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7 Interval Estimation

7.1 Confidence Intervals

Definition 1. Confidence Interval

An interval

$$(l(x_1, \dots, x_n), u(x_1, \dots, xn))$$

is called a 100γ% confidence interval for θ if

$$P[l(x_1, \dots, x_n) < \theta < u(x_1, \dots, xn)] = \gamma$$

where 0 < γ < 1.

The observed values $l(x_1, \dots, x_n)$ and $u(x_1, \dots, x_n)$ are called lower and upper confidence limits, respectively.

Definition 2. One-Sided Confidence Limits If

$$P[l(x_1, \dots, x_n) < \theta] = \gamma$$

then $l(x_1, \dots, x_n)$ is called a one-sided lower $100\gamma\%$ confidence limit for θ .

If

$$P[u(x_1, \dots, x_n) > \theta] = \gamma$$

then $u(x_1, \dots, x_n)$ is called a one-sided upper $100\gamma\%$ confidence limit for θ .

In general, if (θ_L, θ_U) is a $100\gamma\%$ confidence interval for a parameter θ , and if $\tau(\theta)$ is a monotonic increasing function of $\theta \in \Omega$, The $(\tau(\theta_L), \tau(\theta_U))$ is a $100\gamma\%$ confidence interval for $\tau(\theta)$.

Example 1. Consider a random sample of size n from an exponential distribution, $X_i \sim \text{Exp}(\theta)$.

- (a) Construct a one-sided lower $100\gamma\%$ confidence limit for θ .

- (b) Construct a one-sided upper $100\gamma\%$ confidence limit for θ .

- (c) Construct a $100\gamma\%$ confidence interval for θ .

- (d) Find a one-sided lower $100\gamma\%$ confidence limit for $P(X > t) = e^{-t/\theta}$.

IE-Q25b

Example 2.

Consider independent random samples from two gamma distributions, $X \sim \text{gamma}(4, \beta_1)$ and $Y_j \sim \text{gamma}(10, \beta_2); i = 1, \dots, n_1, j = 1, \dots, n_2$.

1. Find the distribution of $\left(\frac{\beta_2}{\beta_1}\right) \left(\frac{5\bar{X}}{2Y}\right)$.
2. Derive a $100(1 - \alpha)\%$ confidence for $\frac{\beta_2}{\beta_1}$.

Example 3.

Consider independent random samples from two normal distributions, $X_i \sim N(40, \theta_1^2)$ and $Y_j \sim N(80, \theta_2^2)$; $i = \dots, n_1$, $j = 1, \dots, n_2$. Derive a $100(1 - \alpha)\%$ confidence interval for $\frac{\theta_2^2}{\theta_1^2}$ based on sufficient statistics.

Example 4.

Consider a random sample of size 32 from a uniform distribution, $X_i \sim U(0, \theta)$, $\theta > 0$, and let $X_{n:n}$ be the largest order statistic. Find the constant c such that $(x_{n:n}, cx_{n:n})$ is a 99% confidence interval for θ .

7.2 Pivotal Quantity Method

Definition 3. Pivotal Quantity

If $Q = q(X_1, \dots, X_n; \theta)$ is a random variable that is a function only of (X_1, \dots, X_n) and θ , then Q is called a pivotal quantity if its distribution does not depend on θ or any other unknown parameters. That is, if $X \sim F(\mathbf{x}|\theta)$, then Q has the same distribution for all values of θ .

Example 5. (Gamma pivot)

Suppose that X_1, \dots, X_n are iid $Exp(\theta)$, find the pivotal quantity based on the sufficient statistics $T = \sum X_i$.

Example 6.

Let X_1, X_2, \dots, X_n be a random sample from a Weibull distribution, $X \sim WEI(\theta, 3)$.

- (a) Show that $Q = 2 \sum_{i=1}^n X_i^3 / \theta^3 \sim \chi^2(2n)$.
- (b) Use Q to derive an equal tailed $100\gamma\%$ confidence interval for θ .

Theorem 1. Let X_1, \dots, X_n be a random sample from a distribution with pdf $f(x; \theta)$ for $\theta \in \Omega$, and assume that an MLE $\hat{\theta}$ exists.

1. If θ is a location parameter, then $Q = \hat{\theta} - \theta$ is a pivotal quantity.
2. If θ is a scale parameter, then $Q = \hat{\theta}/\theta$ is a pivotal quantity.

Theorem 2. Let X_1, \dots, X_n be a random sample from a distribution with location-scale parameters. If MLEs $\hat{\theta}_1$ and $\hat{\theta}_2$ exist, then $(\hat{\theta}_1 - \theta_1)/\hat{\theta}_2$ and $\hat{\theta}_2/\theta_2$ are pivotal quantities for θ_1 and θ_2 , respectively.

Notes:

1. Let $f_0(z)$ be a pdf that is free of unknown parameters (including θ).
2. A parameter θ is a location parameter if the pdf has the form $f(x; \theta) = f_0(x - \theta)$.

3. A parameter θ is a scale parameter if the pdf has the form $f(x; \theta) = \frac{1}{\theta} f_0\left(\frac{x}{\theta}\right)$.
4. In the case of location-scale parameters, say θ_1 and θ_2 , the pdf has the form $f(x; \theta_1, \theta_2) = \frac{1}{\theta_2} f_0\left(\frac{x - \theta_1}{\theta_2}\right)$.

Example 7.

Let X have probability density function

$$f(x) = \begin{cases} \frac{\Gamma(5)x^2(\theta-x)}{\Gamma(3)\theta^4}, & 0 < x < \theta \\ 0, & \text{otherwise} \end{cases}$$

Show that $\frac{X}{\theta}$ is a pivotal quantity and use this pivotal quantity to find a 93% upper confidence limit for θ .

Example 8.

Consider a random sample from a normal distribution, $X \sim N(\mu, \sigma^2)$, where both μ and σ^2 are unknown. If $\hat{\mu}$ and $\hat{\sigma}$ are the MLEs of μ and σ ,

(a) show that $\frac{\hat{\mu} - \mu}{\hat{\sigma}}$ and $\hat{\sigma}/\sigma$ are pivotal quantities;

(b) find a $100(1 - \alpha)\%$ confidence interval for μ .

(c) find a $100(1 - \alpha)\%$ confidence interval for σ^2 .

Example 9.

IE-Q14

Let X_1, \dots, X_n be a random sample from a Poisson distribution with mean λ . Define $Y = \sum X_i$, suppose $Y = y_0$ is observed, show that $1 - \alpha$ confidence interval for λ is

$$\left(\frac{\chi_{1-\alpha/2}^2(2y_0)}{2n} \leq \lambda \leq \frac{\chi_{\alpha/2}^2(2(y_0 + 1))}{2n} \right),$$

by solving λ from the following equations:

$$\sum_{k=0}^{y_0} e^{-n\lambda} \frac{(n\lambda)^k}{k!} = \frac{\alpha}{2} \text{ and } \sum_{k=y_0}^{\infty} e^{-n\lambda} \frac{(n\lambda)^k}{k!}.$$

Consider $n = 15$ and observe $y_0 = \sum x_i = 15$. Find a 95% confidence interval for λ .

It may not always be possible to find a pivotal quantity based on MLEs, but for a sample from a continuous distribution with a single unknown parameter, at least one pivotal quantity can always be derived by use of the probability integral transform.

If

$$X \sim f(x; \theta)$$

and if

$$F(x; \theta)$$

is the CDF of X , then

$$F(X; \theta) \sim U(0, 1)$$

and consequently

$$Y_i \sim -\ln F(X_i, \theta) \sim EXP(1).$$

For a random sample X_1, \dots, X_n , it follows that

$$-2 \sum_{i=1}^n \ln F(X_i; \theta) \sim \chi^2(2n)$$

so that

$$P[\chi_{\alpha/2}^2(2n) < -2 \ln F(X_i; \theta) < \chi_{1-\alpha/1}^2(2n)] = 1 - \alpha$$

and inverting this statement will provide a confidence region for θ .

If the CDF is not in closed form or if it is too complicated, then the inversion may have to be done numerically.

If $F(x; \theta)$ is a monotonic increasing (or decreasing) function of θ , then the resulting confidence region will be an interval.

Notice also that $1 - F(X_i; \theta) \sim U(0, 1)$ and

$$-2 \sum_{i=1}^n \ln[1 - F(X_i; \theta)] \sim \chi^2(2n)$$

Example 10.

IE-Q15 Consider a random sample from a Pareto distribution, $X \sim PAR(\alpha, \theta = 200)$, find a $100(1 - \alpha)\%$ confidence interval for α .

Example 11.

IE-Q19 Let X be a single observation from the $Beta(2\theta, 1)$ distributio. Use a pivotal quantity to set up a 90% confidence interval for θ . If $x = 0.0157$, find the length of the confidence interval.

7.3 Aproximate Confidence Intervals

For discrete distributions, and for some multiparameter problems, a pivotal quantity may not exist. However, an approximate pivotal quantity often can be obtained based on asymptotic results. Let X_1, \dots, X_n be a random sample from a distribution with pdf $f(x; \theta)$. As noted in previous chapter, MLEs are asymptotically normal under certain condition.

Example 12.

Consider a random sample from a Bernoulli distribution, $X \sim BIN(1, p)$. Find an approximate confidence limits for p .

7.4 Credible Interval

A credible interval (or in general, a credible set) is the Bayesian analogue of a confidence interval. A $100(1 - \alpha)\%$ credible set C is a subset of Θ such that

$$\int_C \pi(\theta|\mathbf{x})d\theta = 1 - \alpha$$

If the parameter space Θ is discrete, a sum replaces the integral.

Definition 4.

If a is the $\frac{\alpha}{2}$ posterior quantile for θ , and b is the $1 - \frac{\alpha}{2}$ posterior quantile for θ , then (a, b) is a $100(1 - \alpha)\%$ **equal probability credible interval** for θ .

Note:

$$P(\theta < a|\mathbf{x}) = \frac{\alpha}{2} \text{ and } P(\theta > b|\mathbf{x}) = \frac{\alpha}{2}$$

$$\Rightarrow P(\theta \in (a, b)|\mathbf{x})$$

$$= 1 - P(\theta \notin (a, b)|\mathbf{x})$$

$$= 1 - (P(\theta < a|\mathbf{x}) + P(\theta > b|\mathbf{x})) = 1 - \alpha$$

Example 13.

IE-Q17 The following amounts were paid on a hospital liability policy:

121, 140, 147, 105, 130, 317, 128, 106, 141, 237.

The amount of a single payment has the single-parameter Pareto distribution with $\theta = 103$ and α unknown. The prior distribution has the gamma distribution with $\alpha = 3$ and $\theta = 1$. Determine the 95% equal probability credible interval for α .

Example 14.

IE-Q26b Let X_1, X_2, \dots, X_n be a random sample from a gamma distribution with parameters $\alpha = 4$ and $\theta = \frac{1}{\lambda}$, the prior density of λ is exponential with mean $\frac{1}{\mu}$ where μ is known. Derive a $100(1 - \alpha)\%$ equal probability Bayesian confidence interval for λ in terms of χ^2 random variable.

The equal-tail credible interval approach is ideal when the posterior distribution is symmetric. If $\pi(\theta|\mathbf{x})$ is skewed, a better approach is to create an interval of θ -values having the Highest Posterior Density (HPD).

Definition 5.

A $100(1 - \alpha)\%$ HPD region for θ is a subset $C \in \Theta$ defined by

$$C = \{\theta : \pi(\theta|\mathbf{x}) \geq k\}$$

where k is the largest number such that

$$\int_{\theta: \pi(\theta|\mathbf{x}) \geq k} \pi(\theta|\mathbf{x}) d\theta = 1 - \alpha$$

The value k can be thought of as a horizontal line placed over the posterior density whose intersection(s) with the posterior define regions with probability $1 - \alpha$.

Theorem 3.

If the posterior random variable $\theta|\mathbf{x}$ is continuous and unimodal, then the $100(1 - \alpha)\%$ HPD credible interval is the unique solution to

$$\begin{aligned} \int_a^b \pi(\theta|\mathbf{x}) d\theta &= 1 - \alpha \\ \pi(a|\mathbf{x}) &= \pi(b|\mathbf{x}) \end{aligned}$$

Example 15.

IE-Q16 A Bayesian analysis is performed. The posterior density function is

$$\pi(\theta|\mathbf{x}) = \begin{cases} 0.9\theta & 0 \leq \theta \leq \frac{20}{28} \\ 0.8392 - 0.2707\theta & \frac{20}{28} \leq \theta \leq 3.1 \end{cases}$$

Construct the 93% HPD credibility interval.

Example 16.

IE-Q18 In a Bayes analysis, a parameter θ has a continuous posterior with pdf

$$\pi(\theta) = \begin{cases} ce^{-0.12\theta} & \text{for } 2 < \theta < 5 \\ 0 & \text{otherwise} \end{cases}$$

for an appropriate constant c . Find a 95% "HPD" credible set for θ .

Example 17.
The following amounts were paid on a hospital liability policy:

125 132 141 107 133
319 126 104 145 223

The amount of a single payment has the single-parameter Pareto distribution with $\theta = 100$ and α unknown. The prior distribution has the gamma distribution with $\alpha = 2$ and $\theta = 1$. Determine the 95% HPD credible interval for α .

a=1.1832, b = 3.9384

```
f = function(x){
y = numeric(2)
y[1] = pgamma(x[2],12,4.801121) - pgamma(x[1],12,4.801121) - 0.95
y[2] = dgamma(x[1],12,4.801121) - dgamma(x[2],12,4.801121)
y
}
library(nleqslv)
xstart = c(1,3)
nleqslv(xstart, f, control=list(btol=.01),
method="Newton")
```