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## 3 Properties of a Random Sample

### 3.1 Convergence in Probability

In this section, we formalize a way of saying that a sequence of random variables  $\{X_n\}$  is getting “close” to another random variable  $X$ , as  $n \rightarrow \infty$ .

**Definition 1.** Let  $\{X_n\}$  be a sequence of random variables and let  $X$  be a random variable defined on a sample space. We say that  $\{X_n\}$  converges in probability to  $X$  if, for all  $\epsilon > 0$ ,

$$\lim_{n \rightarrow \infty} P[|X_n - X| \geq \epsilon] = 0,$$

or equivalently

$$\lim_{n \rightarrow \infty} P[|X_n - X| < \epsilon] = 1,$$

If so, we write

$$X_n \xrightarrow{P} X.$$

One way of showing convergence in probability is to use Chebyshev's Theorem.

**Theorem 1.** If  $X$  is a random variable and  $u(x)$  is a nonnegative real-valued function, then for any positive constant  $c > 0$ .

$$P[u(X) \geq c] \leq \frac{E[u(X)]}{c}$$

A special case, known as the **Markov inequality**, is obtained if  $u(x) = |x|^r$  for  $r > 0$ , namely

$$P[|X| \geq c] \leq \frac{E[|X|^r]}{c^r}$$

**Theorem 2. Chebychev inequality** If  $X$  is a random variable with mean  $\mu$  and variance  $\sigma^2$ , then for any  $k > 0$ ,

$$P[|X - \mu| \geq k\sigma] < \frac{1}{k^2}$$

An alternative form is

$$P[|X - \mu| < k\sigma] \geq 1 - \frac{1}{k^2}$$

and if we let  $\epsilon = k\sigma$ , then

$$P[|X - \mu| < \epsilon] \geq 1 - \frac{\sigma^2}{\epsilon^2}$$

and

$$P[|X - \mu| \geq \epsilon] \leq \frac{\sigma^2}{\epsilon^2}$$

**Example 1.**

Suppose that  $X$  is a random variable for which  $E(X) = 19$ ,  $P(X \leq 15) = 0.16$ , and  $P(X \geq 23) = 0.32$ . Prove that  $V(X) > c$  and identify  $c$ .

**Theorem 3.** (Weak Law of Large Numbers). Let  $\{X_n\}$  be a sequence of iid random variables having common mean  $\mu$  and variance  $\sigma^2$ . Let  $\bar{X}_n = \frac{\sum_{i=1}^n X_i}{n}$ . Then

$$\bar{X}_n \xrightarrow{P} \mu.$$

**Example 2.**

Let  $X_1, \dots, X_n$  denote a random sample from a distribution with mean  $\mu$  and variance  $\sigma^2$ . Assume that  $E[X_i^4] < \infty$ . Show that  $\frac{\sum_{i=1}^n X_i^2}{n}$  converges in probability to  $E(X_i^2)$ .

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**Theorem 4.** Suppose  $X_n \xrightarrow{P} X$  and  $Y_n \xrightarrow{P} Y$   
Then  $X_n + Y_n \xrightarrow{P} X + Y$ .



**Theorem 5.** Suppose  $X_n \xrightarrow{P} X$  and  $a$  is a constant. Then  $aX_n \xrightarrow{P} aX$ .

**Theorem 6.** Suppose  $X_n \xrightarrow{P} a$  and the real function  $g$  is continuous at  $a$ . Then  $g(X_n) \xrightarrow{P} g(a)$ .

**Theorem 7.** Suppose  $X_n \xrightarrow{P} X$  and the real function  $g$  is continuous at  $a$ . Then  $g(X_n) \xrightarrow{P} g(X)$

**Example 3.**

Suppose  $X_n \xrightarrow{P} X$  and  $Y_n \xrightarrow{P} Y$ . Then  $X_n Y_n \xrightarrow{P} XY$ .

**Example 4.**

Consider a random sample from a Poisson distribution,  $X_i \sim POI(\mu)$ . Show that  $Y_n = e^{-\bar{X}_n}$  converges in probability to a constant, identify the constant.

**Definition 2.** (Consistency). Let  $X$  be a random variable with cdf  $F(x, \theta)$ ,  $\theta \in \Omega$ . Let  $X_1, \dots, X_n$  be a sample from the distribution of  $X$  and let  $T_n$  denote a statistic. We say  $T_n$  is a consistent estimator of  $\theta$  if

$$T_n \xrightarrow{P} \theta.$$

**Example 5.**

Let  $X_1, \dots, X_n$  denote a random sample from a distribution with mean  $\mu$  and variance  $\sigma^2$ . Assume that  $E[X_i^4] < \infty$ , so that  $V(S^2) < \infty$ . Show that  $S_n^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2$  converges in probability to  $\sigma^2$ .

## 3.2 Convergence in Distribution

**Definition 3.** Let  $X_n$  be a sequence of random variables and let  $X$  be a random variable. Let  $F_{X_n}$  and  $F_X$  be, respectively, the cdfs of  $X_n$  and  $X$ . Let  $C(F_X)$  denote the set of all points where  $F_X$  is continuous. We say that  $X_n$  **converges in distribution** to  $X$  if

$$\lim_{n \rightarrow \infty} F_{X_n} = F_X, \forall x \in C(F_X).$$

We denote this convergence by

$$X_n \xrightarrow{D} X.$$

Notes:

The material on convergence in probability and in distribution comes under what statisticians and probabilists refer to as asymptotic theory. Often, we say that the distribution of  $X$  is the asymptotic distribution or the limiting distribution of the sequence  $\{X_n\}$ .

**Definition 4.** The function  $F_X$  is the CDF of a **degenerate distribution** at value  $x = c$  if

$$F_X = \begin{cases} 0, & x < c \\ 1, & x \geq c \end{cases}$$

In other words,  $F_X$  is the CDF of a discrete distribution that assigns probability one at the value  $x = c$  and zero otherwise.

**Notes:** The following limits are useful in many problems:

1.  $\lim_{n \rightarrow \infty} \left(1 + \frac{c}{n}\right)^{nb} = e^{cb}$
2.  $\lim_{n \rightarrow \infty} \left(1 + \frac{c}{n} + \frac{d(n)}{n}\right)^{nb} = e^{cb}$  if  $\lim_{n \rightarrow \infty} d(n) = 0$

**Example 6.**

Let  $X_1, \dots, X_n$ , be a random sample from a uniform distribution,  $X \sim U(0, 1)$ , and let  $Y_n = X_{n:n}$  the largest order statistic. Find the limiting distribution of  $Y_n$ .



**Example 7.**

Suppose that  $X_1, \dots, X_n$ , is a random sample from a Pareto distribution,  $X \sim PAR(\alpha = 1, \theta = 25)$ . Let  $Y_n = 1/n X_{n:n}$ , find the limiting distribution of  $Y_n$ ,  $F(y)$ , state the distribution and it's parameter, then find  $F(20.2)$ .

**Example 8.**

Let  $Y_3$  denote the third smallest item of a random sample of size  $n$  from a distribution of the continuous type that has cdf  $F_X(x)$  and pdf  $f_X(x) = F'_X(x)$ . Find the limiting distribution of  $W_n = nF_{Y_3}(y)$ .

**Theorem 8.**

If  $X_n$  converges to  $X$  in probability, then  $X_n$  converges to  $X$  in distribution.

**Theorem 9. Slutsky's Theorem**

If  $X_n$  and  $Y_n$  are two sequences of random variables such that  $X_n \xrightarrow{P} c$  and  $Y_n \xrightarrow{D} Y$ , then :

1.  $X_n + Y_n \xrightarrow{D} c + Y$

2.  $X_n Y_n \xrightarrow{D} cY$

3.  $X_n/Y_n \xrightarrow{D} c/Y$

**Theorem 10.**

If  $X_n \xrightarrow{D} X$ , then for any continuous function  $g(x)$ ,  $g(X_n) \xrightarrow{D} g(X)$ . Note that  $g(x)$  is assumed not to depend on  $n$ .

**Example 9.**

Consider a random sample of size  $n$  from a Bernoulli distribution,  $X_i \sim \text{Bin}(1, p)$ .

(a) Show that  $\hat{p} = \frac{\sum_{i=1}^n X_i}{n} \xrightarrow{P} p$ .

(b) Show that  $\hat{p}(1 - \hat{p}) \xrightarrow{P} p(1 - p)$ .

(c) We know that  $\frac{\hat{p}-p}{\sqrt{p(1-p)/n}} \xrightarrow{D} Z \sim N(0, 1)$ ,  
find the limiting distribution of  $\frac{\hat{p}-p}{\sqrt{\hat{p}(1-\hat{p})/n}}$ .

### 3.3 Moment Generating Function Technique

To find the limiting distribution function of a random variable  $X_n$  by using the definition obviously requires that we know  $F_{X_n}(x)$  for each positive integer  $n$ . But it is often difficult to obtain  $F_{X_n}(x)$  in closed form. Fortunately, if it exists, the mgf that corresponds to the cdf  $F_{X_n}(x)$  often provides a convenient method of determining the limiting cdf.

**Theorem 11.** Let  $\{X_n\}$  be a sequence of random variables with mgf  $M_{X_n}(t)$  that exists for  $-h < t < h$  for all  $n$ . Let  $X$  be a random variable with mgf  $M(t)$ , which exists for  $|t| < h_1 < h$ . If  $\lim_{n \rightarrow \infty} M_{X_n}(t) = M(t)$  for  $|t| < h_1$ , then  $X_n \xrightarrow{D} X$ .

**Example 10.**

Let  $Y_n$  have a distribution that is  $\text{Bin}(n, p)$ . Suppose that the mean  $\mu = np$  is the same for every  $n$ ; that is,  $p = \mu/n$ , where  $\mu$  is a constant. Find the limiting distribution of  $Y_n$  using moment generating function technique.

**Example 11.**

Let  $Y_n \sim GAM(\alpha = n, \theta = 2)$ . Find the limiting distribution of  $\frac{Y_n - n}{\sqrt{2n}}$  as  $n \rightarrow \infty$ , using moment generating function.

**Theorem 12.**

**Central Limit Theorem (CLT)** If  $X_1, \dots, X_n$ , is a random sample from a distribution with mean  $\mu$  and variance  $\sigma^2 < \infty$ , then the limiting distribution of

$$Z_n = \frac{\sum_{i=1}^n X_i - n\mu}{\sqrt{n}\sigma}$$

is the standard normal,  $Z_n \rightarrow Z \sim N(0, 1)$  as  $n \rightarrow \infty$ .



**Example 12.** Let  $X_1, X_2, \dots, X_{100}$  be a random sample from an exponential distribution,  $X_i \sim EXP(1)$ , and let  $Y = X_1 + X_2 + \dots + X_{100}$ .

- (a) Give an approximation for  $P[Y > 110]$ . [0.1587](#)
- (b) If  $\bar{X}$  is the sample mean, then approximate  $P[1.1 < \bar{X} < 1.2]$ . [0.1359](#)

**Example 13.**

Let  $X_i \sim U(25, 68)$ , where  $X_1, X_2, \dots, X_{74}$  are independent. Find normal approximation for

$$P \left[ \sum_{i=1}^{74} X_i \leq 3450.0 \right] .$$

**Theorem 13.  $\Delta$ –Method**

If  $\frac{\sqrt{n}(X_n - m)}{c} \xrightarrow{D} Z \sim N(0, 1)$ , and if  $g(x)$  has a nonzero derivative at  $x = m$ ,  $g'(m) \neq 0$ , then

$$\frac{\sqrt{n}[g(X_n) - g(m)]}{|cg'(m)|} \xrightarrow{D} Z \sim N(0, 1)$$

In other words, for large  $n$ , if  $X_n \sim N(m, c^2/n)$ , then approximately

$$g(X_n) \sim N \left( g(m), \frac{c^2[g'(m)]^2}{n} \right)$$

**Example 14.**

Consider a random sample from a Poisson distribution,  $X_i \sim POI(\mu)$ . Find the asymptotic normal distribution of  $Y_n = e^{-\bar{X}_n}$ .

### 3.4 Parameter and Statistic

Consider a set of observable random variables  $X_1, \dots, X_n$ . For example, suppose the variables are a random sample of size  $n$  from a population.

**Definition 5.** A **parameter** is a numerical summary that would be calculated from all of the units in the population.

**Definition 6.** A function of observable random variables,  $T = t(X_1, \dots, X_n)$ , which does not depend on any unknown parameters, is called a **statistic**.

In other words, a **statistic** is a numerical summary that is calculated from all of the units in a sample.

**Theorem 14.** If  $T = t(X_1, \dots, X_n)$ , denotes a random sample from  $f(x)$  with  $E(X) = \mu$  and  $V(X) = \sigma^2$  then.

$$E(\bar{X}) = \mu$$

and

$$V(\bar{X}) = \frac{\sigma^2}{n}$$

### 3.5 Sampling Distributions

A statistic is also a random variable, the distribution of which depends on the distribution of a random sample and on the form of the function  $t(x_1, x_2, \dots, x_n)$ . The distribution of a statistic sometimes is referred to as a **derived distribution** or **sampling distribution**, in contrast to the population distribution.

### 3.5.1 Linear Combinations of Normal Variables

**Theorem 15.** If  $X_i \sim N(\mu_i, \sigma_i^2)$ ;  $i = 1, \dots, n$  denote independent normal variables, then

$$Y = \sum_{i=1}^n a_i X_i \sim N \left( \sum_{i=1}^n a_i \mu_i, \sum_{i=1}^n a_i^2 \sigma_i^2 \right)$$

#### **Theorem 16.**

Let  $X_1, X_2, \dots, X_n$  random sample of size  $n$  from a normal distribution with mean  $\mu$  and variance  $\sigma^2$ . Then

$$\bar{X} = \frac{\sum_{i=1}^n X_i}{n}$$

is normally distributed with mean  $\mu_{\bar{X}} = \mu$  and variance  $\sigma_{\bar{X}}^2 = \frac{\sigma^2}{n}$ .

### 3.6 Chi-Square Distribution

**Definition 7.** The variable  $Y$  is said to follow a chi-square distribution with  $v$  degrees of freedom if

$$Y \sim GAM(\alpha = \frac{v}{2}, \theta = 2).$$

A special notation for this is

$$Y \sim \chi^2(v)$$

**Theorem 17.** If  $Y \sim \chi^2(v)$ , then

- $M_Y(t) = (1 - 2t)^{-v/2}$
- $E(Y^r) = 2^r \frac{\Gamma(v/2+r)}{\Gamma(v/2)}$



**Theorem 18.** If  $X \sim GAM(\alpha, \theta)$ , then

$$Y = \frac{2X}{\theta} \sim \chi^2(2\alpha).$$

**Example 15.** The time to failure (in years) of a certain type of component follows a gamma distribution with  $\alpha = 2$  and  $\theta = 3$ . It is desired to determine a guarantee period for which 90% of the components will survive. Find the guarantee period.

**Theorem 19.** If  $Y_i \sim \chi^2(v_i)$ ;  $i = 1, \dots, n$  are independent chi-square variables, then

$$V = \sum_{i=1}^n Y_i \sim \chi^2 \left( \sum_{i=1}^n v_i \right)$$

**Theorem 20.** If  $Z \sim N(0, 1)$ , then  $Z^2 \sim \chi^2(1)$ .

**Theorem 21.** If  $X_1, \dots, X_n$  denotes a random sample of size  $n$  from  $N(\mu, \sigma^2)$ , then

$$\frac{\sum_{i=1}^n (X_i - \mu)^2}{\sigma^2} \sim \chi^2(n)$$

$$\frac{n(\bar{X} - \mu)^2}{\sigma^2} \sim \chi^2(1)$$

**Theorem 22.** If  $X_1, \dots, X_n$  denotes a random sample from  $N(\mu, \sigma^2)$ , then

- (i)  $\bar{X}$  and the terms  $X_i - \bar{X}$ ,  $i = 1, \dots, n$  are independent.
- (ii)  $\bar{X}$  and  $S^2$  are independent.
- (iii)  $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$ .

**Example 16.** Let  $X$  represent the lifetime in months of a battery, and assume that approximately  $X \sim N(60, 36)$ . Suppose that it was decided to sample 25 batteries, and to reject the claim that  $\sigma^2 = 36$  if  $S^2 \geq 54.63$ , and not reject the claim if  $S^2 < 54.63$ . Under this procedure, what would be the probability of rejecting the claim when in fact  $\sigma^2 = 36$ ?

### 3.7 Student's $t$ Distributions

**Theorem 23.** If  $Z \sim N(0, 1)$  and  $V \sim \chi^2(v)$ , and if  $Z$  and  $V$  are independent, then the distribution of

$$T = \frac{Z}{\sqrt{V/v}}$$

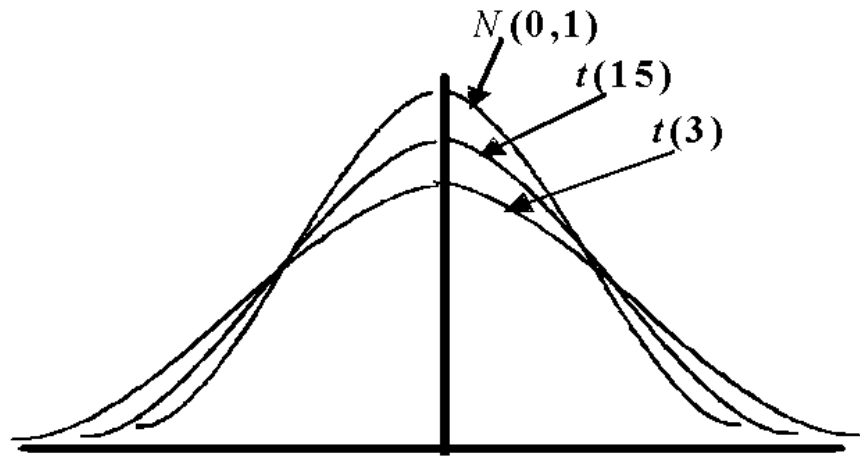
is referred to as **Student's  $t$  distribution** with  $v$  degrees of freedom, denoted by  $T \sim t(v)$ . The pdf is given by

$$f(t) = \frac{\Gamma(\frac{v+1}{2})}{\Gamma(\frac{v}{2})} \frac{1}{\sqrt{v\pi}} \left(1 + \frac{t^2}{2}\right)^{-(v+1)/2}$$



The  $t$  distribution is symmetric about zero, and its general shape is similar to that of the standard normal distribution. Indeed, the  $t$  distribution approaches the standard normal distribution as  $v \rightarrow \infty$ . For smaller  $v$  the  $t$  distribution is flatter with thicker tails and, in fact,  $T \sim CAU(1, 0)$  when  $v = 1$ .

### Various T-distributions



**Theorem 24.** If  $X_1, \dots, X_n$  denotes a random sample from  $N(\mu, \sigma^2)$  then

$$\frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t(n - 1)$$

**Example 17.**

Assume that  $Z$ ,  $V_1$ , and  $V_2$  are independent random variables with  $Z \sim N(0, 1)$ ,  $V_1 \sim \chi^2(5)$ , and  $V_2 \sim \chi^2(9)$ . Find the following:

- (a)  $P[V_1 + V_2 < 8.6]$ .
- (b)  $P[Z/\sqrt{V_1/5} < 2.015]$ .
- (c)  $P[Z > 0.611\sqrt{V_2}]$ .

### 3.8 Snedecor's F Distribution

**Theorem 25.** If  $V_1 \sim \chi^2(v_1)$  and  $V_2 \sim \chi^2(v_2)$  are independent, then the random variable

$$X = \frac{V_1/v_1}{V_2/v_2}$$

has the following pdf for  $x > 0$ :

$$f(x) = \frac{\Gamma(\frac{v_1+v_2}{2})}{\Gamma(\frac{v_1}{2})\Gamma(\frac{v_2}{2})} \left(\frac{v_1}{v_2}\right)^{v_1/2} \left(1 + \frac{v_1}{v_2}x\right)^{-(v_1+v_2)/2}$$

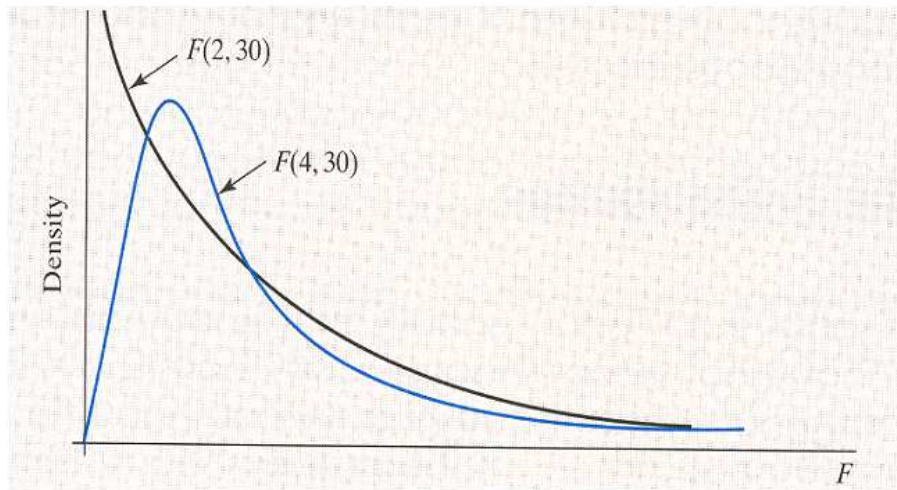
This is known as Snedecor's  $F$  distribution with  $v_1$  and  $v_2$  degrees of freedom, and is denoted by  $X \sim F(v_1, v_2)$ .

#### Properties of the $F$ -distribution

- The total area under the curve is one (as it is a density curve).
- The distribution is skewed to the right.
- The values are non-negative, start at zero, extend to the right—the curve approaches, but never touches, the horizontal axis.

- A different  $F$ -distribution for each different set of degrees of freedom.

### Various F-distributions



**Example 18.** If we take independent samples of size  $n_1 = 6$  and  $n_2 = 10$  from two normal populations with equal population variances, find  $b$  such that  $P\left(\frac{S_1^2}{S_2^2} \leq b\right) = 0.95$

**Example 19.**

Suppose that  $X_i \sim N(\mu, \sigma^2)$ ,  $i = 1, \dots, 15$ ,  $Z_j \sim N(0, 1)$ ,  $j = 1, \dots, 6$ , and  $W_k \sim \chi^2(2)$ ,  $k = 1, \dots, 14$  and all random variables are independent.

(a) Let  $Y_1 = \frac{5 \sum_{i=1}^{15} (X_i - \bar{X})^2}{14\sigma^2 \sum_{j=1}^6 (Z_j - \bar{Z})^2}$ , find  $P[Y_1 \leq 1.75]$ .

(b) Let  $Y_2 = \frac{5 \sum_{k=1}^6 W_k}{12 \sum_{j=1}^6 (Z_j - \bar{Z})^2}$ , find  $P(Y_2 \leq 1.73)$ .

(c) Let  $Y_3 = \frac{\sqrt{30}(\bar{X} - \mu)}{\sigma\sqrt{W_1}}$ , find  $P(Y_3 \leq 0.164)$ .

### 3.9 Beta Distribution

#### Theorem 26.

If  $X$  and  $Y$  be independent random variables with  $X \sim GAM(\alpha_1, 2)$  and  $Y \sim GAM(\alpha_2, 2)$ , then  $U = \frac{X}{X+Y} \sim Beta(a = \alpha_1, b = \alpha_2)$ .



An  $F$  variable can be transformed to have the beta distribution. IF  $X \sim F(v_1, v_2)$  then the random variable

$$Y = \frac{(v_1/v_2)X}{1 + (v_1/v_2)X} \sim \text{Beta}(a = \frac{v_1}{2}, b = \frac{v_2}{2})$$

The pdf of  $Y$  is

$$f(y) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} y^{a-1} (1-y)^{b-1}, 0 < y < 1$$

The  $k^{th}$  raw moment of  $Y$  is

$$E(Y^k) = \frac{a(a+1) \cdots (a+k-1)}{(a+b)(a+b+1) \cdots (a+b+k-1)}$$

**Example 20.**

Suppose  $Y \sim \text{Beta}(a = 8, b = 8)$ , use the relationship between Beta distribution and  $F$  distribution, find  $P[Y > 0.322]$ .

**Example 21.**

Suppose  $Y \sim \text{Beta}(a = 8, b = 6)$ , use the relationship between Beta distribution and  $F$  distribution, find 90<sup>th</sup> percentile of  $Y$ .

**Example 22.**

Suppose that  $X_i \sim N(\mu, \sigma^2), i = 1, \dots, 18, Z_j \sim N(0, 1), j = 1, \dots, 6$ , and  $W_k \sim \chi^2(v), k = 1, \dots, 17$  and all random variables are independent. State the distribution of each of the following variables if it is a "named" distribution. [For example  $X_1 + X_2 \sim N(2\mu, 2\sigma^2)$ ]

$$(a) \frac{Z_i^2/W_1}{1+Z_1^2/W_1}$$

$$(b) \frac{\frac{\sum_{k=1}^6 W_k}{\sum_{j=1}^6 (Z_j - \bar{Z})^2}}{1 + \frac{\sum_{k=1}^6 W_k}{\sum_{j=1}^6 (Z_j - \bar{Z})^2}}$$

$$(c) \frac{W_1}{W_1 + W_2 + W_3 + W_4}$$