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7.1 Reinvestment Rates

Previously, we assume that the reinvestment rate is level throughout the entire period of investment and equals to IRR. In practice, the lender (investor) may be able to reinvest the payments from the borrower at rates higher or lower than the original investment. Hence, the reinvestment rate is usually lower than the yield rate. This is demonstrated with the following types of investments.

1. D for n periods at rate i s.t. the interest is reinvested at rate j:

```
i account
D
|-----|-----|--...--|-----|
0     1     2        n-1   n
     iD    iD       iD    iD
                          AV = D
j account
      iD    iD       iD    iD
|-----|-----|--...--|-----|
0     1     2        n-1   n
                     AV = iDs_(n)|j
```

$$AV = D + iDs_{\overline{n}|j}.$$

2. D at the beginning of each period for n periods at rate i s.t. the interest is reinvested at rate j:

```
i account
D     D     D       D
|-----|-----|--...--|-----|
0     1     2       n-1   n
     iD   2iD   (n-1)iD   niD
                         AV = nD
j account
      iD   2iD  (n-1)iD  niD
|-----|-----|--...--|-----|
0     1     2       n-1   n
                     AV = iD(Is)_(n)|j
```

$$
\begin{aligned}
\text{AV} &= nD + iD(Is)_{\overline{n|}j} \\
&= nD + iD\left[\frac{s_{\overline{n+1|}j} - (n+1)}{j}\right]
\end{aligned}
$$

3. A loan of amount L and receives n periodic payments of R at interest rate i as received, but can reinvest them at rate j. The adjusted yield rate, i', is governed by

$$L(1+i')^n = Rs_{\overline{n}|j}.$$

4. A bond is purchased for P, coupons of Fr are paid at the end of each period for n periods, the bond is redeemed for C at the end of n periods, and coupons are reinvested at rate j. Hence i' satisfies

$$P(1+i')^n = Frs_{\overline{n}|j} + C.$$

Example 1.

1000 is invested at the beginning of year 1 for 15 years at 11%. The interest is reinvested at 5.5%. What is the total AV at the end of 15 years?

Ans:

$$\begin{aligned}\text{AV} &= D + iDs_{\overline{n|}j\%}\\ &= 1000 + (0.11)(1000)s_{\overline{15|}5.5\%}\\ &= 1000 + (0.11)(1000)\left[\frac{1.06^{15}-1}{0.06}\right]\\ &= \boxed{3{,}560.36}\end{aligned}$$

Example 2 (T7Q1).

Payments of 1,000 are invested at the beginning of each year for 10 years. The payments earn interest at 7% effective and the interest can be reinvested at 5% effective. Find the purchase price an investor should pay to produce a yield rate of 8% effective.

Ans:

$$\begin{aligned}
\text{AV} &= nD + iD(Is)_{\overline{n}|j\%} \\
&= nD + iD\left[\frac{s_{\overline{n+1}|j} - (n-1)}{j}\right] \\
&= 10(1000) + 0.07(1000)\left[\frac{s_{\overline{11}|5\%} - 11}{0.05}\right] \\
&= 10(1000) + 0.07(1000)\left[\frac{s_{\overline{11}|5\%} - 11}{0.05}\right] \\
&= 14489.52
\end{aligned}$$

$s_{\overline{11}|5\%} = \frac{1.05^{11}}{0.05} = 14.2068$

Purchase Price $= 14489.52(1{+}0.08)^{10} = 14489.52(1.08^{1}$

$= \boxed{6{,}711.45}$

Example 3 (T7Q2).

A 100 par value 20-year bond with 9% semiannual coupons is selling for 88. If the coupons from the bond can be reinvested at only i1% convertible semiannually. Find the yield rate convertible semiannually taking into account reinvestment rates.

Ans:

$88(1+i')^{40} = 4.5s_{\overline{40}|0.035} + 100$

$s_{\overline{40}|0.035} = \frac{1.035^{40}-1}{0.035} = 84.5503$

$\Rightarrow (1+i')^{40} = \frac{4.5(84.5503)+100}{88} = 5.45996$

$i' = 0.04335$

$i'^{(2)} = 2(0.04335) = \boxed{0.0867}$.

Example 4 (T7Q3)**.**

Sally lends 60,000 to Tim. Tim agrees to pay back the loan over 7 years with monthly payments payable at the end of each month. Sally can reinvest the monthly payments from Tim in a savings account paying interest at 6%, compounded monthy. The yield rate earned on Sally's investment over the 7-period turned out to be 6.45% compounded semiannually. What nominal rate of interest, compounded monthly, did Sally charge Tim on the loan?

Ans:

Let j_1 be Sally's reinvestment monthly effective saving rate, j_2 be the annual effective yield rate earned by Sally's investment, j_3 be the monthly effective rate that Sally charge Tim

$j_1 = \frac{6\%}{12} = 0.0050$

$(1 + j_2) = 1.0322^2$

$$\frac{60{,}000}{a_{\overline{84}|j_3}} s_{\overline{84}|j_1} = 60000(1.0322)^{2(7)}$$

$$s_{\overline{84}|j_1} = \frac{1.0050^{84}}{0.0050} = 104.0739272164164$$

$$a_{\overline{84}|j_3} = \frac{s_{\overline{84}|j_1}}{(1.0322)^{2(7)}} = \frac{104.0739}{(1.0322)^{2(7)}} = 66.7350$$

$$\frac{1-(1+j_3)^{-84}}{j_3} = 66.7350$$

$$f(j_3) = \frac{1-(1+j_3)^{-84}}{j_3} - 66.73496872342591 = 0$$

$$j_3 = [0.00564878]$$

$$i = j_3 \times 12 \times 100 = [0.00564878] \times 12 \times 100 = \boxed{[6.77853172]\%}$$

7.2 Dollar Weighted Interest Rates

Suppose a fund earns a nominal rate of interest of 100% compounded semiannually in the first 6 months. The same fund earns a nominal rate of 20% compounded semiannually in the second 6 months. which of the following schedules of deposits would result in a higher equivalent level effective rate of interest for the year?

(a) \$100 at time 0 and \$10 at time $\frac{1}{2}$.

(b) \$10 at time 0 and \$100 at time $\frac{1}{2}$.

The results under the two schedules of deposits are as follows:

(a) $100(1.5)(1.1) + 10(1.1) = 176$
Using simple interest from the date of each deposit to approximate the equivalent level annual effective rate i:
$100(1 + i) + 10(1 + i/2) = 176$
$i = 62.86\%$

(b) $10(1.5)(1.1) + 100(1.1) = 126.5$

Using simple interest from the date of each deposit to approximate the equivalent level annual effective rate i:

$10(1 + i) + 100(1 + i/2) = 126.5$

$i = 27.5\%$

From the above results, we see that the effective annual rate is very much dependent on the amount of dollars invested. When we use the actual dollars invested in computing the effective rate, the rate is referred to a **dollar-weighted** rate of interest.

Let

- A be the amount in the fund at the beginning of the year;
- B be the amount in the fund at the end of the year; and
- C_t be the deposit in (positive value) or the withdrawal from (negative value) the fund at time t.

Then,

$$A(1+i) + \sum C_t(1+i)^{1-t} = B$$

Assuming simple interest from the date of each deposit or withdrawal to the end of the year, we have:

$$A(1+i) + \sum C_t[1+(1-t)i] = B$$

$$\boxed{i_{DW} = \frac{B - A - \sum C_t}{A + \sum C_t(1-t)}}$$

Notes:

- $\sum C_t$ = Total Deposit - Total Withdrawal
- $B - A - \sum C_t$ = Investment Income

Example 5 (T7Q4)**.**

An investment fund has a value of 5,000 at the beginning and the end of the year. A deposit of 1000 was made at the end of 4 months. A withdrawal of 1500 was made at the end of 7 months. Find the rate of interest earned by the fund assuming simple interest during the year.

Ans:

Total Deposit = 1000.0

Total Withdrawal = 1500.0

$\sum C_t = 1000.0 - 1500.0 = -500.0$

Investment Income $= B - A - \sum C_t = 5000 - 5000 - (-500.0) = 500.0$

$$\begin{aligned} i_{DW} &= \frac{B-A-\sum C_t}{A+\sum C_t(1-t)} \\ &= \frac{500.0}{5000+1000\left(\frac{8}{12}\right)-1500\left(\frac{5}{12}\right)} \\ &= \boxed{0.0992} \end{aligned}$$

Example 6.

An association had a fund balance of 80 on January 1 and 65 on December 31. At the end of every month during the year, the association deposited 13 from membership fees. There were withdrawals of 7 on February 28, 35 on June 30, 98 on October 15, and 45 on October 31. Calculate the dollar-weighted rate of return for the year.

Ans:

Total Deposit $= 12(13) = 156$

Total Withdrawal $= 7 + 35 + 98 + 45 = 185$

$\sum C_t = 156 - 185 = -29$

Investment Income $= B - A - \sum C_t = 65 - 80 - (-29) = 14$

$$
\begin{aligned}
i_{DW} &= \frac{B-A-\sum C_t}{A+\sum C_t(1-t)} \\
&= \frac{14}{80+13\left(\frac{1}{12}+\cdots+\frac{11}{12}\right)-7\left(\frac{10}{12}\right)-35\left(\frac{1}{2}\right)-98\left(\frac{2.5}{12}\right)-45\left(\frac{2}{12}\right)} \\
&= \frac{14}{100.25} \\
&= \boxed{0.1397}
\end{aligned}
$$

7.3 Time-Weighted Interest Rate

The dollar-weighted methods for computing the yield rate earned by an investment fund are sensitive to the amounts of money invested during various subperiods when the investment experience is volatile during the year. A better measure of A fund's underlying performance is to determine the yield rate based on investing a single amount at the beginning of the year, with no further deposits or withdrawals. This rate is called the **time-weighted** rate of interest.

Let

- B_k = Balance just preceding deposits or withdrawals at time t_k
- W_k = Deposit(+) or withdrawal(-) at t_k
- $W_0 = W_n = 0$

Define interest over $[t_{k-1}, t_k]$
$= B_k - (B_{k-1} + W_{k-1})$

Let i_k be the effective rate of interest over the interval $[t_{k-1}, t_k]$
Define i_k:

$$i_k(B_{k-1} + W_{k-1}) = B_k - (B_{k-1} + W_{k-1}))$$

or

$$i_k = \frac{B_k - (B_{k-1} + W_{k-1})}{B_{k-1} + W_{k-1}}$$

and

$$1 + i_k = \frac{B_k}{B_{k-1} + W_{k-1}}$$

The time-weighted rate of return,

$$\boxed{i_{tw} = (1 + i_1)(1 + i_2) \cdots (1 + i_n) - 1}$$

Remarks:

(i) The formula for i_k looks complicated, but it is just "interest earned" in the period divided by balance at beginning of period. Interest earned can be positive, negative, or zero.

Interest earned $= B_k - (B_{k-1} + W_{k-1})$

(ii) If balances are not given (and cannot be determined from other information) you can't calculate time-weighted rate of return.

(iii) time-weighted rate return is always calculated as a rate for the entire period $[0, t_n]$ (not necessarily a year). The subintervals $\{[t_{k-1}, t_k]\}$ are not necessarily equal in length.

(iv) If the time-weighted rate is calculated for a period of n years, then the equivalent annual rate is the solution to $(1+i)^n = (1+i_{tw})$.

Example 7 (T7Q5).

11,000 is invested on January 1, 2020. On July 1, 2020, the balance is 12,100. Immediately after calculation of this balance, 847 is withdrawn from the account. 14,520 is in the account on January 1, 2021. What is the time-weighted rate of return over 2020?

Ans:

```
B0 = 11000      B1 = 12100.0  B2 = 14520.0
   |-------------|-------------|
   1/1/20        1/7/20        1/1/21
                 W1 = -847.0
```

$1 + i_1 = \frac{B_1}{B_0+W_0} = \frac{12{,}100}{11{,}000+0} = 1.1$

$1 + i_2 = \frac{B_2}{B_1+W_1} = \frac{14{,}520}{12{,}100-847} = 1.2903$

$(1 + i_{tw}) = (1 + i_1)(1 + i_2) = (1.1)(1.2903) = 1.4193$

$i_{tw} = \boxed{0.4193}$

Example 8.

An investment account has 0 in it on January 1 and 2000 of new principal deposited. On April 1, the value of the account has increased to 2,600 and an additional deposit of 840.0 is made. On September 1, the value has increased to 3,120 and 680.0 is withdrawn. On the following January 1, the investment account is worth 2,600. what is the time-weighted rates of interest for the year?

Ans:

```
B0 = 0   B1 = 2600.0      B2 = 3120.0 B3 =
   |----------|---------|---------|
   1/1        1/4       1/9       1/1
 W0 = 2000   W1 = 840.0       W2 = -680.0
```

$1+i_1 = \frac{B_1}{B_0+W_0} = \frac{2,600}{2000} = 1.3$

$1+i_2 = \frac{B_2}{B_1+W_1} = \frac{3,120}{2,600+840} = 0.907$

$1+i_3 = \frac{B_3}{B_2+W_2} = \frac{2600.0}{3120.0-680.0} = 1.0656$

$1+i_{tw} = (1+i_1)(1+i_2)(1+i_3) = 1.3(0.907)(1.0656) = 1.2564$

$i_{tw} = \boxed{0.2564}$

Example 9.

An investor deposits 50 in an investment account on January 1. the following summarizes the activity in the account during the year:

Date	Value Immediately Before Deposit	Deposit
March 15	40	20
June 1	80	80
October 1	175	75

On June30, the value of the account is 157.50, On December 31, the value of the account is X. Using the time-weighted method, the equivalent annual yield during the first 6 months is equal to the (time-weighted) annual effective yield during the entire 1-year period. Calculate X. $\boxed{236.25}$

Ans:

6-month yield:

```
B0=0   B1 = 40      B2 = 80   B3 = 157.5
 |-------|-------|---------|
 1/1       15/3     1/6        30/6
W0 = 50   W1 = 20 W2=80
```

$1+i_1 = \frac{B_1}{B_0+W_0} = \frac{40}{50}$
$1+i_2 = \frac{B_2}{B_1+W_1} = \frac{80}{60}$
$1+i_3 = \frac{B_3}{B_2+W_2} = \frac{157.5}{160}$
$1+j = \left(\frac{40}{50}\right)\left(\frac{80}{60}\right)\left(\frac{157.5}{160}\right) = 1.05$
$1+i = (1+j)^2 = 1.05^2$, Thus,
Effective annual yield $= i = 1.05^2 - 1 = 0.1025$
1-year yield:

```
B0=0  B1 = 40     B2 = 80  B3 = 175  B4 = X
 |--------|--------|---------|----------|
 1/1      15/3     1/6      1/10        31/12
W0 = 50  W1 = 20 W2=80    W3=75
```

$1+i_1 = \frac{B_1}{B_0+W_0} = \frac{40}{50}$
$1+i_2 = \frac{B_2}{B_1+W_1} = \frac{80}{60}$
$1+i_3 = \frac{B_3}{B_2+W_2} = \frac{175}{160}$
$1+i_4 = \frac{B_4}{B_3+W_3} = \frac{X}{250}$
$1+i = \left(\frac{40}{50}\right)\left(\frac{80}{60}\right)\left(\frac{175}{160}\right)\left(\frac{X}{250}\right) = 1.1025$
$X = \boxed{236.25}$

Example 10 (T7Q6).

The following table gives information concerning an investment fund (in RM millions):

Calendar Year	2019	2020	2021	2022
Value of fund at 30 June	-	480	520	740
Net cash flow received on 1 July	-	46	47	65
Value of fund at 31 December	410	550	600	X

If the time weighted rate of return earned on the fund during the period from 31 December 2019 to 31 December 2022 is 10% per annum effective, calculate X, the value of the fund on 31 December 2022.

Ans:

```
B0=410    B1=480        B2=520      B3=740      B4=X
    |-----------|----------|----------|----------|
                W1=46     W2=47       W3=65
```

$1 + i_1 = \frac{B_1}{B_0+W_0} = \frac{480}{410}$

$1 + i_2 = \frac{B_2}{B_1+W_1} = \frac{520}{480+46}$

$1 + i_3 = \frac{B_3}{B_2+W_2} = \frac{740}{520+47}$

$1 + i_4 = \frac{B_4}{B_3+W_3} = \frac{X}{740+65}$

$(1 + i_{tw})^3 = (1 + i_1)(1 + i_2)(1 + i_3)(1 + i_4)$

$(1 + 0.1)^3 = \frac{480}{410} \times \frac{520}{480+46} \times \frac{740}{520+47} \times \frac{X}{740+65}$

$X = \boxed{709.33}$