CONTENTS

6 Bonds

One of the major applications of the theory of interest is the determination of prices and values for bonds and other securities, such as preferred stock and common stock. There are three main questions related to securities:

1. Given the desired yield rate of an investor, what price should be paid for a given security?
2. Given the purchase price of a security, what is the resulting yield rate to an investor?
3. What is the value of a security on a given date after it has been purchased?

6.1 Price of a Bond

A bond is basically a loan to a governmental entity or corporation on which we typically receive interest payments called "coupons", and then receive the redemption value on the redemption date.

Formal notations:

- P = the *price* of a bond;
- F = the *par value*, or *face amount*. It is printed on the front of the bond and is often the amount payable at the maturity date. It is customary to quote bond prices in terms of a par value of 100;
- C = the *redemption value* of a bond. It is the amount of money paid at a redemption date to the holder of the bond. Often $C = F$, however there are exceptions.

- r = the *coupon rate*, rate per coupon payment period. The most common frequency for bond coupons in the US is semiannual. E.g. an 8% bond with semiannual coupons has $r = 0.04$. In the international financial markets, there are other coupon frequencies.
- Fr = the amount of the coupon;
- $g = \frac{Fr}{C}$ is the *modified rate* of the bond. We have $Cg = Fr$. (If a bond is redeemable at par, $C = F$ and $g = r$.)
- i = the *yield rate* of the bond, i.e. the rate realized by the investor, or the *internal rate of return*, or the *yield to maturity*.
- n = the number of coupon payment periods;
- $K = Cv^n$ = the present value of the redemption;
- $G = \frac{Fr}{i}$, the base amount of a bond;
- $v = \frac{1}{1+i}$.

6.1.1 Basic Formulas:

The price of a bond to yield an effective rate i is the PV of bond payments at that rate. The PV of the payments is the PV of the coupons plus the PV of the redemption value:

$$P = Fra_{\overline{n}|} + Cv^n$$

or

$$P = Cga_{\overline{n}|} + Cv^n$$

6.1.2 The Premium/Discount Formula

Since $a_{\overline{n}|} = \frac{1-v^n}{i}$, and $v^n = 1 - ia_{\overline{n}|}$:

$$\begin{aligned} P &= Fra_{\overline{n}|} + Cv^n \\ &= Fra_{\overline{n}|} + C(1 - ia_{\overline{n}|}) \\ &= C + (Fr - Ci)a_{\overline{n}|} \\ &= C + (Cg - Ci)a_{\overline{n}|} \end{aligned}$$

Example 1.
A 10-year 3,000 par value bond with 12% semi-annual coupons is purchansed to earn a yield of 11% convertible semiannually. What is the price of the bond?

Example 2.
Find the price to yield 4% effective for a 100 bond with 5.5% annual coupons redeemable at 110 in 10 years, using the basic formula and premium/discount formula. $\boxed{118.92}$

Example 3 (T06Q1)**.**
John buys a bond that is due to mature at par in 2 year. It has a 800 par value and coupons at 7% convertible semiannually. John pays 846.66 to obtain a yield rate i convertible semiannually, $i > 0$. Calculate i.

Example 4.

John purchases a 1000 par value 14-year bond with coupons at 10% convertible semiannually which will be redeemed for C. The purchase price is 582.0 and the present value of the redemption value is 76.03. Calculate C.

Example 5 (T06Q2)**.**

A 1,000 bond with with annual coupons is redeemable at par at the end of 8 years. At a purchase price of 880, the yield rate is i. The coupon rate $i - 0.03$. Calculate i.

Example 6 (T06Q3)**.**

A 1000 par value 24-year bond with annual coupons and redeemable at maturity at 1050 is purchased for P to yield an annual effective rate of 8.67%. The first coupon is 90. Each subsequent coupon is 4% greater than the preceding coupon. Determine P.

Example 7.

Two bonds are purchased for the same price to yield 5%. Bond X has 4% annual coupons and matures for its face value of 100. Bond Y has annual coupons of 3 and matures for 180. Both bonds mature at the end of n years. Calculate n. $\boxed{33}$

Example 8.
A 1,000 par value 3-year bond with annual coupons of 50 for the first year, 70 for the second year and 90 for the third year is bought to yield a force of interest

$$\delta_t = \frac{2t-1}{2(t^2-t+1)} \text{ for } t \geq 0.$$

Calculate the price of this fund. $\boxed{502.4}$

Example 9.
A bond with coupons equal to 40 sells for P. A second bond with the same maturity value and term has coupons equal to 30 and sells for Q. A third bond with the same maturity value and term has coupons equal to 80. All prices are based on the same yield rate, and all coupons paid at the same frequency. Determine the price of the third bond. $\boxed{5P-4Q}$

Example 10.
A 30-year bond has a annual coupon rate of 6% for the first 10 years, 7% for the next ten years, 8% for the last ten years, and matures at its par value of 100. The bond is bought to produce an effective annual yield rate of 7%. Determine an expression for the price of the bond. (All interest functions are at 7%.)

Example 11.
A 100 par value 100-year bond with a redemption value of 100 has annual coupons of 10% for the first 10 years, 9% for the next 10 years, 8% for the next 10 years, ..., 1% for the last 10 years. Show that the price of the bond to yield i is

$$\frac{10s_{\overline{10}|} - a_{\overline{100}|}}{is_{\overline{10}|}} + 100v^{100}$$

Example 12 (T06Q4)**.**

Brian buys a 27-year bond with a par value of 1,700 and annual coupons. The bond is redeemable at par. Brian pays 2,468 for the bond assuming an annual effective yield rate of i. The coupon rate is twice the yield rate. At the end of 6 years, Brian sells the bond for P, which produces the same annual effective yield rate of i for the new buyer. Calculate P.

6.2 Premium and Discount

When a bond is purchased for more than its redemption value, the excess of the price over the redemption value is called the "premium". When a bond is purchased for less than its redemption value, the excess of the price over the redemption value is called the "discount".

The premium/discount formula is ideal for quickly determining whether a bond is sold at a premium or discount:

$$P = C + (Cg - Ci)a_{\overline{n}|}$$

Thus

1. If $g > i$, then $P > C$ and the bond is purchases at a premium equal to $P - C = (Cg - Ci)a_{\overline{n}|}$.
2. If $g < i$, then $P < C$ and the bond is purchases at a discount equal to $C - P = (Cg - Ci)a_{\overline{n}|}$.

Example 13.

A 1,000 bond with 8% semiannual coupons redeemable at 1,050 in 1.5 years, purchased to yield a nominal annual rate of 6% compounded semiannually.

(a) Determine whether the bond is purchased at a premium or discount.

(b) What is the amount of premium/discount?

Example 14.

You are given:

(i) A 10-year 8% semiannual coupon bond is purchased at a discount of X.

(ii) A 10-year 9% semiannual coupon bond is purchased at a premium of Y.

(iii) A 10-year 10% semiannual coupon bond is purchased at a premium of $2X$.

(iv) All bonds were purchased at the same yield rate and have par values of 1000.

Calculate Y. $\boxed{0.5X}$

Example 15.
Find the price of a 1000 par value two-year 8% bond with semiannual coupons bought to yield 6% convertible semiannually if the investor can replace the premium by means of a sinking fund earning 5% convertible semiannually. $\boxed{1036.93}$

Example 16.
Two 24-year bonds with 100 redemption values are each purchased to yield an effective annual interest rate of 7%. The first bond bears annual g% coupons and is purchased at a premium of 11.87. The second bond bears annual $(g+2)$% coupons. What is the purchased price of the second bond?

6.3 Bond Amortization

When a bond is bought at a premium ($g > i$), the book value will gradually be adjusted downward. This process is called amortization of premium or "write down." When a bond is bought at a discount ($g < i$), the book value will gradually be adjusted upward. This process is called accumulation of discount or "write ups." The concept of the bond amortization is the same as the loan amortization, however, bond terminology is different:

(i) "Book value" is used instead of "outstanding loan balance."

(ii) Instead of "principal repaid," the periodic reduction in the book value is called the "amount for amortization of premium." if the bond is purchased at premium and the periodic increase in the book value is called the 'amount for accumulation of discount".

(iii) Instead of "payment amount," coupon is used.

6.3.1 Bond Amortization When a Bond is Purchased At a Premium

Consider a 1,000 bond with 8% semiannual coupons redeemable at 1,050 in 1.5 years, purchased to yield a nominal annual rate of 6% compounded semiannually. We construct the bond a bond amortization schedule:
$B_0 = P = Fra_{\overline{n}|i} + Cv^n = 1000(.04)\left[\frac{1-1.03^{-3}}{.03}\right] + 1050(1.03^{-3}) = 1074.04$

Period t	Coupon	Interest Earned $I_t = iB_{t-1}$	Amount for Amortization of Premium $P_t = \text{Coupon} - I_t$	Book Value $B_t = B_{t-1} - P_t$
0				1,074.04
1	40	32.22	7.78	1,066.26
2	40	31.99	8.01	1,058.25
3	40	31.75	8.25	1,050.00
Totals	120	95.96	24.04	

Notes:

1. Book Value, $B_t = B_{t-1} - P_t$ or
Prospectively:$B_t = Fra_{\overline{n-t}|} + Cv^{n-t}$
retrospectively: $B_t = B_0(1+i)^t - Frs_{\overline{t}|}$

2. $I_t = \boxed{iB_{t-1}}$
$= i[Fra_{\overline{n-t+1}|} + Cv^{n-t+1}]$
$= Fr(1 - v^{n-t+1}) + Civ^{n-t+1}$
$= \boxed{Fr - (Fr - Ci)v^{n-t+1}}$

3. The amounts for amortization of premium (write down),

$$P_t = Fr - I_t$$
$$= (Fr - Ci)v^{n-t+1}$$
$$= (Cg - Ci)v^{n-t+1}$$

4. P_t are also in geometric progression with common ratio $(1+i)$.
$$P_t = (1+i)^{t-1}P_1$$

5. P_t in this case is also called "write down" because the asset value of a bond is "written down" by this amount each period.

6. Total Interest Earned
= Total payments received - Amount invested
$$\sum_{t=1}^{n} I_t = (nFr + C) - P$$
For this bond:
Total payments received
= 3 coupons + redemption value
$= 3 \times 40 + 1,050 = 1,170$
The amount invested = 1,074.04, so,
Total Interest Earned $= 1,170 - 1074.04 = 95.56$

7. The total of the amounts for amortization of premium = Premium.
$$\sum_{t=1}^{n} P_t = P - C$$
Premium $= P - C = (cg - ci)a_{\overline{n}|}$
$= [40 - 1050(0.03)]a_{\overline{3}|}$
$= 24.04$

Example 17.

A 1000 par value 7-year bond with 9% semiannual coupons was bought to yield 8.5% convertible semiannually. Determine the amount of premium amortized(write down) in the 9th coupon payment.

Example 18 (T06Q5)**.**

An actuary finds a 21-year bond that was purchased at a premium has determined the following:

- The bonds pays semiannual interest.
- The amount for amortization of the premium in the 2nd coupon payment was 983.57.
- The amount for amortization of the premium in the 4th coupon payment was 1128.05.

What is the value of the premium?

Example 19.

A 14-year bond with par value of 1000 is purchased to yield 8% convertible semiannually. Par value equals redemption value. The interest paid portion of the first semiannually coupon is 39.84. At what nominal rate of interest(express in %) convertible semiannually are the coupons paid?

Example 20 (T06Q6)**.**

Bryan buys a $2n-$year 1000 par value bond with 7.2% annual coupons at a price P. The price assumes an annual effective yield of 14%. At the end of n years, the book value of the bond, X, is 50.82 greater than the purchase price, P. Assume $v^n_{14\%} < 0.5$. Calculate X.

Example 21.

An $n-$ year 1000 par value bond with 8% annual coupons has an annual effective yield of $i, i > 0$. The book value of the bond at the end of the year 3 is 1099.84 and the book value at the end of year 5 is 1082.27. Calculate the purchase price of the bond. $\boxed{1122.38}$

Example 22. A 10,000 par value 10-year bond with 8% annual coupons is bought at a premium to yield an annual effective rate of 6%. Calculate the interest portion of the 7th coupon.

6.3.2 Bond Amortization When a Bond is Purchased At a Discount

When a bond is bought at a discount ($g < i$), the book value will gradually be adjusted upward. This process is called accumulation of discount or "write ups."
Consider a 1,000 bond with 8% semiannual coupons redeemable at 1,050 in 1.5 years, purchased to yield a nominal annual rate of 10% compounded semiannually. The price of the bond is:

$$\begin{aligned} P &= 1,050 + [40 - (1,050)(0.05)]a_{\overline{3}|} \\ &= 1,050 - 12.5a_{\overline{3}|} \\ &= 1,015.96 \end{aligned}$$

We construct the bond a bond amortization schedule, to avoid using negative numbers, we will write the entries in the principal repaid column as positive amounts and remember to add each of them to the previous book value.

Period t	Coupon	Interest Earned $I_t = iB_{t-1}$	Amount for Accumulation of Discount $P_t = \text{Coupon} - I_t$	Book Value $B_t = B_{t-1} + P_t$
0				1,015.96
1	40	50.80	10.80	1,026.76
2	40	51.34	11.34	1,038.10
3	40	51.90	11.90	1,050.00
Totals	120	154.04	34.04	

Notes:

- Total Interest Earned
  = Total payments received - Amount invested

  $$\sum_{t=1}^{n} I_t = (nFr + C) - P$$

  In this case, $\sum_{t=1}^{n} I_t = 3 \times 40 + 1,050 - 1015.96 = 154.04$

- The total of the "amount for accumulation of discount" is equal to the discount.

  $$\sum P_t = C - P$$

- Another term used for "P_t = amount for accumulation of discount" is "write up" since the asset value of a bond is "written up" by this amount each period.

  $$P_t = (Ci - Cg)v^{n-t+1} = (Ci - Fr)v^{n-t-1}$$

- The amount for accumulation of discount are in geometric progression.

  $$P_t = P_1(1+i)^{t-1}$$

Note that in the case of a bond purchased at a discount, we have defined the amount for accumulation of discount P_t as the negative of the normal principal repaid, to avoid negative signs.

Example 23 (T06Q7).
A 25-year bond with semiannual coupons has a redemption value of 100. It is purchased at a discount to yield 8% compounded semiannually. If the amount for accumulation of discount in the 47th coupon payment is 2.48. Determine the total amount of discount in the original purchase price?

Example 24.

A 36-year 10,000 bond that pays 3% annual coupons matures at par. It is purchased to yield 5% for the first 18 year and 4% thereafter. Calculate the amount for accumulation of discount for year 10.

Example 25 (T06Q8)**.**

Laura buys two bonds at time 0. Bond X is a 5,000 par value 16-year bond with 13% annual coupons. It is bought at a price to yield an annual effective rate of 11%. Bond Y is a 16-year par value bond with 8.775% annual coupons and a face amount of F. Laura pays P for Bond Y to yield an annual effective rate of 11%. During year 8, the write-down in premium (principal adjustment) on bond X is equal to the write-up in discount (principal adjustment) on bond Y. Calculate P.

Example 26.

A 1,000 par value bond bearing 4% annual coupons is purchased at a discount to yield an effective annual rate of 5%, the write-up in value during the first year is 4.36. Determine the purchased price. $\boxed{887.2}$

6.4 Determination of Yield Rates

Up to now, we have usually assumed that the yield rate is known and that the price has to be computed. In this section, the yield rate is unknown, and most of the times the equation of values has to be solve numerically.
In the case of a bond, using the Basic formula, we are looking for the solution of the equation for i:

$$P = Fra_{\overline{n}|} + Cv^n$$

where P, Fr, C and n are known. We have to use the $\boxed{\text{Table}}$ in TI30 calculator to search the solution.

Example 27.

A 100 par value 11-year bond with 10% semiannual coupons is selling for 92. Find the yield rate convertible semiannually.

Example 28 (T06Q9)**.**

A 1,000 16-year 12% bond with semiannual coupons is purchased for 1017. The redemption value is 1,000. The coupons are reinvested at a nominal annual rate of 9%, compounded semiannually. Determined the purchaser's annual effective yield rate over the 16 year period.

6.5 Callable Bonds

Many bonds have the options that permits the borrowers to redeem ("call") the bonds prior to the normal maturity date. The main reason for doing this is to take the advantage of a decline in interest rates. These kind of bonds is called callable bonds.

6.5.1 Rules For Calculating Callable Bond Price

1. If the bond sells at a Premium ($i < r$), the Earliest redemption date is the Worst (**PEW**). Consider a bond selling at a premium, say a 100 bond that sells for 120 with 6% annual coupons. Assume the bond is redeemed in a year,

```
                         100+6
 |---------------------|
 0                     1
120
```


On an investment of 120, we get back only 106 at the end of a year. There is a "loss" of 20, the excess of the price over the redemption value (i.e., the premium). This is offset by the 6 coupon, for a net loss of 14 in one year on an investment of 120. so

Yield rate $= -\frac{14}{120} = -11.67\%$

The more years that we can spread the 20 loss over, the better the yield rate. The yield rate would be negative if the bond were called within 3 years (We would get back less that 120 investment). The yield rate will be positive if the bond were called at the end of 4 years.

Yield rate at the end of 4 year $= \frac{4}{120} = 3.33\%$

2. If the bond sells at a Discount ($i > r$), the Latest redemption date will be the Worst.

   Take any bond selling at a discount, say a 100 bond with 6% annual coups selling for 80. Assume the bond is redeemed in a year,

```
                        6
                      100
|----------------------|
0                      1
80
```

   The "gain" of 20 is received in one year, plus a 6 coupon. Thus, there is a total return of 26 on an investment of 80, for an exact effective yield rate of $\frac{26}{80} = 32.5\%$. If the bond is redeemed later, the discount of 20 is spread over more years, which dilutes the effective rate. Thus, for discount bonds, the **latest** redemption date is worst.

Example 29.
A 100 bond with 6% annual cuopons and a maturity date 20 years from now can be called at par on any coupon due date starting 10 years from now. What is the highest price an investor can pay and still be certain of a yield of at least 4%?

Example 30 (T06Q10)**.**
A 1000 bond with 6% annual cuopons and a maturity date 22 years from now can be called at par on any coupon due date starting 11 years from now. What is the price an investor pay to get a minimum yield rate of 8% effective? If this price is paid, what is the maximum yield rate the investor can earn?

Example 31.
An investor bought a 15-year bond with par value of 100,000 and 8% semiannual coupons. The bond is callable at par on any coupon date beginning with the 24th coupon. Find the price paid that will yield a rate not less that $i^{(2)} = 10\%$.

Example 32. An investor purchases a 1,000 bond redeemable at par that pays 8% semiannual coupons and matures in 10 years. The bond will yield 7% convertible semiannually to maturity. If the bond is called in five years, the minimum redemption value the investor needs to realize the same yield is X. Determine X.

6.5.2 A More General Principle

It is possible to state a more general principle that covers both premium and discount situations. This principle also cover more complex situations where it may not be clear whether the earliest or latest redemption date should be used, because the redemption values are not the same on different call dates.

In the general principle, to ensure that we earn a specified minimum yield rate, we compute the **lowest price** for all of the possible redemption dates at this yield. For example, consider Example 29. Using the premium/discount formula:

$$\begin{aligned}P &= C + (Fr - Ci)a_{\overline{n}|} \\ &= 100 + (6-4)a_{\overline{n}|4\%} \\ &= 100 + 2a_{\overline{n}|}\end{aligned}$$

The possible values of n are from $n = 10$ to $n = 20$. The lowest price is the price when $n = 10$, which has the same result in Example 29.

In Example **??**, using the premium/discount formula:

$$\begin{aligned}P &= C + (Fr - Ci)a_{\overline{n}|} \\ &= 100 + (6-8)a_{\overline{n}|8\%} \\ &= 100 - 2a_{\overline{n}|8\%}\end{aligned}$$

Again, the possible values of n are from $n = 10$ to $n = 20$. the lowest price is the price when $n = 20$.

Example 33.
A 5% semiannual coupon 100 bond maturing in 15 years is callable on any coupon date after the 10th. If called on the 11th through 20th coupon date, the redemption value would be 110. If called on the 21st through 30th coupon date, redemption would be at par. Find the price that would ensure an investor minimum yield of 3% per annum compounded semiannually. $\boxed{117.9}$