CONTENTS

5 Amortization Schedules and Sinking Funds

5.1 Method of Loan Repayment

1. **Amortization Method**:
   In the amortization method, the borrower repays the lender by means of installment payments at periodic intervals. Typically this method is used with individual borrowers. Examples include car loan, mortgage repayment.

2. **Sinking Fung Method**:
   In the sinking fund method, the borrower repays the lender by means of a lump-sum payment at the end of the term of the loan. The borrower pays interest on the loan in installments over this period. It is also assumed that the borrower makes periodic payments into a fund, called a "sinking fund", which will accumulate to the amount of the loan to be repaid at the end of the term of the loan

5.2 Amortizing a Loan

Suppose you borrow an amount of money at 5% effective and you are to repay the loan by making payments of \$1,000 at the end of each year for 5 years.

- What is the original amount of the loan?

- How much interest should you pay the lender at the end of the first year?

- What is the excess of \$1,000 over the interest paid in the first year? What does this excess amount represent?

- What is the remaining amount of the loan at the end of the first year? What does this remaining amount represent?

5.3 Outstanding Balance

Two ways of looking at the **outstanding loan balance**:

1. **Retrospective Method:**
   Suppose that you and the lender agree that you will completely settle the debt at the end of first year by making an additional lump sum payment just after the first regular payment of \$1,000. This lump sum payment would have to be the outstanding loan balance of \$3,545.95, since this is the remaining portion of the original loan that you still owe. The outstanding balance is computed by looking backwards to the original loan amount, the payment of \$1,000 and the principal repaid in that payment. Thus this method is called the **retrospective method**.

Let the outstanding loan balance at time t be B_t. The original loan balance, B_0, is denoted by L and then level payments are R. The prospective method gives

$$\boxed{B_t = \underbrace{L(1+i)^t}_{\text{amount owed at time } t} - \underbrace{Rs_{\overline{t}|}}_{\text{amount paid at time } t}}.$$

Note: Use this method if

- the number of payments is not known; or
- the size of future payments is not known.

2. **Prospective Method:**
As in (1), you are to make an additional lump sum payment at the end of one year to completely settle the debt. But now you and the lender reason that what you still owe are the four remaining payments of \$1,000. If you are to pay off the loan with a lump sum, the amount must be the PV of the four payments, i.e. $1000a_{\overline{4}|} = \$3545.95$. This outstanding balance is computed by looking forward to the remaining payments, thus this method is called the **prospective method**. Outstanding loan balance at any point in time is equal to the present value at that date of the remaining installment payments.

$$\boxed{B_t = \frac{L}{a_{\overline{n}|}} a_{\overline{n-t}|} = R a_{\overline{n-t}|}.}$$

Example 1.
Smith obtains a 36-month automobile loan of \$10,000 at a nominal interest rate of 18%, compounded monthly. Equal payments are due at the end of each month. Find the outstanding balance immediately after the 25th payment,

(i) using retrospective method; $\boxed{36417}$

(ii) using prospective method. $\boxed{3641}$

Example 2.

A loan is to be repaid in level installments payable at the end of each year for 9 years. The effective annual rate of interest on the loan is 6%. After the 5th payment the principal remaining is 10,000. Determine the amount of loan.

Example 3.

Suppose that you get a mortgage of 100,000, to be repaid in equal annual payments over 20 years, at the end of each year, at the annual effective rate of interest of 6%. After 10 years, you get an enormous salary hike and decide to completely repay the loan. How much will you pay (not including the installment at 10 years)?

Example 4 (T05Q01).

John borrows 50,000 that is to be paid back over 10 years with level monthly payments at the end of each month. The interest is charged on the loan at a nominal rate of 10% compunded monthly. On the due date of the 50th payment. John decides to repay the loan in full with a single payment of X. Calculate X.

Example 5 (T05Q02).

A 6,000 loan is being repaid with regular payments of X at the end of each year for as long as necessary plus a smaller payment one year after the final regular payment. Immediately after the 12th payment, the outstanding principal is 4 times the size of the regular payment (that is, $4X$). If the annual interest rate i is 12%, what is the value of X?

Example 6 (T05Q03)**.**

Allan buys a house and takes out a 140,000 30-year mortgage. The interest rate is 12% convertible monthly and Allan makes monthly payments of 1,690 for the first 3 years. Determine how large his monthly payment needs to be for the remaining 27 years in order to pay off the mortgage at the end of the 30-year period.

Example 7 (T05Q04)**.**

You took a mortgage loan of 200,000 on January 1, 2022 which required to pay 30 equal annual payments at 12% interest with the first payment due on January 1, 2023. The bank sold your mortgage to an investor immediately after receiving your 7th payment. The yield to the investor is 8%. Determine the bank's overall return on its investment.

Example 8 (T05Q05).

Steven have a 30-year 200,000 mortgage with an 7% interest rate convertible monthly. Payments are made at the end of the month. Immediate after the 120th payment, he refinance the mortgage. The interest rate is reduced to 5.5%, convertible monthly, and the term is reduced to 20 years (so there are 10 years of payments remaining). He also make an additional payment of 26,667 at the time of refinancing. Calculate his new monthly payment.

5.4 Amortization Schedule

Let

- I_t = interest paid at the end of year t
- P_t = principal repaid at the end of year t
- B_t = outstanding loan balance at the end of year t (just after the loan payment is made)
- R_t = loan payment at the end of year t

Thus we have the followings for the first two years:

Symbol	Amount
	First Year
$I_1 = iB_0$	$I_1 = (0.05)(4,329.48) = 216.47$
$P_1 = R_1 - I_1$	$P_1 = 1,000 - 216.47 = 783.53$
$B_1 = B_0 - P_1$	$B_1 = 4,329.48 - 783.53 = 3545.95$
	Second Year
$I_2 = iB_1$	$I_2 = (0.05)(3,545.95) = 177.30$
$P_2 = R_2 - I_2$	$P_1 = 1,000 - 177.30 = 822.70$
$B_2 = B_1 - P_2$	$B_2 = 3,545.95 - 822.70 = 2,723.25$
	t^{th} Year
$I_t = iB_{t-1}$	
$P_t = R_t - I_t$	
$B_t = B_{t-1} - P_t$	

The process of reducing the outstanding balance of a loan by making payments that are part interest and part principal is known as **amortizing** the loan.

It would be convenient to organize all of the results in a table called an "amortization schedule".

Notes:

- The amount of interest paid decreases with each payment.
- The principal repaid increases with each payment.
- The amount of principal repaid increases in geometric progression with common ratio $(1+i)$.

$$P_t = (1+i)^{t-1}P_1, \text{for } i = 2, 3, \ldots$$

- The total of principal repaid is the original loan amount.

$$L = \sum_{t=1}^{n} P_t$$

- The total of the interest paid is the sum of the payments minus the sum of the principal repaid.

$$\sum_{t=1}^{n} I_t = \sum_{t=1}^{n} R_t - \sum_{t=1}^{n} P_t = nR - L$$

- In general, consider a loan of $a_{\overline{n}|}$ at i, repaid by n payment of 1. The amortization schedule for this loan is:

Duration: t	Payment: R	Interest: $I_t = iB_{t-1}$	Principal: $P_t = R - I_t$	Outstanding Balance: $B_t = B_{t-1} - P_t$
0				$a_{\overline{n}|}$
1	1	$ia_{\overline{n}|} = 1 - v^n$	v^n	$a_{\overline{n}|} - v^n = a_{\overline{n-1}|}$
2	1	$ia_{\overline{n-1}|} = 1 - v^{n-1}$	v^{n-1}	$a_{\overline{n-1}|} - v^{n-1} = a_{\overline{n-2}|}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
t	1	$ia_{\overline{n-t+1}|} = 1 - v^{n-t+1}$	v^{n-t+1}	$a_{\overline{n-t+1}|} - v^{n-t+1} = a_{\overline{n-t}|}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	1	$ia_{\overline{1}|} = 1 - v$	v	$a_{\overline{1}|} - v = 0$
Total	n	$n - a_{\overline{n}|}$	$a_{\overline{n}|}$	

The t^{th} line of the schedule gives us the general formulas for interest paid, principal repaid and outstanding balance for a loan of $a_{\overline{n}|}$.

$$I_t = 1 - v^{n-t+1} \quad P_t = v^{n-t+1} \quad B_t = a_{\overline{n-t}|}$$

In general, the formulas for a loan of L with equal payments, $R = \frac{L}{a_{\overline{n}|}}$ are

$$I_t = R(1 - v^{n-t+1})$$
$$P_t = Rv^{n-t+1}$$
$$B_t = Ra_{\overline{n-t}|}$$

Example 9.

The amount of principal repaid in the first payment of a loan being repaid by level payments at 3% is 100. What is the principal repaid in the 14^{th} payment?

Example 10 (T05Q06).

Mike takes out a 30-year loan on January 1, 2022 for 10,000 at an annual effective interest rate of 5%. Payments are made at the end of each year. On January 1, 2032, Mike takes out a 20-year loan for 5,000 at an annual effective interest rate of 7%. Payments are also made at the end of each year. Calculate the total amount of principal repaid during year 2032 on both loans.

Example 11.

The amount of principal repaid in the first payment of a 14-year loan being repaid by level payments at 5% is 200. What is the amount of loan?

Example 12 (T05Q07)**.**

A loan is being amortized by means of level monthly payments at an annuial effective interest rate of 6%. the amount of principal repaid in the 18-th payment is 4,000 and the amount of principal repaid in the t-th payment is 16000. Calculate t.

Example 13. A loan of a certain amount is being repaid at an effective rate $i = 5\%$ by the following sechedule of payments:

Year	Payment
1,2,...,9,10	100,200,...,900,1000
11,12,13,14,...,17,18,19,20	500,500,600,600,...,800,800,900,900
21,22,23,24,...,39,40	700.400, 700,400,...,700,400
41-60	1,200

Compute the outstanding balance just after the 50^{th} payment.

Example 14.
Kevin takes out a 10-year loan of L, which he repays by the amortization method at an annual effective interest rate of i. Kevin makes payments of 1000 at the end of each year. The total amount of interest repaid during the life of the loan is also equal L. Calculate the amount of interest repaid during the first year of the loan.

Example 15.

James takes out a 35-year loan, which is repaid with annual payments at the end of each year. he repays the loan by making payments which are equal to X during years $1-21$, $3X$ during year $22-28$, and $2X$ during years $29-35$. Interest is charged on the loan at an annual effective rate of i, $i>0$. The amount of interest repaid during year 22 is twice as much as the amount of interest repaid during year 29. Calculate i.

Example 16 (T05Q08)**.**

A 9,000 loan is to be repaid with equal payments at the end of each year for 30 years. The principal portion of the 15th payment is 1.5 times the principal portion of the 5th payment. Calculate the total amount of interest paid on the loan.

Example 17.

Luke is repaying a loan with payments of 2,800 at the end of every two years. If the amount of interest in the 8th installment is 2,610, find the amount of principal in the 12th installment. Assume an annual effective interest rate of 13%.

Example 18 (T05Q09)**.**

Alvin purchases a 150,000 home. Mortgage payments are to made monthly for 40 years, with the first payment to be made one month from now. The annual effective rate of interest is 5%. After 10 years, the amount of each monthly payment is increased by 705.74 in order to repay the mortgage more quickly. Calculate the amount of interest paid over the duration of the loan.

Example 19.

A loan of 86,000 is being repaid by 21 equal annual installments made at the end of each year at 8% interest effective annually. Immediately after the 5-th payment, the loan is renegotiated as follows:

- The borrower will make 16 annual payments of K to repay the loan, with the first payment three years from the date of renegotiation.
- The interest rate is changed to 9.5% effective annually.

Calculate K.

Example 20 (T05Q10).

A loan of 300,000 is being amortized with payments at the end of each year for 20 years. If $v^{10} = 0.869$, find the amount of principal repaid in the first 10 years.

Example 21.

A loan of L is to be repaid with 80 payments of 200 at the end of each quarter. Interest on the loan is charged at a nominal rate $i, 0 < i < 1$, convertible quarterly. The outstanding principal immediately after 62th and 71th payments are 2963.9 and 1622.21, respectively. Calculate the amount of interest repaid in the 24th payment.

Example 22 (T05Q11)**.**

A loan of 120,000 is repaid with unequal annual payments at the end of each year for 45 years. Each of the first 44 payments is equal to two times the amount of interest then due. The final payment repays the remaining loan balance at that time. Interest is charged at an annual effective rate of 9%. Calculate the amount of the final loan payment.

Example 23.

A 43-year loan is to be repaid in equal annual installments. The amount of interest paid in the 16th installment is 123. The amount of interest paid in the 30th installment is 89. Calculate the amount of interest paid in th 37th installment.

5.5 Sinking Funds

Suppose the borrower will pay only the interest due at the end of each year and will repay the original loan amount at the end of n years. This method of payment is perfectly OK except that the lender may be worry that the lender may not be able to payback the original amount of loan. Also, the payment of a big lump-sum at the end of n years could be very disruptive to the borrower's financial position at that time. The borrower can make systematic deposit into a fund that eventually accumulate to the loan amount. In fact, the lender might insist on it to safeguard the return of the of the loan.

A fund that designed to accumulate a specified amount of money in a specified time by making regular deposits is called **sinking fund**.

For example, suppose the interest payments to the lender are at 6%, but the borrower can earn 5% on the. What is the total annual payment made by the borrower for a loan of 1000?

$$\begin{array}{lll}
(1) & \text{Interest payment} & \\
 & \text{to the lender at } 6\% & = 0.06 \times 1000 = 60 \\
(2) & \text{SF deposit} = \frac{1000}{s_{\overline{10}|}} \text{ at } 5\% & = 79.50 \\
\hline
 & \text{Total Annual Payment} & 139.50 \\
\hline
\end{array}$$

In general, for a loan of L, an interest on loan of i and interest on SF deposit of j, we have:

$$\boxed{\begin{aligned}\text{Total annual payment} &= iL + \frac{L}{s_{\overline{n}|j}} \\ &= L\left(i + \frac{1}{s_{\overline{n}|j}}\right)\end{aligned}}$$

Alternatively, suppose we know the **Total Annual Payment**(TAP), we could express the Annual SF deposit as $(TAP - iL)$
To solve for L, we accumulate the SF deposit and equate it to L:

$$\boxed{(TAP - iL)s_{\overline{n}|j} = L}$$

This is useful to solve for L in the case of non-level SF deposit.

5.5.1 General Approach to SF Problem

The general approach to SF problems is to answer two questions:

(1) What is the interest paid to the lender?

(2) What is the SF deposit?

The sum of (1) and (2) is the total periodic payment made by the borrower to repay the loan.

Example 24.

A borrower repays a loan by making annual interest payments to the lender at 9% and by making SF deposits at 8% for 17 years. The borrower pays a total of 37,564 annually. What is the amount of loan?

Example 25.

A borrower makes annual interest payments to the lender at 10% and SF deposits at 8% for 24 years. The total annual payment made by the borrower is 20,000 in the first 17 years and 11,000 in the last 7 years. What is the amount of the loan?

Example 26 (T05Q12)**.**

A borrower and a lender agree to the following arrangement: The borrower will pay annual interest to the lender for 13 years at 5%. The borrower will pay 120% of the original loan amount to the lender at the end of 13 years by making 6 annual deposits in a SF earning 3%. After making the 6 deposits, the SF grows with interest only. The total annual payment made by the borrower in the first 6 years is 14,000. What is the amount of the loan?

Example 27 (T05Q13)**.**

A 20-year loan of 20,000 may be repaid under the following two methods:

- amortization method with equal annual payments at an annual 1 effective rate of 6.5%.
- sinking fund method in which the lender receives an annual effective rate of 8.0% and the sinking fund earns an annual effective rate j.

Both methods require a payment of X to be made at the end of each year for 20 year. Calculate j.

Example 28 (T05Q14)**.**

Justin and Maggie each take out a 20-year loan L. Justin repays his loan using the amortization method, at an annual effective interest rate of i. He makes an annual payment of 800 at the end of each year. Maggie repays her loan using the sinking fund method. She pays interest annually, also at an effective interest rate of i. In addition, Maggie makes level annual deposits at the end of each year for 20 years into a sinking fund. The annual effective rate on the sinking fund is 4.22%, and she pays off the loan after 20 years. Maggie's total payment each year is equal to 9% of the original loan amount. Calculate L.

Example 29.

Suppose that the sinking fund method is used for the repayment of a loan of RM1,000, and that the annual effective rate of the loan and the annual effective rate of the sinking fund are both 8%. Suppose that the term of the loan is 5 years. Calculate the balance of the sinking fund immediately after each of the deposits. □

5.5.2 Outstanding Balance, Interest Paid and Principal Repaid under the Sinking Fund Method

The amount in the SF at the end of year t is:

$$SF_t = \frac{L}{s_{\overline{n}|}} s_{\overline{t}|}$$

Interest earned on $(SF)_t = j(SF)_{t-1}$

Interest paid (net amount of interest to lender and interest earned on SF) is

$$I_t = iL - j(SF)_{t-1}$$

The principal repaid under any method of repaying a loan is equal to the decrease in the outstanding balance. So the principal repaid in the t^{th} period under the SF method can be determined by either of the two methods, both of which represent the increase in the SF:

(1) The interest earned on the SF in the t^{th} period plus the deposit made at the end of that period, i.e.

$$P_t = j(SF)_{t-1} + (SF)$$

(2) or, the AV of the SF at the end of the t^{th} period minus the AV at the end of $(t-1)^{th}$, i.e.

$$P_t = \frac{L}{s_{\overline{n}|j}}(s_{\overline{t}|j} - s_{\overline{t-1}|j})$$

Summary:

- $SF_t = \frac{L}{s_{\overline{n}|j}} s_{\overline{t}|j}$
- Interest earned on $(SF)_t = j(SF)_{t-1}$
- $I_t = iL - j(SF)_{t-1}$
- $P_t = j(SF)_{t-1} + (SF) = \frac{L}{s_{\overline{n}|j}}(s_{\overline{t}|j} - s_{\overline{t-1}|j})$

Example 30.
Consider a 10-year loan of 1,000 is to be repaid with payments at the end of each year consisting of interest on the loan and a sinking fund deposit. Interest on the loan is charged at a 6% annual effective rate. The sinking fund's annual effective interest rate is 6%. Calculate

(a) The amount in the SF at the end of year 2, $(SF)_2$.

(b) I_3.

(c) P_3.

Example 31 (T05Q15).

John borrows X and repays the principal by making 10 annual payments at the end of each year into a sinking fund which earns an annual effective rate of 6%. The interest earned on the sinking fund in the 6th year is 206.31. Calculate X.

Example 32.

A corporation borrows 14,000 for 27 years, at an effective annual rate of 9%. A sinking fund is used to accumulate the principal by means of 27 annual deposits earning an effective annual interest rate of 8%. Calculate the sum of the net amount of interest paid in the 13th installment and the increment in the sinking fund for the 8th year.

5.6 Varying Series of Payments

5.6.1 Amortization

If a loan is being repaid by the amortization method, it is possible that the borrower repays the loan with instalments which are not level. In fact, any series of payments whose present value is equal to the loan amount will repay it.

Assume that the interest conversion period and the payment period are equal and coincide. Consider a loan L to be repaid with n periodic instalments $R_1, \cdots, R_n$. Then we have

$$L = \sum_{i=1}^{n} v^t R_t.$$

We can also use the recursion formulas:

$$I_t = iB_{t-1}$$

$$P_t = R_t - I_t$$

$$B_t = B_{t-1} - P_t$$

Example 33.

A loan of 10,000 at 10% effective is being repaid by payments of 2,000, 3,000, and 7,590 at the end of years 1, 2, and 3, respectively. Construct an amortization schedule for this loan.

Remark: When non-level payments are made, it is possible that one or more of the payments will be less than the interest. Thus, we "owe" the amortization schedule "$I_t - R_t$". We can regard this as an additional loan. This is called **negative amortization**.

Example 34.
A loan of 10,000 at 10% effective is being repaid by payments of 600, 5,000, and 7,084 at the end of years 1, 2, and 3, respectively. Construct an amortization schedule for this loan.

Example 35.

A borrower is repaying a loan at 6% effective with payments at the end of each year for 14 years, such that the first year,s payment is 400, the second year 380.0, and so forth, until the 14th year it is 140.0. Find the amount of the loan.

Example 36 (T05Q16)**.**

A borrower is repaying a loan at 6% effective with payments at the end of each year for 12 years, such that the first year,s payment is 920, the second year 874.0, and so forth, until the 12th year it is 414.0. Find the principal and interest in the 8th payment.

Example 37 (T05Q17)**.**

A loan of 95,000 is being repaid by a 35-year increasing annuity-immediate. The initial payment is K, and each subsequent payment is K larger than the preceding payment. Determine the principal outstanding immediately after the 10th payment, using an annual effective interest rate of 6%.

Example 38 (T05Q18)**.**

A loan is repayable by a decreasing annuity payable annually in arrears for 20 years. The repayment at the end of the first year is 8000 and subsequent repayments reduce by 300 each year. The repayments were calculated using an annual effective rate of interest of 9%. Construct the schedule of armortization for years eight and nine, showing the outstanding balance at the beginning of the year (B_{t-1}), the repayments (R_t), the interest element (I_t), the Principal repayment (P_t)and the outstanding balance at the end of the year (B_t) for $t = 8$ and $t = 9$.

5.6.2 Sinking Funds

Assume that the varying payments with the sinking fund method and the interest paid to the lender is constant each period so that only the sinking fund deposits vary. Assume that the varying payments by the borrower are $R_1, \cdots, R_n$ and $i \neq j$. The amount of the loan is

$$\begin{aligned} L &= (R_1 - iL)(1+j)^{n-1} + (R_2 - iL)(1+j)^{n-2} + \cdots + (R_n - iL) \\ &= \sum_{t=1}^{n} R_t(1+j)^{n-t} - iLs_{\overline{n}|j} \\ &= \frac{\sum_{t=1}^{n} v_j^t R_t}{1+(i-j)a_{\overline{n}|j}} \end{aligned}$$

Remark:

It should be noted that we have implicitly assumed that the sinking fund deposit $R_t - iL$ is positive. If it were negative, then it would mean that the payment in that year is not even sufficient to pay the interest on the loan. We would then have a negative sinking fund deposit, i.e. a withdrawal, from the sinking fund for that year.

Example 39. A borrows a certain amount of money L from B for 15 years. He pays B interest every year on L at an effective rate of 7% for the first 10 years and 6% for the last 5 years. A also makes deposits in a SF that will accumulate to L at the end of 15 years. The SF earn 5% during the first 10 years and 4% thereafter. A's total anuual payments is 10,000 in the first 10 years and 8,000 in the last 5 years. Determine L.

5.7 Equal Principal Repayments

One particular way of repaying a loan that sometimes comes up is to pay a **level amount of principal** and **interest on the outstanding balance** at the end of each year.

Paying a level amount of principal is different from making level payments. In fact the loan payments decrease each year.

```
       L
       |------|-------------|----...----|
       0      1             2          10
Pincipal:     P             P
Interest:     I1 = Li       I2= (L-P)i
Loan Payment: P+I1 = P+Li   P+I2=P+Li-Pi
Outstanding Balance: L-P    L - 2P
```


Example 40.
A loan of 1,000 is repaid by equal annual amounts of principal for 10 years and annual interest of 7% on the outstanding balance.

(a) What is the schedule of loan payments?

(b) What is the PV of the payments in (a) at 7% effective?

(c) What is the purchase price of this loan to yield 5% effective?

Example 41 (T05Q19).

A 14-year loan of 7000 is to be repaid with payments at the end of each year. It can be repaid under the following two options:

(i) Equal annual payments at an annual effective rate of 6.82%.

(ii) Installments of 500.0 each year plus interest on the unpaid balance at an annual effective rate of i.

The sum of the payments under option (i) equals the sum of the payments under option (ii). Determine i.

Example 42.

A loan of 5600 is being repaid in 14 years by semiannual installments of 200.0, plus interest on the unpaid balance at 6% per annum compounded semiannually. The installments and interest payments are reinvested at 7% per annum compounded semiannually. Calculate the annual effective yield rate of the loan.

Example 43 (T05Q20).

Annie borrows 25,200 from Bank X. Annie repays the loan by making 36 equal payments of principal at the end of each months. She also pays interest on the unpaid balance each month at a nominal rate of 12%, compounded monthly. Immediately after the 17th payment is made, Bank X sells the rights to future payments to Bank Y. Bank Y wishes to yield a nominal rate of 14%, compounded semiannually, on its investment. What price does Bank X receive?