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2 Solutions of Problems in Interest

2.1 Equations of Value

Suppose that deposits made today can earn 5% effective over the next year. Which would you rather have, \$1000 today or \$1,050 in a year?

A basic **principle of time value money** in interest theory tells us that it doesn't matter to us whether we have \$1,000 today or \$1,050 in a year (assuming that the effective rate of interest is 5%). In fact, if we are told that the effective rate is 5% for the next 10 years, we will assume that we would be just as happy with \$1,000 today as we would be with $\$1,00(1.05)^t$ in t years, where t is any time from 0 to 10.

The principle of time value money allows us to solve interest problems by setting up **equations of value** of a common **comparison date**.

Notes:

- The same date (the comparison date) must be used to evaluate the deposit and the AV.
- Any date can be used for the comparison date.

Example 1.

Suppose you want to accumulate \$5,000 in two years by making a deposit of X today and another deposit of X in a year. If the effective rate of interest is 6%, determine X

(i) using time 2 as comparison date.

Ans: $X(1.06^2 + 1.06) = 5,000$

$X = \frac{5,000}{1.06^2+1.06} = 2,289.8$

(ii) using time 100 as comparison date.

Ans: $X(1.06)^{100}+X(1.06)^{99} = 5,000(1.06)^{98}$

$X = \frac{5,000(1.06)^{98}}{1.06^{100}+1.06^{99}} = 2,289.8$

Example 2. (T02Q1)

Int-Q42 You are given two loans, with each loan to be repaid by a single payments in the future. Each payment include both principal and interest. The first loan is repaid by a 4300 pyament at the end of 4 years. The interest is accrued at 12% per annum compounded semiannually. The second loan is repaid by a 5300 pyament at the end of 5 years. The interest is accrued at 10% per annum compounded semiannually. These two loans are to be consolidated. The consolidated loan is to be repaid by two equal instalments of X, with interest 14% per annum compounded semiannually. The first payment is due immediately and the second payment is due one year from now. Calculate X.

Ans:

$X\left(1+\left(1+\frac{0.14}{2}\right)^{-2\times 1}\right)=4300\left(1+\frac{0.12}{2}\right)^{-2\times 4}+5300\left(1+\frac{0.1}{2}\right)^{-2\times 5}$

$X(1+1.07^{-2})=4300(1.06^{-2\times 4})+5300(1.05^{-2\times 5})$

$X=\frac{4300(1.06^{-2\times 4})+5300(1.05^{-2\times 5})}{1+1.07^{-2}}=\boxed{3176.84}$

Example 3. (T02Q2)

Int-Q43 Find the nominal rate of interest convertible quarterly which is equivalent to a nominal rate of discount of 12% per annum convertible monthly.

Ans:

$\left(1+\frac{i^{(4)}}{4}\right)^4=\left(1-\frac{0.12}{12}\right)^{-12}$

$1+\frac{i^{(4)}}{4}=(0.99)^{-3}$

$i^{(4)}=4[(0.99)^{-3}-1]=\boxed{0.1224}$

2.2 Time Diagrams

One useful technique in the solutions of equations of value is the **time diagram**, note the following points:

- All deposits are placed below the line and all withdrawals are placed above the line.
- To the left of the time line, the interest period is noted, say ("years"), ("1/2 years") or ("2 years").
- The effective interest rate for one period is marked off, as a reminder of the rate to be used in calculations.
- A vertical arrow is placed at the comparison date chosen.

Example 4.
Draw a time diagram and write an equation of value for the following problem:
Find the present value of a quarterly payments of \$100 for 10 years, first payment 3 months from now, at a nominal rate of interest of 6% compounded semiannually.

(a) Draw the diagram in terms of the **payment period** and determine the equivalent effective rate for this period:

Ans:

```
                 j
               &lt;---->
                   100   100            100
1/4 years      |----|----|----|-- ... --|
               0    1    2    3         40
```

where $(1+j)^4 = (1+.06/2)^2 \Rightarrow j = 1.03^{1/2} - 1$
$PV = 100(v + v^2 + \cdots + v^{40})$

(b) Draw the diagram in terms of the **interest period** given in the problem.

Ans:

```
                3%
              <--->
                    100   100   100         100
1/2 years     |----|----|----|-- ... --|
              0    0.5   1     1.5         20
```

$v = 1.03^{-1}$
$PV = 100(v^{1/2} + v + v^{1.5} + \cdots + v^{20})$

Example 5.
Deposits of 500 are made on January 1 of even years only from 1994 to 2014 inclusive. Find the accumulated value on the date of the last deposit if the nominal rate is 8% compounded quarterly. Draw a time diagram and write an equation of value

(a) Using the payment period.

Ans:

```
                    j
                 <---->          AV
               500   500         500
2 years        |----|-- ... --|
               0     1           10
            1/1/94            1/1/14
                                 AV
```

where $1+j = 1.02^8$, $j = 1.02^8 - 1 = 17.1659\%$
$AV = 500[1 + (1+j) + \cdots + (1+j)^{10}]$

(b) Using the interest period.

Ans:

```
              2%                              |
            <---->                            V
           500          500       500        500
1/4
years      |----|- ... |--- ...|----... --|
           0    1       8      16          80
         1/1/94      1/1/96  1/1/98     1/1/14
                                          AV
```

$AV = 500(1 + 1.02^8 + 1.02^{16} + \cdots + 1.02^{80})$

2.3 Unknown Time

Consider a situation in which several payments made at various points in time are to be replaced by one payment numerically equal to the sum of the other payments. The problem is to

> find the point in time at which the one payment should be made such that it is equivalent in value to the payments made separately.

The fundamental equation of value is

$$(s_1 + s_2 + \cdots + s_n)v^t = s_1 v^{t_1} + \cdots + s_n v^{t_n} \quad (1) \quad \boxed{\texttt{e:2.1}}$$

which is one equation in one unknown t. This equation can readily be solved using logarithms.

ex:MET

Example 6.

Suppose you are schedule to make three payments: 5, 1, 15 at time 1, 3, 10 to your friend at 5% effective rate of interest. Now suppose both of you agree that in lieu of the schedule payments, you will make a single payment to your friend equal to the sum of the schedule payments. When should this single payment be made? $\boxed{7.12}$

Ans:

Let the time of the single payment be t.

Then the following equation of values as of time 0 must be satisfied:

$$21v^t = 5v + v^3 + 15v^{10}$$

$$t = \frac{\ln\left(\frac{5v+v^3+15v^{10}}{21}\right)}{\ln v} = 7.12$$

Example 7. (T02Q3)

Int-Q01

Payments of 850, 950,700 are due at the ends of year 3, 8 and 13 respectively. Assuming an effective rate of interest of 4.00% per annum, determine the point in time, t, at which a payment of RM2500 would be equivalent.

Ans:

$2500(1.04)^{-t} = 850(1.04)^{-3}+950(1.04)^{-8}+700(1.04)^{-}$

$t = \boxed{7.4002}$

Example 8. (T02Q4)

Int-Q46 You invest 6000 today and plan to invest another 3000.0 two years from today. You plan to withdraw 9000.0 in n years and another 9000.0 in $n+5$ years, exactly liquidating your investment account at that time. If the effective rate of discount is equal 7%, find n.

Ans:

```
                     9000.0        9000.0
   |------|--------|--------|-
   0      2        n        n+5
 6000   3000.0
```

$9000.0v^n + 9000.0v^{n+5} = 6000 + 3000.0v^2$
$90.0v^n(1 + v^5) = 60 + 30.0v^2$
$90.0v^n(1+0.9299999999999999^5) = 60+30.0(0.929999$
$v^n = \frac{60+30.0(0.9299999999999999^2)}{90.0(1+0.9299999999999999^5)} = 0.5632$
$n = \frac{\ln(0.5632)}{\ln(0.9299999999999999)} = \boxed{7.9112}$

As a first approximation, we can estimate t by

$$\bar{t} = \frac{s_1t_1 + \cdots + s_nt_n}{s_1 + \cdots + s_n}. \qquad (2)$$ e:2.2

This estimation is called the **method of equated time**. It can be shown that $\bar{t} > t$.

Example 9.

Determine t of Example ex:MET 6 using method of equated time. $\boxed{7.52}$

Ans: $\bar{t} = \frac{5\times1+1\times3+15\times10}{5+1+15} = 7.52$

Example 10.

A annuity provides an infinite series of annual payments of $d, \frac{d^2}{2}, \frac{d^3}{3}, \ldots$, first payment one year from now, where d is the effective rate of discount. In lieu of these payments, a single payment equal to their sum is to be made at time t. Determine t using the method of equated time. $\boxed{i/\delta}$

Ans:

$$\begin{aligned}\textstyle\sum_{i=1}^{\infty} s_i t_i &= (d)(1) + \left(\frac{d^2}{2}\right)(2) + \left(\frac{d^3}{3}\right)(3) + \cdots \\ &= d + d^2 + d^3 + \cdots \\ &= \frac{d}{1-d} \\ &= i\end{aligned}$$

$$\begin{aligned}\textstyle\sum_{i=1}^{\infty} s_i &= d + \frac{d^2}{2} + \frac{d^3}{3} + \cdots \\ &= -\ln(1-d) \\ &= \delta\end{aligned}$$

$$\bar{t} = \frac{i}{\delta}$$

Example 11. (T02Q5)

Int-Q44 Payments of 390, 590, and 790 are made at the end of years 10, 11 and 13, respectively. Interest is accumulated at an annual effective rate of 7%. You are to find the point in time at which single payment of 1770 is equivalent to the above series of payments. You are given:

- X is the point in time calculated by the method of equated time.
- Y is the exact point in time.

Calculate $X + Y$.

Ans:

$$X = \frac{390(10)+590(11)+790(13)}{390+590+790} = 11.6723$$

$$\begin{aligned}1770v^Y &= 390v^5 + 590v^6 + 790v^8 \\ 1770v^Y &= 390(1.07^{-10})+590(1.07^{-11})+790(1.07^{-13}) \\ v^Y &= 0.4556 \\ Y &= 11.6192\end{aligned}$$

$$X + Y = 11.6723 + 11.6192 = \boxed{21.6723}$$

2.4 Unknown Rate of Interest

The unknown rate of interest is usually described by and equation $f(i) = 0$. To solve this, we can use $\boxed{\text{table}}$ function in TI-30 calculator to search the solution if the equation cannot be solve using simple closed form.

eg:2.4

Example 12.

At what interest rate *convertible quarterly* would \$1000 accumulate to \$1600 in 6 years? $\boxed{.0791}$

Ans:

The equation of value is

$1000(1 + \frac{i^{(4)}}{4})^{4\times 6} = 1600$

$\Rightarrow i^{(4)} = 4(\exp(\frac{\ln 1.6}{24}) - 1) = 0.0791$

Example 13.

At what rate of interest will a payment of 1 now and 2 in one year accumulated to 5 in 4 years? The equation of value as of time 4 is:

$\boxed{.1646}$

Ans: $(1 + i)^4 + 2(1 + i)^3 = 5$ Using calculator, $i = 0.1646$

Example 14. (T02Q6)

Int-Q06 Jeff puts 1000 into a fund that pays an effective annual rate of discount of 15% for the first two years and a force of interest of rate $\delta = 2/(8-t), 2 \leq t \leq 4$, for the next two years. At the end of four years, the amount in Jeff's account is the same as what it would have been if he had put 1000 into an account paying interest at the nominal rate of i per annum compounded quarterly for four years. Calculate i.

Ans:

$$\begin{aligned}(1+\tfrac{i}{4})^{4(4)} &= (1-0.15)^{-2} e^{\int_2^4 \frac{2}{8-t}dt} \\ &= 0.85^{-2} e^{-2\ln(8-t)|_2^4} \\ &= 0.85^{-2} e^{-2(\ln 4.0 - \ln 6.0)} \\ &= 0.85^{-2} e^{\ln(6.0/4.0)^2} \\ &= 0.85^{-2}(6.0/4.0)^2\end{aligned}$$

$i = \boxed{0.2943}$

Example 15. (T02Q7)

Int-Q09 At time $t = 0$, John deposit 5,000 into a fund which credits interest at a nominal interest rate of 11% compounded semiannually. At the same time, he deposits P into a different fund which credits interest at a nominal discount rate of 8% compounded monthly. At time $t = 18$, the amount in each fund are equal. What is the annual effective interest rate earned on the total deposit, $5000 + P$, over the 18-year period?

Ans:

$$5,000\left(1+\frac{0.11}{2}\right)^{2\times 18} = P\left(1-\frac{0.08}{12}\right)^{-12\times 18}$$

$P = 8101.78$

$$(13101.779999999999)(1+i)^{18} = 5,000\left(1+\frac{0.11}{2}\right)^{2\times 18}$$

$$8101.78\left(1-\frac{0.08}{12}\right)^{-12\times 18}$$

$i = \boxed{0.0964}$

Example 16. (T02Q8)

Int-Q10 You are given a loan on which interest is charged over 4-year period, as follows:

- an effective rate of discount of 0.064 for the first year;
- a nominal rate of discount of 0.055 compounded every 2 years for the second year;
- a nominal rate of interest of 0.051 compounded semiannually for the third year; and
- a force of interest of 0.078 for the forth year.

Calculate the annual effective rate of interest over the 4-year period.

Ans:

$$\begin{aligned} \text{AV} &= (1-d)^{-1}(1-\tfrac{d^{(1/2)}}{1/2})^{-1/2}(1+\tfrac{i^{(2)}}{2})^2 e^{\int_3^4 \delta dt} \\ &= (1-0.064)^{-1}(1-\tfrac{0.055}{1/2})^{-1/2}(1+\tfrac{0.051}{2})^2 e^{\int_3^4 0.07} \\ &= 1.2876 \end{aligned}$$

$(1+i)^4 = 1.2876$

$i = 1.2876^{1/4} - 1 = \boxed{0.0652}$

Example 17. (T02Q9)

Int-Q24 You are given that $1000d^{(m)} = 60.785$ and $1000d^{(2m)} = 60.94$, and they are equivalent rates. Find $i^{(3m)}$.

Ans:

Given $d^{(m)} = 0.0607851$, $d^{(2m)} = 0.0609398$.

The equation of value is

$$(1-\frac{d^{(m)}}{m})^m = (1-\frac{d^{(2m)}}{2m})^{2m} = (1+\frac{i^{3m}}{(3m)})^{-3m}$$

This implies $1-\dfrac{0.0607851}{m} = \left(1-\dfrac{0.0609398}{2m}\right)^2 = \left(1+\dfrac{i^{(3m)}}{3m}\right)^{-3}$

Let $f(m) = 1-\frac{0.0607851}{m} - \left(1-\frac{0.0609398}{2m}\right)^2$,

Using calculator, the value of m is 6, so

$\left(1+\frac{i^{(3m)}}{3(6)}\right)^{-3} = 1-\frac{0.0607851}{6}$

$i^{(3m)} = \left[\left(1-\frac{0.0607851}{6}\right)^{-1/3} - 1\right][3(6)] = \boxed{0.030573}$

Example 18. (T02Q10)

Int-Q49 At a certain interest rate the present value of the following two payment patterns are equal:

- 200 at the end of 5 years plus 500 at the end of 10 years.
- 400.94 at the end of 5 years.

At the same interest rate, 100 invested now plus 120 invested at the end of 5 years will accumulate to P at the end of 10 years. Calculate P.

Ans:
Using time 5 as comparison date:
$200 + 500v^5 = 400.94$
$v^5 = \frac{400.94-200}{500} = 0.40188$
$P = 100v^{-5\times 2} + 120v^{-5} = 100(0.40188^{-2}) + 120(0.40188^{-1}) = \boxed{917.76}$