MEME16203Linear Models Marking Guide

Assignment 5

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Faculty:	FES	Unit Code:	MEME16203
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Due by:			

MX-Q26 Q1. A study was conducted on human subjectsto measure the effects of 12 different foods on serum glocuse levels. On each of 6 days, 12 subjectse randomly assigned to the 12 food with one subject for each food. Each subject was given an assigned food, and then blood samples were taken from each subject at 4 time points: 15, 30, 45, and 60 minutes after consumption ofvthe food. The same process was repeated each day with 12 different subjects, so a total of 72 subjects were used in the experiment. The level of the protein of interest was measured in each of the 288 blood samples. For $i = 1, \ldots, 6$, $j = 1, \ldots, 12$, and $k = 1, \ldots, 4$, let y_{ijk} be the serum glocuse measurement on day i for food j at time k. For $i = 1, \ldots, 6$, $j = 1, \ldots, 12$, and $k = 1, \ldots, 4$, consider the model

$$y_{ijk} = \mu_{jk} + d_i + e_{ijk},$$

where μ_{jk} terms are unknown fixed parameters and the other terms are random effects defined as follows. Let $\mathbf{d} = [d_1, \ldots, d_6]^T$. For $i = 1, \ldots, 6$ and $j = 1, \ldots, 12$, let $\mathbf{e}_{ij} = [e_{ij1}, \ldots, e_{ij4}]^T$.
Suppose

$$\mathbf{d} \sim N(\mathbf{0}, \sigma_d^2 \mathbf{I}_{6\times 6}),$$

and

$$\mathbf{e}_{ij} \sim N(\mathbf{0}, \mathbf{\Sigma}_e) \text{ for } i = 1, \ldots, 6 \text{ and } j = 1, \ldots, 12,$$

where σ_d^2 is an unknown positive variance parameter and

$$\mathbf{\Sigma}_e = \sigma_e^2 \begin{bmatrix} 1 & \phi & \phi^2 & \phi^3 \\ \phi & 1 & \phi & \phi^2 \\ \phi^2 & \phi & 1 & \phi \\ \phi^3 & \phi^2 & \phi & 1 \end{bmatrix}$$

for some unknown variance parameter $\sigma_e^2 > 0$ and some unknown correlation parameter ϕ. Finally, suppose that $\mathbf{d}$ and $\mathbf{e}_{11}, \ldots, \mathbf{e}_{6,12}$ are all independent. In terms of model parameters, give a simplified expression for the variance of the generalized least squares estimator of each of the following:

(a) μ_{41}

Ans.
$V(\bar{Y}_{.41}) = V(\bar{d}_{.} + \bar{e}_{.41}) = \boxed{\frac{\sigma_d^2}{6} + \frac{\sigma_e^2}{6}}$

(b) $\bar{\mu}_{4.}$

Ans.
$V(\bar{Y}_{.4.}) = V(\bar{d}_{.} + \bar{e}_{.4.}) = \frac{\sigma_d^2}{6} + V(\frac{1}{12}\sum_{i=1}^{6}\sum_{k=1}^{4} e_{i1k}) = \frac{\sigma_d^2}{5} + \frac{1}{12^2}(6\sigma_e^2)\mathbf{1}^T\mathbf{\Sigma_e 1}$
$= \frac{\sigma_d^2}{6} + \frac{6\sigma_e^2[4+6\phi+4\phi^2+\phi^3]}{144}$

(c) $\mu_{14} - \mu_{24}$

Ans.
$V(\bar{Y}_{.14} - \bar{Y}_{.24}) = V(\bar{e}_{.14} - \bar{e}_{.24}) = V(\bar{e}_{.14} + \bar{e}_{.24}) = \frac{\sigma_e^2}{6} + \frac{\sigma_e^2}{6} = \frac{2\sigma_e^2}{6}$

(d) $\mu_{11} - \mu_{14}$

Ans.
$V(\bar{Y}_{.11} - \bar{Y}_{.14}) = V(\bar{e}_{.11} - \bar{e}_{.14}) = V\left[\frac{1}{6}\sum_{i=1}^{6}(e_{i11} - e_{i14})\right] = \frac{\sigma_e^2+\sigma_e^2-2cov(e_{i11},e_{i14})}{6} = \frac{2\sigma_e^2-2\sigma_e^2\phi^3}{6} = \frac{2\sigma_e^2(1-\phi^3)}{6}$

MX-Q28 Q2. A repeated measures study was conducted to examine the effects of 4 different store displays for a household product on sales in 8 successive time period. 32 stores were randomly selected, and 8 were assigned at random to each display. Consider the model

$$Y_{ijk} = \mu + \alpha_i + S_{ij} + \tau_k + \gamma_{ik} + e_{ijk}$$

where y_{ijk} is the sales amount at the k^{th} time point for the j^{th} store with the i^{th} store display, α_i is the fixed store display effect, τ_k is the fixed time effect and γ_{ik} is the fixed store display $\times$ time effect, $S_{ij} \sim NID(0, \sigma_S^2)$ and is independent of $e_{ijk} \sim NID(0, \sigma_e^2)$.

(a) Find $V(\mathbf{Y_{ij}})$, for this model?

Ans.

$$\mathbf{Y_{ij}} = \begin{bmatrix} Y_{ij1} \\ Y_{ij2} \\ \vdots \\ Y_{ij8} \end{bmatrix}$$

$$
\begin{aligned}
V(Y_{ijk}) &= V(\mu + \alpha_i + S_{ij} + \tau_k + e_{ijk}, \mu + \alpha_i + S_{ij} + \tau_l + e_{ijl}) \\
&= V(S_{ij} + e_{ijk}) \\
&= \sigma_S^2 + \sigma_e^2
\end{aligned}
$$

$$
\begin{aligned}
Cov(Y_{ijk}, Y_{ijl}) &= Cov(\mu + \alpha_i + S_{ij} + \tau_k + e_{ijk}, \mu + \alpha_i + S_{ij} + \tau_l + e_{ijl}) \\
&= Cov(S_{ij}, S_{ij}) \\
&= \sigma_S^2 \text{ for } k \neq l
\end{aligned}
$$

$$
\begin{aligned}
V(\mathbf{Y_{ij}}) &= \begin{bmatrix} V(Y_{ij1}) & Cov(Y_{ij1}, Y_{ij2}) & \cdots & Cov(Y_{ij1}, Y_{ij8}) \\ Cov(Y_{ij2}, Y_{ij1}) & V(Y_{ij2}) & \cdots & Cov(Y_{ij2}, Y_{ij8}) \\ \vdots & \ddots & & \vdots \\ Cov(Y_{ij8}, Y_{ij1}) & Cov(Y_{ij8}, Y_{ij2}) & \cdots & V(Y_{ij8}) \end{bmatrix} \\
&= \begin{bmatrix} \sigma_S^2 + \sigma_e^2 & \sigma_S^2 & \cdots & \sigma_S^2 \\ \sigma_S^2 & \sigma_S^2 + \sigma_e^2 & \cdots & \sigma_S^2 \\ \vdots & \ddots & & \vdots \\ \sigma_S^2 & \sigma_S^2 & \cdots & \sigma_S^2 + \sigma_e^2 \end{bmatrix} \\
&= \sigma_e^2 \mathbf{I_8} + \sigma_S^2 \mathbf{J_8}
\end{aligned}
$$

(b) Provide the formulas for the estimator of σ_e^2 and σ_S^2.

Ans.

$\hat{\sigma}_e^2 = MSE$

$\hat{\sigma}_S^2 = \frac{MS(Store) - MSE}{8}$

(c) What is the correlation between observations taken on the same store?

Ans.

$$\rho = \frac{Cov(Y_{ijk}, Y_{ijl})}{\sqrt{V(Y_{ijk})}\sqrt{V(Y_{ijl})}} = \frac{\sigma_S^2}{\sigma_S^2 + \sigma_e^2}$$

(d) Find the estimator of $V(\bar{Y}_{ij.})$ and provide it's degrees of freedom.

Ans.

$$
\begin{aligned}
V(\bar{Y}_{ij.}) &= V\left(\tfrac{1}{8}\textstyle\sum_{k=1}^{8} Y_{ijk}\right) \\
&= \tfrac{1}{8^2} V(Y_{ij1} + Y_{ij2} + \cdots + Y_{ij8}) \\
&= \tfrac{1}{8^2} [\textstyle\sum_{k=1}^{8} V(Y_{ijk}) + 2\sum\sum_{k\neq l} Cov((Y_{ijk}, Y_{ijl})] \\
&= \tfrac{1}{8^2} [\textstyle\sum_{k=1}^{8} (\sigma_S^2 + \sigma_e^2) + 2\binom{8}{2}(\sigma_S^2) \\
&= \tfrac{1}{8^2} [8(\sigma_S^2 + \sigma_e^2) + 2\binom{8}{2}(\sigma_S^2) \\
&= \tfrac{1}{8^2} [8^2 \sigma_S^2 + 8\sigma_e^2] \\
&= \sigma_S^2 + \tfrac{1}{8}\sigma_e^2
\end{aligned}
$$

$$S^2_{\bar{Y}_{ij.}} = \frac{MS(Store)-MSE}{8} + \frac{MSE}{8} = \frac{MS(Store)}{8}$$
$$DF = 8 - 1 = 7$$

(e) Find the estimator of $V(\bar{Y}_{i.k})$ and provide it's Satterthwaith degrees of freedom.

Ans.

$$\begin{aligned} V(\bar{Y}_{i.k}) &= V\left(\frac{1}{32}\sum_{j=1}^{32} Y_{ijk}\right) \\ &= \frac{1}{32^2}\sum_{j=1}^{32} V(Y_{ijk}) \\ &= \frac{1}{32}(\sigma_S^2 + \sigma_e^2) \end{aligned}$$

$$S^2_{\bar{Y}_{i.k}} = \sqrt{\frac{1}{32}\left[\frac{7MSE}{8} + \frac{MS(Store)}{8}\right]}$$
$$D.F = \frac{\frac{1}{32}\left[\frac{7MSE}{8} + \frac{MS(Store)}{8}\right]}{\frac{(\frac{7MSE}{8})^2}{7} + \frac{(\frac{MS(store)}{8})^2}{8}}$$