount in the fund at time t.

Definition 1.

- **Amount function** $A(t)$: accumulated value at time t of an original investment

- **Accumulation function** $a(t)$: amount function for an original investment of 1
- $A(t) = k \cdot a(t)$, where k is the amount of the original investment.

These definitions assume that the fund is only growing through interest, ie., no principal is added. or withdrawn.

Properties of $a(t)$:

1. $a(0) = 1$.
2. $a(t)$ is increasing w.r.t t in general (decreasing happens when losing money).
3. If interest rate is discontinuous, $a(t)$ is discontinuous.

Example 1.

Amy makes deposit of 1000 at time 0. The fund grows with an accumulation function, $a(t) = e^{\frac{t^3}{300}}$, determine

(a) The accumulated value at time $t = 3$. $\boxed{109.42}$

(b) The accumulated value at time $t = 6$. $\boxed{205.44}$

(c) The amount of interest earned from time 3 to time 6. $\boxed{96.0259}$

1.2 Effective Rate of Interest

Suppose we want to measure the rate of growth of a fund in t^{th} year. (Note that t^{th} year runs from time $(t-1)$ to time t, just as the 1^{st} year runs from time 0 to time 1.)

The **amount** of growth in the t^{th} year (i.e., the interest earned is

- $a(t) - a(t-1)$ for original investment 1
- $A(t) - A(t-1)$ for original investment of k

The **rate of growth**, i_t is

$$\boxed{i_t = \frac{A(t) - A(t-1)}{A(t-1)} = \frac{a(t) - a(t-1)}{a(t-1)}}$$

This rate of growth is called the **effective rate of interest**.

Use the formula of i_t to solve $A(t)$ we get

$$\begin{aligned} A(t) &= A(t-1) + i_t[A(t-1)] \\ &= (1 + i_t)A(t-1) \end{aligned}$$

This can be interpreted as the fund at the end of the t^{th} year is equal to the fund at the beginning of the year plus the interest earn during the year.

Example 2.
An investment of \$10,000 is made into a fund at time $t = 0$. The fund develops the following balances over the next four years:

t	$A(t)$
0	10,000.00
1	10,600.00
2	11,130.00
3	11,575.20
4	12,153.96

Find the effective rate of interest for each of the four years.

1.3 The Accumulation Function in Terms of i_t

Suppose we are given that the effective rate of interest is 3% in the first year and 5% in the second year. We invest \$1 at time 0.

1. How much is the fund at the end one year? Obviously, \$1.03
2. How much is the fund at the end two years? We start with \$1.03 at the beginning of 2^{nd} year and we earn interest on it at 5%, so:
$$a(2) = 1.03 + 0.05(1.03) = (1.03)(1.05).$$

In symbol,
$$a(2) = (1 + i_1)(1 + i_2)$$

If we continue this process for t years, we have
$$a(t) = (1 + i_1)(1 + i_2) \cdots (1 + i_t) = \prod_{j=1}^{t}(1 + i_j)$$

Example 3.

Simon deposits 18,000 in a bank. During the first year, the bank credits an annual effective rate of interest i. During the second year, the bank credits an annual effective rate of interest $(i - 5\%)$. At the end of two years, he has 20,160.00 in the bank. Calculate i.

Example 4.

Money accumulates in a fund at an effective annual interest rate of i during the first 7 years, and at an effective annual interest rate of $3.5i$ thereafter. A deposit of 1 is made into the fund at time 0. It accumulates to 3.25 at the end of 20 years and to 13.62 at the end of 25 years. What is the value of deposit at the end of 13 years?

1.4 Compound Interest

When i_j is constant, the chain product of $a(t)$ collapses into:

$$a(t) = (1+i)^t$$

This special case is called **compound interest**.

The effective rate of interest is a constant can also be view from

$$i_t = \frac{(1+i)^t - (1+i)^{t-1}}{(1+i)^{t-1}} = (1+i) - 1 = i.$$

Example 5.
Justin deposits \$10,000 into Fund X. Fund X earns compound interest at the annual rate of 8%. Find the accumulated value at the end of 5 years. $\boxed{14{,}693.28}$

Example 6.
10,000 accumulates to 20,610 in 10 years, what is the effective annual interest rate? $\boxed{0.075}$

Example 7.

Fund A is invested at an effective annual interest rate of 5%. Fund B is invested at an effective annual interest rate of 4%. At the end of 25 years, the total in the two funds is 13,500. At the end of 36 years, the amount in Fund A is twice the amount in Fund B. Calculate the total in the two funds at the end of 12 years.

1.5 Simple Interest

Definition 2.

When the accumulation function,

$$a(t) = 1 + it, t = 1, 2, \cdots ,$$

the accruing of interest according to this pattern is called **simple interest**.

In this case, the effective rate of interest,

$$i_t = \frac{(1 + it) - (1 + i(t-1))}{1 + i(t-1)} = \frac{i}{1 + i(t-1)}.$$

This is a decreasing function of t. So we see that "simple interest" isn't simple at all. A linear accumulation function implies a decreasing effective rate of interest. (in fact, it decrease hyperbolically!)

Example 8.

If the rate of simple interest is 8% per annum.

(a) Find the accumulated value of \$2000 invested for 4 years;

(b) Find the effective rate of interest for each of the four years.

Example 9.

Edward deposits 16,000 in a bank. During the first 14 years, the bank credits a simple annual interest rate of 6% for 14 years, and during the next 14 years, the bank credits an annual compound interest rate of 5%. What would Edward have in the bank at the end of 28 years?

Example 10.

An investor puts 280 into Fund X and 280 into Fund Y. Fund Y earns compound interest at the annual rate of $j > 0$, and Fund X earns simple interest at the annual rate of $1.07j$. At the end of 2 years, the amount in Fund Y is equal to the amount in Fund X. Calculate the amount in Fund Y at the end of 7 years.

Example 11.

Chase age 30 deposit \$10,000 in a fund earning 6% compound interest until retirement at age 60. Find the amount of interest earned between ages 30 and 40, between 40 and 50 and between 50 and 60.

Example 12.
Find the AV of 10,000 at the end of 6 years and 3 months invested at 8% per annum:

(a) Assuming compound interest throughout.

(b) Assuming simple interest during the final fractional period.

1.6 Present Value

We have been talking about the accumulated value of a fund, i.e., how much is in the fund after t years, if we invest a given amount today. Consider the "opposite" question: How much should we invest today in order to have a given amount, say \$1, at the end of t years? The amount that we should invest is called **present value(PV)** of \$1 due in t years.
Consider a general accumulation function $a(t)$. We want to determine how much to invest today in order to have \$1 in t years. We will designate this amount as (PV). Since we require that (PV) grows to \$1 in t years, then:

$$(PV)a(t) = 1$$

$$PV = \frac{1}{a(t)}$$

The function $\frac{1}{a(t)}$ is called the "discount function".

In particular, if $a(t) = (1+i)^t$(the compound interest case), we have

$$PV = \frac{1}{a(t)} = \frac{1}{(1+i)^t} = (1+i)^{-1}$$

Because the term $(1+i)^{-1}$ comes up so often, a special symbol has been used, namely, v. So we have:

$$v = \frac{1}{(1+i)} = (1+i)^{-1}$$

PV of \$1 due in t years $= v^t = (1+i)^{-t}$.

Example 13.

What is the PV of \$1250 due in 8 years at an effective annual interest rate of 5.23%? $\boxed{831.3683}$

Example 14.

What deposit made today will provide for a payment of \$1,000 in 1 year and \$2,000 in 3 years, if the effective rate of interest is 7.5%? $\boxed{2540.15}$

Example 15.

At an annual effective interest rate of i, $i > 0$, the following are all equal:

- the present value of 14,000 at the end of 8 years;
- the sum of the present values of 5,200 at the end of year t and 58,000 at the end of year $2t$; and
- 8,333.22 immediately.

Calculate the present value of a payment of 12,000 at the end of year $t + 2$ using the same annual effective interest rate.

1.7 Effective Rate of Discount

The effective rate of interest tells us how fast a fund is growing based on the amount in the fund at the beginning of the year. But we could also define a rate of growth based on the amount in the fund at the end of the year. This rate is called **effective rate of discount**, d_t:

$$d_t = \frac{a(t) - a(t-1)}{a(t)}$$

The process of finding the price that we would be willing to pay for a promise to receive a future amount is called "discounting". It is just another term for "finding the present value."

For example, suppose a note that promises to pay \$1,000 in one year for sale, what would you be willing to pay for the note if you want to earn an effective rate of discount of 10%?

The amount of interest to be earned in the year $= \$1000 \times 10\% = \$100$.
The present value of the note must be \$1000 - \$100 = \$900.

In general, if we use d, the discounted amount of \$1000 is \$1,000 - 1,000d = \$1,000$(1-d) = 1000v$. Thus we have

$$1 - d = v = \frac{1}{1+i}$$

Solve i in terms of d, we get

$$i = \frac{d}{1-d}$$

Other relationships between i and d are:

$$d = \frac{i}{1+i} = iv$$
$$i - d = i - iv = i(1-v) = id$$
$$\frac{1}{d} - \frac{1}{i} = \frac{1+i}{i} - \frac{1}{i} = 1$$

Remarks:

- When using d to **discount**, the exponent is **positive**, e.g. the PV of 1 due in t years is $(1-d)^t$.
- When using i to **discount**, the exponent is **negative**, e.g. the PV of 1 due in t years is $(1+i)^{-t}$.
- To **accumulate** 1 using d, the exponent is **negative**: $(1-d)^{-t}$.
- To **accumulate** 1 using i, the exponent is **positive**: $(1+i)^t$.

Notes: Always remember that $v = 1-d$. Then the present value is $v^t = (1-d)^t$ and the accumulated value is $v^{-t} = (1-d)^{-t}$.

Example 16.
Show that

$$\frac{d^3}{(1-d)^2} = \frac{(i-d)^2}{1-v}.$$

Example 17.

A deposit of 220 is made into a fund which pays an annual effective interest rate of 6% for 14 years. At the same time, 110 is deposited into another fund which pays an annual effective rate of discount of d for 14 years. The amounts of interest earned over the 14 years are equal for both funds. Calculate d.

1.8 Nominal Rates of Interest

Suppose that a principal of \$100 is invested at the beginning of the year. When we say that interest is accumulated at nominal annual rate of 12%, compounded monthly, this means:

- The amount of interest earned during each month is equal to $\frac{12\%}{12} = 1\%$ of the balance at the beginning of the month t.
- Each month, the amount function grows by $\frac{12\%}{12} = 1\%$
- Therefore the accumulated amount at the end of the year is

$$A(1) = 100(1+.01)^{12} = 100\left(1+\frac{.12}{12}\right)^{12}$$

- The effective annual rate of interest i satisfies the equation

$$1+i = \left(1+\frac{0.12}{12}\right)^{12}$$

$$i = 12.68\%$$

- The nominal rate of interest is denoted $i^{(12)} = 12\%$.
- Assuming compound interest, we say that the interest is earned at nominal rate of interest $i^{(m)}$, convertible $m$thly,if

$$\left(1+\frac{i^{(m)}}{m}\right)^m = 1+i$$

- $i^{(m)} = m[(1+i)^{1/m} - 1]$
- In general, $i^{(m)} < i$.

Example 18.

Jenny deposits 1000 into a bank account. the bank credits interest at a nominal annual rate of 5% covertible semiannually. Determine the accumulated amount at the end of 5 years. $\boxed{1280.08}$

Example 19.

Brian and Jenny each take out a loan of X. You are given:

- Jenny will repay her loan by making one payment of 900 at the end of year 10. Brian will repay his loan by making one payment of 2000 at the end of year 10.
- The nominal rate of interest convertible 6thly being charged to Jenny is exactly one-half the nominal rate of interest convertible 6thly being charged to Brian.

Calculate X.

1.9 Nominal Rates of Discount

Assuming compound interest, we say that the interest is earned at nominal rate of discount $d^{(m)}$, convertible $m$thly, if

$$\left(1 - \frac{d^{(m)}}{m}\right)^m = 1 - d$$

and

$$d^{(m)} = m[1 - (1-d)^{1/m}] = m[1 - v^{1/m}].$$

Note:

- There is a close relationship between nominal rate of interest and nominal rate of discount. The following relationship holds, since both sides of equation aew equal to $1 + i$

$$\left(1 + \frac{i^{(m)}}{m}\right)^m = \left(1 - \frac{d^{(m)}}{m}\right)^{-p}$$

- If $m = p$, then

$$\left(1 + \frac{i^{(m)}}{m}\right) = \left(1 - \frac{d^{(m)}}{m}\right)^{-1}$$

- $i^{(m)} - d^{(m)} = \frac{i^{(m)}d^{(m)}}{m}$
- $\left(1 - \frac{d^{(m)}}{m}\right)^m$ is the PV of 1 due in a year. If we want to determine the accumulated value of 1, we must use a negative exponent.
- If the nominal rate is compounded at unusual frequencies, say once every two years, then $m = \frac{1}{2}$.

Example 20.

Show that $i^{(m)} - d^{(m)} = \frac{i^{(m)}d^{(m)}}{m}$

Example 21.

Find the accumulated value of 800 invested for 9 years at 8% per annum convertible quarterly.

Example 22.

Find the present value of 800 to be paid at the end of 11 years at 8% per annum payable in advance and convertible semiannually.

Example 23.

Find the nominal rate of interest convertible quarterly which is equivalent to a nominal rate of discount of 8% per annum convertible monthly.

Example 24.

Express the following amounts in terms of the given interest and discount rates.

1. AV of 150 at the end of 9 years at a nominal annual rate of interest of 7% compounded semiannually for the first 4 years and 5% compounded quaterly for the next 5 years. □
2. PV of 750 due 7 years from now, at a nominal annual rate of interest of 6% compounded monthly. □
3. AV of 1,000 at the end of 6 years, at a nominal annual rate of discount of 5% compounded semiannually. □

Example 25.

Jeff deposits 20 into a fund today and 40 20-year later. Interest for the first 7 years is credited at a nominal discount rate of d compounded quarterly, and thereafter at a nominal interest rate of 7% compounded semiannually. The accumulated balance in the fund at the end of 43 years is 558. Calculate d.

1.10 Force of Interest and discount

Force of interest measures the intensity of interest at each moment of time.

Definition 3.
The **force of interest** at time t, denoted by δ_t is defined as

$$\delta_t = \lim_{h\to 0} \frac{A(t+h)-A(t)}{hA(t)} = \frac{A'(t)}{A(t)} = \frac{a'(t)}{a(t)}$$
$$= \frac{d}{dt} \ln a(t) = \lim_{m\to\infty} i^{(m)}$$

where $h = \dfrac{1}{m}$, $i^{(m)} = \dfrac{A(t+\frac{1}{m}) - A(t)}{\frac{1}{m}} \times \dfrac{1}{A(t)}$.

Notes: Force of interest is defined for any type of interest:

- For simple interest $a(t) = 1 + it$ we have
$$\delta_t = \frac{i}{1+it}$$
- For compound interest $a(t) = (1+i)^t$ we have
$$\delta_t = \delta = \ln(1+i)$$

For compound interest, the force of interest is equal to the limit of $i^{(m)}$ as $m \to \infty$:

$$\lim_{m\to\infty} i^{(m)} = \delta$$

Intuitively, δ is a nominal rate on interest convertible continuously.

Definition 4.
Analogously, we can define the **force of discount** as

$$\delta_t' = -\frac{\frac{d}{dt}a^{-1}(t)}{a^{-1}(t)}.$$

Using calculus, we can show that "force of discount = force of interest":

$$\delta_t' = \frac{a^{-2}(t)\frac{d}{dt}a(t)}{a^{-1}(t)} = a^{-1}(t) \times a(t)\delta_t = \delta_t.$$

1.11 Accumulation Function and Present Value in Terms of the Force of Interest

Given the force of interest δ_t,

$$a(t) = e^{\int_0^t \delta_y dy}.$$

$$a^{-1} = e^{-\int_0^t \delta_y dy}.$$

In theory, the force of interest may vary instantaneously. However, in practice it is often constant, i.e. $\delta_t = \delta$ (recall the definition of compound interest). In this case,

$$a(t) = (1+i)^t = \exp(\int_0^t \delta dy) = \exp(\delta t)$$
$$\Rightarrow \delta = \ln(1+i).$$

Example 26.

You are given $\delta_t = \frac{2}{1+t}$. A payment of 370 at the end of 5 years and 740 at the end of 10 years has the same present value as a payment of 270 at the end of 4 years and X at the end of 9 years. Calculus X.

Example 27.

You are given:

(i) $\delta_t = \frac{1}{3+t}$; and

(ii) the total interest earned during the first n years on an investment of 1 at time $t = 0$ is 2.1.

Determine n.

Example 28.

It takes 13.612 years for an initial investment to double at a force of interest δ. How long will it take for an initial investment to triple at a nominal rate of interest numerically equal to δ and convertible twice a year?

Example 29.

The force of interest is $\delta_t = 0.018t$, where t is the number of years from January 1, 2021. If 1 is invested on January 1, 2024, how much is in the fund on January 1, 2029?

1.12 Some Useful Series

Recall from your Intermediate Calculus:
$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

Note that the expansion of $\ln(1+x)$:

- starts with x, not 1
- has alternating signs
- has denominator of $1, 2, 3, \ldots$ terms.

In the first series, substitute x with δ, we have

$$e^{\delta} = 1 + \delta + \frac{\delta^2}{2!} + \frac{\delta^3}{3!} + \cdots$$

Since $e^{\delta} = 1 + i$, then

$$\boxed{i = \delta + \frac{\delta^2}{2!} + \frac{\delta^3}{3!} + \cdots}$$

Also, if we substitute x with $-\delta$ into the first series, we have

$$e^{-\delta} = 1 - \delta + \frac{\delta^2}{2!} - \frac{\delta^3}{3!} + \cdots$$

and since $e^{-\delta} = (1+i)^{-1} = v = 1 - d$, then

$$\boxed{d = \delta - \frac{\delta^2}{2!} + \frac{\delta^3}{3!} - \cdots}$$

In the second series, substitute x with i, we have

$$\boxed{\delta = \ln(1+i) = i - \frac{i^2}{2} + \frac{i^3}{3} - \cdots}$$

carefully substitute $(-d)$ for x, we have

$$ln(1-d) = -d - \frac{d^2}{2} - \frac{d^3}{3} - \cdots$$

Multiply both sides by (-1),we have

$$-ln(1-d) = d + \frac{d^2}{2} + \frac{d^3}{3} + \cdots$$

Since $-\ln(1-d) = -\ln v = -\ln(1+i)^{-1} = \ln(1+i) = \delta$, thus,

$$\boxed{\delta = d + \frac{d^2}{2} + \frac{d^3}{3} + \cdots}$$

Summary:

- $i = \delta + \frac{\delta^2}{2!} + \frac{\delta^3}{3!} + \cdots$; $d = \delta - \frac{\delta^2}{2!} + \frac{\delta^3}{3!} - \cdots$
- $\delta = i - \frac{i^2}{2} + \frac{i^3}{3} - \cdots$; $\delta = d + \frac{d^2}{2} + \frac{d^3}{3} + \cdots$

Example 30.
Express δ as a power series in i and d.

1.13 Equivalent Rates

Two rates are said to be equivalent if both rates produce the same results over the same period of time. By "same results" we mean that the accumulated value of an investment, or the present value of a future amount, is the same under both rates.

1.13.1 Constant Rate Case

When the rates are constant, it is generally best to use a one-year period in writing equation.

Example 31.
What nominal rate of interest compounded semi-annually is equivalent to a nominal rate of interest of 3% compounded once every two years?

Example 32.
What nominal annual discount rate compounded semiannually is equivalent to a nominal rate of interest of 12% compounded monthly? $\boxed{11.59\%}$

1.13.2 Varying Rate Case

When one or more of the rates vary with time. The equivalent effective rate of interest for one year period, say $t = 0$ to $t = 1$ is different from the equivalent rate from say $t = 1$ to $t = 2$.

Example 33.
If $\delta_t = 0.02t$, determine the equivalent annual effective rates for the periods $t = 0$ to $t = 1$, $t = 1$ to $t = 2$ and $t = 0$ to $t = 2$.

Example 34.

Which simple interest over six years is closest to being equivalent to the following:

- an effective rate of discount of 3% for the first year.
- an effective rate of discount of 6% for the second year.
- an effective rate of discount of 9% for the third year.
- and an effective rate of interest of 5% for the fourth, fifty, and sixth years?

$\boxed{.06586}$