MEME16203Linear Models Marking Guide

Assignment 1

UNIVERSITI TUNKU ABDUL RAHMAN

Faculty:	FES	Unit Code:	MEME15203
Course:	MAC	Unit Title:	Statistical Inference
Year:	1,2	Lecturer:	Dr Yong Chin Khian
Session:	January 2022		
Due by:			

Q1. Suppose that X and Y have joint probability density function (pdf)

$$f(x,y) = \begin{cases} \frac{2}{3^3}(x+y), & 0 \le x \le y \le 3 \\ 0, \text{otherwise} \end{cases}$$

Find $P[Y < 4X]$.

(5 marks)

Ans.

$P[Y < 4X]$
$= \frac{2}{3^3}\int_0^3 \int_{\frac{y}{4}}^{y} (x+y)dxdy$
$= \frac{2}{3^3}\int_0^3 \left[\frac{x^2}{2} + xy\right]_{\frac{y}{4}}^{y} dy$
$= \frac{2}{3^3}\int_0^3 \left[\frac{y^2}{2} + y^2 - \frac{y^2/4^2}{2} - \frac{y^2}{4}\right] dy$
$= \frac{78}{864}\int_0^3 y^2 dy$
$= \frac{78}{864}\left[\frac{y^3}{3}\right]_0^3$
$= \frac{78}{864}\left[\frac{3^3}{3}\right]$
$= \boxed{0.8125}$

Q2. The random variable X_1 has an exponential distribution with mean 2. The random variable X_2 is related to X_1 in such a way that $E(X_2|x_1) = 2x_1$ and $V(X_2|x_1) = 3x_1^2$. Find $V(5X_1 + 3X_2)$.

(10 marks)

Ans.

$E(X_1) = 2$, $V(X_1) = 2^2 = 4$
$E(X_1^2) = V(X_1) + E^2(X_1) = 2^2 + 2^2 = 2(2^2) = 8$

$E(X_2) = E[E(X_2|x_1)] = E(2X_1) = 2(2) = 4$
$E(X_1X_2) = E[E(X_1X_2|x_1)] = E[X_1E(X_2|x_1)] = E[X_1(2X_1] = 2E(X_1^2) = 2[8] = 16$
$Cov(X_1, X_2) = E(X_1X_2) - E(X_1)E(X_2) = 16 - 2(4) = 8$
$V(X_2) = E[V(X_2|x_1)]+V[E(X_2|x_1)] = E(3X_1^2)+V(2X_1) = 3E(X_1^2)+2^2V(X_1) = 3(8) + 2^2(4) = 40$
$V(5X_1+3X_2) = 5^2V(X_1)+3^2V(X_2)+2(5)(3)Cov(X_1, X_2) = 5^2(4)+(3^2)(40)+2(5)(3)((8)) = \boxed{700}$

Q3. Let X_1, X_2 be two random variables with joint pdf $f(x_1, x_2) = x_1e^{-x_2}$, for $0 < x_1 < x_2 < \infty$, zero otherwise. Determine the joint mgf of X_1, X_2. Does $M(t_1, t_2) = M(t_1, 0)M(0, t_2)$?

(10 marks)

Ans.

$M(t_1, t_2)$
$= E(e^{t_1X_1+t_2X_2})$
$= \int_0^\infty \int_{x_1}^\infty e^{t_1x_1+t_2x_2}x_1e^{-x_2}dx_2dx_1$
$= \int_0^\infty x_1e^{t_1x_1} \int_{x_1}^\infty e^{-x_2(1-t_2)}dx_2dx_1$
$= \int_0^\infty x_1e^{t_1x_1} \frac{e^{-x_1(1-t_2)}}{1-t_2}dx_1$
$= \frac{1}{1-t_2} \int_0^\infty x_1e^{-x_1(1-t_1-t_2)}dx_1$
$= \frac{1}{(1-t_2)(1-t_1-t_2)^2}$
provided that $t_1 + t_2 < 1$ and $t_2 < 1$.
$M_{X_1}(t_1) = M(t_1, 0) = \frac{1}{(1-t_1)^2}$ provided that $t_1 < 1$.
$M_{X_2}(t_2) = M(0, t_2) = \frac{1}{(1-t_2)^3}$ provided that $t_2 < 1$.Thus
$M(t_1, t_2) \neq M(t_1, 0)M(0, t_2)$

Q4. Suppose $P[\mu = 1] = 0.3$ and $P[\mu = 2] = 0.7$, and that conditional on μ, $X|\mu \sim POI(\mu)$. Find $V(4X - 4\mu)$.

(5 marks)

Ans.

$E(\mu) = 1(0.3) + 2(0.7) = 1.7$
$E(X) = E[E(X|\mu)] = E(\mu) = 1.7$
$E(\mu^2) = 1^2(0.3) + 2^(0.7) = 3.1$
$V(\mu) = E(\mu^2) - E^2(\mu) = 3.1 - 1.7^2 = 0.21$
$V(X) = E[V(X|\mu)] + V[E(X|\mu)] = E(\mu) + V(\mu) = 1.7 + 0.21 = 1.91$
$Cov(\mu, X) = E(\mu X) - E(X)E(\mu) = 3.1 - (1.7)(1.7) = 0.21$

$V(4X-4\mu) = 4^2V(X)+4^2V(\mu)-2(4)(4)Cov(X,\mu) = 4^2(1.91)+4^2(0.21)-2(4)(4)(0.21) = \boxed{27.2}$

Q5. Let X and Y have joint pdf $f(x,y) = cy^2e^{-6y}, 0 < x < y < \infty$ and zero otherwise.

(a)Find the joint pdf of $S = X+Y$ and $T = X$.

(b)Find the marginal pdf of T.

(c)Find the marginal pdf of S.

(15 marks)

Ans.

(a) $\int_0^\infty \int_0^y cy^2e^{-6y}dxdy = 1$

$c\int_0^\infty [y^3e^{-6y}]dy = 1$

$c\left(\Gamma(4)\frac{1}{6^4}\right) = 1$

$c = \frac{6^4}{\Gamma(4)}$

$f(x,y) = \frac{6^4}{\Gamma(4)}y^2e^{-6y}, 0 < x < y < \infty$

Let $T = X$ and $S = X+Y$. Then this corresponds to the transformation $X = T$ and $Y = S-T$ which have unique solutions $h_1(t,s) = x = t$ and $h_2(t,s) = y = s-t$,

$J = \begin{vmatrix} \frac{\partial h_1}{\partial t} & \frac{\partial h_1}{\partial s} \\ \frac{\partial h_2}{\partial t} & \frac{\partial h_2}{\partial s} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$

$f_{T,S}(t,s) = f_{X,Y}(t,s-t)|J| = \frac{6^4}{\Gamma(4)}(s-t)^2e^{-6(s-t)},\ 0 < 2t < s < \infty$

(b) $f_T(t) = \int_{2t}^\infty f_{T,S}(t,s)ds$

$= \int_{2t}^\infty \frac{6^4}{\Gamma(4)}(s-t)^2e^{-6(s-t)}ds$

Let $v = s-t$, $dv = ds$

$= \int_t^\infty \frac{6^4}{\Gamma(4)}v^2e^{-6v}dv$

$= \left(\frac{6^4}{\Gamma(4)}\right)\left(\frac{\Gamma(3)}{6^3}\right)\int_t^\infty \frac{6^3}{\Gamma(3)}v^2e^{-6v}dv$

$= \left(\frac{6}{3}\right)P(S_3 > t)$ where $S_3 \sim GAM(\alpha = 3, \theta = \frac{1}{6})$

$= \frac{6}{3}\left[e^{-6t}\left(\sum_{i=0}^2 \frac{(6t)^i}{i!}\right)\right]$

$= \frac{6}{3}\left[e^{-6t}\left(1+6t+\frac{1}{2}(6t)^2\right)\right], t > 0$

(c) $f_S(s) = \int_0^{s/2} f_{T,S}(t,s)dt$

$= \int_0^{s/2} \frac{6^4}{\Gamma(4)}(s-t)^2e^{-6(s-t)}dt$

Let $v = s-t$, then $dv = -dt$

$= \int_s^{s/2} \frac{6^4}{\Gamma(4)}v^2e^{-6v}(-dv)$

$= \left(\frac{6^4}{\Gamma(4)}\right)\left(\frac{\Gamma(3)}{6^3}\right)\int_{s/2}^s \frac{6^3}{\Gamma(3)}v^2e^{-6v}dv$

$= \left(\frac{6}{3}\right)[P(S_3 > s/2) - P(S_3 > s)]$ where $S_3 \sim GAM(\alpha = 3, \theta = \frac{1}{6})$

$$= \left(\frac{6}{3}\right)\left[e^{-6s/2}\left(\sum_{i=0}^{2}\frac{(6s/2)^i}{i!}\right) - e^{-6s}\left(\sum_{i=0}^{2}\frac{(6s)^i}{i!}\right)\right], s > 0$$

Q6. Let X be a random variable with a density function given by

$$f(x) = \begin{cases} \frac{3}{2}x^2, & -1 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the density function of $U_1 = 7X$ using distribution method.

(b) Find the density function of $U_2 = 7 - X$ using one to one transformation.

(10 marks)

Ans.

(a) $F_U(u) = P(U \le u) = P(7X \le u) = P(X \le \frac{u}{7}) = \displaystyle\int_{-1}^{u/7} \frac{3}{2}x^2 dx = \frac{3}{2}\left[\frac{x^3}{3}\right]_{-1}^{u/7} = \frac{1}{2}[(\frac{u}{7})^3 - 1]$

$f_U(u) = F'(u) = \frac{1}{2}\left[3(\frac{u}{7})^2\right] = \frac{3u^2}{686}, -7 < u < 7$, zero otherwise.

(b) $u = 7 - x$ corresponds to a one to one transformation with unique solution of $x = w(u) = 7 - u$, $w'(u) = -1$. $f_U(u) = f_X(w(u))|w'(u)| = \frac{3}{2}(7-u)^2(|-1|) = \frac{3}{2}(7-u)^2, 6 < u < 8$, zero otherwise.

Q7. A member of the power family of distributions has a distribution function given by

$$F(x) = \begin{cases} 0, & x < 0 \\ (\frac{x}{\theta})^\alpha, & 0 \le x \le \theta \\ 1, & x > \theta \end{cases}$$

where $\alpha, \theta > 0$.

(a) For fixed values of α and θ, find a transformation $G(U)$ so that $G(U)$ has a distribution function of F when U possesses a uniform $(0, 1)$ distribution.

(b) Given that a random sample of size 5 from a uniform distribution on the interval $(0, 1)$ yielded the values $u_1 = 0.027$, $u_2 = 0.06901$, $u_3 = 0.01413$, $u_4 = 0.01523$, and $u_5 = 0.03609$, use the transformation derived in the above result to give values associated with a random variable with a power family distribution with $\alpha = 2$, $\theta = 4$.

(10 marks)

Ans.

Let $W = G(U)$, $F_X(x) = u = \frac{x}{\theta})^{\alpha}$, $x = \theta u^{1/\alpha}$. So, $G(u) = \theta U^{1/\alpha}$

Then values are:

- $x_1 = 4(0.027)^{1/2} = 0.65727$
- $x_2 = 4(0.06901)^{1/2} = 1.05079$
- $x_3 = 4(0.01413)^{1/2} = 0.47548$
- $x_4 = 4(0.01523)^{1/2} = 0.49364$
- $x_5 = 4(0.03609)^{1/2} = 0.75989$

Q8. Let X_1 and X_2 be independent random variables with $X_1 \sim GAM(\alpha_1 = a, \theta = 2)$ and $X_2 \sim GAM(\alpha_2 = b, \theta = 2)$, show that $U = \frac{X_1}{X_1+X_2}$ follow a Beta distribution. Suppose $Y_i \sim GAM(\alpha = 7, \theta = 2)$, using the result above, find the distribution of $V = \frac{Y_1}{\sum_{i=1}^{20} Y_i}$.

(10 marks)

Ans.

$$f_{X_1,X_2}(x_1, x_2) = \frac{1}{\Gamma(a)\theta^a} x_1^{a-1} e^{-\frac{x_1}{\theta}} \frac{1}{\Gamma(b)\theta^b} x_2^{b-1} e^{-\frac{x_2}{\theta}}, -\infty < x_1 < \infty < x_2 < \infty.$$

Let $u = x_1/(x_1 + x_2)$, $v = x_1 + x_2$, then $x_1 = uv$, $x_2 = v - uv$

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial u} & \frac{\partial x_1}{\partial v} \\ \frac{\partial x_2}{\partial u} & \frac{\partial x_2}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ -v & 1-u \end{vmatrix} = v$$

The set $(x_1 > 0, x_2 > 0)$ is mapped to the set $(0 < u < 1, v > 0)$

$f_{U,V}(u, v)$

$= f_{X_1,X_2}(uv, v - uv)|J|$

$= \dfrac{1}{\Gamma(a)\theta^a}(uv)^{a-1} e^{-\frac{uv}{\theta}} \dfrac{1}{\Gamma(b)\theta^b}(v - uv)^{b-1} e^{-\frac{u-uv}{\theta}} |v|$

$= \frac{1}{\Gamma(a)\Gamma(b)\theta^{a+b}} v^{a+b-1} e^{-\frac{v}{\theta}} u^{a-1}(1-u)^{b-1}, 0 < u < 1, v > 0$

$f_U(u)$

$= \int_0^{\infty} \frac{1}{\Gamma(a)\Gamma(b)} v^{a+b-1} e^{-\frac{v}{\theta}} u^{a-1}(1-u)^{b-1} dv$

$= \frac{1}{\Gamma(a)\Gamma(b)\theta^{a+b}} [\Gamma(a+b)\theta^{a+b}] u^{a-1}(1-u)^{b-1}$

$$= \begin{cases} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} u^{a-1}(1-u)^{b-1}, & 0 < u, 1 \\ 0, & \text{otherwise} \end{cases}$$

Hence, $U = X_1/(X_1 + X_2)$ is $Beta(a, b)$

$V = \frac{Y_1}{\sum_{i=1}^{20} Y_i} = \frac{Y_1}{Y_1 + \sum_{i=2}^{20} Y_i}$

Here, $Y_1 \sim Gamma(\alpha = 7, 2)$ and $\sum_{i=2}^{20} Y_i \sim Gamma(133, 2)$, thus, $V \sim Beta(a = 7, b = 133)$

Q9. Consider a random sample of size n from an exponential distribution, $X_i \sim EXP(1)$. Derive the pdf of the sample range, $R = Y_n - Y_1$, where $Y_1 = \min(X_1, \ldots, X_n)$ and $Y_n = \max(X_1, \ldots, X_n)$.

(10 marks)

Ans.

$f(x) = e^{-x}, x > 0$
$F(x) = 1 - e^{-x}, x > 0$
$f_{Y_1,Y_n}(y_1, y_n)$
$= \frac{n!}{(n-2)!} f(y_1)[F(y_n) - F(y_1)]^{n-2} f(y_n)$
$= \frac{n!}{(n-2)!} e^{-y_1}[e^{-y_1} - e^{-y_n}]^{n-2} e^{-y_n}, y_1 > 0, y_n > 0$

Making the tranformation $R = Y_n - Y_1$, $S = Y_1$, yields the inverse tranformation $y_1 = s$, $y_n = r + s$, and $|J| = 1$. Thus the joint pdf of R and S is

$f_{R,S}(r, s)$
$= f_{Y_1,Y_n}(s, s + t)|J|$
$= \frac{n!}{(n-2)!} e^{-s}[e^{-s} - e^{-(r+s)}]^{n-2} e^{-(r+s)}$
$= \frac{n!}{(n-2)!} e^{-r} e^{-2s}[e^{-s}(1 - e^{-r})]^{n-2}$
$= \frac{n!}{(n-2)!} e^{-r}[1 - e^{-r})]^{n-2} e^{-ns}, r > 0, s > 0$

$$\begin{aligned} f_R(r) &= \tfrac{n!}{(n-2)!} e^{-r}[1 - e^{-r})]^{n-2} \int_0^\infty e^{-ns} ds \\ &= \tfrac{n!}{(n-2)!} e^{-r}[1 - e^{-r})]^{n-2} \tfrac{1}{n} \\ &= (n-1)e^{-r}[1 - e^{-r})]^{n-2} \end{aligned}$$

Q10. Suppose that $X \sim \chi^2(23)$, $S = X + Y \sim \chi^2(62)$, and X and Y are independent. Use MGFs to find the distrbution of $S - X$.

(5 marks)

Ans.

$S - X = X + Y - X = Y$
$M_X(t) = (1 - 2t)^{-23/2}$,
$M_S(t) = (1 - 2t)^{-62/2}$
$M_S(t) = M_X(t)M_Y(t)$
$(1 - 2t)^{-62/2} = (1 - 2t)^{-23/2} M_Y(t)$
$M_Y(t) = (1 - 2t)^{-39/2}$
$\Rightarrow Y = S - X \sim \chi^2(39)$

Q11. Suppose that $X_1, \ldots, X_n$, is a random sample from a Pareto distribution, $X \sim PAR(\alpha = 1, \theta = 25)$. Let $Y_n = 1/nX_{n:n}$, find the limiting distribution of Y_n, $F(y)$, state the distribution and it's parameter, then find $F(28.6)$.

(5 marks)

Ans. $F_X(x) = 1 - \frac{x}{(x+25)} = \frac{25}{x+25}$
$F_n(y)$
$= P(1/nX_{n:n} \le y)$
$= P(X_{n:n} \le ny)$
$= [F_X(ny)]^n$
$= [\frac{25}{(ny+25)}]^n$
$= \left[1 + \frac{25}{ny}\right]^{-n}$
$\lim_{n\to\infty} F_n(y)$
$$= \lim_{n\to\infty} \left[1 + \frac{25}{ny}\right]^{-n}$$
$= e^{-25/y}, y > 0$
$\Rightarrow F(y) \sim InvEXP(25)$
$F(28.6) = e^{-25/28.6} = \boxed{0.4172}$

Q12. Consider a random sample from a Geometric distribution, $X_i \sim GEO(p)$. Let $W_i = e^{X_i}$ and $V_n = W_1 \times W_2 \times \cdots W_n$. $V_n^{\frac{1}{n}}$ converges in probability to a constant, identify the constant.

(5 marks)

Ans.

$E(\bar{X}_n) = \frac{1}{p}$, $V(\bar{X}_n) = \frac{1}{n}V(X) = \frac{1-p}{np^2}$
$P\left[|\overline{X}_n - \frac{1}{p}| \ge \epsilon\sqrt{\frac{1-p}{np^2}}\sqrt{\frac{np^2}{1-p}}\right] < \frac{(1-p)}{np^2\epsilon^2} \to 0$
$\therefore \bar{X}_n \xrightarrow{P} \frac{1}{p}$
$(V_n)^{1/n} = (W_1 \times W_2 \times \cdots \times W_n)^{1/n} = e^{\bar{X}_n}$
Thus, $(V_n)^{1/n} \xrightarrow{P} e^{\frac{1}{p}}$