

Bayesian Inference and Decision Theory

Unit 8: Bayesian Regression

v8.3b



Learning Objectives for Unit 8

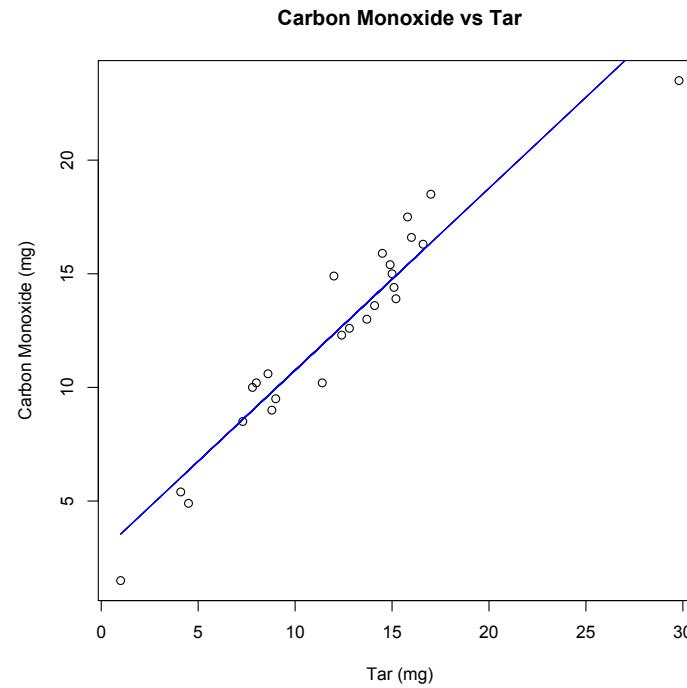
- Define regression
- Compare the Bayesian and frequentist approaches to regression
- For a simple linear regression model with normal errors:
 - Compute the posterior distribution for a noninformative prior distribution
 - Compute the posterior distribution for an informative normal/Gamma conjugate prior distribution
 - Use Gibbs sampling to approximate the posterior distribution for an informative normal/Gamma semi-conjugate prior distribution
- Compute:
 - Credible intervals for regression parameters
 - Predictive distribution for a new observation
- Extend simple linear regression model to multiple regression and linear models, including analysis of variance (ANOVA)
- Extend regression models to non-normal errors
- Analyze regression problems with missing data



What is Regression?

- Regression models the relationship of one or more dependent (or response or outcome) variables to one or more independent (or explanatory) variables
 - Linear or nonlinear
 - One explanatory variable or many
- Objectives:
 - Understand relationship between explanatory variables and dependent variable(s)
 - Predict values of dependent variable(s) given particular values of independent variable(s)
 - Infer cause and effect relationships
 - Estimate systematic relationships and filter out noise

Example taken from <http://www.mste.uiuc.edu/regression/cig.html>



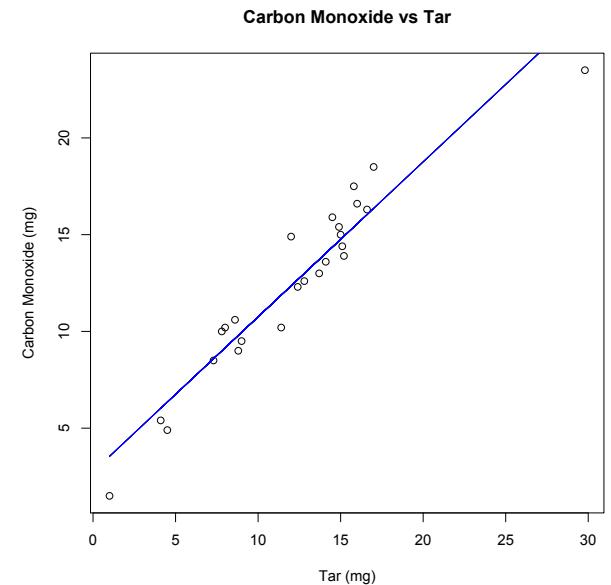
Example: Cigarette Data

- Data set consists of measurements of weight, tar, nicotine and CO content of 25 cigarette brands
- Graph shows scatterplot of data points and linear regression line with coefficients estimated by method of least squares



Simple Linear Regression

- Mean of dependent variable Y is linearly related to a single independent variable X
 - $E[Y | X] = \alpha + \beta X$
 - Equivalently, $Y = \alpha + \beta X + \varepsilon$ where $E[\varepsilon] = 0$
- Data consist of a sample of (Y_i, X_i) pairs
- Objectives of regression analysis:
 - Infer whether or not distribution of Y depends on X (whether slope of line is zero)
 - Estimate coefficients of relationship between Y and X (slope and intercept of line)
 - Find credible intervals for coefficients of the slope and intercept
 - Evaluate how much of the variability in Y is explained by X
 - Predict not-yet-observed Y for a given X value
 - Evaluate adequacy of model (are modeling assumptions met?)



Simple Linear Regression with Normal Homoscedastic* Errors

- Common assumptions for simple linear regression model:

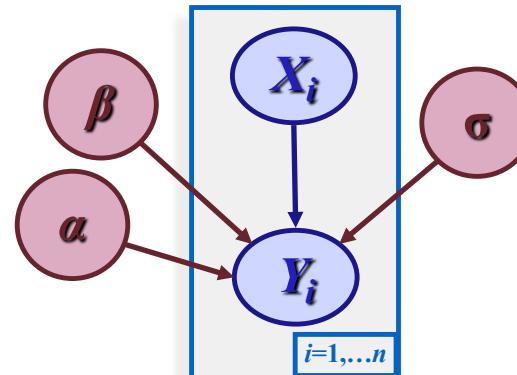
- Y given X is normally distributed
- $E[Y | X] = \alpha + \beta X$
- Observations $Y_{1:n}$ are independent given $X_{1:n}$
- $Var[Y|X] = \sigma^2$ is independent of X

- The likelihood function:

$$f(\underline{y} | \underline{x}, \alpha, \beta, \sigma) = (2\pi)^{-n/2} \sigma^{-n} \exp \left\{ -\frac{1}{2\sigma^2} \sum_i (y_i - \alpha - \beta x_i)^2 \right\}$$

*How would you verify
these assumptions?*

*Homoscedastic (from ancient Greek "homo" (same) and "skedasis" (dispersion) means all observations have same variance



*Plate representation of
normal simple linear
regression model:
Goal is to find posterior
distribution of α , β and σ*



Least Squares Estimation of Regression Coefficients

- The standard estimates for α and β are:

$$b = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2} \quad a = \bar{y} - b\bar{x} \quad \bar{y} = \frac{1}{n} \sum_{i=1}^n y_i, \quad \bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$$

- These are often called *least squares estimates* because they minimize the sum of squared deviations from the regression line (residuals):

$$(a, b) = \arg \min \{S_{ee}\}, \text{ where } S_{ee} = \sum_{i=1}^n (y_i - a - bx_i)^2$$

- If the error term is normally distributed with equal variances, then a and b are the maximum likelihood estimates of α and β



A Non-Informative Prior Distribution

- The model:
 - Observations Y_i are independent and normally distributed
 - $E[Y | X] = \alpha + \beta X$
 - $\text{Var}[Y|X] = \sigma^2$ is independent of X
- Likelihood function:
$$f(\underline{y} | \underline{x}, \alpha, \beta, \sigma) = (2\pi)^{-n/2} \sigma^{-n} \exp\left\{-\frac{1}{2\sigma^2} \sum_i (y_i - \alpha - \beta x_i)^2\right\}$$
- To simplify the calculations we re-parameterize:
 - $\eta = \alpha + \beta \bar{X}$ where $\bar{X} = \frac{1}{n} \sum_i X_i$ is the sample mean
 - Then $E[Y | X] = \eta + \beta(X - \bar{X})$
 - β is called the slope of the regression line
 - η is the transformed intercept
 - $\rho = 1/\sigma^2$ is the precision

(be careful: η is easy to confuse with n)
- A commonly used non-informative prior distribution:
 - $g(\eta, \beta, \rho) \propto \rho^{-1}$ Uniform on η and β ; weakly favoring smaller values of ρ
- Posterior distribution: η and β are independent and normal given ρ ; precision $\rho = 1/\sigma^2$ has a gamma distribution



Finding the Posterior Distribution (1 of 2)

- The likelihood function:

$$f(\underline{y} | \underline{x}, \eta, \beta, \rho) = \left(\frac{\rho}{2\pi} \right)^{n/2} \exp \left\{ -\frac{\rho}{2} \sum_i (y_i - \eta - \beta(x_i - \bar{x}))^2 \right\}$$

- Prior times likelihood:

$$f(\underline{y} | \underline{x}, \eta, \beta, \rho) g(\rho) \propto \rho^{n/2-1} \exp \left\{ -\frac{\rho}{2} \sum_i (y_i - \eta - \beta(x_i - \bar{x}))^2 \right\}$$

- Algebraic manipulation of squared deviations:

$$\sum_i (y_i - \eta - \beta(x_i - \bar{x}))^2 = S_{ee} + n(\eta - \bar{y})^2 + S_{xx}(\beta - b)^2$$

- Re-express prior times likelihood:

$$\begin{aligned} f(\underline{y} | \underline{x}, \eta, \beta, \rho) g(\rho) &\propto \rho^{n/2-1} \exp \left\{ -\frac{\rho}{2} (S_{ee} + n(\eta - \bar{y})^2 + S_{xx}(\beta - b)^2) \right\} \\ &= \left(\rho^{(n-2)/2-1} \exp \left\{ -\frac{1}{2} \rho S_{ee} \right\} \right) \left(\rho^{1/2} \exp \left\{ -\frac{1}{2} \rho n(\eta - \bar{y})^2 \right\} \right) \left(\rho^{1/2} \exp \left\{ -\frac{1}{2} \rho S_{xx}(\beta - b)^2 \right\} \right) \end{aligned}$$

- The first term is proportional to a gamma density function
- Given ρ , the second and third parts are proportional to normal density functions

Notation:

$$S_{xx} = \sum_i (x_i - \bar{x})^2$$

$$S_{yy} = \sum_i (y_i - \bar{y})^2$$

$$S_{xy} = \sum_i (x_i - \bar{x})(y_i - \bar{y})$$

$$S_{ee} = \sum_{i=1}^n (y_i - \bar{y} - b(x_i - \bar{x}))^2$$



Finding the Posterior Distribution (2 of 2)

- Posterior distribution for ρ is Gamma with shape $(n - 2)/2$ and scale $2/S_{ee}$
- η and β are independent and normally distributed conditional on ρ
 - η has mean \bar{y} and precision $n\rho$
 - β has mean b and precision $S_{xx}\rho$
 - α has mean $\bar{y} - b\bar{x}$ and precision $\rho \left(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}} \right)^{-1}$
- Note: the untransformed intercept α is not independent of the slope β
- Distribution of (α, β) conditional on $\rho = \sigma^{-2}$:
 - (α, β) has a bivariate normal distribution
 - Mean is $(\bar{y} - b\bar{x}, b)$
 - Covariance matrix:
 - $Var(\alpha) = Var(\eta - \beta x) = \sigma^2 \left(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}} \right)$
 - $Var(\beta) = \sigma^2 / S_{xx}$
 - $Cov(\alpha, \beta) = E[(\alpha - \bar{y} + b\bar{x})(\beta - b)] = E[\bar{x}(\beta - b)^2] = \bar{x}\sigma^2 / S_{xx}$

$$Cov \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \sigma^2 \begin{bmatrix} \frac{1}{n} + \frac{\bar{x}^2}{S_{xx}} & \frac{\bar{x}}{S_{xx}} \\ \frac{\bar{x}}{S_{xx}} & \frac{1}{S_{xx}} \end{bmatrix}$$



Marginal Posterior Distribution for Parameters

Transformed parameters:

- η has nonstandard t distribution with:
 - Center \bar{y}
 - Spread $(n(n - 2)/S_{ee})^{-1/2} = \sqrt{\frac{S_{ee}}{n(n-2)}}$
 - Degrees of freedom $n - 2$
- β has nonstandard t distribution with:
 - Center b
 - Spread $(S_{xx}(n - 2)/S_{ee})^{-1/2} = \sqrt{\frac{S_{ee}}{n-2} \frac{1}{S_{xx}}}$
 - Degrees of freedom $n - 2$
- η and β are:
 - Independent conditional on ρ
 - Uncorrelated but not independent when we marginalize out ρ

• Original parameters:

- (α, β) has a bivariate t distribution with:

- Center $\begin{pmatrix} \bar{y} - b\bar{x} \\ b \end{pmatrix}$

- Spread matrix $\frac{S_{ee}}{(n-2)} \begin{bmatrix} \frac{1}{n} + \frac{\bar{x}^2}{S_{xx}} & \frac{\bar{x}}{S_{xx}} \\ \frac{\bar{x}}{S_{xx}} & \frac{1}{S_{xx}} \end{bmatrix}$

- Degrees of freedom $n - 2$



Cigarette Example: Posterior Distribution

CO (mg)	Tar (mg)
13.60	14.10
16.60	16.00
23.50	29.80
10.20	8.00
5.40	4.10
15.00	15.00
9.00	8.80
12.30	12.40
16.30	16.60
15.40	14.90
13.00	13.70
14.40	15.10
10.00	7.80
10.20	11.40
9.50	9.00
1.50	1.00
18.50	17.00
12.60	12.80
17.50	15.80
4.90	4.50
15.90	14.50
8.50	7.30
10.60	8.60
13.90	15.20
14.90	12.00

- ρ has a Gamma distribution with
 - Shape $(n - 2)/2 = 11.5$
 - Scale $= 2/S_{ee} = 0.0446$
- Conditional on ρ
 - The transformed intercept η has a normal distribution with mean 12.53 and precision 25ρ
 - The slope β is independent of η and has a normal distribution with mean 0.80 and precision $\rho S_{xx} = 770.43\rho$
- Unconditionally
 - The transformed intercept η has a nonstandard t distribution with
 - Center $\bar{y} = 12.53$
 - Spread $(n(n - 2)/S_{ee})^{-1/2} = 0.28$
 - Degrees of freedom $n - 2 = 23$
 - The slope β has a nonstandard t distribution with
 - Center $b = 0.80$
 - Spread $(S_{xx}(n - 2)/S_{ee})^{-1/2} = 0.05$
 - Degrees of freedom $n - 2 = 23$

\bar{x}	12.22
\bar{y}	12.53
S_{xy}	617.10
S_{xx}	770.43
S_{ee}	44.87
n	25
$\eta = \bar{y}$	12.53
$b = S_{xy}/S_{xx}$	0.80
$s^2 = S_{ee}/(n - 2)$	1.95
s	1.40
a	2.74



Bayesian Interpretation of Standard Regression

- Most statistical software packages provide regression analysis
- We can use these software packages and give the result a Bayesian interpretation
- Results from cigarette regression in R:

```
Call:  
lm(formula = mgCarbonMonoxide ~ mgTar, data = cigdata)  
  
Residuals:  
    Min      1Q  Median      3Q     Max  
-3.1124 -0.7167 -0.3754  1.0091  2.5450  
  
Coefficients:  
            Estimate Std. Error t value Pr(>|t|)  
(Intercept) 2.74328   0.67521  4.063 0.000481 ***  
mgTar        0.80098   0.05032 15.918 6.55e-14 ***  
---  
Signif. codes:  0 '****' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1  
  
Residual standard error: 1.397 on 23 degrees of freedom  
Multiple R-squared: 0.9168, Adjusted R-squared: 0.9132  
F-statistic: 253.4 on 1 and 23 DF,  p-value: 6.552e-14
```

Center of intercept and slope

Spread of intercept and slope

Approximate posterior expectation of observation std dev

Degrees of freedom

Bayesian Regression and Least Squares Regression

- We can give results from standard regression packages a Bayesian interpretation
- Assumptions:
 - Linear regression with normal error
 - Noninformative prior distribution on parameters
- Under these assumptions, confidence intervals for regression coefficients can be interpreted as Bayesian credible intervals
- If a t-test rejects the hypothesis that coefficient is equal to zero, then 0 does not belong to the corresponding posterior credible interval for the coefficient



Predictive Distribution for New Observation (Exact)

- We would like to predict the CO content of a cigarette with given tar content
- Given tar content x_{new} , slope β , transformed intercept η and precision ρ , the mean CO content is normally distributed with:
 - Mean $(x_{new} - \bar{x})\beta + \eta = x_{new}\beta + \alpha$
 - Variance $\sigma^2 = 1/\rho$
- Integrating over β and η , and conditional on ρ , the predictive distribution for x_{new} is normal with
 - Mean $(x_{new} - \bar{x})b + \bar{y} = x_{new}b + a$ *Fitted value*
 - Variance $\left(\frac{(x_{new} - \bar{x})^2}{S_{xx}} + \frac{1}{n} + 1\right)\sigma^2$
- Integrating out the precision, the predictive distribution has a non-standard t distribution with
 - Center $(x_{new} - \bar{x})b + \bar{y} = x_{new}b + a$ *Fitted value*
 - Spread $\sqrt{\left(\frac{(x_{new} - \bar{x})^2}{S_{xx}} + \frac{1}{n} + 1\right)\left(\frac{S_{ee}}{n-2}\right)}$
 - Degrees of freedom $(n - 2)$



Predictive Distribution: Examples

- For tar content 14 mg the CO content $Y_{new}|(X_{new} = 14)$ has a nonstandard t distribution with
 - Center 13.96
 - Spread 1.43
 - Degrees of freedom 23
- For tar content 30 mg the CO content $Y_{new}|(X_{new} = 30)$ has a nonstandard t distribution with
 - Center 26.77
 - Spread 1.68
 - Degrees of freedom 23

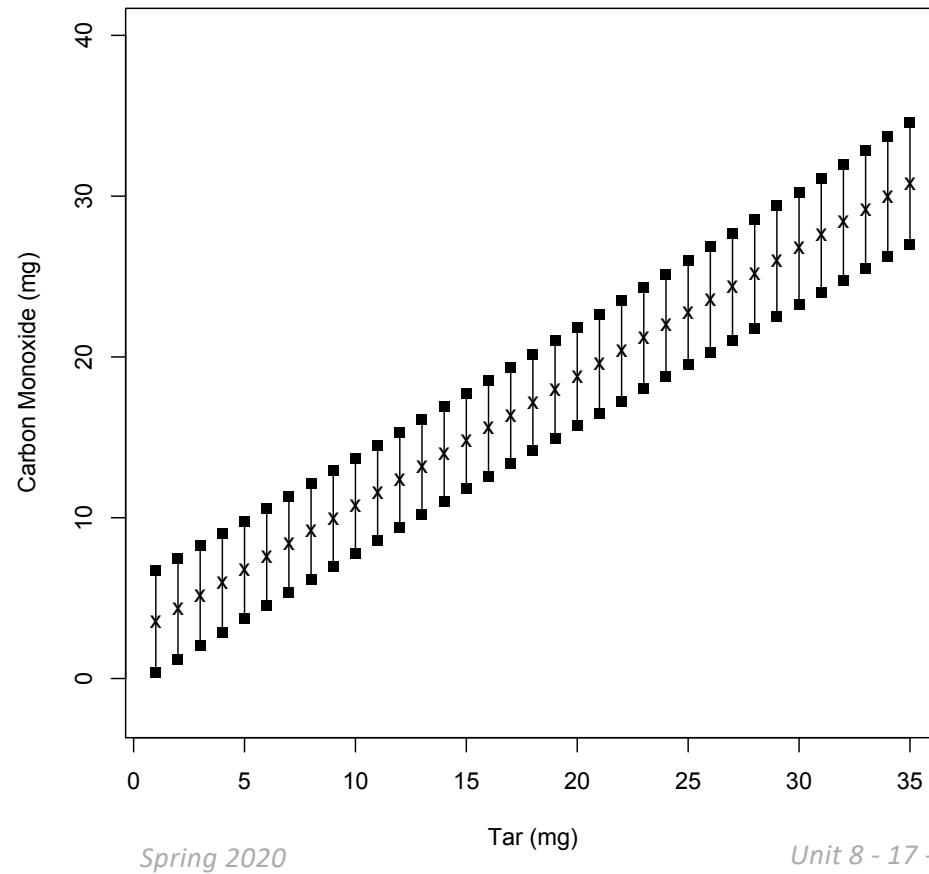


Predictive Distribution for New Observation (Direct MC)

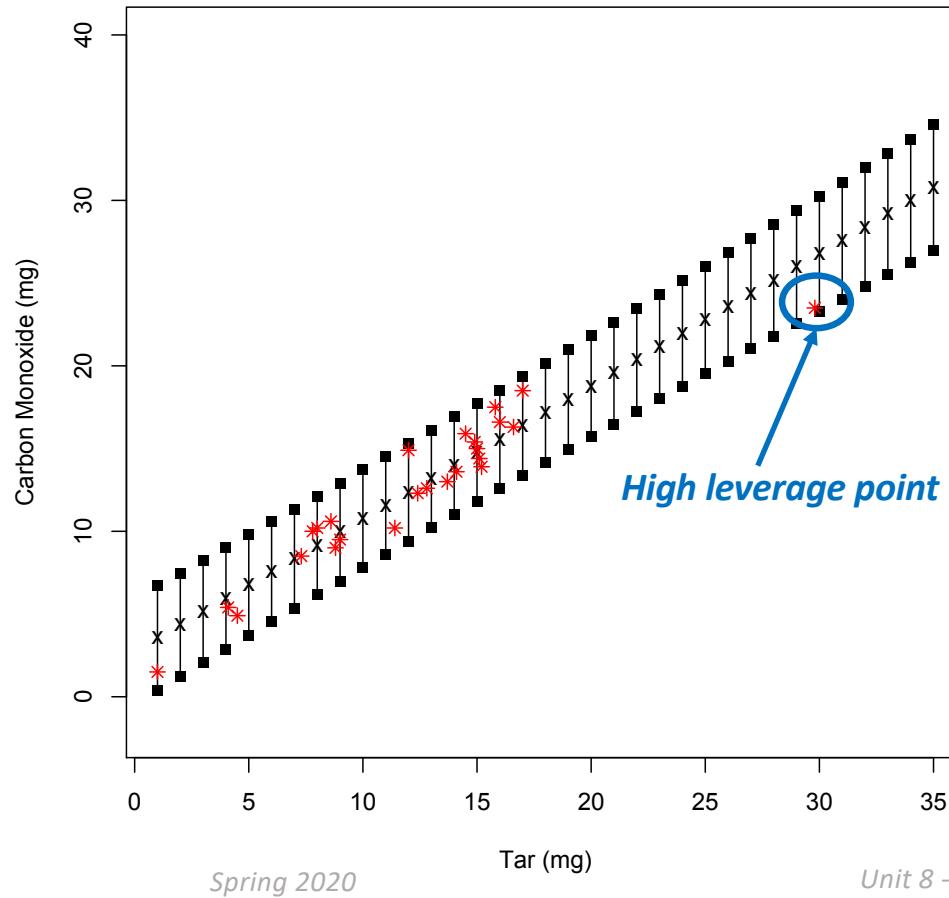
- We can also estimate the predictive distribution by direct Monte Carlo
- The algorithm: For $k = 1, \dots, K$ (K = desired sample size)
 - Simulate $\rho^{(k)}$ from a Gamma distribution with shape $(n - 2)/2$ and scale $2/S_{ee}$
 - Simulate $\eta^{(k)}$ from a Normal distribution with mean \bar{y} and precision $n\rho^{(k)}$
 - Simulate $\beta^{(k)}$ from a Normal distribution with mean b and precision $S_{xx} \rho^{(k)}$
 - Simulate $y_{new}^{(k)}$ from a Normal distribution with mean $\eta^{(k)} + \beta^{(k)}(x_{new} - \bar{x})$ and precision $\rho^{(k)}$
- The sample $y_{new}^{(1)}, \dots, y_{new}^{(K)}$ approximates the posterior predictive distribution of Y given $X = x_{new}$



95% Intervals for Predictive Distribution of New Observation



Posterior Predictive Model Check

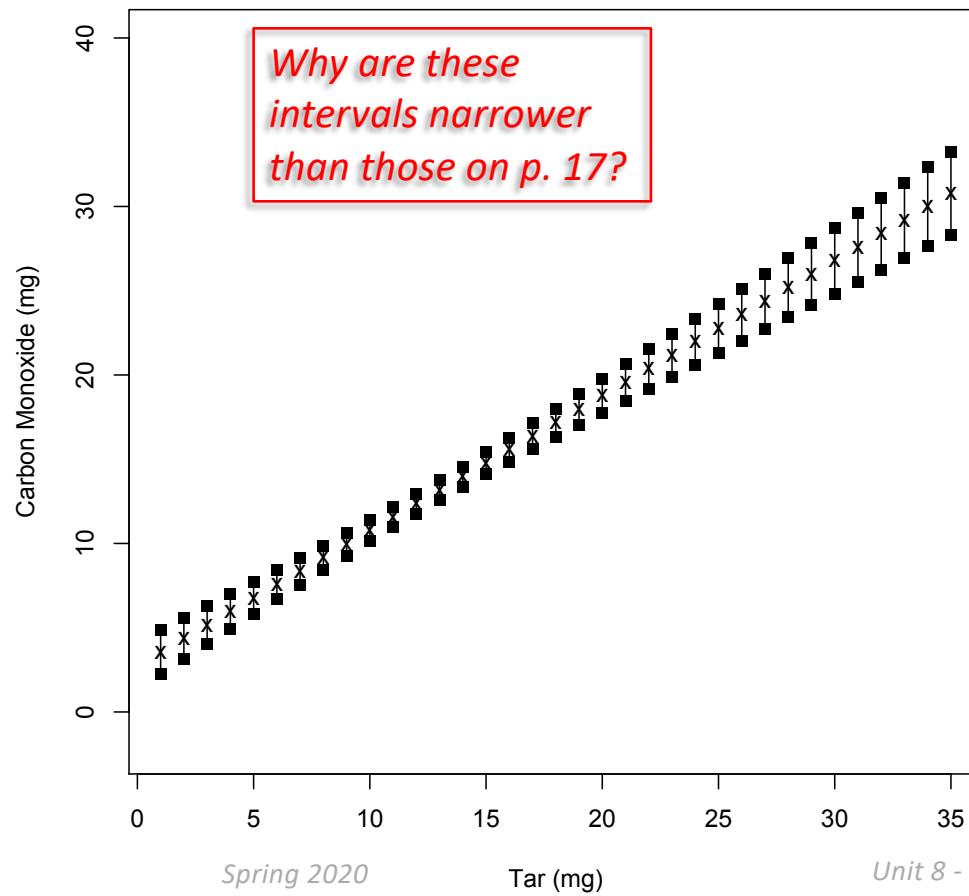


Mean of Predictive Distribution for New Observation

- Given tar content x_{new} , slope β , transformed intercept η and precision ρ , the tar content is normally distributed with:
 - Mean $x_{new}\beta + \alpha$
 - Variance $\sigma^2 = 1/\rho$
- Given the precision ρ , mean $x_{new}\beta + \alpha$ of the predictive distribution is normally distributed with
 - Mean $x_{new}b + \alpha$
 - Variance $\left(\frac{(x_{new}-\bar{x})^2}{S_{xx}} + \frac{1}{n}\right) \sigma^2$
- Integrating out the precision, mean $x_{new}\beta + \alpha$ of the predictive distribution has a non-standard t distribution with
 - Center $x_{new}b + \alpha$
 - Spread $\sqrt{\left(\frac{(x_{new}-\bar{x})^2}{S_{xx}} + \frac{1}{n}\right) \left(\frac{S_{ee}}{n-2}\right)}$
 - Degrees of freedom $(n - 2)$

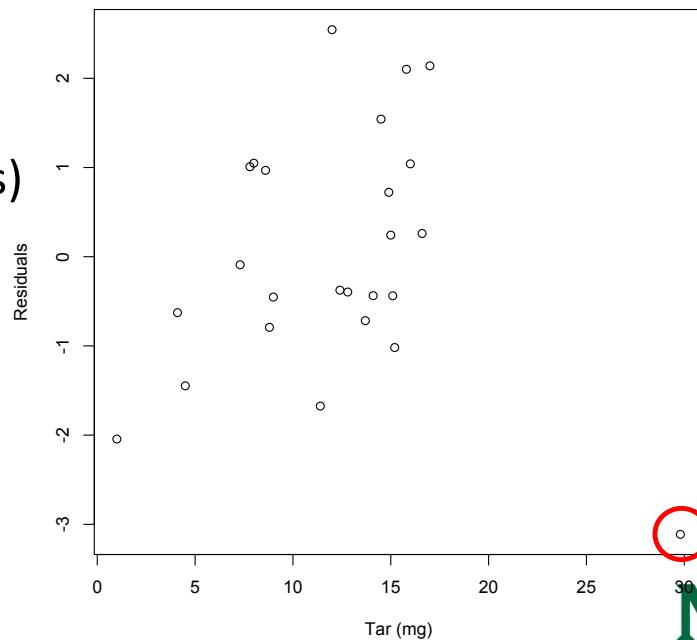


95% Credible Interval for Mean of Predictive Distribution for New Observation

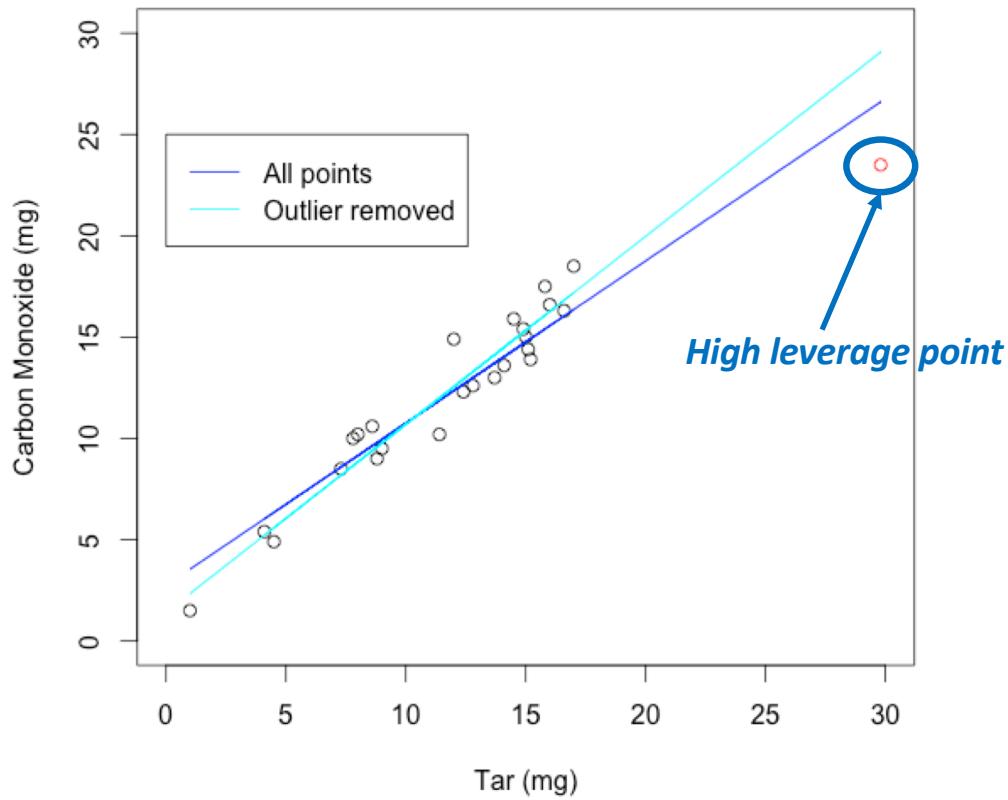


Diagnostics

- Residual plots are a good way to diagnose model inadequacy
- Residuals should show no systematic pattern (“white noise”)
- Things to look for in residuals
 - U or bowl shape (indicates nonlinearity)
 - Outliers (very small or very large residuals)
 - Fan shape (indicates variance is not constant in X)
 - Unusual values



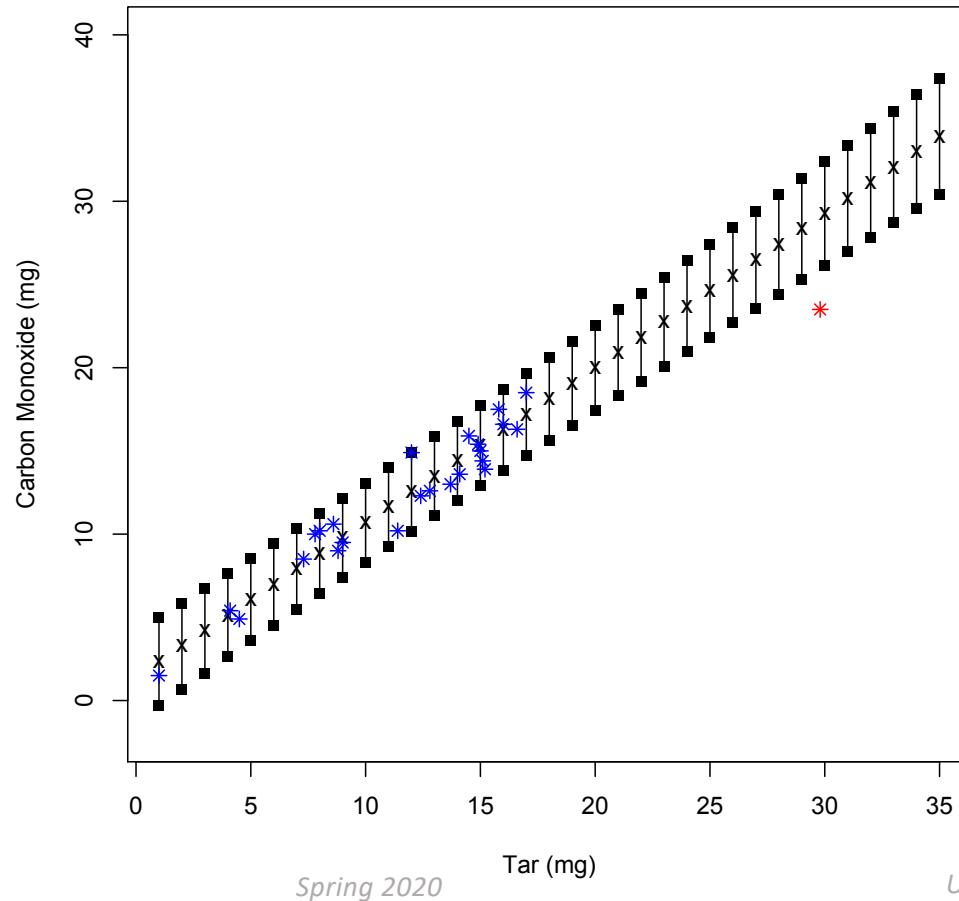
Cigarette Example with Outlier Removed



	All Data	Outlier removed
Slope est	0.80	0.93
Slope SE	0.05	0.05
Intrcp est	2.74	1.41
Intrcp SE	0.68	0.65
Regr SE	1.40	1.12



Predicting Outlier from Model without Outliers



An Informative Normal-Gamma Prior Distribution

- An informative conjugate prior distribution has the same form as the posterior distribution from a regression with a noninformative prior distribution:
 - Given the precision $\rho = 1/\sigma^2$, η and β follow a bivariate normal distribution
 - Mean of η is μ_η and precision of η is $k_\eta \rho$
 - Mean of β is μ_β and precision of β is $k_\beta \rho$
 - η and β are independent (a more general conjugate family allows dependence)
 - Precision $\rho = 1/\sigma^2$ has a Gamma(c, d) distribution
- The posterior distribution given the data $(y_{1:n}, x_{1:n})$ has the same form as the prior distribution:
 - Given the precision ρ , the coefficients η and β are independent and normal
 - Mean of η is $\mu_\eta^* = (k_\eta \mu_\eta + n\bar{y})/(k_\eta + n)$
 - Precision of η is $\rho(k_\eta + n)$
 - Mean of β is $\mu_\beta^* = \frac{k_\beta \mu_\beta + S_{xy}}{k_\beta + S_{xx}}$
 - Precision of β is $\rho(k_\beta + S_{xx})$
 - Precision ρ has a gamma distribution, with shape $c^* = c + \frac{n}{2}$ and scale

$$d^* = \left(d^{-1} + \frac{1}{2} S_{ee} + \frac{k_\eta n}{2(k_\eta + n)} (\bar{y} - \mu_\eta)^2 + \frac{k_\beta S_{xx}}{2(k_\beta + S_{xx})} (b - \mu_\beta)^2 \right)^{-1}$$



Interpreting the Parameters of the Conjugate Distribution

- μ_η : the mean of the transformed intercept term η
- k_η : the ratio of the precision ρk_η of the intercept η to the precision ρ of the deviations from the regression line
- μ_β : the mean of the slope term β
- k_β : the ratio of the precision ρk_β of the slope β to the precision ρ of the deviations from the regression line
- c : the shape parameter for the Gamma prior for the precision ρ of the deviations from the regression line
- d : the scale parameter for the Gamma prior for the precision ρ of the deviations from the regression line



Generalization: Multiple Regression

- In the multiple regression model, a dependent variable Y depends on several independent variables X_1, \dots, X_p

- Multiple linear regression with normal errors:

- Y given X_1, \dots, X_p is normally distributed
- $E[Y|X_1, \dots, X_p] = X_1\beta_1 + \dots + X_p\beta_p$
- Observations Y_i are independent
- $Var[Y|X_1, \dots, X_p] = \sigma^2$, and σ^2 is independent of X_1, \dots, X_p

- If we have n observations, we write $\underline{y} = \underline{X}\underline{\beta} + \underline{\varepsilon}$ where:

- \underline{y} is a $n \times 1$ vector of observations
- \underline{X} is a $n \times p$ matrix of independent variable values
 - Usually the first column consists of 1's, corresponding to a constant term in the regression equation
 - We can represent nonlinear equations by including polynomial terms like X_1^2 , interaction terms like X_1X_2 , or other nonlinear terms like $\log(X_i)$ as predictors
- $\underline{\beta}$ is a $p \times 1$ vector of regression coefficients
- $\underline{\varepsilon}$ is a $n \times 1$ vector of deviations, assumed normal with mean 0 and variance σ^2

$$\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} 1 & \cdots & x_{p1} \\ \vdots & \ddots & \vdots \\ 1 & \cdots & x_{pn} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_p \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{pmatrix}$$

$n \times 1 \qquad n \times p \qquad p \times 1 \qquad n \times 1$



Multivariate Normal Distribution

- The multivariate normal distribution $N(\underline{\theta}, \Sigma)$ is usually parameterized by a mean vector $\underline{\theta} = (\theta_1, \dots, \theta_p)^T$ and covariance matrix $\Sigma = [\sigma_{ij}]$
- The density function for a random vector $X = (X_1, \dots, X_p)^T$ with the $N(\underline{\theta}, \Sigma)$ distribution is:

$$f(\underline{x} | \underline{\theta}, \Sigma) = \frac{1}{|\Sigma|^{\frac{1}{2}} (\sqrt{2\pi})^k} \exp \left\{ -\frac{1}{2} (\underline{x} - \underline{\theta})^T \Sigma^{-1} (\underline{x} - \underline{\theta}) \right\}$$

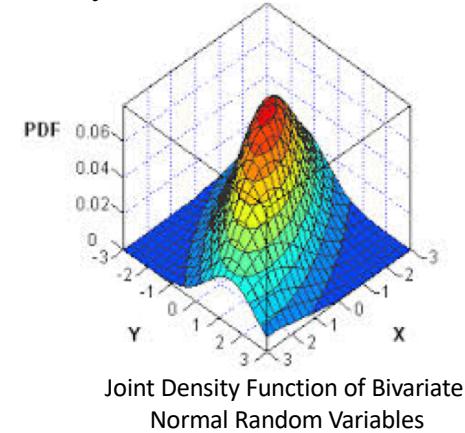
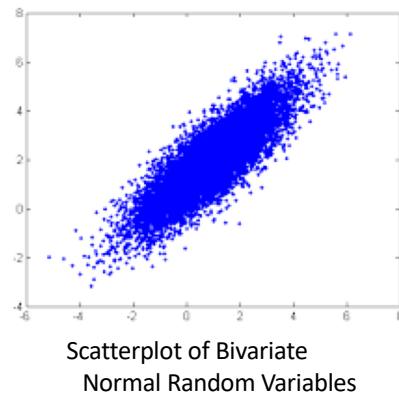
- Bayesian statisticians often use an alternate parameterization that is more convenient for Bayesian updating
- Bayesians often parameterize the multivariate normal distribution with mean vector $\underline{\theta} = (\theta_1, \dots, \theta_p)^T$ and precision matrix $P = [\rho_{ij}]$, where $P = \Sigma^{-1}$
- The multivariate normal density function can be written in terms of $\underline{\theta}$ and P :

$$f(\underline{x} | \underline{\theta}, P) = \frac{|P|^{\frac{1}{2}}}{(\sqrt{2\pi})^k} \exp \left\{ -\frac{1}{2} (\underline{x} - \underline{\theta})^T P (\underline{x} - \underline{\theta}) \right\}$$



Multivariate Normal Distribution

- If a random vector $X = (X_1, \dots, X_p)^T$ has a multivariate normal distribution with mean $\underline{\theta} = (\theta_1, \dots, \theta_p)^T$ and covariance matrix Σ , then
 - Each component X_i has a univariate normal distribution with mean θ_i and variance $\sigma_{ii} = \sigma_i^2$
 - The covariance $\sigma_{ij} = Cov(X_i, X_j) = E[(X_i - \theta_i)(X_j - \theta_j)]$ measures how X_i and X_j vary together
 - X_i and X_j are independent if $\sigma_{ij} = 0$
 - The correlation $Cor(X_i, X_j) = Cov(X_i, X_j)/\sigma_{ij}$, lies between -1 and 1.
 - A correlation of 1 or -1 (perfect correlation) occurs when X_i is a linear function of X_j



Least Squares Estimation of Multiple Regression Coefficients

- The standard estimate for β is $\underline{b} = (X^T X)^{-1} X^T \underline{y}$
- This is often called the *least squares* estimate because it minimizes the sum of squared deviations from the regression line:

$$\underline{b} = \arg \min_{b'} \left\{ \sum_{i=1}^n (y_i - X_i b')^2 \right\}$$

- If the error term is normally distributed with equal variances, then \underline{b} is the maximum likelihood estimate of β

- Simple regression is a special case:

- X is a $nx2$ matrix with first column a vector of 1's and second column the independent variable
- It is straightforward (although tedious) to verify that the least squares estimates for simple regression are

$$\begin{pmatrix} a \\ b \end{pmatrix} = (X^T X)^{-1} X^T \underline{y}$$



Conjugate Prior Distribution for Multiple Linear Regression with Normal Errors

- Observations $\underline{y} \sim_{indep} Normal(\underline{X}\underline{\beta}, \sigma^2 I)$, where I is an $n \times n$ identity matrix
- Normal-Gamma conjugate prior for β and $\rho = 1/\sigma^2$
 - Given the precision ρ , the regression coefficients β are multivariate normal random variables with mean $\underline{\mu}$ and nonnegative definite precision matrix ρK
 - The precision ρ has a gamma distribution with shape c and scale d
- Posterior distribution for β and ρ
 - Given the precision ρ , the regression coefficients β are normally distributed with mean
$$\underline{\mu}^* = (K + X^T X)^{-1} (K \underline{\beta} + X^T \underline{y})$$
 - The precision ρ has a gamma distribution with shape $c^* = c + n/2$ and scale

$$d^* = \left(d^{-1} + \frac{1}{2} \left((\underline{y} - \underline{X}\underline{b})^T (\underline{y} - \underline{X}\underline{b}) + \frac{1}{2} (\underline{b} - \underline{\mu})^T (K^{-1} + (X^T X)^{-1}) (\underline{b} - \underline{\mu}) \right) \right)^{-1}$$

- Marginal distribution for β is multivariate t
- Predictive distribution / marginal likelihood for observations is multivariate t



Marginal Likelihood for Normal-Gamma Conjugate Prior Distribution

- We are given values \underline{x} for the independent variables and want to predict the dependent variable y
 - y given $\underline{x} = (x_1, \dots, x_p)$ is normal with mean $\underline{x}^T \beta$ and precision ρ
 - β is normal with mean μ and precision matrix ρK (a $p \times p$ matrix)
 - ρ is gamma with shape c and scale d
- Given ρ, K and μ
 - y is normally distributed
 - Mean is $\underline{x}^T \mu$
 - Precision is $\rho (\underline{x}^T K^{-1} \underline{x} + 1)^{-1}$
- Integrate out ρ to obtain the predictive distribution:
 - y is nonstandard t
 - Center is $\underline{x}^T \mu$
 - Spread is $\sqrt{\frac{1}{cd} (\underline{x}^T K^{-1} \underline{x} + 1)^{-1}}$
- Typically we use simulation to estimate marginal likelihoods
 - Simulate ρ and β
 - Simulate y given \underline{x}, ρ and β



Zellner's g-Prior Distribution

- It is often difficult to specify prior hyperparameters
- Zellner's g-prior is a commonly used “weakly informative” conjugate prior distribution
 - y given $\underline{x} = (x_1, \dots, x_p)$ is normal with mean $\underline{x}^T \underline{\beta}$ and precision ρ
 - β is normal with mean $\underline{0}$ and precision matrix $\rho g^{-1}(X^T X)^{-1}$ (a $p \times p$ matrix)
 - ρ is gamma with shape c and scale d
- Posterior distribution for β and ρ
 - Given the precision ρ , the regression coefficients β are normally distributed with mean $\underline{\mu}^* = \frac{g}{g+1}(X^T X)^{-1} X^T \underline{y}$ and variance $\nu = \frac{g}{g+1} \sigma^2 (X^T X)^{-1}$
 - The precision ρ has a gamma distribution with shape $c^* = c + n/2$ and scale $d^* = \left(d^{-1} + \frac{1}{2} (\underline{y} - X \underline{\mu}^*)^T (\underline{y} - X \underline{\mu}^*) \right)^{-1}$
- Predictive distribution / marginal likelihood for observations is multivariate t



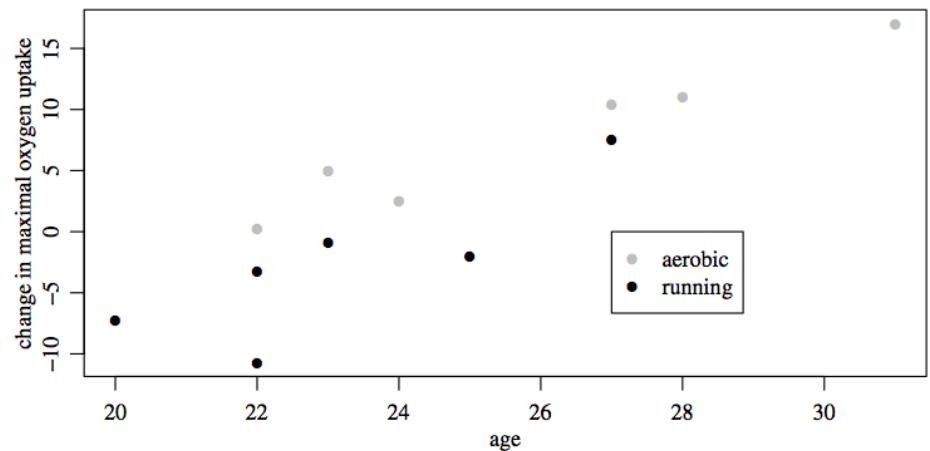
Remarks on the g-Prior

- Weakly informative
- Shrinks toward prior mean $\underline{\mu} = \underline{0}$
 - Precision matrix $\rho g^{-1}(X^T X)^{-1}$ for prior distribution of regression coefficient β depends on observed values of dependent variable X
 - Using this distribution implies that our prior knowledge about β before seeing the response variable y depends (weakly) on the values of the independent variables X
 - Parameter g controls amount of shrinkage
 - Larger g corresponds to less information and less shrinkage
 - Shrinkage toward $\underline{\mu} = \underline{0}$ helps prevent overfitting



Example: Oxygen Uptake (from Hoff Chapter 9)

- Plot shows data from study on oxygen uptake during exercise
 - There are 12 subjects in the study
- Regression equation:
$$Y_i = \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \beta_4 x_{i4} + \varepsilon$$
 - Y_i = change in maximal oxygen uptake after 12-week exercise program
 - $x_{i1} = 1$ for each subject i
 - $x_{i2} = 0$ if subject i is on running program; 1 if on aerobic exercise program
 - x_{i3} = age of subject i
 - $x_{i4} = x_{i2} x_{i3}$ (interaction between age and program)

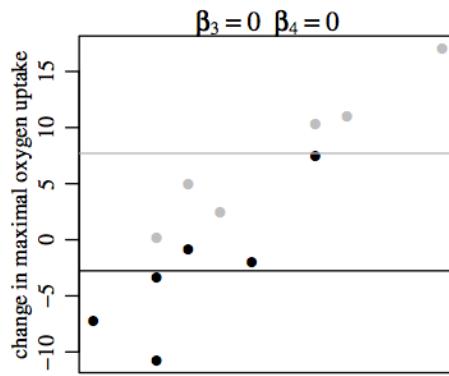


$$E[Y | \underline{x}] = \begin{cases} \beta_1 + \beta_3 \times \text{age} & \text{if running} \\ (\beta_1 + \beta_2) + (\beta_3 + \beta_4) \times \text{age} & \text{if aerobic} \end{cases}$$



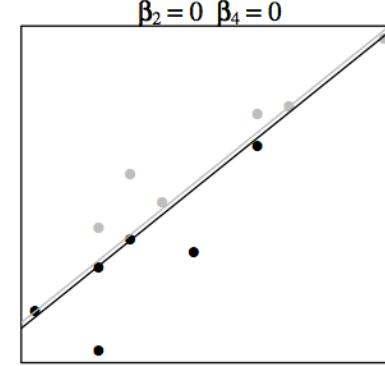
Least Squares Estimates for Alternative Models

No
dependence
on age

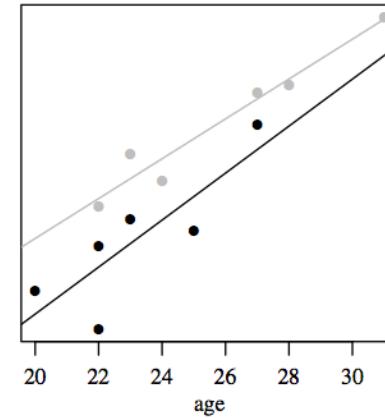
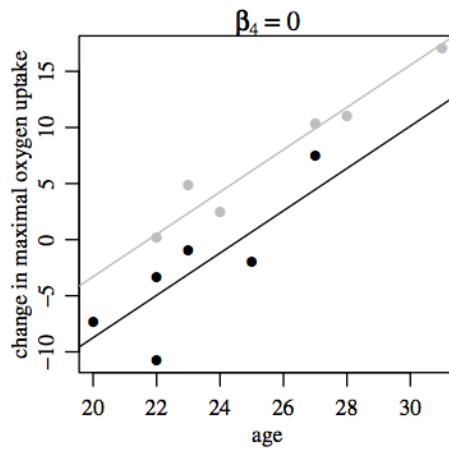


$\beta_2 = 0 \ \beta_4 = 0$

Running
same as
aerobic



Running
slope same
as aerobic
slope but
intercepts
are different



Slopes and
intercepts
are different



Oxygen Uptake: Analysis

$$d^* = \left(\frac{1}{d} + \frac{1}{2} \sum_{i=1}^n (y_i - X_i \mu^*)^2 \right)^{-1} = \left(\frac{s^2 + SSR}{2} \right)^{-1}$$

- Hoff text uses a weakly informative conjugate Zellner g -prior distribution with $g=12$ (about one observation of prior information)
 - Conditional on the precision ρ , the coefficients β are normally distributed with mean $\mu = 0$ and precision ρK , where $K = \frac{1}{12} X^T X$
 - Precision ρ has gamma distribution with shape $c = 1/2$ and scale $d = 2/s^2$, where $s^2 = S_{ee} / (n - p)$ is an unbiased estimate of the variance computed from the least squares sum of squared residuals
 - This is a “unit observation prior” – contains amount of information in a single observation
- Posterior distribution
 - Conditional on precision ρ , the coefficients β are normally distributed with mean μ^* and precision matrix ρK^*
$$\mu^* = \frac{12}{13} (X^T X)^{-1} X^T \underline{y} \quad K^* = \frac{12}{13} (X^T X)^{-1}$$
 - Precision ρ has gamma distribution with shape $c^* = (n + 1)/2$ and scale d^*
$$d^* = \left(\frac{1}{d} + \frac{1}{2} \sum_{i=1}^n (y_i - X_i \mu^*)^2 \right)^{-1} = \left(\frac{1}{2} \left(s^2 + \sum_{i=1}^n (y_i - X_i \mu^*)^2 \right) \right)^{-1}$$
- In this analysis, we use the observed values of the predictors X to specify the prior distribution for β and both the values of X and y to specify the prior distribution for ρ
- The small virtual sample size limits the impact of the prior distribution



More on Oxygen Uptake

```
## Oxygen Uptake - Bayesian analysis with g-prior, g=12

x=cbind(array(1,length(y)), aerobic, age, age*aerobic)
colnames(x) = c("const","aerobic","age","age*aerobic")
n = dim(x)[1]      # number of observations
p = dim(x)[2]      # number of regression coefficients

# Weakly informative prior distribution
## note that sigma(lmfit) is least-squares estimate of variance
## it is the sum of squared residuals divided by n-p
## verify that (sigma(lmfit))^2 =  sum(lmfit$residuals^2)/(n-4)
g = 12             # Zellnear g-prior
shp = 0.5          # shape parameter for prior on precision
scl = 2/(sigma(lmfit))^2 # scale parameter for prior on precision

## posterior mean and precision matrix
g = 12
mu.star = g/(g+1)*solve(t(x) %% x) %% t(x) %% y
K.star = g/(g+1)*solve(t(x) %% x)

## Fitted values and residuals
g.fitted = x %% mu.star
g.resid = y - g.fitted

## posterior shape and scale
shp.star = shp + n - p
scl.star = (1/scl + 0.5*sum(g.resid^2))^-1
```

Coefficient	Posterior mean (g-prior) $\underline{\mu}^* = \frac{12}{13}(X^T X)^{-1} X^T \underline{y}$	Least squares (g=∞) $\underline{b} = (X^T X)^{-1} X^T \underline{y}$
Constant	-47.35	-51.29
Aerobic indicator	12.10	13.11
Age	1.93	-2.09
Age * Aerobic indicator	-0.29	-0.32



Posterior Distribution for β_2 and β_4

- Recall that β_2 and β_4 are the difference in intercept and slope for the two exercise groups
- These plots suggest lack of evidence for a difference between the groups

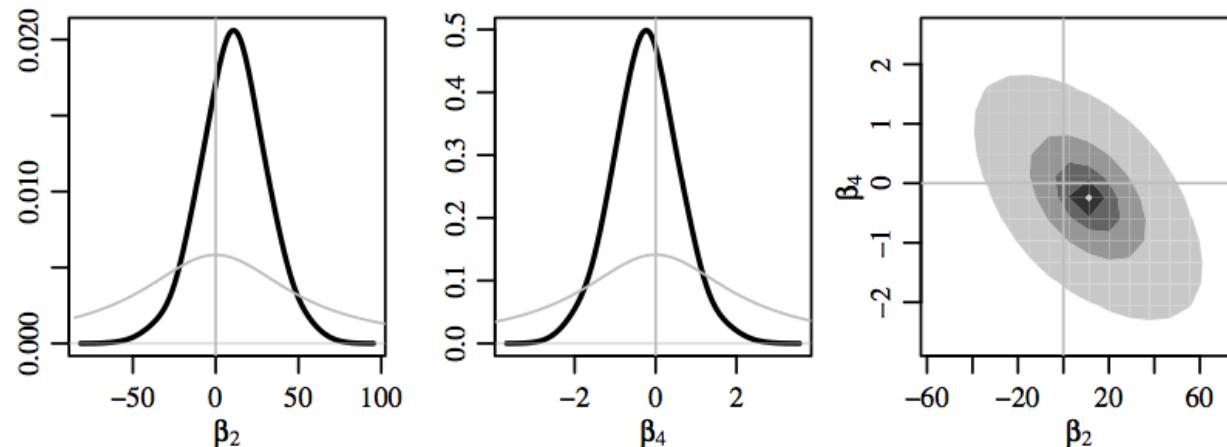


Fig. 9.3. Posterior distributions for β_2 and β_4 , with marginal prior distributions in the first two plots for comparison.



Posterior Distribution for Aerobics Effect by Age

- The difference in expected oxygen uptake for an individual of age x in aerobics versus running program is $\beta_2 + \beta_4 x$
- The plot shows 95% intervals of $\beta_2 + \beta_4 x$ at each age x
- There is reasonably strong evidence of a difference between programs for younger ages

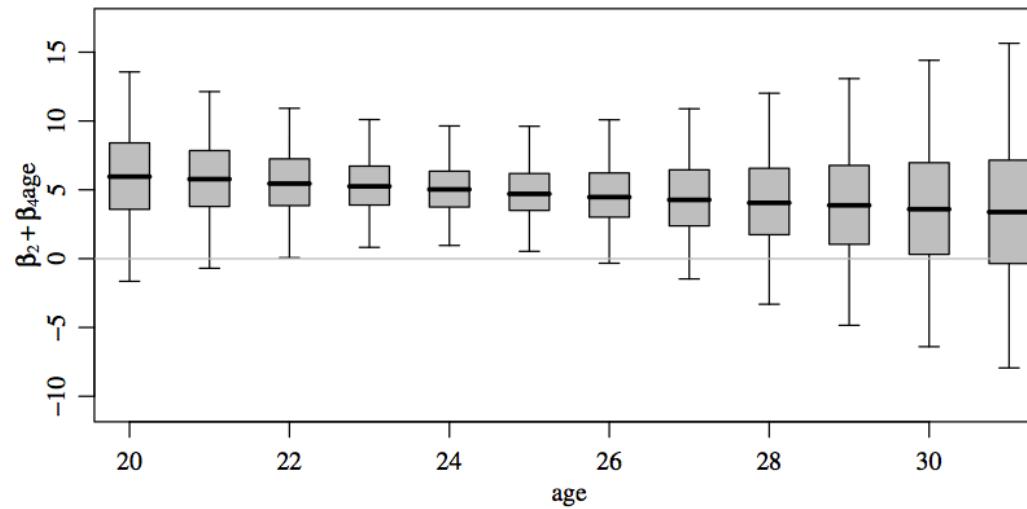
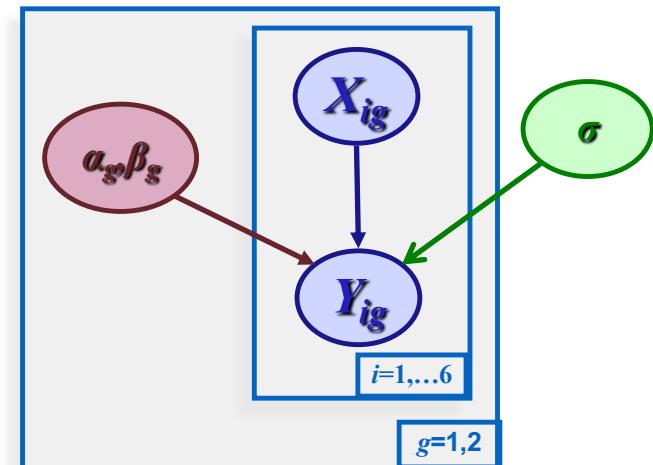


Fig. 9.4. Ninety-five percent confidence intervals for the difference in expected change scores between aerobics subjects and running subjects.

Reframing Oxygen Model as Hierarchical Model

- We could reframe the oxygen model as a hierarchical model with vague prior information on top-level hyperparameters:
 - $Y_{gi} \sim Normal(\alpha_g + \beta_g x_{ig}, \sigma^2)$
 - $\alpha_g \sim iid Normal(0, 100)$
 - $\beta_g \sim iid Normal(0, 100)$
 - $\rho = \sigma^{-2} \sim Gamma(0.5, 1000)$
- We can use JAGS to estimate the parameters of this model
- The model in JAGS:

```
model{  
  for (i in 1:n) {  
    y[i] ~ dnorm((a[2]+b[2]*age[i])*aerobic[i]+(a[1]+b[1]*age[i])*(1-aerobic[i]), rho)  
  }  
  for (i in 1:2) {  
    a[i] ~ dnorm(0, 0.0001)  
    b[i] ~ dnorm(0, 0.0001)  
  }  
  rho ~ dgamma(0.5, 1/1000)  
}
```



Note that JAGS parameterizes normal distribution with precision (not standard deviation) and Gamma distribution with rate (not scale)



Comparison: Least Squares Regression with Hierarchical Model in JAGS

```
Call:  
lm(formula = y ~ age * aerobic)  
  
Residuals:  
    Min      1Q  Median      3Q     Max  
-5.5295 -0.9610  0.3945  2.1717  2.2883  
  
Coefficients:  
            Estimate Std. Error t value Pr(>|t|)  
(Intercept) -51.2939   12.2522  -4.187  0.00305 **  
age          2.0947    0.5264   3.980  0.00406 **  
aerobic      13.1071   15.7620   0.832  0.42978  
age:aerobic  -0.3182    0.6498  -0.490  0.63746  
---  
Signif. codes:  0 '****' 0.001 '***' 0.01 '**' 0.05 '*' 0.1 '.' 1  
  
Residual standard error: 2.923 on 8 degrees of freedom  
Multiple R-squared: 0.9049, Adjusted R-squared: 0.8692  
F-statistic: 25.36 on 3 and 8 DF, p-value: 0.0001938
```

Summary Comparison:

Effect	OLS	JAGS
a _{run}	-51.29	-50.51
a _{aer}	-38.19	-37.57
b _{run}	2.09	2.06
b _{aer}	1.78	1.75
s	2.92	2.80

```
Inference for Bugs model at "Oxygen.model.2.jags", fit using jags,  
3 chains, each with 10000 iterations (first 1000 discarded), n.thin = 9  
n.sims = 3000 iterations saved  
          mu.vect sd.vect 2.5%   25%   50%   75% 97.5% Rhat n.eff  
a[1]    -50.514 12.802 -75.516 -58.575 -50.715 -42.629 -24.519 1.001 2600  
a[2]    -37.566 10.913 -59.713 -44.066 -37.473 -30.650 -16.305 1.001 3000  
b[1]     2.061  0.550  0.921  1.720  2.065  2.412  3.145 1.002 1900  
b[2]     1.753  0.420  0.926  1.500  1.749  2.003  2.608 1.001 3000  
rho     0.132  0.061  0.039  0.086  0.124  0.167  0.265 1.004  690  
deviance 60.746  3.870 55.888  57.952  59.926  62.629  70.124 1.001 3000
```



Example: Educational Testing (Gelman, et al. Section 5.5)

- Educational Testing Service conducted a study of SAT coaching
 - Eight schools with different coaching programs
 - All students had already taken PSAT
 - Some students were coached; some were not
 - Regression estimates were calculated for each school
 - Estimated coaching effects assumed independent for the eight schools
 - Sampling distributions are approximately normal
 - Sampling variances assumed known (over 30 students in each school provides a large sample for estimating variance if equal variances are assumed)
- Why not just compute 8 separate coaching effects estimates?
 - Confidence intervals for the coaching effects have a great deal of overlap
 - It is difficult to distinguish statistically between the experiments
- Why not just assume all coaching effects are equal and compute a pooled estimate?
 - Do we really believe the schools are all the same?
 - The largest school has estimated effect 28 under the separate model and 7 under the pooled model



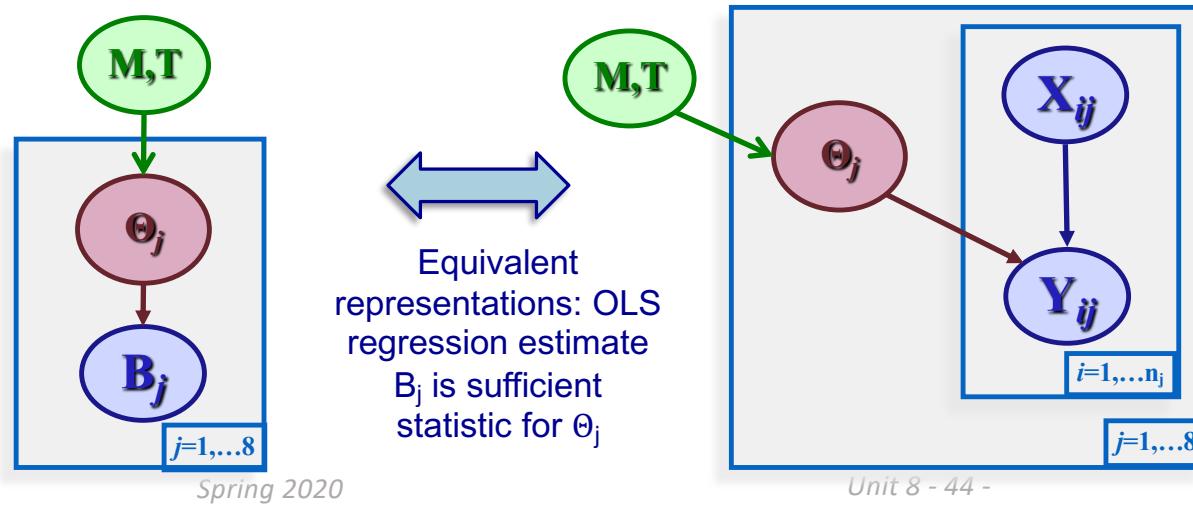
Bayesian Hierarchical Model

- Estimated coaching effect B_j for school j was estimated by linear regression
 - B_j is the least squares regression coefficient
 - Hierarchical model treats B_j as drawn from a common distribution
- Compromises between the extremes of separate estimates and pooling
 - When schools are similar it behaves more like pooling
 - When schools are very different it behaves more like separate estimates
 - When schools are different but not too different it gives results between pooling and separate estimates



The Coaching Effects Model

- B_j has normal distribution with unknown mean Θ_j and known standard deviation σ_j
 - Θ_j is a school-specific mean difference in scores between coached and uncoached students in school j
 - $\sigma_j^2 = \sigma^2/n_j$ is the variance (assumed known) of B_j , equal to the common variance divided by the sample size for the j th school
- Prior distributions for expected coaching effect Θ_j are iid normal with mean M and standard deviation T
- Prior distribution for M and T are uniform



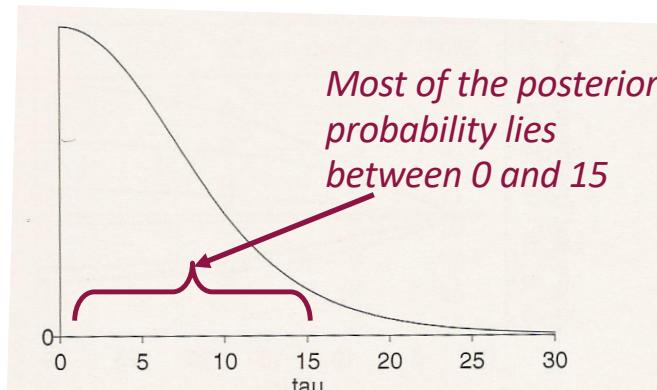
Comments on the Model

- This is the same kind of model as the one we used for the rat experiment
 - Data are from an exponential family
 - Parameter for each experiment has a conjugate prior distribution with experiment-specific parameter
 - Experiment-specific parameters are treated as drawn from a population
 - Distribution for experiment-specific parameter has no conjugate hyperprior
 - We have vague prior information about the parameters of the experiment-specific distribution
 - We can use numerical integration and simulation to estimate the posterior distribution for the hyperparameter
 - We have an analytic solution to the conditional posterior distribution for experiment-specific parameters given hyperparameter
 - We combine numerical integration and simulation to obtain an estimate of the joint posterior distribution
- Comments
 - This kind of model combines tractability and simplicity of conjugate updating with flexibility and power of hierarchical models



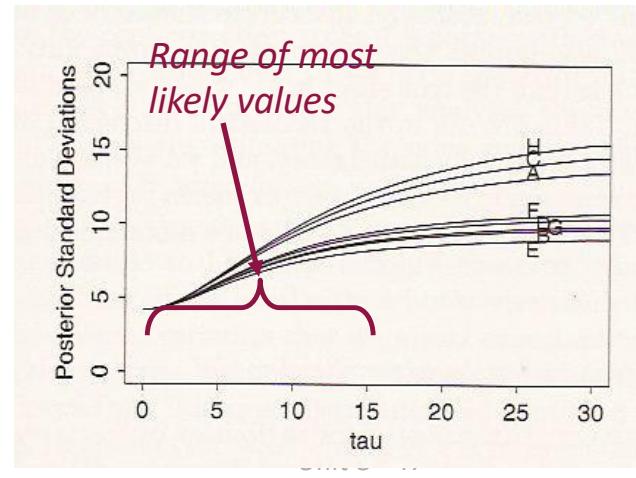
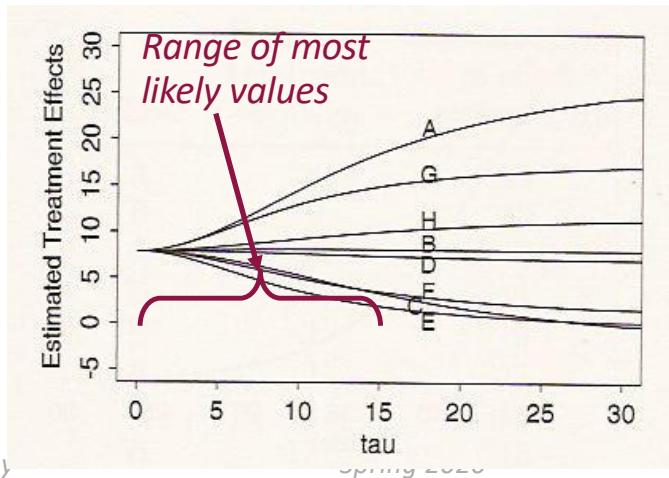
Inferences about T

- The parameter T measures amount of school-to-school variability in coaching effect
- The mode of the posterior distribution is zero (no variation)
- There is considerable uncertainty in the posterior distribution
- It would be a mistake to use maximum *a posteriori* point estimate $T = 0$
- It would also be a mistake to analyze each school completely independently of the others



Posterior Distribution for School Coaching Effect Means

- Distribution of school-level coaching effects depend on coaching effect standard deviation T
 - Difference in school-level mean coaching effects increases with T
 - Standard deviations of school-level coaching effects increase and become more different from each other as T increases
- Conclusions
 - Coaching effects vary from almost none to as high as 15 points and probably differ among schools
 - Raw estimate of 28 points for School A is probably too high
 - Posterior median for School A is 10; 97.5% point is 23



Generalizing Linear Regression with Normal Errors

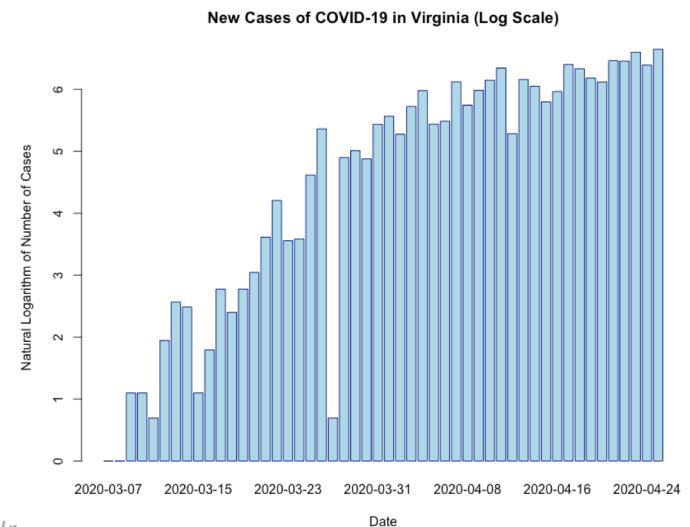
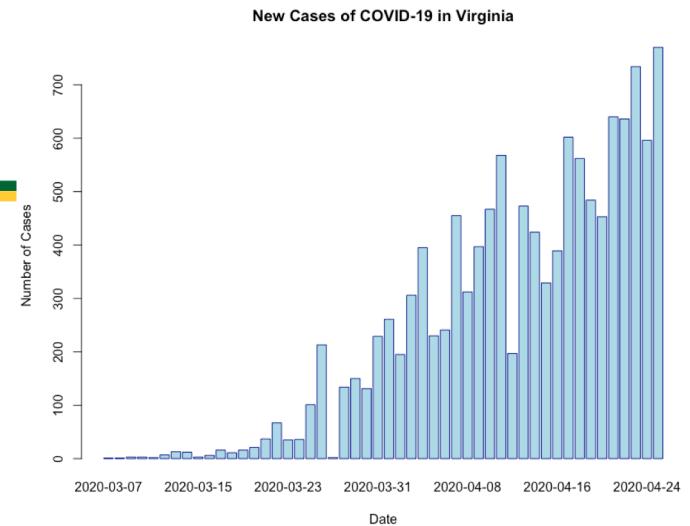
- Ordinary linear regression assumes:
 - Expected value of response variable is a linear function of predictor variables
 - Errors are normally distributed
- We can model many kinds of nonlinear relationship by transforming dependent and/or independent variables
- Generalized linear model (GLM) extends ordinary linear regression model to non-normal errors
 - Linear predictor is transformed by *link function*: $E[Y|X = g^{-1}(X_1\beta_1 + \dots + X_p\beta_p)]$
 - Probability distribution of Y given X is a member of an exponential family of distributions
- Examples:
 - Logistic regression for 0/1 dependent variable
 - Poisson regression for count data
- R fits generalized linear model using `glm` function



Example: New COVID-19 Cases in Virginia

- Data: new cases from date of first case until 4/25/2020
- Question of interest: what is rate of growth?
 - If growth is exponential with constant rate, logarithm of new cases should follow approximately a straight line
 - Is rate of growth decreasing?
 - Will number of new cases start to decline? When?
- With constant growth rate and logarithm link function:
 - $E[y|d] = \exp(\alpha + \beta d)$, where d is the number of days since 3/07/20 (date of first case)
- We will consider a model with a quadratic term:
 - $E[y|d] = \exp(\beta_0 + \beta_1 d + \beta_2 d^2)$
 - If $\beta_2 < 0$ then growth rate is slowing

Data from New York Times COVID-19 tracking database available from
<https://www.nytimes.com/interactive/2020/us/coronavirus-us-cases.html>



Poisson Regression using `glm` in R

`glm` with Poisson family; default link function for Poisson family is logarithm

Call:
`glm(formula = new ~ day + I(day^2), family = "poisson", data = va.cases)`

Deviance Residuals:
Min 1Q Median 3Q Max
-13.6739 -3.2095 -0.8165 1.9502 10.9287

Coefficients:
Estimate Std. Error z value Pr(>|z|)
(Intercept) 8.608e-01 9.826e-02 8.76 <2e-16 ***
day 2.299e-01 5.870e-03 39.17 <2e-16 ***
I(day^2) -2.382e-03 8.486e-05 -28.07 <2e-16 ***

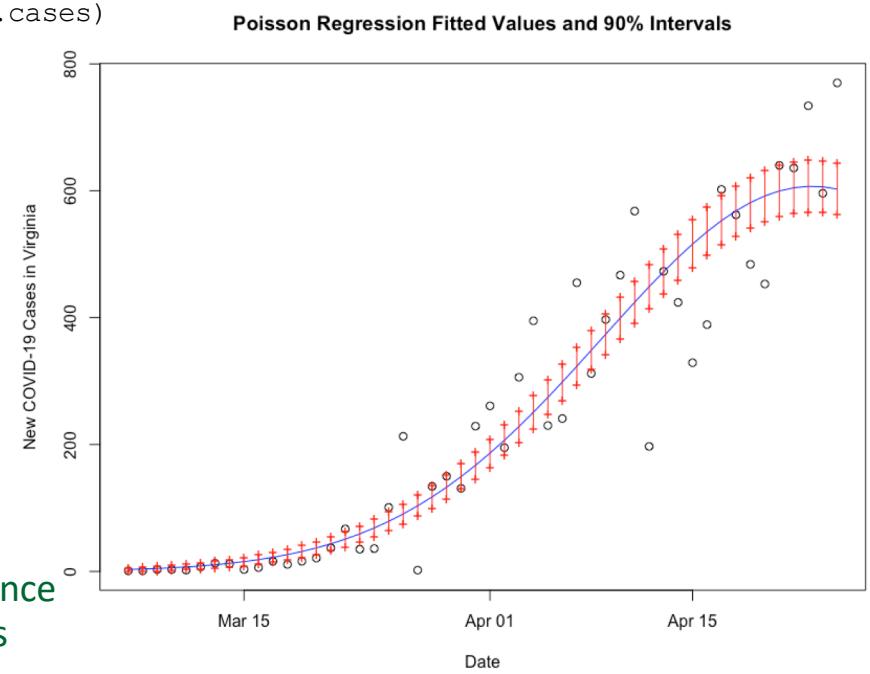
Signif. codes: 0 '****' 0.001 '***' 0.01 '**' 0.05 '*' 0.1 '.' 1

(Dispersion parameter for poisson family taken to be 1)

Null deviance: 12446.3 on 49 degrees of freedom
Residual deviance: 1135.2 on 47 degrees of freedom
AIC: 1454.1

Number of Fisher Scoring iterations: 5

Observations have too high a variance for Poisson distribution (variance is much larger than mean; most data points far outside 90% intervals)



Bayesian Model for New Case Counts

- Observations $y_d, d = 1, \dots, D$ have Poisson distribution with uncertain rate λ_d
- λ_d has gamma distribution with:
 - Shape $\alpha_d = f e^{\beta_0 + \beta_1 d + \beta_1 d^2}$
 - Scale $\beta = \frac{1}{f}$
 - Mean $m_d = \alpha_d \beta = \exp(\beta_0 + \beta_1 d + \beta_1 d^2)$
- Comments:
 - Logarithm of mean of λ_d is a quadratic function of d
 - Factor $0 < f \leq \infty$ controls amount of overdispersion: $f \rightarrow \infty$ corresponds to Poisson regression; f near zero means greater uncertainty about λ_d
- Integrating over λ_d , the marginal distribution of y_d is negative binomial with size α_d and probability $\frac{f}{f+1}$
 - Expected value is $E[y|d] = m_d = \exp(\beta_0 + \beta_1 d + \beta_1 d^2)$
 - Variance is greater than Poisson variance (equal to mean λ_d)



Bayesian Model for New Case Counts: JAGS in R

- Negative binomial observations with size $\alpha_d = f e^{\beta_0 + \beta_1 d + \beta_2 d^2}$ and probability $\frac{f}{f+1}$
 - $E[y|d] = m_d = \exp(\beta_0 + \beta_1 d + \beta_2 d^2)$
 - Parameter f (sfactor in code) controls overdispersion (small values mean large dispersion)

```
#Gibbs sample using JAGS
```

```
nc = va.cases$new
nd=dim(va.cases)[1]

covid.data <- list("nd","nc") # data to pass to JAGS

covid.params <- c("b0","b1","b2","sfactor") # parameters to monitor

covid.inits <- function(){
  list("b0"=0.9, "b1"=0.23, "b2"=-0.002, "sfactor"=0.1)
}
```

```
# The jags function takes data and starting values as input. It automatically writes
# a jags script, calls the model, and saves the simulations for easy access in R.
covid.fit <- jags(data=covid.data, inits=covid.inits, covid.params, n.chains=3,
  n.iter=5500, n.burnin=500,model.file="pandemic.va.negbin.jags",
  n.thin=1)
```

```
model {
  pr <- sfactor/(sfactor+1)
  for(d in 1:nd) {
    ll[d] <- b0 + b1*d + b2*d^2
    elambda[d] <- exp(ll[d])
    sz[d] <- elambda[d]*sfactor
    nc[d] ~ dnegbin(pr,sz[d])
  }
  sfactor ~ dbeta(1,1)T(0.01,0.2) # controls overdispersion
  b0 ~ dnorm(0, .1)T(-0.3,4) # constant term
  b1 ~ dnorm(0, .1)T(0.1,0.34) # rate of increase at day 1
  b2 ~ dnorm(0, .1)T(-0.0045,0.0) # rate of change of increase
}
```

Results of Model Fit: We have a Problem!

Inference for Bugs model at "pandemic.va.negbin.jags", fit using jags,
3 chains, each with 5500 iterations (first 500 discarded)

n.sims = 15000 iterations saved

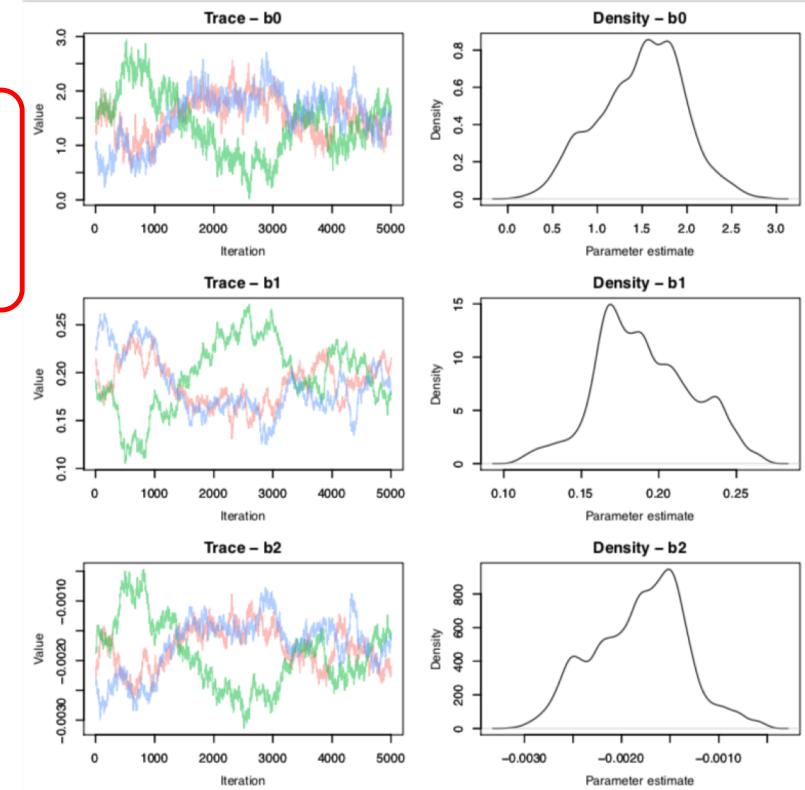
	mu.vect	sd.vect	2.5%	25%	50%	75%	97.5%	Rhat	n.eff
b0	1.483	0.478	0.532	1.158	1.530	1.823	2.377	1.090	62
b1	0.190	0.031	0.129	0.168	0.187	0.212	0.250	1.056	140
b2	-0.002	0.000	-0.003	-0.002	-0.002	-0.001	-0.001	1.058	98
sfactor	0.036	0.008	0.021	0.030	0.035	0.041	0.054	1.002	2300
deviance	529.109	3.187	525.053	526.703	528.413	530.769	537.033	1.054	59

For each parameter, n.eff is a crude measure of effective sample size,
and Rhat is the potential scale reduction factor (at convergence, Rhat=1).

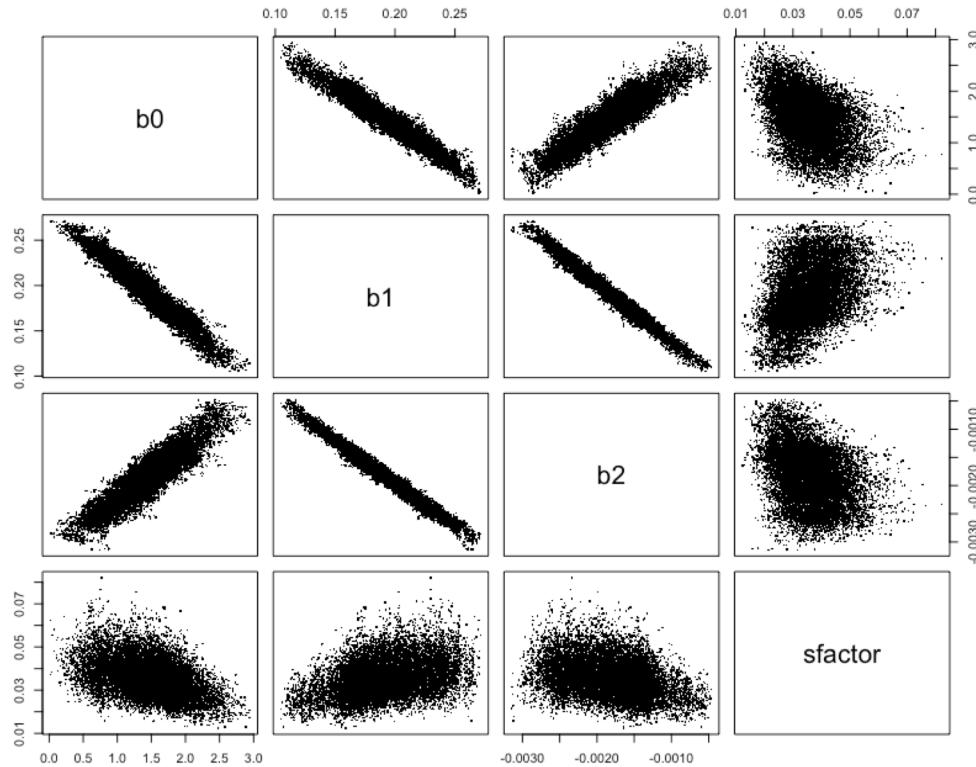
DIC info (using the rule, pD = var(deviance)/2)

pD = 4.9 and DIC = 534.0

DIC is an estimate of expected predictive error (lower deviance is better).



The Problem: Correlated Parameters



- MCMC samplers have high autocorrelation when parameters are highly correlated (“mixing is poor”)
- We can transform our variables to a less correlated representation



Transformed Observations

```
x = va.cases[c("day", "d2")]      # day and day^2
x=cbind(rep(1,nd),x)             # vector of 1's
names(x)=c("const","day","d2")
x=as.matrix(x)
t(x) %% x                        # design matrix for regression
t(x) %% x                         # X-transpose times X
bcov = solve(t(x) %% x)           # Least squares covariance is proportional to this
v=eigen(bcov)$vectors            # eigenvectors of inverse(X-transpose times X)
a=c(b0,b1,b2)%%v                 # convert parameters to less correlated representation
nc = va.cases$new
nd=dim(va.cases)[1]
covid.data <- list("nd","nc","v") # data to pass to JAGS
covid.params <- c("a","sfactor") # parameters to monitor
covid.inits <- function(){
  list("a"=c(0.4, -0.3, 0.004), "sfactor"=0.1)
}
covid.fit.a <- jags(data=covid.data, inits=covid.inits, covid.params, n.chains=3,
                     n.iter=5500, n.burnin=500,model.file="pandemic.va.negbina.jags",
                     n.thin=1)
```

```
model {
  b<- a %*% t(v)
  pr <- sfactor/(sfactor+1)    # Probability for negative binomial
  for(d in 1:nd) {
    ll[d] <- b[1] + b[2]*d + b[3]*d^2
    elambda[d] <- exp(ll[d])      # mean number of cases
    sz[d] <- elambda[d]*sfactor   # Overdispersion parameter
    nc[d] ~ dnegbin(pr,sz[d])     # Negative binomial
  }
  sfactor ~ dbeta(1,1)T(0.01,0.2)  # controls overdispersion
  a[1] ~ dnorm(1, 1)T(-0.83,3.13) # 1st transformed coefficient
  a[2] ~ dnorm(0, .1)T(-0.45,-0.2) # 2nd transformed coefficient
  a[3] ~ dnorm(0, .1)T(0.001,0.006) # 3rd transformed coefficient
}
```



Transformation Improves Mixing

Inference for Bugs model at "pandemic.va.negbina.jags", fit using jags,
3 chains, each with 5500 iterations (first 500 discarded)

n.sims = 15000 iterations saved

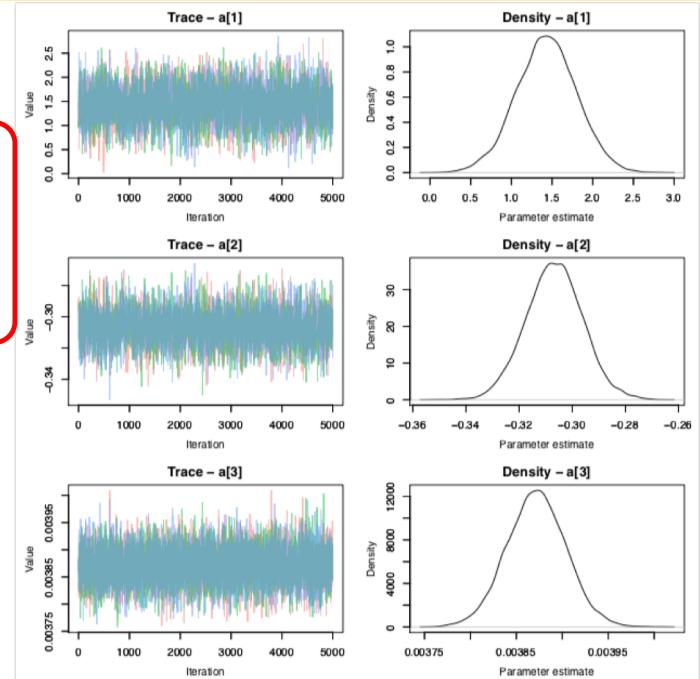
	mu.vect	sd.vect	2.5%	25%	50%	75%	97.5%	Rhat	n.eff
a[1]	1.432	0.370	0.688	1.185	1.432	1.678	2.156	1.001	15000
a[2]	-0.307	0.011	-0.328	-0.314	-0.307	-0.300	-0.286	1.001	11000
a[3]	0.004	0.000	0.004	0.004	0.004	0.004	0.004	1.001	15000
sfactor	0.036	0.008	0.022	0.030	0.035	0.041	0.053	1.002	2900
deviance	528.378	2.749	524.972	526.381	527.740	529.673	535.539	1.001	5600

For each parameter, n.eff is a crude measure of effective sample size,
and Rhat is the potential scale reduction factor (at convergence, Rhat=1).

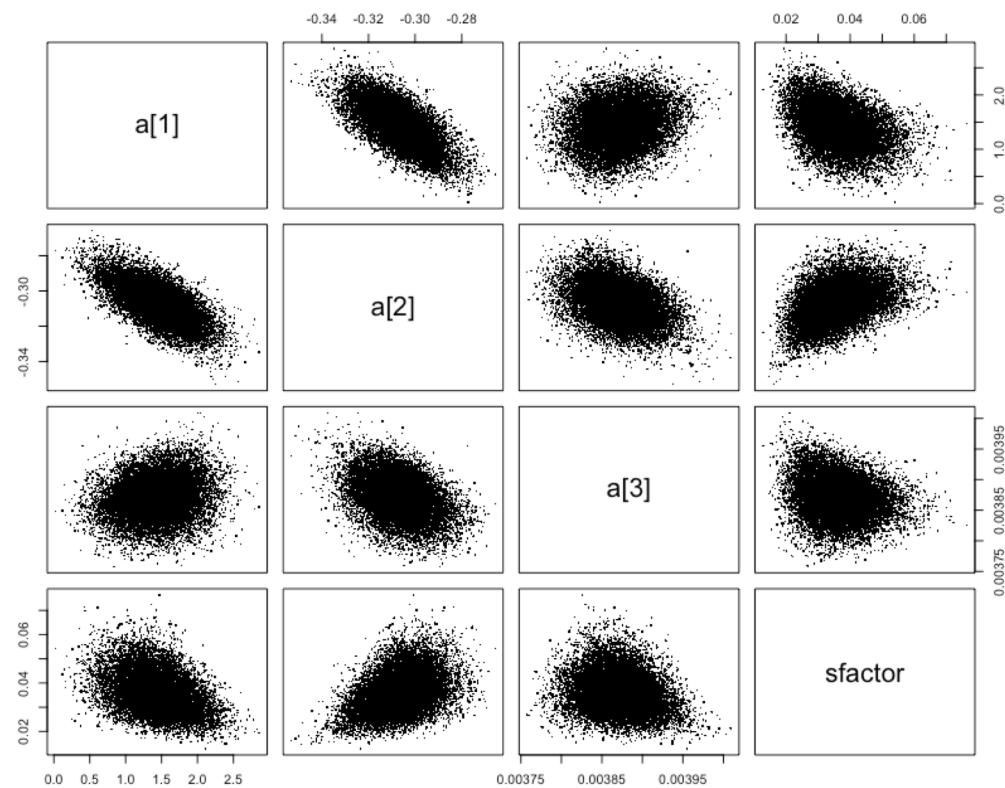
DIC info (using the rule, pD = var(deviance) / 2)

pD = 3.8 and DIC = 532.2

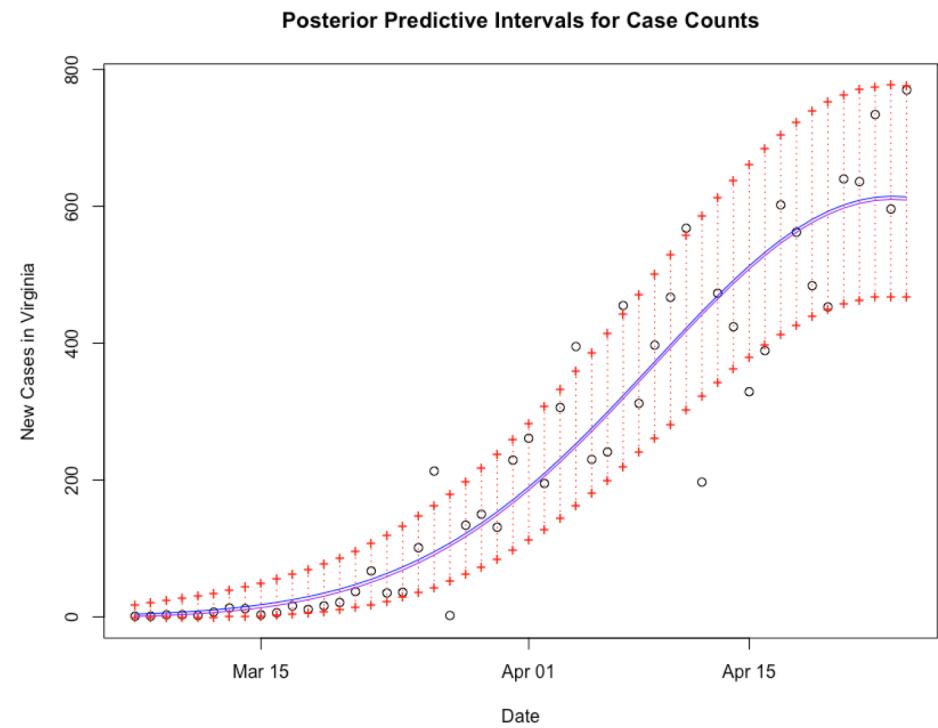
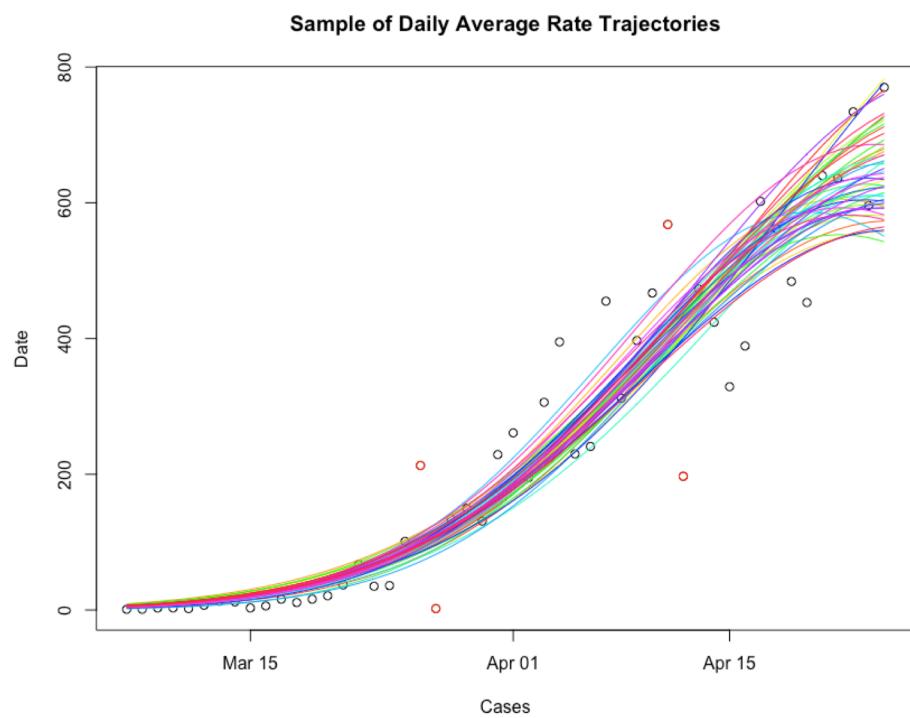
DIC is an estimate of expected predictive error (lower deviance is better).



Transformation Reduces Correlation



Some Results



- Expected peak in new cases is 4/28/20
- Estimated 72.2% probability peak is after 4/25/20 (last data point in sample)
- Estimated 89.3% probability peak is 05/04 or earlier

Missing Data

- Missing observations are common in real-world data sets
- Common approaches to missing data
 - Remove all cases with missing data
 - Reduces power
 - Can bias parameter estimates (depending on why observations are missing)
 - Impute missing values
 - Many statistical packages have methods for imputing (filling in) missing values
 - Simple ad hoc imputation approaches (e.g., use mean of non-missing values) can bias analysis, especially of relationships among variables
 - Regression imputation (use regression model to predict missing observations) gives unbiased estimates of mean but underestimates variability
 - Multiple imputation (use observed data to learn about relationships and randomly generate multiple completions) gives unbiased estimates of both means and variances
 - Model-based methods
 - EM algorithm (iterative approach to find maximum likelihood or maximum a posteriori estimates in presence of missing data)
 - Fully Bayesian model of observed and unobserved data
- It is important to account for mechanism for missing observations



Missing Data: Some Terminology

- Missing Completely at Random (MCAR)
 - The events leading to an observation being missing are not related to any of the random variables being observed or any of the parameters of interest
 - Example – Respondent randomly fails to fill in some items in a questionnaire in a manner unrelated to any of the data being collected or any of the variables being studied
- Missing at Random (MAR)
 - Also known as ignorable non-response
 - Missingness is not random but can be accounted for by the observed data
 - Example – Female respondents are less likely to answer question on job satisfaction, but responses do not depend on job satisfaction after accounting for gender
 - Accounting for gender difference in estimates of job satisfaction can lead to sound statistical estimates
- Missing Not at Random (MNAR)
 - Also known as non-ignorable non-response
 - Whether or not value is missing is related to the reason it is missing in a way that cannot be accounted for by other measurements
 - Example – People who are not satisfied in their job are less likely to answer question on job satisfaction
 - Estimates will be biased unless we can measure something related to reason observations are missing



Missingness Mechanism

- Notation:
 - Full data set: $\underline{Y} = (\underline{Y}_O, \underline{Y}_M)$
 - Observed data: \underline{Y}_O
 - Missing data: \underline{Y}_M
 - Response indicator: \underline{R} (1 if observed, 0 if missing)
 - \underline{R} is a data structure parallel to \underline{Y} , with one entry for each entry in \underline{Y} indicating whether that entry is missing or not
- Missing completely at random (MCAR)
 - $P(r | \underline{Y}_O, \underline{Y}_M) = P(r)$
 - Probability of an observation being missing does not depend on either the observed or unobserved data
 - Deleting all cases with missing values yields unbiased estimates if data are MCAR
- Missing at random (MAR)
 - $P(r | \underline{Y}_O, \underline{Y}_M) = P(r | \underline{Y}_O)$
 - Units with same observed attributes have same distribution for missing attributes
 - EM algorithm and simple Bayesian models assume MAR
- Missing not at random (MNAR)
 - Missingness mechanism is not ignorable
 - Requires joint model of \underline{Y} and \underline{R}



Bayesian Models with Missing Data

- The Bayesian approach handles missing data naturally
 - Specify joint probability distribution $P(\underline{Y}, \underline{\Theta}, \underline{R})$ on full data, parameters, and missing data mechanism
 - Use Bayesian inference to find posterior distribution $P(\underline{\Theta}, \underline{Y}_M | \underline{Y}_O, \underline{R})$ of parameters $\underline{\Theta}$ and missing data YM given observed data YO and indicator R of which observations are missing
 - Calculate and/or approximate desired features of posterior distribution
- JAGS handles missing data
 - NA values are allowed in data variables
 - JAGS will sample values for the NA variables
 - Full probability model $P(\underline{Y}, \underline{\Theta}, \underline{R})$ must be specified
- Examples:
 - Predict future dependent variables for regression given independent variable values
 - Estimate parameters of a regression model when some values of independent variables are missing



JAGS and Missing Data

- Predict future dependent variables for regression given independent variable values
 - Specify distribution for observed \underline{Y} given observed \underline{X} and β
 - Specify prior distribution for β
 - Specify independent variables X_{new} for cases to be predicted
 - Set $Y_{new} = \text{NA}$ for cases to be predicted
 - JAGS will simultaneously sample parameters and future observations from $P(\beta, Y_{new} | \underline{Y}, \underline{X}, X_{new})$
- Estimate parameters of a regression model when some values of independent variables are missing (assume MAR)
 - Specify distribution for observed \underline{Y} given full data \underline{X} and β
 - Specify prior distribution for β
 - Specify prior distribution for \underline{X}
 - Set $\underline{Y}_M = \text{NA}$ for missing observations
 - JAGS will simultaneously sample parameters and missing data from $P(\beta, \underline{Y}_M | \underline{Y}, \underline{X}_O)$ under assumption that observations are missing at random



Example: Oxygen Uptake with Missing Data

JAGS Model:

```
model{
  for (i in 1:n) {
    age[i] ~ dunif(18,35)
    y[i] ~ dnorm((a[2]+b[2]*age[i])*aerobic[i]+(a[1]+b[1]*age[i])*(1-aerobic[i]),rho)
  }
  for (i in 1:2) {
    a[i] ~ dnorm(0,0.0001)
    b[i] ~ dnorm(0,0.0001)
  }
  rho ~ dgamma(0.5,1/1000)
  age13 <- age[13]
}
```

Missing Data:

```
# Add new observation with missing age
age[13]=NA      # Add observation with missing age
aerobic[13]=1   # Person with unknown age was on aerobic program
y[13]=5.5       # Oxygen update for person with unknown age
```

JAGS Output:

```
Inference for Bugs model at "Oxygen.model.missing.jags", fit using jags,
3 chains, each with 10000 iterations (first 1000 discarded), n.thin = 9
n.sims = 3000 iterations saved
          mu.vect sd.vect 2.5%   25%   50%   75% 97.5% Rhat n.eff
a[1]     -49.633 13.566 -76.502 -58.389 -49.579 -40.767 -22.209 1.016 170
a[2]     -35.408 11.652 -59.582 -42.680 -35.159 -27.705 -13.500 1.009 270
age13    24.632  2.231  20.063  23.318  24.646  25.907  29.190 1.001 2600
b[1]      2.024  0.584  0.864  1.642  2.025  2.394  3.189 1.016 170
b[2]      1.668  0.449  0.829  1.372  1.655  1.952  2.598 1.007 320
rho       0.128  0.063  0.038  0.082  0.116  0.162  0.283 1.001 3000
deviance 134.241 4.227 128.039 131.112 133.627 136.572 144.145 1.001 3000
```



Collinearity and Identifiability

- Collinearity means $X^T X$ is singular
 - One of the explanatory variables is a linear combination of other explanatory variables
 - The classical regression estimate $\underline{b} = (X^T X)^{-1} X^T \underline{y}$ does not exist
 - The posterior distribution exists if the prior distribution is informative, but the data provide no information about some linear combinations of the β 's
 - Inferences depend on prior assumptions about these linear combinations
- When $X^T X$ is nearly singular we say the data are nearly collinear
 - Data are not very informative about some linear combinations of the β 's
 - Inferences about these linear combinations are sensitive to the prior distribution
 - Standard errors for some regression coefficients may be very large
 - Posterior distributions for some regression coefficients are highly correlated
- If some β 's cannot be estimated uniquely we say they are not identifiable
- Regression modules in many statistical packages contain diagnostics for collinearity



Indicator Variables

- Sometimes one of the regressors is a categorical variable
- To use a categorical variable with k categories as an explanatory variable, we transform it into $k-1$ indicator variables
- Example: Region variable has possible values North, South, East and West
 - Transform into 3 indicator variables X_1, X_2, X_3
 - North: $X_1 = 1, X_2 = 0, X_3 = 0$
 - South: $X_1 = 0, X_2 = 1, X_3 = 0$
 - East: $X_1 = 0, X_2 = 0, X_3 = 1$
 - West: $X_1 = 0, X_2 = 0, X_3 = 0$
(If we added a fourth categorical variable to have value 1 for West and 0 for other regions, and the regression had a constant term, the data would be collinear)
- Regression coefficient for X_i represents the effect of being in the corresponding region relative to being in the West



ANOVA

- ANalysis Of VAriance (ANOVA) refers to a method for summarizing the results of a normal regression model
- ANOVA partitions the sum of squares into components:
 - $S_{yy} = S_{reg} + S_{ee}$
 - $S_{yy} = \sum_i (y_i - \bar{y})^2$ is the total sum of squares
 - $S_{reg} = \sum_i (a + bx_i - \bar{y})^2$ is the regression sum of squares
 - $S_{ee} = \frac{1}{n} \sum_{i=1}^n (y_i - a - b(x_i - \bar{x}))^2$ is the error (or residual) sum of squares
- Interpretation:
 - S_{reg} measures the variation in the data explained by the regression model
 - S_{ee} measures the unexplained variation
- Most regression software includes ANOVA table
 - F test evaluates whether S_{reg} is larger than expected by chance if α and β are zero



More on ANOVA

- ANOVA is also used to refer to a type of regression model in which the predictor variables are categorical variables
- One Way ANOVA refers to a model in which the data fall into groups with a separate mean for each group and a common variance (e.g., the reaction time data)
- Multi factor ANOVA refers to a model in which
 - There are p factors, each with a finite number of levels
 - The predictor variables consist of the levels of each factor
 - The response variable is normally distributed with mean depending on the factor levels and common variance
- No interaction model: category mean is equal to sum of a coefficient for each factor
- Example: 2-way ANOVA, no interaction
 - Y_{ijk} denotes the k th observation having the i th level on the first factor and the j th level on the second factor
 - $E[Y_{ijk}] = \alpha_i + \beta_j$
 - The ordinary least squares estimates of α_i and β_j are the regression estimates obtained by forming categorical variables for the factors



ANOVA and Regression

- Example: evaluate effect of new teaching method
 - Categorize students by new/old teaching method T, ability level A (categorical), socioeconomic status S (categorical)
 - Measure performance score of i th student who uses method t and has ability level a and socioeconomic status s
- Model:
 - Observation X_{itas} is normally distributed with mean μ_{tas} and standard deviation σ
 - Mean score $\mu_{tas} = \mu + \mu_t + \mu_a + \mu_s$
- We can represent this as a linear regression model with “dummy variables” corresponding to the categories
- Because this kind of model is so common, standard statistical packages provide special-purpose methods for this kind of regression model
- Theory for ANOVA is a special case of linear model theory



Transformation of Variables

- We often need to transform variables to achieve a linear relationship
- Common transformations are power, logarithm, and logit ($\log(x/(1-x))$)
- We can transform the dependent variable and/or the independent variables
- We can include more than one transformation of the same independent variable (e.g., polynomial regression)
- A transformation changes the interpretation of the regression coefficient



Summary and Synthesis

- We studied regression models in which a dependent variable y was related to an independent variable x by a linear equation with normal errors
- We considered the posterior distribution for the coefficients under a noninformative prior distribution
 - Relation to classical estimators
 - Credible intervals for parameters
- We generalized to an informative Normal-Gamma conjugate prior distribution
- We studied the marginal likelihood
 - Used in predicting as-yet-unseen observations
 - Used in hypothesis testing
- We discussed how to generalize these results to multiple regression
- We considered Bayesian hierarchical regression models
- We looked at regression models for count data
- We explored regression models with missing data



References for Unit 8

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