

MEME16203 Linear Models Marking Guide**Assignment 3****UNIVERSITI TUNKU ABDUL RAHMAN**

Faculty:	FES	Unit Code:	MEME16203
Course:	MAC	Unit Title:	Linear Models
Year:	1,2	Lecturer:	Dr Yong Chin Khian
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Due by:			

- Q1. Suppose that \mathbf{y} is $MVN_n(\boldsymbol{\mu}, \sigma^2 \mathbf{I})$ and that \mathbf{P} , \mathbf{Q} , and \mathbf{R} are symmetric $n \times n$ matrices with $\mathbf{PQ} = \mathbf{0}$, $\mathbf{PR} = \mathbf{0}$, and $\mathbf{QR} = \mathbf{0}$. Argue carefully that the three random variables $\mathbf{y}^T \mathbf{P} \mathbf{y}$, $\mathbf{y}^T \mathbf{Q} \mathbf{y}$ and $\mathbf{y}^T \mathbf{R} \mathbf{y}$ are jointly independent.

Ans.

Consider

$$\begin{bmatrix} \mathbf{Py} \\ \mathbf{Qy} \\ \mathbf{Ry} \end{bmatrix} = \begin{bmatrix} \mathbf{P} \\ \mathbf{Q} \\ \mathbf{R} \end{bmatrix} \mathbf{y}$$

This is a multivariate normal distribution and for $i \neq j$

$$Cov(\mathbf{Py}, \mathbf{Qy}) = \mathbf{P} \boldsymbol{\Sigma} \mathbf{Q} = 0$$

$$Cov(\mathbf{Py}, \mathbf{Ry}) = \mathbf{P} \boldsymbol{\Sigma} \mathbf{R} = 0$$

$$Cov(\mathbf{Qy}, \mathbf{Ry}) = \mathbf{Q} \boldsymbol{\Sigma} \mathbf{R} = 0$$

It follows that \mathbf{Py} , \mathbf{Qy} and \mathbf{Ry} are independent random vectors. Since

$$\begin{aligned} \mathbf{y}^T \mathbf{P} \mathbf{y} &= \mathbf{y}^T \mathbf{P} \mathbf{P}^- \mathbf{P} \mathbf{y} \\ &= \mathbf{y}^T \mathbf{P}^T \mathbf{P}^- \mathbf{P} \mathbf{y} \\ &= (\mathbf{Py}) \mathbf{P}^- (\mathbf{Py}) \end{aligned}$$

is a function of \mathbf{Py} . Similarly, $\mathbf{y}^T \mathbf{Q} \mathbf{y}$ is a function of \mathbf{Qy} and $\mathbf{y}^T \mathbf{R} \mathbf{y}$ is a function of \mathbf{Ry} . Thus, $\mathbf{y}^T \mathbf{P} \mathbf{y}$, $\mathbf{y}^T \mathbf{Q} \mathbf{y}$ and $\mathbf{y}^T \mathbf{R} \mathbf{y}$ are jointly independent.

- Q2. Suppose $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$, where $\boldsymbol{\epsilon} \stackrel{iid}{\sim} N(\mathbf{0}, \sigma^2 \mathbf{I})$ for some unknown $\sigma^2 > 0$.

(a) Determine the distribution of $\begin{bmatrix} \hat{\mathbf{Y}} \\ \mathbf{Y} - \hat{\mathbf{Y}} \end{bmatrix}$.

(b) Determine the distribution of $\hat{\mathbf{Y}}^T \hat{\mathbf{Y}}$.

Ans.

MEME16203 Linear Models Marking Guide

$$\begin{aligned}
(a) \quad & \begin{bmatrix} \hat{\mathbf{Y}} \\ \mathbf{Y} - \hat{\mathbf{Y}} \end{bmatrix} = \begin{bmatrix} \mathbf{P}_X \mathbf{Y} \\ \mathbf{Y} - \mathbf{P}_X \mathbf{Y} \end{bmatrix} = \begin{bmatrix} \mathbf{P}_X \\ \mathbf{I} - \mathbf{P}_X \end{bmatrix} \mathbf{Y} \\
& E \left(\begin{bmatrix} \mathbf{P}_X \\ \mathbf{I} - \mathbf{P}_X \end{bmatrix} \mathbf{Y} \right) = \begin{bmatrix} \mathbf{P}_X \\ \mathbf{I} - \mathbf{P}_X \end{bmatrix} E(\mathbf{Y}) = \begin{bmatrix} \mathbf{P}_X \\ \mathbf{I} - \mathbf{P}_X \end{bmatrix} \mathbf{X}\boldsymbol{\beta} = \begin{bmatrix} \mathbf{X}\boldsymbol{\beta} \\ \mathbf{0} \end{bmatrix} \\
& V \left(\begin{bmatrix} \mathbf{P}_X \\ \mathbf{I} - \mathbf{P}_X \end{bmatrix} \mathbf{Y} \right) \\
&= \begin{bmatrix} \mathbf{P}_X \\ \mathbf{I} - \mathbf{P}_X \end{bmatrix} V(\mathbf{Y}) \begin{bmatrix} \mathbf{P}_X & \mathbf{I} - \mathbf{P}_X \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{P}_X \\ \mathbf{I} - \mathbf{P}_X \end{bmatrix} \sigma^2 \mathbf{I} \begin{bmatrix} \mathbf{P}_X & \mathbf{I} - \mathbf{P}_X \end{bmatrix} \\
&= \sigma^2 \begin{bmatrix} \mathbf{P}_X \\ \mathbf{I} - \mathbf{P}_X \end{bmatrix} \begin{bmatrix} \mathbf{P}_X & \mathbf{I} - \mathbf{P}_X \end{bmatrix} \\
&= \sigma^2 \begin{bmatrix} \mathbf{P}_X \mathbf{P}_X & \mathbf{P}_X (\mathbf{I} - \mathbf{P}_X) \\ (\mathbf{I} - \mathbf{P}_X) \mathbf{P}_X & (\mathbf{I} - \mathbf{P}_X) (\mathbf{I} - \mathbf{P}_X) \end{bmatrix} \\
&= \sigma^2 \begin{bmatrix} \mathbf{P}_X & \mathbf{0} \\ \mathbf{0} & (\mathbf{I} - \mathbf{P}_X) \end{bmatrix}
\end{aligned}$$

As a linear transformation of a multivariate normal random variable, it follows that

$$\begin{bmatrix} \hat{\mathbf{Y}} \\ \mathbf{Y} - \hat{\mathbf{Y}} \end{bmatrix} \sim N \left(\begin{bmatrix} \mathbf{X}\boldsymbol{\beta} \\ \mathbf{0} \end{bmatrix}, \sigma^2 \begin{bmatrix} \mathbf{P}_X & \mathbf{0} \\ \mathbf{0} & (\mathbf{I} - \mathbf{P}_X) \end{bmatrix} \right).$$

$$(b) \quad \hat{\mathbf{Y}}^T \hat{\mathbf{Y}} = (\mathbf{P}_X \mathbf{Y})^T (\mathbf{P}_X \mathbf{Y}) = \mathbf{Y}^T \mathbf{P}_X \mathbf{P}_X \mathbf{Y} = \mathbf{Y}^T \mathbf{P}_X \mathbf{Y}$$

Let $\mathbf{A} = \frac{\mathbf{P}_X}{\sigma^2}$ and $\Sigma = \sigma^2 \mathbf{I}$

$\mathbf{A}\Sigma = \mathbf{P}_X$ is clearly idempotent

$DF = \text{rank}(\mathbf{P}_X) = \text{rank}(\mathbf{X})$

$$\lambda = \frac{1}{\sigma^2} \boldsymbol{\beta}^T \mathbf{X}^T \mathbf{P}_X \mathbf{X} \boldsymbol{\beta} = \frac{1}{\sigma^2} \boldsymbol{\beta}^T \mathbf{X}^T \mathbf{X} \boldsymbol{\beta}$$

Therefore, we end up with a scaled non-central chi-square random variable on $\text{rank}(\mathbf{X})$ degrees of freedom: $\hat{\mathbf{Y}}^T \hat{\mathbf{Y}} \sim \sigma^2 \chi_{\text{rank}(\mathbf{X})}^2 \left(\frac{1}{\sigma^2} \boldsymbol{\beta}^T \mathbf{X}^T \mathbf{X} \boldsymbol{\beta} \right)$

Q3. Consider the model

$$Y_{ij} = \gamma_0 + \gamma_1 X_i + \gamma_2 X_i^2 + \alpha_i + \epsilon_{ij}, \quad i = 1, 2, \dots, 11; \quad j = 1, \dots, 5$$

where $\epsilon \sim NID(0, \tau^2)$. This model can be expressed in matrix notation as $\mathbf{Y} = \mathbf{W}\boldsymbol{\gamma} + \boldsymbol{\epsilon}$. Let the matrix \mathbf{Z} be the first 3 columns of the matrix \mathbf{W} , define $\mathbf{P}_Z = \mathbf{Z}(\mathbf{Z}^T \mathbf{Z})^{-1} \mathbf{Z}^T$ and $\mathbf{P}_W = \mathbf{W}(\mathbf{W}^T \mathbf{W})^{-1} \mathbf{W}^T$.

$$(a) \quad \text{Use Cochran's theorem to derive the distribution of } F = \frac{c \mathbf{Y}^T (\mathbf{P}_W - \mathbf{P}_Z) \mathbf{Y}}{\mathbf{Y}^T (\mathbf{I} - \mathbf{P}_W) \mathbf{Y}}.$$

Report c , degrees of freedom and a formula for the noncentrality parameter.

Ans.

Let $\mathbf{A}_1 = \mathbf{P}_Z$, $\mathbf{A}_2 = \mathbf{P}_W - \mathbf{P}_Z$ and $\mathbf{A}_3 = \mathbf{I} - \mathbf{P}_W$. Then,

MEME16203 Linear Models Marking Guide

- \mathbf{A}_1 , \mathbf{A}_2 and \mathbf{A}_3 are all 55×55 symmetric matrices.
- $\mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3 = \mathbf{I}$.
- $\text{rank}(\mathbf{A}_1) + \text{rank}(\mathbf{A}_2) + \text{rank}(\mathbf{A}_3) = 3 + (13 - 3) + (55 - 13) = 55$.

Then, by Cochran's Theorem,

- $\frac{1}{\tau^2} \mathbf{Y}^T \mathbf{A}_k \mathbf{Y} \sim \chi_{r_k}^2 \left(\frac{1}{\tau^2} (\mathbf{X}\boldsymbol{\beta})^T \mathbf{A}_k \mathbf{X} \boldsymbol{\beta} \right)$, where $r_k = \text{rank}(\mathbf{A}_k)$ for $k = 1, 2, 3$
- $\mathbf{Y}^T \mathbf{A}_1 \mathbf{Y}$, $\mathbf{Y}^T \mathbf{A}_2 \mathbf{Y}$ and $\mathbf{Y}^T \mathbf{A}_3 \mathbf{Y}$ are distributed independently.

Now $DF_2 = \text{rank}(\mathbf{A}_2) = 13 - 3 = 10$ and $DF_3 = \text{rank}(\mathbf{A}_3) = 55 - 13 = 42$

- $\lambda_2 = \frac{1}{\tau^2} (\mathbf{W}\boldsymbol{\gamma})^T (\mathbf{P}_W - \mathbf{P}_Z) (\mathbf{W}\boldsymbol{\gamma})$
- $\lambda_3 = \frac{1}{\tau^2} (\mathbf{W}\boldsymbol{\gamma})^T (\mathbf{I} - \mathbf{P}_W) (\mathbf{W}\boldsymbol{\gamma}) = 0$

Hence,

- $\frac{1}{\tau^2} \mathbf{Y}^T (\mathbf{P}_W - \mathbf{P}_Z) \mathbf{Y} \sim \chi_{10}^2(\lambda_2)$ and
- $\frac{1}{\tau^2} \mathbf{Y}^T (\mathbf{I} - \mathbf{P}_W) \mathbf{Y} \sim \chi_{42}^2$

Since $\frac{1}{\tau^2} \mathbf{Y}^T (\mathbf{P}_W - \mathbf{P}_Z) \mathbf{Y}$ and $\frac{1}{\tau^2} \mathbf{Y}^T (\mathbf{I} - \mathbf{P}_W) \mathbf{Y}$ are independent, then

$$F = \frac{c \mathbf{Y}^T (\mathbf{P}_W - \mathbf{P}_Z) \mathbf{Y}}{\mathbf{Y}^T (\mathbf{I} - \mathbf{P}_W) \mathbf{Y}} = \frac{\frac{1}{10\tau^2} \mathbf{Y}^T (\mathbf{P}_W - \mathbf{P}_Z) \mathbf{Y}}{\frac{1}{42\tau^2} \mathbf{Y}^T (\mathbf{I} - \mathbf{P}_W) \mathbf{Y}} = \frac{42 \mathbf{Y}^T (\mathbf{P}_W - \mathbf{P}_Z) \mathbf{Y}}{10 \mathbf{Y}^T (\mathbf{I} - \mathbf{P}_W) \mathbf{Y}} \sim F_{10,42}(\lambda_2).$$

Thus,

- $c = \frac{42}{10}$,
- the degrees of freedom are (10, 42) and
- noncentrality parameter is $\lambda_2 = \frac{1}{\tau^2} (\mathbf{W}\boldsymbol{\gamma})^T (\mathbf{P}_W - \mathbf{P}_Z) (\mathbf{W}\boldsymbol{\gamma})$

- (b) Show that the noncentrality parameter is zero if $\alpha_1 \mathbf{w}_4 + \alpha_2 \mathbf{w}_5 + \cdots + \alpha_{11} \mathbf{w}_{14} = \mathbf{Zc}$ for some vector \mathbf{c} , where \mathbf{w}_j is the j^{th} column of \mathbf{W} .

Ans.

$$\begin{aligned} & \frac{1}{\tau^2} (\mathbf{W}\boldsymbol{\gamma})^T (\mathbf{P}_W - \mathbf{P}_Z) (\mathbf{W}\boldsymbol{\gamma}) \\ &= \frac{1}{\tau^2} [(\mathbf{W}\boldsymbol{\gamma})^T (\mathbf{I} - \mathbf{P}_Z) (\mathbf{W}\boldsymbol{\gamma}) - (\mathbf{W}\boldsymbol{\gamma})^T (\mathbf{P}_Z) (\mathbf{W}\boldsymbol{\gamma})] \\ &= \frac{1}{\tau^2} (\mathbf{W}\boldsymbol{\gamma})^T (\mathbf{I} - \mathbf{P}_Z) (\mathbf{W}\boldsymbol{\gamma}) \end{aligned}$$

This is zero if and only if $(\mathbf{I} - \mathbf{P}_Z) (\mathbf{W}\boldsymbol{\gamma}) = 0$. Note that

$$\begin{aligned} & (\mathbf{I} - \mathbf{P}_Z) (\mathbf{W}\boldsymbol{\gamma}) \\ &= (\mathbf{I} - \mathbf{P}_Z) [\mathbf{Z} \mathbf{w}_4 \mathbf{w}_5 \cdots \mathbf{w}_{14}] \boldsymbol{\gamma} \\ &= [\mathbf{0} (\mathbf{I} - \mathbf{P}_Z) \mathbf{w}_4 (\mathbf{I} - \mathbf{P}_Z) \mathbf{w}_5 \cdots (\mathbf{I} - \mathbf{P}_Z) \mathbf{w}_{14}] \boldsymbol{\gamma} \\ &= [\alpha_1 (\mathbf{I} - \mathbf{P}_Z) \mathbf{w}_4 \alpha_2 (\mathbf{I} - \mathbf{P}_Z) \mathbf{w}_5 \cdots \alpha_{10} (\mathbf{I} - \mathbf{P}_Z) \mathbf{w}_{14}] \boldsymbol{\gamma} \\ &= (\mathbf{I} - \mathbf{P}_Z) [\alpha_1 \mathbf{w}_4 \alpha_2 \mathbf{w}_5 \cdots \alpha_{10} \mathbf{w}_{14}] \boldsymbol{\gamma} \\ &= (\mathbf{I} - \mathbf{P}_Z) \mathbf{Zc} \\ &= 0 \text{ since } \mathbf{P}_Z = \mathbf{Z} \end{aligned}$$

MEME16203 Linear Models Marking Guide

Q4. Consider the model $Y_{ij} = \mu + \alpha_i + \epsilon_{ij}$, where $i = 1, 2, 3$, $j = 1, 2, 3$, and $\mu, \alpha_1, \alpha_2, \alpha_3$, are unknown parameters. Let $\epsilon_{ij} \stackrel{iid}{\sim} N(0, \sigma^2)$, where σ^2 is unknown.

- (a) Determine the distribution of $\frac{\hat{\tau}^2}{35\sigma^2}$ when $\tau = 0$, where $\hat{\tau}$ is the BLUE of $\tau = 2\alpha_1 - 8\alpha_2 + 6\alpha_3$.

Ans.

$$\begin{aligned}\tau &= 2\alpha_1 - 8\alpha_2 + 6\alpha_3 \\ &= 2(\mu + \alpha_1) - 8(\mu + \alpha_2) + 6(\mu + \alpha_3) \\ &= E(2\bar{Y}_{1.} - 8\bar{Y}_{2.} + 6\bar{Y}_{3.})\end{aligned}$$

Hence, τ is estimable.

The BLUE for τ is $2\bar{Y}_{1.} - 8\bar{Y}_{2.} + 6\bar{Y}_{3.}$. Let $\mathbf{Y} = [Y_{11} \ Y_{12} \ Y_{13} \ Y_{21} \ Y_{22} \ Y_{23} \ Y_{31} \ Y_{32} \ Y_{33}]^T$, then

$$\hat{\tau} = 2\bar{Y}_{1.} - 8\bar{Y}_{2.} + 6\bar{Y}_{3.} = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & -\frac{8}{3} & -\frac{8}{3} & -\frac{8}{3} & \frac{6}{3} & \frac{6}{3} & \frac{6}{3} \end{bmatrix} \mathbf{Y} \sim N(\tau = 0, 35\sigma^2).$$

Then,

$$\frac{\hat{\tau} - 0}{\sqrt{35\sigma^2}} \sim N(0, 1) \text{ and } \frac{\hat{\tau}^2}{35\sigma^2} \sim \chi_1^2.$$

- (b) Determine the distribution of $S^2 = \sum_{i=1}^3 \sum_{j=1}^3 (Y_{ij} - \bar{Y}_{i.})^2$.

Ans.

Note that $S^2 = \mathbf{Z}^T \mathbf{Z}$ where

$$\mathbf{Z} = \begin{bmatrix} Y_{11} - Y_{1.} \\ Y_{12} - Y_{1.} \\ Y_{13} - Y_{1.} \\ Y_{21} - Y_{2.} \\ Y_{22} - Y_{2.} \\ Y_{23} - Y_{2.} \\ Y_{31} - Y_{3.} \\ Y_{32} - Y_{3.} \\ Y_{33} - Y_{3.} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} Y_{11} \\ Y_{12} \\ Y_{13} \\ Y_{21} \\ Y_{22} \\ Y_{23} \\ Y_{31} \\ Y_{32} \\ Y_{33} \end{bmatrix} = \mathbf{C}\mathbf{Y}.$$

Then $\mathbf{Z} \sim N(\mathbf{0}, \sigma^2 \mathbf{C})$ and $\mathbf{C}\mathbf{C} = \mathbf{C}$. Let $\mathbf{\Sigma} = \sigma^2 \mathbf{C}$ and $\mathbf{A} = \frac{1}{\sigma^2} \mathbf{I}$. Then $\mathbf{A}\mathbf{\Sigma}\mathbf{A}\mathbf{\Sigma} = \mathbf{C}\mathbf{C} = \mathbf{C} = \mathbf{A}\mathbf{\Sigma}$ which is idempotent. Hence it follows that $\frac{S^2}{\sigma} = \mathbf{Z}^T \mathbf{A} \mathbf{Z} \sim \chi_6^2$ because $\text{rank}(\mathbf{C}) = 6$

- (c) Show that $F = \frac{c\hat{\tau}^2}{S^2}$, where c is a constant, has central F -distribution when $\tau = 0$. Report c .

Ans.

Note that

MEME16203 Linear Models Marking Guide

$$\begin{bmatrix} \hat{\tau} \\ \mathbf{Z} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & -\frac{8}{3} & -\frac{8}{3} & -\frac{8}{3} & \frac{6}{3} & \frac{6}{3} & \frac{6}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} Y_{11} \\ Y_{12} \\ Y_{13} \\ Y_{21} \\ Y_{22} \\ Y_{23} \\ Y_{31} \\ Y_{32} \\ Y_{33} \end{bmatrix} \sim N \left(\begin{bmatrix} \tau \\ \mathbf{0} \end{bmatrix}, \sigma^2 \begin{bmatrix} 35 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{C} \end{bmatrix} \right)$$

Consequently, $\hat{\tau}$ and \mathbf{Z} are independent which implies that $\frac{\hat{\tau}}{35\sigma^2}$ and $\frac{S^2}{\sigma^2}$ are independent central chi-square random variables with 1 and 6 degrees freedom respectively, and

$$F = \frac{\frac{\hat{\tau}^2}{35\sigma^2}}{\frac{S^2}{6\sigma^2}} = \frac{6\hat{\tau}^2}{35\sigma^2} \sim F(1, 3) \text{ and } c = \frac{6}{35}.$$