CONTENTS

5 Amortization Schedules and Sinking Funds

5.1 Method of Loan Repayment

1. **Amortization Method**:
   In the amortization method, the borrower repays the lender by means of installment payments at periodic intervals. Typically this method is used with individual borrowers. Examples include car loan, mortgage repayment.

2. **Sinking Fung Method**:
   In the sinking fund method, the borrower repays the lender by means of a lump-sum payment at the end of the term of the loan. The borrower pays interest on the loan in installments over this period. It is also assumed that the borrower makes periodic payments into a fund, called a “sinking fund”, which will accumulate to the amount of the loan to be repaid at the end of the term of the loan

5.2 Amortizing a Loan

Suppose you borrow an amount of money at 5% effective and you are to repay the loan by making payments of $1,000 at the end of each year for 5 years.

- What is the original amount of the loan?

- How much interest should you pay the lender at the end of the first year?

- What is the excess of $1,000 over the interest paid in the first year? What does this excess amount represent?

- What is the remaining amount of the loan at the end of the first year? What does this remaining amount represent?

5.3 Outstanding Balance

Two ways of looking at the **outstanding loan balance**:

1. **Retrospective Method:**
Suppose that you and the lender agree that you will completely settle the debt at the end of first year by making an additional lump sum payment just after the first regular payment of \$1,000. This lump sum payment would have to be the outstanding loan balance of \$3,545.95, since this is the remaining portion of the original loan that you still owe. The outstanding balance is computed by looking backwards to the original loan amount, the payment of \$1,000 and the principal repaid in that payment. Thus this method is called the **retrospective method**.

Let the outstanding loan balance at time t be B_t. The original loan balance, B_0, is denoted by L and then level payments are R. The prospective method gives

$$\boxed{B_t = \underbrace{L(1+i)^t}_{\text{amount owed at time } t} - \underbrace{Rs_{\overline{t|}}}_{\text{amount paid at time } t}}.$$

Note: Use this method if

- the number of payments is not known; or
- the size of future payments is not known.

2. **Prospective Method:**
As in (1), you are to make an additional lump sum payment at the end of one year to completely settle the debt. But now you and the lender reason that what you still owe are the four remaining payments of \$1,000. If you are to pay off the loan with a lump sum, the amount must be the PV of the four payments, i.e. $1000a_{\overline{4|}} = \$3545.95$. This outstanding balance is computed by looking forward to the remaining payments, thus this method is called the **prospective method**. Outstanding loan balance at any point in time is equal to the present value at that date of the remaining installment payments.

$$\boxed{B_t = \frac{L}{a_{\overline{n|}}} a_{\overline{n-t|}} = R a_{\overline{n-t|}}.}$$

Example 1.
Smith obtains a 36-month automobile loan of \$10,000 at a nominal interest rate of 18%, compounded monthly. Equal payments are due at the end of each month. Find the outstanding balance immediately after the 25th payment,

(i) using retrospective method; 36417

(ii) using prospective method. 3641

Example 2.

A loan is to be repaid in level installments payable at the end of each year for 9 years. The effective annual rate of interest on the loan is 8%. After the 4th payment the principal remaining is 9,000. Determine the amount of loan.

Example 3.

Suppose that you get a mortgage of 100,000, to be repaid in equal annual payments over 20 years, at the end of each year, at the annual effective rate of interest of 6%. After 10 years, you get an enormous salary hike and decide to completely repay the loan. How much will you pay (not including the installment at 10 years)? $\boxed{64168.59}$

Example 4.

John borrows 50,000 that is to be paid back over 10 years with level monthly payments at the end of each month. The interest is charged on the loan at a nominal rate of 10% compunded monthly. On the due date of the 50th payment. John decides to repay the loan in full with a single payment of X. Calculate X.

Example 5.

A 7,000 loan is being repaid with regular payments of X at the end of each year for as long as necessary plus a smaller payment one year after the final regular payment. Immediately after the 10th payment, the outstanding principal is 3 times the size of the regular payment (that is, $3X$). If the annual interest rate i is 16%, what is the value of X?

5.4 Amortization Schedule

Let

- I_t = interest paid at the end of year t
- P_t = principal repaid at the end of year t
- B_t = outstanding loan balance at the end of year t (just after the loan payment is made)
- R_t = loan payment at the end of year t

Thus we have the followings for the first two years:

Symbol	Amount
	First Year
$I_1 = iB_0$	$I_1 = (0.05)(4,329.48) = 216.47$
$P_1 = R_1 - I_1$	$P_1 = 1,000 - 216.47 = 783.53$
$B_1 = B_0 - P_1$	$B_1 = 4,329.48 - 783.53 = 3545.95$
	Second Year
$I_2 = iB_1$	$I_2 = (0.05)(3,545.95) = 177.30$
$P_2 = R_2 - I_2$	$P_1 = 1,000 - 177.30 = 822.70$
$B_2 = B_1 - P_2$	$B_2 = 3,545.95 - 822.70 = 2,723.25$
	t^{th} Year
$I_t = iB_{t-1}$	
$P_t = R_t - I_t$	
$B_t = B_{t-1} - P_t$	

The process of reducing the outstanding balance of a loan by making payments that are part interest and part principal is known as **amortizing** the loan.

It would be convenient to organize all of the results in a table called an "amortization schedule".

Notes:

- The amount of interest paid decreases with each payment.
- The principal repaid increases with each payment.
- The amount of principal repaid increases in geometric progression with common ratio $(1+i)$.

$$P_t = (1+i)^{t-1} P_1, \text{for } i = 2, 3, \ldots$$

- The total of principal repaid is the original loan amount.

$$L = \sum_{t=1}^{n} P_t$$

- The total of the interest paid is the sum of the payments minus the sum of the principal repaid.

$$\sum_{t=1}^{n} I_t = \sum_{t=1}^{n} R_t - \sum_{t=1}^{n} P_t = nR - L$$

- In general, consider a loan of $a_{\overline{n}|}$ at i, repaid by n payment of 1. The amortization schedule for this loan is:

Duration: t	Payment: R	Interest: $I_t = iB_{t-1}$	Principal: $P_t = R - I_t$	Outstanding Balance: $B_t = B_{t-1} - P_t$
0				$a_{\overline{n}|}$
1	1	$ia_{\overline{n}|} = 1 - v^n$	v^n	$a_{\overline{n}|} - v^n = a_{\overline{n-1}|}$
2	1	$ia_{\overline{n-1}|} = 1 - v^{n-1}$	v^{n-1}	$a_{\overline{n-1}|} - v^{n-1} = a_{\overline{n-2}|}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
t	1	$ia_{\overline{n-t+1}|} = 1 - v^{n-t+1}$	v^{n-t+1}	$a_{\overline{n-t+1}|} - v^{n-t+1} = a_{\overline{n-t}|}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	1	$ia_{\overline{1}|} = 1 - v$	v	$a_{\overline{1}|} - v = 0$
Total	n	$n - a_{\overline{n}|}$	$a_{\overline{n}|}$	

The t^{th} line of the schedule gives us the general formulas for interest paid, principal repaid and outstanding balance for a loan of $a_{\overline{n}|}$.

$$\boxed{I_t = 1 - v^{n-t+1} \quad P_t = v^{n-t+1} \quad B_t = a_{\overline{n-t}|}}$$

In general, the formulas for a loan of L with equal payments, $R = \frac{L}{a_{\overline{n}|}}$ are

$$I_t = R(1 - v^{n-t+1})$$
$$P_t = Rv^{n-t+1}$$
$$B_t = Ra_{\overline{n-t}|}$$

Example 6.
A 1000 loan is being repaid by payments of 100 at the end of each quarter for as long as necessary, plus a smaller final payment. If the nominal rate of interest convertible quarterly is 16%, find the amount of principal and interest in the fourth payment. $\boxed{32.51, 67.49}$

Example 7.

The amount of principal repaid in the first payment of a loan being repaid by level payments at 6% is 200. What is the principal repaid in the 10^{th} payment?

Example 8.

The amount of principal repaid in the first payment of a 11-year loan being repaid by level payments at 5% is 100. What is the amount of loan?

Example 9.

The amount of principal repaid in the 5^{th} payment of a 12-year loan at 5% is 264. what is the original loan?

Example 10.

A loan of 1000 is being repaid at 8% effective by level annual payments of 125. What is the outstanding loan balance just after the 8^{th} payment?

$\boxed{521.35}$

Example 11. A loan of a certain amount is being repaid at an effective rate $i = 5\%$ by the following sechedule of payments:

Year	Payment
1,2,...,9,10	100,200,...,900,1000
11,12,13,14,...,17,18,19,20	500,500,600,600,...,800,800,900,900
21,22,23,24,...,39,40	700.400, 700,400,...,700,400
41-60	1,200

Compute the outstanding balance just after the 50^{th} payment. $\boxed{9266}$

Example 12.
Kevin takes out a 10-year loan of L, which he repays by the amortization method at an annual effective interest rate of i. Kevin makes payments of 1000 at the end of each year. The total amount of interest repaid during the life of the loan is also equal L. Calculate the amount of interest repaid during the first year of the loan. $\boxed{754.95}$

Example 13.
A loan is repaid by level payments of 1 at the end of each year for 10 years. At the time of the 4^{th} regular payment, the borrower makes and addition payment equal to the amount of principal that, according to the original payment schedule, would have been repaid in the 5^{th} regular payment. Payments of 1 continue to be made in the 5^{th} and succeeding years until the loan is fully repaid. Determine how much the borrower saved in interest payments over the term of the loan. $\boxed{1 - v^6}$

Example 14.
James takes out a 30-year loan, which is repaid with annual payments at the end of each year. he repays the loan by making payments which are equal to X during years $1 - 10$, $3X$ during year $11 - 20$, and $2X$ during years $21 - 30$. Interest is charged on the loan at an annual effective rate of i, $i > 0$. The amount of interest repaid during year 11 is twice as much as the amount of interest repaid during year 21. Calculate i.

Example 15.
A loan is to be amortized by n level annual payments of X, where $n > 5$. You are given:

- The amount of interest in the first payment is 604.00.
- The amount of interest in the third payment is 593.75.
- The amount of interest in the fifth payment is 582.45.

Calculate X. $\boxed{704}$

Example 16.
Luke is repaying a loan with payments of 4,100 at the end of every two years. If the amount of interest in the 7th installment is 3,957, find the amount of principal in the 12th installment. Assume an annual effective interest rate of 15%.

Example 17.

A loan of 81,000 is being repaid by 17 equal annual installments made at the end of each year at 9% interest effective annually. Immediately after the 6-th payment, the loan is renegotiated as follows:

- The borrower will make 11 annual payments of K to repay the loan, with the first payment three years from the date of renegotiation.
- The interest rate is changed to 10.5% effective annually.

Calculate K.

Example 18.

A loan is repaid in 19 equal annual installments. The first payment is due one year after the loan is made. The effective annual interest rate is 8%. The total amount of interest paid in the 3rd, 4th, and 5th payments combined is 14,858. What is the total amount of principal paid in the 17th, 18th, and 19th payments combined?

Example 19.

A loan of 700,000 is being amortized with payments at the end of each year for 10 years. If $v^5 = 0.915$, find the amount of principal repaid in the first 5 years.

Example 20.

A loan of L is to be repaid with 80 payments of 130 at the end of each quarter. Interest on the loan is charged at a nominal rate $i, 0 < i < 1$, convertible quarterly. The outstanding principal immediately after 66th and 73th payments are 1495.58 and 818.15, respectively. Calculate the amount of interest repaid in the 22th payment.

Example 21. A loan of amount L is to be repaid by ten annual payments beginning one year after the loan is made. The level payment in the final five years is double the level payment in the first five years. The principal outstanding just after the 7th payment is

A) $\frac{2La_{\overline{3}|}}{2a_{\overline{10}|}-a_{\overline{5}|}}$ B) $\frac{La_{\overline{3}|}}{a_{\overline{10}|}+a_{\overline{5}|}}$ C) $\frac{2L\ddot{a}_{\overline{3}|}}{\ddot{a}_{\overline{10}|}+\ddot{a}_{\overline{5}|}}$ D) $\frac{L\ddot{a}_{\overline{4}|}}{2a_{\overline{10}|}-a_{\overline{5}|}}$ E) $\frac{L(1-v^3)}{1+v^5-2v^{10}}$

5.5 Sinking Funds

Suppose the borrower will pay only the interest due at the end of each year and will repay the original loan amount at the end of n years. This method of payment is perfectly OK except that the lender may be worry that the lender may not be able to payback the original amount of loan. Also, the payment of a big lump-sum at the end of n years could be very disruptive to the borrower's financial position at that time. The borrower can make systematic deposit into a fund that eventually accumulate to the loan amount. In fact, the lender might insist on it to safeguard the return of the of the loan.

A fund that designed to accumulate a specified amount of money in a specified time by making regular deposits is called **sinking fund**.

For example, suppose the interest payments to the lender are at 6%, but the borrower can earn 5% on the. What is the total annual payment made by the borrower for a loan of 1000?

(1)	Interest payment to the lender at 6%	$= 0.06 \times 1000 = 60$
(2)	SF deposit $= \frac{1000}{s_{\overline{10}\rceil}}$ at 5%	$= 79.50$
	Total Annual Payment	139.50

In general, for a loan of L, an interest on loan of i and interest on SF deposit of j, we have:

$$\begin{aligned} \text{Total annual payment} &= iL + \frac{L}{s_{\overline{n}\rceil j}} \\ &= L\left(i + \frac{1}{s_{\overline{n}\rceil j}}\right) \end{aligned}$$

Alternatively, suppose we know the **Total Annual Payment**(TAP), we could express the Annual SF deposit as $(TAP - iL)$
To solve for L, we accumulate the SF deposit and equate it to L:

$$\boxed{(TAP - iL)s_{\overline{n}\rceil j} = L}$$

This is useful to solve for L in the case of non-level SF deposit.

5.5.1 General Approach to SF Problem

The general approach to SF problems is to answer two questions:

(1) What is the interest paid to the lender?

(2) What is the SF deposit?

The sum of (1) and (2) is the total periodic payment made by the borrower to repay the loan.

Example 22.

A borrower repays a loan by making annual interest payments to the lender at 5% and by making SF deposits at 4% for 19 years. The borrower pays a total of 59,522 annually. What is the amount of loan?

Example 23.

A borrower makes annual interest payments to the lender at 7% and SF deposits at 5% for 20 years. The total annual payment made by the borrower is 25,000 in the first 14 years and 13,000 in the last 6 years. What is the amount of the loan?

Example 24.

A borrower and a lender agree to the following arrangement: The borrower will pay annual interest to the lender for 15 years at 8%. The borrower will pay 150% of the original loan amount to the lender at the end of 15 years by making 5 annual deposits in a SF earning 6%. After making the 5 deposits, the SF grows with interest only. The total annual payment made by the borrower in the first 5 years is 13,000. What is the amount of the loan?

Example 25.

A 20-year loan of 20,000 may be repaid under the following two methods:

- amortization method with equal annual payments at an annual 1 effective rate of 6.5%.
- sinking fund method in which the lender receives an annual effective rate of 8.0% and the sinking fund earns an annual effective rate j.

Both methods require a payment of X to be made at the end of each year for 20 year. Calculate j.

Example 26.

Justin and Maggie each take out a 15-year loan L. Justin repays his loan using the amortization method, at an annual effective interest rate of i. He makes an annual payment of 500 at the end of each year. Maggie repays her loan using the sinking fund method. She pays interest annually, also at an effective interest rate of i. In addition, Maggie makes level annual deposits at the end of each year for 15 years into a sinking fund. The annual effective rate on the sinking fund is 4.47%, and she pays off the loan after 15 years. Maggie's total payment each year is equal to 10% of the original loan amount. Calculate L.

Example 27.

Suppose that the sinking fund method is used for the repayment of a loan of RM1,000, and that the annual effective rate of the loan and the annual effective rate of the sinking fund are both 8%. Suppose that the term of the loan is 5 years. Calculate the balance of the sinking fund immediately after each of the deposits. □

Example 28.
An investor buys an n-year annuity with a present value of RM1000 computed at 8%. The investor pays a price which will permit the replacement of the original investment in a sinking fund earning 7% and will also produce and overall yield rate of 9% on the entire transaction. Find the price which the investor should pay for the annuity.

5.5.2 Outstanding Balance, Interest Paid and Principal Repaid under the Sinking Fund Method

The amount in the SF at the end of year t is:

$$SF_t = \frac{L}{s_{\overline{n}|}} s_{\overline{t}|}$$

Interest earned on $(SF)_t = j(SF)_{t-1}$

Interest paid (net amount of interest to lender and interest earned on SF) is

$$I_t = iL - j(SF)_{t-1}$$

The principal repaid under any method of repaying a loan is equal to the decrease in the outstanding balance. So the principal repaid in the t^{th} period under the SF method can be determined by either of the two methods, both of which represent the increase in the SF:

(1) The interest earned on the SF in the t^{th} period plus the deposit made at the end of that period, i.e.

$$P_t = j(SF)_{t-1} + (SF)$$

(2) or, the AV of the SF at the end of the t^{th} period minus the AV at the end of $(t-1)^{th}$, i.e.

$$P_t = \frac{L}{s_{\overline{n}|j}}(s_{\overline{t}|j} - s_{\overline{t-1}|j})$$

Summary:

- $SF_t = \frac{L}{s_{\overline{n}|j}} s_{\overline{t}|j}$
- Interest earned on $(SF)_t = j(SF)_{t-1}$
- $I_t = iL - j(SF)_{t-1}$
- $P_t = j(SF)_{t-1} + (SF) = \frac{L}{s_{\overline{n}|j}}(s_{\overline{t}|j} - s_{\overline{t-1}|j})$

Example 29.

Consider a 10-year loan of 1,000 is to be repaid with payments at the end of each year consisting of interest on the loan and a sinking fund deposit. Interest on the loan is charged at a 6% annual effective rate. The sinking fund's annual effective interest rate is 6%. Calculate

(a) The amount in the SF at the end of year 2, $(SF)_2$.

(b) I_3.

(c) P_3.

Example 30.

John borrows X and repays the principal by making 12 annual payments at the end of each year into a sinking fund which earns an annual effective rate of 9%. The interest earned on the sinking fund in the 4th year is 116.75. Calculate X.

Example 31.

A corporation borrows 12,000 for 34 years, at an effective annual rate of 6%. A sinking fund is used to accumulate the principal by means of 34 annual deposits earning an effective annual interest rate of 5%. Calculate the sum of the net amount of interest paid in the 15th installment and the increment in the sinking fund for the 10th year.

5.6 Varying Series of Payments

5.6.1 Amortization

If a loan is being repaid by the amortization method, it is possible that the borrower repays the loan with instalments which are not level. In fact, any series of payments whose present value is equal to the loan amount will repay it.
Assume that the interest conversion period and the payment period are equal and coincide. Consider a loan L to be repaid with n periodic instalments $R_1, \cdots, R_n$. Then we have

$$L = \sum_{i=1}^{n} v^t R_t.$$

We can also use the recursion formulas:

$$I_t = iB_{t-1}$$
$$P_t = R_t - I_t$$
$$B_t = B_{t-1} - P_t$$

Example 32.
A loan of 10,000 at 10% effective is being repaid by payments of 2,000, 3,000, and 7,590 at the end of years 1, 2, and 3, respectively. Construct an amortization schedule for this loan.

Remark: When non-level payments are made, it is possible that one or more of the payments will be less than the interest. Thus, we "owe" the amortization schedule "$I_t - R_t$". We can regard this as an additional loan. This is called **negative amortization**.

Example 33.

A loan of 10,000 at 10% effective is being repaid by payments of 600, 5,000, and 7,084 at the end of years 1, 2, and 3, respectively. Construct an amortization schedule for this loan.

Example 34.

A borrower is repaying a loan at 5% effective with payments at the end of each year for 10 years, such that the first year,s payment is 200, the second year 190.0, and so forth, until the 10th year it is 110.0. Find the amount of the loan.

Example 35.

A borrower is repaying a loan at 8% effective with payments at the end of each year for 11 years, such that the first year,s payment is 1000, the second year 950.0, and so forth, until the 11th year it is 500.0. Find the principal and interest in the 5th payment.

Example 36.

Don takes out a 28-year loan of L, which repays with annual payments at the end of each year using the amortization method. Interest on the loan is charged at an annual effective rate of i. Don repays the loan with a decreasing series of payments. He repays 2,800 in year one, 2,700 in year two, 2,600 in year three, ..., and 100 in year 28. The amount of principal repaid in year three is equal to 1091.46. Calculate L.

5.6.2 Sinking Funds

Assume that the varying payments with the sinking fund method and the interest paid to the lender is constant each period so that only the sinking fund deposits vary. Assume that the varying payments by the borrower are $R_1, \cdots, R_n$ and $i \neq j$. The amount of the loan is

$$
\begin{aligned}
&L \\
&= (R_1 - iL)(1+j)^{n-1} + (R_2 - iL)(1+j)^{n-2} + \cdots + (R_n - iL) \\
&= \sum_{t=1}^{n} R_t(1+j)^{n-t} - iLs_{\overline{n}|j} \\
&= \frac{\sum_{t=1}^{n} v_j^t R_t}{1+(i-j)a_{\overline{n}|j}}
\end{aligned}
$$

Remark:

It should be noted that we have implicitly assumed that the sinking fund deposit $R_t - iL$ is positive. If it were negative, then it would mean that the payment in that year is not even sufficient to pay the interest on the loan. We would then have a negative sinking fund deposit, i.e. a withdrawal, from the sinking fund for that year.

Example 37. A borrows a certain amount of money L from B for 15 years. He pays B interest every year on L at an effective rate of 7% for the first 10 years and 6% for the last 5 years. A also makes deposits in a SF that will accumulate to L at the end of 15 years. The SF earn 5% during the first 10 years and 4% thereafter. A's total anuual payments is 10,000 in the first 10 years and 8,000 in the last 5 years. Determine L.

5.7 Equal Principal Repayments

One particular way of repaying a loan that sometimes comes up is to pay a **level amount of principal** and **interest on the outstanding balance** at the end of each year.

Paying a level amount of principal is different from making level payments. In fact the loan payments decrease each year.

```
             L
             |------|-------------|----...----|
             0      1             2          10
Pincipal:           P             P
Interest:           I1 = Li       I2= (L-P)i
Loan Payment: P+I1 = P+Li   P+I2=P+Li-Pi
Outstanding Balance: L-P    L - 2P
```

Example 38.
A loan of 1,000 is repaid by equal annual amounts of principal for 10 years and annual interest of 7% on the outstanding balance.

(a) What is the schedule of loan payments?

(b) What is the PV of the payments in (a) at 7% effective?

(c) What is the purchase price of this loan to yield 5% effective?

Example 39.

A 14-year loan of 5600 is to be repaid with payments at the end of each year. It can be repaid under the following two options:

(i) Equal annual payments at an annual effective rate of 5.92%.

(ii) Installments of 400.0 each year plus interest on the unpaid balance at an annual effective rate of i.

The sum of the payments under option (i) equals the sum of the payments under option (ii). Determine i.

Example 40.

A loan of 2400 is being repaid in 12 years by semiannual installments of 100.0, plus interest on the unpaid balance at 7% per annum compounded semiannually. The installments and interest payments are reinvested at 8% per annum compounded semiannually. Calculate the annual effective yield rate of the loan.

Example 41.

Annie borrows 24,480 from Bank X. Annie repays the loan by making 36 equal payments of principal at the end of each months. She also pays interest on the unpaid balance each month at a nominal rate of 12%, compounded monthly. Immediately after the 19th payment is made, Bank X sells the rights to future payments to Bank Y. Bank Y wishes to yield a nominal rate of 12%, compounded semiannually, on its investment. What price does Bank X receive?

Example 42.

A borrows 20,000 from B and agrees to repay it with 20 equal annual instalments of principal plus interest on the unpaid balance at 3% effective. After 10 years B sells the right to future payments to C, at a price that yields C 5% effective over the remaining 10 years. Find the price which C should pay to the nearest ringgit.