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CONTENTS

2 Distributions of Functions of a Random Variable

If X is a random variable(r.v.) with cdf $F_X(x)$, then any function of X, $g(X)$ is also a r.v.. We denoted $U = g(X)$ as a new r.v. Since U is a function of X , we can describe the probabilistic behavior of U in terms of X, i.e.

$$P(U \in A) = P(g(X) \in A),$$

which shows that the distribution of U depends on the functions F_X and g.

2.1 The CDF Technique

We will assume that a random variable X has CDF $F_X(x)$ and some functions of X is of interest, say $U = g(X)$. Specifically, for each real u, we can define a set $A_u = \{x|g(X) \leq u\}$. It follows that $[U \leq u]$ and $[x \in A_u]$ are equivalent events, and consequently

$$f_U(u) = P[g(x) \leq u]$$

The probability can be expressed as the integral of the pdf, $f_X(x)$, over the set A_u if X is continuous, or the summation of $F_X(x)$ over x in A_x if X is discrete.
Summary of the CDF technique:
Let U be a function of the random variables $X_1, \ldots, X_n$

1. Find the region $U = u$ in the $(X_1, \ldots, X_n)$ space.
2. Find the region $U \leq u$.
3. Find $F_U(u) = P(U \leq u)$ by integrating $f(X_1, \ldots, X_n)$ over the region $U \leq u$ in the continuous case.
4. Find the density function $f_U(u)$ by differentiating $F_U(u)$. Thus $f_U(u) = dF_U(u)/du$.

Example 1.
Suppose that X has density function given by

$$f_X(x) = \begin{cases} 2x, & 0 \leq x \leq 1 \\ 0, \text{otherwise} & \end{cases}$$

If $U = 3X - 1$, find the probability density function for U.

Example 2.

Suppose $F_X(x) = 1 - e^{-2x}, x > 0$. Find the pdf of $U = e^X$.

Example 3.

Suppose $X \sim N(\mu, \sigma^2)$. Find the distribution of $U = e^X$.

Example 4.
The joint density function of X_1 and X_2 is given by

$$f(x_1, x_2) = \begin{cases} 3x_1, & 0 \le x_2 \le x_1 \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the probability density function for $U = X_1 - X_2$.

2.2 Transformation Methods

Let $u(x)$ be a real-value function of a real variable x. If the equation $u = g(x)$ can be solved uniquely, say $x = w(u)$, then we say the transformation is one-to-one.

2.2.1 One-To-One Transformation

Theorem 1. Discrete Case Suppose that X is a discrete random variable with pdf $f_X(x)$ and that $U = g(X)$ defines a one-to-one transformation. In other words, the equation $u = g(x)$ can be solved uniquely, say $x = w(u)$. The the pdf of U is

$$f_U(u) = f_X(w(u)), u \in B$$

where $B = \{u | f_U(u) > 0\}$.

Example 5.
Let $X \sim GEO(p)$ so that

$$f_X(x) = pq^{x-1} \quad x = 1, 2, 3, \ldots$$

Suppose $U = X - 1$. Find the pdf of U.

Theorem 2. Continuous Case Suppose that X is a continuous random variable with pdf $f_X(x)$ and assume that $U = g(X)$ defines a one-to-one transformation from $A = \{x | f_X(x) > 0\}$ on to $B = \{u | f_U(u) > 0\}$ with inverse transformation $x = w(u)$. If the derivative $\frac{dw(u)}{du}$ is continuous and nonzero on B, then the pdf of U is

$$f_U(u) = f_X(w(u)) \left| \frac{dw(u)}{du} \right|, u \in B$$

Example 6.
Let X have the probability density function given by

$$f_X(x) = \begin{cases} 2x, & 0 \leq x \leq 1, \\ 0, & \text{otherwise} \end{cases}$$

Find the density function of $U = -4X + 3$.

Theorem 3.
Probability Integral Transformation If X is continuous with CDF $F(x)$, then $U = F(x) \sim U(0, 1)$,

Example 7.
If $X \sim Exp(\theta)$, find a random variable U such that $U \sim U(0,1)$.

Example 8.
If $X \sim N(0,1)$, find a random variable U such that $U \sim U(0,1)$.

Theorem 4.
Inverse Probability Integral Transformation

Let $F(x)$ be a continuous cumulative distribution function, and let F^{-1} be its inverse function such that $F^{-1}(u) = \min\{x|F(x) \geq u\} \quad 0 < u < 1$. If $U \sim U(0,1)$, then $F^{-1}(U)$ has F as its CDF.

Example 9.

Let U be a uniform random variable on the interval $(0,1)$. Find a transformation $G(U)$ such that $G(U)$ possesses an exponential distribution with mean θ.

Example 10.
Let X be a continuous random variable with pdf

$$f(x) = \begin{cases} \frac{1}{2}, & 1 < |x-2| < 2 \\ 0, & \text{otherwise} \end{cases}$$

Find $G(u)$.

The Inverse Probability Integral Transformation also call the Inverse Transform Sampling. It works as follows:

1. Generate a random number u from $U \sim U[0,1]$.
2. Find the inverse of the desired CDF, e.g. $F_X^{-1}(x)$.
3. Compute $X = F_X^{-1}(u)$. The computed random variable X has distribution $F_X(x)$.

Example 11.

A member of the power family of distributions has a distribution function given by

$$F(x) = \begin{cases} 0, & x < 0 \\ (\frac{x}{\theta})^{\alpha}, & 0 \le x \le \theta \\ 1, & x > \theta \end{cases}$$

where $\alpha, \theta > 0$.

(a) For fixed values of α and θ, find a transformation $G(U)$ so that $G(U)$ has a distribution function of F when U possesses a uniform $(0,1)$ distribution.

(b) Given that a random sample of size 5 from a uniform distribution on the interval $(0,1)$ yielded the values:

$$u_1 = 0.027,\ u_2 = 0.06901,\ u_3 = 0.01413,$$
$$u_4 = 0.01523, \text{ and } u_5 = 0.03609,$$

use the transformation derived in the above result to give values associated with a random variable with a power family distribution with $\alpha = 2$, $\theta = 4$.

2.2.2 Transformations That Are Not One-To-One

Suppose that the function $g(x)$ is not one-to-one over $A = \{x : f(x) > 0\}$. Although this means that no unique solution to the equation $u = w(x)$ exists, it usually is possible to partition A into disjoint subsets $A_1, A_2, \ldots$ such that $u(x)$ is one-to-one over each A_j. Then, for each u in the range of $w(x)$, the equation $u = g(x)$ has a unique solution $x = w(u)$ over the set A_j. In the discrete case,

$$f_U(u) = \sum_j f_X(w_j(u))$$

In the continuous case,

$$f_U(u) = \sum_j f_X(w_j(u)) \left|\frac{dw_j(u)}{du}\right|$$

Example 12. Let $f(x) = \frac{4}{31}(\frac{1}{2})^x$, $x = -2, -1, 0, 1, 2$, and consider $U = |X|$. Find the pdf of U.

Example 13. Suppose that $X \sim U(-1,1)$ and $U = X^2$. Find the pdf of U.

Example 14.
Let $f(x) = x^2/3, -1 < x < 2$, zero otherwise and $U = X^2$. Find the pdf of U.

2.2.3 Bivariate Joint Transformations

Suppose that X_1 and X_2 are continuous random variables with joint density function $f_{X_1,X_2}(x_1, x_2)$ and that for all (x_1, x_2) such that $f_{X_1,X_2}(x_1, x_2) > 0$

$$u_1 = h_1(x_1, x_2) \text{ and } u_2 = h_2(x1, x2)$$

Is one-to-one transformation form (x_1, x_2) to (u_1, u_2) with inverse

$$x_1 = h_1^{-1}(u_1, u_2) \text{ and } x_2 = h_2^{-1}(u_1, u_2)$$

If $h_1^{-1}(u_1, u_2)$ and $h_2^{-1}(u_1, u_2)$ have continuous partial derivatives with respect to u_1 and u_2 and Jacobian.

$$J = det \begin{bmatrix} \frac{\partial h_1^{-1}}{\partial u_1} & \frac{\partial h_1^{-1}}{\partial u_2} \\ \frac{\partial h_2^{-1}}{\partial u_1} & \frac{\partial h_2^{-1}}{\partial u_2} \end{bmatrix} = \frac{\partial h_1^{-1}}{\partial u_1}\frac{\partial h_2^{-1}}{\partial u_2} - \frac{\partial h_1^{-1}}{\partial u_2}\frac{\partial h_2^{-1}}{\partial u_1} \neq 0$$

Then the joint density of U_1 and U_2 is

$$f_{U_1,U_2}(u_1, u_2) = f_{X_1,X_2}(h_1^{-1}(u_1, u_2), h_2^{-1}(u_1, u_2))|J|$$

where $|J|$ is the absolute value of the Jacobian.

Example 15.

Let X_1 and X_2 have a joint density function given by

$$f_{X_1,X_2}(x_1, x_2) = \begin{cases} e^{-(x_1+x_2)}, & x_1 > 0, x_2 > 0 \\ 0, & \text{otherwise} \end{cases}$$

Find the density function of $U = X_1 + X_2$.

Example 16.

Let X and Y be independent random variables with $X \sim GAM(\alpha_1, \theta)$ and $Y \sim GAM(\alpha_2, \theta)$, show that $U = \frac{X}{X+Y}$ follow a beta distribution. Suppose $W_i \sim Exp(\theta)$, using the above result, find the distribution of $V = \frac{W_1}{\sum_{i=1}^{n} W_i}$.

2.2.4 Multivariate Transformation

Let $(X_1, \ldots, X_n)$ be a random vector with pdf $f_{\mathbf{X}}(x_1, \ldots, x_n)$. Let $\mathbf{A} = \{\mathbf{x} : f_{\mathbf{X}}(\mathbf{x}) > 0\}$. Consider a new random vector $(U_1, \ldots, U_n)$, defined by $U_1 = g_1(X_1, \ldots, X_n)$, $U_2 = g_2(X_1, \ldots, X_n)$, $\ldots$, $U_n = g_n(X_1, \ldots, X_n)$. Suppose that A_0, A_1, $\ldots$, A_k form a partition of $\mathbf{A}$ with these properties. The set A_0, which may be empty, satisfies $P((X_1, \ldots, X_n) \in A_0) = 0$. The transformation $(U_1, \ldots, U_n) = (g_1(\mathbf{X}), \ldots, g_n(\mathbf{X}))$ is a one-to-one transformation from A_i to B for each $i = 1, 2, \ldots, k$. Then for each i, the inverse functions from B to A_i can be found. Denote the ith inverse by $x_1 = h_1(u_1, \ldots, u_n)$, $x_2 = h_2(u_1, \ldots, u_n)$, $\ldots$, $x_n = h_n(u_1, \ldots, u_n)$. This ith inverse gives, for $(u_1, \ldots, u_n) \in B$, the unique $(x_1, \ldots, x_n) \in A_i$ such that $(u_1, \ldots, u_n) = (g_1(x_1, \ldots, x_n), \ldots, g_n(x_1, \ldots, x_n))$. Let J_i denote the Jacobian computed fron the inverse. That is

$$J_i = \begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & \cdots & \frac{\partial x_1}{\partial u_n} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} & \cdots & \frac{\partial x_2}{\partial u_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial u_1} & \frac{\partial x_n}{\partial u_2} & \cdots & \frac{\partial x_n}{\partial u_n} \end{vmatrix} = \begin{vmatrix} \frac{\partial h_{1i}(u)}{\partial u_1} & \frac{\partial h_{1i}(u)}{\partial u_2} & \cdots & \frac{\partial h_{1i}(u)}{\partial u_n} \\ \frac{\partial h_{2i}(u)}{\partial u_1} & \frac{\partial h_{2i}(u)}{\partial u_2} & \cdots & \frac{\partial h_{2i}(u)}{\partial u_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial h_{ni}(u)}{\partial u_1} & \frac{\partial h_{ni}(u)}{\partial u_2} & \cdots & \frac{\partial h_{ni}(u)}{\partial u_n} \end{vmatrix}$$

the determinant of an $n \times n$ matrix. Assuming that these Jacobian do not vanish identically on B, we have the following representation of the joint pdf, $f_{\mathbf{u}}(u_1, \ldots, u_n)$, for $\mathbf{u} \in B$:

$f_{\mathbf{u}}(u_1, \ldots, u_n)$

$= \sum_{i=1}^{k} f_{\mathbf{X}}(h_{1i}(u_1, \ldots, u_n), \ldots, h_{ni}(u_1, \ldots, u_n))|J|.$

Example 17.

Let (X_1, X_2, X_3, X_4) have joint pdf

$$f_{\mathbf{X}}(x_1, x_2, x_3, x_4) = 24e^{-x_1-x_2-x_3-x_4},$$

$$0 < x_1 < x_2 < x_3 < x_4 < \infty$$

Consider the transformation

$U_1 = X_1, U_2 = X_2 - X_1, U_3 = X_3 - X_2, U_4 = X_4 - X_3.$

(a) Find the joint pdf of $\mathbf{U} = (U_1, U_2, U_3, U_4)$

(b) Find the marginal pdf of $U_i, i = 1, 2, 3, 4$

2.3 Sums of Random Variables-Moment Generating Function Method

Sums of independent random variables often arise in practice. A technique based on moment generating functions usually is much more convenient than using transformations for determining the distribution of sums of independent random variables.

Theorem 5.
If $X_1, \ldots, X_n$ are independent random variables with MGFs $M(t)$, then the MGF of $U = \sum_{i=1}^{n} X_i$ is

$$M_U(t) = M_{X_1}(t) \cdots M_{X_n}(t)$$

The MGF of a random variable uniquely determines its distribution. The MGF approach is particularly useful for determining the distribution of a sum of independent random variables, and it often will be much more convenient than trying to carry out a joint transformation.

Example 18.
Let $X_1, \ldots X_k$ be independent binomial random variables with respective parameters n_i, and p, $X_i \sim Bin(n_i, p)$ and let $U = \sum_{i=1}^{k} X_i$. Find the distribution of U.

Example 19.
Let $X_1, \dots X_k$ be independent Poisson-distributed random variables $X_i \sim POI(\mu_i)$ and let $U = \sum_{i=1}^{k} X_i$. Find the distribution of U.

Example 20.
Let $X_1, \dots X_k$ be independent gamma-distributed random variables with respective shape parameters $\alpha_1, \alpha_2, \dots, \alpha_n$ and common scale parameter θ, $X_i \sim GAM(\alpha_i, \theta)$ for $i = 1, \dots, n$ and let $U = \sum_{i=1}^{k} X_i$. Find the distribution of U.

2.4 Order Statistics

Let $X_1, X_2, \ldots, X_n$ denote independent continuous random variables with distribution function $F(x)$ and density $f(x)$. We denote the ordered random variables X_i by $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$, where $X_{(1)} \leq X_{(2)} \leq \ldots X_{(n)}$. Using this notation,

$$X_{(1)} = \min(X_1, X_2, \ldots, X_n)$$

is the minimum of the random variables X_i, and

$$X_{(n)} = \max(X_1, X_2, \ldots, X_n)$$

is the maximum of the random variables X_i.

The probability density functions for $X_{(1)}$ and $X_{(n)}$ can be found using method of distribution functions.

Example 21.

Let $X_1, \ldots, X_8$ be a random sample of size 8 from a distribution $N(140, 50^2)$. Let $U = \max(X_1, X_2, \ldots,$ find the value of the p.d.f. of U at $u = 213.9$.

Example 22. Electronic components of a certain type have a length of life X, with probability density given by

$$f(x) = \begin{cases} (\frac{1}{100}e^{-x/100}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

(Length of life is measured in hours.) Suppose that two such components operate independently and in series in a certain system (hence, the system fails when either component fails). Find the density function for U, the length of life of the system.

Example 23.
Suppose that the components in Example 22 operate in parallel (hence, the system does not fail until both components fail). Find the density function for U, the length of life of the system.

Theorem 6.
If $X_1, X_2, \ldots, X_n$ is a random sample from a population with continuous pdf, $f(x)$, then the joint pdf of the order statistics, $Y_1, Y_2, \ldots, Y_n$ is

$$g(y_1, y_2, \ldots, y_n) = \begin{cases} n!f(y_1)f(y_2)\cdots f(y_n), & y_1 < y_2 \cdots < y_n \\ 0, & \text{otherwise} \end{cases}$$

Theorem 7.
Let $X_1, X_2, \ldots, X_n$ denote independent continuous random variables with common distribution function $F(x)$ and common density functions $f(x)$. If $X_{(k)}$ denotes $k^{th}-$ order statistic, then the density function of $X_{(k)}$ is given by

$$g_{(k)}(x_k) = \frac{n!}{k!(n-k)!}[F(x_k)]^{k-1}[1-F(x_k)]^{n-k}f(x_k),$$

$$x_k \in R$$

If j and k are two integers such that $1 \leq j < k \leq n$, the joint density of $X_{(j)}$ and $X_{(k)}$ is given by

$$\begin{aligned} g_{(j)(k)}(x_j x_k) = & \tfrac{n!}{(j-1)!(k-j-1)!(n-k)!}[F(x_j)]^{j-1} \\ & \times[F(x_k) - F(x_j)]^{k-j-1} \\ & \times[1 - F(x_k)]^{n-k} f(x_j) f(xk) \\ & -\infty < x_j < x_k < \infty \end{aligned}$$

Example 24.
A system is composed of 18 independent components. If the pdf of the time to failure of each component is exponential, $X_i \sim EXP(140)$. Suppose that the 18-component system fails when at least 6 components fail. Give the pdf of the time to failure of the system.

Example 25. Suppose that $X_1, X_2, \ldots, Y_{15}$ denotes a random sample from a uniform distribution defined on the interval $(0, 1)$. That is,

$$f(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the density function for the second-order statistic. Also, give the joint density function for the second- and fourth-order statistics.

Example 26.

Let Y_3 denote the third smallest item of a random sample of size n from a distribution of the continuous type that has cdf $F_X(x)$ and pdf $f_X(x) = F'_X(x)$. Find the probability density function (p.d.f.) of $W_n = nF_{Y_3}(y)$.

The event that the k^{th}-order statistic at most y, $[Y_k \leq y]$ can occur if and only if at least k of the n observations are less than or equal to y . That is, here the probability of "success" on each trial is $F(y)$ and we must have at least k successes. Thus,

$$P(Y_k \leq y) = \sum_{i=k}^{n} \binom{n}{i} [F(y)]^i [1 - F(y)]^{n-i}$$

Example 27.

Let $X_i \sim Exp(90), i = 1, \ldots, 9$ and $Y_1 < Y_2 < \ldots < Y_9$ be the order statistics. Compute the probability that Y_7 is less than 150.3.