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CONTENTS

0 Random Variables and Their Distributions

0.1 Notation and Terminology

- **Experiment** refers to the process of obtaining an observed result of some phenomenon. It could pertain to activities as scientific experiments or games of chance.
- **Trial** of the experiment is a performance of an experiment.
- The set of all possible outcomes of an experiment is called the **sample space**, denoted by S.
- If a sample space S is either finite or countably infinite then it is called a **discrete sample space**.
- An **event** is a subset of the sample space S. If A is an event, then A has occurred if it contains the outcome that occurred.
- **Random variable**, say X, is a function de-

fined over a sample space, S, that associates a real number, $X(e) = x$, with each possible outcome e in S.

Example 1. An experiment consists of tossing two coins, and the observed face of each coin is of interest. The sample space is

Sol:

$$S = \{HH, HT, TH, TT\}$$

Example 2. Suppose that in Example 23 we were not interested in the individual outcomes of the coins, but only in the total number of heads obtained from the two coins. An appropriate sample space is

Sol:

$$S = \{0, 1, 2\}$$

Thus, different sample spaces may be appropriate for the same experiment, depending on the characteristic of interest.

Example 3. A light bulb is placed in service and the time of operation until it burns out is measured, a sample space is

Sol:

$$S = \{t|0 \leq t < \infty\}$$

Example 4. A four-sided die has a different number 1, 2, 3, or 4 affixed to each side. On any given roll, each of the four numbers is equally likely to occur. A game consists of rolling the die twice, and the score is the maximum of the two numbers that occur. Although the score cannot be predicted, we can determine the set of possible values and define a random variable. In particular, if $e = (i, j)$ where $i, j \in 1, 2, 3, 4$, then $X(e) = \max(i, j)$. The sample space, S, and X are

Sol:
$S = \{(1,1),(1,2),(1,3),(1,4),(2,1,),(2,2),(2,3),(2,4),$
$(3,1),(3,2),(3,3),(3,4),(4,1),(4,2),(4,3),(4.4)\}$
$x = 1, 2, 3, 4$

0.2 Discrete Random Variables

Definition 1. If the set of all possible values of a random variable, X, is a countable set, $x_1, x_2, \ldots, x_n$, or $x_1, x_2, \ldots$, then X is called a discrete random variable. The function

$$f(x) = P[X = x] \text{ for } x = x_1, x_2, \ldots$$

that assigns the probability to each possible value x will be called the **discrete probability density function** (discrete pdf).

Note: If it is clear from the context that X is discrete, then we simply will say pdf. Another common terminology is probability mass function (pmf), and the possible values, x, are called mass points of X. Sometimes a subscripted notation, $f_X(x)$, is used.

Theorem 1. A function $f(x)$ is a discrete pdf if and only if it satisfies both of the following properties for at most a countably infinite set of reals $x_1, x_2, \ldots$

$$f(x_i) \geq 0 \text{ for all } x_i,$$

and

$$\sum_{\text{all} x_i} x_i = 1$$

In some problems, it is possible to express the pdf by means of an equation. However, it is sometimes more convenient to express it in tabular form.

Example 5. We roll a red die and a green die. Both dice are fair. Suppose X is the total score from the red and green dice.

(a) What are the possible values of X?

Sol:

$$x = 2, 3, \ldots, 12$$

Here, the set of the possible values of X is finite

(b) Display the distribution of X in a table.

Sol:

x	2	3	4	5	6	7	8	9	10	11	12
$P(X = x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

.

Example 6.

When tossing a fair coin, let X be the number of independent tosses required to observe the first head (H) come up and $P(H) = p$.

(a) What are the possible values of X?

Sol:

$$x = 1, 2, \ldots$$

Here, the set of the possible values of the random variable Y is countably infinite.

(b) Find the distribution of X.

Sol:

$P(X = 1) = p$
$P(X = 2) = qp$
$P(X = 3) = q^2p$
$\vdots$
$P(X = x) = q^{x-1}p$

Example 7.

If we roll a 12-sided die twice. If each face is marked with an integer, 1 through 12, then each value is equally likely to occur on a single roll of the die. Let X be the maximum obtained on the two rolls. Find the pdf of X.

Sol:

It is not hard to see that for each value x there are an odd number, $2x-1$, of ways for that value to occur. Thus, the pdf of X must have the form

$f(x) = c(2x-1)$ for $x = 1, 2, \ldots, 12$

$\sum_{j=1}^{12} f(x) = 1$

$c\sum_{j=1}^{12}(2x-1) = 1$

$c[\frac{2(12)(1+12)}{12} - 12] = 1$

$c = \frac{1}{144}$

Definition 2. The cumulative distribution function (CDF) of a random variable X is defined for any real x by

$$F(x) = P[X \leq x]$$

Theorem 2. Let X be a discrete random variable with pdf $f(x)$ and CDF $F(x)$. If the possible values of X are indexed in increasing order, $x_1 < x_2 < x_3 < \ldots$, then $f(x_1) = F(x_1)$, and for any $i > 1$,

$$f(x_i) = F(x_i) - F(x_{i-1})$$

Furthermore, if $x < x_1$ then $F(x) = 0$, and for any other real x

$$F(x) = \sum_{x_i \leq x} f(x)$$

where the summation is taken over all indices i such that $x_i \leq x$.

The CDF of any random variable must satisfy the properties of the following theorem.

Theorem 3.

A function $F(x)$ is a CDF for some random variable X if and only if it satisfies the following properties:

- $\lim_{x\to-\infty} F(x) = 0$
- $\lim_{x\to\infty} F(x) = 1$
- $\lim_{h\to 0^+} F(x+h) = F(x)$, i.e. $F(x)$ is **continuous from the right**
- $a < b$ implies $F(a) \leq F(b)$, i.e. $F(x)$ is **non-decreasing**

The first two properties say that $F(x)$ can be made **arbitrarily** close to 0 or 1 by taking x arbitrarily large, and negative and positive, respectively.

0.3 Continuous Random Variables

Definition 3. A random variable X is called a continuous random variable if there is a function $f(x)$, called the probability density function (pdf) of X, such that the CDF can be represented as

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

Theorem 4. A function $f(x)$ is a pdf for some continuous random variable X if and only if it satisfies the properties

$$f(x) \geq 0 \text{ for all real } x,$$

and

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

Example 8.

A machine produces copper wire, and occasionally there is a flaw at some point along the wire. The length of wire (in meters) produced between successive flaws is a continuous random variable X with pdf of the form

$$f(x) = \begin{cases} c(1+x)^{-3} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

where c is a constant. Find c and hence $F(x)$.

Sol:

$\int_0^\infty c(1+x)^{-3}dx = 1$

$c\frac{(1+x)^{-2}}{-2}\Big|_0^\infty = 1$

$\frac{1}{2}c = 1$

$c = 2$

$$\begin{aligned} F(x) &= \int_0^x \frac{1}{2}(1+t)^{-3}dt \\ &= 2\frac{(1+t)^{-2}}{-2}\Big|_0^x \\ &= 1-(1+x)^{-2} \end{aligned}$$

0.4 Properties of Random Variables

Definition 4. If X is a discrete random variable with pdf $f(x)$, then the expected value of X is defined by

$$E(X) = \sum_x x f(x)$$

Definition 5. If X is a continuous random variable with pdf $f(x)$, then the expected value of X is defined by

$$E(X) = \int_{-\infty}^{\infty} x f x dx$$

if the integral is absolutely convergent. Otherwise we say that $E(X)$ does not exist.

Other notations for $E(X)$ are μ or μ_X, and the terms mean or expectation of X also are commonly used.

Definition 6. If $0 < p < 1$, then a $100 \times p^{th}$ percentile of the distribution of a continuous random variable X is a solution x to the equation

$$F(x) = p$$

In general, a distribution may not be continuous, and if it has a discontinuity, then there will be some values of p for which equation $F(x) = p$ has no solution. It is possible to state a general definition of percentile by defining a p^{th} percentile of the distribution of X to be a value x_p, such that $P[X \leq x_p] \geq p$ and $P[X \geq x_p] \leq 1 - p$. In essence, x_p is a value such that $100 \times p$ percent of the population values are at most x_p, and $100 \times (1-p)$ percent of the population values are at least x_p. A median of the distribution of X is a 50-th percentile, denoted by $x_{0.5}$ or m.

Example 9. A discrete random variable X has a pdf of the form $f(x) = c(8-x)$ for $x = 0, 1, 2, 3, 4, 5$, and zero otherwise. Find $E(X)$.

Sol:

$c(8+7+6+5+4+3) = 1$

$c = \frac{1}{33}$

$E(X) = \frac{1}{33}(1\times7+2\times6+3\times5+4\times4+5\times3) = \frac{65}{33}$

Example 10.

A continuous random variable X has a pdf of the form $f(x) = \frac{2x}{9}$ for $0 < x < 3$, and zero otherwise.

1. Find a number m such that $P[X \leq m] = P[X \geq m]$.

   Sol:

   $P[X \leq m] = P[X \geq m]$
   $\int_0^m \frac{2x}{9} dx = \int_m^3 \frac{2x}{9} dx$
   $\frac{x^2}{9}{}_0^m = \frac{x^2}{9}{}_m^3$
   $\frac{m^2}{9} = 1 - \frac{m^2}{9}$
   $\frac{2m^2}{9} = 1$
   $m = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$

2. Find $E(X)$.

   Sol:

   $E(X) = \int_0^3 \frac{2x^2}{9} dx = \frac{2x^3}{27}\Big|_0^3 = 2$

Example 11. Consider the distribution of lifetimes, X (in months), of a particular type of component. We will assume that the CDF has the form

$$F(x) = 1 - e^{-(x/3)^2}, X > 0$$

and zero otherwise.

1. Find the median lifetime.

   Sol:
   $F(m) = \frac{1}{2}$
   $1 - e^{-(m/3)^2} = \frac{1}{2}$
   $-\frac{m^2}{9} = \ln\frac{1}{2}$
   $m^2 = 9\ln 2$
   $m = 3\sqrt{\ln 2} = 2.498$ months

2. Find the time t such that 10% of the components fail before t.

Sol:
$F(t) = 0.1$
$1 - e^{-(t/3)^2} = 0.1$
$-\frac{t^2}{9} = \ln 0.9$
$t = 3\sqrt{-\ln 0.9} = 0.974$ months

0.5 Some Properties of Expected Values

Theorem 5. If X is a random variable with pdf $f(x)$ and $u(x)$ is a real valued function whose domain includes the possible values of X, then

$$E[u(X)] = \sum u(x)f(x) \text{ if } X \text{ is discrete}$$
$$E[u(X)] = \int_{-\infty}^{\infty} u(x)f(x)dx \text{ if } X \text{ is continuous}$$

Theorem 6. If X is a random variable with pdf $f(x)$, a and b are constants, and $g(x)$ and $h(x)$ are real valued functions whose domains include the possible values of X, then

$$E[ag(X) + bh(X)] = aE[g(X)] + bE[h(X)]$$

Definition 7. The variance of a random variable X is given by

$$V(X) = E[(X-\mu)^2]$$

Other common notations for the variance are σ^2, σ_X^2, or $V(X)$, and a related quantity, called the standard deviation of X, is the positive square root of the variance, $\sigma = \sigma_X = \sqrt{V(X)}$.
The variance provides a measure of the variability or amount of "spread" in the distribution of a random variable.

Definition 8. The k^{th} moment about the origin of a random variable X is

$$\mu'_k = E(X^k)$$

and the k^{th} moment about the mean is

$$\mu_k = E(X-\mu)^k$$

Theorem 7. If X is a random variable, then

$$V(X) = E(X^2) - \mu^2$$

Sol:

$$\begin{aligned} V(X) &= E(X-\mu)^2 \\ &= E(X^2 - 2\mu X + \mu^2) \\ &= E(X^2) - 2\mu E(X) + \mu^2 \\ &= E(X^2) - 2\mu^2 + \mu^2 \\ &= E(X^2) - \mu^2 \end{aligned}$$

Theorem 8. If X is a random variable and a and b are constants, then

$$V(aX+b) = a^2 V(X)$$

Sol:

$$\begin{aligned} V(aX+b) &= E[aX+b-(a\mu+b)]^2 \\ &= E[aX-a\mu]^2 \\ &= a^2 E(X-\mu)^2 \\ &= a^2 V(X) \end{aligned}$$

This means that the variance is affected by a change of scale, but not by a translation.

Example 12. At a computer store, the annual demand for a particular software package is a discrete random variable X. The store owner orders four copies of the package at \$10 per copy and charges customers \$35 per copy. At the end of the year the package is obsolete and the owner loses the investment on unsold copies. The pdf of X is given by the following table:

x	0	1	2	3	4
$P(X = x)$	.1	.3	.3	.2	.1

(a) Find $E(X)$.

Sol: $E(X) = 1(.1) + 2(.3) + 3(.1) + 4(.1) = 1.9$

(b) Find $V(X)$.

Sol:

$E(X^{)} = 1(.1) + 4(.3) + 9(.1) + 16(.1) = 4.9$
$V(X) = E(X^2) - \mu^2 = 3.8 - 1.9^2 = 1.29$

(c) Express the owner's net profit Y as a linear function of X, and find $E(Y)$ and $V(Y)$.

Sol:

Profit $= 35X - 40$
$E(Profit) = 35E(X) - 40 = 35(1.9) - 40 = 26.5$
$E(Profit^2) = E(35X - 40)^2 = E(35^2X^2 - 2(35)(40)X + 40^2) = 35^2E(X^2) - 2(35)(40)E(X) + 40^2 = 35(4.9) - 2(35)(40)(1.9) + 40^2 = 2282.5$
$V(Profit) = 2282.5 - 26.5^2 = 1580.25$

Example 13. Let X be continuous with pdf $f(x) = 3x^2$ if $0 < x < 1$ and zero otherwise. Find

(a) $E(X)$.

Sol:
$E(X) = \int_0^1 3x^3 dx = \frac{3x^4}{4}|_0^1 = \frac{3}{4}$

(b) $V(X)$

Sol:
$E(X^2) = \int_0^1 3x^4 dx = \frac{3x^5}{5}|_0^1 = \frac{3}{5}$
$V(X) = \frac{3}{5} - \frac{9}{16} = 0.1875$

(c) $E(X^r)$

Sol:
$E(X^r) = \int_0^1 3x^{2+r} dx = \frac{3x^{3+r}}{3+r}|_0^1 = \frac{3}{3+r}$

(d) $E(3X - 5X^2 + 1)$

Sol:
$E(3X - 5X^2 + 1) = 3E(X) - 5E(X^2) + 1 = 3(\frac{3}{4}) - 5(\frac{3}{5}) + 1 = 0.25$

0.6 Moment Generating Functions

Definition 9. If X is a random variable, then the expected value

$$M_X(t) = E(e^{tX})$$

is called the moment generating function (MGF) of X if this expected value exists for all values of t in some interval of the form$-h < t < h$ for some $h > 0$.

Theorem 9. If the MGF of X exists, then

$$E(X^r) = M^r(0) \text{ for all } r = 1, 2, \ldots$$

Example 14. Consider a continuous random variable X with pdf $f(x) = e^{-x}$ if $x > 0$ and zero otherwise.

(a) Find the MGF of X.

(b) Find $E(X^r)$.

(c) Find the mean and variance of X

Sol:

(a) $M_X(t) = E[e^{tX}] = \int_0^\infty e^{tx}e^{-x}dx$
$= \int_0^\infty e^{-(1-t)x}dx = \frac{-e^{-(1-t)}}{1-t}|_0^\infty$
$= \frac{1}{1-t}, t < 1$

(b) $M'_X(t) = \frac{1}{(1-t)^2}$
$M''_x(t) = \frac{2}{(1-t)^3}$
$\vdots$
$M^r_X(t) = \frac{r!}{(1-t)^{r+1}}$
$E(X^r) = M^r_X(0) = r!$

(c) $E(X)1! = 1$
$E(X^2) = 2! = 2$
$V(X) = 2 - 1 = 1$

Example 15. A discrete random variable X has pdf $f(x) = (\frac{1}{2})^{x+1}$ if $x = 0, 1, 2, \ldots,$ and zero otherwise.

(a) Find the MGF of X.

(b) Find the mean of X

Sol:

$M_X(t) = E[e^{tX}] = \sum_{x=0}^{\infty} e^{tx}(\frac{1}{2})^{x+1} = \frac{1}{2}\sum_{i=1}^{\infty}(\frac{e^t}{2})^x$

Using $1 + s + s^2 + \cdots = \frac{1}{1-s}, -1 < s < 1$

$= \frac{1}{2}\left[\frac{1}{1-\frac{1}{2}e^t}\right]$

$= \frac{1}{2-e^t}, t < \ln 2$

$M'_X(t) = \frac{e^t}{(2-e^e)^2}$

$E(X) = M_X(0) = \frac{1}{(2-1)^2} = 1$

Properties of Moment Generating Functions

Theorem 10. If $Y = aX + b$, then $M_Y(t) = e^{bt} M_X(at)$.

Sol:

$$\begin{aligned} M_Y(t) &= E[e^{(aX+b)t}] \\ &= e^{bt} E[e^{(at)X}] \\ &= e^{bt} M_X(at) \end{aligned}$$

Theorem 11. Uniqueness If X_1 and X_2 have respective CDFs $F_1(x)$ and $F_2(x)$, and MGFs $M_1(t)$ and $M_2(t)$, then $F_1(x) = F_2(x)$ for all real x if and only if $M_1(t) = M_2(t)$ for all t in some interval $-h < t < h$ for some $h > 0$

In other words, X_1 and X_2 cannot have the same MGF but different pdf's. Thus, the form of the MGF determines the form of the pdf.

0.7 Probability Generating Function

The probability generating function (PGF) is defined by

$$P_X(z) = E(z^X)$$

It is important to realize that we cannot have intuition about PGFs because they do not correspond to anything which is directly observable.

- PGFs make calculations of expectations and of some probabilities very easy.
  - $P'(1) = E(X)$
  - $P''(1) = E[X(X-1)]$
  - $P^{(3)}(1) = E[X(X-1)(X-2)]$
- PGFs make sums of independent random variables easy to handle. i.e.,

  $$P_{X_1+\ldots+X_n}(z) = [P_X(z)]^n$$

  when $X_i's$ are identically and independently distributed.

0.8 Cumulant Generating Function

The cumulant-generating function $K(t)$, is the natural logarithm of the moment-generating function:

$$K(t) = \ln E(e^{tX}) = \ln M_X(t)$$

The cumulants κ_n are obtained from a power series expansion of the cumulant generating function:

$$K(t) = \sum_{n=1}^{\infty} \kappa_n \frac{t^n}{n!}.$$

This expansion is a Maclaurin series, so that n^{th} cumulant can be obtained by differentiating the above expansion n times and evaluating the result at zero:

$$\kappa_n = K^{(n)}(0).$$

The first cumulant is the expected value; the second and third cumulants are respectively the second and third central moments; but the higher cumulants are neither moments nor central moments.

0.9 Special Discrete Distributions

0.9.1 Bernoulli Distribution

A random variable X has Bernoulli (p) distribution if its pdf is

$$f(x) = P(X = x) = p^x q^(1 - x) \text{ for } x = 0, 1$$

where $0 < p < 1$ is a parameter and $q = 1 - p$.

Example:

(i) Record whether an item is defective ($x = 0$) or nondefective ($x = 1$).

(ii) Record whether an individual is male ($x = 0$) or female ($x = 1$).

In each situation, p stands for $P(X = 1)$.

The mean, variance and MGF of a Bernoulli distribution are:

$$E(X) = p, \sigma^2 = p(1 - p), M_X(t) = pe^t + q$$

0.9.2 Binomial Distribution

A random variable X has Binomial (n, p) distribution if its pdf is

$$f(x) = P(X = x) = \binom{n}{x} p^x q^{n-x} \text{ for } x = 0, 1, \ldots, n$$

The Binomial (n, p) distribution arises as follows. Repeat a Bernoulli experiment independently n times and each time one observes the outcome 0 or 1 and $p = P(X = 1)$.
A short notation to designate that X has the binomial distribution with parameters n and p is $X \sim BIN(n, p)$

The mean, variance and MGF of a Binomial distribution are:

$$E(X) = np, \sigma^2 = np(1-p), M_X(t) = [pe^t + q]^n$$

Notes:

- $x\binom{n}{x} = n\binom{n-1}{x-1}$, $\binom{N}{n} = \frac{N}{n}\binom{N-1}{n-1}$
- Binomial Theorem: $(x+y)^n = \sum_{i=1}^{n} \binom{n}{i} x^i y^{n-i}$

Example 16.

In a 10-question truefalse test:

(a) What is the probability of getting all answers correct by guessing?

(b) What is the probability of getting eight correct by guessing?

Sol:

Let X be the number of questions answer correctly. Then $X \sim Bin(n = 10, p = 0.5$

$P(X = 10) = 0.5^{10}$

$P(X = 8) = \binom{10}{8}(0.5^8)(0.5^2)$

0.9.3 Hypergeometric Distribution

Suppose a population or collection consists of a finite number of items, say N, and there are M items of type 1 and the remaining $N - M$ items are of type 2. Suppose n items are drawn at random without replacement, and denote by X the number of items of type 1 that are drawn. The random variable X is said to have the hypergeometric distribution with parameters N, n and M. Its pdf is

$$f(x) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}},$$

$$x = 0, 1, \ldots, \min(n, M), n - x \leq N - M.$$

We write $X \sim Hyp(n, M, N)$.
The mean and variance of a Hypergeometric distribution are:

$$E(X) = \frac{nM}{N}, \sigma^2 = n\frac{M}{N}\left(1 - \frac{M}{N}\right)\frac{N-n}{N-1}$$

The MGF of Hypergeometric distribution does not exists.

Example 17. A box contained 100 microchips, 80 good and 20 defective. The number of defectives in the box is unknown to a purchaser, who decides to select 10 microchips at random without replacement and to consider the microchips in the box acceptable if the 10 items selected include no more than three defectives. Calculate the probability of accepting a lot.

Sol:

Let X be the number of defective.

$X \sim HYP(n = 10, M = 20, N = 100)$

$f(x) = \frac{\binom{20}{x}\binom{80}{10-x}}{\binom{100}{10}}$

$P[\text{Accepting a lot}]$

$= P[X \leq 3]$

$= \sum_{x=0}^{3} \frac{\binom{20}{x}\binom{80}{10-x}}{\binom{100}{10}}$

$= 0.89$

0.9.4 Geometric Distributions

If we denote the number of trials required to obtain the first success by X, then X is said to have **Geometric distributions**, the discrete pdf of X is given by

$$f(x) = pq^{x-1} \quad x = 1, 2, 3, \ldots$$

We denote $X \sim Geo(p)$
The CDF of X is

$$F(x) = 1 - q^{x} \quad x = 1, 2, 3, \ldots$$

Example 18. A geological exploration may indicate that a well drilled for oil in a region in Texas would strike oil with probability 0.3. Assuming that oil strikes are independent from one drill to another. What is the probability that the first oil strike will occur on the 6th drill?

Sol:

Let X be the number of well drill require to obtain the first oil strike.

$X \sim Geo(0.3)$

$P(X = 6) = 0.3(0.7^6) = \boxed{0.050421}$

Theorem 12. No-Memory Property If

$$X \sim GEO(p),$$

then

$$P[X > j + k | X > j] = P[X > k]$$

Thus, knowing that j trials have passed without a success does not affect the probability of k more trials being required to obtain a success. That is, having several failures in a row does not mean that you are more "due" for a success.

The mean, variance and MGF of a Geometric distribution are:

$$E(X) = \frac{1}{p}, \sigma^2 = \frac{q}{p^2}, M_X(t) = \frac{pe^t}{1 - qe^t}$$

0.9.5 Negative Binomial

In repeated independent Bernoulli trials, let X denote the number of trials required to obtain r successes. Then the probability distribution of X is the negative binomial distribution with discrete pdf given by

$$f(x) = \binom{x-1}{r-1} p^r q^{x-r}, x = r, r+1, \ldots$$

A special notation, which designates that X has the negative binomial distribution

$$X \sim NB(r, p)$$

The mean, variance and MGF of a Negative Binomial distribution are:

$$E(X) = \frac{r}{p}, \sigma^2 = \frac{rq}{p^2}, M_X(t) = \left(\frac{pe^t}{1 - qe^t}\right)^r$$

Example 19. Team A plays team B in a seven-game world series. That is, the series is over when either team wins four games. For each game, $P(A \text{ wins}) = 0.6$, and the games are assumed independent. What is the probability that the series will end in exactly six games?

Sol: Let X and Y be the number of games play until team A and team B wins 4 games respectively.
$X \sim NB(r = 4, p = 0.6)$, $Y \sim NB(r = 4, p = 0.4)$
$P[\text{Team A wins 4 games}]$
$= P[X = 6]$
$= \binom{6-1}{4-1} 0.6^4 (.4^{6-4})$
$= 0.20763$
$P[\text{Team B wins 4 games}]$
$= P[Y = 6]$
$= \binom{6-1}{4-1} 0.4^4 (.4^{6-4})$
$= 0.09216$
$P[\text{series end in six games}] = 0.20763 + 0.09216 = \boxed{0.20736}$

0.9.6 Poisson Distribution

A discrete random variable X is said to have the Poisson distribution with parameter $\mu > 0$ if it has discrete pdf of the form

$$f(x) = \frac{e^{-\mu}\mu^x}{x!} \quad x = 0, 1, 2, \ldots$$

The mean, variance and MGF of a Poisson distribution are:

$$E(X) = \mu, V(X) = \sigma^2 = \mu, M_X(t) = e^{\mu(e^t - 1)}$$

Example 20.

We are inspecting a particular brand of concrete slab specimens for visible cracks. Suppose that the number (X) of cracks per concrete slab has a Poisson distribution with $\mu = 2.5$. What is the probability that a randomly selected slab will have at least 2 cracks?

Sol:

$X \sim POI(2.5)$

$P[X \geq 2]$

$= 1 - P(X < 2)$

$= 1 - P(X = 0) - P(X = 1)$

$= 1 - e^{-2.5} - 2.5e^{-2.5}$

$= \boxed{0.7127}$

0.9.7 Discrete Uniform Distribution

A discrete random variable X has the discrete uniform distribution on the integers $1, 2, \ldots, N$ if it has a pdf of the form

$$f(x) = \frac{1}{N}, X = 1, 2, \ldots, N$$

A special notation for this situation is

$$X \sim DU(N)$$

The mean, variance and MGF of a Discrete Uniform distribution are:

$$E(X) = \frac{N+1}{2}, \sigma^2 = \frac{N^2-1}{12},$$

$$M_X(t) = \frac{1}{N}\frac{e^t - e^{(N+1)^t}}{1-e^t}$$

Example:

- The number obtained by rolling an ordinary six-sided die correspond to $DU(6)$.
- The multiple-choice test on any question, which associate the four choices with the integers 1, 2, 3, and 4, then the response, X, on any given question that is answered at random is $DU(4)$.

0.10 Special Continuous Distributions

0.10.1 Uniform Distribution

A continuous random variable X that assume values only in a bounded interval (a, b), with constant pdf over the interval is known as the **uniform distribution**.

$$f(x) = \frac{1}{b-a}, a < x < b$$

and zero otherwise. A notation that designates that X has pdf of the form above is

$$X \sim U(a, b)$$

The CDF of $X \sim U(a, b)$ has the form

$$F(x) = \begin{cases} 0, & x \leq a \\ \frac{x-a}{b-a}, & a < x < b \\ 1, & b \leq x \end{cases}$$

The mean, variance and MGF of a Discrete Uniform distribution are:

$$E(X) = \frac{a+b}{2}, \sigma^2 = \frac{(b-a)^2}{12},$$

$$M_X(t) = \frac{e^{bt} - e^{at}}{(b-a)t}$$

0.10.2 Gamma Distribution

Definition 10. The gamma function, denoted by $\Gamma(\alpha)$ for all $\alpha > 0$, is given by

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt$$

Theorem 13. The gamma function satisfies the following properties:

- $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$
- $\Gamma(n) = (n-1)!$
- $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

Sol:

- Use $\int udv = uv - \int vdu$
  Let $u = t^{\alpha-1}$, $du = (\alpha-1)t^{\alpha-2}dt$, $dv = e^{-t}dt$, $v = \int e^{-t}dt = -e^{-t}$

  $$\begin{aligned}\Gamma(\alpha) &= \textstyle\int_0^\infty t^{\alpha-1}e^{-t}dt \\ &= t^{\alpha-1}e^{-t}\Big|_0^\infty + (\alpha-1)\textstyle\int_)^\infty t^{\alpha-2}e^{-t}dt \\ &= (\alpha-1)\Gamma(\alpha-1)\end{aligned}$$

- $\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)(n-2)\Gamma(n-2)\cdots(n-1)(n-2)\cdots 1\Gamma(1) = (n-1)!$

- $\Gamma(\frac{1}{2}) = \int_0^\infty t^{-\frac{1}{2}} e^{-t} st$
  $= \text{Let } u = t^{\frac{1}{2}},\ du = \frac{1}{2}u^{-\frac{1}{2}}dt$
  $= \int_0^\infty u^{-1} e^{-u^2} 2udu$
  $= 2\int_0^\infty e^{-u^2} du$

$$\begin{aligned}\left[\Gamma(\tfrac{1}{2})\right]^2 &= \left[\Gamma(\tfrac{1}{2})\right]\left[\Gamma(\tfrac{1}{2})\right] \\ &= \left[2\int_0^\infty e^{-u^2} du\right]\left[2\int_0^\infty e^{-v^2} dv\right]\end{aligned}$$

Let $u = rcos\theta$, $v = rsin\theta$, then $0 \leq r \leq \infty$ and $0 \leq \theta \leq \frac{\pi}{2}$.

$$J = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} cos\theta & -rsin\theta \\ sin\theta & rsin\theta \end{vmatrix}$$

$$= rcos^2\theta + rsin^2\theta$$

$$= r$$

$$\begin{aligned}\left[\Gamma(\tfrac{1}{2})\right]^2 &= 4\int_0^{\pi/2}\int_0^\infty e^{-r^2} r dr d\theta \\ &= 4\int_0^{\pi/2} d\theta \int_0^\infty re^{-r^2} dr \\ &= 4(\pi/2)[-\tfrac{1}{2}e^{-r^2}]_0^\infty \\ &= \pi\end{aligned}$$

Finally, since $e^{-u^2} > 0$ for all $u > 0$, then

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}$$

A continuous random variable X is said to have the **gamma distribution** with parameters $\theta > 0$ and α if it has pdf of the form

$$f(x) = \frac{1}{\theta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} e^{-x/\theta}, x > 0$$

and zero otherwise.
A special notation, which designates that X has pdf given by equation above, is

$$X \sim GAM(\alpha, \theta)$$

The parameter α is called a shape parameter because it determines the basic shape of the graph of the pdf. θ is called the scaled parameter.
The mean, variance and MGF of a Gamma distribution are:

$$E(X) = \alpha\theta, \sigma^2 = \alpha\theta^2, M_X(t) = \left(\frac{1}{1-\theta t}\right)^{\alpha}$$

Theorem 14. If $X \sim GAM(n, \theta)$, where n is a positive integer, then the CDF can be written

$$F(x) = 1 - \sum_{i=0}^{n-1} \frac{(x/\theta)^i}{i!} e^{-x/\theta}$$

Example 21. The daily amount (in inches) of measurable precipitation in a river valley is a random variable $X \sim GAM(\alpha = 6, \theta = 0.2)$. Find the probability that the amount of precipitation will exceed 2 inches.

Sol:
$P(X > 2)$
$= \sum_{i=0}^{5} \frac{10^i e^{-10}}{i!}$
$= \boxed{0.067}$

0.10.3 Exponential Distribution

A continuous random variable X has the exponential distribution with parameter $\theta > 0$ if it has a pdf of the form

$$f(x) = \frac{1}{\theta}e^{-x/\theta}, x > 0$$

and zero otherwise.

The CDF of X is

$$F(x) = 1 - e^{-x/\theta}, x > 0$$

The notation $X \sim GAM(1, \theta)$ could be used to designate that X, but a more common notation is

$$X \sim EXp(\theta)$$

The mean, variance and MGF of an Exponential distribution are:

$$E(X) = \theta, \sigma^2 = \theta^2, M_X(t) = \left(\frac{1}{1 - \theta t}\right)$$

Theorem 15.

no memory property For a continuous random variable X, $X \sim EXP(\theta)$ if and only if

$$P[X > a+t|X > a] = P[X > t]$$

for all$a > 0$ and $t > 0$.

Sol:

$$\begin{aligned} P[X > a+t|X > a] &= \frac{P[X>a+t \text{ and } X>a}{P(X>a)} \\ &= \frac{P[X>a+t]}{P[X>a]} \\ &= \frac{e^{-(a+t)/\theta}}{e^{-a/\theta}} \\ &= P[X > t] \end{aligned}$$

0.10.4 Weibull Distribution

A widely used continuous distribution is named after the physicist W. Weibull, who suggested its use for numerous applications, including fatigue and breaking strength of materials. lt is also a very popular choice as a failure-time distribution. A continuous random variable X is said to have the **Weibull distribution** with parameters $\tau > 0$ and $\theta > 0$ if it has a pdf of the form

$$f(x) = \frac{\tau}{\theta^\tau} x^{\tau-1} e^{-(x/\theta)^\tau}, x > 0$$

and zero otherwise. A notation that designates that X is

$$X \sim WEI(\tau, \theta)$$

The CDF of X is

$$F(x) = 1 - e^{-(x/\theta)^\tau}$$

The mean and variance of a Weibull distribution are:

$$E(X) = \theta\Gamma\left(1 + \frac{1}{\tau}\right), \sigma^2 = \theta^2\left[\Gamma\left(1 + \frac{2}{\tau}\right) - \Gamma^2\left(1 + \frac{1}{\tau}\right)\right]$$

The MGF does not exist.

Example 22. The distance (in inches) that a dart hits from the center of a target may be modeled as a random variable $X \sim WEI(\tau = 2, \theta = 2)$. The probability of hitting within five inches of the center is

Sol:
$P(X \leq 5) = 1 - e^{-(5/10)^2} = \boxed{0.221}$

0.10.5 Pareto Distribution

A continuous random variable X is said to have the Pareto distribution with parameters $\alpha > 0$ and $\theta > 0$ if it has a pdf of the form

$$f(x) = \frac{\alpha\theta^{\alpha}}{(x+\theta)^{\alpha+1}}, x > 0$$

and zero otherwise. A notation to designate that X is

$$X \sim PAR(\alpha, \theta)$$

The CDF is given by

$$F(x) = 1 - (\frac{\theta}{x+\theta})^{\alpha}$$

The mean and variance of a Pareto distribution are:

$$E(X) = \frac{\theta}{\alpha - 1}, \sigma^2 = \frac{\theta^2}{(\alpha-1)^2(\alpha-2)}$$

The MGF does not exist.

0.10.6 Normal Distribution

A random variable X follows the normal distribution with mean μ and variance σ^2 if it has the pdf

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{(x-\mu)^2/2\sigma^2},$$

for $x \in \mathbb{R}$, $\mu \in \mathbb{R}$ and $\sigma > 0$. This denoted by

$$X \sim N(\mu, \sigma^2)$$

Let $Z = \frac{X-\mu}{\sigma}$, then $Z \sim N(0,1)$. Z is called Standard Normal distribution. It pdf is

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, -\infty < z < \infty$$

The standard normal CDF is given by

$$\Phi(x) = \int_{-\infty}^{x} \phi(t) dt$$

The CDF of X is

$$F(x) = \Phi(\frac{x-\mu}{\sigma})$$

The MGF of Normal distribution is

$$M_X(t) = e^{\mu t + \sigma^2/2}$$

Theorem 16. If $X \sim N(\mu, \sigma^2)$ then

1. $E(X-\mu)^{2r} = \frac{(2r)!\sigma^{2r}}{r!2^r}$
2. $E(X-\mu)^{2r-1} = 0$

0.10.7 Log Normal Distribution

A random variable X follows the lognormal distribution with parameters μ and σ if it has the pdf

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} e^{(\ln x - \mu)^2/2\sigma^2},$$

for $x > 0$, $\mu > 0$ and $\sigma > 0$. This denoted by

$$X \sim LN(\mu, \sigma)$$

The CDF of X is

$$F(x) = \Phi(\frac{\ln x - \mu}{\sigma})$$

The k^{th} raw moment of lognormal distribution is

$$E(X^k) = e^{k\mu + \frac{1}{2}k^2\sigma^2}$$

Note: If $X \sim N(\mu, \sigma^2)$, then $u = e^X \sim LN(\mu, \sigma)$

0.10.8 Beta Distribution

The beta family of distributions is a continuous family on $(0, 1)$ indexed by tow parameters. The $beta(a, b)$ pdf is

$$f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1},$$

$$0 < x < 1, a > 0, b > 0.$$

The mean and variance of X are

$$E(X) = \frac{a}{a+b}$$

and

$$V(Y) = \frac{ab}{a+b+1)(a+b)^2}$$

In order to show that the pdf of beta distributions sum to one, we need to find

$$B(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx,$$

where $B(a, b)$ is called the beta function. The beta function is related to the gamma funtion through the following identity:

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Sol:

The Relationship between gamma function and beta function

$\Gamma(a)\Gamma(b)$
$= \int_{u=o}^{\infty} u^{a-1}e^{-u}du \int_{v=o}^{\infty} v^{b-1}e^{-v}dv$
$= \int_{v=o}^{\infty}\int_{u=o}^{\infty} u^{a-1}v^{b-1}e^{-u-v}dudv$
Let $u = f(z,t) = zt$ and $v = g(z,t) = z(1-t)$

$$J(z,t) = \begin{vmatrix} t & z \\ (1-t) & -z \end{vmatrix} = -z$$

$\Gamma(a)\Gamma(b)$
$= \int_{z=0}^{\infty}\int_{t=0}^{1}(zt)^{a-1}[z(1-t)]^{b-1}e^{-z}|J(z,t)|dtdz$
$= \int_{z=0}^{\infty}\int_{t=0}^{1}(zt)^{a-1}[z(1-t)]^{b-1}e^{-z}zdtdz$
$= \int_{z=0}^{\infty} z^{a+b-1}e^{-z}\int_{t=0}^{1} t^{a-1}(1-t)^{b-1}dt$
$= \Gamma(a+b)B(a,b)$
$\therefore B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$

0.11 Location and Scale Parameters

In each of the following definitions, $F_0(z)$ represents a completely specified CDF, and $f_0(z)$ is the pdf.

Definition 11. Location Parameters A quantity η is a location parameter for the distribution of X if the CDF has the form

$$F(x) = F_0(x - \eta)$$

In other words, the pdf has the form

$$f(x) = f_0(x - \eta)$$

For example,
Consider the pdf

$$f_0(z) = e^{-|z|}, -\infty < z < \infty$$

If X has pdf of the form

$$f(x) = e^{-|x-\eta|}, -\infty < x < \infty$$

then η is the location parameter.

Definition 12. Scale Parameter A positive quantity θ is a scale parameter for the distribution of X if the CDF has the form

$$F(x) = F_0\left(\frac{x}{\theta}\right)$$

In other words, he pdf has the form

$$f(x) = f_0\left(\frac{x}{\theta}\right)$$

Notes:

- A frequently encountered example of a random variable, the distribution of which has a scale parameter, is $X \sim EXP(\theta)$.
- The standard deviation, σ, often turns out to be a scale parameter.

Definition 13. Location-Scale Parameter
Quantities η and $\theta > 0$ are called location-scale parameters for the distribution of X if the CDF has the form

$$F(x) = F_0\left(\frac{X-\eta}{\theta}\right)$$

In other words, the pdf has the form

$$f(x) = f_0\left(\frac{x-\eta}{\theta}\right)$$

The normal distribution is the most commonly encountered location-scale distribution, but there are other important examples.

0.11.1 Cauchy Distribution

Consider a pdf of the form

$$f_0(z) = \frac{1}{\pi}\frac{1}{(1+z^2)} \qquad -\infty < z < \infty$$

If X has pdf of the form $\frac{l}{\theta} f_0\left[\frac{x-\eta}{\theta}\right]$, with $f_0(z)$ given by equation above, then X is said to have the **Cauchy distribution** with location scale parameters η and θ, denoted

$$X \sim CAU(\theta, \eta)$$

$$f(x) = \frac{1}{\theta\pi\left[1+\left(\frac{x-\eta}{\theta}\right)^2\right]} \qquad -\infty < x < \infty$$

0.11.2 Two-parameter Exponential Distribution

Another location-scale distribution, which is frequently encountered in life testing applications, has pdf

$$f(x) = \frac{1}{\theta} e^{-\left(\frac{x-\eta}{\theta}\right)} \quad x > \eta$$

and zero otherwise. This is called the **two-parameter exponential distribution**, denoted by

$$X \sim EXP(\eta, \theta)$$

The mean, variance and MGF of an Exponential distribution are:

$$E(X) = \eta + \theta, \sigma^2 = \theta^2, M_X(t) = \left(\frac{e^{\eta t}}{1 - \theta t}\right)$$

0.11.3 Double-Exponential Distribution

If X has pdf of the form

$$f(x) = \frac{1}{2\theta} e^{-|x-\eta|/\theta} \qquad -\infty < x < \infty$$

and zero otherwise.
This location-scale distribution is called the **Laplace** or **double-exponential** distribution, denoted by

$$X \sim DE(\theta, \eta)$$

The mean, variance and MGF of an Exponential distribution are:

$$E(X) = \eta, \sigma^2 = 2\theta^2, M_X(t) = \left(\frac{e^{\eta t}}{1-\theta^2 t^2}\right)$$

0.12 Joint Discrete Distributions

In many applications there will be more than one random variable of interest, say $X_1, X_2, \ldots, X_k$. It is convenient mathematically to regard these variables as components of a $k-$dimensional vector, $X = (X_1, X_2, \ldots, X_k)$, which is capable of assuming values $x = (x_1, x_2, \ldots, x_k)$ in a $k-$ dimensional Euclidean space. Note, for example, that an observed value x may be the result of measuring k characteristics once each, or the result of measuring one characteristic k times.

Definition 14. The joint probability density function (joint pdf) of the $k-$dimensional discrete random variable $X = (X_1, X_2, \ldots, X_k)$ is defined to be

$$f(x_1, x_2, \ldots, x_k) = P[X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k)$$

for all possible values $x = (x_1, x_2, \ldots, x_k)$ of X.

Theorem 17. A function $f(x_1, x_2, \ldots, x_k)$ is the joint pdf for some vector-valued random variable

$$X = (x_1, x_2, \ldots, x_k)$$

if and only if the following properties are satisfied

1. $f(x_1, x_2, \ldots, x_k) > 0$ for all possible values $x_1, x_2, \ldots, x_k$
2. $\sum_{x_1} \sum_{x_2} \cdots \sum_{x_k} f(x_1, x_2, \ldots, x_k) = 1$

Definition 15.

If the $X = (x_1, x_2, \ldots, x_k)$ of discrete random variables has the joint pdf $f(x_1, x_2, \ldots, x_k)$, then the marginal pdf's of X_j is

$$f_j(x_j) = \sum \underset{\text{all} i \neq j}{\cdots} \sum f(x_1, \ldots, x_j, \ldots, x_k)$$

Example 23.

Let the joint pmf of X_1 and X_2 be defined by

$$p(x_1, x_2) = \frac{x_1 + x_2}{32}, \quad x_1 = 1, 2, x_2 = 1, 2, 3, 4.$$

(a) Display the joint probability distribution of X_1 and X_2 in a table.

Sol:

		x_1	
		1	2
x_2	1	$\frac{2}{32}$	$\frac{3}{32}$
	2	$\frac{3}{32}$	$\frac{4}{32}$
	3	$\frac{4}{32}$	$\frac{5}{32}$
	4	$\frac{5}{32}$	$\frac{6}{32}$

(b) Verify that the probability function satisfies Theorem 17.

Sol:

(a) $p(x_1, x_2) \geq 0$ for all x_1 and x_2.

(b) $\sum_{x_1, x_2} p(x_1, x_2)$
$= \frac{2}{32} + \frac{3}{32} + \frac{3}{32} + \frac{4}{32} + \frac{4}{32} + \frac{5}{32} + \frac{5}{32} + \frac{6}{32}$
$= 1$

(c) Find $P(X_1 < X_2)$.

Sol:

$$P(X_1 < X_2)$$
$$= p(1,2) + p(1,3) + p(1,4) + p(2,3) + p(2,4)$$
$$= \frac{3}{32} + \frac{4}{32} + \frac{5}{32} + \frac{5}{32} + \frac{6}{32}$$
$$= \boxed{\frac{23}{32}}$$

(d) Find $P(X_1 + X_2 = 4)$.

Sol:

$$P(X_1 + X_2 = 4)$$
$$= p(1,3) + p(2,2)$$
$$= \frac{4}{32} + \frac{4}{32}$$
$$= \frac{8}{32}$$
$$= \boxed{\frac{1}{4}}$$

Definition 16. Joint CDF The joint cumulative distribution function of the k random variables $X_1, X_2, \ldots X_k$ is the function defined by

$$F(x_1, x_2, \ldots, x_k) = P[X_1 \leq x_1, \ldots, X_k \leq x_k]$$

Theorem 18. A function $F(x_1, x_2)$ is a bivariate CDF if and only if

- $\lim\limits_{x_1 \to -\infty} F(x_1, x_2) = F(-\infty, x_2) = 0 \; \forall \; x_2$
- $\lim\limits_{x_2 \to -\infty} F(x_1, x_2) = F(x_1, -\infty) = 0 \; \forall \; x_1$
- $\lim\limits_{x_1 \to \infty, x_2 \to \infty} F(x_1, x_2) = F(\infty, \infty) = 1 \, \forall \, x_1, x_2$
- $F(b, d) - F(b, c) - F(a, d) + F(a, c) \geq 0 \; \forall \; a < b, c < d$
- $\lim\limits_{h \to 0^+} F(x_1 + h, x_2) = \lim\limits_{h \to 0^+} F(x_1, x_2 + h) = F(x_1, x_2)$

Example 24.

A local supermarket has three checkout counters. Two customers arrive at the counters at different times when the counters are serving no other customers. Each customer chooses a counter at random, independently of the other. Let X_1 denote the number of customers who choose counter 1 and X_2, the number who select counter 2.

(a) Find the joint probability function of X_1 and X_2.

Sol:

Let (i, j) pair denotes the event that the 1st customer choose counter i and the 2nd choose counter j, where $i, j = 1, 2, 3$.

$$S = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$$

		x_2		
		0	1	2
x_1	0	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{1}{9}$
	1	$\frac{2}{9}$	$\frac{2}{9}$	0
	2	$\frac{1}{9}$	0	0

(b) Find $F(-1,2)$, $F(1.5,2)$, and $F(5,7)$.

Sol:

$$F(-1,2) = P(X_1 \leq -1, X_2 \leq 2) = 0$$

$$F(1.5,2) = P(X_1 \leq 1.5, X_2 \leq 2) = \frac{8}{9}$$

$$F(5,7) = P(X_1 \leq 5, X_2 \leq 7) = 1$$

Example 25. If X and Y are discrete random variables with joint pdf

$$f(x,y) = c\frac{2^{x+y}}{x!y!} \quad x = 0,1,2,\ldots; y = 0,1,2,\ldots$$

and zero otherwise.

(a) Find the constant c.

Sol:

$c\sum_{x=0}^{\infty}\sum_{y=0}^{\infty}\frac{2^x 2^y}{x!y!} = 1$

$c\sum_{x=0}^{\infty}\frac{2^x}{x!}\sum_{y=0}^{\infty}\frac{2^y}{y!} = 1$

$ce^2e^2 = 1$

$c = e^{-4}$

(b) Find the marginal pdf's of X and Y.

Sol:

$f(x) = \sum_{y=0}^{\infty}(e^{-4})\frac{2^{x+y}}{x!y!} = (e^{-4})\frac{2^x}{x!}\sum_{y=0}^{\infty}\frac{2^y}{y!}$

$= (e^{-4})\frac{2^x}{x!}e^2$

$= \frac{2^x e^{-2}}{x!}$

Similarly, $f(y) = \frac{2^y e^{-2}}{y!}$

0.13 Joint Continuous Distributions

Definition 17. A $k-$dimensional vector valued random variable $X = (X_1, X_2, \ldots, X_k)$ is said to be continuous if there is a function $f(x_1, x_2, \ldots, x_k)$, called the joint probability density function (joint pdf), of X, such that the joint CDF can be written as
$F(x_1, x_2, \ldots, x_k)$
$= \int_{-\infty}^{x_k} \cdots \int_{-\infty}^{x_1} f(t_1, t_2, \ldots, t_k) dt_1 \cdots dt_k$
$\forall\ x = (x_1, x_2, \ldots, x_k)$.

Theorem 19. Any function $f(x_1, x_2, \ldots, x_k)$ is a joint pdf of a $k-$dimensional random variable if and only if

1. $f(x_1, x_2, \ldots, x_k) \geq 0\ \forall\ x_1, x_2, , \ldots, x_k$
2. $\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, x_2, \ldots, x_k) dx_1 \cdots dx_k = 1$

Example 26.

Let X_1 denote the concentration of a certain substance in one trial of an experiment, and X_2 the concentration of the substance in a second trial of the experiment. Assume that the joint pdf is given by $f(x_1, x_2) = 4x_1x_2; 0 < x_1 < 1, 0 < x_2 < 1$, and zero otherwise.

(a) Find the joint CDF.

Sol:

$$\begin{aligned} F(x_1,x_2) &= \int_{-\infty}^{x_2}\int_{-\infty}^{x_1} f(t_1,t_2)dt_1dt_2 \\ &= \int_0^{x_2}\int_0^{x_1} 4t_1t_2dt_1dt_2 \\ &= 4\int_0^{x_2} t_2[\tfrac{t_1^2}{2}]_0^{x_1}dt_2 \\ &= 2\int_0^{x_2} t_2x_1^2dt_2 \\ &= 2x_1^2[\tfrac{t_2^2}{2}]_0^{x_2} \\ &= x_1^2x_2^2, 0 < x_1 < 1, 0 < x_2 < 1 \end{aligned}$$

$$\begin{aligned} F(x_1,x_2) &= \int_0^{x_2}\int_0^{1} f(t_1,t_2)dt_1dt_2 \\ &= \int_0^{x_2}\int_0^{1} 4t_1t_2dt_1dt_2 \\ &= 4\int_0^{x_2} t_2[\tfrac{t_1^2}{2}]_0^{1}dt_2 \\ &= 2\int_0^{x_2} t_2dt_2 \\ &= 2[\tfrac{t_2^2}{2}]_0^{x_2} \\ &= x_2^2, x_1 > 1, 0 < x_2 < 1 \end{aligned}$$

$$
\begin{aligned}
F(x_1, x_2) &= \int_0^{x_1} \int_0^{1} f(t_1, t_2) dt_1 dt_2 \\
&= \int_0^{x_1} \int_0^{1} 4t_1 t_2 dt_1 dt_2 \\
&= 4 \int_0^{x_1} t_2 [\tfrac{t_1^2}{2}]_0^1 dt_2 \\
&= 2 \int_0^{x_1} t_2 dt_2 \\
&= 2[\tfrac{t_2^2}{2}]_0^{x_1} \\
&= x_1^2, x_2 > 1, 0 < x_1 < 1
\end{aligned}
$$

$$F(x_1, x_2) = \int_0^{\infty} \int_0^{\infty} 4t_1 t_2 dt_1 dt_2 = 1, 0 < x_1, x_2 < \infty$$

(b) Find $P\left[\frac{X_1+X_2}{2} < 0.5\right]$.

Sol:

$$
\begin{aligned}
& P\left[\frac{X_1+X_2}{2} < 0.5\right] \\
&= P(X_1 + X_2 < 1) \\
&= \int_0^1 \int_0^{1-x_2} 4x_1 x_2 dx_1 dx_2 \\
&= \int_0^1 4x_2 [\tfrac{x_1^2}{2}]_0^{1-x_2} dx_2 \\
&= 2 \int_0^1 x_2 (1 - x_2)^2 dx_2 \\
&= 2 \int_0^1 (x_2 - 2x_2^2 + x_2^3) dx_2 \\
&= 2[\tfrac{x_2}{2} - \tfrac{2x_2^3}{3} + \tfrac{x_2^4}{4}]_0^1 \\
&= \tfrac{1}{6}
\end{aligned}
$$

Definition 18.
1f $X = (X_1, X_2, \ldots, X_k)$ is a $k-$dimensional random variable with joint CDF $F(x_1, x_2, \ldots, x_k)$, then the marginal CDF of X is

$$F_j(x_j) = \lim_{x_i \to \infty, \text{all } i \neq j} F(x_1, \ldots, x_j, \ldots, x_k)$$

Furthermore,the marginal pdf is

$$f_j(x_j) = \int \underset{\text{all } i \neq j}{\cdots} \int f(x_1, \ldots, x_j, \ldots, x_k) dx_1 \ldots dx_k$$

Example 27. Let X_1, X_2, and X_3 be continuous with a joint pdf of the form $f(x_1, x_2, x_3) = c; 0 < x_1 < x_2 < x_3 < 1$, and zero otherwise, where c is a constant.

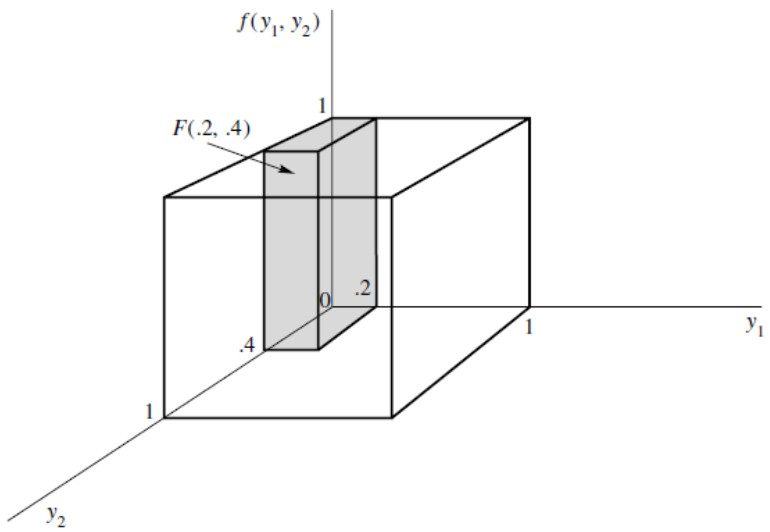
1. Find the marginal distributions of x_3.
2. Find the joint pdf of the pair (X_1, X_2).

Example 28.

Suppose that a radioactive particle is randomly located in a square with sides of unit length. That is, if two regions within the unit square and of equal area are considered, the particle is equally likely to be in either region. Let X_1 and X_2 denote the coordinates of the particle's location. A reasonable model for the relative frequency histogram for X_1 and X_2 is the bivariate analogue of the univariate uniform density function:

$$f(x_1, x_2) = \begin{cases} 1, & 0 \le x_1 \le 1, 0 \le x_2 \le 1, \\ 0, & \text{otherwise} \end{cases}$$

(a) Sketch the probability density surface.



(b) Find $F(.2, .4)$.

Sol:

$$
\begin{aligned}
&F(.2, .4)\\
&= P(X_1 \leq .2, X_2 \leq .4)\\
&= (.2)(.4)\\
&= \boxed{0.08}
\end{aligned}
$$

(c) Find $P(.1 \leq X_1 \leq .3, 0 \leq X_2 \leq .5)$

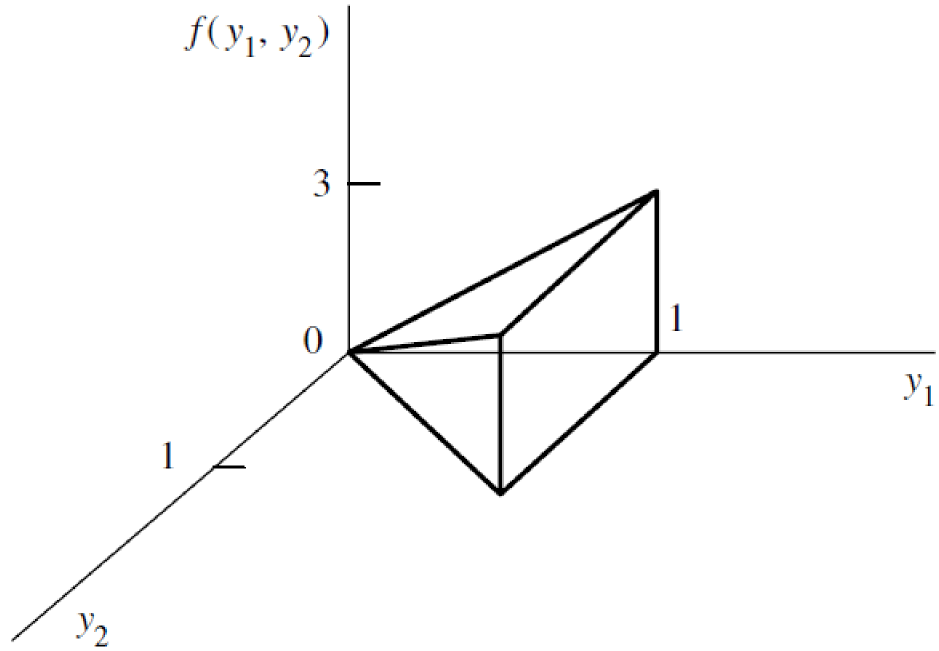
Sol:

$$
\begin{aligned}
&P(.1 \leq X_1 \leq .3, 0 \leq X_2 \leq .5)\\
&= 0.2(0.5)\\
&= \boxed{0.1}
\end{aligned}
$$

Example 29. The joint probability density function of X_1 and X_2 is

$$f(x_1, x_2) = \begin{cases} 3x_1, & 0 \le x_2 \le x_1 \le 1, \\ 0, & \text{otherwise} \end{cases}$$

(a) Sketch the probability density surface.



(b) Find $P(0 \le X_1 \le .5, X_2 \ge 0.25)$.

Sol: $P(0 \le X_1 \le .5, X_2 \ge 0.25)$
$= \int_{.25}^{.5} \int_{.25}^{x_1} 3x_1 dx_2 dx_1$
$= \int_{.25}^{.5} 3x_1 [x_2]_{.25}^{x_1} dx_1$
$= \int_{.25}^{.5} 3x_1 (x_1 - .25) dx_1$
$= 3[\frac{x_1^3}{3} - \frac{.25x_1^2}{2}]_{.25}^{.5}$
$= \boxed{\frac{5}{128}}$

0.14 Conditional Distributions

Definition 19. Conditional pdf If X_1 and X_2 are discrete or continuous random variables with joint pdf $f(x_1, x_2)$, then the conditional probability density function (conditional pdf) of X_2 given $X_1 = x_1$ is defined to be

$$f(x_2|x_1) = \frac{f(x_1, x_2)}{f_1(x_1)}$$

for values x_1 such that $f_1(x_1) > 0$ and zero otherwise.

Similarly, the conditional pdf of X_1 given $X_2 = x_2$ is defined to be

$$f(x_1|x_2) = \frac{f(x_1, x_2)}{f_2(x_2)}$$

for values x_2 such that $f_2(x_2) > 0$ and zero otherwise.

Theorem 20. If X_1 and X_2 are random variables with joint pdf $f(x_1, x_2)$ and marginal pdf's $f_1(x_1)$ and $f_2(x_2)$, then

$$f(x_1, x_2) = f_1(x_1)f(x_2|x_1) = f_2(x_2)f(x_1|x_2)$$

and if X_1 and X_2 are independent, then

$$f(x_2|x_1) = f_2(x_2)$$

and

$$f(x_1|x_2) = f_1(x_1)$$

Example 30.

Let

$$f(x_1, x_2, x_3, x_4) = \begin{cases} \frac{3}{4}(x_1^2 + x_2^2 + x_3^2 + x_4^2), & 0 < x_i < 1, i = 1, 2, 3, 4 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the marginal pdf of (X_1, X_2).

(b) Find the conditional pdf of (X_3, X_4) given $X_1 = \frac{1}{3}$ and $X_2 = \frac{2}{3}$.

Sol:

$f(x_1, x_2)$
$= \int_0^\infty \int_0^\infty f(x_, x_2, x_3, x4) dx_3 dx_4$
$= \int_0^1 \int_0^1 \frac{3}{4}(x_1^2 + x_2^2 + x_3^2 + x_4^2) dx_3 dx_4$
$= \frac{3}{4}(x_1^2 + x_2^2) + \frac{1}{2}$
for $0 < x_1 < 1$ and $0 < x_2 < 1$.

$f(x_3, x_4 | x_1, x_2)$
$= \frac{f(x_1, x_2, x_3, x_4)}{f(x_1, x_2)}$
$= \frac{\frac{3}{4}(x_1^2 + x_2^2 + x_3^2 + x_4^2)}{\frac{3}{4}(x_1^2 + x_2^2) + \frac{1}{2}}$
$= \frac{x_1^2 + x_2^2 + x_3^2 + x_4^2}{x_1^2 + x_2^2 + \frac{2}{3}}$

$f(x_3, x_4 | x_1 = \frac{1}{3}, x_2 = \frac{2}{3})$
$= \frac{(\frac{1}{3})^2 + (\frac{2}{3})^2 + x_3^2 + x_4^2}{(\frac{1}{3})^2 + (\frac{2}{3})^2 + \frac{2}{3}}$
$= \frac{5}{11} + \frac{9}{11}x_3^2 + \frac{9}{11}x_4^2$

Example 31. The joint density function of X_1 and X_2 is given by

$$f(x_1, x_2) = \begin{cases} 30x_1x_2^2, & x_1 - 1 \leq x_2 \leq 1 - x_1, 0 \leq x_1 \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Show that the marginal density of X_1 is a beta density with $a = 2$ and $b = 4$.

Sol:

$f_1(x_1)$
$= \int_{x_1-1}^{1-x_1} 30x_1x_2^2 dx_2$
$= 30x_1[\frac{x_2^3}{3}]_{x_1-1}^{1-x_1}$
$= 10x_1[(1-x_1)^3 + (1-x_1)^3]$
$= 20x_1(1-x_1)^3, 0 \leq x_1 \leq 1$

(b) Derive the marginal density of X_2.

Sol:

For $0 < x_2 < 1$
$f_2(x_2)$
$= \int_0^{1-x_2} 30x_1x_2^2 dx_1$
$= 30x_2^2[\frac{x_1^2}{2}]_0^{1-x_2}$

$= 15x_2^2(1-x_2)^2, 0 < x_2 < 1$

For $-1 < x_2 < 0$
$f_2(x_2)$
$= \int_0^{1+x_2} 30x_1x_2^2dx_1$
$= \int_0^{1+x_2} 30x_1x_2^2dx_1$
$= 30x_2^2[\frac{x_1^2}{2}]_0^{1+x_2}$
$= 15x_2^2(1+x_2)^2, -1 < x_2 < 0$

$$f(x_2) = \begin{cases} 15x_2^2(1-x_2)^2, & 0 < x_2 < 1 \\ 15x_2^2(1+x_2)^2, & -1 < x_2 < 0 \end{cases}$$

(c) Derive the conditional density of X_2 given $X_1 = x_1$.

Sol:

$f(x_2|x_1)$
$= \frac{f(x_1,x_2)}{f_1(x)}$
$= \frac{30x_1x_2^2}{20x_1(1-x_1)^3}$

$= \frac{3x_2^2}{2(1-x_1)^3}, x_1 - 1 \leq x_2 \leq 1 - x_1$

(d) Find $P(X_2 > 0|X_1 = .75)$.

Sol:

$f(x_2|x_1 = 0.75)$
$= \frac{3x_2^2}{2(1-.75)^3}, 0.75 - 1 \leq x_2 \leq 1 - 0.75$
$= 96x_2^2, -.25 \leq x_2 \leq .25$

$P(X_2 > 0|X_1 = .75)$
$= \int_0^{0.25} 96x_2^2 dx_2$
$= 96[\frac{x_2^3}{3}]_0^{0.25}$
$= 32[.25^3]$
$= \boxed{0.5}$

0.15 Independent Random Variables

Definition 20. Independent Random Variables Random variables $X_1, \ldots, X_k$ are said to be independent if for every $a_i < bi$,

$$P(a_1 \leq x_1 \leq b_1, \cdots, a_k \leq x_k \leq b_k) \\ = \prod_{i=1}^{k} P(a_i \leq x_i \leq b_i)$$

Theorem 21. Random variables $X_1, \ldots, X_k$ are independent if and only if the following properties holds:

$$F(x_1, \ldots, x_k) = F_1(x_1) \cdots F_k(x_k)$$

$$f(x_1, \ldots, x_k) = f_1(x_1) \cdots f_k(x_k)$$

where $F_i(x_i)$ and $f_i(x_i)$ are the marginal CDF and pdf of X, respectively,

Theorem 22. Two random variables X_1 and X_2 with joint pdf $f(x_1, x_2)$ are independent if and only if:

1. The "support set" $\{(x_1, x_2)|f(x_1, x_2) > 0]$, is a Cartesian product, $A \times B$, and
2. The joint pdf can be factored into the product of functions of x_1 and x_2, $f(x_1, x_2) = g(x_1)h(x_2)$

Example 32. The joint pdf of a pair X_1 and X_2 is

$$f(x_1, x_2) = 8x_1x_2, 0 < x_1 < x_2 < 1$$

and zero otherwise. Are X_1 and X_2 independent?

Sol: This function can clearly be factored according to part (2) of the theorem, but the support set, $\{(x_1, x_2), 0 < x_1 < x_2 < 1\}$, is a triangular region that cannot be represented as a Cartesian product. Thus, X_1 and X_2 are dependent.

Example 33. Consider now a pair X_1 and X_2 with joint pdf

$$f(x_1, x_2) = x_1 + x_2, 0 < x_1 < 1, 0 < x_2 < 1$$

and zero otherwise. Are X_1 and X_2 independent?

Sol: In this case the support set is $\{(x_1, x_2), 0 < x_1 < 1 \text{ and } 0 < x_2 < 1\}$, which can be represented as $A \times B$, where A and B are both the open interval $(0, 1)$. However, part (2) of the theorem is not satisfied because $x_1 + x_2$ cannot be factored as $g(x_1)h(x_2)$. Thus, X_1 and X_2 are dependent

Example 34. The joint distribution of X_1 and X_2 is given by the entries in the following table.

	x_2	
x_1	0	1
0	0.12	0.28
1	0.18	0.42

Show that X_1 and X_2 are independent.

Sol:

$p_1(0) = 0.12 + 0.28 = 0.4$, $p_1(1) = .18 + .42 = .6$

$p_2(0) = 0.12 + 0.18 = 0.3$, $p_2(1) = .28 + .42 = .7$

$p_1(0)p_2(0) = .4(.3) = .12 = p(0, 0)$

$p_1(0)p_2(1) = .4(.7) = .28 = p(0, 1)$

$p_1(1)p_2(0) = .6(.3) = .18p(1, 0)$

$p_1(1)p_2(1) = .6(.7) = .42 = p(1, 1)$

Example 35.

The joint distribution of X_1 and X_2 is given by the entries in the following table.

	x_2		
x_1	0	1	2
0	1/9	2/9	1/9
1	2/9	2/9	0
2	1/9	0	0

Is X_1 independent of X_2?

Sol:

$p_1(0) = 4/9$; $p_2(0) = 4/9$

$p_1(0)p_2(0) = 16/81 \neq p(0,0)$. Thus X_1 and X_2 are not independent.

Example 36. Let

$$f(x_1, x_2) = \begin{cases} 2, & 0 \le x_2 \le x_1 \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Show that X_1 and X_2 are dependent.

Sol:
$f_1(x_1) = \int_0^{x_1} 2dx_2 = 2x_1, 0 \le x_1 \le 1$ 0 otherwise
$f_2(x_2) = \int_{x_2}^{1} 2dx_1 = 2(1 - x_2), 0 \le x_2 \le 1$ 0 otherwise
$f_1(x_1)f_2(x_2) = 4x_1(1 - x_2) \ne f(x_1x_2)$
Thus X_1 and X_2 are dependent.

0.16 The Expected Value of a Function of Random Variables

Definition 21. If $X = (X_1, \ldots, X_k)$ has a joint pdf $f(x_1, \ldots, x_k)$, and if $Y = u(X_1, \ldots, X_k)$ is a function of X, then $E(Y) = E[u(X_1, \ldots, X_k)]$, where

$$E_X[u(X_1, \ldots, X_k)] = \sum_{x_1} \cdots \sum_{x_k} u(x_1, \ldots, x_k) f(x_1, \ldots, x_k)$$

if X is discrete, and

$$E_X[u(X_1, \ldots, X_k)] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} u(x_1, \ldots, x_k) f(x_1, \ldots, x_k) dx_1 \cdots dx_k$$

if X is continuous.

Theorem 23. If X_1 and X_2 are random variables with joint pdf $f(x_1, x_2)$, then

$$E(X_1 + X_2) = E(X_1) + E(X_2)$$

Sol: We will show this for the continuous case:

$$\begin{aligned}
E(X_1 + X_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x_1 + x_2)f(x_1, x_2)dx_1dx_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x_1)f(x_1, x_2)dx_1dx_2 \\
&+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x_2)f(x_1, x_2)dx_1dx_2 \\
&\int_{-\infty}^{\infty}(x_1)\int_{-\infty}^{\infty} f(x_1, x_2)dx_2dx_1 \\
&+ \int_{-\infty}^{\infty}(x_2)\int_{-\infty}^{\infty} f(x_1, x_2)dx_1dx_2 \\
&= \int_{-\infty}^{\infty}(x_1)f(x_1)dx_1 + \int_{-\infty}^{\infty}(x_2)f(x_2)dx_2 \\
&= E(X_1) + E(X_2)
\end{aligned}$$

It is possible to combine the preceding theorems to show that if $a_1, a_2, \ldots, a_k$, are constants and $X_1, X_2, \ldots, X_k$ are jointly distributed random variables, then

$$E\left(\sum_{i=1}^{k} a_i X_i\right) = \sum_{i=1}^{k} a_i E(X_i)$$

Theorem 24. If X and Y are independent random variables and $g(x)$ and $h(y)$ are functions, then

$$E[g(X)h(Y)] = E[g(X)]E[h(Y)]$$

Sol:

$E[g(X)h(Y)] = E[g(X)]E[h(Y)]$
$= \int \int g(x)h(y)f(x,y)dxdy$
$= \int \int g(x)h(y)f(x)f(y)dxdy$
$= \int g(x)f(x)dx \int h(y)f(y)dy = E[g(X)]E[h(Y)]$

It is possible to generalize this theorem to more than two variables, Specifically, if $X_1, X_2, \ldots, X_k$ are independent random variables, and $u_1(x_1), \ldots, u_k(x_k)$ are functions, then

$$E[u_1(X_1)\cdots u_k(X_k)] = E[u_1(X_1)]\cdots E[u_k(X_k)]$$

Example 37.

The joint distribution of X_1 and X_2 is given by the entries in the following table.

	x_2		
x_1	0	1	2
0	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{1}{9}$
1	$\frac{2}{9}$	$\frac{2}{9}$	0
2	$\frac{1}{9}$	0	0

(a) Find $E(X_1)$

Sol:

$E(X_1) = \sum_{x_1} \sum_{x_2} xp(x_1, x_2) = 0(1/9)+1(2/9)+2(1/9) + 0(2/9) + 1(2/9) + 2(0/9) + 0(1/9) + 1(0) + 2(0) = \frac{2}{3}$

(b) Find $V(X_1)$

Sol:

$E(X_1^2) = \sum_{x_1} \sum_{x_2} x^2 p(x_1, x_2) = 0(1/9) + 1(2/9) + 4(1/9) + 0(2/9) + 1(2/9) + 4(0/9) +$

$0(1/9) + 1(0) + 4(0) = \frac{8}{9}$
$V(X_1) = E(X_1^2) - E^2(X_1) = \frac{8}{9} - (\frac{2}{3})^2 = \frac{4}{9}$

(c) Find $E(X_1X_2)$

Sol:

$E(X_1X_2) = \sum_{x_1}\sum_{x_2} xyp(x_1, x_2) = 0(0)(1/9)+$
$1(0)(2/9)+2(0)(1/9)+0(1)(2/9)+(1)(1)(2/9)+$
$(1)(2)(0/9) + (2)(0)(1/9) + 2(1)(0) + 2(2)(0)$
$= \frac{2}{9}$

Example 38.

Let $f(x_1,x_2) = \begin{cases} 2x_1, & 0 \le x_1 \le 1; 0 \le x_2 \le 1 \\ 0, & \text{otherwise} \end{cases}$

(a) Find $E(X_1)$

Sol:

$E(X_1) = \int_0^1 \int_0^1 2x_1^2 dx_2 dx_1 = \int_0^1 2x_1^2 x_2]_0^1 dx_1$

$= \int_0^1 2x_1^2 dx_1 = \frac{2x_1^3}{3}]_0^1 = \frac{2}{3}$

(b) Find $V(X_1)$

Sol:

$E(X_1^2) = \int_0^1 \int_0^1 2x_1^3 dx_2 dx_1 = \int_0^1 2x_1^3 x_2]_0^1 dx_1$

$= \int_0^1 2x_1^2 dx_1 = \frac{2x_1^4}{4}]_0^1 = \frac{1}{2}$

$V(X_1) = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$

(c) Find $E(X_1^2 X_2)$

Sol:

$E(X_1^2 X_2) = \int_0^1 \int_0^1 2x_1^3 x_2 dx_2 dx_1 = \int_0^1 x_1^3 x_2^2]_0^1 dx_1$

$= \int_0^1 x_1^3 dx_1 = \frac{x_1^4}{4}]_0^1 = \frac{1}{4}$

Example 39.

Let $f(x_1, x_2) = \begin{cases} 2x_1, & 0 \le x_1 \le 1; 0 \le x_2 \le 1 \\ 0, & \text{otherwise} \end{cases}$

(a) Find $E(X_2)$

Sol:

$E(X_2) = \int_0^1 \int_0^1 2x_1 x_2 dx_2 dx_1 = \int_0^1 x_1 x_2^2]_0^1 dx_1$
$= \int_0^1 x_1 dx_1 = x_1^2/2]_0^1 = \frac{1}{2}$

(b) Find $E(X_2 - X_1)$

Sol:

$E(X_2 - X_1) = E(X_2) - E(X_1) = \frac{1}{2} - \frac{2}{3} = -\frac{1}{6}$

(c) Find $E(5X_1 + 6X_2 - 2X_1^2 X_2)$

Sol:

$E(5X_1 + 6X_2 - 2X_1^2 X_2) = 4E(X_1) + 6E(X_2) - 2E(X_1^2 X_2) = 4(\frac{2}{3}) + 6(\frac{1}{2}) - 2(\frac{1}{4}) = \frac{11}{3}$

Example 40.

Suppose X_1 and X_2 are independent random variables, $E(X_1) = 2$ and $E(X_2) = \frac{1}{3}$. Find $E(X_1X_2)$.

Sol:

$E(X_1X_2)$

$= E(X_1)E(X_2)$ Since X_1 and X_2 are independent random variables

$= 2(\frac{1}{3})$

$= \frac{2}{3}$

Definition 22. The covariance of a pair of random variables X and Y is defined by

$$Cov(X,Y) = E[(X-\mu_X))(Y-\mu_Y)]$$

Another common notation for covariance is σ_{XY}.

Theorem 25. If X and Y are random variables and a and b are constants, then

- $Cov(aX, bY) = abCov(X,Y)$
- $Cov(X+a, Y+b) = Cov(X,Y)$
- $Cov(X, aX+b) = aV(X)$

Sol:

- $Cov(aX, bY)$
  $= E(aX - aE(X))(bY - bE(Y))$
  $= aE[X - E(X)]b[Y - E(Y)]$
  $= abE[X - E(X)][Y - E(Y)]$
  $= abCov(X,Y)$
- $Cov(X+a, Y+b)$
  $= E[X+a-E(X+a)][(Y+b-E(Y+b))$

$$= E[X + a - E(X) - a][Y + b - E(Y) - b]$$
$$= E[X - E(X)][Y - E(Y)]$$
$$= Cov(X, Y)$$

- $Cov(X, aX + b)$
$$= E[X - E(X)][aX + b - E(aX + b)]$$
$$= E[X - E(X)][aX + b - E(aX) - b]$$
$$= E[X - E(X)][aX - E(aX)]$$
$$= E[X - E(X)][a(X - E(X)]$$
$$= aE[X - -E(X)][X - E(X)]$$
$$= aV(X)$$

Theorem 26. If X and Y are random variables, then

$$Cov(X, Y) = E(XY) - E(X)E(Y)$$

and $Cov(X, Y) = 0$ whenever X and Y are independent.

Sol:

$$\begin{aligned} Cov(X, Y) &= E[X - E(X)][Y - E(Y)] \\ &= E[XY - XE(Y) - Y(E(X) + E(X)E \\ &= E(XY) - E(X)E(Y) - EY)E(X) + \\ &= E(XY) - E(X)E(Y) \end{aligned}$$

When X and Y are independent, $E(XY) = E(X)E(Y)$, hence $Cov(X, Y) = 0$.

Definition 23. If X and Y are random variables with variances σ_X^2 and σ_Y^2 and covariance $\sigma_{XY} = Cov(X, Y)$, then the correlation coefficient of X and Y is

$$\rho = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

The random variables X and Y are said to be uncorrelated if $\rho = 0$; otherwise they are said to be correlated.

Theorem 27. If ρ is the correlation coefficient of X and Y, then

$$-1 \leq \rho \leq 1$$

and $\rho = \pm 1$ if and only if $Y = aX + b$ with probability 1 for some $a \neq 0$ and b.

Theorem 28. If X_1 and X_2 are random variables with joint pdf $f(x_1, x_2)$, then

$$V(X_1+X_2) = V(X_1)+V(X_2)+2Cov(X_1, X_2)$$

and

$$V(X_1 + X_2) = V(X_1) + V(X_2)$$

whenever X_1 and X_2 are independent.

Sol:

$$\begin{aligned}
V(X_1 + X_2) &= E[X_1 + X_2 - E(X_1 + X_2)]^2 \\
&= E[(X_1 + X_2)^2 - 2(X_1 + X_2)E(X_1 + \\
&= E[(X_1 + X_2)]^2 - 2E^2(X_1 + X_2) + E^2 \\
&= E[(X_1 + X_2)]^2 - E^2(X_1 + X_2) \\
&= E[X_1^2 + X_2^2 + 2X_1X_2 - (E^2(X_1) + E \\
&= E[X_1^2 - E^2(X_1)] + E[X_2^2 - E^2(X_2)] \\
&= V(X_1) + V(X_2) + 2Cov(X_1, X_2)
\end{aligned}$$

Whenever X_1 and X_2 are independent, $Cov(X_1, X2) =$ 0, hence, $V(X_1 + X_2) = V(X_1) + V(X_2)$.

It also can be verified that if $a_1, a_2, \ldots, a_k$, are constants and $X_1, X_2, \ldots, X_k$, are random variables, then

$$V(\sum_{i=1}^{k} a_i X_i) = \sum_{i=1}^{k} a_i^2 V(X_i) + 2\sum\sum_{i<k} a_i a_j Cov(X_i, X_j)$$

and if $X_1, X_2, \ldots, X_k$ are independent, then

$$V(\sum_{i=1}^{k} a_i X_i) = \sum_{i=1}^{k} a_i^2 V(X_i)$$

Example 41. X_1 and X_2 have joint density given by

$$f(x_1, x_2) = \begin{cases} 2x_1, & 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Find $Cov(X_1, X_2)$.

Sol:
Since $f(x_1, x_2) = g(x_1)h(x_2)$ where $g(x_1) = y_1$ and $h(x_2) = 2$. Then X_1 and X_2 are independent. Thus $Cov(X_1, X_2) = 0$

Example 42.

Let $f(x,y) = 6x, 0 < x < y < 1$, and zero otherwise. Find $Cov(X,Y)$.

Sol:

$E(X) = \int_0^1 \int_x^1 6x^2 dydx = \int_0^1 6x^2(1-x)dx = 6\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{2}$

$E(Y) = \int_0^1 \int_x^1 6xydydx = \int_0^1 6x[\frac{y^2}{2}]_x^1 dx = \int_0^1 6x[\frac{1}{2} - \frac{x^2}{2}] = 3\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = \frac{3}{4}$

$E(XY) = \int_0^1 \int_x^1 6x^2ydydx = \int_0^1 6x^2[\frac{y^2}{2}]_x^1 dx = \int_0^1 6x^2[\frac{1}{2} - \frac{x^2}{2}] = 3\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \frac{2}{5}$

$Cov(X,Y) = E(XY) - E(X)E(Y) = \frac{2}{5} - (\frac{1}{2})(\frac{3}{4}) = \boxed{\frac{1}{40}}$

Example 43.

Let X and Y be discrete random variables with joint pdf $f(x,y) = \frac{4}{5xy}$ if $x = 1, 2$ and $y = 2, 3$, and zero otherwise. Find $Cov(X, Y)$.

Sol:

	y		
x	2	3	$f_2(y)$
1	$\frac{4}{10}$	$\frac{4}{15}$	$\frac{2}{3}$
2	$\frac{4}{20}$	$\frac{4}{30}$	$\frac{1}{3}$
$f_1(x)$	$\frac{3}{5}$	$\frac{2}{5}$	

$E(X) = 1(\frac{2}{3}) + 2(\frac{1}{3}) = \frac{4}{3}$
$E(Y) = 2(\frac{3}{5}) + 3(\frac{2}{5}) = 2.4$
$E(XY) = 1(2)(\frac{4}{10}) + 1(3)(\frac{4}{15}) + 2(2)(\frac{4}{20}) + 2(3)(\frac{4}{30}) = 3.2$
$Cov(X,Y) = E(XY) - E(X)E(Y) = 3.2 - (\frac{4}{3})(2.4) = 0$

Example 44. Let X_1 and X_2 be discrete random variables with joint probability distribution as show in table below. Show that X_1 and X_2 are dependent but have zero covariance.

x_1	x_2		
	-1	0	1
-1	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{1}{16}$
0	$\frac{3}{16}$	0	$\frac{3}{16}$
1	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{1}{16}$

Sol:

x_1	x_2			$p(x_1)$
	-1	0	1	
-1	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{1}{16}$	$\frac{5}{16}$
0	$\frac{3}{16}$	0	$\frac{3}{16}$	$\frac{6}{16}$
1	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{1}{16}$	$\frac{5}{16}$
$p(x_2)$	$\frac{5}{16}$	$\frac{6}{16}$	$\frac{5}{16}$	

$p_1(-1)p_2(-1) = (\frac{5}{16}^2 = \frac{25}{256} \neq p(-1,-1)$

Thus X_1 and X_2 are dependent.

$E(X_1) = (-1)(\frac{5}{16}) + 0(\frac{6}{16}) + (1)(\frac{5}{16} = 0 = E(X_2)$

$E(X_1X_2) = (-1)(-1)(\frac{1}{16}) + (-1)(0)(\frac{3}{16}) + (-1)(1)(\frac{1}{16}) + (0)(-1)(\frac{3}{16} + (0)(0)(0) + (0)(1)(\frac{3}{16} + (1)(-1)(\frac{1}{16} + (1)(0)(\frac{3}{16} + (1)(1)(\frac{1}{16}) = 0$

$Cov(X_1, X_2) = 0$