CONTENTS

4 Tests of Hypotheses and Confidence Intervals

4.1 Test of Hypotheses

Consider the linear model with

$$E(\mathbf{Y}) = \mathbf{X}\boldsymbol{\beta} \text{ and } Var(\mathbf{Y}) = \Sigma$$

This can also be expressed as

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$$

where $E(\boldsymbol{\epsilon}) = \mathbf{0}$ and $Var(\boldsymbol{\epsilon}) = \Sigma$.

Typical null hypothesis (H_0)

- is a status quo or prevailing viewpoint about a population
- specifies the values for one or more elements of $\boldsymbol{\beta}$
- specifies the values for some linear functions of the elements of $\boldsymbol{\beta}$

An alternative hypothesis (H_1)

- is an alternative to the null hypothesis – the change in the population that the researcher hopes is true

We may test

$$H_0 : \mathbf{C}\boldsymbol{\beta} = \mathbf{d} \quad \text{vs} \quad H_1 : \mathbf{C}\boldsymbol{\beta} \neq \mathbf{d}$$

where

- $\mathbf{C}$ is an $m \times k$ matrix of constants
- $\mathbf{d}$ is an $m \times 1$ vector of constants

The null hypothesis is rejected if it is shown to be sufficiently incompatible with the observed data.

Failing to reject H_0 is **not** the same as proving H_0 is true.

- too little data to accurately estimate $\mathbf{C}\boldsymbol{\beta}$
- relatively large variation in $\boldsymbol{\epsilon}$ (or $\mathbf{Y}$)
- if $H_0 : \mathbf{C}\boldsymbol{\beta} = \mathbf{d}$ is false, $\mathbf{C}\boldsymbol{\beta} - \mathbf{d}$ may be "small"

You can never be completely sure that you made the correct decision

- Type I error (significance level):
  $P(H_0 \text{ is rejected} | H_0 \text{ is true})$
- Type II error:
  $P(H_0 \text{ is rejected} | H_0 \text{ is true})$

Basic considerations in specifying a null hypothesis $H_0 : \mathbf{C}\boldsymbol{\beta} = \mathbf{d}$

(i) $\mathbf{C}\boldsymbol{\beta}$ should be estimable.

(ii) Inconsistencies should be avoided, i.e., $\mathbf{C}\boldsymbol{\beta} = \mathbf{d}$ should be a consistent set of equations

(iii) Redundancies should be eliminated, i.e., in $\mathbf{C}\boldsymbol{\beta} = \mathbf{d}$ we should have

$$\text{rank}(\mathbf{C}) = \text{number-of-rows-in-}\mathbf{C}$$

4.2 Hypothesis Tests for Estimable Function

Consider the following effects models:

$$Y_{ij} = \mu + \alpha_i + \epsilon_{ij} \quad i = 1, 2, 3 \\ \qquad\qquad\qquad j = 1, \ldots, n_i$$

In this case

$$\begin{bmatrix} Y_{11} \\ Y_{12} \\ Y_{21} \\ Y_{31} \\ Y_{32} \\ Y_{33} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} + \begin{bmatrix} \epsilon_{11} \\ \epsilon_{12} \\ \epsilon_{21} \\ \epsilon_{31} \\ \epsilon_{32} \\ \epsilon_{33} \end{bmatrix}$$

4.2.1 The Mean Response For Any Treatments

By definition

$$E(Y_{ij}) = \mu + \alpha_i \text{ is estimable.}$$

We can test

$$H_0 : \mu + \alpha_1 = 60 \text{ seconds}$$

against

$$H_1 : \mu + \alpha_1 \neq 60 \text{ seconds}$$

(two-sided alternative)

Or we can test

$$H_0 : \mu + \alpha_1 = 60 \text{ seconds}$$

against

$$H_1 : \mu + \alpha_1 < 60 \text{ seconds}$$

(one-sided alternative)

In this case

$$\mu + \alpha_1 = \mathbf{c}^T \boldsymbol{\beta} \quad \text{where} \quad \mathbf{c} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Note that this quantity is estimable, i. e.,

$$\mathbf{c}^T\boldsymbol{\beta} = \mu + \alpha_1 = E\left[\left(\frac{1}{2}\;\frac{1}{2}\;0\;0\;0\;0\right)\mathbf{Y}\right].$$

Then, any solution

$$\mathbf{b} = (\mathbf{X}^T\Sigma^{-1}\mathbf{X})^{-}\mathbf{X}^T\Sigma^{-1}\mathbf{Y}$$

to the generalized least squares estimating equations

$$\mathbf{X}^T\Sigma^{-1}\mathbf{X}\mathbf{b} = \mathbf{X}^T\Sigma^{-1}\mathbf{Y}$$

yields the **same value** for $\mathbf{c}^T\mathbf{b}$ and it is the **unique blue** for $\mathbf{c}^T\boldsymbol{\beta}$.

We will reject $H_0 : \mathbf{c}^T\boldsymbol{\beta} = 60$ if

$$\mathbf{c}^T\mathbf{b} = \mathbf{c}^T(\mathbf{X}^T\Sigma^{-1}\mathbf{X})^{-}\mathbf{X}^T\Sigma^{-1}\mathbf{Y}$$

is too far away from 60.

If $\text{Var}(\mathbf{Y}) = \sigma^2 I$, then any solution

$$\mathbf{b} = (\mathbf{X}^T\mathbf{X})^{-}\mathbf{X}^T\mathbf{Y}$$

to the least squares estimating equations

$$\mathbf{X}^T\mathbf{X}\mathbf{b} = \mathbf{X}^T\mathbf{Y}$$

yields the **same value** for $\mathbf{c}^T\mathbf{b}$, and $\mathbf{c}^T\mathbf{b}$ is the **unique blue** for $\mathbf{c}^T\boldsymbol{\beta}$.

We will reject $H_0 : \mathbf{c}^T\boldsymbol{\beta} = 60$ if

$$\mathbf{c}^T\mathbf{b} = \mathbf{c}^T(\mathbf{X}^T\mathbf{X})^{-}\mathbf{X}^T\mathbf{Y}$$

is too far away from 60.

4.2.2 Difference between the mean response for two treatments

$$\alpha_1 - \alpha_3 = (\mu + \alpha_1) - (\mu + \alpha_3)$$
$$= \left(\frac{1}{2}\ \frac{1}{2}\ 0\ \frac{-1}{3}\ \frac{-1}{3}\ \frac{-1}{3}\right) E(\mathbf{Y})$$

and we can test

$$H_0 : \alpha_1 - \alpha_3 = 0 \quad \text{vs.} \quad H_1 : \alpha_1 - \alpha_3 \neq 0$$

If $\mathrm{Var}(\mathbf{Y}) = \sigma^2 I$, the unique blue for

$$\alpha_1 - \alpha_3 = (0\ 1\ 0\ -1)\boldsymbol{\beta} = \mathbf{c}^T\boldsymbol{\beta}$$

is

$$\mathbf{c}^T\mathbf{b} \text{ for any } \mathbf{b} = (\mathbf{X}^T\mathbf{X})^{-}\mathbf{X}^T\mathbf{Y}$$

Reject $H_0 : \alpha_1 - \alpha_3 = \mathbf{c}^T\boldsymbol{\beta} = 0$ if $\mathbf{c}^T\mathbf{b}$ is too far from 0.

4.2.3 Non Estimable Functions

It would not make much sense to attempt to test

$$H_0 : \alpha_1 = 3 \quad \text{vs.} \quad H_1 : \alpha_1 \neq 3$$

because $\alpha_1 = [0\ 1\ 0\ 0]\boldsymbol{\beta} = \mathbf{c}^T\boldsymbol{\beta}$ is not estimable.

- Although $E(Y_{1j}) = \mu + \alpha_1$ neither μ nor α_1 has a clear interpretation.
- Different solutions to the normal equations produce different values for
$$\hat{\alpha}_1 = \mathbf{c}^T\mathbf{b} = \mathbf{c}^T(\mathbf{X}^T\mathbf{X})^{-}\mathbf{X}^T\mathbf{Y}$$
- To make a statement about α_1, an additional restriction must be imposed on the parameters in the model to give α_1 a precise meaning.

4.3 Consistencies and Redundancies

For $\mathbf{C} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$, consider testing

$$H_0 : \mathbf{C}\boldsymbol{\beta} = \begin{bmatrix} -3 \\ 60 \\ 70 \end{bmatrix} \text{ vs. } H_1 : \mathbf{C}\boldsymbol{\beta} \neq \begin{bmatrix} -3 \\ 60 \\ 70 \end{bmatrix}$$

In this case $\mathbf{C}\boldsymbol{\beta}$ is estimable, but there is an inconsistency. If the null hypothesis is true,

$$\mathbf{C}\boldsymbol{\beta} = \begin{bmatrix} \alpha_1 - \alpha_3 \\ \mu + \alpha_1 \\ \mu + \alpha_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 60 \\ 70 \end{bmatrix}$$

Then $\mu + \alpha_1 = 60$ and $\mu + \alpha_3 = 70$ implies

$$\begin{aligned}(\alpha_1 - \alpha_3) &= (\mu + \alpha_1) - (\mu + \alpha_3) \\ &= 60 - 70 \\ &= \mathbf{-10}\end{aligned}$$

Such inconsistencies should be avoided.

For $\mathbf{C} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

consider testing

$$H_0 : \mathbf{C}\boldsymbol{\beta} = \begin{bmatrix} -10 \\ 60 \\ 70 \end{bmatrix} \text{ vs. } H_1 : \mathbf{C}\boldsymbol{\beta} \neq \begin{bmatrix} -10 \\ 60 \\ 70 \end{bmatrix}$$

In this case $\mathbf{C}\boldsymbol{\beta}$ is estimable and the equations specified by the null hypothesis are consistent.

There is a redundancy

$$[1\ 1\ 0\ 0]\ \boldsymbol{\beta} = \mu + \alpha_1 = 60$$
$$[1\ 0\ 0\ 1]\ \boldsymbol{\beta} = \mu + \alpha_3 = 70$$

imply that

$$\begin{aligned}[0\ 1\ 0\ \ -1]\,\boldsymbol{\beta} &= \alpha_1 - \alpha_3 \\ &= (\mu + \alpha_1) - (\mu + \alpha_3) \\ &= 60 - 70 \\ &= -10\end{aligned}$$

The rows of $\mathbf{C}$ are not linearly independent, i.e., $\text{rank}(\mathbf{C}) <$ number of rows in $\mathbf{C}$.
There are many equivalent ways to remove a redundancy:

$$H_0 : \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \boldsymbol{\beta} = \begin{bmatrix} 60 \\ 70 \end{bmatrix}$$

$$H_0 : \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \boldsymbol{\beta} = \begin{bmatrix} -10 \\ 60 \end{bmatrix}$$

$$H_0 : \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \boldsymbol{\beta} = \begin{bmatrix} -10 \\ 70 \end{bmatrix}$$

$$H_0 : \begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 1 & 0 & 1 \end{bmatrix} \boldsymbol{\beta} = \begin{bmatrix} 50 \\ 130 \end{bmatrix}$$

are all equivalent.

In each case:

- The two rows of $\mathbf{C}$ are linearly independent and

  $$\begin{aligned} \text{rank}(\mathbf{C}) &= 2 \\ &= \text{number of rows in } \mathbf{C} \end{aligned}$$

- The two rows of $\mathbf{C}$ are a basis for the same 2-dimensional subspace of R^4.

  This is the 2-dimensional space spanned by the rows of

  $$\mathbf{C} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

We will only consider null hypotheses of the form

$$H_0 : \mathbf{C}\boldsymbol{\beta} = \mathbf{d}$$

where $\text{rank}(\mathbf{C}) =$ number of rows in $\mathbf{C}$. This leads to the following concept of a "testable" hypothesis.

4.4 Testable Hypothesis

Definition 1.
Consider a linear model

$$E(\mathbf{Y}) = \mathbf{X}\boldsymbol{\beta}$$

where

$$V(\mathbf{Y}) = \Sigma$$

and $\mathbf{X}$ is an $n \times k$ matrix. For an $m \times k$ matrix of constants $\mathbf{C}$ and an $m \times 1$ vector of constants $\mathbf{d}$, we will say that

$$H_0 : \mathbf{C}\boldsymbol{\beta} = \mathbf{d}$$

is **testable** if

(i) $\mathbf{C}\boldsymbol{\beta}$ is estimable

(ii) $\text{rank}(\mathbf{C}) = m =$ number of rows in $\mathbf{C}$

To test $H_0 : \mathbf{C}\boldsymbol{\beta} = \mathbf{d}$

(i) Use the data to estimate $\mathbf{C}\boldsymbol{\beta}$.

(ii) Reject $H_0 : \mathbf{C}\boldsymbol{\beta} = \mathbf{d}$ if the estimate of $\mathbf{C}\boldsymbol{\beta}$ is to far away from $\mathbf{d}$.

- How much of the deviation of the estimate of $\mathbf{C}\boldsymbol{\beta}$ from $\mathbf{d}$ can be attributed to random errors?
- Need a probability distribution for the estimate of $\mathbf{C}\boldsymbol{\beta}$
- Need a probability distribution for a test statistic

4.5 Normal Theory Gauss-Markov Model

$$\mathbf{Y} = \begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix} \sim N(\mathbf{X}\boldsymbol{\beta},\ \sigma^2 I)$$

A least squares estimator $\mathbf{b}$ for $\boldsymbol{\beta}$ minimizes

$$(\mathbf{Y} - \mathbf{X}\mathbf{b})^T(\mathbf{Y} - \mathbf{X}\mathbf{b})$$

For any generalized inverse of $\mathbf{X}^T\mathbf{X}$,

$$\mathbf{b} = (\mathbf{X}^T\mathbf{X})^-\mathbf{X}^T\mathbf{Y}$$

is a solution to the normal equations

$$(\mathbf{X}^T\mathbf{X})\mathbf{b} = \mathbf{X}^T\mathbf{Y} .$$

Result 1. Results for the Gauss-Markov model

For a testable null hypothesis

$$H_0 : \mathbf{C}\boldsymbol{\beta} = \mathbf{d}$$

the OLS estimator for $\mathbf{C}\boldsymbol{\beta}$,

$$\mathbf{C}\mathbf{b} = \mathbf{C}(\mathbf{X}^T\mathbf{X})^-\mathbf{X}^T\mathbf{Y} ,$$

has the following properties:

(i) Since $\mathbf{C}\boldsymbol{\beta}$ is estimable, $\mathbf{C}\mathbf{b}$ is invariant to the choice of $(\mathbf{X}^T\mathbf{X})^-$.

(ii) Since $\mathbf{C}\boldsymbol{\beta}$ is estimable, $\mathbf{C}\mathbf{b}$ is the unique BLUE for $\mathbf{C}\boldsymbol{\beta}$.

(iii) $E(\mathbf{C}\mathbf{b} - \mathbf{d}) = \mathbf{C}\boldsymbol{\beta} - \mathbf{d}$

(iv) $V(\mathbf{Cb}-\mathbf{d}) = V(\mathbf{Cb}) = \sigma^2\mathbf{C}(\mathbf{X}^T\mathbf{X})^-\mathbf{C}^T$

(v) $\mathbf{Cb}-\mathbf{d} \sim N(\mathbf{C}\boldsymbol{\beta}-\mathbf{d}, \sigma^2\mathbf{C}(\mathbf{X}^T\mathbf{X})^-\mathbf{C}^T)$

(vi) When $H_0 : \mathbf{C}\boldsymbol{\beta} = \mathbf{d}$ is true,

$$\mathbf{Cb}-\mathbf{d} \sim N(\mathbf{0}, \sigma^2\mathbf{C}(\mathbf{X}^T\mathbf{X})^-\mathbf{C}^T)$$

(vii) Define

$SS_{H_0} = (\mathbf{Cb}-\mathbf{d})^T[\mathbf{C}(\mathbf{X}^T\mathbf{X})^-\mathbf{C}^T]^{-1}(\mathbf{Cb}-\mathbf{d})$

then

$$\frac{1}{\sigma^2}\mathrm{SS}_{H_0} \sim \chi^2_m(\lambda)$$

where $m = rank(\mathbf{C})$ and

$\lambda = \frac{1}{\sigma^2}(\mathbf{C}\boldsymbol{\beta}-\mathbf{d})^T[\mathbf{C}(\mathbf{X}^T\mathbf{X})^-\mathbf{C}^T]^{-1}(\mathbf{C}\boldsymbol{\beta}-\mathbf{d})$

(viii) $\frac{1}{\sigma^2}SS_{H_0} \sim \chi^2_m$ if and only if $H_0 : \mathbf{C}\boldsymbol{\beta} = \mathbf{d}$ is true.

(ix) $E(SS_{residuals}) = (n-k)\sigma^2$ where
$k = \text{rank}(\mathbf{X}) = \text{rank}(\mathbf{P_X})$ and
$n-k = \text{rank}(\mathbf{I}-\mathbf{P_X})$ and it follows that

$$\mathrm{MS}_{\mathrm{residuals}} = \frac{\mathrm{SS}_{\mathrm{residuals}}}{\mathrm{n-k}}$$

is an unbiased estimator of σ^2.

(x) $\frac{1}{\sigma^2}SS_{residuals} \sim \chi^2_{n-k}$

(xi) SS_{H_0} and $SS_{residuals}$ are independently distributed.

(xii) $$F = \frac{\left(\frac{\mathrm{SS}_{\mathrm{H}_0}}{m\sigma^2}\right)}{\left(\frac{\mathrm{SS}_{\mathrm{residuals}}}{(n-k)\sigma^2}\right)} = \frac{\frac{\mathrm{SS}_{\mathrm{H}_0}}{m}}{\frac{\mathrm{SS}_{\mathrm{residuals}}}{n-k}} = \frac{(n-k)\mathrm{SS}_{\mathrm{H}_0}}{m\mathrm{SS}_{\mathrm{residuals}}}$$

$\sim F_{m,n-k}(\lambda)$

with noncentrality parameter

$$\lambda = \frac{1}{\sigma^2}(\mathbf{C}\boldsymbol{\beta} - \mathbf{d})^T[\mathbf{C}(\mathbf{X}^T\mathbf{X})^-\mathbf{C}^T]^{-1}(\mathbf{C}\boldsymbol{\beta} - \mathbf{d})$$

$$\geq 0$$

and $\lambda = 0$ if and only if $H_0 : \mathbf{C}\boldsymbol{\beta} = \mathbf{d}$ is true.

Example 1.

Consider the linear model $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$ with

$$\mathbf{Y} = \begin{bmatrix} Y_{11} \\ Y_{12} \\ Y_{21} \\ Y_{22} \\ Y_{23} \\ Y_{24} \\ Y_{31} \\ Y_{32} \end{bmatrix} \quad \mathbf{X} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \quad \boldsymbol{\beta} = \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

and $\boldsymbol{\epsilon} \sim N(0, \sigma^2 I)$.

(a) Determine which of the following hypotheses are testable.

i. $H_0 : \alpha_1 = \alpha_2$

ii. $H_0 : \alpha_1 - 2\alpha_2 + 3\alpha_3 = 0$

iii. $H_0 : \alpha_3 = 0$

iv. $H_0 : \mu = 0$

v. $\alpha_1 = \alpha_3$ and $\alpha_1 - 2\alpha_2 + \alpha_3 = 0$

vi. $\alpha_1 = \alpha_3$ and $\alpha_2 = \alpha_3$ and $\alpha_1 + \alpha_2 - 2\alpha_3 = 0$

(b) Suppose

$$H_0 : \mathbf{C}\boldsymbol{\beta} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

against the alternative $H_1 : \alpha_1 \neq \alpha_3$ or $\alpha_1 - 2\alpha_2 + \alpha_3 \neq 0$.

i. Show that H_0 is testable.

ii. Express the numerator and denominator of your F-statistic as two quadratic forms. Show that the quadratic form in the denominator of your F-statistic, has a central chi-square distribution. Report it's degrees of freedom.

iii. Show that the quadratic form in the numerator of your F-statistic, has a non central chi-square distribution. Report it's degrees of freedom and non centrality parameter.

iv. Show that the numerator and denominator of your F-statistic are independently distributed.

v. Show that your F-statistic has a non-central F-distribution. Report it.s degrees of freedom and express the non-centrality parameter as a function of α_1, α_2, α_3.

vi. Show that your test statistic has a central F-distribution when the null hypothesis is true.

4.6 Elements of Hypothesis Test

We perform the test by rejecting

$$H_0 : \mathbf{C}\boldsymbol{\beta} = \mathbf{d}$$

if

$$F > F_{(m,n-k),\alpha}$$

where α is a specified significance level (Type I error level) for the test.

$$\alpha = Pr\left\{rejectH_0 | \; H_0 \text{ is true}\right\}$$

4.6.1 Type I Error Level

$$\alpha = Pr\left\{F > F_{m,n-k,\alpha} \,|\, H_0 \text{ is true}\right\}$$

When H_0 is true,

$$F = \frac{MS_{H_0}}{MS_{residuals}} \sim F_{m,n-k}$$

This is the probability of incorrectly rejecting a null hypothesis that is true.

4.6.2 Type II Error Level

$$\begin{aligned}\beta &= Pr\{\text{Type II error}\} \\ &= Pr\left\{\text{fail to reject } H_0 \;\middle|\; H_0 \text{ is false}\right\} \\ &= Pr\left\{F < F_{m,n-k,\alpha} \;\middle|\; H_0 \text{ is false}\right\}\end{aligned}$$

When H_0 is false,

$$F = \frac{MS_{H_0}}{MS_{residuals}} \sim F_{(m,n-k)}(\lambda)$$

4.6.3 Power of a Test

$$\begin{aligned} power &= 1-\beta \\ &= Pr\{F > F_{m,n-k,\alpha} \mid H_0 \text{ is false}\} \end{aligned}$$

this determines the value of the noncentrality parameter.

For a fixed type I error level (significance level) α, the power of the test increases as the noncentrality parameter increases.

$$\lambda = \frac{1}{\sigma^2}(\mathbf{C}\boldsymbol{\beta}-\mathbf{d})^T[\mathbf{C}(\mathbf{X}^T\mathbf{X})^-\mathbf{C}^T]^{-1}(\mathbf{C}\boldsymbol{\beta}-\mathbf{d})$$

Example 2.

Effects of three diets on blood coagulation times in rats.

Diet factor: Diet 1, Diet 2, Diet 3

Response: blood coagulation time

Model for a completely randomized experiment with n_i rats assigned to the i-th diet.

$$Y_{ij} = \mu + \alpha_i + \epsilon_{ij}$$

where

$$\epsilon_{ij} \sim NID(0, \sigma^2)$$

for $i = 1, 2, 3$ and $j = 1, 2, ..., n_i$.

Here, $E(Y_{ij}) = \mu + \alpha_i$ is the mean coagulation time for rats fed the i-th diet.

Test the null hypothesis that the mean blood coagulation time is the same for all three diets.

..

Example 3.
Suppose we are willing to specify:

(i) α = type I error level = .05
(ii) $n_1 = n_2 = n_3 = n$
(iii) power $\geq$ .90 to detect
(iv) a specific alternative

$$(\mu + \alpha_1) - (\mu + \alpha_3) = 0.5\sigma$$
$$(\mu + \alpha_2) - (\mu + \alpha_3) = \sigma$$

How many observations (in this case rats) are needed?

Example 4.
For the hypotheses testing

$$H_0 : (\mu + \alpha_1) = (\mu + \alpha_2) = \cdots = (\mu + \alpha_k)$$

against

$$H_1 : (\mu + \alpha_1) \neq (\mu + \alpha_j) \text{for some} i \neq j$$

Obtain the test statistic and the corresponding non-centrality parameter.

4.7 Confidence intervals for estimable functions of $\boldsymbol{\beta}$

Definition 2.
Suppose $Z \sim N(0,1)$ is distributed independently of $W \sim \chi^2_v$, and then the distribution of

$$t = \frac{Z}{\sqrt{\frac{W}{v}}}$$

is called the student t-distribution with v degrees of freedom. We will use the notation

$$t \sim t_v$$

For the normal-theory Gauss-Markov model

$$\mathbf{Y} \sim N(\mathbf{X}\boldsymbol{\beta}, \sigma^2 I),$$

the OLS estimator of an estimable function, $\mathbf{c}^T\boldsymbol{\beta}$,

$$\mathbf{c}^T\mathbf{b} = \mathbf{c}^T(\mathbf{X}^T\mathbf{X})^-\mathbf{X}^T\mathbf{Y}$$

follows a normal distribution, i.e.,

$$\mathbf{c}^T\mathbf{b} \sim N(\mathbf{c}^T\boldsymbol{\beta}, \sigma^2\mathbf{c}^T(\mathbf{X}^T\mathbf{X})^-\mathbf{c}).$$

It follows that

$$Z = \frac{\mathbf{c}^T\mathbf{b} - \mathbf{c}^T\boldsymbol{\beta}}{\sqrt{\sigma^2\mathbf{c}^T(\mathbf{X}^T\mathbf{X})^-\mathbf{c}}} \sim N(0,1)$$

From Result 1.(ix), we have

$$\frac{1}{\sigma^2}SSE = \frac{1}{\sigma^2}\mathbf{Y}^T(I - P_\mathbf{X})\mathbf{Y} \sim \chi^2_{(n-k)}$$

where $k = \text{rank}(\mathbf{X})$.

Using the same argument that we used to derive Result 1.(x), we can show that $c^T\mathbf{b}$ is distributed independently of $\frac{1}{\sigma^2}$ SSE.

First note that

$$\begin{bmatrix} \mathbf{c}^T\mathbf{b} \\ (I - P_\mathbf{X})\mathbf{Y} \end{bmatrix} \begin{bmatrix} \mathbf{c}^T(\mathbf{X}^T\mathbf{X})^-\mathbf{X}^T \\ (I - P_\mathbf{X}) \end{bmatrix} \mathbf{Y}$$

has a joint normal distribution under the normal-theory Gauss-Markov model.

Note that

$$
\begin{aligned}
&Cov(\mathbf{c}^T\mathbf{b}, (I-P_{\mathbf{X}})\mathbf{Y}) \\
&\quad = (\mathbf{c}^T(\mathbf{X}^T\mathbf{X})^-\mathbf{X}^T)(V(\mathbf{Y}))(I-P_{\mathbf{X}})^T \\
&\quad = (\mathbf{c}^T(\mathbf{X}^T\mathbf{X})^-\mathbf{X}^T)(\sigma^2)(I-P_{\mathbf{X}}) \\
&\quad = \sigma^2\mathbf{c}^T(\mathbf{X}^T\mathbf{X})^-\underbrace{\mathbf{X}^{\mathbf{T}}(\mathbf{I}-\mathbf{P}_{\mathbf{X}})}_{\text{this is a matrix of zeros}} \\
&\quad = 0
\end{aligned}
$$

Consequently,
$\mathbf{c}^T\mathbf{b}$ is distributed independently of
$\mathbf{e} = (I-P_{\mathbf{X}})\mathbf{Y}$

which implies that

$\mathbf{c}^T\mathbf{b}$ is distributed independently of SSE $= \mathbf{e}^T\mathbf{e}$.

Then,

$$
t = \frac{Z}{\sqrt{\frac{\text{SSE}}{\sigma^2(n-k)}}}
$$

$$
= \frac{\frac{\mathbf{c}^T\mathbf{b}-\mathbf{c}^T\boldsymbol{\beta}}{\sqrt{\sigma^2\mathbf{c}^T(\mathbf{X}^T\mathbf{X})^-\mathbf{c}}}}{\sqrt{\frac{\text{SSE}}{\sigma^2(n-k)}}}
$$

$$
= \frac{\mathbf{c}^T\mathbf{b}-\mathbf{c}^T\boldsymbol{\beta}}{\sqrt{\underbrace{\frac{\text{SSE}}{(n-k)}}_{\frac{\text{SSE}}{n-k}\text{ is the MSE}}\mathbf{c}^T(\mathbf{X}^T\mathbf{X})^-\mathbf{c}}} \sim t_{(n-k)}
$$

It follows that

$$
\begin{aligned}
1-\alpha &= Pr\left\{-t_{(n-k),\alpha/2} \le \frac{\mathbf{c}^T\mathbf{b}-\mathbf{c}^T\boldsymbol{\beta}}{\sqrt{\text{MSE}\,\mathbf{c}^{\mathrm{T}}(\mathbf{X}^{\mathrm{T}}\mathbf{X})^-\mathbf{c}}} \le t_{(n-k),\alpha/2}\right\} \\
&= Pr\left\{\mathbf{c}^T\mathbf{b} - t_{(n-k),\alpha/2}\sqrt{\text{MSE}\,\mathbf{c}^{\mathrm{T}}(\mathbf{X}^{\mathrm{T}}\mathbf{X})^-\mathbf{c}} \le \mathbf{c}^T\boldsymbol{\beta}\right. \\
&\qquad \left. \le \mathbf{c}^T\mathbf{b} + t_{(n-k),\alpha/2}\sqrt{\text{MSE}\,\mathbf{c}^{\mathrm{T}}(\mathbf{X}^{\mathrm{T}}\mathbf{X})^-\mathbf{c}}\right\}
\end{aligned}
$$

and a $(1-\alpha)\times 100\%$ confidence interval for $\mathbf{c}^T\boldsymbol{\beta}$ is

$$\Big(\mathbf{c}^T\mathbf{b} - t_{(n-k),\alpha/2}\sqrt{\mathrm{MSE}\,\mathbf{c}^{\mathrm{T}}(\mathbf{X}^{\mathrm{T}}\mathbf{X})^{-}\mathbf{c}}\,,$$
$$\mathbf{c}^T\mathbf{b} + t_{(n-k),\alpha/2}\sqrt{\mathrm{MSE}\,\mathbf{c}^{\mathrm{T}}(\mathbf{X}^{\mathrm{T}}\mathbf{X})^{-}\mathbf{c}}\Big)$$

For brevity we will also write

$$\mathbf{c}^T\mathbf{b} \pm t_{(n-k),\alpha/2}\; S_{\mathbf{c}^T\mathbf{b}}$$

where

$$S_{\mathbf{c}^T\mathbf{b}} = \sqrt{\mathrm{MSE}\,\mathbf{c}^{\mathrm{T}}(\mathbf{X}^{\mathrm{T}}\mathbf{X})^{-}\mathbf{c}}\,.$$

For the normal-theory Gauss-Markov model with $\mathbf{Y} \sim N(\mathbf{X}\boldsymbol{\beta}, \sigma^2 I)$, the interval

$$\mathbf{c}^T\mathbf{b} \pm t_{(n-k),\alpha/2} S_{\mathbf{c}^T\mathbf{b}}$$

is the **shortest random interval** with probability $(1-\alpha)$ of containing $\mathbf{c}^T\boldsymbol{\beta}$.

4.8 Confidence interval for σ^2:

For the normal-theory Gauss-Markov model with $\mathbf{Y} \sim N(\mathbf{X}\boldsymbol{\beta}, \sigma^2 I)$ we have shown that

$$\frac{\mathrm{SSE}}{\sigma^2} = \frac{\mathbf{Y}^T(I - P_{\mathbf{X}})\mathbf{Y}}{\sigma^2} \sim \chi^2_{(n-k)}$$

Then,

$$\begin{aligned} 1-\alpha &= Pr\left\{\chi^2_{(n-k),1-\alpha/2} \le \frac{\mathrm{SSE}}{\sigma^2} \le \chi_{(n-k),\alpha/2}\right\} \\ &= Pr\left\{\frac{\mathrm{SSE}}{\chi^2_{(n-k),\alpha/2}} \le \sigma^2 \le \frac{\mathrm{SSE}}{\chi_{(n-k),1-\alpha/2}}\right\} \end{aligned}$$

The resulting $(1-\alpha)\times 100\%$ confidence interval for σ^2 is

$$\left(\frac{\mathrm{SSE}}{\chi^2_{(n-k),\alpha/2}}\,,\ \frac{\mathrm{SSE}}{\chi^2_{(n-k),1-\alpha/2}}\right)$$

Example 5.

For the simple regression model

$$Y_i = \beta_0 + \beta_1 \mathbf{X}_{i1} + \epsilon_i,$$

where for $\mathbf{e} = (\epsilon_1, \ldots, \epsilon_i)^T$, $E(\mathbf{e}) = \mathbf{0}$. You are given

x	2	3	4	5
y	4	7	6	8

Suppose that $V(\mathbf{e}) = \sigma^2 I$.

(a) Construct the 95% confidence interval for β_1.

(b) Give 95% two-sided confidence interval for σ^2 in the normal version model.