CONTENTS

7 Interval Estimation

7.1 Confidence Intervals

Definition 1. Confidence Interval
An interval

$$(l(x_1, \ldots, x_n), u(x_1, \ldots, xn))$$

is called a $100\gamma\%$ confidence interval for θ if

$$P[l(x_1, \ldots, x_n) < \theta < u(x_1, \ldots, xn)] = \gamma$$

where $0 < \gamma < 1$.

The observed values $l(x_1, \ldots, x_n)$ and $u(x_1, \ldots, x_n)$ are called lower and upper confidence limits, respectively.

Definition 2. One-Sided Confidence Limits If

$$P[l(x_1, \ldots, x_n) < \theta] = \gamma$$

then $l(x_1, \ldots, x_n)$ is called a one-sided lower $100\gamma\%$ confidence limit for θ.

If

$$P[u(x_1, \ldots, x_n) > \theta] = \gamma$$

then $u(x_1, \ldots, x_n)$ is called a one-sided upper $100\gamma\%$ confidence limit for θ.

In general, if (θ_L, θ_U) is a $100\gamma\%$ confidence interval for a parameter θ, and if $\tau(\theta)$ is a monotonic increasing function of $\theta \in \Omega$, The $(\tau(\theta_L), \tau(\theta_U))$ is a $100\gamma\%$ confidence interval for $\tau(\theta)$.

Example 1. Consider a random sample of size n from an exponential distribution, $X_i \sim Exp(\theta)$.

(a) Construct a one-sided lower $100\gamma\%$ confidence limit for θ.

(b) Construct a one-sided upper $100\gamma\%$ confidence limit for θ.

(c) Construct a $100\gamma\%$ confidence interval for θ.

(d) Find a one-sided lower $100\gamma\%$ confidence limit for $P(X > t) = e^{-t/\theta}$.

Example 2.

IE-Q25b Consider independent random samples from two gamma distributions, $X \sim gamma(4, \beta_1)$ and $Y_j \sim gamma(10, \beta_2)$; $i = 1, \ldots, n_1$, $j = 1, \ldots, n_2$.

1. Find the distribution of $\left(\frac{\beta_2}{\beta_1}\right)\left(\frac{5\bar{X}}{2\bar{Y}}\right)$.
2. Derive a $100(1-\alpha)\%$ confidence for $\frac{\beta_2}{\beta_1}$.

Example 3.

IE-Q24b Consider independent random samples from two normal distributions, $X_i \sim N(40, \theta_1^2)$ and $Y_j \sim N(80, \theta_2^2)$; $i = \dots, n_1$, $j = 1, \dots, n_2$. Derive a $100(1-\alpha)\%$ confidence interval for $\frac{\theta_2^2}{\theta_1^2}$ based on sufficient statistics.

Example 4.

IE-Q08c Consider a random sample of size 32 from a uniform distribution, $X_i \sim U(0, \theta)$, $\theta > 0$, and let $X_{n:n}$ be the largest order statistic. Find the constant c such that $(x_{n:n}, cx_{n:n})$ is a 99% confidence interval for θ.

7.2 Pivotal Quantity Method

Definition 3. Pivotal Quantity

If $Q = q(X_1, \dots, X_n; \theta)$ is a random variable that is a function only of $(X_1, \dots, X_n)$ and θ, then Q is called a pivotal quantity if its distribution does not depend on θ or any other unknown parameters. That is, if $X \sim F(\mathbf{x}|\theta)$, then Q has the same distribution for all values of θ.

Example 5. (Gamma pivot)

Suppose that $X_1, \dots, X_n$ are iid $Exp(\theta)$, find the pivotal quantity based on the sufficient statistics $T = \sum X_i$.

Example 6.

IE-Q02 Let $X_1, X_2, \ldots, X_n$ be a random sample from a Weibull distribution, $X \sim WEI(\theta, 3)$.

(a) Show that $Q = 2\sum_{i=1}^{n} X_i^3/\theta^3 \sim \chi^2(2n)$.

(b) Use Q to derive an equal tailed $100\gamma\%$ confidence interval for θ.

Theorem 1. Let $X_1, \ldots, X_n$ be a random sample from a distribution with pdf $f(x;\theta)$ for $\theta \in \Omega$, and assume that an MLE $\hat{\theta}$ exists.

1. If θ is a location parameter, then $Q = \hat{\theta} - \theta$ is a pivotal quantity.
2. If θ is a scale parameter, then $Q = \hat{\theta}/\theta$ is a pivotal quantity.

Theorem 2. Let $X_1, \ldots, X_n$ be a random sample from a distribution with location-scale parameters. If MLEs $\hat{\theta}_1$ and $\hat{\theta}_2$ exist, then $(\hat{\theta}_1 - \theta_1)/\hat{\theta}_2$ and $\hat{\theta}_2/\theta_2$ are pivotal quantities for θ_1 and θ_2, respectively.

Notes:

1. Let $f_0(z)$ be a pdf that is free of unknown parameters (including θ).
2. A parameter θ is a location parameter if the pdf has the form $f(x;\theta) = f_0(x-\theta)$.

3. A parameter θ is a scale parameter if the pdf has the form $f(x;\theta) = \frac{1}{\theta} f_0\left(\frac{x}{\theta}\right)$.
4. In the case of location-scale parameters, say θ_1 and θ_2, the pdf has the form $f(x;\theta_1,\theta_2) = \frac{1}{\theta_2} f_0\left(\frac{x-\theta_1}{\theta}\right)$.

Example 7.

IE-Q23b Let X have probability density function

$$f(x) = \begin{cases} \frac{\Gamma(5)x^2(\theta-x)}{\Gamma(3)\theta^4}, & 0 < x < \theta \\ 0, & \text{otherwise} \end{cases}$$

Show that $\frac{X}{\theta}$ is a pivotal quantity and use this pivotal quantity to find a 93% upper confidence limit for θ.

Example 8.

Consider a random sample from a normal distribution, $X \sim N(\mu, \sigma^2)$, where both μ and σ^2 are unknown. If $\hat{\mu}$ and $\hat{\sigma}$ are the MLEs of μ and σ,

(a) show that $\frac{\hat{\mu}-\mu}{\hat{\sigma}}$ and $\hat{\sigma}/\sigma$ are pivotal quantities;

(b) find a $100(1-\alpha)\%$ confidence interval for μ.

(c) find a $100(1-\alpha)\%$ confidence interval for σ^2.

Example 9.

IE-Q14 Let $X_1, \ldots, X_n$ be a random sample from a Poisson distribution with mean λ. Define $Y = \sum X_i$, suppose $Y = y_0$ is observed, show that $1-\alpha$ confidence interval for λ is

$$\left(\frac{\chi^2_{1-\alpha/2}(2y_0)}{2n} \le \lambda \le \frac{\chi^2_{\alpha/2}(2(y_0+1))}{2n} \right),$$

by solving λ from the following equations:

$$\sum_{k=0}^{y_0} e^{-n\lambda}\frac{(n\lambda)^k}{k!} = \frac{\alpha}{2} \text{ and } \sum_{k=y_0}^{\infty} e^{-n\lambda}\frac{(n\lambda)^k}{k!}.$$

Consider $n = 15$ and observe $y_0 = \sum x_i = 15$. Find a 95% confidence interval for λ.

It may not always be possible to find a pivotal quatity based on MLEs, but for a sample from a continuous distribution with a single unknown parameter, at least one pivotal quantity can always be derived by use of the probability integral transform.
If
$$X \sim f(x;\theta)$$
and if
$$F(x;\theta)$$
is the CDF of X, then
$$F(X;\theta) \sim U(0,1)$$
and consequently
$$Y_i \sim -\ln F(X_i,\theta) \sim EXP(1).$$
For a random sample $X_1,\ldots,X_n$, it follows that
$$-2\sum_{i=1}^{n} \ln F(X_i;\theta) \sim \chi^2(2n)$$
so that
$$P[\chi^2_{\alpha/2}(2n) < -2\ln F(X_i;\theta) < \chi^2_{1-\alpha/1}(2n)] = 1-\alpha$$

and inverting this statement will provide a confidence region for θ.
If the CDF is not in closed form or if it is too complicated, then the inversion may have to be done numerically.
If $F(x;\theta)$ is a monotonic increasing (or decreasing) function of θ, then the resulting confidence region will be an interval.

Notice also that $1 - F(X_i;\theta) \sim U(0,1)$ and
$$-2\sum_{i=1}^{n} \ln[1 - F(X_i;\theta)] \sim \chi^2(2n)$$

Example 10.

IE-Q15 Consider a random sample from a Pareto distribution, $X \sim PAR(\alpha, \theta = 200)$, find a $100(1-\alpha)\%$ confidence interval for α.

Example 11.

IE-Q19 Let X be a single obervation from the $Beta(2\theta, 1)$ distributio. Use a pivotal quantity to set up a 90% confidence interval for θ. If $x = 0.0157$, find the length of the confidence interval.

7.3 Aproximate Confidence Intervals

For discrete distributions, and for some multiparameter problems, a pivotal quantity may not exist. However, an approximate pivotal quantity often can be obtained based on asymptotic resultys. Let $X_1, \ldots, X_n$ be a random sample from a distribution with pdf $f(x;\theta)$. As noted in previous chapter, MLEs are asymptotically normal under certain condition.

Example 12.
Consider a random sample from a Bernoulli distribution, $X \sim BIN(1,p)$. Find an approximate confidence limits for p.

7.4 Credible Interval

A credible interval (or in general, a credible set) is the Bayesian analogue of a confidence interval. A $100(1-\alpha)\%$ credible set C is a subset of Θ such that

$$\int_C \pi(\theta|\mathbf{x})d\theta = 1-\alpha$$

If the parameter space Θ is discrete, a sum replaces the integral.

Definition 4.
If a is the $\frac{\alpha}{2}$ posterior quantile for θ, and b is the $1-\frac{\alpha}{2}$ posterior quantile for θ, then (a,b) is a $100(1-\alpha)\%$ **equal probability credible interval** for θ.

Note:
$P(\theta < a|\mathbf{x}) = \frac{\alpha}{2}$ and $P(\theta > b|\mathbf{x}) = \frac{\alpha}{2}$
$\Rightarrow P(\theta \in (a,b)|\mathbf{x})$
$= 1 - P(\theta \notin (a,b)|\mathbf{x})$
$= 1 - (P(\theta < a|\mathbf{x}) + P(\theta > b|\mathbf{x})) = 1-\alpha$

Example 13.
IE-Q17 The following amounts were paid on a hospital liability policy:

$$121, 140, 147, 105, 130, 317, 128, 106, 141, 237.$$

The amount of a single payment has the single-parameter Pareto distribution with $\theta = 103$ and α unknown. The prior distribution has the gamma distribution with $\alpha = 3$ and $\theta = 1$. Determine the 95% equal probability credible interval for α.

Example 14.
IE-Q26b Let $X_1, X_2, \ldots, X_n$ be a random sample from a gamma distribution with parameters $\alpha = 4$ and $\theta = \frac{1}{\lambda}$, the prior density of λ is exponential with mean $\frac{1}{\mu}$ where μ is known. Derive a $100(1-\alpha)\%$ equal probabilty Bayesian confidence interval for λ in terms of χ^2 random variable.

The equal-tail credible interval approach is ideal when the posterior distribution is symmetric. If $\pi(\theta|\mathbf{x})$ is skewed, a better approach is to create an interval of $\theta-$values having the Highest Posterior Density (HPD).

Definition 5.
A $100(1-\alpha)\%$ HPD region for θ is a subset $C \in \Theta$ defined by

$$C = \{\theta : \pi(\theta|\mathbf{x}) \geq k\}$$

where k is the largest number such that

$$\int_{\theta:\pi(\theta|\mathbf{x})\geq k} \pi(\theta|\mathbf{x})d\theta = 1-\alpha$$

The value k can be thought of as a horizontal line placed over the posterior density whose intersection(s) with the posterior define regions with probability $1-\alpha$.

Theorem 3.
If the posterior random variable $\theta|\mathbf{x}$ is continuous and unimodal, then the $100(1-\alpha)\%$ HPD credible interval is the unique solution to

$$\int_a^b \pi(\theta|\mathbf{x})d\theta = 1-\alpha$$
$$\pi(a|\mathbf{x}) = \pi(b|\mathbf{x})$$

Example 15.
IE-Q16 A Bayesian analysis is performed. The posterior density funtion is

$$\pi(\theta|\mathbf{x}) = \begin{cases} 0.9\theta & 0 \leq \theta \leq \frac{20}{28} \\ 0.8392 - 0.27078\theta & \frac{20}{28} \leq \theta \leq 3.1 \end{cases}$$

Construct the 93% HPD credibility interval.

Example 16.
IE-Q18 In a Bayes analysis, a parameter θ has a continuous posterior with pdf

$$\pi(\theta) = \begin{cases} ce^{-0.12\theta} & \text{for} 2 < \theta < 5 \\ 0 & \text{otherwise} \end{cases}$$

for an appropriate constant c. Find a 95% "HPD" credible set for θ.

Example 17.

The following amounts were paid on a hospital liability policy:

125	132	141	107	133
319	126	104	145	223

The amount of a single payment has the single-parameter Pareto distribution with $\theta = 100$ and α unknown. The prior distribution has the gamma distribution with $\alpha = 2$ and $\theta = 1$. Determine the 95% HPD credible interval for α.

$\boxed{\text{a=1.1832, b = 3.9384}}$


```
f = function(x){
y = numeric(2)
y[1] = pgamma(x[2],12,4.801121) - pgamma(x[1],12,4.801121) - 0.95
y[2] = dgamma(x[1],12,4.801121) - dgamma(x[2],12,4.801121)
y
}
library(nleqslv)
xstart = c(1,3)
nleqslv(xstart, f, control=list(btol=.01),
method="Newton")
```