Coin Flips: Random Generation from Bernoulli Distribution

When spun on edge 250 times, a Belgian one-euro coin came up heads 140 times and tails 110. It looks very suspicious to me," said Barry Blight, a statistics lecturer at the London School of Economics.

- Let's simulate the coin flip experiments.
- Let's try to find an estimate of the probability of head.

In [1]:
```
import numpy as np
import matplotlib.pyplot as plt
import seaborn as sns;
sns.set_style("darkgrid");

plt.rcParams['figure.dpi'] = 100
plt.rcParams.update({"grid.linewidth":0.5, "grid.alpha":0.5})
plt.style.use("dark_background")

import scipy.stats as stats
```

Simulation

In [2]:
```
p = 0.5
```

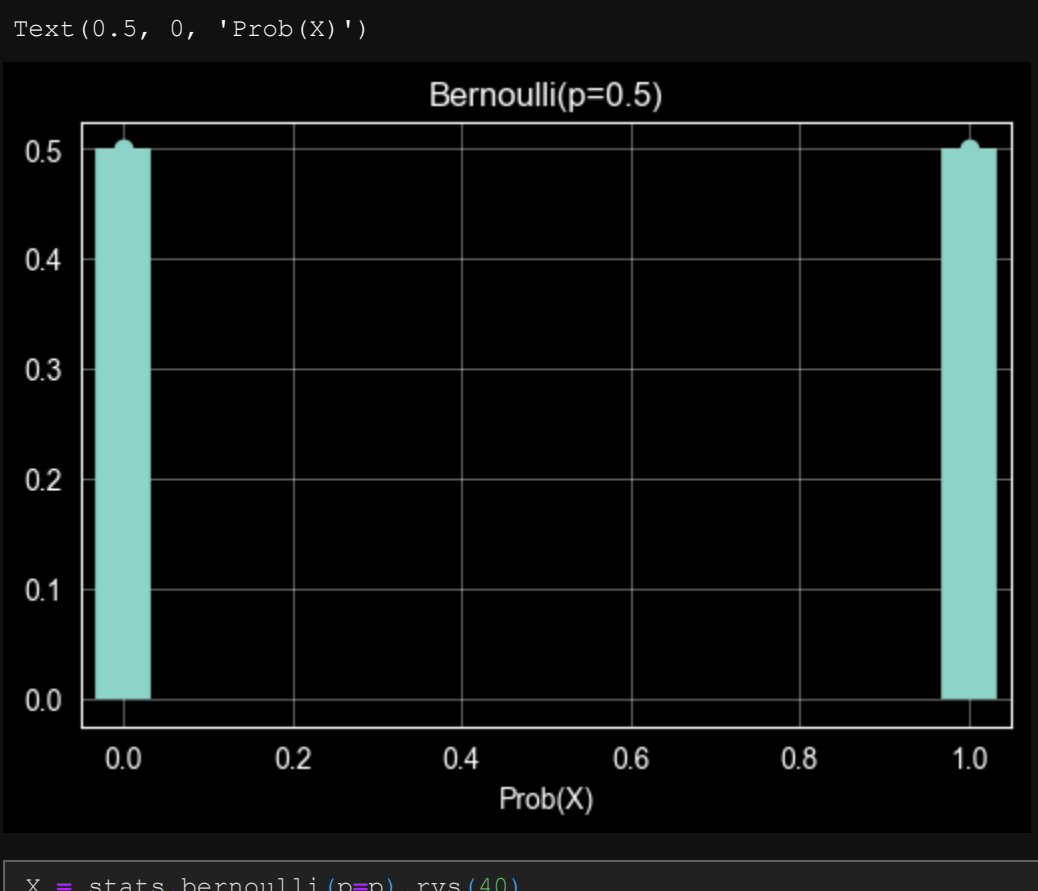
In [3]:
```
u = np.random.uniform(low=0.0, high=1.0, size=1)  # generate from uniform distributio
X = 1 if u <= p else 0

print(X, u)
```

```
1 [0.41599619]
```

In [4]:
```
x = np.array([0, 1])
P = stats.bernoulli(p=p).pmf(x)
print(x, P)
markers, stems, base = plt.stem(x, P);
stems.set_linewidth(20); base.set_linewidth(0)
plt.title(f'Bernoulli(p={p})'); plt.xlabel('Prob(X)');
```

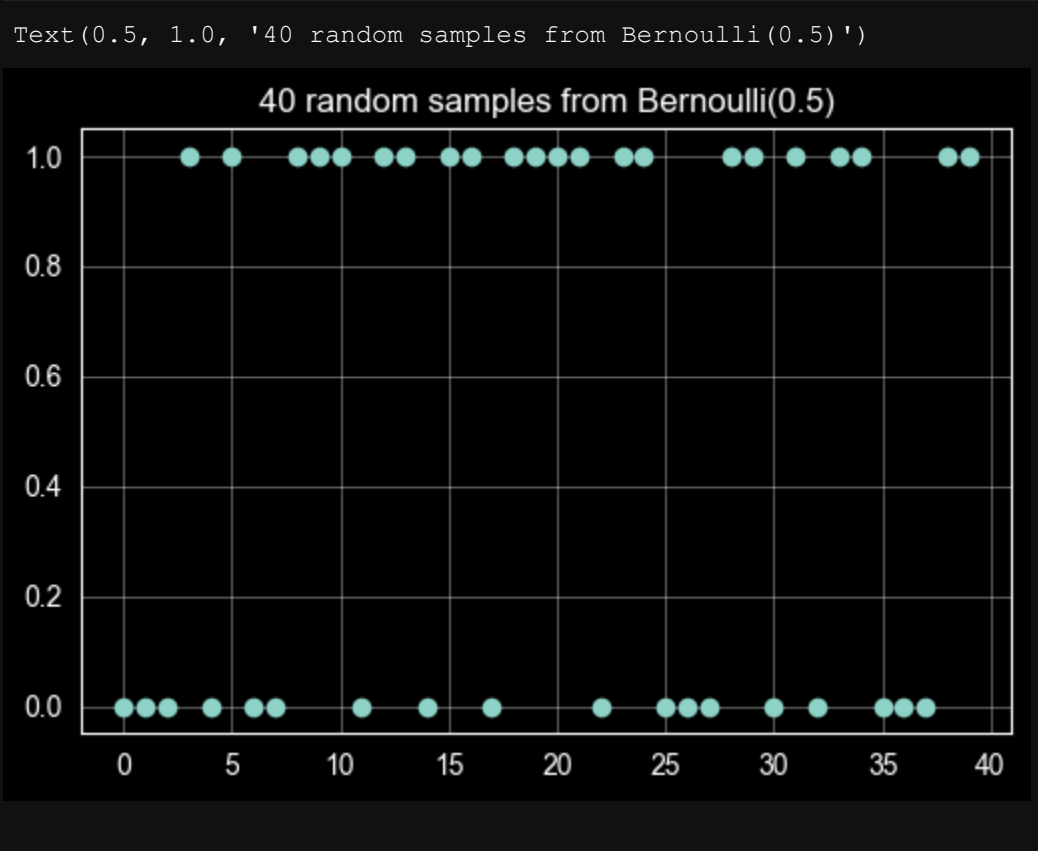
```
[0 1] [0.5 0.5]
```

Out[4]: `Text(0.5, 0, 'Prob(X)')`



In [5]:
```
X = stats.bernoulli(p=p).rvs(40)
plt.plot(X, 'o'); plt.title('40 random samples from Bernoulli(0.5)');
```

Out[5]: `Text(0.5, 1.0, '40 random samples from Bernoulli(0.5)')`



Mean & Std

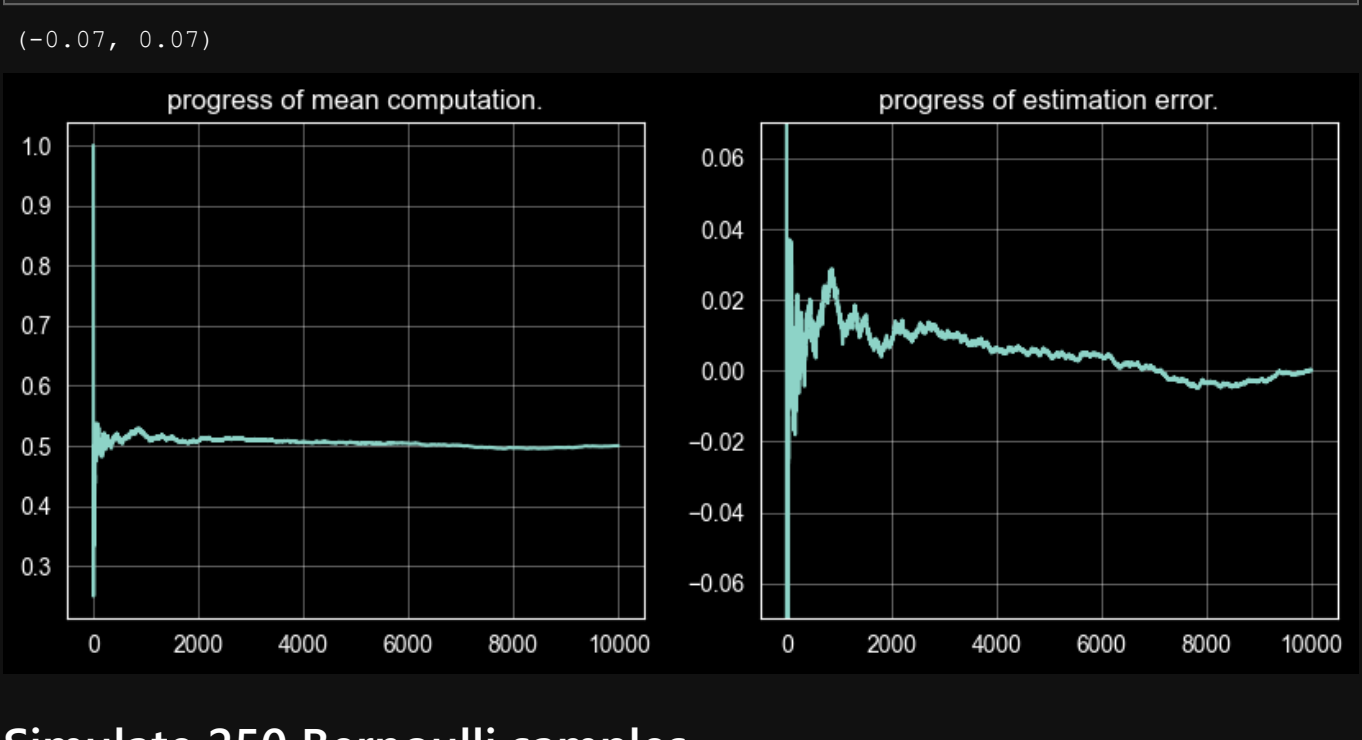
1. Theoretically, $$
   \begin{align}
       \mu &= \mathbb{E}[X] = \sum_{X\in\{0,1\}} X P(X) = p, \\
       \sigma &= \mathbb{E}[(X-\mu)^2] = pq
   \end{align}
   $$
2. Generate many samples and compute the average. It is called `sample mean`.
3. Std = sample standard deviation

In [6]:
```
n_samples = 10000
X = stats.bernoulli(p=0.5).rvs(n_samples)
mean = X.sum()/n_samples
std = np.sqrt( ((X - mean)*(X-mean)).sum() / n_samples )
print(f'mean={mean}  std={std}; or  {X.mean()} & {X.std()}')
```

```
mean=0.5001  std=0.49999998999999995; or  0.5001 & 0.49999998999999995
```

In [7]:
```
csum = X.cumsum();
p_est = np.array([ cs / (k+1) for k, cs in enumerate(csum)])
fig, axes = plt.subplots(1,2, figsize=(10,4))
axes[0].plot(range(n_samples), p_est); axes[0].set_title(f'progress of mean computati
axes[1].plot(range(n_samples), p_est - p); axes[1].set_title('progress of estimation e
axes[1].set_ylim(-0.07, 0.07);
```

Out[7]: `(-0.07, 0.07)`



Simulate 250 Bernoulli samples

https://docs.scipy.org/doc/scipy/reference/generated/scipy.stats.bernoulli.html

In [8]:
```
p = 0.5
ntotal = 250
nheads = 140
```

In [9]:
```
X = stats.bernoulli(p=p).rvs(ntotal); print(f'{X.shape}, Heads: {X.sum()}')
```

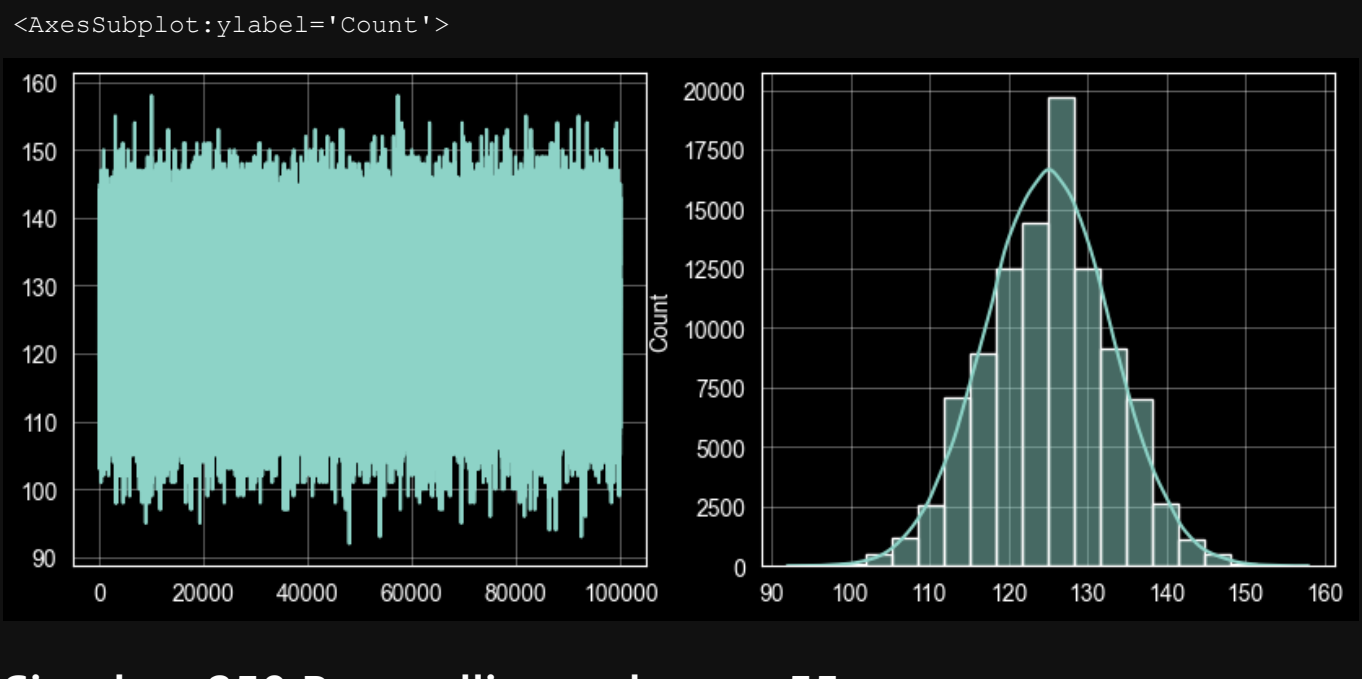
```
(250,), Heads: 132
```

In [10]:
```
repeat = 100000
# slow, do not use
# heads_simul = [stats.bernoulli(p=p).rvs(ntotal).sum() for i in range(repeat)]
```

In [11]:
```
X = stats.bernoulli(p=p).rvs((repeat, ntotal))
heads_simul = X.sum(axis=1)
over_140 = heads_simul >= nheads
print(f'The proportion that # of heads is over 140 is {over_140.sum() * 100 / repeat}
fig, axes = plt.subplots(1,2, figsize=(10,4))
axes[0].plot(heads_simul)
sns.histplot(heads_simul, kde=True, bins=20);
```

```
The proportion that # of heads is over 140 is 3.315%
```

Out[11]: `<AxesSubplot:ylabel='Count'>`



Simulate 250 Bernoulli samples, p=.55

https://docs.scipy.org/doc/scipy/reference/generated/scipy.stats.bernoulli.html

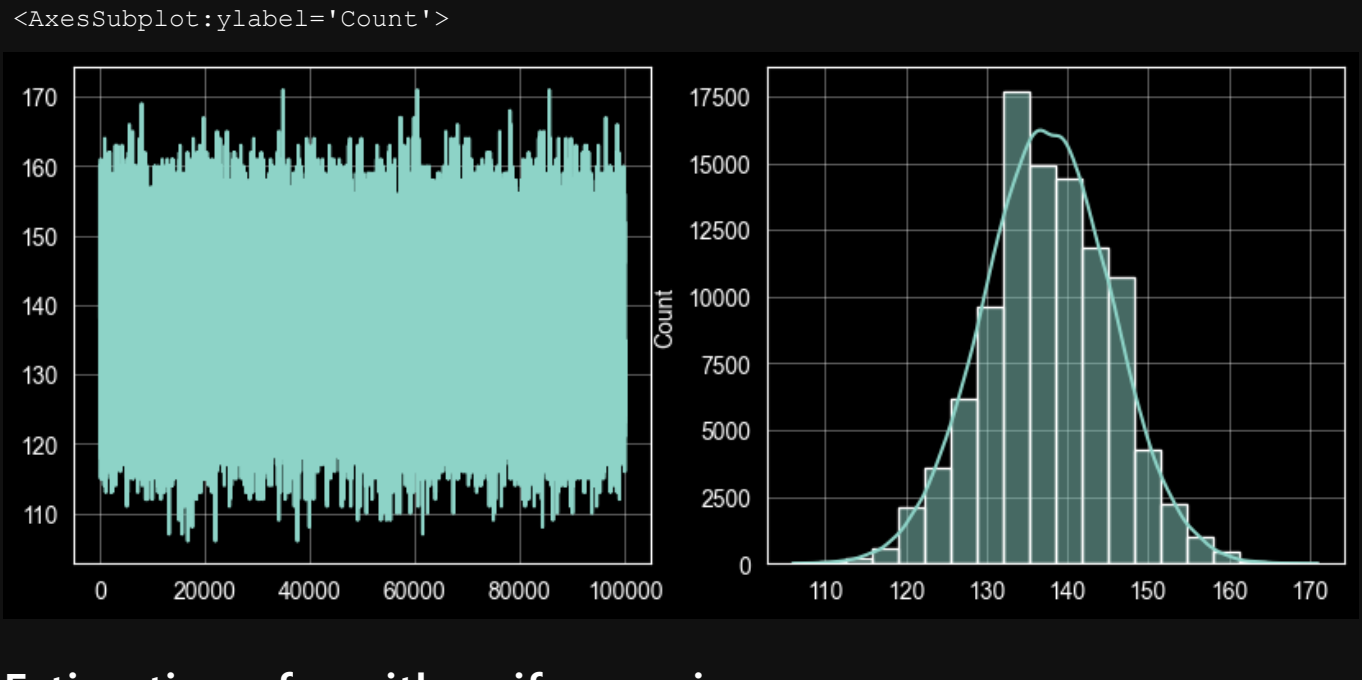
In [12]:
```
p = 0.55
```

In [13]:
```
repeat = 100000
```

In [14]:
```
X = stats.bernoulli(p=p).rvs((repeat, ntotal))
heads_simul = X.sum(axis=1)
over_140 = heads_simul >= nheads
print(f'The proportion that # of heads is over 140 is {over_140.sum() * 100 / repeat}
fig, axes = plt.subplots(1,2, figsize=(10,4))
axes[0].plot(heads_simul)
sns.histplot(heads_simul, kde=True, bins=20);
```

```
The proportion that # of heads is over 140 is 40.062%
```

Out[14]: `<AxesSubplot:ylabel='Count'>`



Estimation of p with uniform prior

In [15]:
```
probs = np.linspace(.49, .60, 12)
prior = np.ones_like(probs) / len(probs)
print(f'candidate p: {probs}\nprior probability: {prior}')
```

```
candidate p: [0.49 0.5  0.51 0.52 0.53 0.54 0.55 0.56 0.57 0.58 0.59 0.6 ]
prior probability: [0.08333333 0.08333333 0.08333333 0.08333333 0.08333333 0.08333333
 0.08333333 0.08333333 0.08333333 0.08333333 0.08333333 0.08333333]
```

In [16]:
```
# for each hypothesis p, we comput
likelihood = np.array([p**140 * (1-p)**110 for p in probs])
likelihood
```

Out[16]:
```
array([2.88516447e-76, 5.52714788e-76, 9.58061651e-76, 1.50315657e-75,
       2.13494218e-75, 2.74465562e-75, 3.19266274e-75, 3.35828999e-75,
       3.19161067e-75, 2.73741752e-75, 2.11595228e-75, 1.47154687e-75])
```

In [17]:
```
posterior_unnormalized = prior * likelihood
posterior = posterior_unnormalized / posterior_unnormalized.sum()
posterior
```

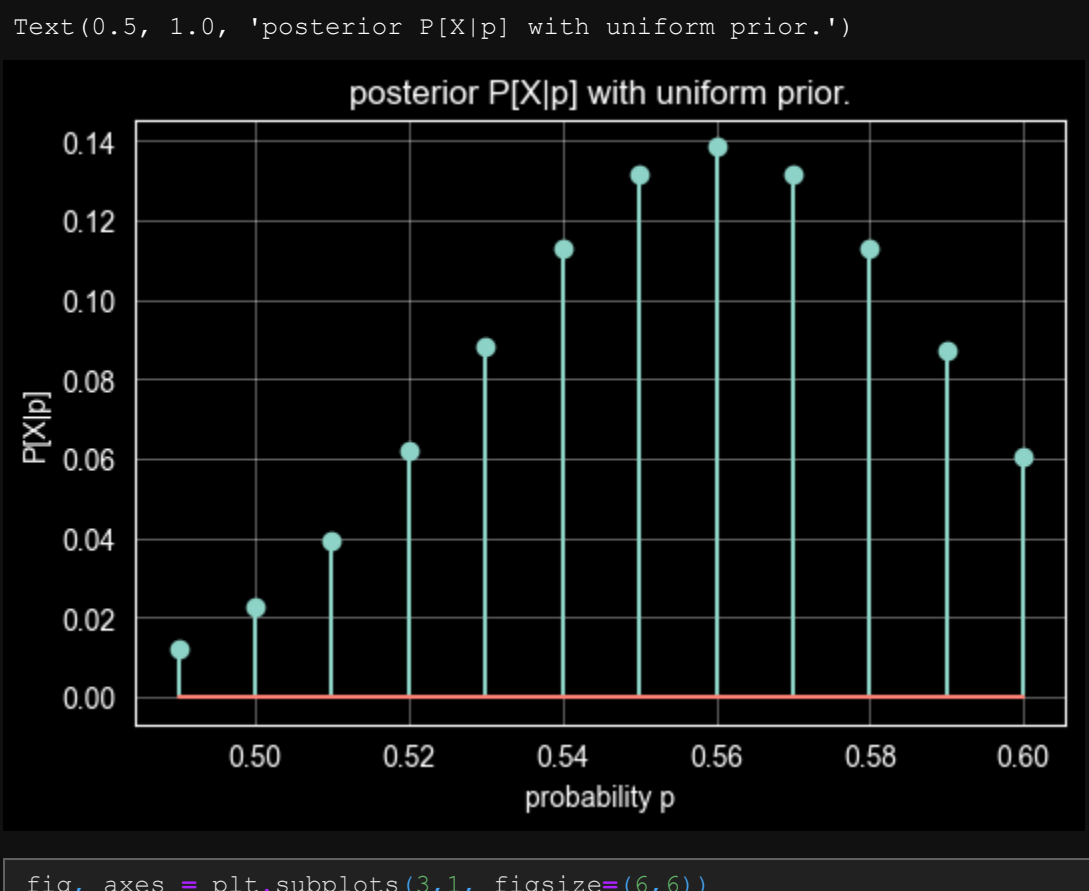
Out[17]:
```
array([0.01189782, 0.02279281, 0.03950847, 0.06198705, 0.08804057,
       0.11318388, 0.13165876, 0.13848888, 0.13161538, 0.1128854 ,
       0.08725746, 0.06068353])
```

In [18]:
```
istar = np.argmax(posterior)
pstar = probs[istar]
print(f'* best estimate is {pstar}')
```

```
* best estimate is 0.5599999999999999
```

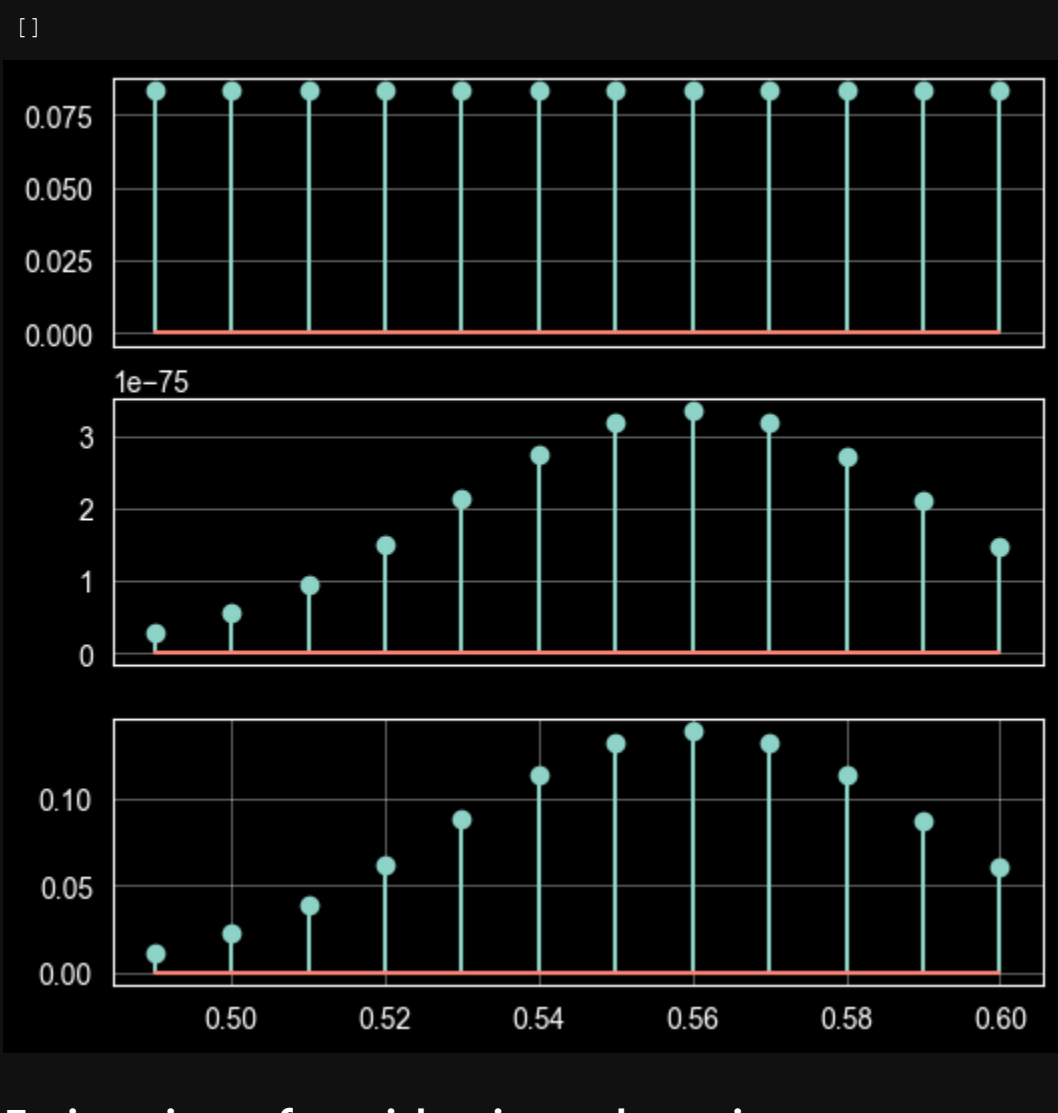
In [19]:
```
plt.stem(probs, posterior);
plt.xlabel('probability p')
plt.ylabel('P[X|p]')
plt.title('posterior P[X|p] with uniform prior.');
```

Out[19]: `Text(0.5, 1.0, 'posterior P[X|p] with uniform prior.')`



In [20]:
```
fig, axes = plt.subplots(3,1, figsize=(6,6))
for ax, pdf in zip(axes, [prior, likelihood, posterior]):
    ax.stem(probs, pdf)
axes[0].set_xticks([]); axes[1].set_xticks([]);
```

Out[20]: `[]`



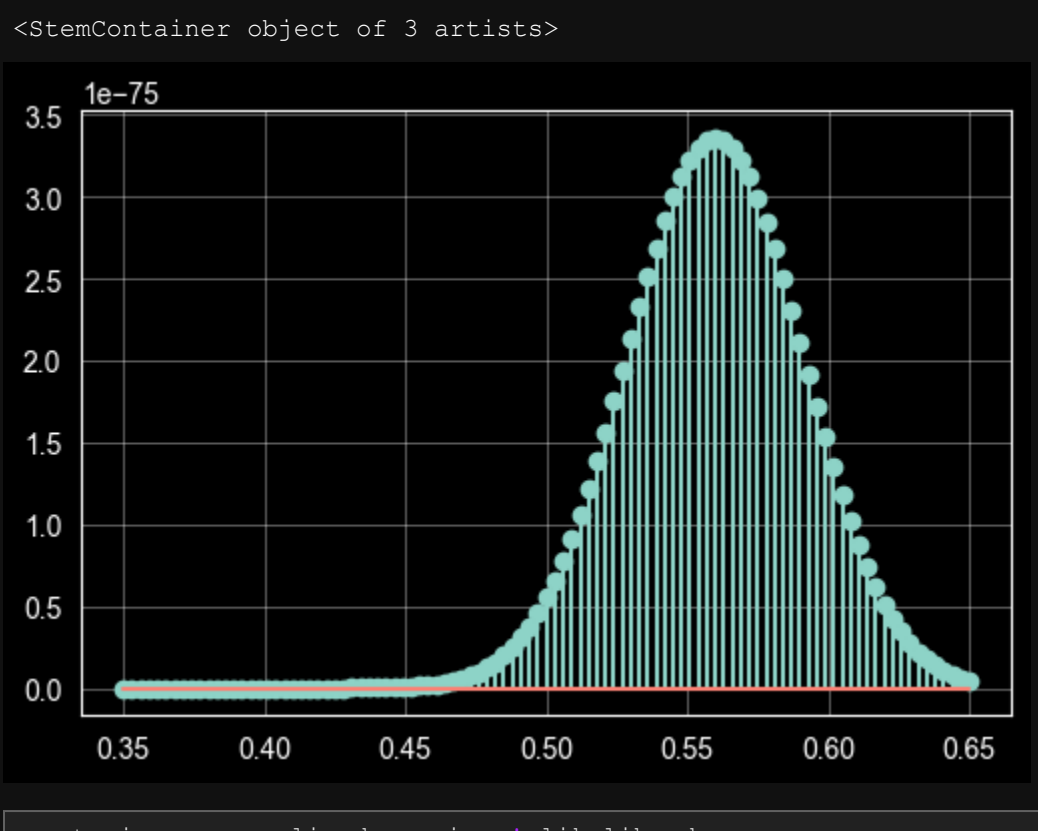
Estimation of p with triangular prior

In [21]:
```
probs = np.linspace(.35, .65, 101)
m = len(probs)//2
pm = probs[m]
prior = np.empty_like(probs)
prior[:m] = np.array([(1 - 0)/(probs[m] - probs[0])*(p - probs[0]) for p in probs[:m]]
prior[m:] = np.array([(0 - 1)/(probs[-1] - probs[m])*(p - probs[m]) + 1 for p in probs
prior = prior / prior.sum()
plt.stem(probs, prior);
```

Out[21]: `<StemContainer object of 3 artists>`

In [22]:
```
# for each hypothesis p, we comput
likelihood = np.array([p**140 * (1-p)**110 for p in probs])
plt.stem(probs, likelihood)
```

Out[22]: `<StemContainer object of 3 artists>`



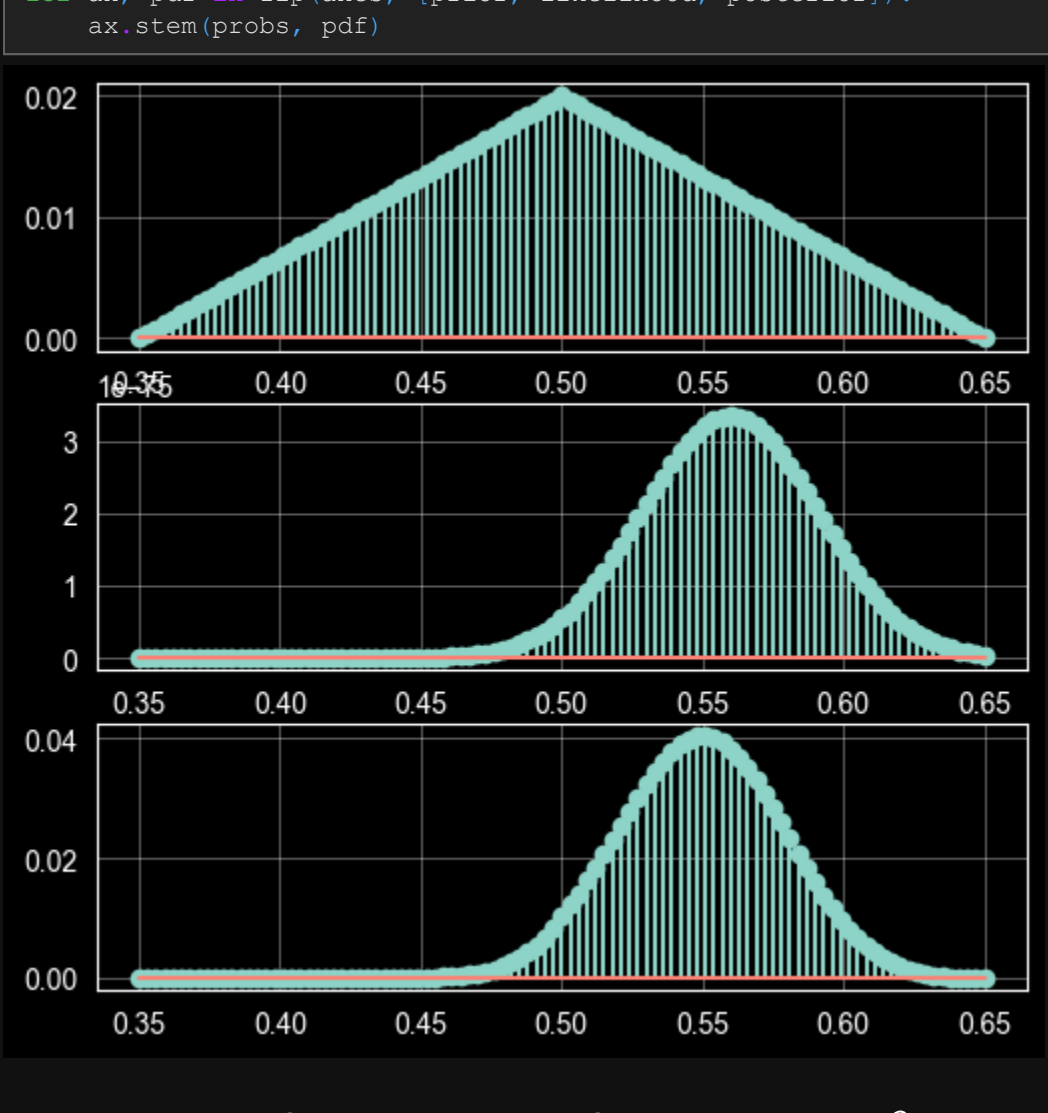
In [23]:
```
posterior_unnormalized = prior * likelihood
posterior = posterior_unnormalized / posterior_unnormalized.sum()
posterior
```

Out[23]:
```
array([0.00000000e+00, 2.93895754e-13, 1.15275975e-12, 3.34958487e-12,
       8.54669506e-12, 2.01996802e-11, 4.52885232e-11, 9.75618594e-11,
       2.03495417e-10, 4.13028818e-10, 8.18564781e-10, 1.58800830e-09,
       3.02126215e-09, 5.64526519e-09, 1.03713388e-08, 1.87516523e-08,
       3.33908746e-08, 5.85969801e-08, 1.01394845e-07, 1.73081578e-07,
       2.91578099e-07, 4.84932563e-07, 7.96463421e-07, 1.29219869e-06,
       2.07147781e-06, 3.28183555e-06, 5.13958273e-06, 7.95782847e-06,
       1.21840389e-05, 1.84495718e-05, 2.76339277e-05, 4.09466634e-05,
       6.00299528e-05, 8.70845752e-05, 1.25021558e-04, 1.77640714e-04,
       2.49835774e-04, 3.47823670e-04, 4.79392709e-04, 6.54160871e-04,
       8.83831431e-04, 1.18242858e-03, 1.56649107e-03, 2.05519758e-03,
       2.67039364e-03, 3.43648772e-03, 4.38018348e-03, 5.53001738e-03,
       6.91567575e-03, 8.56707448e-03, 1.05131963e-02, 1.22794929e-02,
       1.42082850e-02, 1.62860289e-02, 1.84925360e-02, 2.08007747e-02,
       2.31769845e-02, 2.55811390e-02, 2.79677772e-02, 3.02872018e-02,
       3.24870137e-02, 3.45139302e-02, 3.63158069e-02, 3.78437641e-02,
       3.90543020e-02, 3.99112821e-02, 4.03876532e-02, 4.04668103e-02,
       4.01434952e-02, 3.94241725e-02, 3.83268505e-02, 3.68803495e-02,
       3.51230586e-02, 3.31012551e-02, 3.08670876e-02, 2.84763456e-02,
       2.59861482e-02, 2.34526856e-02, 2.09291374e-02, 1.84638743e-02,
       1.60990240e-02, 1.38694542e-02, 1.18021931e-02, 9.91627878e-03,
       8.22300152e-03, 6.72648090e-03, 5.42450595e-03, 4.30955687e-03,
       3.36992630e-03, 2.59086313e-03, 1.95567121e-03, 1.44670922e-03,
       1.04625263e-03, 7.37194597e-04, 5.03577071e-04, 3.30956046e-04,
       2.06614813e-04, 1.19646064e-04, 6.09277144e-05, 2.30186882e-05,
       0.00000000e+00])
```

In [24]:
```
istar = np.argmax(posterior)
pstar = probs[istar]
print(f'* best estimate is {pstar}')
```

```
* best estimate is 0.551
```

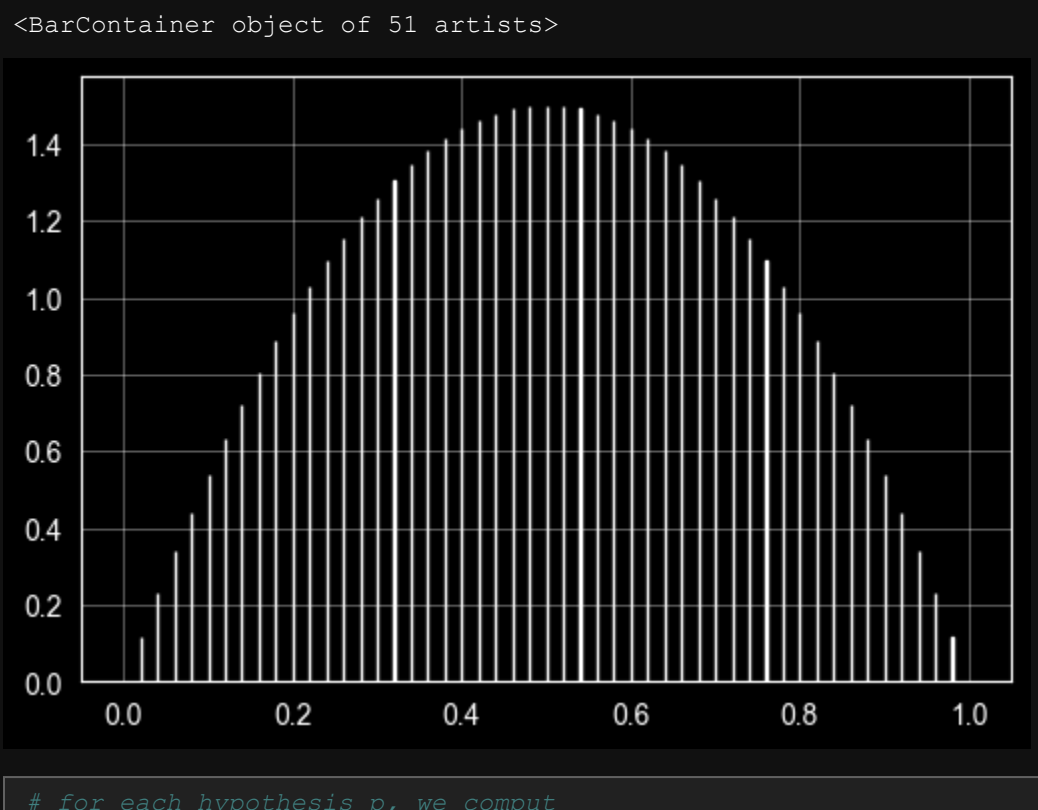
In [25]:
```
fig, axes = plt.subplots(3,1, figsize=(6,6))
for ax, pdf in zip(axes, [prior, likelihood, posterior]):
    ax.stem(probs, pdf)
```



Estimation of p with Grids from Beta(α, β)

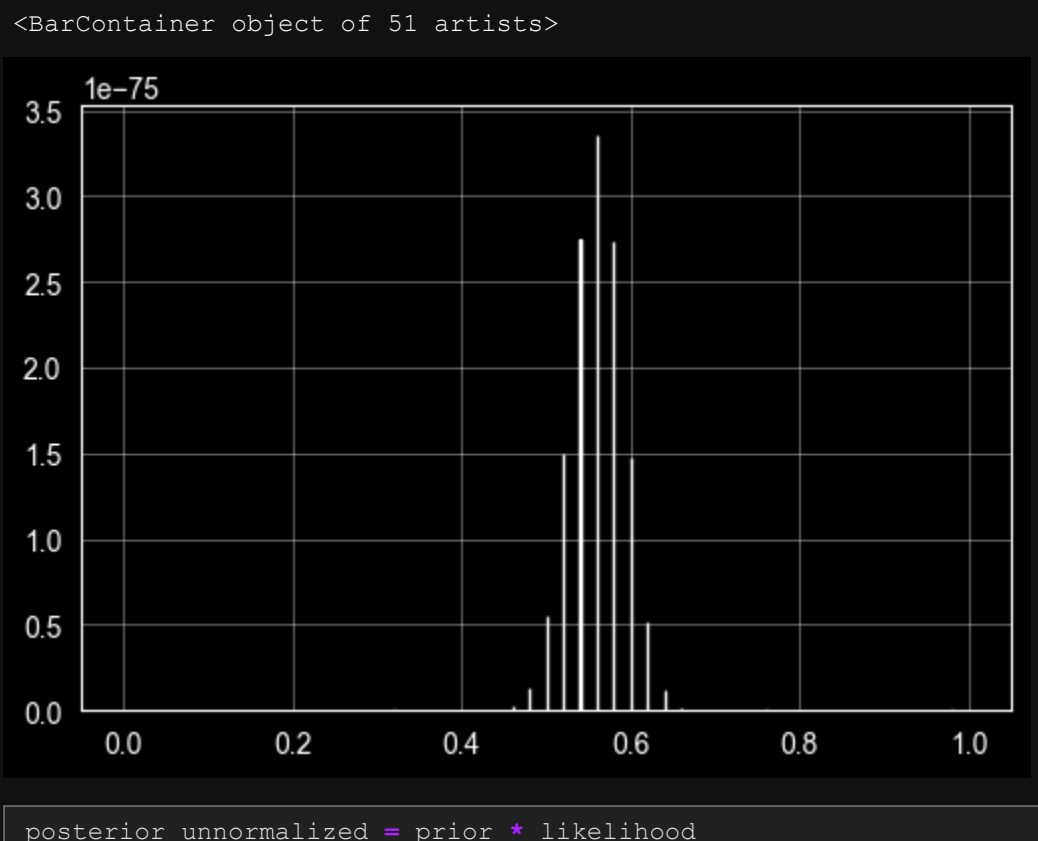
In [26]:
```
probs = np.linspace(.0, 1., 51)

import scipy.stats as stats
alpha, beta = 2, 2
prior = stats.beta(alpha, beta).pdf(probs)

width = 0.0001
plt.bar(probs, prior, width=width);
```

Out[26]: `<BarContainer object of 51 artists>`



In [27]:
```
# for each hypothesis p, we comput
likelihood = np.array([p**140 * (1-p)**110 for p in probs])
plt.bar(probs, likelihood, width=width);
```

Out[27]: `<BarContainer object of 51 artists>`



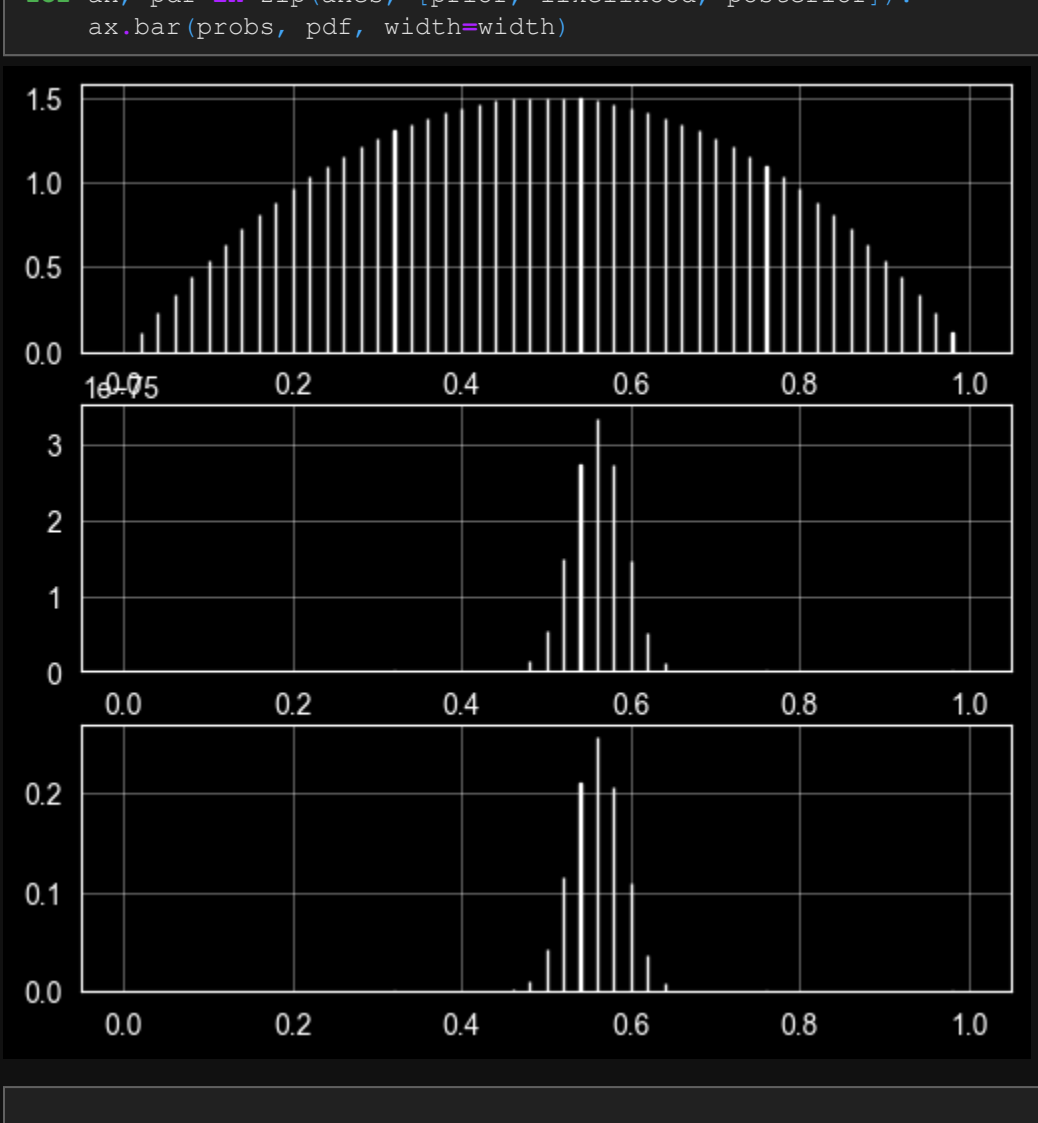
In [28]:
```
posterior_unnormalized = prior * likelihood
posterior = posterior_unnormalized / posterior_unnormalized.sum()
posterior[:10]
```

Out[28]:
```
array([0.00000000e+000, 9.15303747e-167, 2.58706353e-125, 1.68561933e-101,
       6.40292553e-085, 2.57727047e-072, 3.10723250e-062, 6.66227520e-054,
       7.34784112e-047, 8.26042293e-041])
```

In [29]:
```
istar = np.argmax(posterior)
pstar = probs[istar]
print(f'* best estimate is {pstar}')
```

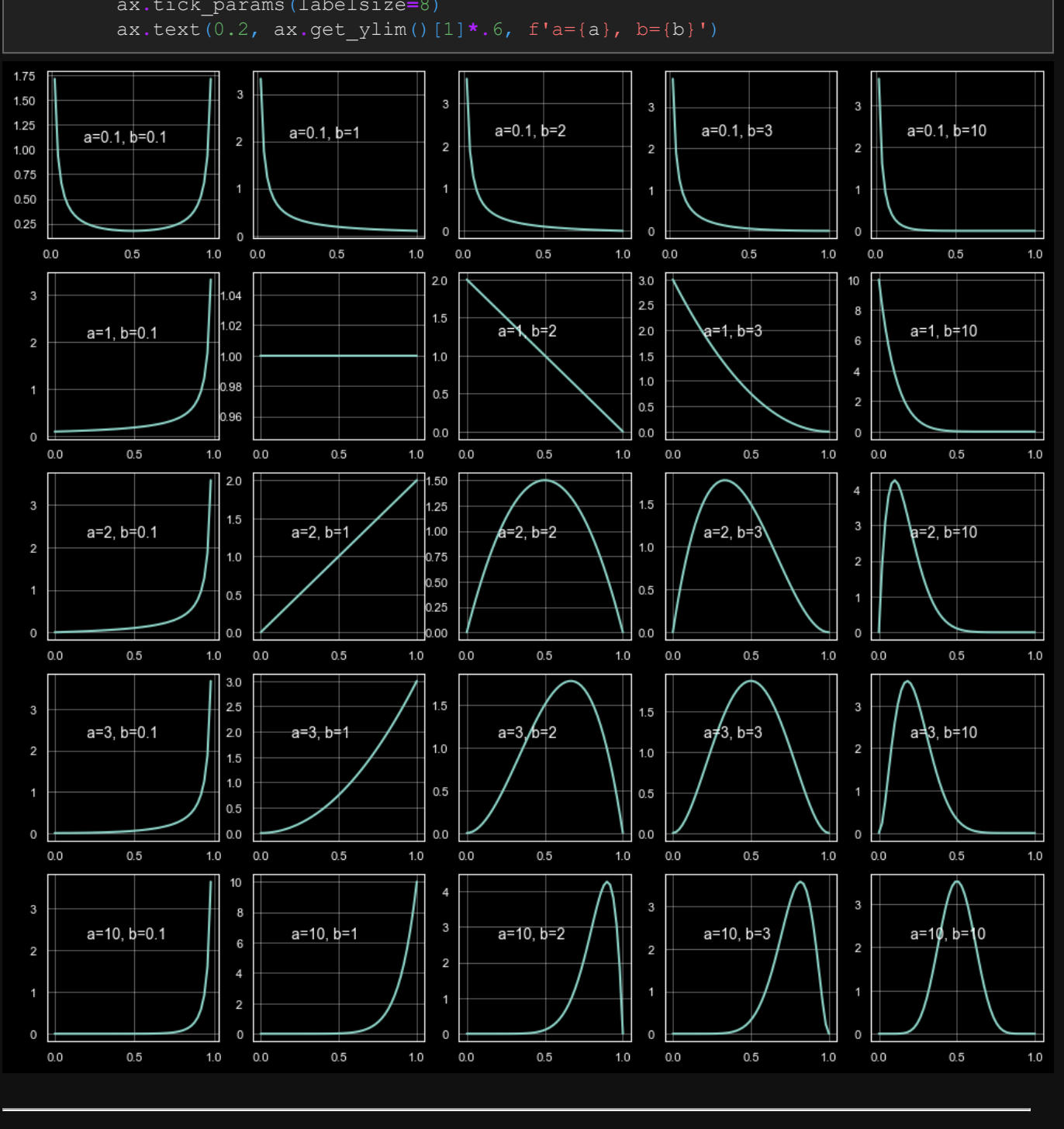
```
* best estimate is 0.56
```

In [30]:
```
fig, axes = plt.subplots(3,1, figsize=(6,6))
for ax, pdf in zip(axes, [prior, likelihood, posterior]):
    ax.bar(probs, pdf, width=width)
```



In [ ]:

In [31]:
```
## Beta distribution
alphas = [0.1, 1, 2, 3, 10]
betas = [0.1, 1, 2, 3, 10]
p = np.linspace(0, 1, 50)
fig, axes = plt.subplots(5, 5, figsize=(12,12))
for i, a in enumerate(alphas):
    for j, b in enumerate(betas):
        ax = axes[i,j]
        ax.plot(p, stats.beta(a,b).pdf(p))
        ax.tick_params(labelsize=6)
        ax.text(0.2, ax.get_ylim()[1]*.8, f'a={a}, b={b}')
```



In [32]:
```
!jupyter nbconvert coin_flips.ipynb --to webpdf --HTMLExporter.theme=dark --allow-chro
```

```
This application is used to convert notebook files (*.ipynb) to various other
formats.

[NbConvertApp] WARNING | pattern 'coin_flips.ipynb' matched no files
WARNING: THE COMMANDLINE INTERFACE MAY CHANGE IN FUTURE RELEASES.

Options
=======
The options below are convenience aliases to configurable class-options,
as listed in the "Equivalent to" description-line of the aliases.
To see all configurable class-options for some <cmd>, use:
    <cmd> --help-all

--debug
    set log level to logging.DEBUG (maximize logging output)
    Equivalent to: [--Application.log_level=10]
--generate-config
    generate default config file
    Equivalent to: [--JupyterApp.generate_config=True]
-y
    Answer yes to any questions instead of prompting.
    Equivalent to: [--JupyterApp.answer_yes=True]
--execute
    Execute the notebook prior to export.
    Equivalent to: [--ExecutePreprocessor.enabled=True]
--allow-errors
    Continue notebook execution even if one of the cells throws an error and include t
he error message in the cell output (the default behaviour is to abort conversion). Th
is flag is only relevant if '--execute' was specified, too.
    Equivalent to: [--ExecutePreprocessor.allow_errors=True]
--stdin
    read a single notebook file from stdin. Write the resulting notebook with default
basename 'notebook.*'
    Equivalent to: [--NbConvertApp.from_stdin=True]
--stdout
    Write notebook output to stdout instead of files.
    Equivalent to: [--NbConvertApp.writer_class=StdoutWriter]
--inplace
    Run nbconvert in place, overwriting the existing notebook (only
    relevant when converting to notebook format)
    Equivalent to: [--NbConvertApp.use_output_suffix=False --NbConvertApp.export_forma
t=notebook --FilesWriter.build_directory=]
--clear-output
    Clear output of current file and save in place,
    overwriting the existing notebook.
    Equivalent to: [--NbConvertApp.use_output_suffix=False --NbConvertApp.export_forma
t=notebook --FilesWriter.build_directory= --ClearOutputPreprocessor.enabled=True]
--no-prompt
    Exclude input and output prompts from converted document.
    Equivalent to: [--TemplateExporter.exclude_input_prompt=True --TemplateExporter.ex
clude_output_prompt=True]
--no-input
    Exclude input cells and output prompts from converted document.
    This mode is ideal for generating code-free reports.
    Equivalent to: [--TemplateExporter.exclude_output_prompt=True --TemplateExporter.e
xclude_input=True]
--allow-chromium-download
    Whether to allow downloading chromium if no suitable version is found on the syste
m.
    Equivalent to: [--WebPDFExporter.allow_chromium_download=True]
--log-level=<Enum>
    Set the log level by value or name.
    Choices: any of [0, 10, 20, 30, 40, 50, 'DEBUG', 'INFO', 'WARN', 'ERROR', 'CRITICA
L']
    Default: 30
    Equivalent to: [--Application.log_level]
--config=<Unicode>
    Full path of a config file.
    Default: ''
    Equivalent to: [--JupyterApp.config_file]
--to=<Unicode>
    The export format to be used, either one of the built-in formats
    ['asciidoc', 'custom', 'html', 'html_ch', 'html_embed', 'html_toc',
    'html_with_lenvs', 'html_with_toclenvs', 'latex', 'latex_with_lenvs',
    'markdown', 'notebook', 'pdf', 'python', 'rst', 'script', 'selectLanguage',
    'slides', 'slides_with_lenvs', 'webpdf'] or a dotted object name that
    represents the import path for an `Exporter` class
    Default: ''
    Equivalent to: [--NbConvertApp.export_format]
--template=<Unicode>
    Name of the template to use
    Default: ''
    Equivalent to: [--TemplateExporter.template_name]
--template-file=<Unicode>
    Name of the template file to use
    Default: None
    Equivalent to: [--TemplateExporter.template_file]
--writer=<DottedObjectName>
    Writer class used to write the  results of the conversion
    Default: 'FilesWriter'
    Equivalent to: [--NbConvertApp.writer_class]
--post=<DottedOrNone>
    PostProcessor class used to write the results of the conversion
    Default: ''
    Equivalent to: [--NbConvertApp.postprocessor_class]
--output=<Unicode>
    overwrite base name use for output files. can only be used when converting
    one notebook at a time.
    Default: ''
    Equivalent to: [--NbConvertApp.output_base]
--output-dir=<Unicode>
    Directory to write output(s) to. Defaults to output to the directory of each
    notebook. To recover previous default behaviour (outputting to the current
    working directory) use . as the flag value.
    Default: ''
    Equivalent to: [--FilesWriter.build_directory]
--reveal-prefix=<Unicode>
    The URL prefix for reveal.js (version 3.x). This defaults to the reveal CDN,
    but can be any url pointing to a copy  of reveal.js.
    For speaker notes to work, this must be a relative path to a local  copy of
    reveal.js: e.g., "reveal.js".
    If a relative path is given, it must be a subdirectory of the current
    directory (from which the server is run).
    See the usage documentation
    (https://nbconvert.readthedocs.io/en/latest/usage.html#reveal-js-html-
    slideshow) for more details.
    Default: ''
    Equivalent to: [--SlidesExporter.reveal_url_prefix]
--nbformat=<Enum>
    The nbformat version to write. Use this to downgrade notebooks.
    Choices: any of [1, 2, 3, 4]
    Default: 4
    Equivalent to: [--NotebookExporter.nbformat_version]

Examples
--------

    The simplest way to use nbconvert is

    > jupyter nbconvert mynotebook.ipynb --to html

    Options include ['asciidoc', 'custom', 'html', 'html_ch', 'html_embed', 'html_to
c', 'html_with_lenvs', 'html_with_toclenvs', 'latex', 'latex_with_lenvs', 'markdown',
'notebook', 'pdf', 'python', 'rst', 'script', 'selectLanguage', 'slides', 'slides_with
_lenvs', 'webpdf'].

    > jupyter nbconvert --to latex mynotebook.ipynb

    Both HTML and LaTeX support multiple output templates. LaTeX includes
    'base', 'article' and 'report'.  HTML includes 'basic' and 'full'. You
    can specify the flavor of the format used.

    > jupyter nbconvert --to html --template lab mynotebook.ipynb

    You can also pipe the output to stdout, rather than a file

    > jupyter nbconvert mynotebook.ipynb --stdout

    PDF is generated via latex

    > jupyter nbconvert mynotebook.ipynb --to pdf

    You can get (and serve) a Reveal.js-powered slideshow

    > jupyter nbconvert myslides.ipynb --to slides --post serve

    Multiple notebooks can be given at the command line in a couple of
    different ways:

    > jupyter nbconvert notebook*.ipynb
    > jupyter nbconvert notebook1.ipynb notebook2.ipynb

    or you can specify the notebooks list in a config file, containing::

        c.NbConvertApp.notebooks = ["my_notebook.ipynb"]

    > jupyter nbconvert --config mycfg.py

To see all available configurables, use `--help-all`.
```