MEME16203 Linear Models Marking Guide

Test 2

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Q1. CO3: Derive the probability distribution of linear and quadratic forms

(a) Consider the model

$$Y_{ij} = \gamma_0 + \gamma_1 X_i + \gamma_2 X_i^2 + \gamma_3 X_i^3 + \alpha_i + \epsilon_{ij}, \quad i = 1, 2, \dots, 8; \quad j = 1, \dots, 8$$

where $\epsilon \sim NID(0, \tau^2)$. This model can be expresses in matrix notation as $\mathbf{Y} = \mathbf{W}\boldsymbol{\gamma} + \boldsymbol{\epsilon}$. Let the matrix $\mathbf{Z}$ be the first 4 columns of the matrix $\mathbf{W}$, define $\mathbf{P_Z} = \mathbf{Z}(\mathbf{Z^TZ})^{-1}\mathbf{Z^T}$ and $\mathbf{P_W} = \mathbf{W}(\mathbf{W^TW})^{-1}\mathbf{W^T}$.

(i) Use Cochran's theorem to derive the distribution of $F = \dfrac{c\mathbf{Y^T}(\mathbf{P_W} - \mathbf{P_Z})\mathbf{Y}}{\mathbf{Y^T}(\mathbf{I} - \mathbf{P_W})\mathbf{Y}}$. Report c, degrees of freedom and a formula for the noncentrality parameter. (10 marks)

Ans.

Let $\mathbf{A_1} = \mathbf{P_Z}$, $\mathbf{A_2} = \mathbf{P_W} - \mathbf{P_Z}$ and $\mathbf{A_3} = \mathbf{I} - \mathbf{P_W}$. Then,

- $\mathbf{A_1}$, $\mathbf{A_2}$ and $\mathbf{A_3}$ are all 64×64 symmetric matrices.
- $\mathbf{A_1} + \mathbf{A_2} + \mathbf{A_3} = \mathbf{I}$.
- $\text{rank}(\mathbf{A_1}) + \text{rank}(\mathbf{A_2}) + \text{rank}(\mathbf{A_3}) = 4 + (11-4) + (64-11) = 64$.

Then, by Cochran's Theorem,

- $\frac{1}{\tau^2}\mathbf{Y^T A_k Y} \sim \chi^2_{r_k}(\frac{1}{\tau^2}(\mathbf{X}\boldsymbol{\beta})^T\mathbf{A_k X}\boldsymbol{\beta})$, where $r_k = \text{rank}(\mathbf{A_k})$ for $k = 1, 2, 3$
- $\mathbf{Y^T A_1 Y}$, $\mathbf{Y^T A_2 Y}$ and $\mathbf{Y^T A_3 Y}$ are distributed independently.

Now $DF_2 = \text{rank}(\mathbf{A_2})$ = 11-4 = 7 and $DF_3 = \text{rank}(\mathbf{A_3})$ = 64-11 = 53

- $\lambda_2 = \frac{1}{\tau^2}(\mathbf{W}\boldsymbol{\gamma})^T(\mathbf{P_W} - \mathbf{P_Z})(\mathbf{W}\boldsymbol{\gamma})$
- $\lambda_3 = \frac{1}{\tau^2}(\mathbf{W}\boldsymbol{\gamma})^T(\mathbf{I} - \mathbf{P_W})(\mathbf{W}\boldsymbol{\gamma}) = 0$

Hence,

- $\frac{1}{\tau^2}\mathbf{Y^T}(\mathbf{P_W} - \mathbf{P_Z})\mathbf{Y} \sim \chi^2_7(\lambda_2)$ and
- $\frac{1}{\tau^2}\mathbf{Y^T}(\mathbf{I} - \mathbf{P_W})\mathbf{Y} \sim \chi^2_{53}$

Since $\frac{1}{\tau^2}\mathbf{Y^T}(\mathbf{P_W} - \mathbf{P_Z})\mathbf{Y}$ and $\frac{1}{\tau^2}\mathbf{Y^T}(\mathbf{I} - \mathbf{P_W})\mathbf{Y}$ are independent, then

$$F = \frac{c\mathbf{Y^T}(\mathbf{P_W} - \mathbf{P_Z})\mathbf{Y}}{\mathbf{Y^T}(\mathbf{I} - \mathbf{P_W})\mathbf{Y}} = \frac{\frac{1}{7\tau^2}\mathbf{Y^T}(\mathbf{P_W} - \mathbf{P_Z})\mathbf{Y}}{\frac{1}{53\tau^2}\mathbf{Y^T}(\mathbf{I} - \mathbf{P_X})\mathbf{Y}} = \frac{53\mathbf{Y^T}(\mathbf{P_X} - \mathbf{P_1})\mathbf{Y}}{7\mathbf{Y^T}(\mathbf{I} - \mathbf{P_X})\mathbf{Y}} \sim F_{7,53}(\lambda$$

Thus,

- $c = \frac{53}{7}$,
- the degrees of freedom are $(7, 53)$ and
- noncentrality parameter is $\lambda_2 = \frac{1}{\tau^2}(\mathbf{W}\boldsymbol{\gamma})^T(\mathbf{P_W} - \mathbf{P_Z})(\mathbf{W}\boldsymbol{\gamma})$

(ii) Show that the noncentrality parameter is zero if $\eta_1\mathbf{w_5} + \eta_2\mathbf{w_6} + \cdots + \eta_8\mathbf{w_{12}} = \mathbf{Zc}$ for some vector $\mathbf{c}$, where $\mathbf{w_j}$ is the j^{th} column of $\mathbf{W}$. (10 marks)

Ans.

$$\begin{aligned}
&\frac{1}{\tau^2}(\mathbf{W}\boldsymbol{\gamma})^T(\mathbf{P_W} - \mathbf{P_Z})(\mathbf{W}\boldsymbol{\gamma}) \\
&= \frac{1}{\tau^2}[(\mathbf{W}\boldsymbol{\gamma})^T(\mathbf{I} - \mathbf{P_Z})(\mathbf{W}\boldsymbol{\gamma}) - (\mathbf{W}\boldsymbol{\gamma})^T(\mathbf{I} - \mathbf{P_W})(\mathbf{W}\boldsymbol{\gamma})] \\
&= \frac{1}{\tau^2}(\mathbf{W}\boldsymbol{\gamma})^T(\mathbf{I} - \mathbf{P_Z})(\mathbf{W}\boldsymbol{\gamma})
\end{aligned}$$

This is zero if and only if $(\mathbf{I} - \mathbf{P_Z})(\mathbf{W}\boldsymbol{\gamma}) = 0$. Note that

$$\begin{aligned}
&(\mathbf{I} - \mathbf{P_Z})(\mathbf{W}\boldsymbol{\gamma}) \\
&= (\mathbf{I} - \mathbf{P_Z})\left[\mathbf{Z}\ \mathbf{w_5}\ \mathbf{w_6}\ \cdots\ \mathbf{w_{12}}\right]\boldsymbol{\gamma} \\
&= \left[\mathbf{0}\ (\mathbf{I} - \mathbf{P_Z})\mathbf{w_5}\ (\mathbf{I} - \mathbf{P_Z})\mathbf{w_6}\ \cdots\ (\mathbf{I} - \mathbf{P_Z})\mathbf{w_{12}}\right]\boldsymbol{\gamma} \\
&= \left[\eta_1(\mathbf{I} - \mathbf{P_Z})\mathbf{w_5}\ \eta_2(\mathbf{I} - \mathbf{P_Z})\mathbf{w_6}\ \cdots\ \eta_8(\mathbf{I} - \mathbf{P_Z})\mathbf{w_{12}}\right] \\
&= (\mathbf{I} - \mathbf{P_Z})\left[\eta_1\mathbf{w_5}\ \eta_2\mathbf{w_6}\ \cdots\ \eta_8\mathbf{w_{12}}\right] \\
&= (\mathbf{I} - \mathbf{P_Z})\mathbf{Zc} \\
&= 0 \text{ since } \mathbf{P_Z} = \mathbf{Z}
\end{aligned}$$

(b) Consider the model $Y_{ij} = \mu + \alpha_i + \epsilon_{ij}$, where $i = 1, 2, 3, 4$, $j = 1, 2$, and μ, α_1, α_2, α_3, α_4, are unknown parameters. Let $\epsilon_{ij} \overset{iid}{\sim} N(0, \sigma^2)$, where σ^2 is unknown.

(i) Determine the distribution of $\frac{\hat{\tau}^2}{19\sigma^2}$ when $\tau = 0$, where $\hat{\tau}$ is the BLUE of $\tau = 3\alpha_1 - 4\alpha_2 + 3\alpha_3 + 2\alpha_4$. (10 marks)

Ans.

$$\begin{aligned}
\tau &= 3\alpha_1 - 4\alpha_2 + 3\alpha_3 + 2\alpha_4 \\
&= 3(\mu + \alpha_1) - 4(\mu + \alpha_2) + 3(\mu + \alpha_3) + 2(\mu + \alpha_4) \\
&= E(3\bar{Y}_{1.} - 4\bar{Y}_{2.1} + 3\bar{Y}_{3.} + 2\bar{Y}_{4.})
\end{aligned}$$

Hence, τ is estimable.

The BLUE for τ is $3\bar{Y}_{1.} - 4\bar{Y}_{2.} + 3\bar{Y}_{3.} + 2\bar{Y}_{4.}$). Let$\mathbf{Y}$ $=$

$\left[Y_{11}\ Y_{12}\ Y_{21}\ Y_{22}\ Y_{31}\ Y_{32}\ Y_{41}\ Y_{42}\right]^T$, then
$\hat{\tau} = 3\bar{Y}_{1.} - 4\bar{Y}_{2.} + 3\bar{Y}_{3.} + 2\bar{Y}_{4.} = \left[\frac{3}{2}\ \frac{3}{2}\ -\frac{4}{2}\ -\frac{4}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{2}{2}\ \frac{2}{2}\right]\mathbf{Y} \sim N(\tau = 0, 19\sigma^2)$.
Then,
$\frac{\hat{\tau}-0}{\sqrt{19\sigma^2}} \sim N(0,1)$ and $\frac{\hat{\tau}^2}{19\sigma^2} \sim \chi^2(1)$.

(ii) Determine the distribution of $S^2 = \sum_{i=1}^4 \sum_{j=1}^2 (Y_{ij} - \bar{Y}_{i.})^2$. (10 marks)

Ans.
Note that $S^2 = \mathbf{Z^T Z}$ where

$$\mathbf{Z} = \begin{bmatrix} Y_{11}-Y_{1.} \\ Y_{12}-Y_{1.} \\ Y_{21}-Y_{2.} \\ Y_{22}-Y_{2.} \\ Y_{31}-Y_{3.} \\ Y_{32}-Y_{3.} \\ Y_{41}-Y_{4.} \\ Y_{42}-Y_{4.} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} Y_{11} \\ Y_{12} \\ Y_{21} \\ Y_{22} \\ Y_{31} \\ Y_{32} \\ Y_{41} \\ Y_{42} \end{bmatrix} = \mathbf{CY}.$$

Then $\mathbf{Z} \sim N(\mathbf{0}, \sigma^2\mathbf{C})$ and $\mathbf{CC} = \mathbf{C}$. Let $\mathbf{\Sigma} = \sigma^2\mathbf{C}$ and $\mathbf{A} = \frac{1}{\sigma^2}\mathbf{I}$. Then $\mathbf{A\Sigma A\Sigma} = \mathbf{CC} = \mathbf{C} = \mathbf{A\Sigma}$ which is idempotent. Hence it follows that $\frac{S^2}{\sigma} = \mathbf{Z^T A Z} \sim \chi^2(4)$ because rank$(\mathbf{C}) = 4$

(iii) Show that $F = \frac{c\hat{\tau}^2}{S^2}$, where c is a constant, has central $F-$distribution when $\tau = 0$. Report c. (10 marks)

Ans.
Note that

$$\begin{bmatrix} \hat{\tau} \\ \mathbf{Z} \end{bmatrix} = \sim N\left(\begin{bmatrix} \tau \\ \mathbf{0} \end{bmatrix}, \sigma^2 \begin{bmatrix} 19 & \mathbf{0^T} \\ \mathbf{0} & \mathbf{C} \end{bmatrix}\right)$$

Consequently, $\widehat{\boldsymbol{\tau}}$ and $\mathbf{Z}$ are independent which implies that $\frac{\hat{\tau}}{19\sigma^2}$ and $\frac{S^2}{\sigma^2}$ are independent central chi-square random variables with 1 and 4 degrees freedom respectively, and

$F = \frac{\frac{\hat{\tau}^2}{19\sigma^2}}{\frac{S^2}{4\sigma^2}} = \frac{4\hat{\tau}^2}{19\sigma^2} \sim F(1,3)$ and $c = \frac{4}{19}$.

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Q2. CO4: Test the hypotheses on estimable functions

(a) Let $\mathbf{Y} \sim N(\mathbf{W}\boldsymbol{\gamma}, \sigma^2 I)$, where

- $\mathbf{W} = \begin{bmatrix}\mathbf{W_1} & \mathbf{W_2} & \mathbf{W_3}\end{bmatrix}$,
- $\mathbf{W_1} = \mathbf{1_{10}}$, $\mathbf{W_2} = \begin{bmatrix}1\\-1\end{bmatrix} \otimes \mathbf{1_5}$, $\mathbf{W_3} = \mathbf{1_2} \otimes \begin{bmatrix}-8\\-4\\0\\4\\8\end{bmatrix}$, and
- $\boldsymbol{\gamma} = \begin{bmatrix}\gamma_1\\\gamma_2\\\gamma_3\end{bmatrix}$

(i) Use Cochran's theorem to find the distributions of

- $\frac{1}{\sigma^2}SSE = \mathbf{e^T e} = \mathbf{Y^T(I - P_W)Y}$, where $\mathbf{P_W} = \mathbf{W(W^TW)^{-1}W^T}$
- $\frac{1}{\sigma^2}R(\gamma_1) = \mathbf{Y^T P_{W_1} Y}$ where $\mathbf{W_1} = \mathbf{1}$ is the first column of $\mathbf{W}$ and $\mathbf{P_{W_1}} = \mathbf{W_1(W_1^T W)^{-1}W_1^T}$.
- $\frac{1}{\sigma^2}R(\gamma_2|\gamma_1) = \mathbf{Y^T(P_{W_2} - P_{W_1})}Y$ where $\mathbf{W_2}$ contains the first two columns of $\mathbf{W}$ and $\mathbf{P_{W_2}} = \mathbf{W_2(W_2^T W_2)^{-1}W_2^T}$.
- $\frac{1}{\sigma^2}R(\gamma_3|\gamma_1\gamma_2) = \mathbf{Y^T(P_W - P_{W_2})Y}$.

(10 marks)

Ans.
Check the conditions of Cochran's Theorem.
1) $\mathbf{(I - P_W)}+\mathbf{P_{W_1}}$ + $\mathbf{(P_{W_2} - P_{W1})}$ +$\mathbf{(P_W - P_{W_2})} = \mathbf{I}$
2) Rank$\mathbf{(I - P_W)}$+Rank$\mathbf{P_{W_1}}$ + Rank$\mathbf{(P_{W_2} - P_{W1})}$ +Rank$\mathbf{(P_W - P_{W_2})} = (10-3)+1+(2-1)+(3-2) = 10 = n$
3) $\mathbf{(I - P_W)}$, $\mathbf{P_{W_1}}$, $\mathbf{(P_{W_2} - P_{W1})}$, $\mathbf{(P_W - P_{W_2})}$ are all symmetric.
Therefore, by Cochran's theorem, the sums of squares are independently distributed as chi-square random, variables with d.f. 7, 1, 1, 1 respectively. (10 marks)

(ii) Report a formula for the non-centrality parameter of the non-central F distribution of

$$F = \frac{R(\gamma_2|\gamma_1)}{SSE/7}$$

Use it to identify the null and alternative hypotheses associated with this test statistic. You are given that: $\mathbf{W^T(P_{W_2} - P_{W_1})W]} = \begin{bmatrix}0 & 0 & 0\\0 & 10 & 0\\0 & 0 & 0\end{bmatrix}$. (10 marks)

Ans.

$\lambda = \frac{1}{\sigma^2}\boldsymbol{\beta}^{\mathbf{T}}\mathbf{W}^{\mathbf{T}}(P_{W_2} - P_{W_1})\mathbf{W}\boldsymbol{\beta} = \frac{1}{\sigma^2}\begin{bmatrix}\gamma_1 & \gamma_2 & \gamma_3\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\ 0 & 10 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\gamma_1\\ \gamma_2\\ \gamma_3\end{bmatrix} =$
$\frac{10}{\sigma^2}\gamma_2^2$

Therefore, $\lambda = 0$ iff $\gamma_2 = 0$ and the hypothesis is $H_0 : \gamma_3 = 0$ against the alternative $H_1 : \gamma_2 \neq 0$.

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(b) An researcher recorded moisture content for three types of cheese made by two different methods. A 3×2 factorial experiment with types of cheese made by two different methods was conducted. The data had unequal replications among the six treatment combinations of the two factors, Cheese and Method. The collected data are given below.

	Cheese		
Method	1	2	3
1	38.04 39.45	39.2 39.1 38.81	38.99 38.72 39.02 39.04
2	38.79 38.87 39.18	39.55	39.36 39.14 39.28

Consider the model $y_{ijk} = \mu_{ij} + \epsilon_{ijk}$, where $\epsilon_{ijk} \sim NID(0, \sigma^2)$, $i = 1, 2$, and $j = 1, 2, 3$ and $k = 1, \ldots, n_{ij}$. This model can be expressed in matrix form as $\mathbf{Y} = \mathbf{D}\boldsymbol{\beta} + \boldsymbol{\epsilon}$. Examine type III sums of squares for these data.

(i) Specify the $\mathbf{C}$ matrix needed to write the null hypothesis associated with the F-test for Method effects in the form $H_0 : \mathbf{C_1}\boldsymbol{\beta} = \mathbf{0}$. (10 marks)

Ans.

$$\mathbf{C_1} = [\mathbf{I}_{a-1} | -\mathbf{1}_{a-1}] \otimes \mathbf{1_b^T} = [\mathbf{I}_1 | -\mathbf{1}_1] \otimes \mathbf{1_3^T} = \begin{bmatrix} 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 & 1 & -1 & -1 & -1 \end{bmatrix}$$

(ii) Present a formula for $SS_{H_{0,1}}$, corresponding to the null hypothesis in part (a), and state it's distribution when the null hypothesis is true. (10 marks)

Ans.

$$SS_{H_{0,1}} = (\mathbf{C_1 b} - \mathbf{0})^{\mathbf{T}} [\mathbf{C_1} (\mathbf{D^T D})^{-1} \mathbf{C_1^T}]^{-1} (\mathbf{C_1 b} - \mathbf{0})$$
$$= \mathbf{y^T D} (\mathbf{D^T D})^{-1} \mathbf{C_1^T} [\mathbf{C_1} (\mathbf{D^T D})^{-1} \mathbf{C_1^T}]^{-1} \mathbf{C_1} (\mathbf{D^T D})^{-1} \mathbf{D^T y}$$

Under H_0, $SSH_{0,1} \sim \frac{1}{\sigma^2}\chi_1^2$

(iii) Compute $SS_{H_{0,1}}$. (10 marks)

Ans.

$$\mathbf{b} = \begin{bmatrix} \bar{Y}_{11.} & \bar{Y}_{12.} & \bar{Y}_{13.} & \bar{Y}_{21.} & \bar{Y}_{22.} & \bar{Y}_{23.} \end{bmatrix}^T = \begin{bmatrix} 38.745 & 39.037 & 38.943 & 38.947 & 39.55 & 39.26 \end{bmatrix}^T$$

$$\mathbf{C_1 b} = \begin{bmatrix} 1 & 1 & 1 & -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 38.745 \\ 39.037 \\ 38.943 \\ 38.947 \\ 39.55 \\ 39.26 \end{bmatrix} = \begin{bmatrix} -1.032 \end{bmatrix}$$

$\mathbf{D^TD^{-1}} = diag(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{3}, 1, \frac{1}{3})$

$$C_1(D^TD)^{-1}C^T] = \begin{bmatrix} 1 & 1 & 1 & -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2.75 \end{bmatrix}$$

$SSH_{01} = \mathbf{(C_1b - 0)^T[C_1(D^TD)^{-1}C_1^T]^{-1}(C_1b - 0)} = \begin{bmatrix} -1.032 \end{bmatrix} \begin{bmatrix} 0.3636 \end{bmatrix} \begin{bmatrix} -1.032 \end{bmatrix} = 0.3873$