**MEME16203Linear Models Marking Guide**

**Assignment 3**

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| Faculty: | FES | Unit Code: | MEME16203 |
| Course: | MAC | Unit Title: | Linear Models |
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| Due by: | | | |

Q1. Suppose that $\mathbf{y}$ is $\text{MVN}_n(\boldsymbol{\mu}, \sigma^2\mathbf{I})$ and that $\mathbf{P}$, $\mathbf{Q}$, and $\mathbf{R}$ are symmetric $n \times n$ matrices with $\mathbf{PQ} = \mathbf{0}$ , $\mathbf{PR} = \mathbf{0}$, and $\mathbf{QR} = \mathbf{0}$ . Argue carefully that the three random variables $\mathbf{y^TPy}$, $\mathbf{y^TQy}$ and $\mathbf{y^TRy}$ are jointly independent.

*Ans.*

Consider

$$\begin{bmatrix} \mathbf{Py} \\ \mathbf{Qy} \\ \mathbf{Ry} \end{bmatrix} = \begin{bmatrix} \mathbf{P} \\ \mathbf{Q} \\ \mathbf{R} \end{bmatrix} \mathbf{y}$$

This is a multivariate normal distribution and for $i \neq j$

$$Cov(\mathbf{Py}, \mathbf{Qy}) = \mathbf{P}\Sigma\mathbf{Q} = 0$$

$$Cov(\mathbf{Py}, \mathbf{Ry}) = \mathbf{P}\Sigma\mathbf{R} = 0$$

$$Cov(\mathbf{Qy}, \mathbf{Ry}) = \mathbf{Q}\Sigma\mathbf{R} = 0$$

It follows that $\mathbf{Py}$, $\mathbf{Qy}$ and $\mathbf{Ry}$ are independent random vectors. Since

$$\begin{aligned} \mathbf{y^TPy} &= \mathbf{y^TPP^-Py} \\ &= \mathbf{y^TP^TP^-Py} \\ &= (\mathbf{Py})\mathbf{P}^-(\mathbf{Py}) \end{aligned}$$

is a function of $\mathbf{Py}$. Similarly, $\mathbf{y^TQy}$ is a function of $\mathbf{Qy}$ and $\mathbf{y^TRy}$ is a function of $\mathbf{Ry}$. Thus, $\mathbf{y^TPy}$, $\mathbf{y^TQy}$ and $\mathbf{y^TRy}$ are jointly independent.

Q2. Suppose $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$, where $\boldsymbol{\epsilon} \overset{iid}{\sim} N(\mathbf{0}, \sigma^2\mathbf{I}$ for some unknown $\sigma^2 > 0$.

(a) Determine the distribution of $\begin{bmatrix} \widehat{\mathbf{Y}} \\ \mathbf{Y} - \widehat{\mathbf{Y}} \end{bmatrix}$.

(b) Determine the distribution of $\widehat{\mathbf{Y}}^\mathbf{T}\widehat{\mathbf{Y}}$.

*Ans.*

(a) $$\begin{bmatrix}\widehat{\mathbf{Y}}\\ \mathbf{Y}-\widehat{\mathbf{Y}}\end{bmatrix} = \begin{bmatrix}\mathbf{P_X Y}\\ \mathbf{Y}-\mathbf{P_X Y}\end{bmatrix} = \begin{bmatrix}\mathbf{P_X}\\ \mathbf{I}-\mathbf{P_X}\end{bmatrix}\mathbf{Y}$$

$$E\left(\begin{bmatrix}\mathbf{P_X}\\ \mathbf{I}-\mathbf{P_X}\end{bmatrix}\mathbf{Y}\right) = \begin{bmatrix}\mathbf{P_X}\\ \mathbf{I}-\mathbf{P_X}\end{bmatrix}E(\mathbf{Y}) = \begin{bmatrix}\mathbf{P_X}\\ \mathbf{I}-\mathbf{P_X}\end{bmatrix}\mathbf{X}\boldsymbol{\beta} = \begin{bmatrix}\mathbf{X}\boldsymbol{\beta}\\ \mathbf{0}\end{bmatrix}$$

$$V\left(\begin{bmatrix}\mathbf{P_X}\\ \mathbf{I}-\mathbf{P_X}\end{bmatrix}\mathbf{Y}\right)$$
$$= \begin{bmatrix}\mathbf{P_X}\\ \mathbf{I}-\mathbf{P_X}\end{bmatrix}V(\mathbf{Y})\begin{bmatrix}\mathbf{P_X} & \mathbf{I}-\mathbf{P_X}\end{bmatrix}$$
$$= \begin{bmatrix}\mathbf{P_X}\\ \mathbf{I}-\mathbf{P_X}\end{bmatrix}\sigma^2\mathbf{I}\begin{bmatrix}\mathbf{P_X} & \mathbf{I}-\mathbf{P_X}\end{bmatrix}$$
$$= \sigma^2\begin{bmatrix}\mathbf{P_X}\\ \mathbf{I}-\mathbf{P_X}\end{bmatrix}\begin{bmatrix}\mathbf{P_X} & \mathbf{I}-\mathbf{P_X}\end{bmatrix}$$
$$= \sigma^2\begin{bmatrix}\mathbf{P_X P_X} & \mathbf{P_X}(\mathbf{I}-\mathbf{P_X})\\ (\mathbf{I}-\mathbf{P_X})\mathbf{P_X} & (\mathbf{I}-\mathbf{P_X})(\mathbf{I}-\mathbf{P_X})\end{bmatrix}$$
$$= \sigma^2\begin{bmatrix}\mathbf{P_X} & \mathbf{0}\\ \mathbf{0} & (\mathbf{I}-\mathbf{P_X})\end{bmatrix}$$

As a linear transformation of a multivariate normal random variable, it follows that
$$\begin{bmatrix}\widehat{\mathbf{Y}}\\ \mathbf{Y}-\widehat{\mathbf{Y}}\end{bmatrix} \sim N\left(\begin{bmatrix}\mathbf{X}\boldsymbol{\beta}\\ \mathbf{0}\end{bmatrix}, \sigma^2\begin{bmatrix}\mathbf{P_X} & \mathbf{0}\\ \mathbf{0} & (\mathbf{I}-\mathbf{P_X})\end{bmatrix}\right).$$

(b) $\widehat{\mathbf{Y}}^{\mathbf{T}}\widehat{\mathbf{Y}} = (\mathbf{P_X Y})^{\mathbf{T}}(\mathbf{P_X Y}) = \mathbf{Y^T P_X P_X Y} = \mathbf{Y^T P_X Y}$

Let $\mathbf{A} = \frac{\mathbf{P_X}}{\sigma^2}$ and $\Sigma = \sigma^2\mathbf{I}$

$\mathbf{A\Sigma} = \mathbf{P_X}$ is clearly idempotent

$DF = rank(\mathbf{P_X}) = rank(\mathbf{X})$

$\lambda = \frac{1}{\sigma^2}\boldsymbol{\beta}^T\mathbf{X}^T P_X \mathbf{X}\boldsymbol{\beta} = \frac{1}{\sigma^2}\boldsymbol{\beta}^T\mathbf{X}^T\mathbf{X}\boldsymbol{\beta}$

Therefore, we end up with a scaled non-central chi-square random variable on $rank(\mathbf{X})$degrees of freedom: $\widehat{\mathbf{Y}}^{\mathbf{T}}\widehat{\mathbf{Y}} \sim \sigma^2\chi^2_{rank(\mathbf{X})}(\frac{1}{\sigma^2}\boldsymbol{\beta}^T\mathbf{X}^T\mathbf{X}\boldsymbol{\beta})$

Q3. Consider the model

$$Y_{ij} = \gamma_0 + \gamma_1 X_i + \gamma_2 X_i^2 + \alpha_i + \epsilon_{ij}, \quad i = 1, 2, \ldots, 11; \quad j = 1, \ldots, 5$$

where $\epsilon \sim NID(0, \tau^2)$. This model can be expresses in matrix notation as $\mathbf{Y} = \mathbf{W}\boldsymbol{\gamma} + \boldsymbol{\epsilon}$. Let the matrix $\mathbf{Z}$ be the first 3 columns of the matrix $\mathbf{W}$, define $\mathbf{P_Z} = \mathbf{Z}(\mathbf{Z^T Z})^{-1}\mathbf{Z^T}$ and $\mathbf{P_W} = \mathbf{W}(\mathbf{W^T W})^{-1}\mathbf{W^T}$.

(a) Use Cochran's theorem to derive the distribution of $F = \dfrac{c\mathbf{Y^T}(\mathbf{P_W}-\mathbf{P_Z})\mathbf{Y}}{\mathbf{Y^T}(\mathbf{I}-\mathbf{P_W})\mathbf{Y}}$. Report $c$, degrees of freedom and a formula for the noncentrality parameter.

*Ans.*

Let $\mathbf{A_1} = \mathbf{P_Z}$, $\mathbf{A_2} = \mathbf{P_W} - \mathbf{P_Z}$ and $\mathbf{A_3} = \mathbf{I} - \mathbf{P_W}$. Then,

- $\mathbf{A_1}$, $\mathbf{A_2}$ and $\mathbf{A_3}$ are all $55 \times 55$ symmetric matrices.
- $\mathbf{A_1} + \mathbf{A_2} + \mathbf{A_3} = \mathbf{I}$.
- rank($\mathbf{A_1}$) + rank($\mathbf{A_2}$)+ rank($\mathbf{A_3}$) $= 3 + (13-3) + (55-13) = 55$.

Then, by Cochran's Theorem,

- $\frac{1}{\tau^2}\mathbf{Y^T A_k Y} \sim \chi^2_{r_k}(\frac{1}{\tau^2}(\mathbf{X}\boldsymbol{\beta})^T\mathbf{A_k}\mathbf{X}\boldsymbol{\beta})$, where $r_k$ = rank($\mathbf{A_k}$) for $k = 1, 2, 3$
- $\mathbf{Y^T A_1 Y}$, $\mathbf{Y^T A_2 Y}$ and $\mathbf{Y^T A_3 Y}$ are distributed independently.

Now $DF_2$ = rank($\mathbf{A_2}$) = 13-3 = 10 and $DF_3$ = rank($\mathbf{A_3}$) = 55-13 = 42

- $\lambda_2 = \frac{1}{\tau^2}(\mathbf{W}\boldsymbol{\gamma})^T(\mathbf{P_W} - \mathbf{P_Z})(\mathbf{W}\boldsymbol{\gamma})$
- $\lambda_3 = \frac{1}{\tau^2}(\mathbf{W}\boldsymbol{\gamma})^T(\mathbf{I} - \mathbf{P_W})(\mathbf{W}\boldsymbol{\gamma}) = 0$

Hence,

- $\frac{1}{\tau^2}\mathbf{Y^T}(\mathbf{P_W} - \mathbf{P_Z})\mathbf{Y} \sim \chi^2_{10}(\lambda_2)$ and
- $\frac{1}{\tau^2}\mathbf{Y^T}(\mathbf{I} - \mathbf{P_W})\mathbf{Y} \sim \chi^2_{42}$

Since $\frac{1}{\tau^2}\mathbf{Y^T}(\mathbf{P_W} - \mathbf{P_Z})\mathbf{Y}$ and $\frac{1}{\tau^2}\mathbf{Y^T}(\mathbf{I} - \mathbf{P_W})\mathbf{Y}$ are independent, then

$$F = \frac{c\mathbf{Y^T}(\mathbf{P_W} - \mathbf{P_Z})\mathbf{Y}}{\mathbf{Y^T}(\mathbf{I} - \mathbf{P_W})\mathbf{Y}} = \frac{\frac{1}{10\tau^2}\mathbf{Y^T}(\mathbf{P_W} - \mathbf{P_Z})\mathbf{Y}}{\frac{1}{42\tau^2}\mathbf{Y^T}(\mathbf{I} - \mathbf{P_X})\mathbf{Y}} = \frac{42\mathbf{Y^T}(\mathbf{P_X} - \mathbf{P_1})\mathbf{Y}}{10\mathbf{Y^T}(\mathbf{I} - \mathbf{P_X})\mathbf{Y}} \sim F_{10,42}(\lambda_2).$$

Thus,

- $c = \frac{42}{10}$,
- the degrees of freedom are $(10, 42)$ and
- noncentrality parameter is $\lambda_2 = \frac{1}{\tau^2}(\mathbf{W}\boldsymbol{\gamma})^T(\mathbf{P_W} - \mathbf{P_Z})(\mathbf{W}\boldsymbol{\gamma})$

(b) Show that the noncentrality parameter is zero if $\alpha_1\mathbf{w_4} + \alpha_2\mathbf{w_5} + \cdots + \alpha_{11}\mathbf{w_{14}} = \mathbf{Zc}$ for some vector $\mathbf{c}$, where $\mathbf{w_j}$ is the $j^{th}$ column of $\mathbf{W}$.

*Ans.*

$\frac{1}{\tau^2}(\mathbf{W}\boldsymbol{\gamma})^T(\mathbf{P_W} - \mathbf{P_Z})(\mathbf{W}\boldsymbol{\gamma})$
$= \frac{1}{\tau^2}[(\mathbf{W}\boldsymbol{\gamma})^T(\mathbf{I} - \mathbf{P_Z})(\mathbf{W}\boldsymbol{\gamma}) - (\mathbf{W}\boldsymbol{\gamma})^T(\mathbf{I} - \mathbf{P_W})(\mathbf{W}\boldsymbol{\gamma})]$
$= \frac{1}{\tau^2}(\mathbf{W}\boldsymbol{\gamma})^T(\mathbf{I} - \mathbf{P_Z})(\mathbf{W}\boldsymbol{\gamma})$

This is zero if and only if $(\mathbf{I} - \mathbf{P_Z})(\mathbf{W}\boldsymbol{\gamma}) = 0$. Note that

$(\mathbf{I} - \mathbf{P_Z})(\mathbf{W}\boldsymbol{\gamma})$
$= (\mathbf{I} - \mathbf{P_Z})\left[\mathbf{Z}\ \mathbf{w_4}\ \mathbf{w_5}\ \cdots\ \mathbf{w_{11}}\right]\boldsymbol{\gamma}$
$= \left[\mathbf{0}\ (\mathbf{I} - \mathbf{P_Z})\mathbf{w_4}\ (\mathbf{I} - \mathbf{P_Z})\mathbf{w_5}\ \cdots\ (\mathbf{I} - \mathbf{P_Z})\mathbf{w_{14}}\right]\boldsymbol{\gamma}$
$= \left[\alpha_1(\mathbf{I} - \mathbf{P_Z})\mathbf{w_4}\ \alpha_2(\mathbf{I} - \mathbf{P_Z})\mathbf{w_5}\ \cdots\ \alpha_{10}(\mathbf{I} - \mathbf{P_Z})\mathbf{w_{14}}\right]$
$= (\mathbf{I} - \mathbf{P_Z})\left[\alpha_1\mathbf{w_4}\ \alpha_2\mathbf{w_5}\ \cdots\ \alpha_{10}\mathbf{w_{14}}\right]$
$= (\mathbf{I} - \mathbf{P_Z})\mathbf{Zc}$
$= 0$ since $\mathbf{P_Z} = \mathbf{Z}$

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Q4. Consider the model $Y_{ij} = \mu + \alpha_i + \epsilon_{ij}$, where $i = 1, 2, 3$, $j = 1, 2, 3$, and $\mu$, $\alpha_1$, $\alpha_2$, $\alpha_3$, are unknown parameters. Let $\epsilon_{ij} \overset{iid}{\sim} N(0, \sigma^2)$, where $\sigma^2$ is unknown.

(a) Determine the distribution of $\frac{\hat{\tau}^2}{35\sigma^2}$ when $\tau = 0$, where $\hat{\tau}$ is the BLUE of $\tau = 2\alpha_1 - 8\alpha_2 + 6\alpha_3$.

*Ans.*

$$\begin{aligned}\tau &= 2\alpha_1 - 8\alpha_2 + 6\alpha_3 \\ &= 2(\mu + \alpha_1) - 8(\mu + \alpha_2) + 6(\mu + \alpha_3) \\ &= E(2\bar{Y}_{1.} - 8\bar{Y}_{2.} + 6\bar{Y}_{3.})\end{aligned}$$

Hence, $\tau$ is estimable.

The BLUE for $\tau$ is $2\bar{Y}_{1.} - 8\bar{Y}_{2.} + 6\bar{Y}_{3.}$. Let $\mathbf{Y} = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{21} & Y_{22} & Y_{23} & Y_{31} & Y_{32} & Y_{33} \end{bmatrix}^T$, then

$\hat{\tau} = 2\bar{Y}_{1.} - 8\bar{Y}_{2.} + 6\bar{Y}_{3.} = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & -\frac{8}{3} & -\frac{8}{3} & -\frac{8}{3} & \frac{6}{3} & \frac{6}{3} & \frac{6}{3} \end{bmatrix} \mathbf{Y} \sim N(\tau = 0, 35\sigma^2)$.

Then,

$\frac{\hat{\tau} - 0}{\sqrt{35\sigma^2}} \sim N(0, 1)$ and $\frac{\hat{\tau}^2}{35\sigma^2} \sim \chi^2_1$.

(b) Determine the distribution of $S^2 = \sum_{i=1}^{3} \sum_{j=1}^{3} (Y_{ij} - \bar{Y}_{i.})^2$.

*Ans.*

Note that $S^2 = \mathbf{Z^T Z}$ where

$$\mathbf{Z} = \begin{bmatrix} Y_{11} - Y_{1.} \\ Y_{12} - Y_{1.} \\ Y_{13} - Y_{1.} \\ Y_{21} - Y_{2.} \\ Y_{22} - Y_{2.} \\ Y_{23} - Y_{2.} \\ Y_{31} - Y_{3.} \\ Y_{32} - Y_{3.} \\ Y_{33} - Y_{3.} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} Y_{11} \\ Y_{12} \\ Y_{13} \\ Y_{21} \\ Y_{22} \\ Y_{23} \\ Y_{31} \\ Y_{32} \\ Y_{33} \end{bmatrix} = \mathbf{CY}.$$

Then $\mathbf{Z} \sim N(\mathbf{0}, \sigma^2\mathbf{C})$ and $\mathbf{CC} = \mathbf{C}$. Let $\mathbf{\Sigma} = \sigma^2\mathbf{C}$ and $\mathbf{A} = \frac{1}{\sigma^2}\mathbf{I}$. Then $\mathbf{A\Sigma A\Sigma} = \mathbf{CC} = \mathbf{C} = \mathbf{A\Sigma}$ which is idempotent. Hence it follows that $\frac{S^2}{\sigma} = \mathbf{Z^T A Z} \sim \chi^2_6$ because rank$(\mathbf{C}) = 6$

(c) Show that $F = \frac{c\hat{\tau}^2}{S^2}$, where $c$ is a constant, has central $F-$distribution when $\tau = 0$. Report c.

*Ans.*

Note that

$$\begin{bmatrix} \hat{\tau} \\ \mathbf{Z} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & -\frac{8}{3} & -\frac{8}{3} & -\frac{8}{3} & \frac{6}{3} & \frac{6}{3} & \frac{6}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} Y_{11} \\ Y_{12} \\ Y_{13} \\ Y_{21} \\ Y_{22} \\ Y_{23} \\ Y_{31} \\ Y_{32} \\ Y_{33} \end{bmatrix} \sim N\left( \begin{bmatrix} \tau \\ \mathbf{0} \end{bmatrix}, \sigma^2 \begin{bmatrix} 35 & \mathbf{0^T} \\ \mathbf{0} & \mathbf{C} \end{bmatrix} \right)$$

Consequently, $\widehat{\boldsymbol{\tau}}$ and $\mathbf{Z}$ are independent which implies that $\frac{\hat{\tau}}{35\sigma^2}$ and $\frac{S^2}{\sigma^2}$ are independent central chi-square random variables with 1 and 6 degrees freedom respectively, and

$F = \frac{\frac{\hat{\tau}^2}{35\sigma^2}}{\frac{S^2}{6\sigma^2}} = \frac{6\hat{\tau}^2}{35\sigma^2} \sim F(1,3)$ and $c = \frac{6}{35}$.