MEME15203 Statistical Inference

Assignment 1

UNIVERSITI TUNKU ABDUL RAHMAN

Faculty:	FES	Unit Code:	MEME15203
Course:	MAC	Unit Title:	Statistical Inference
Year:	1,2	Lecturer:	Dr Yong Chin Khian
Session:	January 2023		
Due by:	2/3/2023		

Q1. Suppose the joint probability function of X_1 and X_2 is given by

$$p(x_1, x_2) = k, \quad \textbf{for} \quad x_1 = 1, 2, \ldots, 10; x_2 = 1, 2, \ldots, x_1$$

(a) Find k

(b) Find $P(X_1 = 10) + P(X_2 = 7)$.

(c) Find the conditional mean of X_2 given $X_1 = 7$, i.e. find $E(X_2|X_1 = 7)$.

(4 marks)

Q2. The joint density function of X_1 and X_2 is given by

$$f(x_1, x_2) = \begin{cases} cx_1^5 x_2^6, & x_1 - 1 \le x_2 \le 1 - x_1, 0 \le x_1 \le 1 \\ 0, & \text{otherwise} \end{cases},$$

(a)Find c.

(b)Show that the marginal density of X_1 is a beta density with $a = 6$ and $b = 8$.

(c)Derive the conditional density of X_2 given $X_1 = x_1$.

(d)Find $P(X_2 > 0|X_1 = 0.53)$.

(e)Derive the marginal density of X_2.

(10 marks)

Q3. Given that the nonnegative function $g(x)$ has the property that

$$\int_0^\infty g(x)dx = 1,$$

show that

$$f(x_1, x_2) = \frac{2g(\sqrt{x_1^2 + x_2^2})}{\pi\sqrt{x_1^2 + x_2^2}}, 0 < x_1 < \infty, 0 < x_2 < \infty,$$

zero elsewhere, satisfies the conditions for a pdf of two continuous-type random variables X_1 and X_2. *Hint:* Use polar coordinates

(3 marks)

Q4. Suppose X and Y are continuous random variables with joint pdf $f(x, y) = cx^3y^3$ if $x > 0, y > 0$, and $x + y < 1$, and zero otherwise, where c is a constant.

(a) Find c.

(b) find $V(5X + 8Y)$.

(6 marks)

Q5. Let X_1, X_2 be two random variables with joint pdf $f(x_1, x_2) = \frac{1}{8!(50^{10})}x_1^8e^{-x_2/50}$, for $0 < x_1 < x_2 < \infty$, zero otherwise.

(a) Determine the joint mgf of X_1, X_2, $M_{X_1,X_2}(t_1, t_2)$.

(b) Determine the marginal distribution of X_1.

(c) Determine the marginal distribution of X_2.

(7 marks)

Q6. Suppose that $X \sim \chi^2(25)$, $S = X + Y \sim \chi^2(60)$, and X and Y are independent. Use MGFs to find the distrbution of $S - X$.

(4 marks)

Q7. Consider a random sample of size n from an exponential distribution, $X_i \sim EXP(1)$. Derive the pdf of the sample range, $R = Y_n - Y_1$, where $Y_1 = \min(X_1, \ldots, X_n)$ and $Y_n = \max(X_1, \ldots, X_n)$.

(8 marks)

Q8. Let X_1 and X_2 be a random sample of size 2 from a distribution $N(\theta, 2^2)$, and let

$$U = X_1 + X_2 \text{ and } W = X_1 - X_2.$$

(a)Find the joint pdf of U and W.

(b)Find the marginal pdf of U.

(c)Find the marginal pdf of W.

(d)Show that U and W are independent.

(10 marks)

Q9. Let $X_1, \ldots, X_4$ be a random sample of size 4 from a distribution $N(270, 50^2)$. Let $U = \max(X_1, X_2, \ldots, X_4)$, find the value of the p.d.f. of U at $u = 363.75$.

(3 marks)

Q10. Consider a random sample from a Poisson distribution, $X_i \sim POI(\mu)$. Show that $\bar{X}_n e^{-\bar{X}_n}$ converges in probability to a constant, identify the constant.

(3 marks)

Q11. Let $X_1, \ldots, X_n$, be a random sample from a uniform distribution, $X \sim U(0, \theta)$, and let $Y_n = X_{n:n}$ the largest order statistic. Find the limiting distribution of $Z_n = n(\theta - Y_n)$.

(3 marks)

Q12. Consider a random sample from a Exponential distribution, $X_i \sim Exp(\theta)$. Find the asymtotic normal distribution of $Y_n = [\ln(\bar{X}_n)]^4$.

(3 marks)

Q13. Suppose that $W_1, W_2, \ldots$ are iid $Lognormal(\mu, \sigma)$. Let $V_n = W_1 \times W_2 \times \cdots \times W_n$. Both $(V_n)^{1/n}$ and $(V_n)^{1/n^2}$ converge in probability to constants. Identify those constants. .

(3 marks)

Q14. Let the random variable Y_n have a distribution that is $Bin(n, p)$. Prove that $\left(\frac{Y_n}{n}\right)\left(1 - \frac{Y_n}{n}\right)$ converges in probability to a constant, identify the constant.

(3 marks)

Q15. Let $\bar{X}_n$ denote the mean of a random sample of size n from a Poisson distribution with parameter μ. Determine the limiting distribution of $Y_n = \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sqrt{\mu}}$.

(3 marks)

Q16. Let $Y_n \sim \chi^2(n)$. Find the limiting distribution of $\frac{Y_n - n}{\sqrt{2n}}$ as $n \to \infty$, using moment generating function.

(3 marks)

Q17. Suppose that $X_i \sim N(\mu, \sigma^2), i = 1, \ldots, 21$ and $Z_i \sim N(0, 1), i = 1, \ldots, 28$, $W_i \sim \chi^2(11), i = 1, \ldots, 11$, $Y_i \sim EXP(130), i = 1, \ldots, 7$, and all variables are independent. State the distribution of each of the following variables if it is a "named" distribution or otherwise state "unknown."

(a) $\frac{3X_1 + 5X_2 - 8\mu}{\sigma S_Z \sqrt{34}}$

(b) $\frac{11 Z_1^2}{W_1}$

(c) $\frac{\sqrt{588}(\bar{X} - \mu)}{\sigma \sqrt{\sum_{i=1}^{28} Z_i^2}}$

(d) $\frac{\sum_{i=1}^{21}(X_i - \mu)^2}{\sigma^2} + \sum_{i=1}^{28}(Z_i - \bar{Z})^2 + \sum_{i=1}^{11} W_i$

(e) $\frac{(27)\sum_{i=1}^{21}(X_i - \bar{X})^2}{(20)\sigma^2 \sum_{i=1}^{28}(Z_i - \bar{Z})^2}$

(f) $\frac{2\sigma^2 (20) \sum_{i=1}^{7} Y_i}{130 \sum_{i=1}^{21}(X_i - \bar{X})^2}$

(12 marks)

Q18. Suppose $Y \sim Beta(a = 8, b = 6)$, use the relationship between Beta distribution and F distribution, find $P[Y > 0.388]$.

(3 marks)

Q19. Suppose $Y \sim Beta(a = 4, b = 6)$, use the relationship between Beta distribution and F distribution, find 90^{th} percentile of Y.

(3 marks)

Q20. Suppose that $X_i \sim N(\mu, \sigma^2), i = 1, \ldots, 17,\ Z_j \sim N(0, 1), j = 1, \ldots, 28$, and $W_k \sim \chi^2(v), k = 1, \ldots, 16$ and all random variables are independent. State the distribution of each of the following variables if it is a "named" distribution. [For example $X_1 + X_2 \sim N(2\mu, 2\sigma^2)$]

(a) $\frac{27\sum_{i=1}^{17}(X_i - \bar{X})^2}{16\sigma^2\sum_{j=1}^{28}(Z_j - \bar{Z})^2}$.

(b) $\frac{W_1}{\sum_{k=1}^{28} W_k}$

(c) $\frac{\bar{X}}{\sigma^2} + \frac{\sum_{j=1}^{28} Z_i}{28}$

(6 marks)