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8 Practical Applications

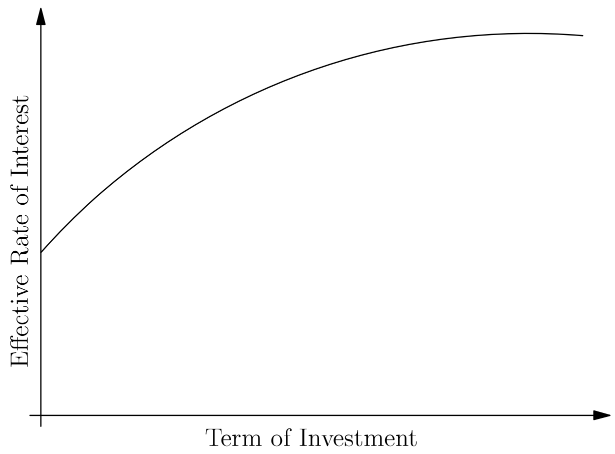
8.1 Yield Curve

In most of all the previous chapters, we have assumed one level interest throughout the period of investment. However, in practice, a variety of financial instruments such as certificates of deposit, mortgage loans, bonds, of different terms have different short-term and long-term rates of interest. The phenomenon in which rates of interest differ depending on the term of otherwise identical financial instruments is called the **term structure of interest rates**. The **term structure of interest rates** is a relationship between rates of interest and the term of the investment. The graph that displays this relationship is called a **yield curve**.

- If the yield curve has a positive slope, it is called a **normal yield curve**.
- If the yield curve has a negative slope, it is called a **inverted yield curve**.

- If the yield curve is almost constant over any major portion of the term structure, it is called a **flat yield curve**.

A typical yield curve looks like this.



A pattern of yield rates for zero coupon bonds can looks something like the following table.

Table 6.1: Yield rates on a Zero Coupon bonds

Years to Maturity	Yield Rates on a Zero Coupon Bond
1	5.00%
2	5.35%
3	5.65%
4	5.90%
5	6.05%
⋮	⋮
10	6.50%

8.2 Spot Rates

A spot rate is the yield rate for a zero coupon bond with a given term to maturity, or the yield rate for a similar investment that makes a single lump sum payment to the investor. The interest rates on the yield curve are often called **spot rate**. The spot rate for a term of length of t is denoted by s_t.

Example 1.

Suppose you want to buy an investment that will return 1,000 at the end of the next 3 years. What should you pay for this investment base on the spot rates shown in Table 6.1. $\boxed{2701.39}$

Example 2.

What is the yield rate on the investment in Example 1 if it is bought at the computed price? $\boxed{5.4313\%}$

Example 3.

An investment will return 2000 in 1-year and 4000 in 2-year. It sells for 5586.56. The one year spot rate is 3.63%. Determine the 2-year spot rate.

Example 4 (T8Q1)**.**

The one-year spot rate is 5.5%. A two year 100 bond maturing at par, with 7% annual coupon, is currently selling for its par value. Determine the two-year spot rate.

Example 5.

The yield to maturity on a 5% annual coupon bonds maturing at 1000 par is as follows:

Term	Yield to Maturity
1 year	6.00%
2 year	6.50
3 year	7.00

Determine the one, two, and three year spot rates.

Example 6.

The yield rate on a one year zero-coupon bond is currently 7% and the yield rate on a two year zero-coupon bond is currently 8%. The treasury plans to issue a two year bond with 9% annual coupons, maturing at 100 par value. Determine the yield to maturity of the two year coupon bond.

Example 7 (T8Q2).

You are given the following information about two bonds that will mature in 8-years at par:

	Bond A	Bond B
Par value	1000	1200
Annual coupon rate	12%	6%
Price	667.0	840.0

Determine the 8-year spot rate.

8.3 Forward Rates

A **forward rate** is an interest rate that will be earned on an investment made in a future point in time.

For example, if we are told that "the one-year forward rate, deferred two years is 5%", it means that we can make an investment two years from now that will earn 5% in the next 1-year period (i.e., in the year from time 2 to time 3).

Let ${}_nf_t$ be n **years forward rate deferred** t **years** which comes existence at time t and covers the interval from time t to $t+n$. We have

$$\boxed{(1+s_{t+n})^{t+n} = (1+s_t)^t(1+{}_n f_t)^n.}$$

Thus,

$$\boxed{{}_nf_t = \left[\frac{(1+s_{t+n})^{t+n}}{(1+s_t)^t}\right]^{\frac{1}{n}} - 1}$$

Note that ${}_1f_0 = s_1$. Thus, we can readily determine one-year forward rates deferred 0 years from a set of spot rates.

We can also determine t-year spot in terms of t one-year forward rates, deferred t years:

$$\boxed{(1+s_t)^t = (1+{}_1f_0)(1+{}_1f_1)\cdots(1+{}_1f_{t-1}).}$$

Example 8.

You are given the following selected values from a yield curve:

Term(years)	1	2	3	4	5
Spot rate (%)	7.00	8.00	8.75	9.25	9.5

Determine all one-year forward rates, deferred t years for $t = 0, 1, \ldots, 4$.

Example 9 (T8Q3)**.**

The following are the prices of 100 zero-coupon bonds redeemable at par:

Term to Maturity	Price
1	94.91
2	88.71
3	84.65
4	78.98

Determine the one-year forward rate deferred 3 years.

Example 10 (T8Q4)**.**

The n-year spot rate of interest, s_n, is given by:

$$s_n = 0.05 + \frac{n^2}{1000} \text{ for } n = 1, 2, \ldots$$

Calculate the one-year forward rates applicable at times $t = 7$ and $t = 8$.

Example 11.

A 1,000 par value bond with 10% annual coupons matures at par in two years. You are given that the one-year spot rate is 9% and the one-year forward rate deferred one year is 10.5%. Determine the price of the bond. $\boxed{1005.02}$

8.4 Duration of a Single Cash Flow

Consider two zero coupon bonds where $F = C = 1000$, $i = 10\%$:

- Bond 1: $n = 10$
- Bond 2: $n = 20$

what are the prices for these two bonds?

- $P_{\text{Bond 1}} =$
- $P_{\text{Bond 2}} =$

Suppose i increased to $i' = 10.1\%$, what would happen to these prices? Using your intuition, which bond do you think will have a relative greater impact? (Note: Relative impact means the change in price as a percentage of the original price.)

- $P_{\text{Bond 1}} =$
- $P_{\text{Bond 2}} =$

- Percentage change of Bond 1 =
- Percentage change of Bond 2 =

The percentage change in price is called **price sensitivity** of a bond to a change in the interest rate.
This example illustrates that the greater the term of a zero coupon bond, the more sensitive its price is to changes in the interest.
The time remaining to a single cash inflow or cash outflow is called its **duration**.

8.5 Macaulay Duration

It was easy to define duration in the case of a single cash flow. It's simply the time remaining until the cash flow. As we have seen, the greater the duration, the more sensitive the PV is to changes in then interest rate. But what if an asset with multiple cash flows, such as a coupon bond or a mortgage? F.R. Macaulay introduced **Macaulay Duration**. Note that Macaulay duration is often called just **duration** in the financial literature.

Let A_t be cash inflow from an asset at time t. Macaulay duration is defined as:

$$\text{MacD} = \frac{\sum (v^t A_t)(t)}{\sum v^t A_t}$$

Note that some of the values of A_t may be zero, i.e., cash flows may not occur at all times t.

Example 12.
Determine the Macaulay duration for each of the following assets at an effective annual rate of interest of 5%.

(a) A perpetuity-immediate with level annual payments

(b) A 10 year zero coupon bond

(c) A 10 year bond with 5% annual coupon maturing at par

(d) A 10 year mortgage with level annual payments

Example 13 (T8Q5)**.**

A perpetuity-immediate has annual payments of $1.06, 1.06^2, 1.06^3, \ldots$. Determine the duration of this perpetuity at an effective rate of 12%.

Example 14.

A n-year bond with annual coupons sells at its par value. Show thta the Macaulay duration of the bond is $\ddot{a}_{\overline{n}|}$.

8.6 Macaulay Duration as a Measure of Price Sensitivity

Macaulay duration can also be defined in terms of price sensitivity (i.e., the sensitivity of PV of the cash flow to a change in the interest rate). Note that Price sensitivity with respect to the force of interest is

$$\text{Price sensitivity} = -\frac{\frac{dP_A}{d\delta}}{P_A}$$

where P_A is the PV of the cash flow.

Recall, $v = e^{-\delta}$, thus
$P_A(\delta) = \sum e^{-\delta t} A_t$ and $\frac{dP_A}{d\delta} = -\sum te^{-\delta t} A_t$

Thus,price sensitivity (with respect to the force of interest)
$= \frac{\sum te^{-\delta t} A_t}{\sum e^{-\delta t} A_t} = \frac{\sum tv^t A_t}{\sum v^t A_t} = \text{MacD}$

8.7 Modified Duration

Most of the time we are interested in the effect on the price of a change in i, not a change in δ. Thus
Price sensitivity (with respect to a change in i
$= -\frac{\frac{d}{di} P_A}{P_A}$
$= -\frac{\frac{d}{di} \sum v^t A_t}{\sum v^t A_t}$
$= -\frac{\sum tv^{t+1} A_t}{\sum v^t A_t}$
$= v \left[\frac{\sum tv^t A_t}{\sum v^t A_t} \right]$
$= v\text{MacD}$

This measure of price sensitivity is called **modified duration**.

$$\text{ModD} = v\text{MacD}$$

Note: Modified duration is sometimes called "volatility."

By definition:

$$\text{ModD} = -\frac{\frac{d}{di}}{P}$$

The first derivative of the price P is:

$$P'(i) - \lim_{\Delta i \to 0} \frac{P(i + \Delta i) - P(i)}{\Delta i} = \lim_{\Delta i \to 0} \frac{\Delta P(i)}{\Delta i}$$

where Δi is the change in the variable i. For small Δi, the first derivative is approximately:

$$P'(i) \approx \frac{\Delta P}{\Delta i}$$

Substituting in the definition of ModD, we have

$$(ModD) \approx -\frac{\frac{\Delta P}{\Delta i}}{P}$$

If the modified duration is given, the change in price, ΔP for a small change in the interest rate is

$$\Delta P \approx -(\text{ModD})(P)(\Delta i)$$

Example 15.

A 15-year 1,000 bond with 8% annual coupons sells at par. What is the price of the bond at an effective rate of 7.92%. Using modified duration to approximate the change in price? How does this compare to exact change in price?

Example 16 (T8Q6)**.**

A company makes a loan and receives level annual repayments from the borrowers at the end of each year for 10 years. The effective rate of interest is 6.96%. What is the modified duration of the loan repayments?

Example 17.

A 100 par value bond with 7% annual coupons and maturing at par in 4 years sells at a price to yield 6%. Determine the modified duration of the bond. $\boxed{3.43}$

8.8 Duration of a Portfolio

Suppose a company has a portfolio of assets, such as a group of bonds with different remaining terms, different rates and different maturing values. Then the duration of the entire portfolio is:

$$MacD = \frac{P_1(MacD_1)+P_2(MacD_2)+\cdots+P_k(MacD_k)}{P_1+P_2+\cdots+P_k}$$

Example 18.
There are 3 bonds in a portfolio of assets. the Macaulay duration of the entire portfolio is 10 years. The duration of the first bond is 8 years and the duration of the second bond is 6 years. The price of the first bond is twice the price of the second bond and half the price of the third bond. What is the duration of the third bond? 12

8.9 Convexity

One problem with using duration to estimate changes in price is that the estimate is consistently under-estimate the exact price at the new interest rates. Similar to the definition of sensitivity, we introduce the notion of **convexity**.

$$\text{Convexity} = \frac{P''(i)}{P(i)}.$$

The analysis of the present value of a set of cash flows $P(i)$ to changes in the rate of interest can be made more accurate as follows:

$$P(i+\Delta i) \approx P(i) + \Delta i P'(i) + \frac{(\Delta i)^2}{2} P''(i).$$

Multiply and divide the last two terms on the right by the price:

$$P(i+\Delta i) \approx P(i) + \left[\frac{\Delta i P'(i)}{P(i)} + \frac{1}{2}(\Delta i)^2 \frac{P''(i)}{P(i)}\right] P(i)$$

By definition of MacD and convexity, we have

$P(i+\Delta i) \approx P(i)+\left[-\Delta i(ModD) + \frac{1}{2}(\Delta i)^2(convexity)\right] P(i)$
By subtracting $P(i)$ from each side, we can express the approximate change in $P(i)$ for a change in i of Δi as follows:

$$P(i+\Delta i) - P(i) = \Delta P$$
$$\approx \left[-\Delta i(ModD) + \tfrac{1}{2}(\Delta i)^2(convexity)\right] P(i)$$

To actually calculate convexity, we have to differentiate $P(i)$ twice:

$$P_A(i) = \sum v^t A_t$$

$$P'_A = -\sum t v^{t+1} A_t$$

$$P''_A = \sum t(t+1) v^{t+2} A_t$$

$$\text{Convexity} = \frac{\sum t(t+1)v^{t+2}A_t}{\sum v^t A_t}$$

Example 19.
An asset will provide two cash inflows: 10,000 in two years and 25,000 in 10 years. The asset is currently priced at 6% effective.

(a) what is the price of the assets?

(b) What is the modified duration of the asset?

(c) What is the convexity of the assets?

(d) Estimate the price if the interest rate changes to 5.9% using only modified duration.

(e) Estimate the price if the interest rate changes to 5.9% using both modified duration and convexity.

(f) Determine the exact price at $i = 5.9\%$.

Example 20 (T8Q7)**.**

An investment will return 1,000 in two years and 5,000 in five years. Determine the ratio of the convexity of the payments to their modified duration, evaluated at $i = 7.5\%$.

8.10 Redington Immunization

Suppose we have purchased a portfolio of bonds that we will use to pay liabilities. At the start, everything is in balance, i.e., the PV of cash inflows from the bond is equal to the PV of the liability cash outflows at a specified interest rate $i = i_0$. If the interest rates change, this can impair our ability to pay off the liabilities.

The process of protecting a financial enterprise from changes in interest rates is known as **immunization**. A British actuary, F.M. Redington laid out the principles for protecting an enterprise from **small** changes in interest rate, either up or down. There are three conditions for what has come to be known as **Redington immunization**. These conditions must hold for the interest rate $i = i_0$ at which we want to immunize the enterprise:

1. PV of assets = PV of liabilities (This condition assures us that if the interest rate does not

change from i_0, the assets will be sufficient to pay the liabilities.)

2. Duration of assets = Duration od liabilities (either MacD or ModD)
3. Convexity of assets > convexity of liabilities

Consider the second condition, using ModD:

$$-\frac{P'_A}{P_A} = -\frac{P'_L}{P_L}$$

$\Rightarrow P'_A = P'_L$ since by condition 1, $P_A = P_L$

similarly, the third condition is equivalent to:

$$P''_A > P''_L$$

To sum up, another way to express the three conditions for Redington immunization at $i = i_0$ is:

1. $P_A = P_L$
2. $P'_A = P'_L$
3. $P''_A = P''_L$

Let P be the net present value, i.e.,

$$P = P_A - P_L = \sum (A_t - L_t)v^t$$

The three conditions then become (at $i = i_0$)

1. $P = 0$ (The NPV of assets and liabilities is 0)
2. $P' = 0$ (The first derivative of the NPV is 0)
3. $P'' = 0$ (The second derivative of the NPV is greater than 0)

Conditions (2) and (3) are the conditions for a relative minimum at $i = i_0$. If the NPV of assets and liabilities has a relative minimumat i_0, it means that a small change on either side of $i_0$will result in an increase in the NPV. In other words, Redington immunization requires that the NPV of assets and liabilities be concave up wards at i_0.

Example 21 (T8Q8)**.**

A company must make payments of 100 annually in the form of a 12-year annuity-immediate. It plans to buy two zero coupon bonds to fund these payments. The first bond matures in 2 years and the second bond matures in 9 years, and both are purchased to yield 11% effective. What face amount of each bond should the company buy in order to be immunized from small changes in the interest rate?

8.11 Full Immunization

Full immunization means that the company is protected against any change in the interest rate, no matter how large. The three conditions for full immunization are:

1. PV of Assets = PV of liabilities
2. Duration of Assets = Liabilities of Assets
3. There is one asset cash inflow before the liability cash outflow and one after it.

Example 22 (T8Q9)**.** A company must pay a liability of 1,000 in 2-years. Zero coupon bonds with terms of 1 years and 4 years are available for investment. The effective rate of interest is 7.5%.

(a) How much of each bond should the company buy in order to achieve full immunization?

(b) Show empirically that immunization has been achieved even for large changes in the interest rate. Take as an example a decrease in the interest rate to 0% and an increase to 100%.

8.12 Immunization by Exact Matching

Suppose that the asset cash inflow at each time t is equal to the corresponding liability cash outflow at that time,i.e., $A_t = L_t$ for all t, the the changes in interest rate will not affect the ability of the assets to pay for the liabilities. This exact matching of assets and liabilities is also called **dedication**.

Example 23 (T8Q10)**.**
A company expects to have liability cash outflows in one, two, three and four years of 200, 400, 600, and 500 respectively. The only investments available are the following bonds, all with annual coupons and all redeemable at par:

Term of Bond	Coupon Rate
1 year	7%
2 years	4%
3 years	5%
4 years	6%

How much of each bond should the company buy

in order to exactly match the liability cash outflows?