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7 Interval Estimation

7.1 Confidence Intervals

Definition 1. Confidence Interval

An interval

$$(l(x_1, \dots, x_n), u(x_1, \dots, xn))$$

is called a 100γ% confidence interval for θ if

$$P[l(x_1, \dots, x_n) < \theta < u(x_1, \dots, xn)] = \gamma$$

where 0 < γ < 1.

The observed values  $l(x_1, \dots, x_n)$  and  $u(x_1, \dots, x_n)$  are called lower and upper confidence limits, respectively.

**Definition 2.** One-Sided Confidence Limits If

$$P[l(x_1, \dots, x_n) < \theta] = \gamma$$

then  $l(x_1, \dots, x_n)$  is called a one-sided lower  $100\gamma\%$  confidence limit for  $\theta$ .

If

$$P[u(x_1, \dots, x_n) > \theta] = \gamma$$

then  $u(x_1, \dots, x_n)$  is called a one-sided upper  $100\gamma\%$  confidence limit for  $\theta$ .

In general, if  $(\theta_L, \theta_U)$  is a  $100\gamma\%$  confidence interval for a parameter  $\theta$ , and if  $\tau(\theta)$  is a monotonic increasing function of  $\theta \in \Omega$ , The  $(\tau(\theta_L), \tau(\theta_U))$  is a  $100\gamma\%$  confidence interval for  $\tau(\theta)$ .

**Example 1.** Consider a random sample of size  $n$  from an exponential distribution,  $X_i \sim \text{Exp}(\theta)$ .

- (a) Construct a one-sided lower  $100\gamma\%$  confidence limit for  $\theta$ .

(b) Construct a one-sided upper  $100\gamma\%$  confidence limit for  $\theta$ .

(c) Construct a  $100\gamma\%$  confidence interval for  $\theta$ .

- (d) Find a one-sided lower  $100\gamma\%$  confidence limit for  $P(X > t) = e^{-t/\theta}$ .

**Example 2.**

Consider independent random samples from two gamma distributions,  $X \sim \text{gamma}(6, \beta_1)$  and  $Y_j \sim \text{gamma}(12, \beta_2); i = 1, \dots, n_1, j = 1, \dots, n_2$ .

- (a) Find the distribution of  $\left(\frac{\beta_2}{\beta_1}\right) \left(\frac{6\bar{X}}{3\bar{Y}}\right)$ .

- (b) Derive a  $100(1 - \alpha)\%$  confidence for  $\frac{\beta_2}{\beta_1}$ .

**Example 3.**

Consider a random sample of size 50 from a uniform distribution,  $X_i \sim U(0, \theta)$ ,  $\theta > 0$ , and let  $X_{n:n}$  be the largest order statistic. Find the constant  $c$  such that  $(x_{n:n}, cx_{n:n})$  is a 97% confidence interval for  $\theta$ .

**7.2 Pivotal Quantity Method****Definition 3. Pivotal Quantity**

If  $Q = q(X_1, \dots, X_n; \theta)$  is a random variable that is a function only of  $(X_1, \dots, X_n)$  and  $\theta$ , then  $Q$  is called a pivotal quantity if its distribution does not depend on  $\theta$  or any other unknown parameters. That is, if  $X \sim F(\mathbf{x}|\theta)$ , then  $Q$  has the same distribution for all values of  $\theta$ .

**Example 4.** (Gamma pivot)

Suppose that  $X_1, \dots, X_n$  are iid  $\text{Exp}(\theta)$ , find the pivotal quantity based on the sufficient statistics  $T = \sum X_i$ .

**Example 5.**

Consider a random sample from a normal distribution,  $X \sim N(\mu, \sigma^2)$ , where both  $\mu$  and  $\sigma^2$  are unknown. If  $\hat{\mu}$  and  $\hat{\sigma}$  are the MLEs of  $\mu$  and  $\sigma$ ,

(a) show that  $\frac{\hat{\mu} - \mu}{\hat{\sigma}}$  and  $\hat{\sigma}/\sigma$  are pivotal quantities;

(b) find a  $100(1 - \alpha)\%$  confidence interval for  $\mu$ .

(c) find an equal tail  $100(1 - \alpha)\%$  confidence interval for  $\sigma^2$ .

**Example 6.**

Let  $X_1, X_2, \dots, X_n$  be a random sample from a Weibull distribution,  $X \sim WEI(\theta, 4)$ .

- (a) Show that  $Q = 2 \sum_{i=1}^n X_i^4 / \theta^4 \sim \chi^2(2n)$ .  
(b) Use  $Q$  to derive an equal tailed  $100\gamma\%$  confidence interval for  $\theta$ .

**Example 7.**

Let  $X$  have probability density function

$$f(x) = \begin{cases} \frac{\Gamma(5)x^2(\theta-x)}{\Gamma(3)\theta^4}, & 0 < x < \theta \\ 0, & \text{otherwise} \end{cases}$$

Show that  $\frac{X}{\theta}$  is a pivotal quantity and use this pivotal quantity to find a 92% upper confidence limit for  $\theta$ .



**Example 8.**

Let  $X$  be a single observation from the  $Beta(8\theta, 1)$  pdf.

- (a) Prove that  $T = X^{8\theta}$  is a pivotal quantity.
- (b) Find the shortest 97% pivotal interval.

**Example 9.**

Let  $X_1, \dots, X_n$ , be a random sample from a gamma distribution with parameters  $\alpha = 6$  and unknown  $\theta$ .

- (a) Find a pivotal quantity for the parameter  $\theta$  based on the sufficient statistic.
- (b) Derive an equal tail 95% confidence interval for  $\theta$  based on the pivotal quantity from part (a).

It may not always be possible to find a pivotal quantity, but for a sample from a continuous distribution with a single unknown parameter, at least one pivotal quantity can always be derived by use of the probability integral transform.

If

$$X \sim f(x; \theta)$$

and if

$$F(x; \theta)$$

is the CDF of  $X$ , then

$$F(X; \theta) \sim U(0, 1)$$

and consequently

$$Y_i \sim -\ln F(X_i, \theta) \sim EXP(1).$$

For a random sample  $X_1, \dots, X_n$ , it follows that

$$-2 \sum_{i=1}^n \ln F(X_i; \theta) \sim \chi^2(2n)$$

so that

$$P[\chi_{\alpha/2}^2(2n) < -2 \ln F(X_i; \theta) < \chi_{1-\alpha/1}^2(2n)] = 1 - \alpha$$

and inverting this statement will provide a confidence region for  $\theta$ .

If the CDF is not in closed form or if it is too complicated, then the inversion may have to be done numerically.

If  $F(x; \theta)$  is a monotonic increasing (or decreasing) function of  $\theta$ , then the resulting confidence region will be an interval.

Notice also that  $1 - F(X_i; \theta) \sim U(0, 1)$  and

$$-2 \sum_{i=1}^n \ln[1 - F(X_i; \theta)] \sim \chi^2(2n)$$

**Example 10.**

Consider a random sample from a Pareto distribution,  $X \sim PAR(\alpha, \theta = 700)$ , find a  $100(1 - \alpha)\%$  confidence interval for  $\alpha$ .

**7.3 Aproximate Confidence Intervals**

For discrete distributions, and for some multiparameter problems, a pivotal quantity may not exist. However, an approximate pivotal quantity often can be obtained based on asymptotic results. Let  $X_1, \dots, X_n$  be a random sample from a distribution with pdf  $f(x; \theta)$ . As noted in previous chapter, MLEs are asymptotically normal under certain condition.

**Example 11.**

Consider a random sample from a Bernoulli distribution,  $X \sim \text{BIN}(1, p)$ . Find an approximate confidence limits for  $p$ .

**7.4 Credible Interval**

A credible interval (or in general, a credible set) is the Bayesian analogue of a confidence interval. A  $100(1 - \alpha)\%$  credible set  $C$  is a subset of  $\Theta$  such that

$$\int_C \pi(\theta|\mathbf{x})d\theta = 1 - \alpha$$

If the parameter space  $\Theta$  is discrete, a sum replaces the integral.

**Definition 4.**

If  $a$  is the  $\frac{\alpha}{2}$  posterior quantile for  $\theta$ , and  $b$  is the  $1 - \frac{\alpha}{2}$  posterior quantile for  $\theta$ , then  $(a, b)$  is a  $100(1 - \alpha)\%$  **equal probability credible interval** for  $\theta$ .

**Example 12.**

The following amounts were paid on a hospital liability policy:

122, 137, 143, 103, 131, 318, 121, 106, 149, 226.

The amount of a single payment has the single-parameter Pareto distribution with  $\theta = 108$  and  $\alpha$  unknown. The prior distribution has the gamma distribution with  $\alpha = 2$  and  $\theta = 1$ . Determine the 94% equal probability credible interval for  $\alpha$ .

The equal-tail credible interval approach is ideal when the posterior distribution is symmetric. If  $\pi(\theta|\mathbf{x})$  is skewed, a better approach is to create an interval of  $\theta$ -values having the Highest Posterior Density (HPD).

**Definition 5.**

A  $100(1 - \alpha)\%$  HPD region for  $\theta$  is a subset  $C \in \Theta$  defined by

$$C = \{\theta : \pi(\theta|\mathbf{x}) \geq k\}$$

where  $k$  is the largest number such that

$$\int_{\theta: \pi(\theta|\mathbf{x}) \geq k} \pi(\theta|\mathbf{x}) d\theta = 1 - \alpha$$

The value  $k$  can be thought of as a horizontal line placed over the posterior density whose intersection(s) with the posterior define regions with probability  $1 - \alpha$ .

**Theorem 1.**

If the posterior random variable  $\theta|\mathbf{x}$  is continuous and unimodal, then the  $100(1 - \alpha)\%$  HPD credible interval is the unique solution to

$$\int_a^b \pi(\theta|\mathbf{x})d\theta = 1 - \alpha$$
$$\pi(a|\mathbf{x}) = \pi(b|\mathbf{x})$$

**Example 13.**

You are given the following:

$$f(x|\theta) = \frac{6x^5}{\theta^6}, 0 < x < \theta.$$

$$\pi(\theta) = \frac{4}{\theta^5}, \theta > 1.$$

Three observations were observed: 100, 800, 1300.  
Find a 99% "HPD" credible set for  $\theta$ .

**Example 14.**  
The following amounts were paid on a hospital liability policy:

125	132	141	107	133
319	126	104	145	223

The amount of a single payment has the single-parameter Pareto distribution with  $\theta = 100$  and  $\alpha$  unknown. The prior distribution has the gamma distribution with  $\alpha = 2$  and  $\theta = 1$ . Determine the 95% HPD credible interval for  $\alpha$ .

a=1.1832, b = 3.9384

```
f = function(x){
y = numeric(2)
y[1] = pgamma(x[2],12,4.801121) - pgamma(x[1],12,4.801121) - 0.95
y[2] = dgamma(x[1],12,4.801121) - dgamma(x[2],12,4.801121)
y
}
library(nleqslv)
xstart = c(1,3)
nleqslv(xstart, f, control=list(btol=.01),
method="Newton")
```