MEME16203 Linear Models Marking Guide

Test 1

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Faculty:	FES	Unit Code:	MEME16203
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Show your workings. If no workings are shown, ZERO is awarded.

Q1. Suppose $\boldsymbol{X}$ is an $n \times p$ matrix. Prove that $\mathcal{C}(\boldsymbol{X}) = \mathcal{C}(\boldsymbol{P_X})$.

(10 marks)

Ans.

Key:
1. $\boldsymbol{a} \in \mathcal{C}(\boldsymbol{X}) \iff \boldsymbol{a} = \boldsymbol{Xb}$ for some $\boldsymbol{b}$.
2. $\boldsymbol{P_X X} = \boldsymbol{X}$ by property of projection matrix

Prove that $\mathcal{C}(\boldsymbol{X}) = \mathcal{C}(\boldsymbol{P_X})$:

$$
\begin{aligned}
\boldsymbol{a} \in \mathcal{C}(\boldsymbol{X}) &\iff \boldsymbol{a} = \boldsymbol{Xb} \text{ for some } \boldsymbol{b} \text{ by key 1} \\
&\iff \boldsymbol{a} = \underbrace{\boldsymbol{P_X X}}_{\boldsymbol{X}} \boldsymbol{b} \text{ for some } \boldsymbol{b} \text{ by key 2} \\
&\iff \boldsymbol{a} = \boldsymbol{P_X} \underbrace{\boldsymbol{Xb}}_{n\times 1} \text{ treat as } \boldsymbol{P_X} \text{ product a } n \times 1 \text{ vector} \\
&\iff \boldsymbol{a} = \boldsymbol{P_X k} \text{ for some } \boldsymbol{k} = \boldsymbol{Xb} \\
&\implies \boldsymbol{a} \in \mathcal{C}(\boldsymbol{P_X}) \text{ by key 1}
\end{aligned}
$$

So, $\mathcal{C}(\boldsymbol{X}) \subseteq \mathcal{C}(\boldsymbol{P_X})$

Then similarly,

$$
\begin{aligned}
\boldsymbol{g} \in \mathcal{C}(\boldsymbol{P_X}) &\iff \boldsymbol{g} = \boldsymbol{P_X h} \text{ for some } \boldsymbol{h} \text{ by key 1} \\
&\iff \boldsymbol{g} = \underbrace{\boldsymbol{X}(\boldsymbol{X}^{\boldsymbol{X}})^{-}\boldsymbol{XT}}_{\boldsymbol{P_X}} \boldsymbol{h} \text{ for some } \boldsymbol{h} \\
&\iff \boldsymbol{g} = \boldsymbol{X}\, \underbrace{\boldsymbol{X}(\boldsymbol{X}^{\boldsymbol{T}}\boldsymbol{X})^{-}\boldsymbol{X}^{\boldsymbol{T}}\boldsymbol{X}}_{p\times 1} \text{ treat as } \boldsymbol{X} \text{ product a } p \times 1 \text{ vector} \\
&\iff \boldsymbol{g} = \boldsymbol{Xq} \text{ for some } \boldsymbol{q} = (\boldsymbol{X}^{\boldsymbol{T}}\boldsymbol{X})^{-}\boldsymbol{Xh} \\
&\implies \boldsymbol{q} \in \mathcal{C}(\boldsymbol{X}) \text{ by key 1}
\end{aligned}
$$

So, $\mathcal{C}(\boldsymbol{P_X}) \subseteq \mathcal{C}(\boldsymbol{X})$

And hence,

$\mathcal{C}(\boldsymbol{X}) = \mathcal{C}(\boldsymbol{P_X})$

Q2. Suppose $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ is an $n \times r$ matrix. Prove that if $rank(\mathbf{A}) = n$, then $\mathbf{B}^{-}\mathbf{A}^{-}$ is a generalized inverse of $\mathbf{AB}$.

(15 marks)

Ans.

$\mathbf{AB}(\mathbf{B}^{-}\mathbf{A}^{-})\mathbf{AB}$
$= \mathbf{ABB^{-}IB}$ Since $rank(\mathbf{A}) = n$ and hence $\mathbf{A}^{-} = \mathbf{A}^{-1}$ exist and $\mathbf{A}^{-}\mathbf{A} = \mathbf{I}$
$= \mathbf{ABB^{-}B}$
$= \mathbf{AB}$ Since $\mathbf{BB^{-}B} = \mathbf{B}$
Hence $\mathbf{B}^{-}\mathbf{A}^{-}$ is a generalized inverse of $\mathbf{AB}$.

Q3. Show tthat if $\mathbf{X}_{n\times k}$ has rank$(\mathbf{X}) < k$,and if $\mathbf{G} = (\mathbf{X}^T\mathbf{X})^{-}$ is a generalized inverse of $\mathbf{X}^T\mathbf{X}$, then

$$\mathbf{b} = (\mathbf{X}^T\mathbf{X})^{-}\,\mathbf{X}^T\mathbf{y} = \mathbf{G}\mathbf{X}^T\mathbf{y}$$

is a solution of the normal equations.

(15 marks)

Ans.

If $(\mathbf{X^TX})\mathbf{G}(\mathbf{X^TX}) = (\mathbf{X^TX})$, then
$(\mathbf{X^TX})\mathbf{G}(\mathbf{X^TX})\mathbf{b} = (\mathbf{X^TX})\mathbf{b}$, when
$(\mathbf{X^TX})\mathbf{b} = (\mathbf{X^Ty})$, this gives
$(\mathbf{X^TX})\mathbf{GX^Ty} = (\mathbf{X^Ty})$, i.e
$(\mathbf{X^TX})[\mathbf{GX^Ty}] = (\mathbf{X^Ty})$,hence
$\mathbf{b} = \mathbf{GX^Ty}$ is a solution of
$(\mathbf{X^TX})\mathbf{b} = (\mathbf{X^Ty})$.

Q4. Consider a problem of quadratic regression in one variable, $\mathbf{X}$. In particular, suppose that $n = 5$ values of a response $\mathbf{y}$ are related to values $x = 0, 1, 2, 3, 4$ by a linear model $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$ for

$$\mathbf{y} = \begin{bmatrix} 4 \\ 4 \\ 6 \\ 6 \\ 16 \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix}, \quad \boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix}, \text{ and } \quad \boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \end{bmatrix}$$

Define

$$\mathbf{W} = \begin{bmatrix} 1 & -2 & 2 \\ 1 & -1 & -1 \\ 1 & 0 & -2 \\ 1 & 1 & -1 \\ 1 & 2 & 2 \end{bmatrix}$$

Show that $\mathbf{y} = \mathbf{W}\boldsymbol{\gamma} + \boldsymbol{\epsilon}$ is reparameterization of $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$, where $\boldsymbol{\gamma}^{\boldsymbol{T}} = [\gamma_1, \gamma_2, \gamma_3]$. Find the OLS estimate of $\boldsymbol{\gamma}$ in the model $\mathbf{y} = \mathbf{W}\boldsymbol{\gamma} + \boldsymbol{\epsilon}$ and then OLS estimate of $\boldsymbol{\beta}$ in the original model. (Find numerical values.)

(15 marks)

Ans.

$\mathbf{W} = \mathbf{XF}$ where

$$\mathbf{F} = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$\mathbf{X} = \mathbf{WG}$ where

$$\mathbf{G} = \begin{bmatrix} 1 & 2 & 6 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$\mathbf{W^TW} = diag(5, 10, 14)$
$(\mathbf{W^TW})^{-1} = diag(\frac{1}{5}, \frac{1}{10}, \frac{1}{14})$

Then, $\hat{\boldsymbol{\gamma}}_{OLS} = (\mathbf{W^TW})^{-1}\mathbf{W^Ty} = diag(\frac{1}{5}, \frac{1}{10}, \frac{1}{14}) \begin{bmatrix} 36 \\ 26 \\ 18 \end{bmatrix} = \begin{bmatrix} 7.2 \\ 2.6 \\ 1.2857 \end{bmatrix}$

Since $\mathbf{W}\hat{\boldsymbol{\gamma}}_{OLS} = \mathbf{XF}\boldsymbol{\gamma}_{OLS}$, we must have $\hat{\boldsymbol{\beta}}_{OLS} = \mathbf{F}\boldsymbol{\gamma}_{OLS}$.

$$\boldsymbol{\beta}_{OLS} = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 7.2 \\ 2.6 \\ 1.2857 \end{bmatrix} = \begin{bmatrix} 4.5714 \\ -2.5428 \\ 1.2857 \end{bmatrix}$$

Q5. In a small scale study of persistence, an experimenter gave 5 subjects a very difficult task. Data on the age of the subject (x) and on the number of attempts to accomplish the task before giving up (y) follow:

Subject i	1	2	3	4	5
Age x_i :	37	29	29	27	34
Number of attempts y :	18	14	14	13	16

Suppose

$$y_i = \beta x_i + \epsilon,$$

where for $\mathbf{e} = (\epsilon_1, \epsilon_2, \cdots, \epsilon_5)^T$, $E(\mathbf{e}) = \mathbf{0}$ and $V(\mathbf{e}) = \sigma^2 diag(4, 9, 16, 25, 36)$. Evaluate an appropriate BLUE of β under the model assumptions.

(15 marks)

Ans.

Let $\mathbf{V}^{-\frac{1}{2}} = diag(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6})$.

Then $\mathbf{Z} = \mathbf{V}^{-\frac{1}{2}}\mathbf{y} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{6} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} \frac{y_1}{2} \\ \frac{y_2}{3} \\ \frac{y_3}{4} \\ \frac{y_4}{5} \\ \frac{y_5}{6} \end{bmatrix} = \begin{bmatrix} 9.0 \\ 4.67 \\ 3.5 \\ 2.6 \\ 2.67 \end{bmatrix}$

$$\begin{aligned} E(\mathbf{Z}) &= \mathbf{V^{-1/2}}E(\mathbf{y}) \\ &= \mathbf{V^{-1/2}X}\boldsymbol{\beta} \\ &= \mathbf{W}\boldsymbol{\beta} \end{aligned}$$

where

$$\mathbf{W} = \mathbf{V^{-1/2}X} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{6} \end{bmatrix} \begin{bmatrix} 37.0 \\ 29.0 \\ 29.0 \\ 27.0 \\ 34.0 \end{bmatrix} = \begin{bmatrix} 18.5 \\ 9.67 \\ 7.25 \\ 5.4 \\ 5.67 \end{bmatrix}$$

$$\begin{aligned} V(\mathbf{Z}) &= \mathbf{V^{-1/2}}^T Var(\mathbf{y})\mathbf{V^{-1/2}} \\ &= \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{6} \end{bmatrix} \sigma^2 \begin{bmatrix} 4 & & & & \\ & 9 & & & \\ & & 16 & & \\ & & & 25 & \\ & & & & 36 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{6} \end{bmatrix} \\ &= \sigma^2\mathbf{I} \end{aligned}$$

Thus, $\mathbf{Z}$ follows a Gauss-Markov model with model matrix $\mathbf{W}$.
So, the BLUE of $\boldsymbol{\beta}$ is $\hat{\boldsymbol{\beta}} = [(\mathbf{W^TW})^{-1} W^T\mathbf{Z}]$

$$(\mathbf{W^TW}) = \begin{bmatrix} 18.5 & 9.67 & 7.25 & 5.4 & 5.67 \end{bmatrix} \begin{bmatrix} 18.5 \\ 9.67 \\ 7.25 \\ 5.4 \\ 5.67 \end{bmatrix} = 549.6303$$

$$\mathbf{W^T Z} = \begin{bmatrix}18.5 & 9.67 & 7.25 & 5.4 & 5.67\end{bmatrix}\begin{bmatrix}9.0\\4.67\\3.5\\2.6\\2.67\end{bmatrix} = 266.21279999999996$$

so, $\hat{\boldsymbol{\beta}} = (549.6303^{-1})(266.21279999999996) = \boxed{0.4843}$

Q6. The response time in milliseconds was determined for four different types of circuits that could be used in an automatic valve shutoff mechanism. The results are shown in the following table.

Circuits Type	Response Time			
1	9	12	10	11
2	20	21	23	
3	6	5	8	10
3	36	35	38	

Consider the linear model

$$y_{ij} = \mu + \tau_i + \epsilon_{ij}, \text{ for } i = 1, 2, 3, 4 \text{ and } j = 1, 2, n_i$$

where where μ is the overall mean, τ_i is the circuit type content effects and $\epsilon_{ij} \sim N(0, \sigma^2)$ is the random error.

(a) Write model above in the form $\mathbf{y} = X\boldsymbol{\beta} + \boldsymbol{\epsilon}$. Do not impose any restriction on the parameters.

Ans.

$$\begin{bmatrix}y_{11}\\y_{12}\\y_{13}\\y_{14}\\y_{21}\\y_{22}\\y_{23}\\y_{31}\\y_{32}\\y_{33}\\y_{34}\\y_{41}\\y_{42}\\y_{43}\end{bmatrix} = \begin{bmatrix}1&1&0&0&0\\1&1&0&0&0\\1&1&0&0&0\\1&1&0&0&0\\1&0&1&0&0\\1&0&1&0&0\\1&0&1&0&0\\1&0&0&1&0\\1&0&0&1&0\\1&0&0&1&0\\1&0&0&1&0\\1&0&0&0&1\\1&0&0&0&1\\1&0&0&0&1\end{bmatrix}\begin{bmatrix}\mu\\\tau_1\\\tau_2\\\tau_3\\\tau_4\end{bmatrix} + \begin{bmatrix}e_{11}\\e_{12}\\e_{13}\\e_{14}\\e_{21}\\e_{22}\\e_{23}\\e_{31}\\e_{32}\\e_{33}\\e_{34}\\e_{41}\\e_{42}\\e_{43}\end{bmatrix}$$

(b) Obtain one of the generalized inverse of $\mathbf{X^TX}$, $\mathbf{G}$.

Ans.

$$\mathbf{X^TX} = \begin{bmatrix} 14 & 4 & 3 & 4 & 3 \\ 4 & 4 & 0 & 0 & 0 \\ 4 & 0 & 3 & 0 & 0 \\ 4 & 0 & 0 & 4 & 0 \\ 4 & 0 & 0 & 0 & 3 \end{bmatrix}$$

Let $\mathbf{W} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$, $\mathbf{W^{-1}} = \begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{bmatrix}$

So, one of the generalized inverse of $\mathbf{X^TX}$, G is

$$\mathbf{G} = (\mathbf{X^TX})^{-} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} \end{bmatrix}$$

(c) Use the generalized inverse you obtained in part(b) to compute a solution to the normal equations, $\hat{\boldsymbol{\beta}} = \begin{bmatrix} \hat{\mu} \\ \hat{\tau}_1 \\ \hat{\tau}_2 \\ \hat{\tau}_3 \end{bmatrix}$.

Ans.

$$\mathbf{X^Ty} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 9 \\ 12 \\ 10 \\ 11 \\ 20 \\ 21 \\ 23 \\ 6 \\ 5 \\ 8 \\ 10 \\ 36 \\ 35 \\ 38 \end{bmatrix} = \begin{bmatrix} 244 \\ 42 \\ 64 \\ 29 \\ 109 \end{bmatrix}$$

$$\hat{\boldsymbol{\beta}} = (\mathbf{X^T X})^{-}\mathbf{X^T y} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 244 \\ 42 \\ 173 \\ 29 \\ 109 \end{bmatrix} = \begin{bmatrix} 0 \\ 10.5 \\ 21.33 \\ 7.25 \\ 36.33 \end{bmatrix}$$

(d) Using your solution $\hat{\boldsymbol{\beta}}$ to the normal equation from part (c), estimates $3\mu + \tau_1 + \tau_2 + \tau_3$.

Ans.

$$3\mu + \tau_1 + \tau_2 + \tau_3 = c^T\boldsymbol{\beta} = \begin{bmatrix} 3 & 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mu \\ \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \end{bmatrix}$$

$$3\mu + \widehat{\tau_1 + \tau_2} + \tau_3 = c^T\hat{\boldsymbol{\beta}} = \begin{bmatrix} 3 & 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{\mu} \\ \hat{\tau}_1 \\ \hat{\tau}_2 \\ \hat{\tau}_3 \\ \hat{\tau}_4 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 10.5 \\ 21.33 \\ 7.25 \\ 36.33 \end{bmatrix} =$$

39.08

(e) Is your estimate in part(d) the unique BLUE? Explain.

Ans.

$$\begin{aligned} E(y_{11} + y_{21} + y_{31}) &= (\mu + \tau_1) + (\mu + \tau_2) + (\mu + \tau_3) \\ &= 3\mu + \tau_1 + \tau_2 + \tau_3 \\ &= c^T\boldsymbol{\beta} \end{aligned}$$

Thus $3\mu + \tau_1 + \tau_2 + \tau_3$ is estimable and hence $c^T\hat{\boldsymbol{\beta}}$ is the BLUE of $c^T\boldsymbol{\beta}$ Hence by Gauss Markov Theorem, $3\mu + \widehat{\tau_1 + \tau_2} + \tau_3 = c^T\hat{\boldsymbol{\beta}}$ is the unique BLUE.

(30 marks)