**MEME15203 Statistical Inference Marking Guide**

**Test 1**

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| Faculty: | FES | Unit Code: | MEME15203 |
| Course: | MAC | Unit Title: | Statistical Inference |
| Year: | 1,2 | Lecturer: | Dr Yong Chin Khian |
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Show your workings. If no workings are shown, ZERO is awarded.

Q1. Let $X$ and $Y$ have joint pdf $f(x,y) = cy^3e^{-4y}, 0 < x < y < \infty$ and zero otherwise.

(a) Find the joint pdf of $S = X + Y$ and $T = X$. (10 marks)

(b) Find the marginal pdf of $T$. (10 marks)

*Ans.*

(a) $\int_0^\infty \int_0^y cy^3e^{-4y}dxdy = 1$

$c\int_0^\infty [y^4e^{-4y}]dy = 1$

$c\left(\Gamma(5)\frac{1}{4^5}\right) = 1$

$c = \frac{4^5}{\Gamma(5)}$

$f(x,y) = \frac{4^5}{\Gamma(5)}y^3e^{-4y}, 0 < x < y < \infty$

Let $T = X$ and $S = X + Y$. Then this corresponds to the transformation $X = T$ and $Y = S - T$ which have unique solutions $h_1(t,s) = x = t$ and $h_2(t,s) = y = s - t$,

$J = \begin{vmatrix} \frac{\partial h_1}{\partial t} & \frac{\partial h_1}{\partial s} \\ \frac{\partial h_2}{\partial t} & \frac{\partial h_2}{\partial s} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$

$f_{T,S}(t,s) = f_{X,Y}(t, s-t)|J| = \frac{4^5}{\Gamma(5)}(s-t)^3e^{-4(s-t)}$, $0 < 2t < s < \infty$

(b) $f_T(t) = \int_{2t}^\infty f_{T,S}(t,s)ds$

$= \int_{2t}^\infty \frac{4^5}{\Gamma(5)}(s-t)^3e^{-4(s-t)}ds$

Let $v = s - t$, $dv = ds$

$= \int_t^\infty \frac{4^5}{\Gamma(5)}v^3e^{-4v}dv$

$= \left(\frac{4^5}{\Gamma(5)}\right)\left(\frac{\Gamma(4)}{4^4}\right)\int_t^\infty \frac{4^4}{\Gamma(4)}v^3e^{-4v}dv$

$= \left(\frac{4}{4}\right)P(S_4 > t)$ where $S_4 \sim GAM(\alpha = 4, \theta = \frac{1}{4})$

$= \frac{4}{4}\left[e^{-4t}\left(\sum_{i=0}^3 \frac{(4t)^i}{i!}\right)\right]$

Q2. The waiting time $X$ until delivery of a new component for an industrial operation is uniformly distributed over the interval from 1 to 8 days. The cost of this delay is given by $U = 2X^2 + 8$. Find the probability density function for $U$ using distribution method. (10 marks)

*Ans.*

$X \sim U(1,8)$
$f_X(x) = \frac{1}{8-1}, 1 < x < 8$
$F_X(x) = \int_1^x \frac{1}{8-1} dt = \frac{x-1}{8-1} = \frac{1}{7}(x-1)$
$F_U(u) = P(U \leq u) = P(2X^2 + 8 \leq u) = P(X \leq \sqrt{\frac{u-8}{2}}) = \frac{1}{7}\left(\sqrt{\frac{u-8}{2}} - 1\right)$
$f_U(u) = F'_U(u) = \frac{1}{7}\left(\frac{1}{2}\sqrt{\frac{2}{u-8}}\right)\frac{1}{2} = \frac{1}{28}\left(\sqrt{\frac{2}{u-8}}\right), 10 < u < 136$, zero otherwise.

Q3. Let $X_1$ and $X_2$ be a random sample of size $n = 2$ from a continuous distribution with pdf of the form $f(x) = 2x^1$ if $0 < x < 1$ and zero otherwise.

(a) Find the joint pdf of $Y_1 = \min(X_1, X_2)$ and $Y_2 = \max(X_1, X_2)$. (5 marks)

(b) Find the pdf of the sample range $R = Y_2 - Y_1$. (10 marks)

*Ans.*

(a) $f_{Y_1,Y_2}(y_1, y_2) = 2!f(y_1)f(y_2) = 2!(2y_1)(2y_2) = 8y_1y_2, 0 < y_1 < y_2 < 1$

(b) Making the tranformation $R = Y_2 - Y_1$, $S = Y_1$, yields the inverse tranformation $y_1 = s$, $y_2 = r + s$, and $|J| = 1$. Thus the joint pdf of $R$ and $S$ is
$f_{R,S}(r, s) = f_{Y_1,Y_2}(s, r + s)|J| = 8s(r + s), 0 < s < 1 - r, 0 < r < 1$

$$\begin{aligned}
f_R(r) &= \int_0^{1-r} 8s(r+s)ds \\
&= 8[\tfrac{s^2 r}{2} + \tfrac{s^3}{3}]_0^{1-r} \\
&= 8[\tfrac{(1-r)^2 r}{2} + \tfrac{(1-r)^3}{3}] \\
&= 8[\tfrac{3(1-r)^2 r}{6} + \tfrac{2(1-r)^3}{6}] \\
&= \tfrac{8}{6}(r+2)(1-r)^2
\end{aligned}$$

Q4. Suppose that $X \sim POI(28)$, $S = X + Y \sim POI(60)$, and $X$ and $Y$ are independent. Use MGFs to find the distrbution of $S - X$. (10 marks)

*Ans.*

$S - X = X + Y - X = Y$
$M_X(t) = e^{28(e^t-1)}$,
$M_S(t) = e^{60(e^t-1)}$
$M_S(t) = M_X(t)M_Y(t)$
$e^{60(e^t-1)} = e^{28(e^t-1)}M_Y(t)$
$M_Y(t) = \frac{e^{60(e^t-1)}}{e^{28(e^t-1)}} = e^{(60-28)(e^t-1)} = e^{32(e^t-1)}$
$\Rightarrow Y = S - X \sim POI(32)$

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Q5. Let $X$, $Y$ be two random variables with joint pdf $f(x,y) = \frac{1}{7!}x^7e^{-y}$, for $0 < x < y < \infty$, zero otherwise. Determine the joint mgf of $X, Y$, $M_{X,Y}(t_1, t_2)$. (10 marks)

*Ans.*

$M_{X,Y}(t_1,t_2)$
$= E(e^{t_1X+t_2Y})$
$= \int_0^\infty \int_x^\infty e^{t_1x+t_2y}(\frac{1}{7!})x^7e^{-y}dydx$
$= \int_0^\infty (\frac{1}{7!})x^7e^{t_1x}\int_x^\infty e^{-y(1-t_2)}dydx$
$= \int_0^\infty (\frac{1}{7!})x^7e^{t_1x}\frac{e^{-x(1-t_2)}}{1-t_2}dx$
$= (\frac{1}{7!})(\frac{1}{1-t_2})\int_0^\infty x^7e^{-x(1-t_1-t_2)}dx$
$= \frac{7!}{7!(1-t_2)(1-t_1-t_2)^8}$
$= \frac{1}{(1-t_2)(1-t_1-t_2)^8}$
provided that $t_1 + t_2 < 1$ and $t_2 < 1$.

Q6. Suppose $P[\theta = 1] = 0.3$ and $P[\theta = 2] = 0.7$, and that conditional on $\theta$, $X|\theta \sim BIN(n = 50, \theta)$. Find $V(2X + 6\theta)$. (15 marks)

*Ans.*

$E(\theta) = 1(0.3) + 2(0.7) = 1.7$
$E(X) = E[E(X|\theta)] = E(50\theta) = 50(1.7) = 85.0$
$E(\theta^2) = 1^2(0.3) + 2^2(0.7) = 3.1$
$V(\theta) = E(\theta^2) - E^2(\theta) = 3.1 - 1.7^2 = 0.21$
$V(X) = E[V(X|\theta)]+V[E(X|\theta)] = E[50\theta(1-\theta)]+V(50\theta) = 50[E(\theta)-E(\theta^2)]+ 50^2V(\theta) = 50[1.7 - 3.1] + 50^2(0.21) = 455.0$
$E(\theta X) = E[E(\theta X|\theta)] = E[\theta E(X|\theta)] = E[\theta(50)\theta] = 50E(\theta^2) = 50(3.1) = 155.0$
$Cov(\theta, X) = E(\theta X) - E(X)E(\theta) = 155.0 - (85.0)(1.7) = 10.5$
$V(2X + 6\theta) = 2^2V(X) + 6^2V(\theta) + 2(2)(6)Cov(X, \theta) = 2^2(455.0) + 6^2(0.21) + 2(2)(6)(10.5) = \boxed{2079.56}$

Q7. Consider a random sample from a Poisson distribution, $X_i \sim POI(\mu)$. Let $W_i = e^{X_i}$ and $V_n = W_1 \times W_2 \times \cdots W_n$. Both $V_n^{\frac{1}{n}}$ and $V_n^{\frac{1}{n^2}}$ converge in probability to constants, identify those constants. (10 marks)

*Ans.*

$E(\bar{X}_n) = \mu$, $V(\bar{X}_n) = \frac{1}{n}V(X) = \frac{\mu}{n}$
$P\left[|\overline{X_n} - \mu| \geq \epsilon\sqrt{\frac{\mu}{n}}\sqrt{\frac{n}{\mu}}\right] < \frac{\mu}{n\epsilon^2} \to 0$

$\therefore \bar{X}_n \xrightarrow{P} \mu$
and $\frac{1}{n}\bar{X}_n \xrightarrow{P} 0$
$(V_n)^{1/n} = (W_1 \times W_2 \times \cdots \times W_n)^{1/n} = e^{\bar{X}_n}$
Thus, $(V_n)^{1/n} \xrightarrow{P} e^{\mu}$ and
$(V_n)^{1/n^2} = (W_1 \times W_2 \times \cdots \times W_n)^{1/n^2} = e^{\frac{1}{n}\bar{X}_n}$
and hence, $(V_n)^{1/n} \xrightarrow{P} e^0 = 1$

Q8. Let $X_1, \ldots, X_n$, be a random sample from a uniform distribution, $X \sim U(0, \theta)$, and let $Y_n = X_{n:n}$ the largest order statistic. Find the limiting distribution of $Z_n = n(\theta - Y_n)$. (10 marks)

*Ans.* $F_X(x) = \frac{x}{\theta}$
$F_n(y) = P[Y_n \leq y] = [F_X(y)]^n = \left[\frac{y}{\theta}\right]^n$
$F_n(z) = P[Z_n \leq z] = P[Y_n > \theta - z/n] = 1 - \left[\frac{\theta - z/n}{\theta}\right]^n = 1 - \left[1 - \frac{z/\theta}{n}\right]^n$
$$\lim_{n\to\infty} F_n(z) = 1 - \lim_{n\to\infty}\left[1 - \frac{z/\theta}{n}\right]^n = 1 - e^{-y/\theta}, y > 0$$
$\Rightarrow F(z) \sim EXP(\theta)$