

CONTENTS

| | | |
|----------|---|----------|
| 1 | The Black-Scholes Formula | 2 |
| 1.1 | Binary Options | 2 |
| 1.2 | The Black-Scholes Formula | 10 |
| 1.3 | Greek Letters and Elasticity | 23 |
| 1.4 | The Delta-Gamma-Theta Approx- imation | 41 |
| 1.5 | The Mean Return and Volatility of a Derivative | 45 |
| 1.6 | Delta-hedging a Portfolio | 53 |
| 1.7 | Gamma Neutrality | 62 |
| 1.8 | Implied Volatility | 64 |
| 1.9 | Historical Volatility | 67 |
| 1.10 | Expected Rate of Appreciation | 69 |

1 The Black-Scholes Formula**1.1 Binary Options**

Consider a **cash or nothing call**, which pays 1 at time T if $S(T) > K$ and nothing otherwise.

Thus, the payoff function is

$$I(S(T) > K)$$

Recall that under the true measure,

$$E[I(S(T) > K)|S(t)] = P(S(T) > K|S(t)) = N(\hat{d}_2)$$

where

$$\hat{d}_2 = \frac{\ln[S(t)/K] + (\alpha - \delta - \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}$$

To price a cash or nothing option, we use the risk-neutral measure. Replacing α with r , and let

$$d_2 = \frac{\ln[S(t)/K] + (r - \delta - \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}},$$

the expected payoff under risk-neutral measure is given by

$$E^*[I(S(T) > K|S(t))] = N(d_2)$$

and the time t price of the option is

$$F_{t,T}^P[I(S(T) > K)] = e^{-r(T-t)}N(d_2)$$

Similarly, under the **cash or nothing put**, the payoff function is

$$I(S(T) < K)$$

and under the true measure,

$$E[I(S(T) < K)|S(t)] = P(S(T) < K|S(t)) = N(-\hat{d}_2)$$

Under the risk-neutral measure

$$E^*[I(S(T) < K)|S(t)] = N(-d_2)$$

and the time t price of the put option is

$$F_{t,T}^P[I(S(T) < K)] = e^{-r(T-t)}N(-d_2)$$

Next, we consider an **asset or nothing call**, which pays $S(t)$ at time T if $S(T) > K$ and nothing otherwise. Thus, the payoff function is

$$S(T)I(S(T) > K)$$

Recall that under the true measure,

$$E[S(T)I(S(T) > K)|S(t)] = E[S(T)|S(t)]N(\hat{d}_1)$$

where

$$\hat{d}_1 = \frac{\ln[S(t)/K] + (\alpha - \delta + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

To price the option, the risk-neutral measure must be used. Thus, we replace α with r and let

$$d_1 = \frac{\ln[S(t)/K] + (r - \delta + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$

the expected payoff under the risk-neutral measure can be expressed as

$$E^*[S(T)I(S(T) > K)|S(t)] = E^*[S(T)|S(t)]N(d_1).$$

As a result, the time t price of the option is

$$\begin{aligned} F_{t,T}^P[S(T)I(S(T) > K)] \\ &= e^{-r(T-t)}S(t)e^{(r-\delta)(T-t)}N(d_1) \\ &= S(t)e^{-\delta(T-t)}N(d_1). \end{aligned}$$

For **asset or nothing put**, which pays $S(t)$ at time T if $S(T) < K$ and nothing otherwise, the time t price of this option is

$$F_{t,T}^P[S(T)I(S(T) < K)] = S(t)e^{-\delta(T-t)}N(-d_1).$$

The Binary options are summarized as follows:

| Binary Option | Payoff at Maturity | Time- t Price |
|-----------------------|--------------------|-------------------------------|
| Cash-or-nothing call | $I(S(T) > K)$ | $e^{-r(T-t)}N(d_2)$ |
| Cash-or-nothing put | $I(S(T) < K)$ | $e^{-r(T-t)}N(-d_2)$ |
| Asset-or-nothing call | $S(T)I(S(T) > K)$ | $S(t)e^{-\delta(T-t)}N(d_1)$ |
| Asset-or-nothing put | $S(T)I(S(T) < K)$ | $S(t)e^{-\delta(T-t)}N(-d_1)$ |

Notes:

- The time- t prices are of the form $F_{t,T}^P(S)N(\pm d_1)$ or $F_{t,T}^P(K)N(\pm d_2)$.
- $\pm d_1$ is used for asset-or-nothing options, and $\pm d_2$ is used for cash-or-nothing options.
- Negative sign is attached to d_1 or d_2 for put options.

Example 1.

You are given:

- A nondividend paying stock has a current price of 100.
 - The volatility of the stock is 25%.
 - The continuous compounded risk free interest rate is 4%.
- (a) Calculate the price of a 1-year 100-strike cash-or-nothing call.

- (b) Calculate the price of a 1-year 100-strike asset-or-nothing call.

Example 2.

Consider a stock that follows a geometric Brownian motion. You are given:

- The current stock price is 100.
- The dividend yield is 1%.
- The stock volatility is 0.20.
- The continuously compounded risk-free interest rate is 4%.

Calculate the price of the following options, which all mature at $T = 0.5$ and have strike of $K = 90$:

- (a) A cash-or-nothing put. [0.213](#)

(b) An Asset-or-nothing put. [17.79](#)

1.2 The Black-Scholes Formula

The payoff of a European call option on a stock, $[S(T) - K]_+$ can be decomposed into the payoff of cash-or-nothing and asset or-nothing call, i.e.,

$$\begin{aligned} [S(T) - K]_+ &= [S(T) - K]I(S(T) > K) \\ &= S(T)I(S(T) > K) - KI(S(T) > K) \end{aligned}$$

Thus, the **Black-Scholes formula for European call options** is

$$c[S(t), K] = S(t)e^{-\delta(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2)$$

where

$$d_1 = \frac{\ln[S(t)/K] + (r - \delta + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \text{ and}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

Similarly, the payoff of a European put option on a stock, $[K - S(T)]_+$ can be decomposed into the payoff of cash-or-nothing and asset or-nothing call, i.e.,

$$\begin{aligned} [K - S(T)]_+ &= [K - S(T)]I(S(T) < K) \\ &= KI(S(T) < K) - S(T)I(S(T) < K) \end{aligned}$$

Thus, the Black-scholes formula for European call options is

$$p[S(t), K] = Ke^{-r(T-t)}N(-d_2) - S(t)e^{-\delta(T-t)}N(-d_1)$$

Example 3 (T02Q1).

You are considering the purchase of a 3-month European call option on a stock. You are given the following information:

- The strike price is 77.
- The current stock price is 79.
- The annual risk-free interest rate is 11% compounded continuously.
- The stock pays continuous dividend proportional to its price at a rate of 4%.
- The annual volatility of the stock is 36%.
- The stock follows the Black-Scholes framework.

Calculate the price of the option.

Example 4 (T02Q2).

You are considering the purchase of a three-month European put option on a nondividend paying stock. You are given the following information:

- The strike price is 58.
- The current stock price is 63.
- The annual risk-free interest rate is 14% compounded continuously.
- The annual volatility of the stock is 30%.
- The stock follows the Black-Scholes framework.

Calculate the price of the option.

Prepaid Forward Version of the Black-Scholes Formula

Recall that $F_{t,T}^P(S)$ is the time- t prepaid forward price of the stock for delivery at time T . Similarly, $F_{t,T}^P(K)$ is the time- t prepaid forward price to delivery K dollars at time T . Thus,

$$c(S(t), K, T) = F_{t,T}^P(S)N(d_1) - F_{t,T}^P(K)N(d_2)$$

and

$$p(S(t), K, T) = F_{t,T}^P(K)N(-d_2) - F_{t,T}^P(S)N(-d_1)$$

where

$$d_1 = \frac{\ln[F_{t,T}^P(S)/F_{t,T}^P(K)] + 0.5\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$

and

$$d_2 = \frac{\ln[F_{t,T}^P(S)/F_{t,T}^P(K)] - 0.5\sigma^2(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T}$$

By replacing $F_{t,T}^P(S)$ accordingly, we can use the prepaid forward version of Black-Scholes formula for pricing options on other assets.

• Options on stocks with discrete dividends

In this case,

$$F_{T,t}^P(S) = S_t - PV_{t,T}(Div)$$

where $PV_{t,T}(Div)$ is the present value of the (Discrete) dividends payable over the life of the option.

Example 5 (T02Q3).

For a 1-year European call option on a stock:

- The strike price is 76.
- The stock's current price is 81.
- The continuously compounded risk-free interest rate is 0.07.
- The stock pays a dividend of 3 every 3 months, starting immediately after the call option is written. The dividend at the end of one year is paid before the option may be exercised.
- The annual volatility of a prepaid forward on the stock is 0.31.

- The stock follows the Black-Scholes framework.

Calculate the price of the option.

• Options on Currencies

For a currency option, suppose a foreign currency has a time- t exchange rate of $x(t)$ units of domestic currency per unit of foreign currency. If the foreign risk-free interest rate is r_f . The prepaid forward price is

$$F_{t,T}^P(x) = x(t)e^{-r_f(T-t)}.$$

Since

$$\ln F_{t,T}^P(x) = \ln x(t) - r_f(T-t)$$

and

$$\ln F_{t,T}^P(K) = \ln K e^{-r(T-t)} = \ln K - r(T-t)$$

Thus,

$$\begin{aligned} d_1 &= \frac{\ln[F_{t,T}^P(x)/F_{t,T}^P(K)] + 0.5\sigma^2(T-t)}{\sigma\sqrt{T-t}} \\ &= \frac{\ln(x(t)/K) + (r - r_f + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \end{aligned}$$

$$c(S(t), K) = F_{t,T}^P(x)N(d_1) - F_{t,T}^P(K)N(d_2)$$

$$p(S(t), K) = F_{t,T}^P(K)N(-d_2) - F_{t,T}^P(x)N(-d_1)$$

Example 6 (T02Q4).

Suppose that the spot exchange rate is \$1.3/£. The exchange rate has a volatility of 0.25. Assume that the US dollar interest rate is 0.052 and the pounds-denominated interest rate is 0.029. Calculate the Black-Scholes price (in US) of a call option to buy 100£ with 108.33 USD 6 months from now.

• Options on Future

For Futures,

$$F_{T,t}^P(F) = F(t)e^{-r(T-t)}$$

and

$$F_{T,t}^P(K) = Ke^{-r(T-t)}$$

Thus

$$\begin{aligned} d_1 &= \frac{\ln[F_{t,T}^P(S)/F_{t,T}^P(K)] + 0.5\sigma^2(T-t)}{\sigma\sqrt{T-t}} \\ &= \frac{\ln(F(t)/K) + (\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \end{aligned}$$

$$c(F(t), K) = F_{t,T}^P(F)N(d_1) - F_{t,T}^P(K)N(d_2)$$

$$p(F(t), K) = F_{t,T}^P(K)N(-d_2) - F_{t,T}^P(F)N(-d_1)$$

Example 7 (T02Q5).

Suppose that 6-month futures price for a certain stock is 48. The futures prices follows a geometric Brownian motion and has a volatility of 0.31. Consider a European call option on the future contract. The option expires 6 months from now and has a strike price of 48. Assume that the risk-free interest rate is 0.03, calculate the price of the option.

Example 8 (T02Q6).

Let $S(t)$ denote the price at time- t of a stock that pays no dividends. The Black-Scholes framework holds. Consider a European call option with exercise date T , $T > 0$, and exercise price $S(0)e^{rT}$, where r is the continuously compounded risk-free interest rate. You are given:

- $S(0) = 260$
- $T = 11$
- $V[\ln S(t)] = 0.35t, t > 0$.

Determine the price of the call option.

Example 9 (T02Q7).

You are given:

- The time- t price of a stock, $S(t)$, where t is measured in years, follows the risk-neutral process

$$d(\ln S(t)) = 0.039dt + 0.25d\tilde{Z}(t)$$

where $\tilde{Z}(t)$ is a standard Brownian motion in the risk-neutral measure.

- $S(0) = 8.9$.
- The continuously compounded risk-free interest rate is 0.07.

An option pays $\max(0, 843 - S(1)^3)$ at the end of one year. Calculate the value of the option.

1.3 Greek Letters and Elasticity

In a Black-Scholes framework, the price of any derivative security, $V(S, t)$, depends on the following six factors:

| Stock | Option | Environment |
|----------|--------|-------------|
| S | t | r |
| σ | K | |
| δ | | |

One way to quantify the risk of a derivative is to measure how sensitive $V(S, t)$ is when S or t changes.

- **Delta** (Δ) measures the change in the price of a derivative when the stock price increase by \$1.

$$\Delta = \frac{\partial V}{\partial S}$$

A large delta $\Rightarrow V$ is very sensitive to price.

Delta for call and put are:

$$\begin{aligned}\Delta_{\text{call}} &= e^{-\delta(T-t)} N(d_1) \\ \Delta_{\text{put}} &= -e^{-\delta(T-t)} N(-d_1)\end{aligned}$$

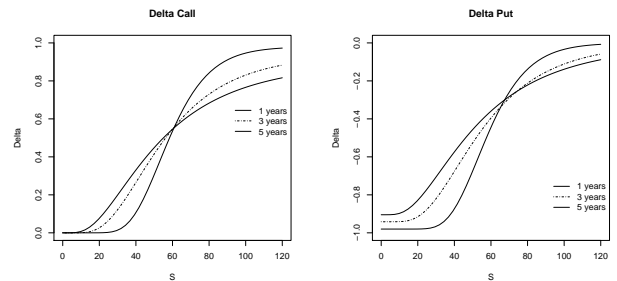
From put-call parity,

$$C(S, K) - P(S, K) = Se^{-\delta(T-t)} - Ke^{-r(T-t)}.$$

Differentiating with respect to S , we have

$$\Delta_{\text{call}} - \Delta_{\text{put}} = e^{-\delta(T-t)}$$

Properties of Δ



$K = 60$; $r = 0.1$, $\sigma = 0.3$, $\delta = 0.02$

Notes:

- Calls Δ is positive and bounded by 0 and $e^{-\delta(T-t)}$.
- Puts Δ is negative and bounded by $-e^{-\delta(T-t)}$ and 0.
- Δ calls $\rightarrow 0$ for low S and $\rightarrow e^{\delta(T-t)}$ for high S .
- Δ puts $\rightarrow 0$ for high S and $\rightarrow -e^{\delta(T-t)}$ for low S .

Example 10.

Assume the Black-Scholes framework. Compute the time- t delta for a cash-or-nothing call.

Example 11 (T02Q8).

For a stock that follows a geometric Brownian motion, you are given that

- The current stock price is 26.
- The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 8%.
- The expected stock price after 1 year is 30.82.
- The variance of the stock price after 1 year is 158.64.
- The delta of a 1-year at-the-money European put option is -0.352723.

Find the price of the put option.

- **Gamma**(Γ) measures the change in delta when the stock price increases by \$1.

$$\Gamma = \frac{\partial^2 V}{\partial S^2} = \frac{\partial \Delta}{\partial S}$$

Gamma for call and put are:

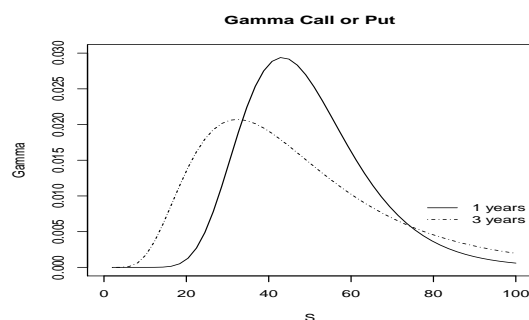
$$\Gamma = \frac{e^{-\delta(T-t)}\phi(d_1)}{S\sigma\sqrt{T-t}}$$

where $\phi(d_1) = \frac{1}{\sqrt{2\pi}}e^{-d_1^2/2}$

Differentiating both sides of the put call parity equation with respect to S , we get

$$\Gamma_{\text{call}} - \Gamma_{\text{put}} = 0$$

Properties of Γ



Notes:

- $\Gamma_{\text{call}} = \Gamma_{\text{put}}$ for same K and T .
- Γ is positive for long position of calls and puts.
- $\Gamma \rightarrow 0$ when S is very low or very high.

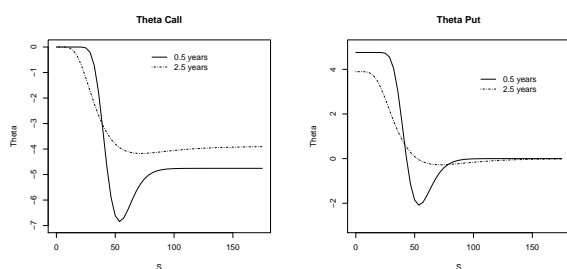
- **Theta** (θ) measures the change in price of a derivative when there is a decrease in the time to expiration t .

$$\theta = \frac{\partial V}{\partial t}$$

$$\theta_{\text{Call}} = S\delta e^{-\delta(T-t)}N(d_1) - Kre^{-r(T-t)}N(d_2) - \frac{S\sigma e^{-\delta(T-t)}\phi(d_1)}{2\sqrt{T-t}}$$

$$\theta_{\text{Put}} = Kre^{-r(T-t)}N(-d_2) - S\delta e^{-\delta(T-t)}N(-d_1) - \frac{S\sigma e^{-\delta(T-t)}\phi(d_1)}{2\sqrt{T-t}}$$

Properties of Θ



$$r = 0.1, \sigma = 0.3, K = 50, \delta = 0$$

Notes:

- The value of θ can be positive or negative. It is usually negative because call and put prices tend to drop as time passes.
- If time to expiration is measured in years, theta will be the annualized change in the option value. To obtain a per-day theta, divide by 365.

Example 12 (T02Q9).

For a 1-month European put option, you are given:

- (i) Theta is -0.0151 per day.
- (ii) The underlying stock price is 54.
- (iii) The strike price is 51.0.
- (iv) The stock's continuous dividend rate is 0.02.
- (v) The continuously compounded risk-free annual interest rate is 0.067.

Calculate theta per day for a 1-month European call option on the same stock with the same strike price.

• **Relation Between Delta, Gamma and Theta**

Recall the Black-Scholes equation

$$V_t + (r - \delta)SV_S + \frac{1}{2}\sigma^2 S^2 V_{SS} = rV$$

As

$$V_t = \frac{\partial V}{\partial t} = \theta;$$

$$V_S = \frac{\partial V}{\partial S} = \Delta$$

and

$$V_{SS} = \frac{\partial^2 V}{\partial S^2} = \Gamma,$$

we have

$$\theta + (r - \delta)S\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma = rV$$

- **Vega** (v) measures the change in price of a derivative when there is an increase in volatility.

$$v = \frac{\partial V}{\partial \sigma}$$

$$v_{\text{Call}} = S e^{-\delta(T-t)} \sqrt{T-t} \phi(d_1) = v_{\text{Put}}$$

Properties of Vegas

- Vegas are positive for both long calls and long puts.
- Vegas for calls and puts with the same strike and time to expiration are the same.
- Vega tends to be (but not always) greater for at-the-money options, and greater for options with longer times to expiration.
- The shape of vega is asymmetric hump, peak similar to Γ .
- It is common to report vega as the change in option price per percentage point change in volatility. This requires dividing the vega formula above by 100.

- **Psi** (ψ) measures the change in price of a derivative when there is an increase in the continuously dividend yield.

$$\psi = \frac{\partial V}{\partial \delta}$$

$$\psi_{\text{Call}} = -(T-t) S e^{-\delta(T-t)} N(d_1)$$

$$\psi_{\text{Put}} = (T-t) S e^{-\delta(T-t)} N(-d_1)$$

Properties of Psi

- Psi is negative for calls and positive for puts.
- A higher dividend yield would lower the prepaid forward price of the stock but not the present value of K , when the dividend yield increases, the price of a European call, which has a payoff of $[S(T) - K]_+$, will fall, while the price of a European put, which has a payoff of $[K - S(T)]_+$, will rise.
- The shape of Psi is a decreasing curve (Negative for calls, positive for puts)
- To interpret psi as price change per percentage point change in the dividend yield, divide 100.

- **Rho** (ρ) measures the change in price of a derivative when there is an increase in the risk free interest rate.

$$\rho = \frac{\partial V}{\partial r}$$

$$\rho_{\text{Call}} = (T - t)K e^{-r(T-t)} N(d_2)$$

$$\rho_{\text{Put}} = -(T - t)K e^{-r(T-t)} N(-d_2)$$

Properties of Rho

- We will use the above formula divided by 100. That is, the change of option value per 1% change in interest rate.
- Rho is positive for calls and negative for puts.
- A higher interest rate would lower the present value of K but not the prepaid forward price of $S(T)$. Therefore, when the present value of K decreases, call prices rise, while put prices fall.
- The shape of Rho is an increasing curve (Positive for calls, negative for puts)

- The sign of the six Greek letters can be summarized as below:

| Greek | Call | Put |
|----------|-----------------|-----|
| δ | + | - |
| Γ | + | + |
| θ | +or-(Usually -) | |
| v | + | + |
| ψ | - | + |
| ρ | + | - |

• Greek Letters of a Portfolio of Derivatives

Suppose that an investor forms a portfolio with n derivatives written on the same underlying stock S . The investor take a position of w_i units of the i -th derivative, whose price is denoted by V_t . If $w_i > 0$, then it is a long position, and vice versa. The value of the portfolio is given by

$$P = \sum_{i=1}^n w_i V_i.$$

Hence, the delta of the portfolio is

$$\frac{\partial P}{\partial S} = \sum_{i=1}^n w_i \frac{\partial V_i}{\partial S} = \sum_{i=1}^n w_i \Delta_i$$

Similarly,

$$\begin{aligned}\frac{\partial^2 P}{\partial S^2} &= \sum_{i=1}^n w_i \Gamma_i \\ \frac{\partial P}{\partial t} &= \sum_{i=1}^n w_i \theta_i \\ \frac{\partial P}{\partial r} &= \sum_{i=1}^n w_i v_i \\ \frac{\partial P}{\partial \delta} &= \sum_{i=1}^n w_i \psi_i \\ \frac{\partial P}{\partial \sigma} &= \sum_{i=1}^n w_i \rho_i\end{aligned}$$

Example 13.

A stock has a continuously compounded dividend yield of 0.06. Delta for a 6-month European put option on the stock is -0.79. Determine delta for a 6-month European call option on the stock with the same strike price. [0.1804](#)

Example 14.

Assume the Black-Scholes framework. You are given that:

- A nondividend paying stock has a current price of 10 and a volatility of 40%.
- A T -year K -strike European put option on S has a price of 2.4954 and a theta of -0.3903.
- A T -year K -strike European call option written on S has a delta of 0.4480 and a gamma of 0.09889.

Find r the continuously compounded risk-free interest rate. [0.05](#)

Example 15 (T02Q10).

You are given the following information on two derivatives:

| Derivative | Price | Delta | Gamma | Vega |
|------------|--------|---------|--------|---------|
| A | 1.1553 | 2.3917 | 2.373 | 0.9488 |
| B | 0.3403 | -0.8427 | -0.111 | -0.0436 |

You form derivative C by taking positions on derivative A and B. If derivative C has a zero delta and a gamma of 0.5, calculate its vega.

1.4 The Delta-Gamma-Theta Approximation

Delta, gamma, and theta can be used to approximate the change of the price of a derivative, $V(S, t)$ when there is a small change of the price of a stock. By the Taylor's theorem, we have

$$V(S + \epsilon, t) \approx V(S, t) + V_S(S, t)\epsilon + \frac{1}{2}V_{SS}(S, t)\epsilon^2$$

Thus, we have the **Delta-Gamma** Approximation:

$$V(S + \epsilon, t) \approx V(S, t) + \Delta\epsilon + \frac{1}{2}\Gamma\epsilon^2$$

If we drop the gamma term, we have the **delta** approximation:

$$V(S + \epsilon, t) \approx V(S, t) + \Delta\epsilon$$

A more comprehensive description is that the stock price changes from $S(t)$ to $S(t + h)$ when time proceeds from t to $t + h$. We can model this with a delta-gamma-theta approximation, which derived from the multivariate version of Taylor's

theorem. The **Delta-Gamma-Theta Approximation** is

$$V(S(t+h), t+h) \approx V(S(t), t) + \Delta\epsilon + \frac{1}{2}\Gamma\epsilon^2 + \theta h$$

where $\epsilon = S(t+h) - S(t)$ and the three Greeks are evaluated at $S(t)$ and t .

Example 16.

Assume the Black-Scholes framework holds. The price of a nondividend paying stock is \$30. The price of a put option on this stock is \$4.00. You are given $\Delta = -0.28$ and $\Gamma = 0.10$. Using the delta-gamma approximation, determine the price of the put option if the stock price changes to \$31.50. [3.6925](#)

Example 17 (T02Q11).

Let $c(S, T, K, r)$ and $p(S, T, K, r)$ be the prices of call and put when the stock price is S , the time until expiration is T , the strike price is K , and the continuously compounded risk-free interest rate is r . You are given:

- The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 0.03.
- $c(7, 0.5, 8, 0.09) = 0.33799$
- $\frac{\partial}{\partial r} \Big|_{r=0.09} c(7, 0.5, 8, r) = 1.1399$

Approximate the value of $p(7, 0.5, 8, 0.079)$.

1.5 The Mean Return and Volatility of a Derivative

The SDE for $V(S(t), t)$ is

$$\frac{dV(S(t), t)}{V(S(t), t)} = m_V dt + s_V dZ(t)$$

where m_V is the mean return $V(S(t), t)$ and s_V is the volatility of $V(S(t), t)$.

To compute s_V , we use Ito's lemma:

$$\begin{aligned} dV(S(t), t) &= V_t dt + V_S dS + \frac{1}{2} V_{SS} (dS)^2 \\ &= V_t dt + V_S [(\alpha - \delta) S dt + \sigma S dZ(t)] \\ &\quad + (\dots) dt \\ &= (\dots) dt + S V_S \sigma dZ(t) \end{aligned}$$

As a result,

$$\frac{dV}{V} = (\dots) dt + \frac{S V_S}{V} \sigma dZ(t).$$

and thus

$$s_V = \frac{S \Delta}{V} \sigma$$

Define

$$\Omega = \frac{S \Delta}{V}$$

then,

$$s_V = \Omega \sigma$$

Note:

- Depending on the sign of Δ , s_V may be positive or negative. It is customary to report the magnitude of s_V only.
- Ω is called the elasticity of a derivative.
- Since

$$|s_V| = |\Omega| \sigma$$

\Rightarrow If $|\Omega| > 1$, then $s_V > \sigma$, and hence the option is riskier than the underlying asset. As a result, elasticity is a measure of leverage.

- Another interpretation of the option is based on the option delta:

$$\Omega = \frac{S \partial V}{V \partial S} = \frac{\frac{\partial V}{V}}{\frac{\partial S}{S}}$$

which means that the elasticity is the percentage change in the derivative price relative to the percentage change in the stock price.

Properties of Elasticity of Calls and Puts

- $\Omega \geq 1$ for a call.
- $\Omega \leq 0$ for a put since $\Delta_{\text{put}} \leq 0$.
- $|\Omega|$ is an increasing function of t .
- $|\Omega|$ increases when $S \rightarrow K$
- Suppose that an investor forms a portfolio with n derivatives written on the same underlying stock S . the investor takes a position of w_i units of the i th derivative, whose price is denoted by V_i . The price of the portfolio is given by

$$P = \sum_{i=1}^n w_i V_i$$

By definition,

$$\begin{aligned} \Omega_{\text{portfolio}} &= \frac{S}{P} \frac{\partial P}{\partial S} \\ &= \sum_{i=1}^n \frac{S w_i}{P} \Delta_i \\ &= \sum_{i=1}^n \left(\frac{S w_i}{P} \right) \left(\frac{V_i \Omega_i}{S} \right) \\ &= \sum_{i=1}^n \left(\frac{w_i V_i}{P} \right) \Omega_i \end{aligned}$$

Thus, the option elasticity of a portfolio is the **value weighted average of the option**

To obtain m_V , we use the equality of Shape ratios,

$$\frac{m_V - r}{s_V} = \frac{\alpha - r}{\sigma}$$

$$\begin{aligned} m_V &= \frac{\alpha - r}{\sigma} s_V + r \\ &= \frac{\alpha - r}{\sigma} (\sigma \Omega) + r \\ &= (\alpha - r) \Omega + r \\ &= \Omega \alpha + (1 - \Omega) r \end{aligned}$$

\Rightarrow the instantaneous expected return on a derivative is a weighted average of α and r .

$$\boxed{m_V = \Omega \alpha + (1 - \Omega) r}$$

Example 18 (T02Q12).

Assume the Black-Scholes framework. You are given that

- The current stock price is 28.
- The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 0.033.
- The volatility of the stock is 0.3.
- The continuously compounded risk-free interest rate is 0.144.

Calculate the current volatility of a 6-month 29.0-strike European call option on the stock.

Example 19 (T02Q13).

You are given:

- For a stock whose time- t price is $S(t)$, the risk-neutral process is

$$d[\ln S(t)] = 0.023dt + 0.25d\tilde{Z}(t), S(0) = 110.0$$

where $\tilde{Z}(t)$ is a standard Brownian motion under the risk-neutral measure.

- The true stochastic process is

$$dS(t) = cS(t)dt + 0.25S(t)dZ(t),$$

where $Z(t)$ is a standard Brownian motion under the true probability measure and c is a constant.

- The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 1.2%.

Consider an option that pays 1 one year from now if $S(1) < 115.5$. If the expected instantaneous return of the option at time-0 is -57.0% , find c .

Example 20 (T02Q14).

Let $S(t)$ be time- t price of a nondividend-paying stock and $C(S(t), t)$ be the time- t price of a 0.5-year at the money European call option written on the stock, when the time- t stock price is $S(t)$. You are given that

- $S(0.25) = 50$.
- The true stock price process is

$$dS(t) = 0.22S(t)dt + 0.35S(t)dZ(t)$$

where $Z(t)$ is a standard Brownian motion under the true measure.

- The true stochastic process satisfied by the call option is

$$dC(S(t), t) = a(S(t), t)dt + b(S(t), t)dZ(t)$$

for some a and b .

- The risk-neutral stochastic process satisfied by the call option is

$$dC(S(t), t) = 0.06C(s(t), t)dt + f(S(t), t)d\tilde{Z}(t)$$

where f is a function and $\tilde{Z}(t)$ is a standard Brownian motion under the risk-neutral measure.

Calculate $a(50, 0.25)$.

1.6 Delta-hedging a Portfolio

A market maker is a broker or dealer who sells or buy options. A market maker is not interested in speculating on the market but rather would like to make money through bid-ask spreads. Thus if he sells a call option, he must hedge the investment by buying something that would go up in value if the call goes up in value.

If there are two risky assets X and Y with SDEs

$$\frac{dX(t)}{X(t)} = m_X dt + s_X dZ(t), \quad \frac{dY(t)}{Y(t)} = m_Y dt + s_Y dZ(t)$$

Suppose we have 1 unit of X at time t . To hedge the risk, we purchase $N = -\frac{s_X X(t)}{s_Y Y(t)}$ units of Y and hold a cash position of $W = -X(t) - NY(t)$.

Let $X(t) = V(S(t), t)$ and $Y(t) = S(t)$, then, we purchase

$$N = -\frac{s_V V}{\sigma S} = -\frac{\frac{S\Delta}{V}\sigma V}{\sigma S} = -\Delta$$

shares of S and hold a cash position of

$$W = -V + S\Delta.$$

Example 21.

For a nondividend paying stock, you are given that

- $S(t)$ is the time- t stock price.
- $S(t)$ satisfies the SDE:

$$dS(t) = 0.25S(t)dt + 0.45S(t)dZ(t), S(0) = 60$$
- The continuously compounded risk-free interest rate is 10%.

Justin has just sold 100 unit of 9-month 65-strike call option. To hedge the risk, Justin immediately hedges his position by purchasing the hedge portfolio.

- (a) Calculate the components in Justin's hedge portfolio at $t = 0$.

- (b) Suppose that the stock price after 1 month is 65, compute the profit and loss of
- (i) shorting 100 unit call;
 - (ii) Justin's position, assuming that Justin can borrow or lend at the risk-free interest rate.

Example 22.

You are given that

- $S(t)$ is the time- t stock price.
- $S(t)$ satisfies the SDE:

$$dS(t) = 0.25S(t)dt + 0.45S(t)dZ(t), S(0) = 60$$
- The stock pays dividend continuously at a rate proportional to its price. The dividend yield is 4%.
- The continuously compounded risk-free interest rate is 10%.

Justin has just bought 100 unit of 9-month 60-strike put option. To hedge the risk, Justin immediately hedges his position by purchasing the hedge portfolio.

- (a) Calculate the components in Justin's hedge portfolio at $t = 0$.

- (b) Suppose that the stock price after 1 month is 65, compute the profit and loss of
- (i) owing 100 units of put;
 - (ii) Justin's position, assuming that Justin uses the stock dividend to purchase extra shares, and that Justin can borrow or lend at the risk-free interest rate.

Example 23 (T02Q15).

Let $S(t)$ be the time- t price of a nondividend paying stock. You are given that $S(t)$ follows the stochastic differential equation

$$dS(t) = 0.08S(t)dt + 0.27d\tilde{Z}(t), S(0) = 3,$$

where $\tilde{Z}(t)$ is a standard Brownian motion under the risk-neutral measure.

A market maker has just written a contingent claim that pays the $S^3(3)$ after 3 years. He then immediately delta-hedge his position by trading stocks and cash(W). Calculate W.

Example 24 (T02Q16).

You are given that:

- For a stock whose time- t price is $S(t)$, the risk-neutral process is

$$d[\ln S(t)] = 0.083dt + 0.2d\tilde{Z}(t), S(0) = 22$$

where $\tilde{Z}(t)$ is a standard Brownian motion under the risk-neutral measure.

- The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 1.7%.
- A market-maker has sold 100 6-month 24-strike calls on the stock. He immediately hedges his position by buying shares and risk-free bonds. The dividends received are invested by purchasing extra shares.
- The current Black-Scholes price of the call is 0.8968.

After 1 month, when the stock price is 15 and the Black-Scholes price for the call becomes 0.0003,

the maker-maker rebalances his hedge portfolio by trading shares and risk-free bonds. The maker-maker invests or repays dividends by purchasing or shorting extra shares. Compute the 1-month profit.

1.7 Gamma Neutrality

Delta-hedge can only protect the market-maker against small changes in stock prices. If the gamma of the portfolio is negative, a big change in the stock price will lead to a big hedge loss.

To solve this problem, one can try to make the gamma of the hedge position zero. A portfolio with a zero gamma is called a gamma-neutral position. A position in the underlying asset itself has a zero gamma: $\Gamma = \frac{\partial^2 S}{\partial S^2} = 0$. Thus, it cannot be used to change the gamma of the portfolio. To adjust the gamma of a portfolio, we must make use of instruments such as options that are not linearly dependent on the underlying asset.

Example 25 (T02Q17).

You are given:

- A stock has price 45.
- A market-maker writes put option I on the stock with price 2.59, delta -0.45 and gamma 0.06.
- The market-maker delta-gamma-hedges the option with the stock and with put option II having price 4.47, delta -0.58, and gamma 0.05.

Determine the number of shares of stock to buy to implement the hedge.

1.8 Implied Volatility

Implied volatility is a term commonly used by stock analysts. It is the volatility implied by the **market price** of an option. For example, suppose that the price of a call on a nondividend-paying stock is 1.875 when $S = 21$, $K = 20$, $r = 10\%$, and $T = 0.25$. the implied volatility is the value of σ that gives $C = 1.875$ when it is substituted into the Black-Scholes formula.

In general, to solve for the implied volatility, we need computer software to solve it. However, under the following situation, we can obtain the volatility:

1. At the money stock options with $r = \delta$.
2. At the money future option.
3. The delta of the option is given.

In situation (1) and (2), $d_1 = -d_2 = \frac{\sigma\sqrt{T}}{2}$.

Example 26 (T02Q18).

Assume the Black-Scholes framework. For a stock, you are given that:

- The current stock price 120.
- The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 2.1%.
- The continuously compounded risk-free interest rate is 2.1%.

The current price of a 3-month 120-strike European call on this stock is 9.0. Calculate the implied volatility of this stock.

Example 27 (T02Q19).

Let $Z(t)$ be a standard Brownian motion under the risk-neutral measure. For a stock, you are given:

- The time- t stock price is $S(t)$.
- The stock price process in the risk-neutral measure is

$$dS(t) = 0.021S(t)dt + \sigma S(t)dZ(t), \quad S(0) = 100,$$

where a is a constant that is less than 0.1922.

- The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 2.5%. The delta of a 2-year 100-strike put option on this stock is -0.3655.

Calculate σ .

1.9 Historical Volatility

Historical volatility is an estimate of volatility from historical stock prices. To obtain the estimated volatility:

1. Let $u_i = \ln \frac{S_i}{S_{i-1}}$, $i = 1, 2, \dots, n$, be the continuously compounded rate of return (not annualized) for the i -th time interval. In the Black-Scholes framework,

$$u_i \sim N[(\alpha - \delta - \frac{1}{2}\sigma^2)h, \sigma^2h].$$

2. Compute $\bar{u} = \frac{\sum_{i=1}^n u_i}{n}$.
3. Compute $s_u^2 = \frac{\sum_{i=1}^n (u_i - \bar{u})^2}{n-1}$.
4. Since s_u^2 is the estimate of $\sigma^2 h$, thus $\hat{\sigma}^2 = \frac{s_u^2}{h}$.
5. Thus, $\hat{\sigma} = \frac{s_u}{\sqrt{h}}$.

Example 28.

You are to estimate a nondividend-paying stock's annualized volatility using its prices in the past nine months.

| Month | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|-------------|----|----|----|----|----|-----|----|----|----|
| Stock Price | 80 | 64 | 80 | 64 | 80 | 100 | 80 | 64 | 80 |

Calculate the historical volatility for this stock over the period. [0.8265](#)

1.10 Expected Rate of Appreciation

To estimate the expected rate of return α , we use the following equation:

$$\hat{\alpha} = \frac{\bar{u}}{h} + \delta + \frac{\hat{\sigma}^2}{2}.$$

To estimate the expected rate of appreciation, use

$$\hat{\alpha} - \delta$$

Example 29 (T02Q20).

You are given the following historical prices of a nondividend-paying stock:

| | | | | | | |
|-------------|-----|----|----|----|-----|-----|
| Week | 1 | 2 | 3 | 4 | 5 | 6 |
| Stock Price | 110 | 96 | 93 | 97 | 104 | 103 |

Let α be the stock’s expected rate of return and σ be the stock’s volatility. Estimate $\alpha + \sigma$.