MEME15203 Statistical Inference Marking Guide

Test 1

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Due by:			

Show your workings. If no workings are shown, ZERO is awarded.

Q1. Let $X_1, X_2, \ldots, X_8$ be a random sample of size 8 from a distribution $N(250, 30^2)$. Let $U = \max(X_1, X_2, \ldots, X_8)$, find the value of the p.d.f. of U at $u = 250$.

(8 marks)

Ans.

$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}(x-\mu)^2}, x \in \mathbb{R}$

$$\begin{aligned} f_U(u) &= n[F_X(u)]^{n-1} f_X(u) \\ &= n\left[\Phi\left(\tfrac{u-\mu}{\sigma}\right)\right]^{n-1} \phi\left(\tfrac{u-\mu}{\sigma}\tfrac{1}{\sigma}\right) \end{aligned}$$

$$\begin{aligned} f_U(250) &= 8\left[\Phi\left(\tfrac{250-250}{30}\right)\right]^7 \phi\left(\tfrac{250-250}{30}(\tfrac{1}{30})\right) \\ &= 8\left[\Phi(0.0)\right]^7 \phi(0.0)\left(\tfrac{1}{30}\right) \\ &= 8\left[0.5\right]^7 \tfrac{1}{\sqrt{2\pi}} e^{-0.0^2/2}(\tfrac{1}{30}) \\ &= 8\left[0.5\right]^7 (0.3989)(\tfrac{1}{30}) \\ &= \boxed{0.0008} \end{aligned}$$

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Q2. Suppose that $X_i \sim N(\mu, \sigma^2), i = 1, \ldots, 15$, $Z_j \sim N(0,1), j = 1, \ldots, 22$, and $W_k \sim \chi^2(v), k = 1, \ldots, 14$ and all random variables are independent. State the distribution of each of the following variables if it is a "named" distribution. [For example $X_1 + X_2 \sim N(2\mu, 2\sigma^2)$]

(a) $\frac{21\sum_{i=1}^{15}(X_i-\bar{X})^2}{14\sigma^2\sum_{j=1}^{22}(Z_j-\bar{Z})^2}$.

(b) $\frac{W_1}{\sum_{k=1}^{22} W_k}$

(c) $\frac{\bar{X}}{\sigma^2} + \frac{\sum_{j=1}^{22} Z_j}{22}$

(21 marks)

Ans.

(a) $\frac{\sum_{i=1}^{15}(X_i-\bar{X})^2}{\sigma^2} \sim \chi^2(14)$

$\sum_{j=1}^{22}(Z_j - \bar{Z})^2 \sim \chi^2(21)$

$\frac{\frac{\sum_{i=1}^{15}(X_i-\bar{X})^2}{14\sigma^2}}{\sum_{j=1}^{22}(Z_j-\bar{Z})^2/21} \sim F(14, 21)$

Thus, $\boxed{\frac{21\sum_{i=1}^{15}(X_i-\bar{X})^2}{14\sigma^2\sum_{j=1}^{22}(Z_j-\bar{Z})^2} \sim F(14, 21)}$.

(b) Let $\frac{W_1}{\sum_{k=1}^{22} W_k} = \frac{W_1}{W_1+\sum_{k=2}^{22} W_k}$, then

$W_1 \sim GAM(\frac{v}{2}, 2)$ and $\sum_{k=2}^{22} W_k \sim GAM(\frac{21v}{2}, 2)$.

Thus, $\boxed{\frac{W_1}{\sum_{k=1}^{22} W_k} \sim BETA(\frac{v}{2}, \frac{21v}{2})}$.

(c) $\bar{X} \sim N(\mu, \frac{\sigma^2}{15})$, then $\frac{\bar{X}}{\sigma^2} \sim N(\frac{\mu}{\sigma^2}, \frac{1}{15\sigma^2})$

$\sum_{i=1}^{22} Z_j \sim N(0, 22)$, and $\frac{\sum_{i=1}^{22} Z_i}{22} \sim N(0, \frac{1}{22})$.

Thus $\boxed{\frac{\bar{X}}{\sigma^2} + \frac{\sum_{j=1}^{22} Z_j}{22} \sim N(\frac{\mu}{\sigma^2}, \frac{1}{15\sigma^2} + \frac{1}{22})}$.

Q3. Let X_1, X_2 be two random variables with joint pdf $f(x_1, x_2) = \frac{1}{4!(30^6)} x_1^4 e^{-x_2/30}$, for $0 < x_1 < x_2 < \infty$, zero otherwise.

(a) Determine the joint mgf of X_1, X_2, $M_{X_1,X_2}(t_1, t_2)$.

Ans.

$M_{X_1,X_2}(t_1,t_2)$
$= E(e^{t_1X_1+t_2X_2})$
$= \int_0^\infty \int_{x_1}^\infty e^{t_1x_1+t_2x_2}(\frac{1}{4!(30^6)})x_1^4 e^{-x_2/30} dx_2 dx_1$
$= \int_0^\infty (\frac{1}{4!(30^6)})x_1^4 e^{t_1x_1} \int_{x_1}^\infty e^{-x_2(1/30-t_2)} dx_2 dx_1$
$= \int_0^\infty (\frac{1}{4!(30^6)})x_1^4 e^{t_1x_1} \frac{30e^{-x_1\left(\frac{1-30t_2}{30}\right)}}{1-30t_2} dx_1$
$= (\frac{30}{4!(30^6)})(\frac{1}{1-30t_2}) \int_0^\infty x_1^4 e^{-x_1\left(\frac{1-30t_1-30t_2}{30}\right)} dx_1$
$= (\frac{30}{4!(30^6)})(\frac{1}{1-30t_2}) \frac{4!(30^5)}{(1-30t_1-30t_2)^5}$
$= \frac{1}{(1-30t_2)(1-30t_1-30t_2)^5}$
provided that $30t_1 + 30t_2 < 1$ and $30t_2 < 1$.

(b) Determine the marginal distribution of X_1.

Ans.

$M_{X_1}(t_1, 0) = \frac{1}{(1-30(0))(1-30t_1-30(0))^5} = \frac{1}{(1-30t_1)^5}$
$\Rightarrow X_1 \sim GAM(\alpha = 5, \theta = 30)$

(c) Determine the marginal distribution of X_2.

Ans.

$M_{X_2}(0, t_2) = \frac{1}{(1-30t_2)(1-30(0)-30t_2)^5} = \frac{1}{(1-30t_2)^6}$
$\Rightarrow X_2 \sim GAM(\alpha = 6, \theta = 30)$

(16 marks)

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Q4. The joint density function of X_1 and X_2 is given by

$$f(x_1, x_2) = \begin{cases} cx_1^4 x_2^6, & x_1 - 1 \le x_2 \le 1 - x_1, 0 \le x_1 \le 1 \\ 0, & \text{otherwise} \end{cases},$$

(a)Find c.

Ans.
$\int_0^1 \int_{x_1-1}^{1-x_1} f(x_1, x_2) dx_2 dx_1 = 1$
$\int_0^1 \int_{x_1-1}^{1-x_1} cx_1^4 x_2^6 dx_2 dx_1 = 1$
$c \int_0^1 x_1^4 \left[\frac{x_2^7}{7}\right]_{x_1-1}^{1-x_1} dx_1 = 1$
$\frac{2c}{7} \int_0^1 x_1^4 (1 - x_1)^7 = 1$
$\left[\frac{2c}{7}\right] \left[\frac{\Gamma(5)\Gamma(8)}{\Gamma(5+8)}\right] = 1$
$c = 13860.0$

(b)Show that the marginal density of X_1 is a beta density with $a = 5$ and $b = 8$.

Ans.
$f_1(x_1)$
$= \int_{x_1-1}^{1-x_1} 13860 x_1^4 x_2^6 dx_2$
$= 13860 x_1^4 [\frac{x_2^{6+1}}{6+1}]_{x_1-1}^{1-x_1}$
$= \frac{13860}{7} x_1^4 [(1 - x_1)^7 + (1 - x_1)^7]$
$= \frac{13860}{7} x_1^4 [2(1 - x_1)^7]$
$= 3,960 x_1^4 (1 - x_1)^7, 0 \le x_1 \le 1$
$\Rightarrow X_1 \sim Beta(a = 5, b = 8)$

(c)Derive the conditional density of X_2 given $X_1 = x_1$.

Ans.
$f(x_2|x_1) = kx_2^6, x_1 - 1 \le x_2 \le 1 - x_1$
$k \int_{x_1-1}^{1-x_1} x_2^6 dx_2 = 1$
$k \left[\frac{x_2^7}{7}\right]_{x_1-1}^{1-x_1} = 1$
$k \left[\frac{2(1-x_1)^7}{7}\right] = 1$
$k = \frac{7}{2(1-x_1)^7}$
$\therefore f(x_2|x_1) = \frac{7x_2^6}{2(1-x_1)^7}, x_1 - 1 \le x_2 \le 1 - x_1$

(24 marks)

Q5. Consider a random sample from a Exponential distribution, $X_i \sim Exp(\theta)$. Find the asymptotic normal distribution of $Y_n = [\ln(\bar{X}_n)]^7$.

(8 marks)

Ans.

$E(\bar{X}_n) = \theta$, $V(\bar{X}_n) = \frac{1}{n}V(X) = \frac{\theta^2}{n}$

By CLT, $\bar{X}_n \sim N\left(\theta, \frac{\theta^2}{n}\right)$

$g(\theta) = (\ln\theta)^7$, $g'(\theta) = \frac{7}{\theta}(\ln\theta)^6$, $[g'(\theta)]^2 = \frac{49}{\theta^2}(\ln\theta)^{12}$, thus, by Theorem 11,

$\frac{c^2[g'(m)]^2}{n} = \frac{49\theta^2}{n\theta^2}(\ln\theta)^{12} = \frac{49}{n}(\ln\theta)^{12}$

$Y_n \sim N\left([\ln(\theta)]^7, \frac{49}{n}(\ln\theta)^{12}\right)$

Q6. Let $Y_n \sim GAM(\alpha n, \theta)$. Find the limiting distribution of $Z_n = \frac{Y_n - \alpha n\theta}{\sqrt{\alpha n}\theta}$ as $n \to \infty$, using moment generating function.

(8 marks)

Ans.

$M_{Y_n}(t) = (1-\theta t)^{-\alpha n}$

$$\begin{aligned}
M_{Z_n}(t) &= M_{\frac{Y_n-\alpha n\theta}{\sqrt{\alpha n}\theta}}(t) \\
&= e^{-\frac{\alpha n}{\sqrt{\alpha n}}t} M_{Y_n}\left(\tfrac{1}{\sqrt{\alpha n}\theta}t\right) \\
&= e^{-\frac{\alpha n}{\sqrt{\alpha n}}t}\left(1-\frac{1}{\sqrt{\alpha n}}t\right)^{-\alpha n} \\
&= e^{-\frac{\alpha n}{\sqrt{\alpha n}}t} e^{-\alpha n \ln\left(1-\frac{1}{\sqrt{\alpha n}}t\right)} \\
&= e^{-\frac{\alpha n}{\sqrt{\alpha n}}t} e^{-\alpha n\left[-\frac{1}{\sqrt{\alpha n}}t-\frac{1}{2\alpha n}t^2-\frac{1}{3\alpha n^{3/2}}t^3-\cdots\right]} \\
&= e^{-\frac{\sqrt{2\alpha n}}{2}t+\frac{\sqrt{2\alpha n}}{2}t+\frac{t^2}{2}+\frac{t^3}{3\sqrt{\alpha n}}-\cdots} \\
&= e^{\frac{t^2}{2}+\frac{t^3}{3\sqrt{\alpha n}}-\cdots}
\end{aligned}$$

$$\lim_{\alpha n\to\infty} M_{\frac{Y_n-\alpha n}{\sqrt{2\alpha n}}}(t) = \lim_{n\to\infty} e^{\frac{t^2}{2}+\frac{t^3}{3\sqrt{\alpha n}}-\cdots} = e^{\frac{t^2}{2}}$$

$\Rightarrow Z_n = \frac{Y_n-\alpha n\theta}{\sqrt{\alpha n}\theta} \xrightarrow{d} N(0,1)$

Q7. Let X_1 and X_2 be a random sample of size 2 from a distribution $N(20,1)$, and let

$$U = X_1 + X_2 \quad \text{and} \quad W = X_1 - X_2.$$

(a)Find the joint pdf of U and W.

(b)Show that U and W are independent.

(15 marks)

Ans.

(a) This transformation corresponds to $u = x_1 + x_2$ and $w = x_1 - x_2$ which has unique solution $x_1 = \frac{u+w}{2}$ and $x_2 = \frac{u-w}{2}$. The support set for uw-plaine is determined by the inequalities: $-\infty < x_1 < \infty \Rightarrow -\infty < \frac{u+w}{2} < \infty$ and $-\infty < x_2 < \infty \Rightarrow -\infty < \frac{u-w}{2} < \infty$

Thus, $B = [(u,w)| - \infty < u < \ \infty, -\infty < w < \infty]$

$J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$

$$\begin{aligned} f_{X_1,X_2}(x_1,x_2) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x_1-20)^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x_2-20)^2} \\ &= \frac{1}{2\pi} e^{-\frac{1}{2}[(x_1-20)^2+(x_2-20)^2]} \end{aligned}$$

Note that

$$\begin{aligned} (x_1-20)^2 + (x_2-20)^2 &= [(x_1-20)-(x_2-20)]^2 + 2(x_1-20)(x_2-20) \\ &= (x_1-x_2)^2 + 2[x_1x_2 - (x_1+x_2)(20) + 20^2] \end{aligned}$$

In terms of u, w, $x_1x_2 = \frac{u+w}{2}\frac{u-w}{2} = \frac{1}{4}(u^2-w^2)$

$f_{U,W}(u,w)$

$$= \begin{cases} f_{X_1,X_2}(\frac{u+w}{2}, \frac{u-w}{2})|J|, & (u,w) \in B \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{1}{2\pi} e^{-\frac{1}{2}[w^2+\frac{1}{2}(u^2-w^2)-2u(20)+2(20^2)]}(\frac{1}{2}), & (u,w) \in B \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{1}{4\pi} e^{-\frac{1}{2}[\frac{w^2}{2}+\frac{1}{2}(u^2-4u(20)+4(20^2))]}, & (u,w) \in B \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{1}{4\pi} e^{-\frac{1}{2}[\frac{w^2}{2}+\frac{1}{2}(u-40)^2]}, & (u,w) \in B \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{1}{4\pi} e^{-\frac{1}{2(\sqrt{2})^2}[w^2+(u-40)^2]}, & (u,w) \in B \\ 0, & \text{otherwise} \end{cases}$$

(b) $$f_{U,W}(u,w) = \begin{cases} \frac{1}{\sqrt{4\pi}} e^{-\frac{1}{2(\sqrt{2})^2}[(u-40)^2]} \frac{1}{\sqrt{4\pi}} e^{-\frac{1}{2(\sqrt{2})^2}[w^2]}, & (u,w) \in B \\ 0, & \text{otherwise} \end{cases}$$

Since $f_{U,W}(u,w)$ can be factor into $f_U(u)$ and $f_W(w)$ and the support B is a cartisian product, thus U and W are independent.