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2 Distributions of Functions of a Random Variable

If X is a random variable(r.v.) with cdf $F_X(x)$, then any function of X , $g(X)$ is also a r.v.. We denoted $U = g(X)$ as a new r.v. Since U is a function of X , we can describe the probabilistic behavior of U in terms of X , i.e.

$$P(U \in A) = P(g(X) \in A),$$

which shows that the distribution of U depends on the functions F_X and g .

2.1 The CDF Technique

We will assume that a random variable X has CDF $F_X(x)$ and some functions of X is of interest, say $U = g(X)$. Specifically, for each real u , we can define a set $A_u = \{x|g(X) \leq u\}$. It follows that $[U \leq u]$ and $[x \in A_u]$ are equivalent events, and consequently

$$F_U(u) = P[g(x) \leq u]$$

The probability can be expressed as the integral of the pdf, $f_X(x)$, over the set A_u if X is continuous, or the summation of $F_X(x)$ over x in A_x if X is discrete.

Summary of the CDF technique:

Let U be a function of the random variables X_1, \dots, X_n

1. Find the region $U = u$ in the (X_1, \dots, X_n) space.
2. Find the region $U \leq u$.
3. Find $F_U(u) = P(U \leq u)$ by integrating $f(X_1, \dots, X_n)$ over the region $U \leq u$ in the continuous case.
4. Find the density function $f_U(u)$ by differentiating $F_U(u)$. Thus $f_U(u) = dF_U(u)/du$.

Example 1.

Suppose that X has density function given by

$$f_X(x) = \begin{cases} 2x, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

If $U = 3X - 1$, find the probability density function for U .

Example 2.

Suppose $F_X(x) = 1 - e^{-2x}, x > 0$. Find the pdf of $U = e^X$.

Example 3.

Suppose $X \sim N(\mu, \sigma^2)$. Find the distribution of $U = e^X$.

Example 4.

The joint density function of X_1 and X_2 is given by

$$f(x_1, x_2) = \begin{cases} 3x_1, & 0 \leq x_2 \leq x_1 \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the probability density function for $U = X_1 - X_2$.

2.2 Transformation Methods

Let $u(x)$ be a real-value function of a real variable x . If the equation $u = g(x)$ can be solved uniquely, say $x = w(u)$, then we say the transformation is one-to-one.

2.2.1 One-To-One Transformation

Theorem 1. Discrete Case Suppose that X is a discrete random variable with pdf $f_X(x)$ and that $U = g(X)$ defines a one-to-one transformation. In other words, the equation $u = g(x)$ can be solved uniquely, say $x = w(u)$. The the pdf of U is

$$f_U(u) = f_X(w(u)), u \in B$$

where $B = \{u | f_U(u) > 0\}$.

Example 5.

Let $X \sim GEO(p)$ so that

$$f_X(x) = pq^{x-1} \quad x = 1, 2, 3, \dots$$

Suppose $U = X - 1$. Find the pdf of U .

Theorem 2. Continuous Case Suppose that X is a continuous random variable with pdf $f_X(x)$ and assume that $U = g(X)$ defines a one-to-one transformation from $A = \{x | f_X(x) > 0\}$ on to $B = \{u | f_U(u) > 0\}$ with inverse transformation $x = w(u)$. If the derivative $\frac{dw(u)}{du}$ is continuous and nonzero on B , then the pdf of U is

$$f_U(u) = f_X(w(u)) \left| \frac{dw(u)}{du} \right|, u \in B$$

Example 6.

Let X have the probability density function given by

$$f_X(x) = \begin{cases} 2x, & 0 \leq x \leq 1, \\ 0, & \text{otherwise} \end{cases}$$

Find the density function of $U = -4X + 3$.

Theorem 3.

Probability Integral Transformation If X is continuous with CDF $F(x)$, then $U = F(x) \sim U(0, 1)$,

Example 7.

If $X \sim \text{Exp}(\theta)$, find a random variable U such that $U \sim U(0, 1)$.

Example 8.

If $X \sim N(0, 1)$, find a random variable U such that $U \sim U(0, 1)$.

Theorem 4.**Inverse Probability Integral Transformation**

FRV-Q22 Let $F(x)$ be a continuous cumulative distribution function, and let F^{-1} be its inverse function such that $F^{-1}(u) = \min\{x | F(x) \geq u\}$ $0 < u < 1$. If $U \sim U(0, 1)$, then $F^{-1}(U)$ has F as its CDF.

Example 9.

Let U be a uniform random variable on the interval $(0, 1)$. Find a transformation $G(U)$ such that $G(U)$ possesses an exponential distribution with mean θ .

Example 10.

Let X be a continuous random variable with pdf

$$f(x) = \begin{cases} \frac{1}{2}, & 1 < |x - 2| < 2 \\ 0, & \text{otherwise} \end{cases}$$

Find $G(u)$.

The Inverse Probability Integral Transformation also call the Inverse Transform Sampling. It works as follows:

1. Generate a random number u from $U \sim U[0, 1]$.
2. Find the inverse of the desired CDF, e.g. $F_X^{-1}(x)$.
3. Compute $X = F_X^{-1}(u)$. The computed random variable X has distribution $F_X(x)$.

Example 11.

FRV-Q08

A member of the power family of distributions has a distribution function given by

$$F(x) = \begin{cases} 0, & x < 0 \\ (\frac{x}{\theta})^\alpha, & 0 \leq x \leq \theta \\ 1, & x > \theta \end{cases}$$

where $\alpha, \theta > 0$.

- (a) For fixed values of α and θ , find a transformation $G(U)$ so that $G(U)$ has a distribution function of F when U possesses a uniform $(0, 1)$ distribution.

(b) Given that a random sample of size 5 from a uniform distribution on the interval $(0, 1)$ yielded the values:

$$u_1 = 0.027, u_2 = 0.06901, u_3 = 0.01413, \\ u_4 = 0.01523, \text{ and } u_5 = 0.03609,$$

use the transformation derived in the above result to give values associated with a random variable with a power family distribution with $\alpha = 2, \theta = 4$.

2.2.2 Transformations That Are Not One-To-One

Suppose that the function $g(x)$ is not one-to-one over $A = \{x : f(x) > 0\}$. Although this means that no unique solution to the equation $u = w(x)$ exists, it usually is possible to partition A into disjoint subsets A_1, A_2, \dots such that $u(x)$ is one-to-one over each A_j . Then, for each u in the range of $w(x)$, the equation $u = g(x)$ has a unique solution $x = w(u)$ over the set A_j . In the discrete case,

$$f_U(u) = \sum_j f_X(w_j(u))$$

In the continuous case,

$$f_U(u) = \sum_j f_X(w_j(u)) \left| \frac{dw_j(u)}{du} \right|$$

Example 12. Let $f(x) = \frac{4}{31}(\frac{1}{2})^x$, $x = -2, -1, 0, 1, 2$, and consider $U = |X|$. Find the pdf of U .

Example 13. Suppose that $X \sim U(-1, 1)$ and $U = X^2$. Find the pdf of U .

Example 14.

Let $f(x) = x^2/3$, $-1 < x < 2$, zero otherwise and $U = X^2$. Find the pdf of U .

2.2.3 Bivariate Joint Transformations

Suppose that X_1 and X_2 are continuous random variables with joint density function $f_{X_1, X_2}(x_1, x_2)$ and that for all (x_1, x_2) such that $f_{X_1, X_2}(x_1, x_2) > 0$

$$u_1 = h_1(x_1, x_2) \quad \text{and} \quad u_2 = h_2(x_1, x_2)$$

Is one-to-one transformation from (x_1, x_2) to (u_1, u_2) with inverse

$$x_1 = h_1^{-1}(u_1, u_2) \quad \text{and} \quad x_2 = h_2^{-1}(u_1, u_2)$$

If $h_1^{-1}(u_1, u_2)$ and $h_2^{-1}(u_1, u_2)$ have continuous partial derivatives with respect to u_1 and u_2 and Jacobian.

$$J = \det \begin{bmatrix} \frac{\partial h_1^{-1}}{\partial u_1} & \frac{\partial h_1^{-1}}{\partial u_2} \\ \frac{\partial h_2^{-1}}{\partial u_1} & \frac{\partial h_2^{-1}}{\partial u_2} \end{bmatrix} = \frac{\partial h_1^{-1}}{\partial u_1} \frac{\partial h_2^{-1}}{\partial u_2} - \frac{\partial h_1^{-1}}{\partial u_2} \frac{\partial h_2^{-1}}{\partial u_1} \neq 0$$

Then the joint density of U_1 and U_2 is

$$f_{U_1, U_2}(u_1, u_2) = f_{X_1, X_2}(h_1^{-1}(u_1, u_2), h_2^{-1}(u_1, u_2)) |J|$$

where $|J|$ is the absolute value of the Jacobian.

Example 15.

Let X_1 and X_2 have a joint density function given by

$$f_{X_1, X_2}(x_1, x_2) = \begin{cases} e^{-(x_1+x_2)}, & x_1 > 0, x_2 > 0 \\ 0, & \text{otherwise} \end{cases}$$

Find the density function of $U = X_1 + X_2$.

Example 16.

FRV-Q20 Let X and Y be independent random variables with $X \sim GAM(\alpha_1, 2)$ and $Y \sim GAM(\alpha_2, 2)$, show that $U = \frac{X}{X+Y}$ follow a beta distribution.

2.2.4 Multivariate Transformation

Let (X_1, \dots, X_n) be a random vector with pdf $f_{\mathbf{X}}(x_1, \dots, x_n)$. Let $\mathbf{A} = \{\mathbf{x} : f_{\mathbf{X}}(\mathbf{x}) > 0\}$. Consider a new random vector (U_1, \dots, U_n) , defined by $U_1 = g_1(X_1, \dots, X_n)$, $U_2 = g_2(X_1, \dots, X_n)$, \dots , $U_n = g_n(X_1, \dots, X_n)$. Suppose that A_0, A_1, \dots, A_k form a partition of \mathbf{A} with these properties. The set A_0 , which may be empty, satisfies $P((X_1, \dots, X_n) \in A_0) = 0$. The transformation $(U_1, \dots, U_n) = (g_1(\mathbf{X}), \dots, g_n(\mathbf{X}))$ is a one-to-one transformation from A_i to B for each $i = 1, 2, \dots, k$. Then for each i , the inverse functions from B to A_i can be found. Denote the i th inverse by $x_1 = h_1(u_1, \dots, u_n)$, $x_2 = h_2(u_1, \dots, u_n)$, \dots , $x_n = h_n(u_1, \dots, u_n)$. This i th inverse gives, for $(u_1, \dots, u_n) \in B$, the unique $(x_1, \dots, x_n) \in A_i$ such that $(u_1, \dots, u_n) = (g_1(x_1, \dots, x_n), \dots, g_n(x_1, \dots, x_n))$. Let J_i denote the Jacobian computed from the inverse. That is

$$J_i = \begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & \dots & \frac{\partial x_1}{\partial u_n} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} & \dots & \frac{\partial x_2}{\partial u_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial u_1} & \frac{\partial x_n}{\partial u_2} & \dots & \frac{\partial x_n}{\partial u_n} \end{vmatrix} = \begin{vmatrix} \frac{\partial h_{1i}(u)}{\partial u_1} & \frac{\partial h_{1i}(u)}{\partial u_2} & \dots & \frac{\partial h_{1i}(u)}{\partial u_n} \\ \frac{\partial h_{2i}(u)}{\partial u_1} & \frac{\partial h_{2i}(u)}{\partial u_2} & \dots & \frac{\partial h_{2i}(u)}{\partial u_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial h_{ni}(u)}{\partial u_1} & \frac{\partial h_{ni}(u)}{\partial u_2} & \dots & \frac{\partial h_{ni}(u)}{\partial u_n} \end{vmatrix}$$

the determinant of an $n \times n$ matrix. Assuming that these Jacobian do not vanish identically on B , we have the following representation of the joint pdf, $f_{\mathbf{u}}(u_1, \dots, u_n)$, for $\mathbf{u} \in B$:

$$f_{\mathbf{u}}(u_1, \dots, u_n) \\ = \sum_{i=1}^k f_{\mathbf{X}}(h_{1i}(u_1, \dots, u_n), \dots, h_{ni}(u_1, \dots, u_n)) |J|.$$

Example 17.

Let (X_1, X_2, X_3, X_4) have joint pdf

$$f_{\mathbf{X}}(x_1, x_2, x_3, x_4) = 24e^{-x_1-x_2-x_3-x_4},$$

$$0 < x_1 < x_2 < x_3 < x_4 < \infty$$

Consider the transformation

$$U_1 = X_1, U_2 = X_2 - X_1, U_3 = X_3 - X_2, U_4 = X_4 - X_3.$$

- (a) Find the joint pdf of $\mathbf{U} = (U_1, U_2, U_3, U_4)$
- (b) Find the marginal pdf of $U_i, i = 1, 2, 3, 4$

2.3 Sums of Random Variables-Moment Generating Function Method

Sums of independent random variables often arise in practice. A technique based on moment generating functions usually is much more convenient than using transformations for determining the distribution of sums of independent random variables.

Theorem 5.

If X_1, \dots, X_n are independent random variables with MGFs $M(t)$, then the MGF of $U = \sum_{i=1}^n X_i$ is

$$M_U(t) = M_{X_1}(t) \cdots M_{X_n}(t)$$

The MGF of a random variable uniquely determines its distribution. The MGF approach is particularly useful for determining the distribution of a sum of independent random variables, and it often will be much more convenient than trying to carry out a joint transformation.

Example 18.

Let X_1, \dots, X_k be independent binomial random variables with respective parameters n_i , and p , $X_i \sim \text{Bin}(n_i, p)$ and let $U = \sum_{i=1}^k X_i$. Find the distribution of U .

Example 19.

Let X_1, \dots, X_k be independent Poisson-distributed random variables $X_i \sim POI(\mu_i)$ and let $U = \sum_{i=1}^k X_i$. Find the distribution of U .

Example 20.

Let X_1, \dots, X_k be independent gamma-distributed random variables with respective shape parameters $\alpha_1, \alpha_2, \dots, \alpha_n$ and common scale parameter θ , $X_i \sim GAM(\alpha_i, \theta)$ for $i = 1, \dots, n$ and let $U = \sum_{i=1}^k X_i$. Find the distribution of U .

Example 21.

FRV-Q36

In each of the following, random variable X and Y are independent with the given probability density functions (pdfs). Use moment generating function method to find the probability density functions (pdfs) for $V = X + Y$.

$$(a) \begin{aligned} f(x) &= \binom{15}{x} 0.9^x 0.1^{15-x}, x = 0, 1, \dots, 15 \\ f(y) &= \binom{13}{y} 0.9^y 0.1^{13-y}, y = 0, 1, \dots, 13 \end{aligned}$$

$$(b) \begin{aligned} f(x) &= \frac{e^{-1.0} 1.0^x}{x!}, x = 0, 1, \dots \\ f(y) &= \frac{e^{-3.2} 3.2^y}{y!}, y = 0, 1, \dots \end{aligned}$$

$$(c) \begin{aligned} f(x) &= \frac{1}{4\sqrt{2\pi}} e^{-\frac{1}{32}(x-14)^2}, -\infty < x < \infty \\ f(y) &= \frac{1}{6\sqrt{2\pi}} e^{-\frac{1}{72}(y-6)^2}, -\infty < y < \infty \end{aligned}$$

$$(d) \begin{aligned} f(x) &= \frac{1}{\Gamma(5)3^5} x^4 e^{-\frac{x}{3}}, x > 0 \\ f(y) &= \frac{1}{\Gamma(3)3^5} y^2 e^{-\frac{y}{3}}, y > 0 \end{aligned}$$

2.4 Order Statistics

Let X_1, X_2, \dots, X_n denote independent continuous random variables with distribution function $F(x)$ and density $f(x)$. We denote the ordered random variables X_i by $X_{(1)}, X_{(2)}, \dots, X_{(n)}$, where $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)}$. Using this notation,

$$X_{(1)} = \min(X_1, X_2, \dots, X_n)$$

is the minimum of the random variables X_i , and

$$X_{(n)} = \max(X_1, X_2, \dots, X_n)$$

is the maximum of the random variables X_i .

The probability density functions for $X_{(1)}$ and $X_{(n)}$ can be found using method of distribution functions.

Example 22.

FRV-Q28b

Let X_1, \dots, X_7 be a random sample of size 7 from a distribution $N(210, 10^2)$. Let $U = \max(X_1, X_2, \dots, X_7)$. Find the value of the p.d.f. of U at $u = 229.91$.

ex19

Example 23. Electronic components of a certain type have a length of life X , with probability density given by

$$f(x) = \begin{cases} (\frac{1}{100}e^{-x/100}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

(Length of life is measured in hours.) Suppose that two such components operate independently and in series in a certain system (hence, the system fails when either component fails). Find the density function for U , the length of life of the system.

Example 24.

Suppose that the components in Example ^{ex19}23 operate in parallel (hence, the system does not fail until both components fail). Find the density function for U , the length of life of the system.

Theorem 6.

If X_1, X_2, \dots, X_n is a random sample from a population with continuous pdf, $f(x)$, then the joint pdf of the order statistics, Y_1, Y_2, \dots, Y_n is

$$g(y_1, y_2, \dots, y_n) = \begin{cases} n! f(y_1) f(y_2) \cdots f(y_n), & y_1 < y_2 < \cdots < y_n \\ 0, & \text{otherwise} \end{cases}$$

Theorem 7.

Let X_1, X_2, \dots, X_n denote independent continuous random variables with common distribution function $F(x)$ and common density functions $f(x)$. If $X_{(k)}$ denotes k^{th} – order statistic, then the density function of $X_{(k)}$ is given by

$$g_{(k)}(x_k) = \frac{n!}{k!(n-k)!} [F(x_k)]^{k-1} [1-F(x_k)]^{n-k} f(x_k),$$

$$x_k \in R$$

If j and k are two integers such that $1 \leq j < k \leq n$, the joint density of $X_{(j)}$ and $X_{(k)}$ is given by

$$g_{(j)(k)}(x_j x_k) = \frac{n!}{(j-1)!(k-j-1)!(n-k)!} [F(x_j)]^{j-1} \\ \times [F(x_k) - F(x_j)]^{k-j-1} \\ \times [1 - F(x_k)]^{n-k} f(x_j) f(x_k) \\ -\infty < x_j < x_k < \infty$$

FRV-Q24

Example 25.

A system is composed of 18 independent components. If the pdf of the time to failure of each component is exponential, $X_i \sim EXP(140)$. Suppose that the 18-component system fails when at least 6 components fail. Give the pdf of the time to failure of the system.

Example 26. Suppose that X_1, X_2, \dots, Y_{15} denotes a random sample from a uniform distribution defined on the interval $(0, 1)$. That is,

$$f(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the density function for the second-order statistic. Also, give the joint density function for the second- and fourth-order statistics.

Example 27.

FRV-Q13b

Let X_1 and X_2 be a random sample of size $n = 2$ from a continuous distribution with pdf of the form $f(x) = 4x^3$ if $0 < x < 1$ and zero otherwise.

- (a) Find the joint pdf of $Y_1 = \min(X_1, X_2)$ and $Y_2 = \max(X_1, X_2)$.
- (b) Find the pdf of the sample range $R = Y_2 - Y_1$.

The event that the k^{th} -order statistic at most y , $[Y_k \leq y]$ can occur if and only if at least k of the n observations are less than or equal to y . That is, here the probability of “success” on each trial is $F(y)$ and we must have at least k successes. Thus,

$$P(Y_k \leq y) = \sum_{i=k}^n \binom{n}{i} [F(y)]^i [1 - F(y)]^{n-i}$$

Example 28.

FRV-Q30 Let $X_i \sim \text{Exp}(60)$, $i = 1, \dots, 10$ and $Y_1 < Y_2 < \dots < Y_{10}$ be the order statistics. Compute the probability that Y_8 is less than 130.8.