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2 Models for Claim Frequencies

2.1 Discrete Distributions

Discrete distributions are useful for modeling frequency. In an insurance context, counting distributions describe the number of events such as losses to the insured or claims to the insurance company. Counting distributions are discrete distributions with probabilities only on the nonnegative integers.

The probability function (pf) p_k denotes the probability that exactly k events occur. Let N be a random variable representing the number of such events. Then

$$p_k = P(N = k), k = 0, 1, 2, \ldots$$

The probability generating function (pgf) of a discrete random variable N with pf p_k is

$$P(z) = P_N(z) = E(z^N) = \sum_{k=0}^{\infty} p_k z^k.$$

In particular,

$$P'(1) = E(N)$$

and

$$P''(1) = E[N(N-1)].$$

$$P^{(m)}(z) = E\left(\frac{d^m}{dz^m} z^N\right)$$

$$= E[N(N-1)\ldots(N-m+1)z^{N-m}]$$

$$= \sum_{k=m}^{\infty} k(k-1)\ldots(k-m+1)z^{k-m}p_k$$

$$= m(m-1)\cdots 1(p_m)+$$
$$(m+1)(m)(m-1)\cdots 2(z)(p_{m+1})$$
$$+\cdots$$

$$P^{(m)}(0) = m!p_m$$

$$\implies p_m = \frac{P^{(m)}(0)}{m!} \text{ and } p_0 = P(0)$$

Example 1 (T2Q1)**.**

A random variable follows a Poisson distribution with $\lambda = 0.84$. Calculate the third raw moment of the distribution.

Example 2.

Suppose N is a random variable with probability generating function

$$P_N(z) = \frac{1}{3}z + \frac{1}{6}z^2 + \frac{1}{2}z^3.$$

Calculate the coefficient of variation of N. [0.4142]

Example 3.

Suppose N is a random variable with probability generating function

$$P_N(z) = \frac{1}{6} + \frac{1}{2}z^2 + \frac{1}{3}z^3.$$

Calculate the skewness of N. [-1]

2.2 Three Basic Distributions

2.2.1 The Poisson Distributions

The pf for the Poisson distribution is

$$p_k = P(N = k) = \frac{e^{-\lambda}\lambda^k}{k!}, k = 0, 1, 2, \ldots$$

The probability generating function is

$$\boxed{P(z) = e^{\lambda(z-1)}}$$

The moment generating function is

$$M_N(t) = e^{\lambda(e^t - 1)}$$

Theorem 1.
Let $N_1, \ldots, Nn$ be independent Poisson variables with parameters $\lambda_1, \ldots, \lambda_n$. Then $N = N_1 + \ldots + N_n$ has a Poisson distribution with parameters $\lambda_1 + \cdots + \lambda_n$.

Example 4 (T2Q2)**.**

Dental Insurance Company sells a policy that covers two types of dental procedures: root canals and fillings. There is a limit of 1 root canal per year and a separate limit of 2 fillings per year. The number of root canals a person needs in a year follows a Poisson distribution with $\lambda = 1.0$, and the number of fillings a person needs in a year follows a Poisson distribution with $\lambda = 2.0$. The company is considering replacing the single limits with a combined limit of 3 claims per year, regardless of the type of claim. Determine the change in the expected number of claims per year if the combined limit is adopted.

2.2.2 The Negative Binomial Distribution

Like Poisson distribution, negative binomial distribution has positive probabilities on the nonnegative integers. Because it has two parameters, it has more flexibility in shape than the Poisson.

The pf for the negative Binomial distribution is

$$p_k = \binom{k + r - 1}{k} \left(\frac{1}{1 + \beta}\right)^r \left(\frac{\beta}{1 + \beta}\right)^k$$
$$k = 0, 1, 2, \ldots, r > 0, \beta > 0$$

The mean and variance are

$$E(N) = r\beta$$

and

$$V(N) = r\beta(1 + \beta)$$

Because β is positive, the variance of the negative binomial distribution exceed the mean. This relationship is in contrast to the Poisson distribution for which the variance is equal to the mean. Thus, for a particular set of data, if the observed variance is larger than the observed mean, the negative binomial might be better candidate than Poisson distribution as a model to be used.

The probability generating function is
$$P(z) = [1 - \beta(z - 1)]^{-r}$$
and the moment generating function is
$$M_N(t) = [1 - \beta(e^t - 1)]^{-r}$$

Example 5.
You are given the following claim frequency data:

Frequency	0	1	2	3	4	5+
Number of Insureds	60	22	11	5	2	0

The data are fitted to a negative binomial distribution using method of moments. Determine the resulting estimate of the probability of zero claims. 0.5767

Notes:
The negative binomial distribution is a generalization of the Poisson:

(i) Mixed Poisson distribution with gamma mixing distribution, i.e.
If
$$X|\lambda \sim POI(\lambda)$$
and
$$\Lambda \sim Gamma(\alpha, \beta),$$
then
$$X \sim NB(r = \alpha, \beta = \beta)$$

(ii) A compound Poisson with a logarithmic secondary distribution, i.e.
$S = M_1 + M_2 + \cdots M_N \sim NB\left(r = \frac{\lambda}{\ln(1+\beta)}, \beta\right)$,
where
$N \sim POI(\lambda)$, $M \sim logarithmic(\beta)$

(iii) The geometric distribution is the special case of the negative binomial distribution when $r = 1$.
$$p_0 = \frac{1}{1 + \beta}, p_k = \left(\frac{\beta}{1 + \beta}\right) p_{k-1},$$
so, the probabilities follow a geometric progression with ratio $\frac{\beta}{1+\beta}$. Thus
$$P(N \geq n) = \left(\frac{\beta}{1+\beta}\right)^n$$

(iv) Poisson distribution is a limiting case of the negative binomial distribution, i.e.

Let
$$r \longrightarrow \infty, \beta \longrightarrow 0,$$
and
$$r\beta = \lambda,$$
then using
$$\lim_{n \to \infty} \left(1 + \frac{c}{n}\right)^{nb} = e^{cb},$$
we obtained
$$\lim_{r \to \infty} \left[1 - \frac{\lambda(z-1)}{r}\right]^{-r} = exp[\lambda(z-1)]$$

2.2.3 The Binomial Distribution

The binomial distribution is another counting distribution that arises naturally in claim number modeling. Its variance is smaller than its mean, making its useful for data sets in which observed sample variance is less than the sample mean.

The pf for the Binomial distribution is

$$p_k = \binom{m}{k} q^k (1-q)^{m-k},$$
$$k = 0, 1, 2, \ldots, m.$$

The mean and variance are

$$E(N) = mq$$
and
$$V(N) = mq(1-q)$$

The probability generating function is
$$\boxed{P(z) = [1 + q(z-1)]^m}$$
and the moment generating function is
$$\boxed{M(t) = [1 + q(e^t - 1)]^m}$$

Note:
For large m and small q, binomial probabilities can be approximated as Poisson probabilities with $\lambda = mq$.

Example 6.
A portfolio of 10,000 risks yields the following:

Frequency	0	1	2	3	4
Number of Insureds	6,070	3,022	764	126	18

Based on the portfolio's sample moments, which of the following distributions provide the best fit to the portfolio's number of claims?
A. Binomial   B. Poisson
C. Negative Binomial   D. Lognormal
E. Pareto

2.3 The $(a, b, 0)$ Class

Once we are given the mean and variance of a distribution, we can determine the parameters and furthermore, we can tell which is the appropriate distribution by comparing the mean with the variance. If the variance is greater than the mean, it is negative binomial; if it is equal, then it is Poisson; and if it is less, then it is binomial. These three distributions are the complete set of distributions in the $(a, b, 0)$ class. This class is defined by the following property. Writing

$$\frac{p_k}{p_{k-1}} = a + \frac{b}{k}, k = 1, 2, 3, \ldots$$

Panjer and Willmot shown that the only possible distributions satisfying this recursive formula are the Poisson, binomial, and negative binomial distributions.

2.3.1 $(a, b, 0)$ for the Poisson distribution

- $p_k = P(N = k) = \frac{e^{-\lambda}\lambda^k}{k!}$,

- $\frac{p_1}{p_0} = a + b = \frac{\lambda e^{-\lambda}}{e^{-\lambda}} = \lambda$ ——(1)

- $\frac{p_2}{p_1} = a + \frac{b}{2} = \frac{\lambda^2 e^{-\lambda}}{2\lambda e^{-\lambda}} = \frac{\lambda}{2}$ ——(2)

- $(1) - (2), \frac{b}{2} = \frac{\lambda}{2}, \Longrightarrow b = \lambda$ and $a = 0$.

2.3.2 $(a, b, 0)$ for the Negative Binomial Distribution

- $p_k = \binom{k+r-1}{k} \left(\frac{1}{1+\beta}\right)^r \left(\frac{\beta}{1+\beta}\right)^k$,

- $\frac{p_1}{p_0} = a + b = \frac{r\left(\frac{1}{1+\beta}\right)^r\left(\frac{\beta}{1+\beta}\right)}{\left(\frac{1}{1+\beta}\right)^r} = \frac{r\beta}{1+\beta}$ —— (1)

- $\frac{p_2}{p_1} = a + \frac{b}{2} = \frac{\frac{r(r+1)}{2}\left(\frac{1}{1+\beta}\right)^r\left(\frac{\beta}{1+\beta}\right)^2}{r\left(\frac{1}{1+\beta}\right)^r\left(\frac{\beta}{1+\beta}\right)} = \frac{(r+1)\beta}{2(1+\beta)}$ —— (2)

- $(1) - (2), \frac{b}{2} = \frac{(r-1)}{2}\frac{\beta}{1+\beta}$
  $\Longrightarrow b = (r-1)\left(\frac{\beta}{1+\beta}\right)$
  $a = \frac{r\beta}{1+\beta} - (r-1)\left(\frac{\beta}{1+\beta}\right) = \left(\frac{\beta}{1+\beta}\right)$

2.3.3 $(a, b, 0)$ for the Binomial Distribution

- $p_k = \binom{m}{k}q^k(1-q)^{m-k}$,

- $\frac{p_1}{p_0} = a + b = \frac{mq(1-q)^{m-1}}{(1-q)^m} = \frac{mq}{1-q}$ — (1)

- $\frac{p_2}{p_1} = a + \frac{b}{2} = \frac{\frac{m!}{2!(m-2)!}q^2(1-q)^{m-2}}{mq(1-q)^{m-1}} = \frac{(m-1)q}{2(1-q)}$ — (2)

- $(1) - (2), \left(m - \frac{m-1}{2}\right)\left(\frac{q}{1-q}\right) = \frac{b}{2}$

$\Longrightarrow b = (m+1)\left(\frac{q}{1-q}\right)$,
$a = \frac{mq}{1-q} - (m+1)\left(\frac{q}{1-q}\right) = -\frac{q}{1-q}$

Since $a = \frac{\beta}{1+\beta} > 0$, $a = 0$, and $a = -\frac{q}{1-q} < 0$ for the Negative Binomial, Poisson, and Binomial Distribution respectively. Thus, we can use the sign of a to determine the distribution once we are given that the distribution belongs to the $(a, b, 0)$ class.

The recursive formula can be rewritten as

$$k\frac{p_k}{p_{k-1}} = ak + b, k = 1, 2, 3, \ldots$$

This indicates that $k\frac{p_k}{p_{k-1}}$ is a linear function of k. The value of slope, a, should be an indication of which the models should be selected. This relationship suggests a graphical way of indicating which of the three distributions might be selected for fitting to data. Begin by plotting

$$k\frac{\hat{p}_k}{\hat{p}_{k-1}} = k\frac{n_k}{n_{k-1}}$$

against k. The observed values should form approximately a straight line if one of these models is to be selected, and the value of slope should be an indication of which the models should be selected. Note that this cannot be done if any of the n_k are 0. Hence this procedure is less useful for a small number of observations.

For example, consider the accident data below,

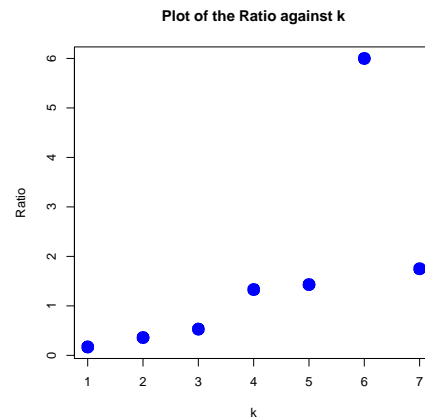| Number of accidents, k	Number of policies, n_k	$k\frac{n_k}{n_{k-1}}$
0	7,840	
1	1,317	0.17
2	239	0.36
3	42	0.53
4	14	1.33
5	4	1.43
6	4	6.00
7	1	1.75
8+	0	

Plot of the Ratio against k

Figure above plots the value of $k\frac{n_k}{n_{k-1}}$ against k. It can be seen that the slope is positive and the data appear approximately linear, suggesting that the negative binomial distribution is an appropriate model.

For a distribution in the $(a, b, 0)$ class, the values of p_k for 3 consecutive values of k enable us to form two equations connecting the a and b which can then be used to determine the exact distribution.

We summarize the formulas in the table below.

Distribution	p_k
Poisson	$e^{-\lambda}\frac{\lambda^k}{k!}$
Binomial	$\binom{m}{k}q^k(1-q)^{m-k}$
Negative Binomial	$\frac{r(r+1)...(r+k-1)\beta^k}{k!(1+\beta)^{r+k}}$
Geometric	$\frac{\beta^k}{(1+\beta^{1-k}}$

Distribution	Mean	Variance	a	b
Poisson	λ	λ	0	λ
Binomial	mq	$mq(1-q)$	$-\frac{q}{1-q}$	$(m+1)\frac{q}{1-q}$
Negative Binomial	$r\beta$	$r\beta(1+\beta)$	$\frac{\beta}{1+\beta}$	$(r-1)\frac{\beta}{1+\beta}$
Geometric	β	$\beta(1+\beta)$	$\frac{\beta}{1+\beta}$	0

Example 7.
Claim frequency follows a distribution in the $(a, b, 0)$ class. You are given that

- the probability of 4 claims is 0.2734375
- the probability of 5 claims is 0.21875
- the probability of 6 claims is 0.109375

Calculate the probability of no claims. $\boxed{0.00390625}$

Example 8 (T2Q3)**.**
For a certain $(a, b, 0)$ distribution,

- $a = 0.77876$,
- $b = 1.55752$, and
- $1000p_0 = 10.829$.

Calculate the probability of exactly 3 events occurring times 1000, i.e. $1000p_3$.

Example 9.
For a distribution in the $(a, b, 0)$ class, you are given that $p_2 = 0.25p_1$ and $p_4 = 0.225p_3$. Determine p_2. $\boxed{0.05367}$

Example 10.

X is a discrete random variable with a probability function which is a member of the $(a, b, 0)$ class of distributions. You are given $P(X = 0) = P(X = 1) = 0.25$, and $P(X = 2) = 0.1875$. Calculate $P(X = 3)$. $\boxed{0.125}$

Example 11.

You are given the following claim frequency data:

Frequency	0	1	2	3	4
Number of Insureds	20	14	10	6	3

Which probability distribution is suggested by this data based on (i) successive ratios of probabilities, and (ii) moments?

(A.) (i) Binomial, (ii) Binomial

(B.) (i) Poisson, (ii) Binomial

(C.) (i) Negative Binomial, (ii) Poisson

(D.) (i) Negative Binomial, (ii) Negative Binomial

(E.) None of the above

Example 12.

For a random variable in the $(a, b, 0)$ class

$$\frac{p_3}{p_2} = 1.6; \quad \frac{p_8}{p_7} = 0.975$$

Determine the mode of this random variable. $\boxed{7}$

Example 13 (T2Q4)**.**

For a discrete probability distribution, you are given the recursion relation

$$p_k = \left(\frac{0.68}{k} + 0.68\right) p_{(k-1)}), k = 1, 2, \ldots$$

Determine p_6.

Example 14 (T2Q8)**.**

A discrete distribution has the following properties:

- $p_k = c(\frac{1}{6k} - \frac{1}{48})p_{k-1}$ for $k = 1, 2, \ldots, c > 0$

- $p_0 = 0.209715$

Calculate c.

2.4 The $(a, b, 1)$ class

2.4.1 Truncation and Modification at Zero

Often special treatment must be given to the probability of zero claims, for which purpose the three distributions of the $(a, b, 0)$ class may prove to be inadequate as they may give inappropriate probability to 0 claims. The $(a, b, 1)$ class consists of distributions for which $p_0 = P(N = 0)$ is arbitrary, but the $(a, b, 0)$ relationship holds for $k > 1$, i.e.,

$$\frac{p_k}{p_{k-1}} = a + \frac{b}{k}, k = 2, 3, 4, \ldots$$

We distinguish between situations in which $p_0 = 0$ and those where $p_0 > 0$. The first subclass is called zero-truncated distributions. The second subclass is referred to as zero-modified distributions.

In order to obtain $(a, b, 1)$ distributions, we modify the $(a, b, 0)$ distribution at 0 by truncating its value at 0 and scaling up the other probabilities appropriately so that they add up to 1.

Let $p_k = P(N = k), k = 0, 1, \ldots$ be the probabilities from the $(a, b, 0)$ class distribution that we start with, p_k^T be the probabilities of the zero-truncated distribution and p_k^M be the probabilities of the zero-modified distribution.

Let

$$P(z) = \sum_{k=0}^{\infty} p_k z^k$$

denote the pgf of a member of the $(a, b, 0)$ class,

$$P^M(z) = \sum_{k=0}^{\infty} p_k^M z^k$$

denote the pgf of the corresponding member of $(a, b, 1)$ class,

$$p_k^M = cp_k, k = 1, 2, 3, \ldots,$$

and p_0^M is an arbitrary number.

Then

$$P^M(z) = p_0^M + \sum_{k=1}^{\infty} p_k^M z^k$$
$$= p_0^M + c \sum_{k=1}^{\infty} p_k z^k$$
$$= p_0^M + c[P(z) - p_0]$$

Because
$P^M(1) = P(1) = 1$, thus
$P^M(1) = p_0^M + c[P(1) - p_0]$
$1 = p_0^M + c[1 - p_0]$

$$\Longrightarrow \boxed{c = \frac{1 - p_0^M}{1 - p_0}} \text{ or } \boxed{p_0^M = 1 - c(1 - p_0)}$$

Thus, $\boxed{p_k^M = cp_k = \frac{1 - p_0^M}{1 - p_0} p_k, k = 1, 2, \ldots}$

For zero-truncated distribution, we let

$$p_0^M = 0,$$

thus,

$$p_k^T = \frac{p_k}{1 - p_0}, k = 1, 2, \ldots,$$

$$\Longrightarrow p_k^M = (1 - p_0^M) p_k^T$$

2.4.2 The mean and variance of the zero modified distributions

As $p_k^M = cp_k = \frac{1-p_0^M}{1-p_0}p_k$

$$E(N^M)^i = \sum_{k=1}^{\infty} k^i p_k^M$$
$$= \sum_{k=1}^{\infty} k^i cp_k$$
$$= c\sum_{k=1}^{\infty} kp_k$$
$$= cE(N^i)$$

So,

$$E(N^M) = cE(N) = \frac{1-p_0^M}{1-p_0}E(N)$$

$$E[(N^M)^2] = cE(N^2) = \frac{1-p_0^M}{1-p_0}E(N^2),$$

$$V(N^M) = cE(N^2) - c^2 E^2(N)$$
$$= \frac{1-p_0^M}{1-p_0}E(N^2) - \left(\frac{1-p_0^M}{1-p_0}\right)^2 E^2(N)$$

To obtain the mean and variance of zero truncated distribution, set $p_0^M = 0$.

Example 15.

For a distribution from $(a, b, 1)$ class, $p_1 = 0.4$, $p_2 = 0.2$, $p_3 = 0.1$. Determine p_0. $\boxed{0.2}$

Example 16 (T2Q5).

For a zero-modified Poisson distribution, $p_1 = 0.25$, $p_2 = 0.1$, calculate the probability of 0.

Example 17.

A zero-truncated Geometric distribution has a mean of 3, calculate the probability of 5. $\boxed{0.06584}$

Example 18.
A claim count distribution has zero-truncated binomial distribution with $m = 4$, $q = 0.2$. Determine the probability of 2 or more claims. $\boxed{0.3062}$

Example 19 (T2Q6)**.**
For a zero-modified negative binomial distribution, $r = 2$ and β, $p_0 = 0.33$, and $p_1 = 0.0447$. Find β.

Example 20 (T2Q7)**.**
You are given:

- p_k denotes the probability that the number of claims, N equals k for $k = 0, 1, 2, \ldots$
- $P(N = k) = p_k = (1.89/k + 0.63)p_{(k-1)}, k = 1, 2, \ldots$

Using the corresponding zero-modified claim count distribution with $p_0^M = 0.19$, calculate the variance of the distribution.

Example 21 (T2Q9)**.**
N^M is a discrete random variable with probability function which is a member of the $(a, b, 1)$ class of distributions. You are given

$$P(z) = 0.24 + 0.76 \left[\frac{[(1 - 5.60(z - 1)]^{-3} - (6.60)^{-3}}{1 - (6.60)^{-3}} \right]$$

Calculate $P(N^M = 2)$.

2.4.3 Extended Truncated Negative Binomial (ETNB)

In addition to modifications of $(a, b, 0)$ class, the $(a, b, 1)$ class admits additional distributions. The (a, b) parameter space can be expanded to admit and extension of the negative binomial distribution to included cases where $-1 < r < 0$; the negative binomial distribution only allows $r > 0$. This extended distribution is called the ETNB (Extended truncated negative binomial), even through it may be zero-modified rather than zero-truncated.

When $r = 0$, the limiting case of the ETNB is the logarithmic distribution with

$$p_k^T = \frac{[\beta/(1+\beta)]^k}{k ln(1+\beta)}, k = 1, 2, 3, \ldots$$

The pgf of the logarithmic distribution is

$$P^T(z) = 1 - \frac{ln[1 - \beta(z-1)]}{ln(1+\beta)}$$

The zero-modified logarithmic distribution is created by assigning an arbitrary probability at zero and reducing the remaining probabilities.

Example 22.
Determine the first four probabilities for an ETNB distribution with $r = -0.5$ and $\beta = 1$. Do this both for the truncated version and for the modified version with $p_0^M = 0.6$ set arbitrary.

Example 23 (T2Q10).
For a zero-modified ETNB distribution, you are given: (i) $p_1 = 0.766496$, (ii) $p_2 = 0.038972$ and $p_3 = 0.005876$. Determine the probability of 0.

2.5 Compound Frequency Models

2.5.1 Compounding

A larger class of distributions can be created by the processes of compounding any two discrete distributions. The term **compounding** reflects the idea that the pgf of new distribution $P(z)$ is written as

$$P(z) = P_N[P_M(z)]$$

where $P_N(z)$ and $PM(z)$ are called the primary and secondary distributions, respectively.

Let N be a counting random variable with pgf $P_N(z)$, and let $M1, M2, \ldots$ be iid random variables with pgf $P_M(z)$. Assuming that the $M_j s$ do not depend on N, then the random sum

$$S = M_1 + M_2 + \ldots + M_N,$$

is the compound random variable with

$$P_S(z) = P_N[P_M(z)].$$

The probability of exactly k claims can be written as

$$P(S = k) = \sum_{n=0}^{\infty} P(S = k|N = n)P(N = n)$$
$$= \sum_{n=0}^{\infty} P(M_1 + \cdots + M_N = k|N = n)P(N = n)$$

Letting $g_n = P(S = n)$, $p_n = P(N = n)$, and $f_n = P(M = n)$, this is written as

$$g_k = \sum_{n=0}^{\infty} p_n f_k^{*n}$$

where $f_k^{*n}, k = 0, 1, \ldots$, is the "n-fold convolution" of the function $f_k, k = 0, 1, \ldots$

When $P_N(z)$ is chosen to be a member of $(a, b, 0)$ or $(a, b, 1)$ classes, then a simple recursive formula can be used to avoids the use of convolutions.

Example 24. Show that negative binomial distribution is a generation of the Poisson, i. e., a compound Poisson with a logarithmic secondary distribution.

Example 25 (T2Q11)**.**

Suppose S is a compound frequency distribution with primary and secondary distributions N and M, respectively. N and M are Poisson with parameters $\lambda_1 = 8.5$ and $\lambda_2 = 2.20$, respectively. Find $1000P(S = 2)$.

2.5.2 The Recursive Formula

Theorem 2.
If the primary distribution is a member of the $(a, b, 0)$ class, the recursive formula is

$$g_k = \frac{1}{1 - af_0} \sum_{j=1}^{k} \left(a + \frac{bj}{k} \right) f_j g_{k-j}, k = 1, 2, \ldots$$

Theorem 3.
For any compound distribution, $g_0 = P_N(f_0)$, where $P_N(z)$ is the pgf of the primary distribution and f_0 is the probability that the secondary distribution takes on the value zero.

Example 26.
Calculate g_3 for the Poisson-ETNB distribution where $\lambda = 3$ for the Poisson distribution and $r = -0.5$, $\beta = 1$ for the ETNB distribution.

Example 27 (T2Q12)**.**
Suppose the probability generating function (pgf) of the primary distribution is

$$P(z) = e^{6.7(z-1)}$$

and the pgf of the secondary distribution is

$$P(z) = [1 - \beta(z - 1)]^{-1},$$

and the probability of no claims equals 0.79. Calculate 1000β.

2.6 The Impact of Deductible on Claim Frequency

2.6.1 The Impact of Coverage Modifications on $(a, b, 0)$ class

An important component in analyzing the effect of policy modifications pertains to the change in the frequency distribution of payments when the deductible is imposed or change. When a deductible is imposed or increased, there will be fewer payments per period, while if a deductible is lowered, there will be more payments.

Let X_j be the severity represents the ground-up loss on j^{th} such loss and there are no coverage modifications.

Let N^L denote the number of losses.
v be the probability that a loss will result in a payment when there is a coverage modification.

For example, if there is a deductible of d, $v = P(X > d)$.

Define

$$I_j = \begin{cases} 1, & \text{if the } j^{th} \text{result in a payment} \\ 0, \text{otherwise} \end{cases}$$

Then

$I_j \sim Bernoulli(v)$ and

$$P_{I_j} = 1 - v + vz$$

Let

$$N^P = I_1 + I_2 + \cdots + I_{N^L}$$

represents the number of payments.

If $I_1, I_2, \ldots$ are mutually independent and are also independent of N^L, then N^P has a compound distribution with N^L as the primary distribution and a Bernoulli secondary distribution.

$$P_{N^p}(z) = P_{N^L}[P_{I_j}(z)] = P_{N^L}[1 + v(z - 1)]$$

In the important special case in which the distribution of N^L depends on a parameter θ such that

$$P_{N^L}(z) = P_{N^L}(z; \theta) = B[\theta(z - 1)],$$

then

$P_{N^p} = B[\theta(1 + vz - v - 1)]$

$$= B[v\theta(z - 1)]$$
$$= P_{N^L}(z; v\theta)$$

This result implies that N^L and N^P are both from the same parametric family and only the parameter θ need be change.

- For Poisson distribution:

$$N^L \sim POI(\lambda)$$

$$P_{N^L}(z) = e^{\lambda(z-1)}$$

$$P_{N^p}(z) = P_{N^L}(z; v\theta) = e^{v\lambda(z-1)}$$

$$\therefore N^P \sim POI(v\lambda)$$

- For Binomial distribution:

$$N^L \sim Bin(m, q)$$

$$P_{N^L}(z) = (1 + q(z - 1))^m$$

$$P_{N^p}(z) = P_{N^L}(z; v\theta) = (1 + vq(z - 1))^m$$

$$\therefore N^P \sim Bin(m, vq)$$

- For Negative Binomial Distribution:

$$N^L \sim NB(r, \beta)$$

$$P_{N^L}(z) = (1 + \beta(z - 1))^{-r}$$

$$P_{N^p}(z) = P_{N^L}(z; v\theta) = (1 + v\beta(z - 1))^{-r}$$

$$\therefore N^P \sim NB(r, v\beta)$$

Example 28.

You are given:

n	$P(N = n)$	x	$P(X = x)$
1	0.8	100	0.2
2	0.2	200	0.7
		500	0.1

An insurance for the losses has an ordinary deductible of 100 per loss. Calculate the probability of 2 or more losses occur. [0.128]

Example 29.

The number of annual losses has a Poisson distribution with a mean of 5. The size of each loss has a two-parameter Pareto distribution with $\theta = 10$ and $\alpha = 2.5$. An insurance for the losses has a ordinary deductible of 5 per loss. Calculate the expected value of the number of losses.

Example 30 (T2Q13)**.**

The number of losses (N) follows a Poisson distribution with mean $\lambda = 2.1$. Loss sizes follow a gamma distribution with $\alpha = 3$ and $\theta = 180$. Let N^P be the number of losses for amount less than 360.0. Determine the third raw moment of N^P.

Example 31.

The frequency distribution for the number of losses when there is no deductible is Binomial with $m = 3$ and $q = 0.8$. Loss amounts have a Weibull distribution with $\tau = 0.3$ and $\theta = 1,000$. Determine the expected number of payments when a deductible of 200 is applied. [1.2936]

Example 32 (T2Q14)**.**

The losses on an auto comprehension coverage have a Pareto distribution with parameters $\alpha = 4$ and $\theta = 1380$. The number of losses has a Bernoulli distribution with an average of 0.35 losses per year. Loss sizes are affected by 5.5% inflation. A 100 deductible is imposed. Calculate 1000 times the variance of the frequency of claims after inflation and the deductible.

2.6.2 The Impact of Coverage Modifications on $(a, b, 1)$ class

The result may be generalized for zero-modified and zero-truncated distributions.

Suppose N^L depends on parameters θ and α such that

$$P_{N^L}(z) = P_{N^L}(z; \theta, \alpha)$$

$$= \alpha + (1 - \alpha)\frac{B[\theta(z-1)] - B(-\theta)}{1 - B(-\theta)}$$

Note that $\alpha = P_{N^L}(0) = P(N^L = 0)$, and so is the modified probability at zero.

$$P_{N^P}(z) = P_{N^L}(1 + v(z - 1); \theta, \alpha^*)$$
$$= \alpha^* + (1 - \alpha^*)\frac{B[\theta(1+v(z-1)-1)] - B(-\theta)}{1 - B(-\theta)}$$
$$= \alpha^* + (1 - \alpha^*)\frac{B[v\theta(z-1)] - B(-\theta)}{1 - B(-\theta)}$$
$$= P_{N^L}(z; v\theta, \alpha^*)$$

where $\alpha^* = P(N_P = 0) = P_{N^P}(0) = P_{N^L}(1 + v(0 - 1); \theta, \alpha) = P_{N^L}(1 - v; \theta, \alpha)$

The following table indicates how the parameters change when moving from N^L to N^P.

N^L	Parameters for N^P
Poisson	$\lambda^* = v\lambda$
ZM Poisson	$p_0^{M*} = p_0^M + (1 - p_0^M)\left[\frac{e^{\lambda(1-v)} - 1}{e^\lambda - 1}\right]$, $\lambda^* = v\lambda$
Binomial	$q^* = vq$
ZM Binomial	$p_0^{M*} = p_0^M + (1 - p_0^M)\left[\frac{[1-vq]^m - (1-q)^m}{1 - (1-q)^m}\right]$, $q^* = vq$
Negative Binomial	$\beta^* = v\beta$, $r^* = r$
ZM Negative Binomial	$p_0^{M*} = p_0^M + (1 - p_0^M)\left[\frac{(1+v\beta)^{-r} - (1+\beta)^{-r}}{1 - (1+\beta)^{-r}}\right]$
	$\beta^* = v\beta$, $r^* = r$
ZM logarithmic	$p_0^{M*} = p_0^M + (1 - p_0^M)\left[1 - \frac{\ln(1+v\beta)}{\ln(1+\beta)^{-r}}\right]$, $\beta^* = v\beta$

Example 33.

Let the frequency distribution be zero-modified negative binomial with $r = 2$, $\beta = 3$, and $p_0^M = 0.4$. Determine the distribution of N^P when a deductible of 250 is imposed on a Pareto distribution with $\alpha = 3$ and $\theta = 1,000$. □

Example 34 (T2Q15)**.**

Let losses occur following a zero modified binomial distribution with $q = 0.82, m = 4$ and $p_0^M = 0.63$. Suppose a deductible is imposed such that the probability of a payment resulting from a loss is now 0.78 rather than 1. Determine the variance of the number of payments made.

Example 35 (T2Q16)**.**

Number of claims follows a zero modified binomial distribution with $q = 0.7, m = 6$ and $p_0^M = 0.73$. Suppose a deductible is imposed such that the probability of a payment resulting from a loss is now 0.86 rather than 1. Determine the probability that the number of payments exceed 4.

2.7 Exposure Modifications

Assume that the current portfolio consists of n entities, each of which could produce claims. Let N_j be the number of claims produced by the jth entity. Then $N = N_1 + \ldots + N_n$. If we assume that the $N_j's$ are iid, then

$$P_N(z) = [P_{N_1}(z)]^n$$

Now suppose the portfolio is expected to expand to n^* entities with frequency N^*. Then

$$P_{N^*}(z) = [P_{N_1}(z)]^{n^*} = [P_N(z)]^{n^*/n}$$

Example 36.

It has been determined from past studies that the number of workers compensation claims for a group of 300 employees in a certain occupation class has negative binomial distribution with $\beta = 0.3$ and $r = 10$. Determine the frequency distribution for a group of 500 such individuals.

Example 37.
Suppose claims on a portfolio of 500 policies are independent and identically distributed. Each has a geometric distribution with parameter $\beta = 0.001$. If 250 additional policies are put on the books, what is the new probability of no claims during the years? $\boxed{0.47254}$

Example 38.
For an employee health coverage got 50 individuals, the aggregate claims frequency distribution is negative binomial with mean 10 and variance 20. The group then expands to 60. Calculate the probability of 10 claims for the expanded group.

Example 39 (T2Q17)**.**

Aggregate claim frequency for an employee dental coverage covering **20** individuals follows a negative binomial distribution with mean 8 and variance 16. Loss size has an exponential distribution with mean **425**. The group expands to 60 individuals and a deductible of 127 is imposed. Calculate the probability of 2 or more claims from the group after these revisions times 1000.