MEME16203Linear Models Marking Guide

Assignment 1

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Faculty:	FES	Unit Code:	MEME16203
Course:	MAC	Unit Title:	Linear Models
Year:	1,2	Lecturer:	Dr Yong Chin Khian
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Q1. In an experiment to investigate the effect of color paper (yellow, green and red) on response rates for questionnaires distributed by the "windshield method" in supermarket parking lots, 12 representative supermarket parking lots were chosen in a metropolitan area and each color was assigned random to three of the lots. The reponse rates (in percent) follow.

		j			
	i	1	2	3	4
1	Yellow	28	26	31	33
2	Green	24	29	25	26
3	Red	33	38	36	39

Consider the linear model

$$y_{ij} = \mu + \tau_i + \epsilon_{ij}, \text{ for } i = 1,2,3 \text{ and } j = 1,2,3,4$$

where

- y_{ij} is the observed response rate for the i^{th} color paper assigned to the j^{th} parking lot.
- τ_i corresponds to the effect of i^{th} color.
- $\epsilon_{ij} \sim N(0, \sigma^2)$.

(a) Write model above in the form $\mathbf{y} = X\boldsymbol{\beta} + \boldsymbol{\epsilon}$. Do not impose any restriction on the parameters. (5 marks)

Ans.

y_{11}
y_{12}
y_{13}
y_{14}
y_{21}
y_{22}
y_{23}

y_{24}
y_{31}
y_{32}
y_{33}
y_{34}
$= 1100$
1100
1100
1100
1010
1010
1010
1010
1001
1001
1001
1001
μ
τ_1
τ_2
$\tau_3 + e_{11}$
e_{12}
e_{13}
e_{14}
e_{21}
e_{22}
e_{23}
e_{24}
e_{31}
e_{32}
e_{33}
e_{34}

(b) Obtain one of the generalized inverse of $\mathbf{X^TX}$, $\mathbf{G}$. (5 marks)

Ans.
$\mathbf{X^TX} = 12444$
4400
4040
4004
Let $\mathbf{W} = 400$
040

004, $\mathbf{W^{-1}} = \frac{1}{4}00$
$0\frac{1}{4}0$
$00\frac{1}{4}$
So, one of the generalized inverse of $\mathbf{X^TX}$, G is
$\mathbf{G} = (\mathbf{X^TX})^- = 0000$
$0\frac{1}{4}00$
$00\frac{1}{4}0$
$000\frac{1}{4}$

(c) Use the generalized inverse you obtained in part(b) to compute a solution to the normal equations, $\hat{\boldsymbol{\beta}} = \begin{bmatrix} \hat{\mu} \\ \hat{\tau}_1 \\ \hat{\tau}_2 \end{bmatrix}$. (5 marks)

Ans.
$\mathbf{X^Ty} = [cccccccccccc111111111111111100000000000011110000000000001111]$ 28
26
31
33
24
29
25
326
33
38
36
39 = 368
118
104
146
$\hat{\boldsymbol{\beta}} = (\mathbf{X^TX})^-\mathbf{X^Ty} = 0000$
$0\frac{1}{4}00$
$00\frac{1}{4}0$
$000\frac{1}{4}368$
118
104
146 = 0.0
29.5
26.0
36.5

(d) Using your solution $\hat{\boldsymbol{\beta}}$ to the normal equation from part (c), estimates $2\tau_1 + \tau_2 - 3\tau_3$. (5 marks)

Ans.

$\tau_1 - \tau_2 = c^T\boldsymbol{\beta} = 021 - 3\mu$

τ_1

τ_2

τ_3

$\widehat{2\tau_1 + \tau_2 - 3\tau_3} = c^T\hat{\boldsymbol{\beta}} = 021 - 3\hat{\mu}$

$\hat{\tau}_1$

$\hat{\tau}_2 = 021 - 30.0$

29.5

26.0

36.5 = -24.5

(e) Is your estimate in part(d) the unique BLUE? Explain. (5 marks)

Ans.

$E(2Y_{11} + Y_{21} - 3Y_{31}) = 2(\mu + \tau_1 + \epsilon_{11}) + (\mu + \tau_2 + \epsilon_{21}) - 3(\mu + \tau_3 + \epsilon_{31})$
$= 2\tau_1 + \tau_2 - 3\tau_3$

Hence $2\tau_1 + \tau_2 - 3\tau_3$ is estimable. Thus by Gauss Markov Theorem, $\widehat{2\tau_1 + \tau_2 - 3\tau_3} = c^T\hat{\boldsymbol{\beta}}$ is the unique BLUE.

Q2. An engineer is designing a battery for use is a device that will be subjected to some extreme variations in temperature. The only design parameter that he can select at this point is the plate material. When the device is manufactured and is shipped to the field, the engineer has no control over the temperature extremes that the device will encounter, and he knows from experience that temperature will probably affect the battery life. However, temperature can be controlled in the product development laboratory for the purpose of a test. The engineer decides to test two plate materials at three temperature levels: 10, 60, and 100$°F$ because these temperature levels are consistent with the product end-use environment. Due to errors made by the laboratory technicians, observations were only obtained from 9 of the 12 batteries. The resulting observed battery life data are given below:

Material Type	Temperature					
	$10°F$		$60°F$		$100°F$	
1	y_{111}	y_{112}	y_{121}	y_{122}	y_{131}	y_{132}
2	y_{211}		y_{221}	y_{222}		

Consider a Gauss-Markov model

$$y_{ijk} = \mu + \alpha_i + \gamma_i X_j + \epsilon_{ijk} \qquad \text{(model A)}$$

where

- y_{ijk} is the battery life time for the k^{th} battery assigned to the j^{th} level of the temperature and the i^{th} level of the battery type,
- X_j denote the level of temperature administered to the battery,
- ϵ_{ijk} denotes a random error with $\epsilon_{ijk} \sim NID(0, \sigma^2)$ where $\sigma^2 > 0$, and
- $\boldsymbol{\beta} = (\mu, \alpha_1, \alpha_2, \alpha_3, \gamma_1, \gamma_2)^T$ is a vector of unknown non-random parameters.

Use this model to answer the following questions. (You may express your answers in matrix notation, but define any new notation that you introduce.)

(a) Write model A in the form $\mathbf{y} = X\boldsymbol{\beta} + \boldsymbol{\epsilon}$. Do not impose any restriction on the parameters. (5 marks)

Ans.

$$\begin{bmatrix} y_{111} \\ y_{112} \\ y_{121} \\ y_{122} \\ y_{131} \\ y_{132} \\ y_{211} \\ y_{221} \\ y_{222} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 10 & 0 \\ 1 & 1 & 0 & 0 & 10 & 0 \\ 1 & 1 & 0 & 0 & 60 & 0 \\ 1 & 1 & 0 & 0 & 60 & 0 \\ 1 & 1 & 0 & 0 & 100 & 0 \\ 1 & 1 & 0 & 0 & 100 & 0 \\ 1 & 0 & 1 & 0 & 0 & 10 \\ 1 & 0 & 1 & 0 & 0 & 60 \\ 1 & 0 & 1 & 0 & 0 & 60 \end{bmatrix} \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \gamma_1 \\ \gamma_2 \end{bmatrix} + \begin{bmatrix} \epsilon_{111} \\ \epsilon_{112} \\ \epsilon_{121} \\ \epsilon_{122} \\ \epsilon_{131} \\ \epsilon_{132} \\ \epsilon_{211} \\ \epsilon_{221} \\ \epsilon_{222} \end{bmatrix}$$

(b) Define what it means for $\mathbf{C}\boldsymbol{\beta}$ to be estimable. (5 marks)

Ans.

For a linear model $E(\mathbf{y}) = X\boldsymbol{\beta}$ with $V(\mathbf{y}) = \boldsymbol{\Sigma}$, a linear function of the parameters $\mathbf{c}^{\mathbf{T}}\boldsymbol{\beta}$ is estimable if there is a vector of constants $\mathbf{a}$ such that $E(\mathbf{a}^{\mathbf{T}}\mathbf{y}) = \mathbf{c}^{\mathbf{T}}\boldsymbol{\beta}$

$\mathbf{C}\boldsymbol{\beta}$ is said to be **estimable** if all of its elements

$$\mathbf{C}\boldsymbol{\beta} = \begin{bmatrix} \mathbf{c}_1^T \\ \mathbf{c}_2^T \\ \vdots \\ \mathbf{c}_r^T \end{bmatrix} \boldsymbol{\beta} = \begin{bmatrix} \mathbf{c}_1^T\boldsymbol{\beta} \\ \mathbf{c}_2^T\boldsymbol{\beta} \\ \vdots \\ \mathbf{c}_r^T\boldsymbol{\beta} \end{bmatrix}$$

are estimable.

(c) For each of the following quantities, determine whether or not it is estimable. Justify your answer. (10 marks)

(i) $\mu + \alpha_1 + \alpha_2$

(ii) γ_1

(iii) $\mu + \alpha_2$

Ans.

(i) let $\mathbf{d^T} = \begin{bmatrix}1 & -1 & -1 & -1 & 0 & 0\end{bmatrix}$, then $\mathbf{Xd} = \mathbf{0}$.
Let $\mathbf{c} = \begin{bmatrix}1 & 1 & 1 & 0 & 0 & 0\end{bmatrix}$, then $\mathbf{c^T}\boldsymbol{\beta} = \mu + \alpha_1 + \alpha_2$, but $\mathbf{c^T d} = -1 \neq 0$, i.e., there exists a vector $\mathbf{d}$ such that $\mathbf{Xd} = \mathbf{0}$ but $\mathbf{c_1^T}\boldsymbol{\beta} \neq 0$. Thus, $\mu + \alpha_1 + \alpha_2$ is not estimable.

(ii) $\frac{E[y_{121} - y_{111}]}{50} = \gamma_1$, Thus γ_1 is estimable.

(iii) $\frac{6E(y_{211}) - E(y_{221})}{5} = \mu + \alpha_2$. Thus $\mu + \alpha_2$ is estimable.

(d) Suppose $\mathbf{C}\boldsymbol{\beta}$ is estimable and $\mathbf{b} = (\mathbf{X^TX})^{-}\mathbf{X^Ty}$ is a solution to the normal equations. Show that $\mathbf{Cb}$ is unbiased and $V(\mathbf{Cb}) = \sigma^2\mathbf{C}(\mathbf{X^TX})^{-}\mathbf{C^T}$.

(5 marks)

Ans.

$$
\begin{aligned}
E(\mathbf{Cb}) &= \mathbf{C}(\mathbf{X^TX})^{-}\mathbf{X^T}E(\mathbf{y}) \\
&= \mathbf{C}(\mathbf{X^TX})^{-}\mathbf{X^TX}\boldsymbol{\beta} \\
&= \mathbf{A^TX}(\mathbf{X^TX})^{-}\mathbf{X^TX}\boldsymbol{\beta} \text{ (since } \mathbf{C}\boldsymbol{\beta} \text{ is estimable and hence } \mathbf{C^T} = \mathbf{A^TX}) \\
&= \mathbf{A^TP_XX}\boldsymbol{\beta} \text{ As } (\mathbf{P_X} = \mathbf{X}(\mathbf{X^TX})^{-}\mathbf{X^T}) \\
&= \mathbf{A^TP_XX}\boldsymbol{\beta} \\
&= \mathbf{A^TX}\boldsymbol{\beta} \text{ (As } \mathbf{P_XX} = \mathbf{X}) \\
&= \mathbf{C}\boldsymbol{\beta}
\end{aligned}
$$

$V(\mathbf{Cb})$

$= V(\mathbf{C}(\mathbf{X^TX})^{-}\mathbf{X^Ty})$

$= \mathbf{C}(\mathbf{X^TX})^{-}\mathbf{X^T}V(\mathbf{y})(\mathbf{C}(\mathbf{X^TX})^{-}\mathbf{X^T})^\mathbf{T}$

$= \mathbf{C}(\mathbf{X^TX})^{-}\mathbf{X^T}\sigma^2\mathbf{I}(\mathbf{X}(\mathbf{X^TX})^{-}\mathbf{C^T})$

$= \mathbf{a^TX}(\mathbf{X^TX})^{-}\mathbf{X^T}\sigma^2\mathbf{I}(\mathbf{X}(\mathbf{X^TX})^{-}(\mathbf{a^TX})^\mathbf{T})$ since $\mathbf{C}\boldsymbol{\beta}$ is estimable and hence $\mathbf{C^T} = \mathbf{a^TX}$

$= \sigma^2\mathbf{a^TX}(\mathbf{X^TX})^{-}\mathbf{X^TX}(\mathbf{X^TX})^{-}\mathbf{X^Ta}$

$= \sigma^2\mathbf{a^TP_XP_Xa} = \sigma^2\mathbf{a^TP_Xa}$ since $\mathbf{P_X}$ is idempotent

$= \sigma^2\mathbf{a^TX}(\mathbf{X^TX})^{-}\mathbf{X^Ta}$

$= \sigma^2\mathbf{C}(\mathbf{X^TX})^{-}\mathbf{C^T}$

Q3. Let the 3×1 random vector $\mathbf{y}$ follows a multivariate normal distribution with mean vector $\boldsymbol{\mu} = [9 \quad 13 \quad 6]^T$ and covariance matrix $\boldsymbol{\Sigma}$ where

$$
\boldsymbol{\Sigma} = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 5 & 1 \\ -1 & 1 & 3 \end{bmatrix}
$$

Consider the vector $\mathbf{w}$ where

$$\mathbf{w} = \begin{bmatrix} 4y_1 - 2y_2 + 1y_3 - 20 \\ 2y_1 + 1y_2 - 5y_3 - 13 \end{bmatrix}.$$

Find the mean vector and covariance matrix of $\mathbf{W}$. (10 marks)

Ans.

$\mathbf{w} = A\mathbf{y} + \mathbf{d}$

where

$$A = \begin{bmatrix} 4 & -2 & 1 \\ 2 & 1 & -5 \end{bmatrix} \quad \text{and} \quad \mathbf{d} = \begin{bmatrix} -20 \\ -13 \end{bmatrix}$$

$$E(\mathbf{w}) = A\boldsymbol{\mu} + \mathbf{d} = \begin{bmatrix} 4 & -2 & 1 \\ 2 & 1 & -5 \end{bmatrix} \begin{bmatrix} 9 \\ 13 \\ 6 \end{bmatrix} + \begin{bmatrix} -20 \\ -13 \end{bmatrix} = \begin{bmatrix} -4 \\ -12 \end{bmatrix}$$

$$\begin{aligned} Cov(\mathbf{w}) &= A\Sigma A^T \\ &= \begin{bmatrix} 4 & -2 & 1 \\ 2 & 1 & -5 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 0 & 5 & 1 \\ -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ -2 & 1 \\ 1 & -5 \end{bmatrix} \\ &= \begin{bmatrix} 43 & 20 \\ 20 & 98 \end{bmatrix} \end{aligned}$$

Q4. Suppose $\mathbf{Z} = \underset{5\times 1}{\mathbf{1}}$, $\mathbf{G} = 36$, $\mathbf{R} = 25\underset{5\times 5}{\mathbf{I}}$. If $\mathbf{\Sigma} = \mathbf{Z}\mathbf{G}\mathbf{Z}^{\mathbf{T}} + \mathbf{R}$, find $\mathbf{\Sigma}^{-1}$. (10 marks)

Ans.

$$\mathbf{\Sigma} = \mathbf{Z}\mathbf{G}\mathbf{Z}^{\mathbf{T}} + \mathbf{R} = \underset{5\times 1}{\mathbf{1}}(36)\underset{1\times 5}{\mathbf{1}^{\mathbf{T}}} + 25\underset{5\times 5}{\mathbf{I}} = 36\underset{5\times 1}{\mathbf{1}}\underset{1\times 5}{\mathbf{1}^{\mathbf{T}}} + 25\underset{5\times 5}{\mathbf{I}} = 36\underset{5\times 5}{\mathbf{J}} + 25\underset{5\times 5}{\mathbf{I}}$$

$$\mathbf{\Sigma}^{-1} = [25\underset{5\times 5}{\mathbf{I}} + 36\underset{5\times 5}{\mathbf{J}}]^{-1} = \frac{1}{25}\left[\underset{5\times 5}{\mathbf{I}} - \frac{36}{25+5\times 36}\underset{5\times 5}{\mathbf{J}}\right] = \frac{1}{25}\begin{bmatrix} \frac{169}{205} & \frac{-36}{205} & \cdots & \frac{-36}{205} \\ \frac{-36}{205} & \frac{169}{205} & \cdots & \frac{-36}{205} \\ \vdots & \vdots & \ddots & \\ \frac{-36}{205} & \frac{-36}{205} & \cdots & \frac{169}{205} \end{bmatrix}$$

Q5. Suppose $\mathbf{X}$ is an $n \times p$ matrix and $\mathbf{B}$ is a $p \times p$ non-singular matrix. Define $\mathbf{P_X} = \mathbf{X}(\mathbf{X^T X})^{-}\mathbf{X^T}$, prove that $\mathcal{C}(\mathbf{P_X}) = \mathcal{C}(\mathbf{P_X B^{-1}})$. (15 marks)

Ans.

Prove that $\mathcal{C}(\mathbf{P_X}) = \mathcal{C}(\mathbf{P_X B^{-1}})$:

$\mathbf{a} \in \mathcal{C}(\mathbf{P_X}) \iff \mathbf{a} = \mathbf{P_X}\mathbf{b}$ for some $\mathbf{b}$

$$\begin{aligned} &\iff \mathbf{a} = \mathbf{P_X I b} \text{ for some } \mathbf{b} \\ &\iff \mathbf{a} = \mathbf{P_X B^{-1}} \underbrace{\mathbf{Bb}}_{p\times 1} \text{ treat as } \mathbf{P_X B^{-1}} \text{ product a } p\times 1 \text{ vector} \\ &\implies \mathbf{a} \in \mathcal{C}(\mathbf{P_X B^{-1}}) \end{aligned}$$

So, $\mathcal{C}(\mathbf{P_X}) \subseteq \mathcal{C}(\mathbf{P_X B^{-1}})$
Then similarly,

$$\begin{aligned} \mathbf{g} \in \mathcal{C}(\mathbf{P_X B^{-1}}) &\iff \mathbf{g} = \mathbf{P_X B^{-1} h} \text{ for some } \mathbf{h} \\ &\iff \mathbf{g} = \mathbf{P_X} \underbrace{\mathbf{B^{-1}h}}_{p\times 1} \text{ treat as } \mathbf{P_X} \text{ product a } p\times 1 \text{ vector} \\ &\implies \mathbf{g} \in \mathcal{C}(\mathbf{P_X}) \end{aligned}$$

So, $\mathcal{C}(\mathbf{P_X B^{-1}}) \subseteq \mathcal{C}(\mathbf{P_X})$
And hence,
$\mathcal{C}(\mathbf{P_X}) = \mathcal{C}(\mathbf{P_X B^{-1}})$

Q6. A useful result from linear algebra (that you may use it without proof) is as follows:

$$\text{rank}(\mathbf{AB}) \le \min[\text{rank}(\mathbf{A}), \text{rank}(\mathbf{B})]$$

for any two matrices $\mathbf{A}$ and $\mathbf{B}$ with dimensions that allow multiplication (number of columns of $\mathbf{A}$ equals the number of rows of $\mathbf{B}$). In words, this result says that the rank of a product of matrices is no greater than the rank of any matrix in the product. Show that f $\mathbf{X}$ is an $n \times p$ matrix and $\mathbf{W}$ is a matrix with n columns satisfying $\mathbf{WP_X} = \mathbf{W}$, then $\text{rank}(\mathbf{WX}) = \text{rank}(\mathbf{W})$. (15 marks)

Ans.

We are given that for any two matrices $\mathbf{A}$ and $\mathbf{B}$ that allow for the product matrix $\mathbf{AB}$,

$$\text{rank}(\mathbf{AB}) \le \min[\text{rank}(\mathbf{A}), \text{rank}(\mathbf{B})]$$

This says rank(AB) is no larger than the smaller of the two quantities rank($\mathbf{A}$) and rank($\mathbf{B}$),which implies

$$\text{rank}(\mathbf{AB}) \le \text{rank}(\mathbf{A}) \text{ and } \text{rank}(\mathbf{AB}) \le \text{rank}(\mathbf{B}).$$

(i) $\text{rank}(\mathbf{WX}) \le \text{rank}\mathbf{W}$
(ii)

$$\begin{aligned} \text{rank}(\mathbf{W}) &= \text{rank}(\mathbf{WP_X}) \text{ since } \mathbf{WP_X} = \mathbf{W} \\ &\le \text{rank}(\mathbf{WX}(\mathbf{X^T X})^{-}\mathbf{X^T}) \\ &\le \text{rank}(\mathbf{WX}) \end{aligned}$$

$\therefore \text{rank}(\mathbf{WX}) = \text{rank}(\mathbf{P_W})$.