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CONTENTS

1 The Black-Scholes Formula

1.1 Binary Options

Consider a **cash or nothing call**, which pays 1 at time T if $S(T) > K$ and nothing otherwise. Thus, the payoff function is

$$I(S(T) > K)$$

Recall that under the true measure,

$$E[I(S(T) > K)|S(t)] = P(S(T) > K|S(t)) = N(\hat{d}_2)$$

where

$$\hat{d}_2 = \frac{\ln[S(t)/K] + (\alpha - \delta - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

To price a cash or nothing option, we use the risk-neutral measure. Replacing α with r, and let

$$d_2 = \frac{\ln[S(t)/K] + (r - \delta - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$

the expected payoff under risk-neutral measure is given by

$$E^*[I(S(T) > K|S(t)] = N(d_2)$$

and the time t price of the option is

$$F_{t,T}^{P}[I(S(T) > K] = e^{-r(T-t)}N(d_2)$$

Similarly, under the **cash or nothing put**, the payoff function is

$$I(S(T) < K)$$

and under the true measure,

$$E[I(S(T) < K)|S(t)] = P(S(T) < K|S(t)) = N(-\hat{d}_2)$$

Under the risk-neutral measure

$$E^{*}[I(S(T) < K|S(t)] = N(-d_2)$$

and the time t price of the put option is

$$F_{t,T}^{P}[I(S(T) < K] = e^{-r(T-t)}N(-d_2)$$

Next, we consider an **asset or nothing call**, which pays $S(t)$ at time T if $S(T) > K$ and nothing otherwise. Thus, the payoff function is

$$S(T)I(S(T) > K)$$

Recall that under the true measure,

$$E[S(T)I(S(T) > K)|S(t)] = E[S(T)|S(t)]N(\hat{d}_1)$$

where

$$\hat{d}_1 = \frac{\ln[S(t)/K] + (\alpha - \delta + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

To price the option, the risk-neutral measure must be used. Thus, we replace α with r and let

$$d_1 = \frac{\ln[S(t)/K] + (r - \delta + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$

the expected payoff under the risk-neutral measure can be expressed as

$$E^{*}[S(T)I(S(T) > K))|S(t)] = E^{*}[S(T)|S(t)]N(d_1).$$

As a result, the time t price of the option is

$$\begin{aligned}&F_{t,T}^{P}[S(T)I(S(T) > K)]\\&= e^{-r(T-t)}S(t)e^{(r-\delta)(T-t)}N(d_1)\\&= S(t)e^{-\delta(T-t)}N(d_1).\end{aligned}$$

For **asset or nothing put**, which pays $S(t)$ at time T if $S(T) < K$ and nothing otherwise, the time t price of this option is

$$F_{t,T}^{P}[S(T)I(S(T) < K] = S(t)e^{-\delta(T-t)}N(-d_1).$$

The Binary options are summarized as follows:

Binary Option	Payoff at Maturity	Time-t Price
Cash-or-nothing call	$I(S(T) > K)$	$e^{-r(T-t)}N(d_2)$
Cash-or-nothing put	$I(S(T) < K)$	$e^{-r(T-t)}N(-d_2)$
Asset-or-nothing call	$S(T)I(S(T) > K)$	$S(t)e^{-\delta(T-t)}N(d_1)$
Asset-or-nothing put	$S(T)I(S(T) < K)$	$S(t)e^{-\delta(T-t)}N(-d_1)$

Notes:

- The time-t prices are of the form $F^P_{t,T}(S)N(\pm d_1)$ or $F^P_{t,T}(K)N(\pm d_2)$.
- $\pm d_1$ is used for asset-or-nothing options, and $\pm d_2$ is used for cash-or-nothing options.
- Negative sign is attached to d_1 or d_2 for put options.

Example 1.

You are given:

- A nondividend paying stock has a current price of 100.
- The volatility of the stock is 25%.
- The continuous compunded risk free interest rate is 4%.

(a) Calculate the price of a 1-year 100-strike cash-or-nothing call.

(b) Calculate the price of a 1-year 100-strike asset-or-nothing call.

Example 2.

Consider a stock that follows a geometric Brownian motion. You are given:

- The current stock price is 100.
- The dividend yield is 1%.
- The stock volatility is 0.20.
- The continuously compounded risk-free interest rate is 4%.

Calculate the price of the following options, which all mature at $T = 0.5$ and have strike of $K = 90$:

(a) A cash-or-nothing put. $\boxed{0.213}$

(b) An Asset-or-nothing put. 17.79

1.2 The Black-Scholes Formula

The payoff of a European call option on a stock, $[S(T)-K]_+$ can be decomposed into the payoff of cash-or-nothing and asset or-nothing call, i.e.,

$$\begin{aligned}[S(T)-K]_+ &= [S(T)-K]I(S(T)>K) \\ &= S(T)I(S(T)>K) - KI(S(T)>K)\end{aligned}$$

Thus, the **Black-Scholes formula for European call options** is

$$\boxed{c[S(t),K] = S(t)e^{-\delta(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2)}$$

where
$d_1 = \frac{\ln[S(t)/K]+(r-\delta+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$, and
$d_2 = d_1 - \sigma\sqrt{T-t}$.

Similarly, the payoff of a European put option on a stock, $[K - S(T)]_+$ can be decomposed into the payoff of cash-or-nothing and asset or-nothing call, i.e.,

$$\begin{aligned}[K - S(T)]_+ &= [K - S(T)]I(S(T) < K) \\ &= KI(S(T) < K) - S(T)I(S(T) < K)\end{aligned}$$

Thus, the Black-scholes formula for European call options is

$$\boxed{p[S(t), K] = Ke^{-r(T-t)}N(-d_2) - S(t)e^{-\delta(T-t)}N(-d_1)}$$

Example 3 (T02Q1)**.**

You are considering the purchase of a 3-month European call option on a stock. You are given the following information:

- The strike price is 77.
- The current stock price is 79.
- The annual risk-free interest rate is 11% compounded continuously.
- The stock pays continuous dividend proportional to its price at a rate of 4%.
- The annual volatility of the stock is 36%.
- The stock follows the Black-Scholes framework.

Calculate the price of the option.

Example 4 (T02Q2).

You are considering the purchase of a three-month European put option on a nondividend paying stock. You are given the following information:

- The strike price is 58.
- The current stock price is 63.
- The annual risk-free interest rate is 14% compounded continuously.
- The annual volatility of the stock is 30%.
- The stock follows the Black-Scholes framework.

Calculate the price of the option.

Prepaid Forward Version of the Black-Scholes Formula

Recall that $F^P_{t,T}(S)$ is the time-t prepaid forward price of the stock for delivery at time T. Similarly, $F^P_{t,T}(K)$ is the time-t prepaid forward price to delivery K dollars at time T. Thus,

$$\boxed{c(S(t), K, T) = F^P_{t,T}(S)N(d_1) - F^P_{t,T}(K)N(d_2)}$$

and

$$\boxed{p(S(t), K, T) = F^P_{t,T}(K)N(-d_2) - F^P_{t,T}(S)N(-d_1)}$$

where

$$\boxed{d_1 = \frac{\ln[F^P_{t,T}(S)/F^P_{t,T}(K)]+0.5\sigma^2(T-t)}{\sigma\sqrt{T-t}}}$$

and

$$\boxed{d_2 = \frac{\ln[F^P_{t,T}(S)/F^P_{t,T}(K)]-0.5\sigma^2(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T}}$$

By replacing $F^P_{t,T}(S)$ accordingly, we can use the prepaid forward version of Black-Scholes formula for pricing options on other assets.

- **Options on stocks with discrete dividends**

  In this case,

$$F^P_{T,t}(S) = S_t - PV_{t,T}(Div)$$

where $PV_{t,T}(Div)$ is the present value of the (Discrete) dividends payable over the life of the option.

Example 5 (T02Q3)**.**

For a 1-year European call option on a stock:

– The strike price is 76.

– The stock's current price is 81.

– The continuously compounded risk-free interest rate is 0.07.

– The stock pays a dividend of 3 every 3 months, starting immediately after the call option is written. The dividend at the end of one year is paid before the option may be exercised.

– The annual volatility of a prepaid forward on the stock is 0.31.

– The stock follows the Black-Scholes framework.

Calculate the price of the option.

- **Options on Currencies**

For a currency option, suppose a foreign currency has a time-t exchange rate of $x(t)$ units of domestic currency per unit of foreign currency. If the foreign risk-free interest rate is r_f. The prepaid forward price is

$$F^P_{t,T}(x) = x(t)e^{-r_f(T-t)}.$$

Since

$$\ln F^P_{t,T}(x) = \ln x(t) - r_f(T-t)$$

and

$$\ln F^P_{t,T}K = \ln Ke^{-r(T-t)} = \ln K - r(T-t)$$

Thus,

$$\begin{aligned} d_1 &= \frac{\ln[F^P_{t,T}(x)/F^P_{t,T}(K)]+0.5\sigma^2(T-t)}{\sigma\sqrt{T-t}} \\ &= \frac{\ln(x(t)/K+(r-rf+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \end{aligned}$$

$$\begin{aligned} c(S(t),K) &= F^P_{t,T}(x)N(d_1) - F^P_{t,T}(K)N(d_2) \\ p(S(t),K) &= F^P_{t,T}(K)N(-d_2) - F^P_{t,T}(x)N(-d_1) \end{aligned}$$

Example 6 (T02Q4)**.**

Suppose that the spot exchange rate is \$1.3/£. The exchange rate has a volatility of 0.25. Assume that the US dollar intrest rate is 0.052 and and the pounds-denominated intrest rate is 0.029. Calculate the Black-Scholes price (in US) of a call option to buy 100£ with 108.33 USD 6 months from now.

- **Options on Future**

For Futures,

$$F^P_{T,t}(F) = F(t)e^{-r(T-t)}$$

and

$$F^P_{T,t}(K) = Ke^{-r(T-t)}$$

Thus

$$d_1 = \frac{\ln[F^P_{t,T}(S)/F^P_{t,T}(K)]+0.5\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$
$$= \frac{\ln(F(t)/K+(\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$c(F(t),K) = F^P_{t,T}(F)N(d_1) - F^P_{t,T}(K)N(d_2)$$
$$p(F(t),K) = F^P_{t,T}(K)N(-d_2) - F^P_{t,T}(F)N(-d_1)$$

Example 7 (T02Q5)**.**

Suppose that 6-month futures price for a certain stock is 48. The futures prices follows a geometric Brownian motion and has a volatility of 0.31. Consider a European call option on the future contract. The option expires 6 months from now and has a strike price of 48. Assume that the risk-free interest rate is 0.03, calculate the price of the option.

Example 8 (T02Q6).

Let $S(t)$ denote the price at time-t of a stock that pays no dividends. The Black-Scholes framework holds. Consider a European call option with exercise date T , $T > 0$, and exercise price $S(0)e^{rT}$, where r is the continuously compounded risk-free interest rate. You are given:

- $S(0) = 260$
- $T = 11$
- $V[\ln S(t)] = 0.35t, t > 0.$

Determine the price of the call option.

Example 9 (T02Q7).

You are given:

- The time-t price of a stock, $S(t)$, where t is measured in years, follows the risk-neutral process

$$d(\ln S(t)) = 0.039dt + 0.25d\tilde{Z}(t)$$

  where $\tilde{Z}(t)$ is a standard Brownian motion is the risk-neutral measure.
- $S(0) = \mathbf{8.9}$.
- The continuously compounded risk-free interest rate is 0.07.

An option pays $\max(0, 843 - S(1)^3)$ at the end of one year. Calculate the value of the option.

1.3 Greek Letters and Elasticity

In a Black-Scholes framework, the price of any derivative security, $V(S,t)$, depends on the following six factors:

Stock	Option	Environment
S	t	r
σ	K	
δ		

One way to quantify the risk of a derivative is to measure how sensitive $V(S,t)$ is when S or t changes.

- **Delta** (Δ) measures the change in the price of a derivative when the stock price increase by \$1.

$$\Delta = \frac{\partial V}{\partial S}$$

A large delta $\Rightarrow V$ is very sensitive to price.

Delta for call and put are:

$$\boxed{\begin{aligned} \Delta_{\text{call}} &= e^{-\delta(T-t)}N(d_1) \\ \Delta_{\text{put}} &= -e^{-\delta(T-t)}N(-d_1) \end{aligned}}$$

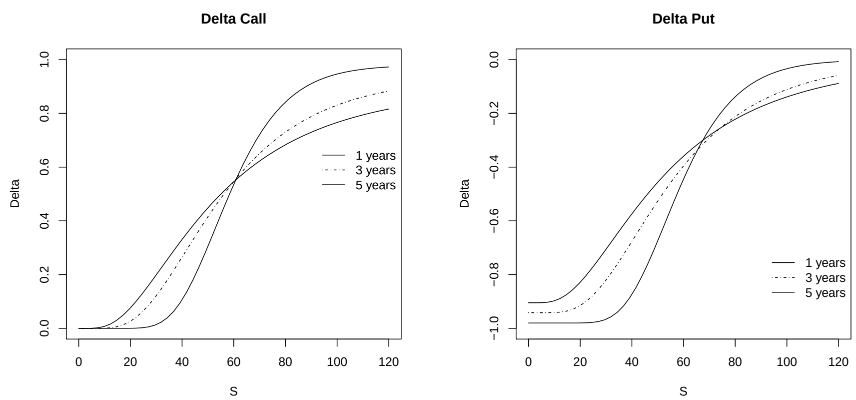
From put-call parity,

$$C(S,K)-P(S,K) = Se^{-\delta(T-t)}-Ke^{-r(T-t)}.$$

Differentiating with respect to S, we have

$$\Delta_{\text{call}} - \Delta_{\text{put}} = e^{-\delta(T-t)}$$

Properties of Δ



$K = 60$; $r = 0.1$, $\sigma = 0.3$, $\delta = 0.02$

Notes:

- Calls Δ is positive and bounded by 0 and $e^{-\delta(T-t)}$.
- Puts Δ is negative and bounded by $-e^{-\delta(T-t)}$. and 0.
- Δ calls $\to 0$ for low S and $\to e^{\delta(T-t}$ for high S .
- Δ puts $\to 0$ for high S ans $\to -e^{\delta(T-t)}$ for low S.

Example 10.

Assume the Black-Scholes framework. Compute the time-t delta for a cash-or-nothing call.

Example 11 (T02Q8)**.**

For a stock that follows a geometric Brownian motion, you are given that

- The current stock price is 26.
- The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 8%.
- The expected stock price after 1 year is 30.82.
- The variance of the stock price after 1 year is 158.64.
- The delta of a 1-year at-the-money European put option is -0.352723.

Find the price of the put option.

- **Gamma**(Γ) measures the change in delta when the stock price increases by \$1.

$$\boxed{\Gamma = \frac{\partial^2 V}{\partial S^2} = \frac{\partial \Delta}{\partial S}}$$

Gamma for call and put are:

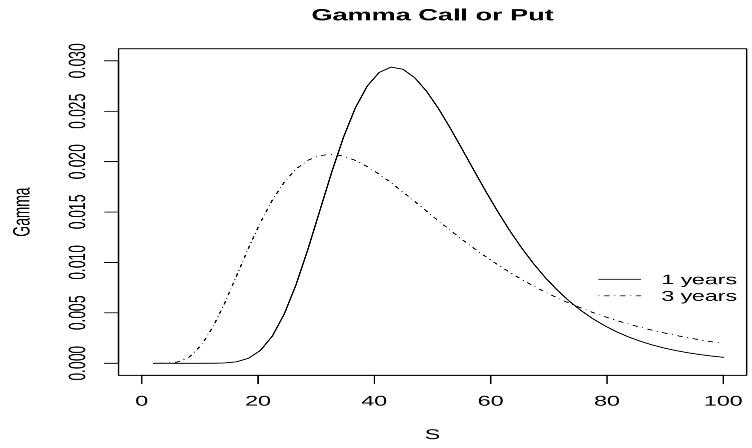
$$\boxed{\Gamma = \frac{e^{-\delta(T-t)}\phi(d_1)}{S\sigma\sqrt{T-t}}}$$

where $\phi(d_1) = \frac{1}{\sqrt{2\pi}}e^{-d_1^2/2}$

Diferentiating both sides of the put call parity equation with respect to S, we get

$$\boxed{\Gamma_{\text{call}} - \Gamma_{\text{put}} = 0}$$

Properties of Γ



Notes:

- $\Gamma_{\text{call}} = \Gamma_{\text{put}}$ for same K and T.
- Γ is positive for long position of calls and puts.
- $\Gamma \to 0$ when S is very low or very high.

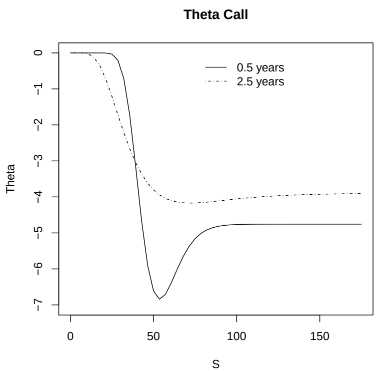
- **Theta** (θ) measures the change in price of a derivative when the is a decrease in the time to expiration t.

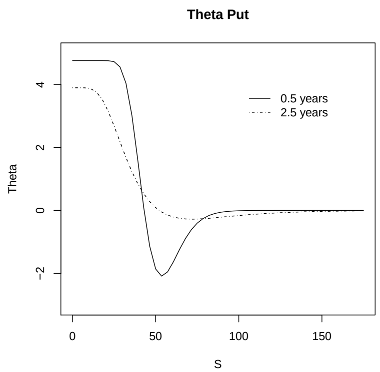
$$\boxed{\theta = \frac{\partial V}{\partial t}}$$

$$\boxed{\theta_{\text{Call}} = S\delta e^{-\delta(T-t)}N(d_1) - Kre^{-r(T-t)}N(d_2) - \frac{S\sigma e^{-\delta(T-t)}\phi(d_1)}{2\sqrt{T-t}})}$$

$$\boxed{\theta_{\text{Put}} = Kre^{-r(T-t)}N(-d_2) - S\delta e^{-\delta(T-t)}N(-d_1) - \frac{S\sigma e^{-\delta(T-t)}\phi(d_1)}{2\sqrt{T-t}})}$$

Properties of Θ





$r = 0.1$, $\sigma = 0.3$, $K = 50$, $\delta = 0$

Notes:

- The value of θ can be positve or negative. It is usually negative because call and put prices tend to drop as time passes.
- If time to expiration is measured in years, theta will be the annualized change in the option value. To obtain a per-day theta, divide by 365.

Example 12 (T02Q9).

For a 1-month European put option, you are given:

(i) Theta is -0.0151 per day.

(ii) The underlying stock price is 54.

(iii) The strike price is 51.0.

(iv) The stock's continuous dividend rate is 0.02.

(v) The continuously compounded risk-free annual interest rate is 0.067.

Calculate theta per day for a 1-month European call option on the same stock with the same strike price.

- **Relation Between Delta, Gamma and Theta**

Recall the Black-Scholes equation

$$V_t + (r-\delta)SV_S + \frac{1}{2}\sigma^2 S^2 V_{SS} = rV$$

As

$$V_t = \frac{\partial V}{\partial t} = \theta;$$

$$V_S = \frac{\partial V}{\partial S} = \Delta$$

and

$$V_{SS} = \frac{\partial^2 V}{\partial S^2} = \Gamma,$$

we have

$$\boxed{\theta + (r-\delta)S\Delta + \tfrac{1}{2}\sigma^2 S^2 \Gamma = rV}$$

- **Vega** (υ) measures the change in price of a derivative when there is an increase in volatility.

$$\boxed{\upsilon = \frac{\partial V}{\partial \sigma}}$$

$$\boxed{\upsilon_{\text{Call}} = Se^{-\delta(T-t)}\sqrt{T-t}\phi(d_1) = \upsilon_{\text{Put}}}$$

Properties of Vegas

- Vegas are positive for both long calls and long puts.
- Vegas for calls and puts with the same strike and time to expiration are the same.
- Vega tends to be (but not always) greater for at-the-money options, and greater for options with longer times to expiration.
- The shape of vega is asymmetric hump, peak similar to Γ.
- It is common to report vega as the change in option price per percentage point change in volatility. This requires dividing the vega formula above by 100.

- **Psi** (ψ) measures the change in price of a derivative when there is an increase in the continuously dividend yield.

$$\boxed{\psi = \frac{\partial V}{\partial \delta}}$$

$$\boxed{\psi_{\text{Call}} = -(T-t)Se^{-\delta(T-t)}N(d_1)}$$

$$\boxed{\psi_{\text{Put}} = (T-t)Se^{-\delta(T-t)}N(-d_1)}$$

Properties of Psi

- Psi is negative for calls and positive for puts.
- A higher dividend yield would lower the prepaid forward price of the stock but not the present value of K, when the dividend yield increases, the price of a European call, which has a payoff of $[S(T) - K]_+$, will fall, while the price of a European put, which has a payoff of $[K - S(T)]_+$, will rise.
- The shape of Psi is a decreasing curve (Negative for calls, positive for puts)
- To interpret psi as price change per percentage point change in the dividend yield, divide 100.

- **Rho** (ρ) measures the change in price of a derivative when there is an increase in the risk free interest rate.

$$\rho = \frac{\partial V}{\partial r}$$

$$\boxed{\rho_{\text{Call}} = (T-t)Ke^{-r(T-t)}N(d_2)}$$

$$\boxed{\rho_{\text{Put}} = -(T-t)Ke^{-r(T-t)}N(-d_2)}$$

Properties of Rho

- We will use the above formula divided by 100. That is, the change of option value per 1% change in interest rate.
- Rho is positive for calls and negative for puts.
- A higher interest rate would lower the present value of K but not the prepaid forward price of $S(T)$. Therefore, when the present value of K decreases, call prices rise, while put prices fall.
- The shape of Rho is an increasing curve (Positive for calls, negative for puts)

- The sign of the six Greek letters can be summarized as below:

Greek	Call	Put
δ	+	-
Γ	+	+
θ	+or-(Usually -)	
υ	+	+
ψ	-	+
ρ	+	-

- **Greek Letters of a Portfolio of Derivatives**
Suppose that an investor forms a portfolio with n derivatives written on the same underlying stock S. The investor take a position of w_i units of the i-th derivative, whose price is denoted by V_t. If $w_i > 0$, then it is a long position, and vice versa. The value of the portfolio is given by

$$P = \sum_{i=1}^{n} w_i V_i.$$

Hence, the delta of the portfolio is

$$\frac{\partial P}{\partial S} = \sum_{i=1}^{n} w_i \frac{\partial V_i}{\partial S} = \sum_{i=1}^{n} w_i \Delta_i$$

Similarly,

$\frac{\partial^2 P}{\partial S^2} = \sum_{i=1}^{n} w_i \Gamma_i$

$\frac{\partial P}{\partial t} = \sum_{i=1}^{n} w_i \theta_i$

$\frac{\partial P}{\partial r} = \sum_{i=1}^{n} w_i \upsilon_i$

$\frac{\partial P}{\partial \delta} = \sum_{i=1}^{n} w_i \psi_i$

$\frac{\partial P}{\partial \sigma} = \sum_{i=1}^{n} w_i \rho_i$

Example 13.

A stock has a continuously compounded dividend yield of 0.06. Delta for a 6-month European put option on the stock is -0.79. Determine delta for a 6-month European call option on the stock with the same strike price. $\boxed{0.1804}$

Example 14.

Assume the Black-Scholes framework. You are given that:

- A nondividend paying stock has a current price of 10 and a volatility of 40%.
- A T-year K-strike European put option on S has a price of 2.4954 and a theta of -0.3903.
- A T-year K-strike European call option written on S has a delta of 0.4480 and a gamma of 0.09889.

Find r the continuously compounded risk-free interest rate. $\boxed{0.05}$

Example 15 (T02Q10)**.**

You are given the following information on two derivatives:

Derivative	Price	Delta	Gamma	Vega
A	1.1553	2.3917	2.373	0.9488
B	0.3403	-0.8427	-0.111	-0.0436

You form derivative C by taking positions on derivative A and B. If derivative C has a zero delta and a gamma of 0.5, calculate its vega.

1.4 The Delta-Gamma-Theta Approximation

Delta, gamma, and theta can be used to approximate the change of the price of of a derivative, $V(S,t)$ when there is a small changes of the price of a stock. By the Taylor's theorem, we have

$$V(S+\epsilon,t) \approx V(S,t)+V_S(S,t)\epsilon+\frac{1}{2}V_{SS}(S,t)\epsilon^2$$

Thus, we have the **Delta-Gamma** Approximation:

$$V(S+\epsilon,t) \approx V(S,t)+\Delta\epsilon+\frac{1}{2}\Gamma\epsilon^2$$

If we drop the gamma term, we have the **delta** approximation:

$$V(S+\epsilon,t) \approx V(S,t)+\Delta\epsilon$$

A more comprehensive description is that the stock price changes from $S(t)$ to $S(t+h)$ when time proceeds from t to $t+h$. We can model this with a delta-gamma-theta approximation, which derived from the multivariate version of Taylor's

theorem. The **Delta-Gamma-Theta Approximation** is

$$V(S(t+h),t+h) \approx V(S(t),t)+\Delta\epsilon+\frac{1}{2}\Gamma\epsilon^2+\theta h$$

where $\epsilon = S(t+h)-S(t)$ and the three Greeks are evaluted at $S(t)$ and t.

Example 16.

Assume the Black-Scholes framework holds. The price of a nondividend paying stock is \$30. The price of a put option on this stock is \$4.00. You are given $\Delta = -0.28$ and $\Gamma = 0.10$. Using the delta-gamma approximation, determine the price of the put option if the stock price changes to \$31.50. $\boxed{3.6925}$

Example 17 (T02Q11).

Let $c(S, T, K, r)$ and $p(S, T, K, r)$ be the prices of call an dput when the stock price is S, the time until expiration is T, the strike price is K, and the continuously compounded risk-free interest rate is r. You are given:

- The stock pays dividends continously at a rate proportional to its price. The dividend yield is 0.03.
- $c(7, 0.5, 8, 0.09) = 0.33799$
- $\frac{\partial}{\partial r}\Big|_{r=0.09} c(7, 0.5, 8, r) = 1.1399$

Approximate the value of $p(7, 0.5, 8, 0.079)$.

1.5 The Mean Return and Volatility of a Derivative

The SDE for $V(S(t),t)$ is

$$\frac{dV(S(t),t)}{V(S(t),t)} = m_V dt + s_V dZ(t)$$

where m_V is the mean return $V(S(t),t)$ and s_V is the volatility of $V(S(t),t)$.
To compute s_V, we use Ito's lemma:

$$\begin{aligned} dV(S(t),t) &= V_t dt + V_s dS + \tfrac{1}{2}V_{SS}(dS)^2 \\ &= V_t dt + V_S[(\alpha-\delta)Sdt + \sigma S dZ(t)] \\ &\quad +(\ldots)dt \\ &= (\ldots)dt + SV_S\sigma dZ(t) \end{aligned}$$

As a result,

$$\frac{dV}{V} = (\ldots)dt + \frac{SV_S}{V}\sigma dZ(t).$$

and thus

$$s_V = \frac{S\Delta}{V}\sigma$$

Define

$$\Omega = \frac{S\Delta}{V}$$

then,

$$\boxed{s_V = \Omega\sigma}$$

Note:

- Depending on the sign of Δ, s_V may be positive or negative. It is customary to report the magnitude of s_V only.
- Ω is called the elasticity of a derivative.
- Since

  $$|s_V| = |\Omega|\sigma$$

  $\Rightarrow$ If $|\Omega| > 1$, then $s_V > \sigma$, and hence the option is riskier than the underlying asset. As a result, elasticity is a measure of leverage.
- Another interpretation of the option is based on the option delta:

  $$\Omega = \frac{S}{V}\frac{\partial V}{\partial S} = \frac{\frac{\partial V}{V}}{\frac{\partial S}{S}}$$

  which means that the elasticity is the percentage change in the derivative price relative to the percentage change in the stock price.

Properties of Elasticity of Calls and Puts

- $\Omega \geq 1$ for a call.
- $\Omega \leq 0$ for a put since $\Delta_{\text{put}} \leq 0$.
- $|\Omega|$ is an increasing function of t.
- $|\Omega|$ increases when $S \to K$
- Suppose that an investor forms a portfolio with n derivatives written on the same underlying stock S. the investor takes a position of w_i units of the ith derivative, whose price is denoted by V_i. The price of the portfolio is given by

$$P = \sum_{i=1}^{n} w_i V_i$$

By definition,

$$\begin{aligned}\Omega_{\text{portfolio}} &= \frac{S}{P}\frac{\partial P}{\partial S} \\ &= \sum_{i=1}^{n} \frac{Sw_i}{P}\Delta_i \\ &= \sum_{i=1}^{n} \left(\frac{Sw_i}{P}\right)\left(\frac{V_i\Omega_i}{S}\right) \\ &= \sum_{i=1}^{n} \left(\frac{w_iV_i}{P}\right)\Omega_i\end{aligned}$$

Thus, the option elasticity of a portfolio is the **value weighted average of the option**

To obtain m_V, we use the equality of Shape ratios,

$$\frac{m_V - r}{s_V} = \frac{\alpha - r}{\sigma}$$

$$\begin{aligned}m_V &= \frac{\alpha - r}{\sigma}s_V + r \\ &= \frac{\alpha - r}{\sigma}(\sigma\Omega) + r \\ &= (\alpha - r)\Omega + r \\ &= \Omega\alpha + (1 - \Omega)r\end{aligned}$$

$\Rightarrow$ the instantaneous expected return on a derivative is a weighted average of α and r.

$$\boxed{m_V = \Omega\alpha + (1 - \Omega)r}$$

Example 18 (T02Q12).

Assume the Black-Scholes framework. You are given that

- The current stock price is 28.
- The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 0.033.
- The volatility of the stock is 0.3.
- The continuously compounded risk-free interest rate is 0.144.

Calculate the current volatility of a 6-month 29.0-strike European call option on the stock.

Example 19 (T02Q13).

You are given:

- For a stock whose time-t price is $S(t)$, the risk-neutral process is

  $d[\ln S(t)] = 0.023dt + 0.25d\tilde{Z}(t), S(0) = 110.0$

  where $\tilde{Z}(t)$ is a standard Brownian motion under the risk-neutral measure.
- The true stochastic process is

  $$dS(t) = cS(t)dt + 0.25S(t)dZ(t),$$

  where $Z(t)$ is a standard Brownian motion under the true probability measure and c is a constant.
- The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 1.2%.

Consider an option that pays 1 one year from now if $S(1) < 115.5$. If the expected instantaneous return of the option at time-0 is –57.0%, find c.

Example 20 (T02Q14)**.**

Let $S(t)$ be time-t price of a nondividend-paying stock and $C(S(t), t)$ be the time-t price of a 0.5-year at the money European call option written on the stock, when the time-t stock price is $S(t)$. You are given that

- $S(0.25) = 50$.
- The true stock price process is
$$dS(t) = 0.22S(t)dt + 0.35S(t)dZ(t)$$
where $Z(t)$ is a standard Brownian motion under the true measure.
- The true stochastic process satisfied by the call option is
$$dC(S(t), t) = a(S(t), t)dt + b(S(t), t)dZ(t)$$
for some a and b.
- The risk-neutral stochastic process satisfied by the call option is
$$dC(S(t), t) = 0.06C(s(t), t)dt + f(S(t), t)d\tilde{Z}(t)$$

where f is a function and $\tilde{Z}(t)$ is a standard Brownian motion under the risk-neutral measuare.

Calculate $a(50, 0.25)$.

1.6 Delta-hedging a Portfolio

A market maker is a broker or dealer who sells or buy options. A market maker is not interested in speculating on the market but rather would like to make money through bid-ask spreads. Thus if he sells a call option, he must hedge the investment by buying something that would go up in value if the call goes up in value.

If there are two risky assets X and Y with SDEs

$$\frac{dX(t)}{X(t)} = m_X dt + s_X dZ(t), \quad \frac{dY(t)}{Y(t)} = m_Y dt + s_Y dZ(t)$$

Suppose we have 1 unit of X at time t. To hedge the risk, we purchase $N = -\frac{s_X X(t)}{s_Y Y(t)}$ units of Y and hold a cash position of $W = -X(t) - NY(t)$.

Let $X(t) = V(S(t), t)$ and $Y(t) = S(t)$, then, we purchase

$$\boxed{N = -\frac{s_V V}{\sigma S} = -\frac{\frac{S\Delta}{V}\sigma V}{\sigma S} = -\Delta}$$

shares of S and hold a cash position of

$$\boxed{W = -V + S\Delta.}$$

Example 21.
For a nondividend paying stock, you are given that

- $S(t)$ is the time-t stock price.
- $S(t)$ satisfies the SDE:

  $$dS(t) = 0.25S(t)dt + 0.45S(t)dZ(t), S(0) = 60$$

- The continuously compounded risk-free interest rate is 10%.

Justin has just sold 100 unit of 9-month 65-strike call option. To hedge the risk, Justin immediately hedges his position by purchasing the hedge portfolio.

(a) Calculate the components in Justin's hedge portfolio at $t = 0$.

(b) Suppose that the stock price after 1 month is 65, compute the profit and loss of

(i) shorting 100 unit call;

(ii) Justin's position, assuming that Justin can borrow or lend at the risk-free interest rate.

Example 22.
You are given that

- $S(t)$ is the time-t stock price.
- $S(t)$ satisfies the SDE:
  $dS(t) = 0.25S(t)dt + 0.45S(t)dZ(t), S(0) = 60$
- The stocks pays dividend continuously at a rate proportional to its price. The dividend yield is 4%.
- The continuously compounded risk-free interest rate is 10%.

Justin has just bought 100 unit of 9-month 60-strike put option. To hedge the risk, Justin immediately hedges his position by purchasing the hedge portfolio.

(a) Calculate the components in Justin's hedge portfolio at $t = 0$.

(b) Suppose that the stock price after 1 month is 65, compute the profit and loss of

(i) owing 100 units of put;

(ii) Justin's position, assuming that Justin uses the stock dividend to purchase extra shares, and that Justin can borrow or lend at the risk-free interest rate.

Example 23 (T02Q15).

Let $S(t)$ be the time$-t$ price of a nondividend paying stock. You are given that $S(t)$ follows the stochastic differential equation

$$dS(t) = 0.08S(t)dt + 0.27d\tilde{Z}(t), S(0) = 3,$$

where $\tilde{Z}(t)$ is a standard Brownian motion under the risk-neutral measure.

A market maker has just written a contingent claim that pays the $S^3(3)$ after 3 years. He then immediately delta-hedge his position by trading stocks and cash(W). Calculate W.

Example 24 (T02Q16).

You are given that:

- For a stock whose time$-t$ price is $S(t)$, the risk-neutral process is

$$d[\ln S(t)] = 0.083dt + 0.2d\tilde{Z}(t), S(0) = 22$$

  where $\tilde{Z}(t)$ is a standard Brownian motion under the risk-neutral measure.
- The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 1.7%.
- A market-maker has sold 100 6-month 24-strike calls on the stock. He immediately hedges his position by buying shares and risk-free bonds. The dividends received are invested by purchasing extra shares.
- The current Black-Scholes price of the call is 0.8968.

After 1 month, when the stock price is 15 and the Black-Scholes price for the call becomes 0.0003, the maker-maker rebalances his hedge portfolio by trading shares and risk-free bonds. The maker-maker invests or repays dividends by purchasing or shorting extra shares. Compute the 1-month profit.

1.7 Gamma Neutrality

Delta-hedge can only protect the market-maker against small chamges in stock prices. If the gamma of the portfolio is negative, a big change in the stock price will lead to a big hedge loss.

To solve this problem, one can try to make the gamma of the hedge position zero. A portfolio with a zero gamma is called a gamma-neutral position. A position in the underlying asset itself has a zero gamma: $\Gamma = \frac{\partial^2 S}{\partial S} = 0$. Thus, it cannot be used to change the gamma of the portfolio. To adjust the gamma of a portfolio, we must make use of instruments such as options that are not linearly dependent on the underlying asset.

Example 25 (T02Q17)**.**

You are given:

- A stock has price 45.
- A market-maker writes put option I on the stock with price 2.59, delta -0.45 and gamma 0.06.
- The market-maker delta-gamma-hedges the option with the stock and with put option II having price 4.47, delta -0.58, and gamma 0.05.

Determine the number of shares of stock to buy to implement the hedge.

1.8 Implied Volatility

Implied volatility is a term commonly used by stock analysts. It is the volatility implied by the **market price** of an option. For eaxmple, suppose that the price of a caqll on a nondividend-paying stock is 1.875 when $S = 21$, $K = 20$, $r = 10\%$, and $T = 0.25$. the implied volatility is the value of σ that gives $C = 1.875$ when it is substituted into the Black-Scholes formula.

In general, to solve for the implied volatility, we need computer software to solve it. however, under the following situation, we can obtain the volatility:

1. At the money stock options with $r = \delta$.
2. At the money future option.
3. The delta of the option is given.

In situation (1) and (2), $d_1 = -d_2 = \frac{\sigma\sqrt{T}}{2}$.

Example 26 (T02Q18).

Assume the Black-Scholes framework. For a stock, you are given that:

- The current stock price 120.
- The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 2.1%.
- The continuously compounded risk-free interest rate is 2.1%.

The current price of a 3-month 120-strike European call on this stock is 9.0. Calculate the implied volatility of this stock.

Example 27 (T02Q19).

Let $Z(t)$ be a standard Brownian motion under the risk-neutral measure. For a stock, you are given:

- The time-t stock price is $S(t)$.
- The stock price process in the risk-neutral measure is

  $$dS(t) = 0.021S(t)dt + \sigma S(t)dZ(t), \quad S(0) = 100,$$

  where a is a constant that is less than 0.1922.
- The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 2.5%. The delta of a 2-year 100-strike put option on this stock is -0.3655.

Calculate σ.

1.9 Historical Volatility

Historical volatility is an estimate of volatility from historical stock prices. To obtain the estimated volatility:

1. Let $u_i = \ln \frac{S_i}{S_{i-1}}$, $i = 1, 2, \ldots, n$, be the continuously compounoded rate of return (not annualized) for the i-th time interval. In the Black-Scholes framework,
$$u_i \sim N[(\alpha - \delta - \frac{1}{2}\sigma^2)h, \sigma^2 h].$$
2. Compute $\bar{u} = \frac{\sum_{i=1}^{n} u_i}{n}$.
3. Compute $s_u^2 = \frac{\sum_{i=1}^{n}(u_i - \bar{u})^2}{n-1}$.
4. Since s_u^2 is the estimate of $\sigma^2 h$, thus $\hat{\sigma}^2 = \frac{s_u^2}{h}$.
5. Thus, $\hat{\sigma} = \frac{s_u}{\sqrt{h}}$.

Example 28.
You are to estimate a nondividend-paying stock's annualized volatility using its prices in the past nine months.

Month	1	2	3	4	5	6	7	8	9
Stock Price	80	64	80	64	80	100	80	64	80

Calculate the historical volatility for this stock over the period. $\boxed{0.8265}$

1.10 Expected Rate of Appreciation

To estimate the expected rate of return α, we use the following equation:

$$\hat{\alpha} = \frac{\bar{u}}{h} + \delta + \frac{\hat{\sigma}^2}{2}.$$

To estimate the expected rate of appreciation,use

$$\hat{\alpha} - \delta$$

Example 29 (T02Q20)**.**

You are given the following historical prices of a nondividend-paying stock:

Week	1	2	3	4	5	6
Stock Price	110	96	93	97	104	103

Let α be the stock's expected rate of return and σ be the stock's volatility. Estimate $\alpha + \sigma$.