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CONTENTS

1 Brownian Motions and Ito Lemma

1.1 Brownian Motions

1.1.1 Random Walk

A random walk can be generated by flipping a coin each period and moving one step, with the direction determined by whether the coin is heads or tails. Let our position at time t be Z_t, and Y_t be the outcome of toss at time t , then we have

$$Y_t = \begin{cases} +1 \text{ with probability } 0.5 \\ -1 \text{ with probability } 0.5 \end{cases}$$

and

$$Z_t = \sum_{k=0}^{t} Y_K$$

where $Z_0 = 0$.

We call Z_n a **random walk**. The random walk may be considered a primary building block of financial modeling. Note that the binomial tree

model is a variant of the random walk model with asymmetric probabilities.

1.1.2 Brownian Motion (Wiener Process)

Brownian motion is a random walk occurring in continuous time, with movements that are continuous rather than discrete . To generate Brownian motion, we would flip the coins infinitely fast and take infinitesimally small steps at each point.

Suppose that each Δt time unit we take a step of size Δy either to the left or the right with equal probabilities. If we let $Z(t)$ denote the position at time t then

$$Z(t) = \Delta y(Y_1 + Y_2 + \cdots + Y_{\lfloor t/\Delta t \rfloor})$$

where

$$Y_k = \begin{cases} +1 \text{ if the } i\text{-th step of length } \Delta y \text{ is to the right} \\ -1 \text{ if it is to the left} \end{cases}$$

and $\lfloor t/\Delta t \rfloor$ is the largest integer less than or equal to $t/\Delta t$, and where the Y_k are assumed independent with

$$P(Y_k = 1) = P(Y_k = -1) = \frac{1}{2}$$

As

$$E(Y_k) = 0 \quad \text{and} \quad V(Y_k) = E(Y_k^2) = 1$$

we have

$$E[Z(t)] = 0 \quad \text{and} \quad V[Z(t)] = (\Delta y)^2 \left[\frac{t}{\Delta t}\right]$$

We shall now let Δy and Δt go to 0. However, we must do it in a way such that the resulting limiting process is nontrivial. Let $\Delta y = \sigma\sqrt{\Delta t}$, then when $\Delta t \to 0$

$$E[Z(t)] = 0 \quad \text{and} \quad V[Z(t)] = (\Delta y)^2 \to \sigma^2 t$$

By Central Limit Theorem,

$$Z(t) \to N(0, \sigma^2 t)$$

In addition, because the changes of value of the random walk in nonoverlapping time intervals are

independent, we have $Z(t), t \geq 0$ **has independent increments**, i.e. for all $t_1 < t_2 < \cdots < t_n$,

$$Z(t_n) - Z(t_{n-1}), Z(t_{n-1}) - Z(t_{n-2}), \ldots, Z(t_2) - Z(t_1), Z(t_1)$$

are independent.

Finally, because the distribution of the change in position of the random walk over any time interval depends only on the length of that interval, it would appear that $Z(t), t \geq 0$ **has stationary increments**, i.e. the distribution of $Z(t+s) - Z(t)$ does not depend on t and depends only on s.

When $\sigma = 1$, the process is called a **Standard Brownian motion**.

Properties of a Standard Brownian Motion

1. $Z(0) = 0$
2. $Z(t) \sim N(0, t)$

3. $Z(t), t \geq 0$ **has independent increments**, i.e. for all $t_1 < t_2 < \cdots < t_n$,
   $$Z(t_n) - Z(t_{n-1}), Z(t_{n-1}) - Z(t_{n-2}), \ldots, Z(t_2) - Z(t_1), Z(t_1)$$
4. $Z(t), t \geq 0$ **has stationary increments**, i.e. the distribution of $Z(t+s) - Z(t)$ does not depend on t and depends only on s.

From property (4), $Z(t+s) - Z(t)$ has the same distribution as $Z(s) - Z(0)$. Since $Z(0) = 0$, we have

$$Z(t+s) - Z(t) \sim N(0, s)$$

by property (3), $Z(t+s) - Z(t)$ does not depend on $Z(t)$, so we have

$$Z(t+s) - Z(t) | Z(t) \sim N(0, s)$$

In general, $Z(t+s) - Z(t)$ does not depend on the history of the Brownian motion up to and including time t, which means

$$Z(t+s) - Z(t) | \{Z(u) : 0 \leq u \leq t\} \sim N(0, s)$$

Note that given the history of $\{Z(u) : 0 \leq u \leq t\}$, $Z(t)$ is a constant not a random variable. By adding $Z(t)$ both sides, we have

$$Z(t+s) | \{Z(u) : 0 \leq u \leq t\} \sim N(Z(t), s)$$

Example 1.
Calculate the followings:

(a) $P(Z(4) \leq 4)$. $\boxed{0.9772}$

(b) $P(-0.4 < Z(3) < 0.4)$. $\boxed{0.182}$

(c) $P(-6 \leq Z(5) < 3)$. $\boxed{0.9062}$

Example 2.
Find the expected value and variance of $Z(8) - Z(5)$. $\boxed{3}$

Example 3. Calculate the following:

(a) $Cov[Z(s), Z(t)]$ for $0 \leq s < t$. $\boxed{s}$

(b) $E[Z(s)Z(t)]$ for $0 \leq s < t$. $\boxed{0.1}$

Example 4.
Calculate the followings:

a $E[Z(5)Z(2)]$

b $V[Z(1) + Z(3)]$

c $E[Z(8)|Z(5)]$

d $V[Z(8)|Z(5)]$

1.1.3 Arithmetic Brownian Motions(ABM)

Random walk model and Standard Brownian motions is not appropriate to model stock prices because:

- Stock prices cannot be negative because shareholders have limited liability.
- The stock should have a positive return, but the mean of random walk and Standard Brownian Motions are both zero.

Thus, to construct a model for stock prices, we extend standard BM to Arithmetic Brownian Motions. We say that $\{X(t) : t \geq 0\}$ is an arithmetic Brownian motion with **drift coefficient** μ and **variance parameter** σ^2 (or volatility σ) if

$$X(t) = \mu t + \sigma Z(t).$$

The drift coefficient μ is also called the **instantaneous mean per unit time** and the variance parameterσ^2 is called the **instantaneous variance per unit time**. If $\mu = 0$, The ABM is said to be **driftless**.

Since $\{Z(t) : t \geq\}$ has stationary and independent increments, thus, $\{X(t) : t \geq\}$ also has stationary and independent increments.
As

$$E[X(t)] = E[\mu t + \sigma Z(t)] = \mu t + \sigma E[Z(t)] = \mu t$$
$$V[X(t)] = V[\mu t + \sigma Z(t)] = V[\sigma Z(t)] = \sigma^2 t$$

Thus,

$$\boxed{X(t) \sim N(\mu t, \sigma^2 t)}$$

Example 5.
Let $\{X(t) : t \geq 0\}$ be an arithmetic Brownian Motion with $\mu = 1$ and $\sigma = 4$. Find the probability that $0 < X(2) < 1$ given $X(1) = 2$.

Example 6.
The price of a stock follows arithmetic Brownian motion of the form $X(t) = X(0) + t + 0.2Z(t)$. The current price of the stock is 40. Determine the probability that the price of the stock at time 4 is less than 43. $\boxed{0.0062}$

1.1.4 Geometric Brownian Motions (GBM)

Arithmetic Brownian motion has several drawbacks:

- There is nothing to prevent $X(t)$ from becoming negative, so it is a poor model for stock prices.
- The mean and variance of changes in dollar terms are independent of the level of the stock price .

To eliminate these criticisms, we consider geometric Brownian motion. let $\{X(t) : t \geq 0\}$ be a Brownian motion with drift ans $Y(0)$ be a contant. Then

$$\boxed{Y(t) = Y(0)e^{X(t)} = Y(0)e^{\mu t + \sigma Z(t)}}$$

is called a geometric Brownian motion.

By taking logarithm,

$\ln Y(t) = \ln Y(0) + X(t) = \ln Y(0) + \mu t + \sigma Z(t)$

$$E[\ln Y(t)] = \ln Y(0) + \mu t$$
$$V[\ln Y(t)] = \sigma^2 t$$

Thus

$$\boxed{\ln Y(t) \sim N(\ln Y(0) + \mu t, \sigma^2 t)}$$

Since $\ln Y(t)$ follows a normal distribution, then $Y(t)$ follows a lognormal(LN) distribution with parameters $\mu* = \ln Y(0) + \mu t$ and $\sigma^* = \sigma\sqrt{t}$, i.e.

$$\boxed{Y(t) \sim LN(\mu* = \ln Y(0) + \mu t, \sigma^* = \sigma\sqrt{t})}$$

Notes:

- If $X \sim N(\mu, \sigma^2)$, then
  - $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, x, \mu \in \mathbb{R}, \sigma > 0$
  - $E(X) = \mu$
  - $V(X) = \sigma^2$
- If $X \sim LN(\mu, \sigma)$
- $f(x) = \frac{1}{x\sqrt{2\pi}\sigma} e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}, x, \mu, \sigma > 0$ $F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$
- $E(X^k) = e^{k\mu + \frac{1}{2}k^2\sigma^2}$

Example 7.
Let $\{Z(t) : t \geq 0\}$ be a standard Brownian motion and $Y(t)$ be a time-t price of stock. It is given that

$$Y(t) = 100e^{0.035t+0.3Z(t)}$$

(a) Find $P(98 \leq Y(2) \leq 103)$. $\boxed{0.0434}$

(b) Find $P(98 \leq Y(2) \leq 103|Y(1) = 102)$. $\boxed{0.0668}$

1.2 Stochastic Calculus

1.2.1 Stochastic Differential Equations

Suppose that the rate of change in $x(t)$ depends on the time t and the value of $x(t)$ itself, i.e.

$$\frac{dx(t)}{dt} = f(t, x(t))$$

.

We "multiply" both sides of the equation by dt to obtain

$$dx(t) = f(t, x(t))dt$$

This equation is a differential equation, which says that the change in x over a very short time interval $[t, t + dt]$ is given by $f(t, x(t))$.
Now suppose that

$$dX(t) = a[t, X(t)]dt + b[t, X(t)]dZ(t)$$

or in shorthand notation

$$dX = adt + bdZ$$

This is called a **stochastic differential equation** (SDE) and X is said to be a diffusion.

- $dZ(t)$ is the change in the Brownian motion over $[t, t+dt]$. We may view $dZ(t)$ as $Z(t+dt) - Z(t)$ and hence $dZ(t) \sim N(0, dt)$.
- $dX(t)$ the change in X over $[t, t+dt]$. We may view $dX(t)$ as $X(t+dt) - X(t)$,
- It follows from independent increments that $dZ(t)$ is independent of the history $\{Z(u) : 0 \leq u \leq t\}$. In particular, $dZ(t)$ and $Z(t)$ are independent.
- Given the value of $X(t)$, The terms $a[t, X(t)]$ and $b[t, X(t)]$ are no longer random, and hence
$$E[dX(t)] = a[t, X(t)]dt$$
and
$$V[dX(t)] = b^2[t, X(t)]dt$$

$a(x,t)$ and $b(x,t)$ are called the drift and volatility. Thus

- for standard Brownian motion, $a(x,t) = 0$ and $b(x,t) = 1$
- for ABM, $a(x,t) = \mu$ and $b(x,t) = \sigma$

1.2.2 Itô's Lemma

In many problems involving Itô processes and geometric Brownian motion, we will encounter situations where we need to multiply dZ by itself, dt by itself, or dZ by dt. Rules for these products are known as multiplication rules. But first we define the following: For $\alpha > 1$ we will define $(dt)^\alpha = 0$. This, makes senses since dt represents a very small number.

Example 8.

(a) Show that $E[(dZ)^2] = dt$.

(b) Show that $Var[(dZ)^2] = 0$.

(c) Show that $(dZ)^2 = dt$.

Example 9.

(a) Show that $E[dZ \times dt] = 0$.

(b) Show that $E[(dZ \times dt)^2] = 0$.

(c) Show that $Var[dZ \times dt] = 0$.

(d) Show that $dZ \times dt = 0$.

Example 10.
Let Z and Z' be two standard Brownian motions. Show that $dZ \times dZ' = \rho dt$ where $\rho = E[Z(t)Z'(t)]$ is also known as the correlation of the underlying assets driven by the different Brownian motions.

Summarizing, the **multiplication rules** are:

- $dt \times dt = 0$,
- $dt \times dZ = 0$,
- $dZ \times dZ = dt$, and
- if $Z(t)$ and $Z'(t)$ are two Brownian motions having correlation ρ, then $dZ \times dZ' = \rho dt$

Theorem 1. Itô's Lemma
Let
$$Y(t) = f(t, X(t))$$
and
$$dX(t) = a[t, X(t)]dt + b[t, X(t)]dZ(t),$$
then
$$dY(t) = \frac{\partial Y}{\partial t}dt + \frac{\partial Y}{\partial X}dX(t) + \frac{1}{2}\left(\frac{\partial^2 Y}{\partial X^2}\right)(dX(t))^2$$
To obtain $(dX)^2$, we use the multiplication rule.

Thus
$$(dX)^2 = b^2(t, X)dt.$$

Notes:If $X(t) = Z(t)$, then

- $a(t, X(t)) = 0$,
- $b(t, X(t))) = 1$, and
- $(dX)^2 = dt$.

Example 11.

Let $Z(t)$ be a standard Brownian motion. Find $dY(t)$ for the following:

(a) $Y(t) = Z^2(t)$. $\boxed{2Z(t)dZ(t) + dt}$

(b) $Y(t) = tZ^2(t)$. $\boxed{(Z^2(t) + t)dt + 2tZ(t)dZ(t)}$

Example 12.

Let $Z(t)$ be a standard Brownian motion and $Y(0)$ be a constant. Find $dY(t)$ for the following:

(a) $Y(t) = Y(0) + \mu t + \sigma Z(t)$. $\boxed{\mu dt + \sigma dZ(t)}$

(b) $Y(t) = Y(0)e^{(\mu - \frac{\sigma^2}{2})t + \sigma Z(t)}$.

$\boxed{\mu Y(t)dt + \sigma Y(t)dZ(t)}$

Example 13. (T01Q1)

Suppose that X follows the stochastic differential equation

$$dX(t) = -12dt + 4dZ(t)$$

where $Z(t)$ is a standard Brownian motion.
Let $W(t) = e^{3tX(t)}$. If

$$dW(t) = a[W(t), t]dt + b[W(t), t]dZ(t),$$

find $a(6, 9)$.

Example 14. (T01Q2)

Assume that $S(t)$ follows an arithmetic Brownian motion:

$$dS(t) = \alpha dt + \sigma dZ(t).$$

If

$$d(14S^3(t) + 4t) = a[S(t), t]dt + b[S(t), t]dZ(t).$$

Find $a(20, 3)$ for $\alpha = 0.012$ and $\sigma = 0.21$.

1.2.3 An Integral Representation

Consider the SDE

$$dX(t) = a[t, X(t)]dt + b[t, X(t)]dZ(t)$$

Integrating both sides from 0 to t, we get

$$X(t) - X(0) = \int_0^t a[s, X(s)]ds + \int_0^t b[s, X(s)]dZ(s)$$

Here $\int_0^t b[s, X(s)]dZ(s)$ is called a stochastic integral.

Both equations above are equivalent ways to express the dynamics of X. Let $a(t, x) = 0$, the equivalent between the two equations means

$$dX(t) = b[t, X(t)]dZ(t) \iff X(t) - X(0) = \int_0^t b[s, X(s)]dZ(s)$$

and

$$dX(t) - dX(0) = d\int_0^t b[s, X(s)]dZ(s)$$

Since $dX(0) = 0$, then

$$\boxed{d\int_0^t b[s, X(s)]dZ(s) = b[t, X(t)]dZ(t)}$$

Example 15.

Let $Y(t) = \int_0^t e^{-\lambda(t-s)} dZ(s)$. Find $dY(t)$.

Example 16. (T01Q3)

Let $Z(t)$ be a standard Brownian motion. You are given:

- $R(t) = R(0)e^{-2.3000000000000003t} + 0.054(1 - e^{-2.3000000000000003t}) + 0.12\int_0^t e^{-2.3000000000000003(t-s)}\sqrt{R(s)}dZ(s)$,
- $Y(t) = [R(t)]^4$.

Suppose that $dY(t) = a[t, Y(t)]dt + b[t, Y(t)]dZ(t)$. Find $dY(t)$ and hence calculate $\frac{a(2,0.15)}{b(2,0.15)}$.

1.2.4 Solutions of Three SDEs

(I) **Type 1: Arithmetic Brownian motions**

The SDE

$$\boxed{dY(t) = \alpha dt + \sigma dZ(t)}$$

has a solution of the form of an arithmetic Brownian motion with drift (including an intercept):

$$\boxed{Y(t) = Y(0) + \alpha t + \sigma Z(t)}$$

(II) **Type 2: Geometric Brownian motions** The SDE

$$\boxed{dY(t) = (\alpha - \delta)Y(t)dt + \sigma Y(t)dZ(t)}$$

or

$$\boxed{\frac{dY(t)}{Y(t)} = (\alpha - \delta)dt + \sigma dZ(t)}$$

has a solution of the form of an geometric Brownian motion:

$$\boxed{Y(t) = Y(0)e^{[(\alpha-\delta-\frac{\sigma^2}{2})t+\sigma Z(t)]}}$$

Geometric Brownian motions: The followings are equivalent representations:

(i) $Y(t)$ follows a geometric Brownian motion with rate of appreciation $\alpha-\delta$ and volatility σ.

(ii) $dY(t) = (\alpha-\delta)Y(t)dt + \sigma Y(t)dZ(t)$ or $\frac{dY(t)}{Y(t)} = (\alpha-\delta)dt + \sigma dZ(t)$

(iii) $d[\ln Y(t)] = \left(\alpha-\delta-\frac{\sigma^2}{2}\right)dt + \sigma dZ(t)$ or $\ln\frac{Y(t)}{Y(0)} = \left(\alpha-\delta-\frac{\sigma^2}{2}\right)t + \sigma Z(t)$

(iv) $Y(t) = Y(0)e^{(\alpha-\delta-\sigma^2/2)t+\sigma Z(t)}$

(v) $\ln Y(t)|Y(0) \sim N\left(\ln Y(0) + (\alpha-\delta-\frac{1}{2}\sigma^2)t, \sigma^2 t\right)$

Example 17.

Suppose

$$dY(t) = (\alpha-\delta)Y(t)dt + \sigma Y(t)dZ(t).$$

Show that

$$d[\ln Y(t)] = \left(\alpha-\delta-\frac{\sigma^2}{2}\right)dt + \sigma dZ(t).$$

Example 18. (T01Q4)

Stock prices follow geometric Brownian motion:

$$d\ln S(t) = 0.036dt + 0.27dZ(t)$$

Suppose $S(0) = 48$. Calculate $P[S(2) < 46)$.

Example 19. You are given:

- $dS(t) = (\alpha-\delta)S(t)dt + \sigma S(t)dZ(t)$
- $F(t) = S(t)e^{(r-\delta)(T-t)}$, and
- $F^P(t) = S(t)e^{(-\delta)(T-t)}$.

Find $dF(t)$ and $dF^P(t)$.

Process for a Forward

The forward process is

$$F_{t,T}(S(T)) = S(t)e^{(r-\delta)(T-t)}$$

If $dS(t) = (\alpha - \delta)S(t)dt + \sigma dZ(t)$, then

$$\frac{dF_{t,T}}{F_{t,T}} = (\alpha - r)dt + \sigma dZ(t)$$

So the forward follows GBM with the same volatility as the stock, and with rate of growth $\alpha - r$. Similarly,

$$\frac{dF^P_{t,T}}{F^P_{t,T}} = \alpha dt + \sigma dZ(t)$$

Example 20. (T01Q5)

You are given:

- $S(t) = S(0)e^{0.15t+0.26Z(t)}$
- $\delta = 0.04$
- $F_{t,T}$ is a forward on the stock.
- $r = 0.06$

$d(\ln F)$ follows the process $\alpha dt + \sigma dZ(t)$. Determine α.

(III) **Type 3: The Ornstein-Uhlenback (OU) process** The SDE

$$dY(t) = \lambda[\alpha - Y(t)]dt + \sigma dZ(t)$$

has a solution of the form:

$$Y(t) = e^{-\lambda t}Y(0) + \alpha(1 - e^{-\lambda t}) + \sigma \int_0^t e^{-\lambda(t-s)} dZ(s)$$

Suppose that $\lambda > 0$. Then the drift term is $\lambda[\alpha - Y(t)]$.

- When $Y(t) < \alpha$, then the drift is positive, pulling $Y(t)$ up toward α.
- When $Y(t) > \alpha$, then the drift is negative, pulling $Y(t)$ down toward α.
- When $Y(t) = \alpha$, then the drift is zero.

We say that Y exhibits **mean reversion**, with λ being the **speed of the reversion**, and α being the **equilibrium level**.

Example 21. (T01Q6)

Let $Z(t)$ be a standard Brownian motion. You are given that

$$Y(t) = 4 + 4e^{-0.61t} + 0.29 \int_0^t e^{-0.61(t-s)} dZ(s).$$

Let $X(t) = Y^3(t)$. Suppose

$$dX(t) = a(t, X(t))dt + b(t, X(t))dZ(t).$$

Find $a(1, 4)$.

Example 22. (T01Q7)

Interest rates $r(t)$ satisfy the SDE

$$dr(t) = 0.2(0.13 - r(t))dt + 0.21\sqrt{r(t)}dZ(t)$$

A solution for $r(t)$ is

$$r(t) = A + (r(0) + B)e^{Ct} + De^{Et}\int_0^t e^{Fs}\sqrt{r(s)}dZ(s)$$

where A, B, C, D, E and F are constants. Determine $B + C + D + E + F$.

1.3 Modeling stock price Dynamics

1.3.1 Modeling Stock Prices with a GBM

Let $S(t)$ be the time-t price of a stock. Then

$$\frac{dS(t)}{S(t)}$$

is the instantaneous percentage change of stock price at time t. This ratio may be assumed to follow a normal distribution. In particular, we may consider the following process:

$$\frac{dS(t)}{S(t)} = (\alpha - \delta)dt + \sigma dZ(t)$$

We interpret α as **expected rate of return per annum**, $(\alpha - \delta)$ as the **continuously compounded rate of appreciation** on the stock, and σ as the **volatility per annum**.

The solution to the above SDE is

$$S(t) = S(0)e^{[(\alpha-\delta-\sigma^2/2)t+\sigma Z(t)]},$$

which implies that

$$S(t)|S(0) \sim LN(\mu^* = \ln S(0) + (\alpha - \delta - \sigma^2/2)t, \sigma^* = \sigma\sqrt{t})$$

Note that if $Y \sim LN(\mu, \sigma)$, then

$$E(Y^k) = e^{k\mu + \frac{1}{2}k^2(\sigma)^2}$$

So,

$$\begin{aligned} E[S^k(t)] &= e^{k(\ln S(0)+\alpha-\delta-\frac{1}{2}\sigma^2)t+\frac{1}{2}k^2\sigma^2 t} \\ &= \boxed{[S^k(0)]e^{k(\alpha-\delta)t+\frac{1}{2}k(k-1)\sigma^2 t}} \end{aligned}$$

The mean and variance are

$$E[S(t)] = S(0)e^{(\alpha-\delta)t}$$

$$V[S(t)] = E^2[S(t)][(e^{\sigma^2 t} - 1)$$

For $t < u$, let

$$Q_1 = \frac{S(t)}{S(0)} \text{ and} Q_2 = \frac{S(u)}{S(t)},$$

then Q_1 and Q_2 cover nonoverlapping intervals and are independent.

$E(Q_1^k) = e^{k(\alpha-\delta)t+\frac{1}{2}k(k-1)\sigma^2 t}$;

$E(Q_2^k) = e^{k(\alpha-\delta)(u-t)+\frac{1}{2}k(k-1)\sigma^2(u-t)}$

$$\begin{aligned} E[S(t)S(u)] &= S^2(0)E[Q_1^2 Q_2] \\ &= S^2(0)e^{2(\alpha-\delta)+\frac{1}{2}\sigma^2)t}e^{(\alpha-\delta)(u-t)} \end{aligned}$$

Example 23. (T01Q8)

Let $S(t)$ be the time-t price of a nondividend-paying stock, you are given that:

- The stock price process is

  $d[\ln S(t)] = -0.01505dt + 0.39dZ(t), S(0) = 2$

  where $Z(t)$ is a standard Brownian motion under the true probability measure.

Calculate $Cov(S^2(3), S^3(5))$.

Example 24. (T01Q9)

The price of a stock follows the stochastic differential equation:

$$\frac{dS(t)}{S(t)} = 0.028dt + 0.22dZ(t)$$

where $Z(t)$ is a standard Brownian motion. Consider the geometric average

$$G = [S(1)S(4)S(7)]^{\frac{1}{3}}.$$

Find the variance of $\ln G$.

1.3.2 Lognormality of Stock Price

For

$S(t) \sim LN(\mu = \ln S(0)+(\alpha-\delta-\sigma^2/2)t, \sigma' = \sigma\sqrt{t})$,

then,

$\ln S(t) \sim N(\mu = \ln S(0)+(\alpha-\delta-\sigma^2/2)t, \sigma' = \sigma\sqrt{t})$,

$$f(s) = \frac{1}{\sqrt{2\pi}\sigma' s} e^{-\frac{(\ln s-\mu)^2}{2(\sigma')^2}}$$

$$\begin{aligned} P[S(t) > k] &= P[\ln S(t) > \ln k] \\ &= P[Z > \tfrac{\ln k-\ln S(0)-(\alpha-\delta-\sigma^2/2)t}{\sigma\sqrt{t}}] \\ &= P[Z > -\tfrac{\ln S(0)/\ln K+(\alpha-\delta-\sigma^2/2)t}{\sigma\sqrt{t}}] \\ &= P[Z > -\hat{d}_2] \\ &= N(\hat{d}_2) \end{aligned}$$

Similarly, we can obtain

$$P[S(t) \leq k] = N(-\hat{d}_2)$$

1.3.3 The Conditional Expected Price

For

$S(t) \sim LN(\mu = \ln S(0)+(\alpha-\delta-\sigma^2/2)t, \sigma' = \sigma\sqrt{t})$,

$$\boxed{E[S(t)I(S(t) > k)] = E[S(t)]N(\hat{d}_1)}$$

where

$$\boxed{\hat{d}_1 = \frac{\ln\frac{S(0)}{k}+(\alpha-\delta+\frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}}$$

Similarly,

$$\boxed{E[S(t)I(S(t) < k)] = E[S(t)]N(-\hat{d}_1)}$$

As

$$P[S(t) \leq k] = N(-d_2)$$

and

$$P[S(t) > k] = N(d_2),$$

we have

$$\boxed{E[(S(t)|S(t) < k] = \frac{E[S(t)]N(-\hat{d}_1)}{N(-\hat{d}_2)}}$$

and

$$\boxed{E[(S(t)|S(t) > k] = \frac{E[S(t)]N(\hat{d}_1)}{N(\hat{d}_2)}}$$

Note that $\hat{d}_2 = \frac{\ln\frac{S(0)}{k}+(\alpha-\delta-\frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}} = \hat{d}_1 - \sigma\sqrt{t}$

Example 25.

You consider a stock with the following characteristics:

- The current price of the stock is 30.
- The stock pays no dividends.
- The stock's volatility is 20%.
- $P[S(2) < 20] = 0.025$

(i) Calculate the expected return α on the stock. $\boxed{0.09445}$

(ii) Calculate $E[S(2)|S(2) < 20]$. $\boxed{18.1189}$

Example 26. (T01Q10)

You are given:

- $S(t)$ is the time-t price of a nondividend-paying stock.
- $S(t)$ follows a geometric Brownian motion.
- The current stock price is 31.
- The expected return on the stock is 0.23.
- The stock's volatility is 0.28.

Calculate $E[S(5)|S(5) > 31]$.

Example 27. (T01Q11)

The time-t price of a stock, $S(t)$, follows the Ito process

$$\frac{dS(t)}{S(t)} = 0.14dt + 0.33dZ(t).$$

The initial price of the stock, $S(0)$, is 60. A 9-month European call option on the stock has strike price 70.0. Calculate the expected payoff of the call option, given that it is pays off.

Example 28. (T01Q12)

For a nondivedend paying stock with price $S(t)$ at time t, you are given:

- $S(0) = 70$
- The continuously compounded expected annual rate of return is 0.13.
- The volatility is 0.23.
- $G = \left[S\left(\frac{1}{12}\right) S\left(\frac{2}{12}\right) S\left(\frac{3}{12}\right)\right]^{\frac{1}{3}}$

Calculate $E[(G - 74.6)I(G > 74.6)|G > 74.6]$.

1.4 The Sharpe Ratio and Black-Scholes Equation

1.4.1 The Black-Scholes Framework

By Black-Scholes framework, we mean the following:

- The underlying asset $S(t)$ follows a geometric Brownian motion.
- The underlying asset is either nondividend-paying or pays dividends continuously at a level proportional to its price.
- The risk free interest rate is constant.
- There are no transaction cost or taxes.
- It is possible to purchase or short-sell any units of the underlying asset.
- The borrowing rate and the lending rate are both equal to the risk-free interest rate.
- There are no arbitrage opportunities.

The most important assumption is GBM. The five statements below are equivalent:

1. $S(t)$ is a GBM with drift $\alpha-\delta$ and volatility σ.
2. $\frac{dS(t)}{S(t)} = (\alpha-\delta)dt + \sigma dZ(t)$ where $Z(t)$ is a standard Brownian motion.
3. $d[\ln S(t)] = (\alpha-\delta-\frac{\sigma^2}{2})dt + \sigma dZ(t)$.
4. $S(t) = S(0)e^{\alpha-\delta-\frac{\sigma^2}{2})t+\sigma Z(t)}$.
5. $\ln S(t)$ is normal distributed with mean $\ln S(0) + (\alpha-\delta-\frac{\sigma^2}{2})t$ and variance $\sigma^2 t$.

1.4.2 The Sharpe Ratio

Let $X(t)$ be the price of an asset. The Sharpe ratio of X at time t is defined as the ratio of the instantaneous average risk premium to the instantaneous volatility. In other word, the Sharpe ratio is a measure of the risk return trade off. We assume the following:

(i) The dynamic of $X(t)$ follow

$$\frac{dX(t)}{X(t)} = mdt + sdZ(t)$$

Here $m = m(X(t),t)$ and $s = s(X(t),t)$ can depend on t and the time-t price of X.

(ii) The assets pays dividends continuously at a rate proportional to its price. The continuous dividend yield is δ. The dollar amount of dividend over an infinitesimally short time period $(t, t+dt)$ is

$$X(t)\delta dt$$

(iii) $\frac{dX(t)}{X(t)}$ is instantaneous return due to capital gains.

(iv) $mdt = E\left[\frac{dX(t)}{X(t)}\right]$ is the expected instantaneous return due to capital gains.

(v) $s^2 dt = Var\left[\frac{dX(t)}{X(t)}\right]$ is the variance of the instantaneous return due to capital gains.

The total return on X is the sum of capital gains and dividends. Thus, the instantaneous total return is $m+\delta$, and the instantaneous risk premium is $m+\delta-r$. The Sharpe ratio is defined by the

ratio of the instantaneous risk premium to the instantaneous standard deviation.

If $\frac{dX(t)}{X(t)} = mdt + sdZ(t)$ and the continuous dividend yield δ, the the Sharpe ratio is

$$\phi = \frac{m+\delta-r}{s}$$

Example 29.
Consider the Black-Scholes model for a stock price $S(t)$. Find the Sharpe ratio.

Example 30.
The time-t price of an asset $S(t)$ follows $S(t) = 10e^{0.01t+0.2Z(t)}$ where $Z(t)$ is a standard Brownian motion. The asset pays dividends continuously at a rate proportional to its price. The dividend yield is 2%. If the continuously compounded risk free interest rate is 4%, find the Sharpe ratio of the stock. $\boxed{0.05}$

1.4.3 Equality of Sharpe Ratios

The Sharpe ratios of two assets driven by the same Brownian motion must be the same. In particular, any contingent claim written on a stock that follows a GBM must have a Sharpe ratio of $\phi = \frac{\alpha - r}{\sigma}$.

Example 31. (T01Q13)
Consider two assets X and Y. There is a single source of uncertainty which is captured by a standard Brownian motion $\{Z(t)\}$. The stochastic process of X satisfies the stochastic differential equations

$$d[\ln X(t)] = 0.02dt + \sigma dZ(t), \sigma > 0$$

while the price of Y stisfies $Y(t) = 100e^{0.08t+0.25Z(t)}$
You are also given that

- Y pays dividends continuously at a rate proportional to its price. The dividend yield is 0.04.
- X is nondividend-paying.
- The continuously compounded risk-free interest rate is 0.06

Determine σ.

Example 32. (T01Q14)

Let $S(t)$ be the time-t price of a nondividend-paying stock. The price process for $S(t)$ is

$$dS(t) = 0.22S(t)dt + 0.038S(t)dZ(t)$$

where $Z(t)$ is a standard Brownian motion. The continuously compounded risk-free rate is 0.1. For another nondividend-paying stock whose time-t price is $Q(t)$:

- $Q(0) = 90$
- $P(Q(1) \geq 100) = 0.95$
- $dQ(t) = AQ(t)dt + BQ(t)dZ(t)$ with $B < 1$.

Determine A.

1.5 The Black-Scholes Equation

Consider a derivative whose time-t price when the stock price is $S(t)$ is $V[S(t), t]$

- By using Itô's lemma, obtain $dV[S(t), t]$.

- Find m and s.

- Since the stock and derivative has Brownian motion $z(t)$. They have the same sharpe ratio, using this fact, derive the Black Schole Equation.

The Black-Scholes Equation for a derivative is

$$\boxed{V_t + (r-\delta)SV_s + \frac{1}{2}\sigma^2 S^2 V_{ss}] = rV}$$

where $V_t = \frac{\partial V}{\partial t}$, $V_s = \frac{\partial V}{\partial s}$, and $V_{ss} = \frac{\partial^2 V}{\partial s^2}$.

The pricing formula for any derivative must satisfy the Black-Scholes equation.

Example 33.
Assume the Black-Scholes framework.

(a) Show that $V[S(t),t] = S(t)e^{-\delta(T-t)}, 0 \le t \le T$, is the price of a certain derivative.

(b) What derivative does V represent?

Example 34. (T01Q15)
Let $S(t)$ be the time-t price of a stock, and $V(t)$ is the time-t price of a derivative security of the stock. You are given:

- $V(t) = e^{0.05t}(0.023t + \ln s(t))$.
- The continuously compounded risk free interest rate is 0.05.
- The stock pays dividends of $0.051S(t)dt$ between times t and $t+dt$.
- The derivative security does not pay dividends.

Determine σ^2, the square of volatility of the stock.

1.6 Risk-neutral Valuation

Recall in the single-period binomial model, we used risk-neutral valuation to price derivatives. The probability of up move is

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d}$$

The most important point in risk-neutral valuation is that α is not involved in the calculation. The quantity α only indicates the degree of risk aversion in the market.

To study risk-neutral valuation in continuous time, we rearrange the dynamics of S so that the drift of S becomes $(r-\delta)S(t)$.

$$\begin{aligned}\frac{dS(t)}{S(t)} &= (\alpha-\delta)dt + \sigma dZ(t) \\ &= (r-\delta)dt + \sigma dZ(t) + (\alpha - r)dt \\ &= (r-\delta)dt + \sigma\left[dZ(t) + \tfrac{\alpha-r}{\sigma}dt\right] \\ &= (r-\delta)dt + \sigma d[Z(t) + \phi t]\end{aligned}$$

Thus, we move to a world where

$$\tilde{Z}(t) = Z(t) + \phi t$$

Let $Z(t)$ be a standard Brownian motion in P, and ϕ be the Sharpe ratio os S. then

$$\tilde{Z}(t) = Z(t) + \phi t$$

is a standard Brownian motion in Q.

The following table summarizes how the distribution of $S(t)$ changes from the true measure to the risk-neutral measure.

	True measure	risk-neutral measure
Binomial model	$p = \frac{e^{(\alpha-\delta)h}-d}{u-d}$	$p^* = \frac{e^{(r-\delta)h}-d}{u-d}$
Black-Scholes model	$Z(t)$ is is a standard BM	$\tilde{Z}(t) = Z(t) + \phi t$ is a standard BM

Notice that $\tilde{Z}(t)$ is an arithmetic Brownian motion with drift ϕ and unit volatility under P, and $Z(t)$ is an arithmetic Brownian motion with drift $-\phi$ and unit volatility under Q.

To compute the time-t price of a European derivative:

- Step 1: Compute the risk-neutral expected payoff, $E^*[V(S(T),T)|S(t)]$.
- Stpe 2: Discount the risk-neutral expected payoff using risk free rate r

$$V(s(t),t) = e^{-r(T-t)}E^*[V(S(T),T)|S(t)]$$

A useful method to compute $E^*[V(S(T),T)|S(t)]$ is to compute $E[V(S(T),T)|S(t)]$ under the P measure. Then replace α with r.

Example 35. (T01Q16)

You are given:

- $S(t)$ is the time-t price of a stock.
- The stock pays dividend continuously at a constant rate proportional to its price.
- The true stock price process is given by
$$\frac{dS(t)}{S(t)} = cdt + \sigma dZ(t)$$
where $Z(t)$ is a standard Brownian motion under the true probability measure, and c and σ are constant.
- The risk-neutral stock price process is given by
$$\frac{dS(t)}{S(t)} = 0.054dt + 0.1d\tilde{Z}(t)$$
where $\tilde{Z}(t)$ is a standard Brownian motion under the risk-neutral measure.
- $Z(9) = \tilde{Z}(9) - 3.1500000000000004$.

Find c.

Example 36.

Assume the Black-Scholes framework. By using the risk-neutral valuation, find the time-t price of a prepaid forward contract that pays one share at T.

1.7 Valuing a Claim on S^a

For a claim that has the time T payoff $V(S(T),T)$, we can compute it time$-t$ price, $V(S(t),t)$ as

$$\boxed{V(S(t),t) = e^{-r(T-t)}E_0^*[V(S(T),T)]}$$

where E_0^* represents an expectation under the risk-neutral distributon.
Suppose a stock with an expected instantaneous return of α, dividend yield of δ, and instantaneous volatility σ follows geometric Brownian motion

$$dS(t) = (\alpha-\delta)S(t)dt + \sigma S(t)dZ(t)$$

Consider a claim maturing at time T that pays

$$V[S(T),T] = S^a(T).$$

If S follows

$$dS(t) = (\alpha-\delta)S(t)dt + \sigma S(t)dZ(t),$$

using Itô's Lemma, we obtain

$$\begin{aligned} dS^a &= as^{a-1}dS + \tfrac{1}{2}a(a-1)s^{a-2}(\sigma S)^2dt \\ &= aS^a\tfrac{dS}{S} + \tfrac{1}{2}a(a-1)S^a\sigma^2dt \\ &= aS^a[(\alpha-\delta)dt + \sigma dZ(t)] + \tfrac{1}{2}a(a-1)S^a\sigma^2dt \\ &= [a(\alpha-\delta) + \tfrac{1}{2}a(a-1)\sigma^2]S^adt + a\sigma S^adZ(t) \end{aligned}$$

Thus, S^a follows geometric Brownian motion with drift $a(\alpha-\delta)+\frac{1}{2}a(a-1)\sigma^2$ and volatility $a\sigma$.

If α is the expected return for S, if we let γ be the total return on $V=S^a$, then

$$\gamma = a(\alpha-r)+r$$

and the risk premium is $a(\alpha-r)$.

Another way to obtain the drift term is by writing

$$S^a(T) = S^a(t)e^{a(\alpha-\delta-0.5\sigma^2)(T-t)+a\sigma Z(T-t)}$$

and hence

$$E[S^a(T)] = S^a(t)e^{a(\alpha-\delta)(T-t)+\frac{1}{2}a(a-1)\sigma^2(T-t)}$$

The time-t price of an option whose payoff is $S^a(T)$ at time T is

$$\begin{aligned} V(S(t),t) &= e^{-r(T-t)}E^*[V(S(T),T)] \\ &= e^{-r(T-t)}E^*[(S^a(T))] \\ &= e^{-r(T-t)}S^a(t)e^{[a(r-\delta)+\frac{1}{2}a(a-1)\sigma^2](T-t)} \\ &= S^a(t)e^{[-r+a(r-\delta)+\frac{1}{2}a(a-1)\sigma^2](T-t)} \end{aligned}$$

$$\boxed{V(S(t),t) = F_{t,T}^P(S^a) = S^a(t)e^{[-r+a(r-\delta)+\frac{1}{2}a(a-1)\sigma^2](T-t)}}$$

Example 37. (T01Q17)
Let $S(t)$ be the time-t price of a nondividend-paying stock. You are given

- The true stochastic process of $S(t)$ is

  $$d[\ln S(t)] = 0.15dt + 0.29dZ(t)$$

  where $Z(t)$ is a standard Brownian motion under the true probability measure.
- For $0 \le t \le T$, the time$-t$ prepaid forward price for a delivery of 1 share of S^4 at time T if $F_{t,T}^P(S^4)$. The risk-neutral stochastic process of $F_{t,T}^P(S^4$ is

  $$d[\ln F_{t,T}^P(S^4) = gdt + hd\tilde{Z}(t)$$

  where $\tilde{Z}(t)$ is a standard Brownian motion under the risk neutral measure.
- The continuously compounded risk-free interest rate is 0.08.

Find g

Example 38. (T01Q18)

Assume the Black-Scholes framework for a nondividend-paying stock whose volatility is 39%. A contingent claim pays $\frac{S(5)S(4)}{S^2(3)}$ at time 5. Calculate the time-2 price of the contingent claim.

Example 39. (T01Q19)

Let $S(t)$ be the time-t price of a nondividend-paying stock, you are given that:

- The stock price process under the true probability measure is

  $$d[\ln S(t)] = 0.00275dt + 0.35000000000000003dZ(t), S(0) = 1$$

  where $Z(t)$ is a standard Brownian motion under the true probability measure.
- The sharpe ratio stock price risk is 0.06.

Compute the price of a contingent claim that pays $\sqrt[2]{S(5)}$ at time 5.

Example 40. (T01Q20)

Let $S(t)$ be the time-t price of a nondividend-paying stock, you are given that:

- The stock price process is

  $$d[\ln S(t)] = 0.35000000000000003dZ(t)$$

  where $Z(t)$ is a standard Brownian motion under the true probability measure.
- The continuously compounded risk-free of interest is 0.044

If $F_{0,2}^P(S^3) = e^{-\gamma}E[S^3(2)]$, find γ.