UNIVERSITI TUNKU ABDUL RAHMAN

Department of Mathematics and Actuarial Science

CONTENTS

3 Properties of a Random Sample

3.1 Convergence in Probability

In this section, we formalize a way of saying that a sequence of random variables $\{X_n\}$ is getting "close" to another random variable X, as $n \to \infty$.

Definition 1. Let $\{X_n\}$ be a sequence of random variables and let X be a random variable defined on a sample space. We say that $\{X_n\}$ converges in probability to X if, for all $\epsilon > 0$,

$$\lim_{n\to\infty} P[|X_n - X| \geq \epsilon] = 0,$$

or equivalently

$$\lim_{n\to\infty} P[|X_n - X| < \epsilon] = 1,$$

If so, we write

$$X_n \xrightarrow{P} X.$$

One way of showing convergence in probability is to use Chebyshev's Theorem.

Theorem 1. If X is a random variable and $u(x)$is a nonnegative real-valued function, then for any positive constant $c > 0$.

$$P[u(X) \geq c] \leq \frac{E[u(X)]}{c}$$

A special case, known as the **Markov inequality**, is obtained if $u(x) = |x|^r$ for $r > 0$, namely

$$P[|X| \geq c] \leq \frac{E[|X|^r]}{c^r}$$

Theorem 2. **Chebychev inequality** If X is a random variable with mean μ and variance σ^2, then for any $k > 0$,

$$P[|X - \mu| \geq k\sigma] < \frac{1}{k^2}$$

An alternative form is

$$P[|X - \mu| < k\sigma] \geq 1 - \frac{1}{k^2}$$

and if we let $\epsilon = k\sigma$, then

$$P[|X - \mu| < \epsilon] \geq 1 - \frac{\sigma^2}{\epsilon^2}$$

and

$$P[|X - \mu| \geq \epsilon] \leq \frac{\sigma^2}{\epsilon^2}$$

Example 1.

Suppose that X is a random variable for which $E(X) = 10$, $P(X \leq 7) = 0.17$, and $P(X \geq 13) = 0.4$. Prove that $V(X) > c$ and identify c.

Theorem 3. (Weak Law of Large Numbers). Let $\{Xn\}$ be a sequence of iid random variables having common mean μ and variance σ^2. Let $\bar{X}_n = \frac{\sum_{i=1}^{n} X_i}{n}$. Then

$$\bar{X}_n \xrightarrow{P} \mu.$$

Example 2.
Let $X_1, \ldots, X_n$ denote a random sample from a distribution with mean μ and variance σ^2. Assume that $E[X_i^4] < \infty$. Show that $\frac{\sum_{i=1}^{n} X_i^2}{n}$ converges in probability to $E(X_i^2)$.

Theorem 4. Suppose $X_n \stackrel{P}{\rightarrow} X$ and $Y_n \stackrel{P}{\rightarrow} Y$ Then $X_n + Y_n \stackrel{P}{\rightarrow} X + Y$.

Theorem 5. Suppose $X_n \xrightarrow{P} X$ and a is a constant. Then $aX_n \xrightarrow{P} aX$.

Theorem 6. Suppose $X_n \xrightarrow{P} a$ and the real function g is continuous at a. Then $g(X_n) \xrightarrow{P} g(a)$.

Theorem 7. Suppose $X_n \xrightarrow{P} X$ and the real function g is continuous at a. Then $g(X_n) \xrightarrow{P} g(X)$

Example 3.

Suppose $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$. Then $X_n Y_n \xrightarrow{P} XY$.

Example 4.

Consider a random sample from a Poisson distribution, $X_i \sim POI(\mu)$. Show that $Y_n = e^{-\bar{X}_n}$ converges in probability to a constant, identify the constant.

Definition 2. (Consistency). Let X be a random variable with cdf $F(x, \theta)$, $\theta \in \Omega$. Let $X_1, \ldots, X_n$ be a sample from the distribution of X and let T_n denote a statistic. We say T_n is a consistent estimator of θ if

$$T_n \xrightarrow{P} \theta.$$

Example 5.

Let $X_1, \ldots, X_n$ denote a random sample from a distribution with mean μ and variance σ^2. Assume that $E[X_i^4] < \infty$, so that $V(S^2) < \infty$. Show that $S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$ converges in probability to σ^2 .

3.2 Convergence in Distribution

Definition 3. Let X_n be a sequence of random variables and let X be a random variable. Let F_{X_n} and F_X be, respectively, the cdfs of X_n and X. Let $C(F_X)$ denote the set of all points where F_X is continuous. We say that X_n **converges in distribution** to X if

$$\lim_{n\to\infty} F_{X_n} = F_X, \forall x \in C(F_X).$$

We denote this convergence by

$$X_n \xrightarrow{D} X.$$

Notes:

The material on convergence in probability and in distribution comes under what statisticians and probabilists refer to as asymptotic theory. Often, we say that the distribution of X is the asymptotic distribution or the limiting distribution of the sequence $\{X_n\}$.

Definition 4. The function F_X is the CDF of a **degenerate distribution** at value $x = c$ if

$$F_X = \begin{cases} 0, & x < c \\ 1, & x \geq c \end{cases}$$

In other words, $F_X$is the CDF of a diccrete distribution that assigns probability one at the value $x = c$ and zero otherwise.

Notes: The following limits are useful in many problems:

1. $\lim_{n\to\infty} \left(1 + \frac{c}{n}\right)^{nb} = e^{cb}$
2. $\lim_{n\to\infty} \left(1 + \frac{c}{n} + \frac{d(n)}{n}\right)^{nb} = e^{cb}$
   if $\lim_{n\to\infty} d(n) = 0$

Example 6.
Let $X_1, \ldots, X_n$, be a random sample from a uniform distribution, $X \sim U(0, 1)$, and let $Y_n = X_{n:n}$ the largest order statistic. Find the limiting distribution of Y_n.

Example 7.

Suppose that $X_1, \ldots, X_n$, is a random sample from a Pareto distribution, $X \sim PAR(\alpha = 1, \theta = 20)$. Let $Y_n = 1/nX_{n:n}$, find the limiting distribution of Y_n, $F(y)$, state the distribution and it's parameter, then find $F(17.6)$.

Example 8.

Let Y_3 denote the third smallest item of a random sample of size n from a distribution of the continuous type that has cdf $F_X(x)$ and pdf $f_X(x) = F'_X(x)$. Find the limiting distribution of $W_n = nF_{Y_3}(y)$.

Theorem 8.
If X_n converges to X in probability, then X_n converges to X in distribution.

Theorem 9. Slutky's Theorem
If X_n and Y_n are two sequences of random variables such that $X_n \xrightarrow{P} c$ and $Y_n \xrightarrow{D} Y$, then :

1. $X_n + Y_n \xrightarrow{D} c + Y$
2. $X_n Y_n \xrightarrow{D} cY$
3. $X_n/Y_n \xrightarrow{D} c/Y$

Theorem 10.
If $X_n \xrightarrow{D} X$, then for any continuous function $g(x)$, $g(X_n) \xrightarrow{D} g(X)$. Note that $g(x)$is assumed not to depend on n.

Example 9.
Consider a random sample of size n from a Bernoulli distribution, $X_i \sim Bin(1, p)$.

(a) Show that $\hat{p} = \frac{\sum_{i=1}^{n} X_i}{n} \xrightarrow{P} p$.

(b) Show that $\hat{p}(1-\hat{p}) \xrightarrow{P} p(1-p)$.

(c) We know that $\frac{\hat{p}-p}{\sqrt{p(1-p)/n}} \xrightarrow{D} Z \sim N(0,1)$, find the limiting distribution of $\frac{\hat{p}-p}{\sqrt{\hat{p}(1-\hat{p})/n}}$.

3.3 Moment Generating Function Technique

To find the limiting distribution function of a random variable X_n by using the definition obviously requires that we know $F_{X_n}(x)$ for each positive integer n. But it is often difficult to obtain $F_{X_n}(x)$ in closed form. Fortunately, if it exists, the mgf that corresponds to the cdf $F_{X_n}(x)$ often provides a convenient method of determining the limiting cdf.

Theorem 11. Let $\{X_n\}$ be a sequence of random variables with mgf $M_{X_n}(t)$ that exists for $-h < t < h$ for all n. Let X be a random variable with mgf $M(t)$, which exists for $|t| < h_1 < h$. If $\lim_{n\to\infty} M_{X_n}(t) = M(t)$ for $|t| < h_1$, then $X_n \xrightarrow{D} X$.

Example 10.
Let Y_n have a distribution that is $Bin(n, p)$. Suppose that the mean $\mu = np$ is the same for every n; that is, $p = \mu/n$, where μ is a constant. Find the limiting distribution of Y_n using moment generating function technique.

Example 11.

Let $Y_n \sim GAM(\alpha = n, \theta = 2)$. Find the limiting distribution of $\frac{Y_n - n}{\sqrt{2n}}$ as $n \to \infty$, using moment generating function.

Theorem 12.

Central Limit Theorem (CLT) If $X_1, \ldots, X_n$, is a random sample from a distribution with mean μ and variance $\sigma^2 < \infty$, then the limiting distribution of

$$Z_n = \frac{\sum_{i=1}^n X_i - n\mu}{\sqrt{n}\sigma}$$

is the standard normal, $Z_n, \to Z \sim N(0, 1)$ as $n \to \infty$.

Example 12. Let $X_1, X_2, \ldots, X_{100}$ be a random sample from an exponential distribution, $X_i \sim EXP(1)$,and let $Y = X_1 + X_2 + \cdots + X_{100}$.

(a) Give an approximation for $P[Y > 110]$. $\boxed{0.1587}$

(b) If $\bar{X}$ is the sample mean, then approximate $P[1.1 < \bar{X} < 1.2]$. $\boxed{0.1359}$

Example 13.

Let $X_i \sim U(24, 54)$, where $X_1, X_2, \ldots, X_{66}$ are independent. Find normal approximation for

$$P\left[\sum_{i=1}^{66} X_i \leq 2579.0\right].$$

Theorem 13. $\Delta-$**Method**
If $\frac{\sqrt{n}(X_n-m)}{c} \xrightarrow{D} Z \sim N(0,1)$, and if $g(x)$ has a nonzero derivative at $x=m$, $g'(m) \neq 0$, then

$$\frac{\sqrt{n}[g(X_n)-g(m)]}{|cg'(m)|} \xrightarrow{D} Z \sim N(0,1)$$

In other words, for large n, if $X_n \sim N(m, c^2/n)$, then approximately

$$g(X_n) \sim N\left(g(m), \frac{c^2[g'(m)]^2}{n}\right)$$

Example 14.

Consider a random sample from a Poisson distribution, $X_i \sim POI(\mu)$. Find the asymtotic normal distribution of $Y_n = e^{-\bar{X}_n}$.

3.4 Parameter and Statistic

Consider a set of observable random variables $X_1, \ldots, X_n$. For example, suppose the variables are a random sample of size n from a population.

Definition 5. A **parameter** is a numerical summary that would be calculated from all of the units in the population.

Definition 6. A function of observable random variables, $T = t(X_1, \ldots, X_n)$, which does not depend on any unknown parameters, is called a **statistic**.

In other words, a **statistic** is a numerical summary that is calculated from all of the units in a sample.

Theorem 14. If ,$T = t(X_1, \ldots, X_n)$, denotes a random sample from $f(x)$ with $E(X) = \mu$ and $V(X) = \sigma^2$ then.

$$E(\bar{X}) = \mu$$

and

$$V(\bar{X}) = \frac{\sigma^2}{n}$$

3.5 Sampling Distributions

A statistic is also a random variable, the distribution of which depends on the distribution of a random sample and on the form of the function $t(x_1, x_2, \ldots, x_n)$. The distribution of a statistic sometimes is referred to as a **derived distribution** or **sampling distribution**, in contrast to the population distribution.

3.5.1 Linear Combinations of Normal Variables

Theorem 15. If $X_i \sim N(\mu_i, \sigma_i^2)$; $i = 1, \ldots, n$ denote independent normal variables, then

$$Y = \sum_{i=1}^{n} a_i X_i \sim N\left(\sum_{i=1}^{n} a_i \mu_i, \sum_{i=1}^{n} a_i^2 \sigma_i^2\right)$$

Theorem 16.

Let $X_1, X_2, \ldots, X_n$ random sample of size n from a normal distribution with mean μ and variance σ^2. Then

$$\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}$$

is normally distributed with mean $\mu_{\bar{X}} = \mu$ and variance $\sigma_{\bar{X}}^2 = \frac{\sigma^2}{n}$.

3.6 Chi-Square Distribution

Definition 7. The variable Y is said to follow a chi-square distribution with v degrees of freedom if

$$Y \sim GAM(\alpha = \frac{v}{2}, \theta = 2).$$

A special notation for this is

$$Y \sim \chi^2(v)$$

Theorem 17. If $Y \sim \chi^2(v)$, then

- $M_Y(t) = (1 - 2t)^{-v/2}$
- $E(Y^r) = 2^r \frac{\Gamma(v/2+r)}{\Gamma(v/2)}$

Theorem 18. If $X \sim GAM(\alpha, \theta)$, then

$$Y = \frac{2X}{\theta} \sim \chi^2(2\alpha).$$

Example 15. The time to failure (in years) of a certain type of component follows a gamma distribution with $\alpha = 2$ and $\theta = 3$ It is desired to determine a guarantee period for which 90% of the components will survive. Find the guarantee period.

Theorem 19. If $Y_i \sim \chi^2(v_i)$; $i = 1, \ldots, n$ are independent chi-square variables, then

$$V = \sum_{i=1}^{n} Y_i \sim \chi^2\left(\sum_{i=1}^{n} v_i\right)$$

Theorem 20. If $Z \sim N(0, 1)$, then $Z^2 \sim \chi^2(1)$.

Theorem 21. If $X_1, \ldots, X_n$ denotes a random sample of size n from $N(\mu, \sigma^2)$, then

$$\frac{\sum_{i=1}^{n}(X_i - \mu)^2}{\sigma^2} \sim \chi^2(n)$$

$$\frac{n(\bar{X} - \mu)^2}{\sigma^2} \sim \chi^2(1)$$

Theorem 22. If $X_1, \ldots, X_n$ denotes a random sample from $N(\mu, \sigma^2)$, then

(i) $\bar{X}$ and the terms $X_i - \bar{X}$, $i = 1, \ldots, n$ are independent.

(ii) $\bar{X}$ and S^2 are independent.

(iii) $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$.

Example 16. Let X represent the lifetime in months of a battery, and assume that approximately $X \sim N(60, 36)$. Suppose that it was decided to sample 25 batteries, and to reject the claim that $\sigma^2 = 36$ if $S^2 \geq 54.63$, and not reject the claim if $S^2 < 54.63$. Under this procedure, what would be the probability of rejecting the claim when in fact $\sigma^2 = 36$?

3.7 Student's t Distributions

Theorem 23. If $Z \sim N(0,1)$ and $V \sim \chi^2(v)$, and if Z and V are independent, then the distribution of

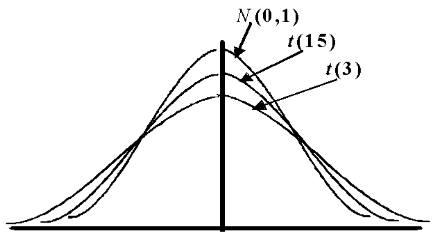
$$T = \frac{Z}{\sqrt{V/v}}$$

is referred to as **Student's t distribution** with v degrees of freedom, denoted by $T \sim t(v)$. The pdf is given by

$$f(t) = \frac{\Gamma(\frac{v+1}{2})}{\Gamma(\frac{v}{2})} \frac{1}{\sqrt{v\pi}} \left(1 + \frac{t^2}{2}\right)^{-(v+1)/2}$$

The t distribution is symmetric about zero, and its general shape is similar to that of the standard normal distribution. Indeed, the t distribution approaches the standard normal distribution as $v \to \infty$. For smaller v the t distribution is flatter with thicker tails and, in fact, $T \sim CAU(1, 0)$ when $v = 1$.

Various T-distributions



Theorem 24. If $X_1, \ldots, X_n$ denotes a random sample from $N(\mu, \sigma^2)$ then

$$\frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t(n-1)$$

Example 17.

Assume that Z, V_1, and V_2 are independent random variables with $Z \sim N(0,1)$, $V_1 \sim \chi^2(5)$, and $V_2 \sim \chi^2(9)$. Find the following:

(a) $P[V_1 + V_2 < 8.6]$.

(b) $P[Z/\sqrt{V_1/5} < 2.015]$.

(c) $P[Z > 0.611\sqrt{V_2}]$.

3.8 Snedecor's F Distribution

Theorem 25. If $V_1 \sim \chi^2(v_1)$ and $V_2 \sim \chi^2(v_2)$ are independent, then the random variable

$$X = \frac{V_1/v_1}{V_2/v_2}$$

has the following pdf for $x > 0$:

$$f(x) = \frac{\Gamma(\frac{v_1+v_2}{2})}{\Gamma(\frac{v_1}{2})\Gamma(\frac{v_2}{2})}\left(\frac{v_1}{v_2}\right)^{v_1/2}\left(1 + \frac{v_1}{v_2}x\right)^{-(v_1+v_2)/2}$$

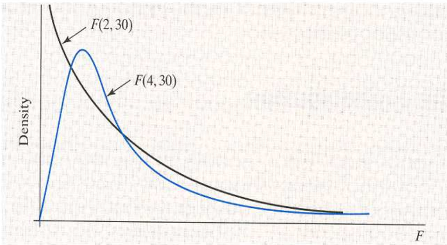
This is known as Snedecor's F distribution with v_1 and v_2 degrees of freedom, and is denoted by $X \sim F(v_1, v_2)$.

Properties of the F-distribution

- The total area under the curve is one (as it is a density curve).
- The distribution is skewed to the right.
- The values are non-negative, start at zero, extend to the right—the curve approaches, but never touches, the horizontal axis.

- A different F-distribution for each different set of degrees of freedom.

Various F-distributions



Example 18. If we take independent samples of size $n_1 = 6$ and $n_2 = 10$ from two normal populations with equal population variances, find b such that $P\left(\frac{S_1^2}{S_2^2} \leq b\right) = 0.95$

Example 19.

Suppose that $X_i \sim N(\mu, \sigma^2), i = 1, \ldots, 35$ and $Z_i \sim N(0,1), i = 1, \ldots, 28$ and all variables are independent. State the distribution of each of the following variables if it is a "named" distribution or otherwise state "unknown."

(a) $\dfrac{\sqrt{35}(\bar{X} - \mu)}{\sigma S_Z}$

(b) $\dfrac{\sum_{i=1}^{35}(X_i - \mu)^2}{\sigma^2} + \sum_{i=1}^{28}(Z_i - \bar{Z})^2$

(c) $\dfrac{\bar{X}}{\sigma^2} + \dfrac{\bar{Z}}{\sigma}$

(d) $\bar{Z}^2$

3.9 Beta Distribution

Theorem 26.

If X and Y be independent random variables with $X \sim GAM(\alpha_1, 2)$ and $Y \sim GAM(\alpha_2, 2)$, then $U = \frac{X}{X+Y} \sim Beta(a = \alpha_1, b = \alpha_2)$.

An F variable can be transformed to have the beta distribution. IF $X \sim F(v_1, v_2)$ then the random variable

$$Y = \frac{(v_1/v_2)X}{1+(v_1/v_2)X} \sim Beta(a = \frac{v_1}{2}, b = \frac{v_2}{2})$$

The pdf of Y is

$$f(y) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} y^{a-1}(1-y)^{b-1}, 0 < y < 1$$

The k^{th} raw moment of Y is

$$E(Y^k) = \frac{a(a+1)\cdots(a+k-1)}{(a+b)(a+b+1)\cdots(a+b+k-1)}$$

Example 20.

Suppose $Y \sim Beta(a = 6, b = 10)$, use the relationship between Beta distribution and F distribution, find $P[Y > 0.488]$.

Example 21.

Suppose $Y \sim Beta(a = 6, b = 8)$, use the relationship between Beta distribution and F distribution, find 92^{th} percentile of Y.

Example 22.

Suppose that $X_i \sim N(\mu, \sigma^2), i = 1, \ldots, 20$, $Z_j \sim N(0, 1), j = 1, \ldots, 5$, and $W_k \sim \chi^2(v), k = 1, \ldots, 19$ and all random variables are independent. State the distribution of each of the following variables if it is a "named" distribution. [For example $X_1 + X_2 \sim N(2\mu, 2\sigma^2)$]

(a) $\frac{Z_i^2/W_1}{1+Z_1^2/W1}$

(b) $\frac{\frac{\sum_{k=1}^5 W_k}{\sum_{j=1}^5 (Z_j - \bar{Z})^2}}{1+\frac{\sum_{k=1}^5 W_k}{\sum_{j=1}^5 (Z_j - \bar{Z})^2}}$

(c) $\frac{W_1}{W_1+W_2+W_3+W_4}$