**MEME15203 Statistical Inference Marking Guide**

**Assignment 3**

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| Faculty: | FES | Unit Code: | MEME15203 |
| Course: | MAC | Unit Title: | Statistical Inference |
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DR-Q05e

Q1. Consider a random sample of size $n$ from a single-parameter pareto distribution, $X_i \sim SP(\alpha = 2, \theta)$. Find the UMVUE of the $p^{th}$ percentile.

(10 marks)

*Ans.*

$f(x_i) = \frac{2\theta^2}{x_i^3}, x_i > \theta$

$$f(x_, x_2, \ldots, x_n) = 2^n \theta^{2n} \prod_{i=1}^n x_i^{-3}, x_{(1)} > \theta$$
$$= g(s,\theta) h(x_1, \ldots, x_n)$$

where $g(s,\theta) = \theta^{2n} I(X_{(1)}) > \theta)$ and $h(x_1, \ldots, x_n) = 2^n \prod_{i=1}^n x_i^{-3}$,
By factorization Theorem, $S = X_{(1)}$ is a sufficient statistic for $\theta$.

$f_S(s) = n f_X(s)[1 - F_X(s)]^{(n-1)} = \frac{2n\theta^2}{s^3} \left(\frac{\theta}{s}\right)^{2(n-1)} = \frac{2n\theta^{2n}}{s^{2n+1}}, s > \theta$
$\Rightarrow S \sim SP(\alpha = 2n, \theta)$
$E[u(S)] = \int_\theta^\infty u(s) \frac{2n\theta^{2n}}{s^{2n+1}} ds = 0 \ \forall \theta$
$\Rightarrow \int_\theta^\infty u(s) s^{-2n} ds = 0 \ \forall \theta$
$\frac{d}{d\theta} \int_\theta^\infty u(s) s^{-2n} ds = u(\theta)\theta^{-2n} = 0 \ \forall \theta$
This implies $u(\theta) = 0$ for all $\theta$, so $S = X_{(1)}$ is complete.

$E(S) = \frac{2n\theta}{2n-1}$

$P(X > \pi_p) = 1 - p$
$\left(\frac{\theta}{\pi_p}\right)^2 = 1 - p$
$\pi_p = \frac{\theta}{\sqrt{1-p}}$

Let $S_1 = \frac{(2n-1)S}{2n\sqrt{1-p}}$, then
$E(S_1) = \frac{(2n-1)E(S)}{2n\sqrt{1-p}} = \frac{\theta}{\sqrt{1-p}}$
Since $S_1 = \frac{(2n-1)S}{2n\sqrt{1-p}}$ is a function of css for $\theta$ and unbisaed for $\pi_p$. Then, $S_1 = \frac{(2n-1)S}{2n\sqrt{1-p}}$ is the UMVUE of the $p^{th}$ percentile.

DR-Q06c Q2. Suppose that $X_1, \ldots, X_n$ is a random sample from a Negative Binomial distribution, $X_i \sim$ NB(r = 6, $\theta$),

a
(a) Show that the p.d.f. of $X$ belongs to the regular exponential family.

(b) Based on the answer in a ??, find a complete and sufficient statistic for $\theta$.

(c) Find the UMVUE of $\left[\frac{4\theta}{1-4(1-\theta)}\right]^{6n}$.

(15 marks)

*Ans.*

(a) $f(x) = \binom{x-1}{r-1}\theta^6(1-\theta)^{x-6} = \binom{x-1}{r-1}\left(\frac{\theta}{1-\theta}\right)^6 e^{x\ln(1-\theta)} = c(\theta)h(x)e^{q(\theta)t(x)}$
where $c(\theta) = \left(\frac{\theta}{1-\theta}\right)^6$, $h(x) = \binom{x-1}{r-1}$, $q(\theta) = \ln(1-\theta)$, and $t(x) = x$.
Thus the p.d.f. of $X$ belongs to the regular exponential family.

(b) Since the p.d.f. of $X$ belongs to the regular exponential family, thus by the theorem, $S = \sum_{i=1}^{n} X_i$ is a c.s.s of $\theta$

(c) Let $E(t^S) = \left[\frac{4\theta}{1-4(1-\theta)}\right]^{6n}$
As $S \sim NB(6n, \theta)$,
$E(t^S) = E(e^{S\ln t}) = \left(\frac{t\theta}{1-t(1-\theta)}\right)^{6n}$
$\Rightarrow t = 4$
Since $4^S$ is a function of the c.s.s. of $\theta$
which is an UE of $\left(\frac{4\theta}{1-4(1-\theta)}\right)^{6n}$,
thus $4^S$ is the UMVUE of $\left(\frac{4\theta}{1-4(1-\theta)}\right)^{6n}$.

DR-Q09d Q3. Consider a random sample of size $n$ from a gamma distribution $X_i \sim GAM(\alpha, \theta)$ and let $\bar{X} = (1/n)\sum X_i$ and $\tilde{X} = (\prod X_i)^{1/n}$ be the sample mean and geometric mean respectively.

(a) Show that $\bar{X}$ and $\tilde{X}$ are jointly complete and sufficient for $\theta$ and $\alpha$.

(b) Find the UMVUE of $\mu = \alpha\theta$.

(c) Find the UMVUE of $\mu^n$..

(d) Show that $\bar{X}$ and $T$ are stochastically independent random variables.

(20 marks)

*Ans.*

(a) $f(x_i) = \frac{1}{\Gamma(\alpha)\theta^\alpha} x_i^{\alpha-1} e^{-x_i/\theta}$
$= \frac{1}{\Gamma(\alpha)\theta^\alpha} x_i^{\alpha} e^{\alpha\ln x_i} e^{-x_i/\theta}$

$= \frac{1}{\Gamma(\alpha)\theta^\alpha} x_i^{-1} e^{\alpha \ln x_i - x_i/\theta}$

Thus $X_i \sim GAM(\alpha, \theta)$ belongs to REC familly where $c(\theta) = \frac{1}{\Gamma(\alpha)\theta^\alpha}$, $h(x) = x_i^{-1}$, $t_1(X_i) = \ln X_1$, $t_2(X_i) = X_1$, $q_1(\theta) = \alpha$ and $q_2(\theta) = \theta$.

Thus, $\sum \ln X_i$ and $\sum X_i$ are jointly complete and sufficient of $\alpha$ and $\theta$. Since $\bar{X}$ is a function of $\sum X_i$ and $\tilde{X}$ is a function of $\sum \ln X_i$, thus $\bar{X}$ and $\tilde{X}$ are jointly complete and sufficient for $\theta$ and $\alpha$.

(b) $\sum X_i \sim GAM(n\alpha, \theta)$
$2\sum X_i/\theta \sim \chi^2(2n\alpha)$
$E(2n\bar{X}/\theta) = 2n\alpha$
$E(\bar{X}) = \alpha\theta$
Since $\bar{X}$ is a CSS of $\theta$ and $\alpha$, thus, $\bar{X}$ is the UMVUE of $\mu = \alpha\theta$.

(c) $E(\prod X_i) = (EX_i)^n = \mu^n$
Thus, $\prod X_i$ is an UE of $\mu$.
Since $\prod X_i = \tilde{X}^n$, then $\prod X_i$ is the UMVUE of $\mu^n$.

(d)
$$\begin{aligned} T &= u(X_1, X_2, \ldots, X_n) \\ &= u(cX_1, cX_2, \ldots, cX_n) \\ &= \frac{(cX_1 + cX_2 + \cdots, +cX_n)/n}{(cX_1 \cdot cX_2 \cdots cX_n)^{1/n}} \\ &= \frac{c(X_1 + X_2 + \cdots, +X_n)/n}{c(X_1 \cdot X_2 \cdots X_n)^{1/n}} \\ &= \frac{\bar{X}}{\tilde{X}} \end{aligned}$$

Since $\bar{X}$ is a CSS of $\theta$ and the distribution of $T$ does not depend on $\theta$. thus by BASU theorem, $\bar{X}$ and $T$ are stochastically independent random variables.

DR-Q24d Q4. %%%%% Suppose that $X_1, \ldots, X_n$ is a random sample from a Poisson distribution, $X_i \sim \text{POI}(\lambda)$. Find the UMVUE of $P(X_1 + X_2 = 0 \text{ or } 1) = (1 + 2\lambda)e^{-2\lambda}$ using Rao-Blackwell theorem.

(10 marks)

*Ans.*

$f(x) = \frac{1}{lambda} e^{-x/\lambda} = c(\lambda)h(x)e^{t(x)q(\lambda)}$ which is in a member of REC. Hence $S = \sum X_i$ is a CSS of $\lambda$.

Let

$$T = \begin{cases} 1, & X_1 + X_2 = 0 \text{ or } 1 \\ 0, & \text{otherwise} \end{cases}.$$

$E(T) = P(X_1 + X_2 = 0 \text{ or } 1) = e^{-2\lambda} + 2\lambda e^{-2\lambda} = (1+2\lambda)e^{-2\lambda}$. Thus $T$ is and unbiased estimator of $(1+2\lambda)e^{-2\lambda}$. Since $S$ is CSS of $\lambda$. Hence by Rao-Blackwell theorem, $T^* = E(T|S)$ is an UMVUE of $(1+2\lambda)e^{-2\lambda}$.

$$E\left[T|\sum_{i=0}^{n} X_i = s\right]$$
$$= 1 \cdot P[X_1 + X_2 = 0 \text{ or } 1 | X_1 + X_2 + \cdots X_n = s]$$
$$= \frac{P(X_1 + X_2 = 0, X_3 + \cdots + X_n = s)}{P(X_1 + \cdots + X_n = s)} + \frac{P(X_1 + X_2 = 1, X_3 + \cdots + X_n = s-1)}{P(X_1 + \cdots + X_n = s)}$$
$$= \frac{P(X_1 + X_2 = 0) \times P(X_3 + \cdots + X_n = s)}{P(X_1 + \cdots + X_n = s)} + \frac{P(X_1 + X_2 = 1) \times P(X_3 + \cdots + X_n = s-1)}{P(X_1 + \cdots + X_n = s)}$$
Since $X_1, X_2, \ldots, X_n$ are independent.
$$= \frac{e^{-2\lambda}[(n-2)\lambda]^s e^{-(n-2)\lambda}}{(n\lambda)^s s! e^{-n\lambda}/s!} + \frac{2\lambda e^{-2\lambda}[(n-2)\lambda]^{s-1} e^{-(n-2)\lambda}}{(n\lambda)^s (s-1)! e^{-n\lambda}/s!}$$
$$= \left(\frac{n-2}{n}\right)^s + \left(\frac{n-2}{n}\right)^s \left(\frac{2s}{n-2}\right)$$

DR-Q34 Q5. Consider a random sample of size $n$ from a distribution with pdf

$$f(x;\theta) = \frac{(\ln\theta)^x}{\theta x!}, x = 1, 1, \ldots; \theta > 1, \text{ zero, otherwise.}$$

(a)Find a complete sufficient statistic for $\theta$.

(b)Find the UMVUE of $\ln\theta$.

(c)Find the UMVUE of $(\ln\theta)^2$.

(15 marks)

*Ans.*

(a)Let $\mu = \ln\theta$

$f(x;\theta) = \frac{\mu^x}{e^{\mu} x!} = e^{-\mu}\frac{1}{x!}e^{x\ln(\mu)} = c(\mu)h(x)e^{q(\mu)t(x)}$

where $c(\mu) = e^{-\mu}$, $h(x) = \frac{1}{x!}$, $q(\mu) = \ln(\mu)$; $t(x) = x$. Thus, $f(x;\theta)$ belong to $REC(\theta)$ and hence $S = \sum X_i$ is a complete sufficient statistic for $\mu = \ln\theta$.

(b)$X \sim POI(\mu = \ln\theta)$, $E(X) = \ln\theta$ and $E\left(\frac{S}{n}\right) = E(\bar{X}) = \ln\theta$. Since $\bar{X}$ is a function of CSS and unbiased for $\ln\theta$, therefore, by Lehmann Scheffe theorem, $\bar{X}$ is the UMVUE of $\ln\theta$.

(c)$E\left[\bar{X}^2 - \frac{\bar{X}}{n}\right] = \frac{\mu}{n} + \mu^2 - \frac{\mu}{n} = \mu^2$. Since $\bar{X}^2 - \frac{\bar{X}}{n}$ is a function of CSS and unbiased for $\mu^2 = (\ln\theta)^2$, therefore, by Lehmann Scheffe theorem, $\bar{X}^2 - \frac{\bar{X}}{n}$ is the UMVUE of $(\ln\theta)^2$.

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DR-Q33 Q6. Show that $X \sim N(0, \theta)$ is not a complete family.

(10 marks)

*Ans.*

$X \sim N(0, \theta)$

Let $U(X) = X$, Then $E[U(S)] = E(X) = 0$, but $X \neq 0$. Thus, $X \sim N(0, \theta)$ is not a complete family.

DR-Q36b Q7. Let $X_1, X_2, \ldots, X_n$ be random sample of size $n$ from a Gamma distribution with probability density function

$$\frac{1}{\theta^2} x e^{-x/\theta}, x > 0$$

zero otherwise. Find the UMVUE of $\gamma = P(X > t)$ using Rao-Blackwell theorem.

(20 marks)

*Ans.*

Let

$$T = \begin{cases} 1, & X_1 > t \\ 0, & \text{otherwise} \end{cases}.$$

Then, $E(T) = P(X > t)$. Thus $T$ is and unbiased estimator of $\gamma$.

$f(x) = \frac{1}{\theta^2} e^{-x/\theta} = c(\theta) h(x) e^{q(\theta) t(x)}$, where $c(\theta) = \frac{1}{\theta^2}$, $h(x) = 1$, $q(\theta) = \frac{1}{\theta}$, and $t(x) = x$, hence $f(x)$ is a member of $REC(\theta)$ and $S = \sum_{i=1}^{n} X_i$ is a complete sufficient statistics for $\theta$.

Thus, by Rao-Blackwell therem, $T^* = E(T|S)$ is an UMVUE of $\gamma$.

$$\begin{aligned} f_{X_1,S}(x_1, s) &= f_{X_1,S_1}(x_1, s - x_1) \text{ where } S_1 = X_2 + \cdots, X_n \sim gamma(2n-2, \theta) \\ &= f_{X_1}(x_1) f_{S_1}(s - x_1) \\ &= \tfrac{1}{\theta^2} x_1 e^{-x_1/\theta} \tfrac{1}{\Gamma(2n-2)\theta^{2n-2}} (s - x_1)^{2n-3} e^{-(s-x_1)/\theta} \\ &= \tfrac{1}{\Gamma(2n-2)\theta^{2n}} x_1 (s - x_1)^{2n-3} e^{-s/\theta} \end{aligned}$$

$f_{X_1|S}(x_1) = k x_1 (s - x_1)^{2n-3}, 0 < x_1 < s$

Let $z = s - x_1$, $dz = -dx$

$\int_s^0 k(s - z) z^{2n-3} dx_1 = 1$

$\int_s^0 k(s z^{2n-3} - z^{2n-2}) dz = 1$

$k \left[ \frac{s z^{2n-2}}{2n-2} - \frac{z^{2n-1}}{2n-1} \right]_0^s = 1$

$k \left[ \frac{s^{2n-1}}{2n-2} - \frac{s^{2n-1}}{2n-1} \right] = 1$

$k \left[ \frac{(2n-1)s^{2n-1} - (2n-2)s^{2n-1}}{(2n-1)(2n-2)} \right] = 1$

$k \left[ \frac{s^{2n-1}}{(2n-1)(2n-2)} \right] = 1$

$k = (2n-2)(2n-1) s^{1-2n}$

$\therefore f_{X_1|S}(x_1) = (2n-2)(2n-1) s^{1-2n} x_1 (s - x_1)^{2n-3}, 0 < x_1 < s$

$$
\begin{aligned}
E[T|S] &= P[X_1 > t|s] \\
&= \int_t^s (2n-2)(2n-1)s^{1-2n}x_1(s-x_1)^{2n-3}dx_1
\end{aligned}
$$

Let $z = s - x_1$, $dz = -dx$

$$
\begin{aligned}
&= (2n-2)(2n-1)s^{1-2n}\int_{s-t} t^0(s-z)z^{2n-3}(-dz) \\
&= (2n-2)(2n-1)s^{1-2n}\left[\frac{sz^{2n-2}}{2n-2} - \frac{z^{2n-1}}{2n-1}\right]_0^{s-t} \\
&= (2n-2)(2n-1)s^{1-2n}\left[\frac{s(s-t)^{2n-2}}{2n-2} - \frac{(s-t)^{2n-1}}{2n-1}\right] \\
&= (2n-2)(2n-1)\left[\frac{\left(\frac{s-t}{s}\right)^{2n-2}}{2n-2} - \frac{\left(\frac{s-t}{s}\right)^{2n-1}}{2n-1}\right] \\
&= (2n-2)(2n-1)\left[\frac{(2n-1)\left(\frac{s-t}{s}\right)^{2n-2} - (2n-2)\left(\frac{s-t}{s}\right)^{2n-1}}{(2n-2)(2n-1)}\right] \\
&= (2n-1)\left(\frac{s-t}{s}\right)^{2n-2} - (2n-2)\left(\frac{s-t}{s}\right)^{2n-1}
\end{aligned}
$$