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CONTENTS

5 Simulation

5.1 Simulation of Normal Random Variables

In Financial Economics, we are interested in the simulation of sequences of normal and lognormal random variables. To simulate a sequence of independent $N(0,1)$ observations $\{z_1, z_2, \ldots, z_n\}$, we use the following procedure:

- Step 1: Generate n independent uniform random numbers $u_1, u_2, \ldots, u_n$, where $u_i \sim U(0,1)$.
- Step 2: Compute $z_1 = N^{-1}(u_1)$, $z_2 = N^{-1}(u_2)$,. . . , $z_n = N^{-1}(u_n)$, where $N^{-1}(.)$ is the inverse of the standard normal cdf.

Alternatively, since $E(\sum_{i=1}^{12} u_i) = \frac{12}{2} = 6$ and $E(\sum_{i=1}^{12} u_i) = 1$, thus to simulate a standard

normal random number z_j, we use

$$z_j = \sum_{i=1}^{12} u_i - 6$$

This method is called **"sum of uniformly distributed random variables"**.

If we want to simulate a sequence of $N(\mu, \sigma^2)$ observations $\{x_1, x_2, \ldots x_n\}$, we can perform one more step:

- Step 3: Compute $x_1 = \mu + \sigma z_1, x_2 = \mu + \sigma z_2, \ldots, x_n = \mu + \sigma z_n$.

Example 1.
By using the following 5 uniform random numbers, simulate 5 observations from $N(0.2, 0.09)$.

$$0.5003, 0.0641, 0.8554, 0.1486, 0.5627.$$

Example 2.
You are given the following 12 standard normal random variates:

$$-0.18, -0.15, 1.75, 1.51, -0.04, 0.31$$

$$-2.06, -0.06, -0.10, 0.46, 1.95, 1.08$$

The values above are obtained by using the inverse cumulative normal distribution. Let $u_i, i = 1, 2 \ldots, 12$, be the uniform random numbers on which the above 12 realizations are based. On the basis of $u_i, i = 1, 2, \ldots, 12$, compute a standard normal random variate using the method of "sums of uniformly distributed random variables."

5.2 Simulating Stock Prices

To simulate stock prices under the Black-Scholes framework. Recall

$$S(T) = S(0)e^{(\alpha-\delta-\sigma^2/2)T+\sigma Z(T)}.$$

We multiply Z by $\sqrt{T}$ to get $Z(T)$.

To calculate $S(T)$, use

$$\boxed{S(T) = S(0)e^{(\alpha-\delta-\sigma^2/2)T+\sigma\sqrt{T}Z}}$$

Example 3. [T5Q1]
The price of a stock is to be estimated using simulation. It is known that:

- The time$-t$ stock price, S_t, follows the lognormal distribution:
$$\ln\frac{S_t}{S_0} \sim N((\alpha - \frac{1}{2}\sigma^2)t, \sigma^2 t)$$
- $S_0 = 70$, $\alpha = 0.18$, and $\sigma = 0.25$.

The following are three uniform $(0, 1)$ random numbers

0.982 0.0324 0.7899

Use each of these three numbers to simulate a time-2 stock price. Calculate the mean of the three simulated prices.

5.3 Simulating Path of Stock Prices

Sometimes we may need to simulate a path of stock prices. For example, to simulate the payoff of a Geometric average-price call with 1 year maturity and observation period of 3 months, the terminal payoff is

$$[\sqrt[4]{S(0.25)S(0.5)S(0.75)S(1)} - K]_+$$

To simulate a single terminal payoff, we need to simulated $S(0.25)$, $S(0.5)$, $S(0.75)$ and $S(1)$. In the Black-Scholes model, this can be achieved by using

$$S(t_2) = S(t_1)e^{(\alpha-\delta-\sigma^2/2)(t_2-t_1)+\sigma(Z(t_2)-Z(t_1))}.$$

Since $Z(t_2) - Z(t_1) \sim N(0, t_2 - t_1)$, then

$$Z(t_2) - Z(t_1) = \sqrt{t_2 - t_1}Z.$$

Thus,

$$\boxed{S(t_2) = S(t_1)e^{(\alpha-\delta-\sigma^2/2)(t_2-t_1)+\sigma\sqrt{t_2-t_1}Z}.}$$

To get a single payoff, use the following steps:

- Step 1: Simulate $Z_1 \sim N(0,1)$, and calculate
$$S(0.25) = S(0)e^{(\alpha-\delta-\sigma^2/2)(0.25)+\sigma\sqrt{0.25}Z_1}.$$
- Step 2: Simulate $Z_2 \sim N(0,1)$, and calculate
$$S(0.5) = S(0.25)e^{(\alpha-\delta-\sigma^2/2)(0.25)+\sigma\sqrt{0.25}Z_2}.$$
- Step 3: Simulate $Z_3 \sim N(0,1)$, and calculate
$$S(0.75) = S(0.5)e^{(\alpha-\delta-\sigma^2/2)(0.25)+\sigma\sqrt{0.25}Z_3}.$$
- Step 4: Simulate $Z_4 \sim N(0,1)$, and calculate
$$S(1) = S(0.75)e^{(\alpha-\delta-\sigma^2/2)(0.25)+\sigma\sqrt{0.25}Z_4}.$$
- Step 5: Calculate the Payoff:
$$[\sqrt[4]{S(0.25)S(0.5)S(0.75)S(1)} - K]_+.$$

Example 4. [T5Q2]
Assume the Black-Scholes framework for a nondividend-paying stock. You are given that

- The current stock price is 30.
- The expected rate of appreciation is 24%.
- The stock's volatility is 25%.

using the following three uniform random numbers, simulate the payoff from a geometric average strike option:

$$0.5003, 0.0641, 0.8554$$

$\boxed{2.4611}$

5.4 Monte-Carlo Simulation

In risk-neutral pricing, for any derivative who payoff is paid at maturity T, we have

$$V = e^{-rT}E^*[\text{payoff at } T].$$

Sometimes it is difficult to compute $E^*[\text{payoff at } T]$ exactly. We can **estimate** $E^*[\text{payoff at } T]$ by simulating n independent payoffs and the taking average. This method is called Monte-Carlo simulation. After estimating the expectation, we discount it at the risk-free rate to obtain an estimate of the price. Note that we need to replace α by r because we are now estimating the **risk-neutral** expectation.

Let s^2 be the sample variance of n independent payoffs with n being the number of simulations. Then

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}[g(S_i) - \bar{g}]^2,$$

where $g(S_i)$ is the i^{th} simulated payoff and $\bar{g}$ is the estimate of $E^*[\text{payoff at } T]$. Note that s^2 and $\bar{g}$ can be obtained from stat mode of the calculator.

To assess the accuracy of the Monte-Carlo estimates, we compute the variance of the estimate which is

$$\frac{s^2}{n}.$$

The variance of the Monte-Carlo estimate of the price is

$$\begin{aligned} V[e^{-rT}E^*(\text{payoff at } T)] &= e^{-2rT}V[E^*(\text{payoff at } T)] \\ &= e^{-2rT}\frac{s^2}{n} \end{aligned}$$

Example 5. [T5Q3]
Assume the Black-Scholes framework for a nondividend-paying stock. You are given

- The current stock price is 41.
- The stock's volatility is 30%.
- The continuously compunded risk-free interest rate is 8%.

You estimate the price of a 40-strike 1-year call on the stock. Using the following 6 uniform random numbers arranged in ascending order

$$0.6918, 0.5425, 0.3339, 0.1397, 0.8512, 0.1228$$

Estimate the price of the European call using Monte-Carlo simulation. $\boxed{4.804}$

Example 6. Compute the standard deviation of the estimate of Example 5. $\boxed{2.7382}$

Example 7. [T5Q4]
You are given the following regarding dollars and pounds:

- The dollars/pounds exchange rate follows the Black-Scholes framework.
- The spot exchange rate is \$1.47/£.
- The continuously compounded risk-free rate for dollars is 0.07.
- The continuously compounded risk-free rate for pounds is 0.08.
- The volatility of the exchange rate between the two currencies is 0.11.

A dollar-denominated Asian arithmetic average strike put option on pounds has a payoff has a payoff based on the average exchange rate at the end of each of four months from the date of purchase. The price of the option is calculated using naive Monte Carlo valuation. Using the following standard normal random numbers for one trial:

$$-0.27 \quad 0.06 \quad 0.12 \quad -1.14$$

Determine the value of the option in this trial. $\boxed{0.0375}$

5.5 Variance Reduction

The simulation described above need a very large number of runs for a reasonable accuracy. The following three methods can lead to a dramatic decrease in the number of neccessarty trials.

5.5.1 Stratified Sampling

Stratified sampling involves a division of the desired probability distribution into n intervals. Simulations are performed by sampling from each interval.

Suppose we need to simulate 15 points that are uniformly distributed over the unit interval [0,1]. While we know that on average there should be 5 points in each of the subintervals $[0, 1/3]$, $[1/3, 2/3]$ and $[2/3, 1]$. Stratified sampling is a method to make sure that each subinterval contains the correct number of points. We do the following:

- Step 1: Simulate 15 uniform (0, 1) random numbers $U_1, U_2, \ldots, U_{15}$.

- Step 2: Apply the stratified sampling method to the random numbers so that U_i, U_{i+3}, U_{i+6},U_{i+9} and U_{i+12} are transformed to random numbers V_i, V_{i+3}, V_{i+6},V_{i+9} and V_{i+12} that are uniformly distributed over the interval $\left(\frac{(i-1)}{3}, \frac{i}{3}\right)$, $i = 1, 2, 3$. In each of the three intervals, a smaller value of U results in a smaller value of V, where $V_i == \frac{i-1+U_{i \text{ or } i+3 \text{ or} \cdots \text{or } i+12}}{3}$.
- Step 3: Compute 15 standard normal random variates by $Z_i = N^{-1}(V_i)$, where N^{-1} is the inverse of the cumulative standard normal distribution function.

After we have made the uniform numbers "more uniform", we can use N^{-1} to turn them into "more normal" standard normal variables.

Example 8. [T5Q5]
Suppose that $S(0) = 5$, $\delta = 0.02$, $\sigma = 0.3$, $r = 0.08$. You obtained the following 6 uniform random numbers:

0.1017, 0.5028, 0.2381, 0.9542, 0.6739, 0.7806

Estimate the price of an ATM 6-month European put on S by using Monte-Carlo simulation. Apply the stratified sampling method to the random numbers so that U_i and U_{i+3} are transformed to random numbers V_i and V_{i+3} that are uniformly distributed over the interval $\left(\frac{(i-1)}{3}, \frac{i}{3}\right)$, $i = 1, 2, 3$. $\boxed{0.324}$

Example 9. [T5Q6]
Michael uses the following method to simulate 8 standard normal random variates:

- Step 1: Simulate 8 uniform (0, 1) random numbers $U_1, U_2, \ldots, U_8$.
- Step 2: Apply the stratified sampling method to the random numbers so that U_i and U_{i+4} are transformed to random numbers V_i and V_{i+4} that are uniformly distributed over the interval $\left(\frac{(i-1)}{4}, \frac{i}{4}\right)$, $i = 1, 2, 3, 4$. In each of the four quartiles, a smaller value of U results in a smaller value of V.
- Step 3: Compute 8 standard normal random variates by $Z_i = N^{-1}(V_i)$, where N^{-1} is the inverse of the cumulative standard normal distribution function.

Michael draws the following 8 uniform (0, 1) random numbers:

i	1	2	3	4	5	6	7	8
U_i	0.4641	0.7873	0.8679	0.4482	0.3172	0.8998	0.5134	0.3051

Find the difference between the largest and the smallest simulated normal random variates. 2.5

5.5.2 Antithetic Variate

- Step 1: Simulate the price V_1 of the derivative based on n trials.
- Flip the signs to all normal random numbers used in Step 1 and recalculate another simulated price V_2.
- Calculate $V_3 = \frac{V_1+V_2}{2}$.

the idea behind this method is that each random number can lead to two different estimate (V_1, V_2). This can obviously save time. Moreover, there can be a gain in efficiency, because the two estimate are negatively correlated.

Example 10. Assume the Black-Scholes framework for a nondividend-paying stock. You are given

- The current stock price is 41.
- The stock's volatility is 30%.
- The continuously compunded risk-free interest rate is 8%.

You estimate the price of a 40-strike 1-year call on the stock. Using the following 6 uniform random numbers arranged in ascending order

$$0.6918, 0.5425, 0.3339, 0.1397, 0.8512, 0.1228$$

Estimate the price of the European call using the antithetic variate method. 6.1122

5.5.3 Control Variate

The control variate method is useful for valuing an arithmetic Asian option. A geometric Asian option can be valued using a formula, but an arithmetic Asian option cannot be. However, these two options are highly correlated, so an geometric Asian option can be used as a control variate for simulation of an arithmetic Asian option. Let

- A_i be the i^{th} simulated discounted payoff from arithmetic Asian Average price option.
- G_i be the i^{th} simulated discounted payoff from geometric Asian Average price optio, calculated using the **same** set of simulated stock prices when computing A_i.
- $A_{sim} = \frac{\sum_{i=1}^{n} A_i}{n}$ be the Monte-Carlo estimate of the price of the arithmetic Asian average price option.
- $G_{sim} = \frac{\sum_{i=1}^{n} G_i}{n}$ be the Monte-Carlo estimate of the price of the goemetic Asian average price option.

Suppose $E(A)$ and $E(G)$ are the true prices of two options. A_{sim} and G_{sim} are their unbiased estimator. Suppose we want to estimate $E(A)$, and we have a closed form formula fo$E(G)$ where the random variables A and G are highly correlated. The errors in the simulated arithmeticand geometric prices are approximately equal. Thus,

$$A_i - E(A) \approx G_i - E(G)$$

To estimate $E(A)$, let A^* be the the estimate of $E(A)$, then

$$A^* = A_{sim} - G_{sim} + E(G)$$

A^* is called a control variate estimator. The mean and variace of A^* are

$$E(A^*) = E(A_{sim}) - E(G_{sim}) + E(G) = E(A) + E(G) - E(G) = E(A)$$

$$V(A^*) = V(A_{sim}) + V(G_{sim}) - 2cov(A_{sim}, G_{sim})$$

For example, if A_{sim} and G_{sim} both have the same variance σ^2 and correlation coefficient ρ, then

$$V(A*) = 2\sigma^2 - 2\rho\sigma^2 = 2\sigma^2(1-\rho) < \sigma^2$$

so variance has been reduced. If ρ close to 1, variance reduction is very high.
$E(G)$ is the **control variate** and this variace reduction method is called the **control variate method**.

Example 11. [T5Q7]
You are to perform one trial of a simulation to estimate the price of an arithmetic average price Asian call option using the control variate method. The option pays the excess over 40, if any, of the arithmetic average of the prices of stock at the end of three consecutive months. You are given:

- The current stock price is 40.
- The continuously compounded expected rate of return from the stock is 0.2.
- The stock pays no dividends.
- The continuously compounded risk-free interest rate is 0.09.
- The volatility of the stock is 0.32.
- The following standard normal random numbers are 2.1, 0.46, -2.1.

The closed form formula for geometric average price Asian call option for the same stock over the same period with the same strike price calculates

a price of 2.078 for this option. Calculate the simulated price of arithmetic average price option in this trial. $\boxed{2.2283}$

A more sophiscated approach to control variates is that the error associated with the simulation of A is directly proportional to the error associated with G. Then

$$A_i - E(A) \approx \beta(G_i - E(G_i))$$

and the control variate estimator

$$A^* = A_{sim} + \beta(E[G] - G_{sim}).$$

is an unbiased estimator of $E(A)$ for any β. where

$$\beta = \frac{cov(A_{sim}, G_{sim})}{V(G_{sim})}$$

The variance is

$$V(A^*) = V(A_{sim}) + \beta^2 V(G_{sim}) - 2\beta Cov(A_{sim}, G_{sim}$$

It is then easy to show that

- $V(A^*)$ is minimized when
$$\beta = \frac{Cov(A_{sim}, G_{sim})}{V(G_{sim})} = \frac{Cov(A_i, G_i)}{V(G_i)}$$
- With this β, $V(A^*) = \sigma^2_{A_{sim}}(1-\rho^2)$.

In practice, $Cov(A_i, G_i)$ is not known, and there may also not a formula fo $V(G_i)$. Thus

$$\widehat{Cov(A_i, G_i)} = \frac{1}{n-1}\sum_{i=1}^{n}(A_i - \bar{A})(G_i - \bar{G})$$

and

$$\widehat{V(G_i)} = \frac{1}{n-1}\sum_{i=1}^{n}(G_i - \bar{G})$$

Thus,

$$\hat{\beta} = \frac{\sum_{i=1}^{n}(A_i - \bar{A})(G_i - \bar{G})}{\sum_{i=1}^{n}(G_i - \bar{G})}$$

which is the slope coefficient of the linear regression of A_i on G_i: $A_i = \alpha + \beta G_i$)

Example 12. [T5Q8]
Let $C(K)$ denote the Black-Scholes price for a 1-year $K-$strike European call option on a nondividend-paying stock.
Let $\hat{C}(K)$ denote the Monte Carlo price for a 1-year $K-$strike European call option on the stock, calculated by using 10 random 1-year stock prices simulated under the risk neutral probability measure.
Suppose that $S(0) = 10$, $\delta = 0$, $\sigma = 0.25$, and $r = 0.1$. Alan knows that the Black-Scholes price of a 1-year 11-strike European call on S is 1.0163 but he does not know the Black-Scholes formula. To find the price of a 1-year 12-strike call on S, he simulates 10 stock prices after 1 year under the risk-neutral measure using the following numbers drawn from a standard normal distribution:

$$0.52,\ -2.3321,\ 1.385,\ 0.9541,\ 0.1659$$

$$1.2348,\ -1.7698,\ -1.1921,\ -0.3358,\ -0.3137$$

He estimates the price of a 1-year 12-strike European call option on the stock using the formula

$$C^*(12) = \hat{C}(12) + \beta[C(11) - \hat{C}(11)],$$

where the coefficient β is such that the variance of $C^*(12)$ is minimized. Obtain the control variate estimates of the price of the 12-strike call. $\boxed{0.6507}$

Example 13. Compute the standard deviation of the estimate of Example 12. $\boxed{0.6057}$