MEME15203 Statistical Inference Marking Guide

Assignment 3

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Q1. Suppose $X \overset{iid}{\sim} POI(\mu)$ and $\gamma = P(X > 0) = 1 - P(X = 0) = 1 - e^{-\mu}$. Find the UMVUE for γ.

(10 marks)

Ans.

$f(x; p) = \frac{e^{-\mu}\mu^x}{x!} = e^{-\mu}\frac{1}{x!}e^{x\ln(\mu)} = c(\mu)h(x)e^{q(\mu)t(x)}$, which is REC(q) with $q(\mu) = e^{-\mu}$ and $t(x) = x$. Hence $S = \sum X_i$ is a complete sufficient statistic for μ.

Let $T = \begin{cases} 1, & x > 0 \\ 0, & \text{otherwise} \end{cases}$

$E(T) = P(X > 0) = 1 - e^{-\mu}$, thus T is an unbiased estimator of γ. Thus, by Rao-Blackwell theorem, $T^* = E(T|S)$ is an UMVUE of γ.

$$\begin{aligned} E(T|S) &= \tfrac{P[X_1>0, S=s]}{P[S=s]} \\ &= 1 - \tfrac{P[X_1=0, X_2+\cdots+X_n=s]}{P[S=s]} \\ &= 1 - \tfrac{e^{-\mu}e^{-(n-1)\mu}[(n-1)\mu]^s/x!}{e^{-n\mu}(n\mu)^s/s!} \quad \text{As } S \sim POI(n\mu) \text{ and } X_2 + \cdot + X_n \sim POI((n-1)\mu) \\ &= 1 - \left(\tfrac{n-1}{n}\right)^s \end{aligned}$$

Q2. Let $X_1, X_2, \ldots, X_n$ be a random sample from $X_i \sim Beta(1, 7\theta)$. Find the UMVUE of θ,

(10 marks)

Ans.

$f(x; \theta) = 7\theta(1-x)^{7\theta-1} = 7\theta e^{(7\theta-1)\ln(1-x)} = c(\theta)h(x)e^{q(\theta)t(x)}$ which is a member of REC(θ) with $q(\theta) = 7\theta - 1$ and $t(x) = \ln(1-x)$. Thus, $T = \sum \ln(1 - X_i)$ is a complete sufficient statistic for θ.

Let $v_i = -ln(1 - x_i)$.Thus $0 < v_i < \infty$. This correspond to a 1-1 transformation of $x_i = 1 - e^{-v_i}$.

Thus, $h^{-1}(v_i) = 1 - e^{-v_i}$
$f_V(v_i) = f_X(h^{-1}(v_i))\left|\frac{dh^{-1}(v_i)}{dv_i}\right| = 7\theta(e^{-(7\theta-1)v_i})e^{-v_i} = 7\theta e^{-7\theta v_i}$
$\Rightarrow V_i \sim EXP(1/7\theta)$ and
$U = -\sum_{i=1}^{n} \ln(1-x_i) = \sum_{i=1}^{n} V_i \sim gamma(\alpha = n, \beta = \frac{1}{7\theta})$

$$\begin{aligned} E(U^{-1}) &= \int_0^\infty u^{-1}\frac{(7\theta)^n}{\Gamma(n)}u^{n-1}e^{-7\theta u}du \\ &= \frac{(7\theta)^n}{\Gamma(n)}\int_0^\infty u^{n-2}e^{-7\theta u}du \\ &= \frac{(7\theta)^n}{\Gamma(n)}\left[\frac{\Gamma(n-1)}{(7\theta)^{n-1}}\right] \\ &= \frac{7\theta}{n-1} \end{aligned}$$

$E\left(\frac{n-1}{7}U^{-1}\right) = \theta$
Thus by Lehmann Scheffie Theorem, $\frac{n-1}{7U} = -\frac{n-1}{7\sum_{i=1}^{n}\ln(1-x_i)}$ is an UMVUE of θ

Q3. Consider a random sample of size n from a Poisson distribution, $X_i \sim POI(\theta)$. Find the UMVUE of θ^2

(10 marks)

Ans.

$f(x_i;\theta) = \frac{e^{-\theta}\theta^{x_i}}{x_i!} = e^{-\theta}\frac{1}{x_i!}e^{x_i\ln(\theta)} = c(\theta)h(x)e^{q(\theta)t(x)}$
where $c(\theta) = e^{-\theta}$, $h(x) = \frac{1}{x_i!}$, $q(\theta) = \ln\theta$; $t(x) = x$. Thus, $f(x;\theta)$ belong to $REC(\theta)$ and hence $S = \sum X_i$ is a complete sufficient statistic for θ.
$E\left[\bar{X}^2 - \frac{\bar{X}}{n}\right] = \frac{\theta}{n} + \theta^2 - \frac{\theta}{n} = \theta^2$. Since $\bar{X}^2 - \frac{\bar{X}}{n}$ is a function of CSS and unbiased for θ^2, therefore, by Lehmann Scheffe theorem, $\bar{X}^2 - \frac{\bar{X}}{n}$ is the UMVUE of θ^2.

Q4. Suppose that $X_1, \ldots, X_{39}$ is a random sample from a Poisson distribution, $X_i \sim$ POI(θ). Find the UMVUE of $e^{-9\theta}$ using Rao-Blackwell theorem.

(10 marks)

Ans.

$f(x) = \frac{1}{\theta}e^{-x/\theta} = c(\theta)h(x)e^{t(x)q(\theta)}$ which is in a member of REC. Hence $S = \sum X_i$ is a CSS of θ.
Let

$$T = \begin{cases} 1, & X_1 + \cdots + X_9 = 0 \\ 0, & \text{otherwise} \end{cases}.$$

$E(T) = P(X_1 + \cdots + X_9 = 0) = e^{-9\theta}$. Thus T is and unbiased estimator of $e^{-9\theta}$. Since S is CSS of θ. Hence by Rao-Blackwell theorem, $T^* = E(T|S)$ is

an UMVUE of $e^{-9\theta}$.
$E\left[T|\sum_{i=0}^{n} X_i = s\right]$
$= 1 \cdot P[X_1 + \cdots + X_9 = 0 | X_1 + X_2 + \cdots X_n = s]$
$= \dfrac{P(X_1 + \cdots + X_9 = 0, X_{10} + \cdots + X_n = s)}{P(X_1 + \cdots + X_n = s)}$
$= \dfrac{P(X_1 + \cdots + X_9 = 0) \times P(X_{10} + \cdots + X_n = s)}{P(X_1 + \cdots + X_n = s)}$ Since $X_1, X_2, \ldots, X_n$ are independent.
$= \dfrac{e^{-9\theta}[(n-9)\theta]^s e^{-(n-9)\theta}}{(n\theta)^s s! e^{-n\theta}/s!}$
$= \left(\frac{n-9}{n}\right)^s$
$= \left(\frac{39-9}{39}\right)^s$
$= \left(\frac{30}{39}\right)^s$

Q5. Let $X_1, X_2, \ldots, X_n$ be random sample of size n from a Gamma distribution with probability density function

$$\frac{1}{\theta^2} x e^{-x/\theta}, x > 0$$

zero otherwise. Find the UMVUE of $\gamma = P(X > t)$ using Rao-Blackwell theorem.

(10 marks)

Ans.

Let

$$T = \begin{cases} 1, & X_1 > t \\ 0, & \text{otherwise} \end{cases}.$$

Then, $E(T) = P(X > t)$. Thus T is and unbiased estimator of γ.
$f(x) = \frac{1}{\theta^2} e^{-x/\theta} = c(\theta)h(x)e^{q(\theta)t(x)}$, where $c(\theta) = \frac{1}{\theta^2}$, $h(x) = 1$, $q(\theta) = \frac{1}{\theta}$, and $t(x) = x$, hence $f(x)$ is a member of $REC(\theta)$ and $S = \sum_{i=1}^{n} X_i$ is a complete sufficient statistics for θ.
Thus, by Rao-Blackwell theorem, $T^* = E(T|S)$ is an UMVUE of γ.

$f_{X_1,S}(x_1, s) = f_{X_1,S_1}(x_1, s - x_1)$ where $S_1 = X_2 + \cdots, X_n \sim gamma(2n-2, \theta)$
$= f_{X_1}(x_1) f_{S_1}(s - x_1)$
$= \frac{1}{\theta^2} x_1 e^{-x_1/\theta} \frac{1}{\Gamma(2n-2)\theta^{2n-2}} (s - x_1)^{2n-3} e^{-(s-x_1)/\theta}$
$= \frac{1}{\Gamma(2n-2)\theta^{2n}} x_1 (s - x_1)^{2n-3} e^{-s/\theta}$

$f_{X_1|S}(x_1) = k x_1 (s - x_1)^{2n-3}, 0 < x_1 < s$
Let $z = s - x_1$, $dz = -dx$
$\int_s^0 k(s-z) z^{2n-3} dx_1 = 1$
$\int_s^0 k(s z^{2n-3} - z^{2n-2}) dz = 1$

$k\left[\frac{sz^{2n-2}}{2n-2}-\frac{z^{2n-1}}{2n-1}\right]_0^s=1$
$k\left[\frac{s^{2n-1}}{2n-2}-\frac{s^{2n-1}}{2n-1}\right]=1$
$k\left[\frac{(2n-1)s^{2n-1}-(2n-2)s^{2n-1}}{(2n-1)(2n-2)}\right]=1$
$k\left[\frac{s^{2n-1}}{(2n-1)(2n-2)}\right]=1$
$k=(2n-2)(2n-1)s^{1-2n}$
$\therefore f_{X_1|S}(x_1)=(2n-2)(2n-1)s^{1-2n}x_1(s-x_1)^{2n-3}, 0<x_1<s$

$$
\begin{aligned}
E[T|S] &= P[X_1>t|s] \\
&= \int_t^s (2n-2)(2n-1)s^{1-2n}x_1(s-x_1)^{2n-3}dx_1
\end{aligned}
$$

Let $z=s-x_1$, $dz=-dx$

$$
\begin{aligned}
&= (2n-2)(2n-1)s^{1-2n}\int_{s-t}t^0(s-z)z^{2n-3}(-dz) \\
&= (2n-2)(2n-1)s^{1-2n}\left[\frac{sz^{2n-2}}{2n-2}-\frac{z^{2n-1}}{2n-1}\right]_0^{s-t} \\
&= (2n-2)(2n-1)s^{1-2n}\left[\frac{s(s-t)^{2n-2}}{2n-2}-\frac{(s-t)^{2n-1}}{2n-1}\right] \\
&= (2n-2)(2n-1)\left[\frac{\left(\frac{s-t}{s}\right)^{2n-2}}{2n-2}-\frac{\left(\frac{s-t}{s}\right)^{2n-1}}{2n-1}\right] \\
&= (2n-2)(2n-1)\left[\frac{(2n-1)\left(\frac{s-t}{s}\right)^{2n-2}-(2n-2)\left(\frac{s-t}{s}\right)^{2n-1}}{(2n-2)(2n-1)}\right] \\
&= (2n-1)\left(\frac{s-t}{s}\right)^{2n-2}-(2n-2)\left(\frac{s-t}{s}\right)^{2n-1}
\end{aligned}
$$

Q6. Let $X_1,\dots,X_n$ be a sample form a population with density $f(x,\theta)$ given by

$$
f(x,\theta)=\begin{cases}\frac{1}{\sigma}\exp\left[-\left(\frac{x-\mu}{\sigma}\right)\right], & \text{if } x\ge\mu \\ 0 & \text{otherwise}\end{cases}
$$

(a) Identify a two-dimensional sufficient statistic for the parameter vector $\boldsymbol{\theta}=(\mu,\sigma)$ with $-\infty<\mu<\infty$, $\sigma>0$, and carefully argue that it is sufficient.

(b) Suppose $\sigma=1$, find a complete sufficient statistic for μ.

(c) Find the UMVUE for μ.

(d) Use Basu's Theorem to show that $X_{1:n}$ and $W=\frac{(X_i-\bar{X})^2}{n}$ are independent.

(20 marks)

Ans.

(a) $f(x_1,\dots,x_n)=I(\mu\le x_{(1)})\left(\frac{1}{\sigma}\right)^n e^{-\frac{\sum x_i}{\sigma}+\frac{n\mu}{\sigma}}=g(x_{(1)},\sum x_i,\mu,\sigma)h(x_1,\dots,x_n)$
where $g(t_1,t_2,\mu,\sigma)=I(\mu\le t_1)\left(\frac{1}{\sigma}\right)^n e^{-\frac{t_2}{\sigma}+\frac{n\mu}{\sigma}}$ and $h(x_1,\dots,x_n)=1$
Thus, by factorization Theorem,

$$(X_{1:n},\sum X_i)$$

is jointly sufficient for (σ, μ).

(b) When $\sigma = 1$ is known, then $T(X) = X_{1:n}$ is sufficient statistic for μ.
$f_T(t) = f_X(t)[1 - F_X(t)]^{(n-1)}n = e^{-(t-\mu)}[e^{-(t-\mu)}]^{n-1}n = ne^{-n(t-\mu)}, t > \mu$
$E[u(T)] = \int_\mu^\infty u(t)ne^{-n(t-\mu)}dt = 0\ \forall\mu$
$\Rightarrow \int_\mu^\infty u(t)ne^{-nt}dt = 0\ \forall\mu$

$\frac{d}{d\mu}\int_\mu^\infty u(t)e^{-nt}dt = -u(\mu)e^{-n\mu} = 0\ \forall\mu$

This implies $u(\mu) = 0$ for all μ, so $T(X) = X_{1:n}$ is a complete sufficient statistic for μ.

(c) $E(T) = \int_\mu^\infty tne^{-n(t-\mu)}dt$
Let $u = t - \mu$, $du = st$
$= n\int_0^\infty (u+\mu)e^{-nu}du$
$= n[\int_0^\infty ue^{-nu}du + \mu\int_0^\infty e^{-nu}du$
$= n[(1/n)^2 + \mu(1/n)]$
$= \mu + \frac{1}{n}$
Thus $T - \frac{1}{n}$ is an UE of μ which is a function of the CSS of μ. Thus $X_{1:} - \frac{1}{n}$ is the UMVUE of μ.

(d) Basu's Theorem says that if $X_{1:n}$ is a complete sufficient statistic for μ, then $X_{1:n}$ is independent of any ancillary statistic.
Let $X_i = Z_i + \mu$, where $Z_1, \ldots, Z_n$ is a random sample from $f(x|0)$. Then
$(X_i - \bar{X})^2 = [(Z_i + \mu) - (\bar{Z} + \mu)]^2 = [Z_i - \bar{Z}]^2$
Because W is a function of only $Z_1, \ldots, Z_n$, the distribution of W does not depend on μ; that is, W is ancillary. Therefore, by Basu's theorem, W is independent of $X_{1:n}$.

Q7. Suppose that $X_1, \ldots, X_{30}$ is a random sample from a Gamma distribution, $X_i \sim$ GAM$(\alpha = 6, \theta)$,

(a) Show that the p.d.f. of X belongs to the regular exponential family.

(b) Find a complete and sufficient statistic for θ.

(c) Find the UMVUE for $\left(\frac{1}{1-9\theta}\right)^{180}$.

(15 marks)

Ans.

(a) $f(x) = \frac{1}{\Gamma(6)\theta^6}x^{6-1}e^{-x/\theta} = c(\theta)h(x)e^{q(\theta)t(x)}$

where $c(\theta) = \theta^{-6}$, $h(x) = 1/\Gamma(6)x^{6-1}$, $q(\theta) = 1/\theta$, and $t(x) = x$.
Thus the p.d.f. of X belongs to the regular exponential family.

(b) Since the p.d.f. of X belongs to the regular exponential family, thus by the theorem, $S = \sum_{i=1}^{30} X_i$ is a c.s.s of θ

(c) $S \sim GAM(180, \theta)$
$E(e^{9S}) = \left(\frac{1}{1-9\theta}\right)^{180}$
Thus e^{9S} is an UE of $\left(\frac{1}{1-9\theta}\right)^{180}$
Since e^{9S} is a function of the c.s.s. of θ and an UE of $\left(\frac{1}{1-9\theta}\right)^{180}$, thus e^{9S} is the UMVUE of $\left(\frac{1}{1-9\theta}\right)^{180}$.

Q8. Suppose $X_1, \ldots, X_n$ is a random sample from a normal distribution, $X_i \sim N(\mu, 16)$. Use the Rao-Blackwell theorem to find the UMVUE of $\nu = P[X \leq c]$.

(15 marks)

Ans.

Let

$$T = \begin{cases} 1, & x_1 \leq c \\ 0, & \text{otherwise} \end{cases}$$

Then $E(T) = P(X_1 \leq c)$ is an unbiased estimator of $\nu = P(X_i \leq c)$.
$f(x_i) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x_i-\mu)^2} = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x_i^2-2\mu x_i+\mu^2)}$ which belongs to $REC(\mu, \sigma^2)$ with $t_1(x_i) = x_i$ and $t_2(x_i) = x_i^2$. Thus, $S_1 = \sum X_i = n\bar{X}$ and $S_2 = \sum X_i^2$ are jointly CSS for (μ, σ^2).
Hence $T^* = E(T|\bar{x})$ is the UMVUE of ν by Rao-Blackwell theorem.
By Basu's theorem, the ancillary statistic $W(X) = (X_1 - \bar{X})$ is independent of $\bar{X}$, then
$E(T|\bar{x})$
$=P(X_1 \leq c|\bar{x})$
$= P(W + \bar{X} \leq c|\bar{x})$
$= P(W \leq c - \bar{X}|\bar{x})$
$= P(W \leq c - \bar{x})$
Note that W is a linear combinations of normal distributions,
$E(W) = E(X_1) - E(\bar{X}) = 0$ and
$V(W) = V\left(\frac{(n-1)X_1 - \sum_{i=2}^{n} X_i}{n}\right) = \frac{(n-1)^2}{n^2}V(X_1) + \frac{n-1}{n^2}V(X_i) = 16\left(\frac{n-1}{n}\right)$
Hence, the UMVUE of ν is
$\Phi\left(\frac{c-\bar{X}}{4\sqrt{\frac{n-1}{n}}}\right) = \Phi\left(\frac{\sqrt{n}(c-\bar{X})}{4\sqrt{n-1}}\right)$