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# 3 Aggregate Loss Models

## 3.1 Introduction

Aggregate losses are the total losses paid by an insurer for a defined portfolio of insureds in one period, say a year. They are the sum of the individual losses for the year.

**Definition 1.**
The **collective risk model(Compound Distribution**, $S = X_1 + X_2 + \ldots + X_N$, is the aggregate loss variable, with $N$ as the random variable representing the number of losses in the year. $X_j's$ are independent and identically (i.i.d.) random variables. The independence assumptions are

(i) Conditional on $N = n$, the random variables $X_1 + X_2 + \ldots + X_n$ are i.i.d. random variables.

(ii) Conditional on $N = n$, the common distribution of the random variables $X_1 + X_2 + \ldots + X_n$ does not depend on $n$.

(iii) The distribution of $N$ does not depend in any way on the values of $X_1 + X_2 + \ldots$.





## 3.2 The Recursive Method

Suppose that the severity distribution $f_X(x)$ is define on $0, 1, 2, \ldots, m$ representing multiples of some convenient monetary unit. The number $m$ represents the largest possible payment and could be infinite. Further suppose that the frequency distribution, $p_k$, is a member of the $(a, b, 1)$ class and therefore satisfies

$$p_k = \left(a + \frac{b}{k}\right) p_{k-1}, k = 2, 3, 4, \ldots.$$

Then, the following result holds.

**Theorem 1.** For the $(a, b, 1)$ class,

$$f_S(k) = g_k = \frac{[p_1^M - (a+b)p_0^M] f_k + \sum_{j=1}^{k} \left(a + \frac{bj}{k}\right) f_j g_{k-j}}{1 - a f_0}$$





**Corollary:**

For the $(a, b, 0)$ class,

$$f_S(k) = g_k = \frac{\sum_{j=1}^{k} \left(a + \frac{bj}{k}\right) f_j g_{k-j}}{1 - a f_0}$$

The starting value of the recursive is

$$\begin{aligned} f_S(0) &= P(S = 0) \\ &= P_N[P_X(0)] \\ &= P_N[f_0]. \end{aligned}$$

**Example 1.**

The number of claims in a period has a Negative Binomial distribution with parameters $r = 5$ and $\beta = 7$. The amount of each claim $X$ follows $P(X = x) = 0.25$, $x = 0, 1, 2, 3$. The number of claims and claim amounts are independent. $S$ is the aggregate claim amount in the period. Using recursive formula, calculate $F_S(3)$.



**Example 2.**

The number of claims on an insurance coverage follows a binomial distribution with parameters $m = 3$, $q = 0.2$. The size of each claim has the following distribution:

| $x$ | 0 | 1 | 2 |
|---|---|---|---|
| $P(X = x)$ | 0.50 | 0.35 | 0.15 |

Calculate the probability of aggregate claims of 3 or more. $\boxed{0.0147}$





**Example 3.**

Taxis arrive at an airport in a Poisson process at a rate of 4 per minute. Each taxi picks up 1 to 4 passengers, with the following probabilities:

| Number of Passengers | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| Probability, $P(X = x)$ | 0.7 | 0.2 | 0.05 | 0.05 |

Calculate the probability that in one minute 4 or more passengers leave the airport by taxi. $\boxed{0.1671}$





**Example 4.**

For an insurance coverage, the number of claims has a zero-modified negative binomial distribution with parameters $r = 4$, $\beta = 1$ and $p_0^M = 0.5$. Claim size is distributed as follows:

| Claim Size | 1 | 2 | 3 |
|---|---|---|---|
| Probability | 0.50 | 0.40 | 0.10 |

Calculate $F_S(3)$. $\boxed{0.63125}$

## 3.3 The Compound Model for Aggregate Claims: Convolution Method

The random sum, $S = X_1 + X_2 + \ldots + X_N$, has distribution function

$$\begin{aligned} F_S(x) &= P(S \le x) \\ &= \sum_{n=0}^{\infty} p_n P(S \le x | N = n) \\ &= \sum_{n=0}^{\infty} p_n F_x^{*n}(x) \end{aligned}$$

$S$ had pdf

$$g_n = f_S(x) = \sum_{n=0}^{\infty} p_n f_X^{*n}(x),$$

for $x > 0$ if $X$ is continuous

$$g_n = f_S(x) = \sum_{n=0}^{\infty} P(S = x | N = n),$$

for $x = 0, 1, \ldots$



**Example 5.**
Let $p_n = P[N = j-1] = j/10$ for $j = 1, 2, 3, 4$, and let $f_X(1) = P(X = 1) = 0.4$, $f_X(2) = P(X = 2) = f_X(2) = 0.6$. $f_S(x)$ and $F_S(x)$ can be calculated as follows. □





**Example 6** (T3Q1)**.**
An aggregate loss distribution has a compound Poisson distribution with expected number of claims equal to 4.28. Individual claim amount can take only the values 1, 2, or 3, with equal probability. Determine the probability that aggregate losses exceed 3.





**Example 7** (T3Q2)**.**
The number of claims in a period has a geometric distribution with mean 6.4. The amount of each claim is distributed as follows

| Claim Amounts, X | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| Probability | 0.38 | 0.25 | 0.16 | 0.16 | 0.05 |

The number of claims and claim amounts are independent. $S$ is the aggregate claim amount in the period. Calculate $F_S(3)$.

**Example 8** (T3Q3).

The number of claims on an insurance coverage follows a binomial distribution with parameters $m = 3$, $q = 0.2$. The size of each claim has the following distribution:

| Claim Size, $x$ | 0 | 1 | 2 |
|---|---|---|---|
| Probability, $P(X = x)$ | 0.50 | 0.35 | 0.15 |

Calculate the probability of aggregate claims of 3 or more.



**Example 9** (T3Q4).

Claim counts and sizes on an insurance coverage are independent and have the following distribution:

| Number of claims | Probability |
|---|---|
| 0 | 0.55 |
| 1 | 0.27 |
| 2 | 0.18 |

| Claim Size | Probability |
|---|---|
| 200 | 0.41 |
| 400 | 0.27 |
| 600 | 0.22 |
| 900 | 0.10 |

Let $S$ be the aggregate claims. Calculate $F_S(600)$.





**Example 10** (T3Q5).

The number of claims on an insurance coverage follows a zero modified Poisson distribution with mean $\lambda = 5$ and $p_0^M = 0.29$. The size of each claim has the following distribution:

| Claim Size, $x$ | 0 | 3 | 6 | 9 |
|---|---|---|---|---|
| Probability, $P(X = x)$ | 0.53 | 0.2 | 0.11 | 0.16 |

Calculate the probability of aggregate claims of 9 or more.





**Example 11** (T3Q6).

The number of claims has a Poisson distribution with mean $\lambda = 1.8$. The distribution of the amount of claims(in thousand) is

| Amount of claims | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Probability | 0.2 | 0.32 | 0.15 | 0.1 | 0.06 | 0.17 |

The number of claims and the amount of claims are independent. Determine the expected total amount of claims given that at least 4 thousand have been claimed. .

**Example 12** (T3Q7)**.**

For an insurance coverage, you are given:

- Claim frequency ($N^M$), before application of deductibles, follows a zero modified geometric distribution with parameters $\beta = 5$ and $P(N^M = 0) = 0.84$.
- Claim size ($X^M$), before application of deductibles, follows a zero modified Poisson distribution with parameters $\lambda = 4$ and $P(X^M = 0) = 0.75$.
- Claim frequency and claim size are independent.
- There is a deductible of 3 per loss.

Calculate the probability that the number of payments is greater than 8.



**Example 13** (T3Q8)**.**

Losses follow a compound distribution with both frequency and severity having discrete distribution.
For frequency

$$P_N(z) = 0.37 + 0.63\left[\frac{(1+0.64(z-1))^3 - (1-0.64)^3}{1-(1-0.64)^3}\right]$$

For Severity

$$P_X(z) = 0.28 + 0.20z + 0.20z^2 + 0.16z^3 + 0.16z^4$$

Calculate the probability that aggregate losses is exactly 3.





## 3.4 Approximating Distribution

Calculating the distribution function for the aggregate distribution is difficult. An alternative is to use an approximating distribution. The aggregate distribution can be approximated with a normal distribution by the Central Limit Theorem when the sample size is large.

If the severity is discrete, then the aggregate loss distribution is discrete, and a continuity correction is required. This means adding or subtracting 1/2 from the bound. For example, if you are asked for the probability that $S$ is greater than 100, then we would calculate the probability that the normal variable is greater that 100.5, i.e.

- $P(S > s) = P(S > s + 0.5)$
- $P(S \geq s) = P(S \geq s - 0.5)$
- $P(S < s) = P(S < s - 0.5)$
- $P(S \leq s) = P(S \leq s + 0.5)$





$$\begin{aligned} E(S) &= E[E(S|N=n)] \\ &= E(X_1 + X_2 + \ldots + X_n) \\ &= E[nE(X)] \\ &= E(N)E(X) \\ &= \mu_N \mu_X \end{aligned}$$

The moments of $S$ can be obtained in terms of the moments of $N$ and the $X_j s$.

$$\begin{aligned} V(S) &= E[V(S|N=n)] + V[E(S|N=n)] \\ &= E[nV(X)] + V[nE(X)] \\ &= E(N)V(X) + V(N)E^2(X) \\ &= \mu_N \sigma_X^2 + \sigma_N^2 \mu_X^2 \end{aligned}$$

By central Limit Theorem, $S$ can be approximated by a normal distribution.

As $n \to \infty$,
$S \to N(\mu = \mu_N \mu_X, \sigma^2 = \mu_N \sigma_X^2 + \sigma_N^2 \mu_X^2)$

Probability Generating Function of $S$ is given by

$$P_S(z) = P_N[P_X(z)]$$

The pgf is typically used when $S$ is discrete.

Moment Generating Function:

$$M_S(z) = E(e^{zS}) = P_N[M_X(z)]$$

**Example 14** (T3Q9)**.**

For insurance coverage, you are given:

- The number of claims for each insured follows a Binomial distribution with parameters $m = 3$ and $q$.
- $q$ varies by insured according to beta distribution with parameters $a = 10$ and $b = 4$
- Claim size, before application to claims limits, follows a gamma distribution with parameters $\alpha = 4$, $\theta = 930$.
- Coverage is subject to claim limit of 1,880.
- Number of claims and claim sizes are independent.

Calculate the probability that aggregate losses will be greater than 775, using the normal approximation.



**Example 15.**

For a group insurance policy, the number of claims from a group has a binomial distribution with mean 100 and variance 20. The size of each claim has the following distribution:

| Claim Size | probability |
|---|---|
| 1 | 0.50 |
| 2 | 0.35 |
| 3 | 0.10 |
| 4 | 0.05 |

Using approximating normal distribution, calculate the probability that aggregate claims from this group will be greater than 180. $\boxed{0.1762}$





**Example 16.**

For aggregate losses $S = X_1 + X_2 + \ldots + X_N$, you are given:

(i) $N$ has a Poisson distribution with mean 500.

(ii) $X_1, X_2, \ldots$ have mean 100 and variance 100.

(iii) $N, X_1, X_2, \ldots$ are mutually independent.

(iv) For a portfolio of insurance policies, the loss ratio is the ratio of aggregate losses to aggregate premiums collected.

(v) The premium collected is 1.1 times the expected aggregate losses.

Using the normal approximation to the compound Poisson distribution, calculate the probability that the loss ratio exceeds 0.95. $\boxed{0.1584}$





**Example 17** (T3Q10)**.**

On an insurance coverage, loss size has the following distribution:

$$F(x) = 1 - (210/x)^{5.00}, x \geq 210$$

- The number of claims has a Poisson with mean $\lambda = 5$.
- Claim counts and loss sizes are independent.
- A deductible of 260 is applied to each claim.

Calculate the probability that aggregate payments at least 180.0, using the normal approximation.

**Example 18** (T3Q11).

Losses follow a compound distribution with both frequency and severity having discrete distribution.
For frequency

$$P_N(z) = 0.42 + 0.58\left[\frac{e^{2.03z}-1}{e^{2.03}-1}\right]$$

For Severity

$$P_X(z) = 0.45 + 0.27z + 0.17z^2 + 0.06z^3 + 0.05z^4$$

Calculate the Variance of the aggregate losses.



**Example 19** (T3Q12).

On an auto collision coverage, the number of claims per year follows a Poisson distribution with mean 0.5. Loss size is exponentially distributed with mean 1180. An ordinary deductible of 480 is applied to each loss. Loss size and claim counts are independent. Calculate the probability that aggregate claim payments for a year will be greater than 369.93, using normal approximation.





## 3.5 Analytic Results

For certain combination of choices, simple analytic results are available, thus reducing the computational problems considerably.

### 3.5.1 Gamma Severity

If $X_j \sim gamma(\alpha, \theta)$ , then

$$F_S(x) = \sum_{n=0}^{\infty} p_n F^{*n}(x)$$

Here, $F^{*n}(x) = P\left[\sum_{j=1}^{n} X_j \le x\right]$

where $\sum_{j=1}^{n} X_j = S_{\alpha n} \sim gamma(\alpha^* = \alpha n, \theta)$

So,

$$F^{*n} = 1 - \sum_{j=0}^{\alpha n-1} e^{-x/\theta}\frac{(x/\theta)^j}{j!}.$$





Substituting, we get

$$\begin{aligned} F_S(x) &= \sum_{n=0}^{\infty} p_n \sum_{j=0}^{\alpha n-1}\left(1 - \frac{(x/\theta)^j e^{-x/\theta}}{j!}\right) \\ &= 1 - \sum_{n=1}^{\infty} p_n \sum_{j=0}^{\alpha n-1}\left(\frac{(x/\theta)^j e^{-x/\theta}}{j!}\right) \\ &= 1 - \sum_{n=1}^{\infty} p_n \sum_{j=1}^{\alpha n-1} a_j \end{aligned}$$

where $a_j = \frac{(x/\theta)^j e^{-x/\theta}}{j!}$ and $p_n = P[N = n]$

This is still an infinite sum, but if the frequency distribution is bounded, such as binomial, it is a finite sum.

$$F_S(x) = 1 - \sum_{n=1}^{m}\binom{m}{n} q^n (1-q)^{m-n} \times \sum_{j=0}^{\alpha(n)-1}\left(\frac{(x/\theta)^j e^{-x/\theta}}{j!}\right)$$

**Example 20.**
Claim counts have a binomial distribution with $m = 2$, $q = 0.2$. Claim sizes are exponential with mean 1000. Calculate the probability that aggregate claims are less than their mean. $\boxed{0.74796}$



**Example 21.**
Claim sizes follow a gamma distribution with parameters $\alpha = 2$ and $\theta = 100$. Claim counts are independent of claim sizes, and have the following distribution:

| $n$ | 0 | 1 | 2 |
|---|---|---|---|
| $p_n$ | 0.6 | 0.3 | 0.1 |

Calculate the probability that aggregate claims are less than 120. $\boxed{0.7046}$





### 3.5.2 Compound Negative Binomial Exponential

$N \sim NB(r, \beta)$; $X \sim exp(\theta)$

$$
\begin{aligned}
M_S(z) &= P_N[M_X(z)] \\
&= P_N[(1-\theta z)^{-1}] \\
&= \{1 - \beta[(1-\theta z)^{-1} - 1]\}^{-r} \\
&= \left(1 + \tfrac{\beta}{1+\beta}\{[1-\theta(1+\beta)z]^{-1} - 1\}\right)^{r} \\
&= P_N^*[M_X^*(z)]
\end{aligned}
$$

where
$P_N^*(z) = [1 + \frac{\beta}{1+\beta}(z-1)]^r$ and $M_X^*(z) = (1 - \theta(1+\beta)z)^{-1}$

$\Longrightarrow N^* \sim Binomial\left[r, \frac{\beta}{1+\beta}\right]$ and $X^* \sim exp[\theta(1+\beta)]$

Thus, $F_S(x) = \sum_{j=0}^{\infty} p_n F^{*n}(x)$
where
$p_n = \binom{r}{n}\left(\frac{\beta}{1+\beta}\right)^n\left(\frac{1}{1+\beta}\right)^{n-r}$ and





$$F^{*n}(x) = P\left[\sum_{i=1}^{n} X_i^* \le x\right]$$

where $\sum_{i=1}^{n} X_i^* \sim gamma(\alpha = n, \theta^* = \theta(1+\beta))$

Thus,

$$
\begin{aligned}
F_S(x) &= 1 - \sum_{n=1}^{r}\binom{r}{n}\left(\frac{\beta}{1+\beta}\right)^n\left(\frac{1}{1+\beta}\right)^{r-n} \\
&\quad \times \sum_{j=0}^{n-1}\frac{[x\theta^{-1}(1+\beta)^{-1}]^j e^{-x/[\theta(1+\beta)]}}{j!} \\
&= 1 - \sum_{n=1}^{r} p_n \sum_{j=0}^{n-1} a_j
\end{aligned}
$$

where $p_n = \binom{r}{n}\left(\frac{\beta}{1+\beta}\right)^n\left(\frac{1}{1+\beta}\right)^{r-n}$;
$a_j = \frac{[x\theta^{-1}(1+\beta)^{-1}]^j e^{-x/[\theta(1+\beta)]}}{j!}$
If $r = 1$, $S$ has a compound geometric distribution, then

$$F_S(x) = 1 - \tfrac{\beta}{1+\beta} exp\left[-\tfrac{x}{\theta(1+\beta}\right]$$

i.e. if $r = 1$, $S$ is a two-point mixture of a degenerate distribution at 0 with weight $\frac{1}{1+\beta}$ and an exponential distribution with mean $\theta(1+\beta)$, weight $\frac{\beta}{1+\beta}$.

**Example 22** (T3Q13).

Let the frequency distribution be negative binomial with $r = 4$ and $\beta = 7$. Let the severity distribution has the exponential distribution with mean 43. Determine $F_S(51)$



### 3.6 Discretization:Method of Rounding

The recursive and convolution methods for calculating the aggregate distribution require a discrete distribution. Usually the severity distribution is continuous. We will pick a span, the distance between the points that will have a positive probability in the discretized distribution.

$f_0 = P(X < h/2) = F_X(h/2 - 0)$

$$\begin{aligned} f_{jh} &= P(jh - h/2 \le X < jh + h/2) \\ &= F_X(jh + h/2 - 0) - F_X(jh - h/2 - 0), \\ &= F_x\left[\tfrac{(2j+1)h}{2} - 0\right] - F_x\left[\tfrac{(2j-1)h}{2} - 0\right] \\ & j = 1, 2, \ldots \end{aligned}$$

**Notes:**

- $h$ is the span of the distribution
- $-0$ indicates that the lower bound is included but the upper bound isn't.





**Example 23.**

$X$ has an exponential distribution with mean 1. Calculate $p_2$ of the distribution discretized using the method of rounding with a span of 1. $\boxed{0.1410}$





**Example 24.**

Loss sizes follow a Pareto distribution with $\alpha = 2$, $\theta = 3$. The distribution will be discretized by the method of rounding with a span of 4. Calculate the resulting probabilities of 0, 4, 8, and 12; $f_0$, $f_4$, $f_8$, and $f_{12}$. $\boxed{0.64, 0.24889, 0.05786, 0.02211}$

**Example 25** (T3Q14).

A random variable has an exponential distribution with mean 40. It is to be discretized using the method of rounding with span 50. Determine the mean of the discretized distribution.



## 3.7 Stop-loss Premium

**Definition 2.**
Insurance on the aggregate losses, subject to a deductible, is called stop-loss insurance. The expected cost of this insurance is called the net stop-loss premium and can be computed as $E[(S-d)_+]$, where $d$ is the deductible and the notation $(.)_+$ means to use the value in parentheses if it is positive but use zero otherwise.

For any aggregate distribution,

$$\begin{aligned} E[(S-d)_+] &= \int_d^\infty [1-F_S(x)dx \\ &= \int_d^\infty (x-d)f_S(x)dx \\ &\text{if } X \text{ is continuous} \end{aligned}$$

$$\begin{aligned} E[(S-d)_+] &= \int_d^\infty [1-F_S(x)dx \\ &= \sum_{k=d}^\infty (x-d)g_k \\ &\text{if } X \text{ is discrete} \end{aligned}$$

Note that
$E[(S-d)_+] = E[S] - E[S \wedge d]$
Since $E[S] = E[N]E[X]$, we only have to deal with $E[S \wedge d]$.





### 3.7.1 Using the definition of $E[S \wedge d]$

For a discrete distribution in which the only possible values are multiples of $h$, it becomes

$$E[S \wedge d] = \sum_{j=0}^{[d/h]-1} h_j g_{hj} + dP[(S \geq d)]$$





**Example 26** (T3Q15).

Prescription drug losses, $S$, are modeled assuming the number of claims has a negatine binomial distribution with parameters $r = 2$ and $\beta = 4.0$, and the amount of each prescription is 252.0. Calculate $E[(S-630)_+]$.

**Example 27** (T3Q16)**.**

A company provides insurance to a concert hall for losses due to power failure. You are given:

- The number of power failures in a year has a Binomial distribution with parameters $m = 6$ and $q = 0.73$.
- The distribution of loss amount due to a single power failure is:

| Loss Amount | 10 | 20 | 30 | 40 |
|---|---|---|---|---|
| Probability | 0.33 | 0.27 | 0.24 | 0.16 |

- There is an annual deductible of 26.

Calculate the expected amount of claims paid by the insurer in one year.



**Example 28.**

You are given:

- The number of claims follows a binomial distribution with $m = 3$, $q = 0.2$.
- Claim sizes follow the following distribution:

| Claim size | Claim probability |
|---|---|
| 0 | 0.2 |
| 1 | 0.5 |
| 2 | 0.2 |
| 3 | 0.1 |

- A reinsurance policy has an aggregate deductible of 6.

Determine the expected aggregate amount paid by the reinsurer. $\boxed{0.000336}$





**Example 29** (T3Q17)**.**

Claim counts follow a Poisson distribution with mean 3. Claim sizes follow an exponential distribution with $\theta = 300$. This severity distribution is discretized using the method of rounding with span 50. Claim counts and claim sizes are independent. A stop-loss reinsurance contract has a deductible of 95. Calculate expected losses paid by the reinsurance contract.





**Example 30** (T3Q18)**.**

A company provides insurance to a concert hall for losses due to power failure. You are given:

- The number of power failures in a year has a Binomial distribution with parameters $m = 2$ and $q = 0.65$.
- The distribution of loss amount due to a single power failure follows a gamma distribution $\alpha = 2$ and $\theta = 11$.
- There is an annual deductible of 25.

Calculate the expected amount of claims paid by the insurer in one year.

**Example 31** (T3Q19).

Let the frequency distribution be negative binomial with $r = 2$ and $\beta = 6$. Let the severity distribution has the exponential distribution with mean 40. Determine $E(S \wedge 200)$.



### 3.7.2 Using Linear Interpolation

For the case where $d$ is non-integral, we can use the following more general result that applies to continuous as well as discrete aggregate distributions.

Let the deductible $d$ satisfy $a < d < b$, where $P(a < S < b) = 0$. Then $E[(S-d)_+]$ is obtained by linear interpolation between $E[(S-a)_+]$ and $E[(S-b)_+]$, producing

**Theorem 2.**
Suppose $P(a < S < b) = 0$. Then for $a \leq d \leq b$,

$$\boxed{E[(S-d)_+] = \frac{b-d}{b-a} \cdot E[(S-a)_+] + \frac{d-a}{b-a} \cdot E[(S-b)_+].}$$

Proof:





**Example 32** (T3Q20).

A stop-loss reinsurance pays 90% of the excess of aggregate claims above 1,060, subject to maximum payment of 468. For aggregate claims, S, you are given:

- $E[(S-1,060)_+] = 430$
- $E[(S-2,120)_+] = 215$
- The probability of an aggregate claim amount between 1,060 and 2,120 is zero.

Determine the total amount of claims the reinsurer expects to pay.





**Example 33.**
One way to establish the market view of probabilities is to consider the market price placed on contingent events in an efficient market. For example, suppose we have the following values from a competitive market:

$$E[(S-5,000)_+] = 2000$$

$$E[(S-10,000)_+] = 1500$$

Suppose also that loss amounts between 5,000 and 10,000 are impossible. According to this information, what is the probability that total losses will exceed 5,000? $\boxed{0.1}$