MEME16203Linear Models

Assignment 1

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Q1. Let $\mathbf{A}$ be an $n \times n$ symmetric matrix with rank $(\mathbf{A}) = r$. Here r may be smaller than n. Let

$$\mathbf{A} = \mathbf{L}\begin{bmatrix} \boldsymbol{\Delta}_r & 0 \\ 0 & 0 \end{bmatrix}\mathbf{L^T}$$

represent the spectral decomposition of A. Then, $\boldsymbol{\Delta}_r$ is an $r \times r$ diagonal matrix containing the positive eigenvalues of $\mathbf{A}$, and $\mathbf{L}$ is an $n \times n$ orthogonal matrix where the columns are eignenvectors of $\mathbf{A}$. Show that

$$\mathbf{G} = \mathbf{L}\begin{bmatrix} \boldsymbol{\Delta}_r^{-1} & 0 \\ 0 & 0 \end{bmatrix}\mathbf{L^T}$$

satisfies the definition of the Moore-Penrose inverse of $\mathbf{A}$.

Q2. Suppose $\mathbf{X}$ and $\mathbf{W}$ are any two matrices with n rows for which $C(\mathbf{X}) = C(\mathbf{W})$. Show that $\mathbf{P_X} = \mathbf{P_W}$, where $\mathbf{P_X} = \mathbf{X}(\mathbf{X^T X})^{-}\mathbf{X^T}$ and $\mathbf{P_W} = \mathbf{W}(\mathbf{W^T W})^{-}\mathbf{W^T}$.

Q3. Suppose $\mathbf{X}$ is an 45×8 matrix. Prove that $\mathcal{C}(\mathbf{X}) = \mathcal{C}(\mathbf{X}(\mathbf{X^T X})^{-}\mathbf{X^T})$.

Q4. Suppose $\mathbf{Z} = \underset{5\times 1}{\mathbf{1}}$, $\mathbf{G} = 36$, $\mathbf{R} = 49\underset{5\times 5}{\mathbf{I}}$. If $\boldsymbol{\Sigma} = \mathbf{ZGZ^T} + \mathbf{R}$, find $\boldsymbol{\Sigma}^{-1}$.

Q5. Show that the matrix $\underset{n\times n}{\mathbf{A}} = \mathbf{I_n} - \frac{1}{n}\mathbf{J_n}$ is singular.

Q6. A useful result from linear algebra (that you may use it without proof) is as follows:

$$\operatorname{rank}(\mathbf{UV}) \le \min[\operatorname{rank}(\mathbf{U}), \operatorname{rank}(\mathbf{V})]$$

for any two matrices $\mathbf{U}$ and $\mathbf{V}$ with dimensions that allow multiplication (number of columns of $\mathbf{U}$ equals the number of rows of $\mathbf{V}$). In words, this result says that the rank of a product of matrices is no greater than the rank of any matrix in the product. Show that for any matrix $\mathbf{X}$, $\operatorname{rank}(\mathbf{X}) = \operatorname{rank}(\mathbf{P_X})$, where $\mathbf{P_X} = \mathbf{X}(\mathbf{X^T X})^{-1}\mathbf{X^T}$.