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## 2 Distributions of Functions of a Random Variable

If  $X$  is a random variable(r.v.) with cdf  $F_X(x)$ , then any function of  $X$ ,  $g(X)$  is also a r.v.. We denoted  $U = g(X)$  as a new r.v. Since  $U$  is a function of  $X$ , we can describe the probabilistic behavior of  $U$  in terms of  $X$ , i.e.

$$P(U \in A) = P(g(X) \in A),$$

which shows that the distribution of  $U$  depends on the functions  $F_X$  and  $g$ .

### 2.1 The CDF Technique

We will assume that a random variable  $X$  has CDF  $F_X(x)$  and some functions of  $X$  is of interest, say  $U = g(X)$ . Specifically, for each real  $u$ , we can define a set  $A_u = \{x | g(X) \leq u\}$ . It follows that  $[U \leq u]$  and  $[x \in A_u]$  are equivalent events, and consequently

$$F_U(u) = P[g(x) \leq u]$$

The probability can be expressed as the integral of the pdf,  $f_X(x)$ , over the set  $A_u$  if  $X$  is continuous, or the summation of  $F_X(x)$  over  $x$  in  $A_x$  if  $X$  is discrete.

Summary of the CDF technique:

Let  $U$  be a function of the random variables  $X_1, \dots, X_n$

1. Find the region  $U = u$  in the  $(X_1, \dots, X_n)$  space.
2. Find the region  $U \leq u$ .
3. Find  $F_U(u) = P(U \leq u)$  by integrating  $f(X_1, \dots, X_n)$  over the region  $U \leq u$  in the continuous case.
4. Find the density function  $f_U(u)$  by differentiating  $F_U(u)$ . Thus  $f_U(u) = dF_U(u)/du$ .

**Example 1.**

Suppose that  $X$  has density function given by

$$f_X(x) = \begin{cases} 2x, & 0 \leq x \leq 1 \\ 0, \text{ otherwise} \end{cases}$$

If  $U = 3X - 1$ , find the probability density function for  $U$ .

**Example 2.**

Suppose  $F_X(x) = 1 - e^{-2x}$ ,  $x > 0$ . Find the pdf of  $U = e^X$ .

**Example 3.**

Suppose  $X \sim N(\mu, \sigma^2)$ . Find the distribution of  $U = e^X$ .

**Example 4.**

The joint density function of  $X_1$  and  $X_2$  is given by

$$f(x_1, x_2) = \begin{cases} 3x_1, & 0 \leq x_2 \leq x_1 \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the probability density function for  $U = X_1 - X_2$ .

## 2.2 Transformation Methods

Let  $u(x)$  be a real-value function of a real variable  $x$ . If the equation  $u = g(x)$  can be solved uniquely, say  $x = w(u)$ , then we say the transformation is one-to-one.

### 2.2.1 One-To-One Transformation

**Theorem 1. Discrete Case** Suppose that  $X$  is a discrete random variable with pdf  $f_X(x)$  and that  $U = g(X)$  defines a one-to-one transformation. In other words, the equation  $u = g(x)$  can be solved uniquely, say  $x = w(u)$ . Then the pdf of  $U$  is

$$f_U(u) = f_X(w(u)), u \in B$$

where  $B = \{u | f_U(u) > 0\}$ .

**Example 5.**

Let  $X \sim GEO(p)$  so that

$$f_X(x) = pq^{x-1} \quad x = 1, 2, 3, \dots$$

Suppose  $U = X - 1$ . Find the pdf of  $U$ .

**Theorem 2. Continuous Case** Suppose that  $X$  is a continuous random variable with pdf  $f_X(x)$  and assume that  $U = g(X)$  defines a one-to-one transformation from  $A = \{x|f_X(x) > 0\}$  on to  $B = \{u|f_U(u) > 0\}$  with inverse transformation  $x = w(u)$ . If the derivative  $\frac{dw(u)}{du}$  is continuous and nonzero on  $B$ , then the pdf of  $U$  is

$$f_U(u) = f_X(w(u)) \left| \frac{dw(u)}{du} \right|, u \in B$$

**Example 6.**

Let  $X$  have the probability density function given by

$$f_X(x) = \begin{cases} 2x, & 0 \leq x \leq 1, \\ 0, & \text{otherwise} \end{cases}$$

Find the density function of  $U = -4X + 3$ .

**Theorem 3.**

**Probability Integral Transformation** If  $X$  is continuous with CDF  $F(x)$ , then  $U = F(x) \sim U(0, 1)$ ,

**Example 7.**

If  $X \sim Exp(\theta)$ , find a random variable  $U$  such that  $U \sim U(0, 1)$ .

**Example 8.**

If  $X \sim N(0, 1)$ , find a random variable  $U$  such that  $U \sim U(0, 1)$ .

**Theorem 4.****Inverse Probability Integral Transformation**

FRV-Q22 Let  $F(x)$  be a continuous cumulative distribution function, and let  $F^{-1}$  be its inverse function such that  $F^{-1}(u) = \min\{x|F(x) \geq u\}$   $0 < u < 1$ . If  $U \sim U(0, 1)$ , then  $F^{-1}(U)$  has  $F$  as its CDF.

**Example 9.**

Let  $U$  be a uniform random variable on the interval  $(0, 1)$ . Find a transformation  $G(U)$  such that  $G(U)$  possesses an exponential distribution with mean  $\theta$ .

**Example 10.**

Let  $X$  be a continuous random variable with pdf

$$f(x) = \begin{cases} \frac{1}{2}, & 1 < |x - 2| < 2 \\ 0, & \text{otherwise} \end{cases}$$

Find  $G(u)$ .

The Inverse Probability Integral Transformation also call the Inverse Transform Sampling. It works as follows:

1. Generate a random number  $u$  from  $U \sim U[0, 1]$ .
2. Find the inverse of the desired CDF, e.g.  $F_X^{-1}(x)$ .
3. Compute  $X = F_X^{-1}(u)$ . The computed random variable  $X$  has distribution  $F_X(x)$ .

## Example 11.

FRV-Q08 A member of the power family of distributions has a distribution function given by

$$F(x) = \begin{cases} 0, & x < 0 \\ (\frac{x}{\theta})^\alpha, & 0 \leq x \leq \theta \\ 1, & x > \theta \end{cases}$$

where  $\alpha, \theta > 0$ .

- (a) For fixed values of  $\alpha$  and  $\theta$ , find a transformation  $G(U)$  so that  $G(U)$  has a distribution function of  $F$  when  $U$  possesses a uniform  $(0, 1)$  distribution.

(b) Given that a random sample of size 5 from a uniform distribution on the interval  $(0, 1)$  yielded the values:

$$u_1 = 0.027, u_2 = 0.06901, u_3 = 0.01413,$$
$$u_4 = 0.01523, \text{ and } u_5 = 0.03609,$$

use the transformation derived in the above result to give values associated with a random variable with a power family distribution with  $\alpha = 2, \theta = 4$ .

## 2.2.2 Transformations That Are Not One-To-One

Suppose that the function  $g(x)$  is not one-to-one over  $A = \{x : f(x) > 0\}$ . Although this means that no unique solution to the equation  $u = w(x)$  exists, it usually is possible to partition  $A$  into disjoint subsets  $A_1, A_2, \dots$  such that  $u(x)$  is one-to-one over each  $A_j$ . Then, for each  $u$  in the range of  $w(x)$ , the equation  $u = g(x)$  has a unique solution  $x = w(u)$  over the set  $A_j$ . In the discrete case,

$$f_U(u) = \sum_j f_X(w_j(u))$$

In the continuous case,

$$f_U(u) = \sum_j f_X(w_j(u)) \left| \frac{dw_j(u)}{du} \right|$$

**Example 12.** Let  $f(x) = \frac{4}{31}(\frac{1}{2})^x$ ,  $x = -2, -1, 0, 1, 2$ , and consider  $U = |X|$ . Find the pdf of  $U$ .

**Example 13.** Suppose that  $X \sim U(-1, 1)$  and  $U = X^2$ . Find the pdf of  $U$ .

**Example 14.**

Let  $f(x) = x^2/3, -1 < x < 2$ , zero otherwise and  $U = X^2$ . Find the pdf of  $U$ .

### 2.2.3 Bivariate Joint Transformations

Suppose that  $X_1$  and  $X_2$  are continuous random variables with joint density function  $f_{X_1, X_2}(x_1, x_2)$  and that for all  $(x_1, x_2)$  such that  $f_{X_1, X_2}(x_1, x_2) > 0$

$$u_1 = h_1(x_1, x_2) \text{ and } u_2 = h_2(x_1, x_2)$$

Is one-to-one transformation form  $(x_1, x_2)$  to  $(u_1, u_2)$  with inverse

$$x_1 = h_1^{-1}(u_1, u_2) \text{ and } x_2 = h_2^{-1}(u_1, u_2)$$

If  $h_1^{-1}(u_1, u_2)$  and  $h_2^{-1}(u_1, u_2)$  have continuous partial derivatives with respect to  $u_1$  and  $u_2$  and Jacobian.

$$J = \det \begin{bmatrix} \frac{\partial h_1^{-1}}{\partial u_1} & \frac{\partial h_1^{-1}}{\partial u_2} \\ \frac{\partial h_2^{-1}}{\partial u_1} & \frac{\partial h_2^{-1}}{\partial u_2} \end{bmatrix} = \frac{\partial h_1^{-1}}{\partial u_1} \frac{\partial h_2^{-1}}{\partial u_2} - \frac{\partial h_1^{-1}}{\partial u_2} \frac{\partial h_2^{-1}}{\partial u_1} \neq 0$$

Then the joint density of  $U_1$  and  $U_2$  is

$$f_{U_1, U_2}(u_1, u_2) = f_{X_1, X_2}(h_1^{-1}(u_1, u_2), h_2^{-1}(u_1, u_2)) |J|$$

where  $|J|$  is the absolute value of the Jacobian.

**Example 15.**

Let  $X_1$  and  $X_2$  have a joint density function given by

$$f_{X_1, X_2}(x_1, x_2) = \begin{cases} e^{-(x_1+x_2)}, & x_1 > 0, x_2 > 0 \\ 0, & \text{otherwise} \end{cases}$$

Find the density function of  $U = X_1 + X_2$ .

## Example 16.

FRV-Q20 Let  $X$  and  $Y$  be independent random variables with  $X \sim GAM(\alpha_1, 2)$  and  $Y \sim GAM(\alpha_2, 2)$ , show that  $U = \frac{X}{X+Y}$  follow a beta distribution.

## 2.2.4 Multivariate Transformation

Let  $(X_1, \dots, X_n)$  be a random vector with pdf  $f_{\mathbf{X}}(x_1, \dots, x_n)$ . Let  $\mathbf{A} = \{\mathbf{x} : f_{\mathbf{X}}(\mathbf{x}) > 0\}$ . Consider a new random vector  $(U_1, \dots, U_n)$ , defined by  $U_1 = g_1(X_1, \dots, X_n)$ ,  $U_2 = g_2(X_1, \dots, X_n)$ , ...,  $U_n = g_n(X_1, \dots, X_n)$ . Suppose that  $A_0, A_1, \dots, A_k$  form a partition of  $\mathbf{A}$  with these properties. The set  $A_0$ , which may be empty, satisfies  $P((X_1, \dots, X_n) \in A_0) = 0$ . The transformation  $(U_1, \dots, U_n) = (g_1(\mathbf{X}), \dots, g_n(\mathbf{X}))$  is a one-to-one transformation from  $A_i$  to  $B$  for each  $i = 1, 2, \dots, k$ . Then for each  $i$ , the inverse functions from  $B$  to  $A_i$  can be found. Denote the  $i$ th inverse by  $x_1 = h_1(u_1, \dots, u_n)$ ,  $x_2 = h_2(u_1, \dots, u_n)$ , ...,  $x_n = h_n(u_1, \dots, u_n)$ . This  $i$ th inverse gives, for  $(u_1, \dots, u_n) \in B$ , the unique  $(x_1, \dots, x_n) \in A_i$  such that  $(u_1, \dots, u_n) = (g_1(x_1, \dots, x_n), \dots, g_n(x_1, \dots, x_n))$ . Let  $J_i$  denote the Jacobian computed from the inverse. That is

$$J_i = \begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & \dots & \frac{\partial x_1}{\partial u_n} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} & \dots & \frac{\partial x_2}{\partial u_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial u_1} & \frac{\partial x_n}{\partial u_2} & \dots & \frac{\partial x_n}{\partial u_n} \end{vmatrix} = \begin{vmatrix} \frac{\partial h_{1i}(u)}{\partial u_1} & \frac{\partial h_{1i}(u)}{\partial u_2} & \dots & \frac{\partial h_{1i}(u)}{\partial u_n} \\ \frac{\partial h_{2i}(u)}{\partial u_1} & \frac{\partial h_{2i}(u)}{\partial u_2} & \dots & \frac{\partial h_{2i}(u)}{\partial u_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial h_{ni}(u)}{\partial u_1} & \frac{\partial h_{ni}(u)}{\partial u_2} & \dots & \frac{\partial h_{ni}(u)}{\partial u_n} \end{vmatrix}$$

the determinant of an  $n \times n$  matrix. Assuming that these Jacobian do not vanish identically on  $B$ , we have the following representation of the joint pdf,  $f_{\mathbf{u}}(u_1, \dots, u_n)$ , for  $\mathbf{u} \in B$ :

$$f_{\mathbf{u}}(u_1, \dots, u_n)$$

$$= \sum_{i=1}^k f_{\mathbf{X}}(h_{1i}(u_1, \dots, u_n), \dots, h_{ni}(u_1, \dots, u_n)) |J|.$$

**Example 17.**

Let  $(X_1, X_2, X_3, X_4)$  have joint pdf

$$f_{\mathbf{X}}(x_1, x_2, x_3, x_4) = 24e^{-x_1-x_2-x_3-x_4},$$

$$0 < x_1 < x_2 < x_3 < x_4 < \infty$$

Consider the transformation

$$U_1 = X_1, U_2 = X_2 - X_1, U_3 = X_3 - X_2, U_4 = X_4 - X_3.$$

- (a) Find the joint pdf of  $\mathbf{U} = (U_1, U_2, U_3, U_4)$
- (b) Find the marginal pdf of  $U_i, i = 1, 2, 3, 4$

## 2.3 Sums of Random Variables-Moment Generating Function Method

Sums of independent random variables often arise in practice. A technique based on moment generating functions usually is much more convenient than using transformations for determining the distribution of sums of independent random variables.

### Theorem 5.

If  $X_1, \dots, X_n$  are independent random variables with MGFs  $M(t)$ , then the MGF of  $U = \sum_{i=1}^n X_i$  is

$$M_U(t) = M_{X_1}(t) \cdots M_{X_n}(t)$$

The MGF of a random variable uniquely determines its distribution. The MGF approach is particularly useful for determining the distribution of a sum of independent random variables, and it often will be much more convenient than trying to carry out a joint transformation.

**Example 18.**

Let  $X_1, \dots, X_k$  be independent binomial random variables with respective parameters  $n_i$ , and  $p$ ,  $X_i \sim Bin(n_i, p)$  and let  $U = \sum_{i=1}^k X_i$ . Find the distribution of  $U$ .

**Example 19.**

Let  $X_1, \dots, X_k$  be independent Poisson-distributed random variables  $X_i \sim POI(\mu_i)$  and let  $U = \sum_{i=1}^k X_i$ . Find the distribution of  $U$ .

**Example 20.**

Let  $X_1, \dots, X_k$  be independent gamma-distributed random variables with respective shape parameters  $\alpha_1, \alpha_2, \dots, \alpha_n$  and common scale parameter  $\theta$ ,  $X_i \sim GAM(\alpha_i, \theta)$  for  $i = 1, \dots, n$  and let  $U = \sum_{i=1}^k X_i$ . Find the distribution of  $U$ .

**Example 21.**

**FRV-Q36** In each of the following, random variable  $X$  and  $Y$  are independant with the given probability density functions (pdfs). Use moment generating function method to find the probability density functions (pdfs) for  $V = X + Y$ .

(a)  $f(x) = \binom{15}{x} 0.9^x 0.1^{15-x}, x = 0, 1, \dots, 15$   
 $f(y) = \binom{13}{y} 0.9^y 0.1^{13-y}, y = 0, 1, \dots, 13$

(b)  $f(x) = \frac{e^{-1.0} 1.0^x}{x!}, x = 0, 1, \dots$   
 $f(y) = \frac{e^{-3.2} 3.2^y}{y!}, y = 0, 1, \dots$

(c)  $f(x) = \frac{1}{4\sqrt{2\pi}} e^{-\frac{1}{32}(x-14)^2}, -\infty < x < \infty$   
 $f(y) = \frac{1}{6\sqrt{2\pi}} e^{-\frac{1}{72}(y-6)^2}, -\infty < y < \infty$

(d)  $f(x) = \frac{1}{\Gamma(5)3^5} x^4 e^{-\frac{x}{3}}, x > 0$   
 $f(y) = \frac{1}{\Gamma(3)3^5} y^2 e^{-\frac{y}{3}}, y > 0$

## 2.4 Order Statistics

Let  $X_1, X_2, \dots, X_n$  denote independent continuous random variables with distribution function  $F(x)$  and density  $f(x)$ . We denote the ordered random variables  $X_i$  by  $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ , where  $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)}$ . Using this notation,

$$X_{(1)} = \min(X_1, X_2, \dots, X_n)$$

is the minimum of the random variables  $X_i$ , and

$$X_{(n)} = \max(X_1, X_2, \dots, X_n)$$

is the maximum of the random variables  $X_i$ .

The probability density functions for  $X_{(1)}$  and  $X_{(n)}$  can be found using method of distribution functions.

## Example 22.

FRV-Q28b Let  $X_1, \dots, X_7$  be a random sample of size 7 from a distribution  $N(210, 10^2)$ . Let  $U = \max(X_1, X_2, \dots, X_7)$ , find the value of the p.d.f. of  $U$  at  $u = 229.91$ .

ex19

**Example 23.** Electronic components of a certain type have a length of life  $X$ , with probability density given by

$$f(x) = \begin{cases} (\frac{1}{100})e^{-x/100}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

(Length of life is measured in hours.) Suppose that two such components operate independently and in series in a certain system (hence, the system fails when either component fails). Find the density function for  $U$ , the length of life of the system.

**Example 24.**

Suppose that the components in Example ~~23~~<sup>ex19</sup> operate in parallel (hence, the system does not fail until both components fail). Find the density function for  $U$ , the length of life of the system.

**Theorem 6.**

If  $X_1, X_2, \dots, X_n$  is a random sample from a population with continuous pdf,  $f(x)$ , then the joint pdf of the order statistics,  $Y_1, Y_2, \dots, Y_n$  is

$$g(y_1, y_2, \dots, y_n)$$

$$= \begin{cases} n!f(y_1)f(y_2) \cdots f(y_n), & y_1 < y_2 < \cdots < y_n \\ 0, & \text{otherwise} \end{cases}$$

**Theorem 7.**

Let  $X_1, X_2, \dots, X_n$  denote independent continuous random variables with common distribution function  $F(x)$  and common density functions  $f(x)$ . If  $X_{(k)}$  denotes  $k^{th}$ -order statistic, then the density function of  $X_{(k)}$  is given by

$$g_{(k)}(x_k) = \frac{n!}{k!(n-k)!} [F(x_k)]^{k-1} [1-F(x_k)]^{n-k} f(x_k),$$

$$x_k \in R$$

If  $j$  and  $k$  are two integers such that  $1 \leq j < k \leq n$ , the joint density of  $X_{(j)}$  and  $X_{(k)}$  is given by

$$g_{(j)(k)}(x_j x_k) = \frac{n!}{(j-1)!(k-j-1)!(n-k)!} [F(x_j)]^{j-1}$$

$$\times [F(x_k) - F(x_j)]^{k-j-1}$$

$$\times [1 - F(x_k)]^{n-k} f(x_j) f(x_k)$$

$$-\infty < x_j < x_k < \infty$$

FRV-Q24

## Example 25.

A system is composed of 18 independent components. If the pdf of the time to failure of each component is exponential,  $X_i \sim EXP(140)$ . Suppose that the 18-component system fails when at least 6 components fail. Give the pdf of the time to failure of the system.

**Example 26.** Suppose that  $X_1, X_2, \dots, Y_{15}$  denotes a random sample from a uniform distribution defined on the interval  $(0, 1)$ . That is,

$$f(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the density function for the second-order statistic. Also, give the joint density function for the second- and fourth-order statistics.

## Example 27.

FRV-Q13b Let  $X_1$  and  $X_2$  be a random sample of size  $n = 2$  from a continuous distribution with pdf of the form  $f(x) = 4x^3$  if  $0 < x < 1$  and zero otherwise.

- (a) Find the joint pdf of  $Y_1 = \min(X_1, X_2)$  and  $Y_2 = \max(X_1, X_2)$ .
- (b) Find the pdf of the sample range  $R = Y_2 - Y_1$ .

The event that the  $k^{th}$ -order statistic at most  $y$ ,  $[Y_k \leq y]$  can occur if and only if at least  $k$  of the  $n$  observations are less than or equal to  $y$ . That is, here the probability of “success” on each trial is  $F(y)$  and we must have at least  $k$  successes. Thus,

$$P(Y_k \leq y) = \sum_{i=k}^n \binom{n}{i} [F(y)]^i [1 - F(y)]^{n-i}$$

### Example 28.

FRV-Q30 Let  $X_i \sim Exp(60)$ ,  $i = 1, \dots, 10$  and  $Y_1 < Y_2 < \dots < Y_{10}$  be the order statistics. Compute the probability that  $Y_8$  is less than 130.8.