

# Blackwell-Monotone Information Costs

University of Mannheim

---

**Xiaoyu Cheng**

Florida State University

**Yonggyun (YG) Kim**

Florida State University

October 1, 2024

# Introduction

---

# Introduction

- Agenda: integration of costly information across various fields
- Question: Which information cost function *should* or *could* be used
  - Information cost: a map from a statistical experiment to a cost
- Examples
  - Entropy Costs: Sims (2003); Matějka, McKay (2015)
  - Posterior Separable Costs: Caplin, Dean, Leahy (2022); Denti (2022)
  - Log-Likelihood Ratio Costs: Pomatto, Strack, Tamuz (2023)
- Common Principle: Blackwell Monotonicity
  - More informative in Blackwell's order  $\Rightarrow$  higher cost
  - **Minimum requirement** for plausible information costs
  - However, conditions for Blackwell monotonicity remain underexplored

# Introduction

- Agenda: integration of costly information across various fields
- Question: Which information cost function *should* or *could* be used
  - Information cost: a map from a statistical experiment to a cost
- Examples
  - Entropy Costs: Sims (2003); Matějka, McKay (2015)
  - Posterior Separable Costs: Caplin, Dean, Leahy (2022); Denti (2022)
  - Log-Likelihood Ratio Costs: Pomatto, Strack, Tamuz (2023)
- Common Principle: Blackwell Monotonicity
  - More informative in Blackwell's order  $\Rightarrow$  higher cost
  - **Minimum requirement** for plausible information costs
  - However, conditions for Blackwell monotonicity remain underexplored

# Introduction

- Agenda: integration of costly information across various fields
- Question: Which information cost function *should* or *could* be used
  - Information cost: a map from a statistical experiment to a cost
- Examples
  - Entropy Costs: Sims (2003); Matějka, McKay (2015)
  - Posterior Separable Costs: Caplin, Dean, Leahy (2022); Denti (2022)
  - Log-Likelihood Ratio Costs: Pomatto, Strack, Tamuz (2023)
- Common Principle: Blackwell Monotonicity
  - More informative in Blackwell's order  $\Rightarrow$  higher cost
  - **Minimum requirement** for plausible information costs
  - However, conditions for Blackwell monotonicity remain underexplored

# Introduction

- Agenda: integration of costly information across various fields
- Question: Which information cost function *should* or *could* be used
  - Information cost: a map from a statistical experiment to a cost
- Examples
  - Entropy Costs: Sims (2003); Matějka, McKay (2015)
  - Posterior Separable Costs: Caplin, Dean, Leahy (2022); Denti (2022)
  - Log-Likelihood Ratio Costs: Pomatto, Strack, Tamuz (2023)
- Common Principle: Blackwell Monotonicity
  - More informative in Blackwell's order  $\Rightarrow$  higher cost
  - **Minimum requirement** for plausible information costs
  - However, conditions for Blackwell monotonicity remain underexplored

## Illustration: Blackwell Monotonicity

- Consider consumers seeking to acquire information about their COVID-19 status
- Two tests are available in the competitive market:

		signal	
		$n$	$p$
state	—	80%	20%
	+	20%	80%

Test A (\$10)

		signal	
		$n$	$p$
state	—	60%	40%
	+	15%	85%

Test B (\$12)

- A producer can make an arbitrage by replicating test B using test A
  - When  $n$  is realized, toss a coin twice and replace it with  $p$  if both are heads

## Illustration: Blackwell Monotonicity

- Consider consumers seeking to acquire information about their COVID-19 status
- Two tests are available in the competitive market:

		signal	
		$n$	$p$
state	—	80%	20%
	+	20%	80%

Test A (\$10)

		signal	
		$n$	$p$
state	—	60%	40%
	+	15%	85%

Test B (\$12)

- A producer can make an arbitrage by replicating test B using test A
  - When  $n$  is realized, toss a coin twice and replace it with  $p$  if both are heads



## Illustration: Blackwell Monotonicity

- Consider consumers seeking to acquire information about their COVID-19 status
- Two tests are available in the competitive market:

		signal		
		$n_{oth.}$	$n_{HH}$	$p$
state	—	60%	20%	20%
	+	15%	5%	80%

Test A (\$10)

		signal	
		$n$	$p$
state	—	60%	40%
	+	15%	85%

Test B (\$12)

- A producer can make an arbitrage by replicating test B using test A
  - When  $n$  is realized, toss a coin twice and replace it with  $p$  if both are heads

## Illustration: Blackwell Monotonicity

- Consider consumers seeking to acquire information about their COVID-19 status
- Two tests are available in the competitive market:

		signal		=			signal	
		$n_{oth.}$	$p$				$n$	$p$
state	—	60%	20%		state	—	60%	40%
	+	15%	80%			+	15%	85%
Test A (\$10)					Test B (\$12)			

- A producer can make an arbitrage by replicating test B using test A
  - When  $n$  is realized, toss a coin twice and replace it with  $p$  if both are heads

# Illustration: Blackwell Monotonicity

- **Blackwell's Theorem**

- $A$  is more informative than  $B \Leftrightarrow B$  is a *garbling* of  $A$

- **Blackwell Monotonicity**

- $A$  should be more costly than  $B$  whenever  $A$  is Blackwell more informative than  $B$

- **Goals**

- identify elementary necessary and sufficient conditions for Blackwell monotonicity
- characterize a practical and tractable class of information cost functions

► Literature

# Illustration: Blackwell Monotonicity

- **Blackwell's Theorem**

- $A$  is more informative than  $B \Leftrightarrow B$  is a *garbling* of  $A$

- **Blackwell Monotonicity**

- $A$  should be more costly than  $B$  whenever  $A$  is Blackwell more informative than  $B$

- **Goals**

- identify elementary necessary and sufficient conditions for Blackwell monotonicity
- characterize a practical and tractable class of information cost functions

► Literature

# Illustration: Blackwell Monotonicity

- **Blackwell's Theorem**

- $A$  is more informative than  $B \Leftrightarrow B$  is a *garbling* of  $A$

- **Blackwell Monotonicity**

- $A$  should be more costly than  $B$  whenever  $A$  is Blackwell more informative than  $B$

- **Goals**

- identify elementary necessary and sufficient conditions for Blackwell monotonicity
- characterize a practical and tractable class of information cost functions

► Literature

# Roadmap

1. Preliminaries
2. Binary Experiments
3. Finite Experiments: more than two signals
4. Likelihood Separable Costs
5. Application: Costly Persuasion

# Preliminaries

---

# Experiments

- $\Omega = \{\omega_1, \dots, \omega_n\}$ : a finite set of states
- $\mathcal{S} = \{s_1, \dots, s_m\}$ : a finite set of signals
- A *statistical experiment*  $f : \Omega \rightarrow \Delta(\mathcal{S})$  can be represented by an  $n \times m$  matrix:

$$f = \begin{bmatrix} f_1^1 & \dots & f_1^m \\ \vdots & \ddots & \vdots \\ f_n^1 & \dots & f_n^m \end{bmatrix},$$

where  $f_i^j = \Pr(s_j | \omega_i)$ , thus,  $f_i^j \geq 0$  and  $\sum_{j=1}^m f_i^j = 1$

- $\mathcal{E}_m \subset \mathbb{R}^{n \times m}$ : the space of all experiments with  $m$  possible signals



# Blackwell Informativeness

- $f \succeq_B g$ :  $f$  is *Blackwell more informative* than  $g$   
iff  $g$  is a garbling of  $f$ :  $\exists$  a stochastic matrix  $M$  s.t.  $g = f M$

- Examples of garbling under binary signal

1. **Signal Replacement**: for some  $\epsilon > 0$ ,

$$M = \begin{bmatrix} 1 - \epsilon & \epsilon \\ 0 & 1 \end{bmatrix}$$

meaning that  $s_1$  is replaced with  $s_2$  with probability  $\epsilon$

2. **Permutation**:

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

meaning that signals are relabeled

iff  $f \succeq_B f \circ P$ : relabeling signals does not change the informativeness

# Blackwell Informativeness

- $f \succeq_B g$ :  $f$  is *Blackwell more informative* than  $g$   
iff  $g$  is a garbling of  $f$ :  $\exists$  a stochastic matrix  $M$  s.t.  $g = f M$

- Examples of garbling under binary signal

1. **Signal Replacement**: for some  $\epsilon > 0$ ,

$$M = \begin{bmatrix} 1 - \epsilon & \epsilon \\ 0 & 1 \end{bmatrix}$$

meaning that  $s_1$  is replaced with  $s_2$  with probability  $\epsilon$

2. **Permutation**:

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

meaning that signals are relabeled

- $f \simeq_B f P$ : relabeling signals does not change the informativeness

# Blackwell Informativeness

- $f \succeq_B g$ :  $f$  is *Blackwell more informative* than  $g$   
iff  $g$  is a garbling of  $f$ :  $\exists$  a stochastic matrix  $M$  s.t.  $g = f M$

- Examples of garbling under binary signal

1. **Signal Replacement**: for some  $\epsilon > 0$ ,

$$M = \begin{bmatrix} 1 - \epsilon & \epsilon \\ 0 & 1 \end{bmatrix}$$

meaning that  $s_1$  is replaced with  $s_2$  with probability  $\epsilon$

2. **Permutation**:

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

meaning that signals are relabeled

- $f \simeq_B f P$ : relabeling signals does not change the informativeness

# Blackwell Informativeness

- $f \succeq_B g$ :  $f$  is *Blackwell more informative* than  $g$   
iff  $g$  is a garbling of  $f$ :  $\exists$  a stochastic matrix  $M$  s.t.  $g = f M$

- Examples of garbling under binary signal

1. **Signal Replacement**: for some  $\epsilon > 0$ ,

$$M = \begin{bmatrix} 1 - \epsilon & \epsilon \\ 0 & 1 \end{bmatrix}$$

meaning that  $s_1$  is replaced with  $s_2$  with probability  $\epsilon$

2. **Permutation**:

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

meaning that signals are relabeled

- $f \simeq_B f P$ : relabeling signals does not change the informativeness

# Information Costs and Blackwell Monotonicity

- **Information Costs**

- $C : \mathcal{E}_m \rightarrow \mathbb{R}_+$  : an information cost function
- $\mathcal{C}_m$ : the set of all absolutely continuous information cost functions defined over  $\mathcal{E}_m$
- Absolute continuity ensures that a derivative exists a.e. and is integrable
- In the talk, assume that  $C$  is differentiable and the gradient exists

- **Blackwell Monotonicity**

- An information cost function  $C \in \mathcal{C}_m$  is **Blackwell monotone** if for all  $f, g \in \mathcal{E}_m$ ,  $C(f) \geq C(g)$  whenever  $f \succeq_B g$ .

- **Permutation Invariance**

- Any Blackwell-monotone information cost function is **permutation invariant**, i.e.,  $C(f) = C(f P)$  for any permutation matrix  $P$

# Information Costs and Blackwell Monotonicity

- **Information Costs**

- $C : \mathcal{E}_m \rightarrow \mathbb{R}_+$  : an information cost function
- $\mathcal{C}_m$ : the set of all absolutely continuous information cost functions defined over  $\mathcal{E}_m$
- Absolute continuity ensures that a derivative exists a.e. and is integrable
- In the talk, assume that  $C$  is differentiable and the gradient exists

- **Blackwell Monotonicity**

- An information cost function  $C \in \mathcal{C}_m$  is **Blackwell monotone** if for all  $f, g \in \mathcal{E}_m$ ,  $C(f) \geq C(g)$  whenever  $f \succeq_B g$ .

- **Permutation Invariance**

- Any Blackwell-monotone information cost function is **permutation invariant**, i.e.,  $C(f) = C(f P)$  for any permutation matrix  $P$

# Information Costs and Blackwell Monotonicity

- **Information Costs**

- $C : \mathcal{E}_m \rightarrow \mathbb{R}_+$  : an information cost function
- $\mathcal{C}_m$ : the set of all absolutely continuous information cost functions defined over  $\mathcal{E}_m$
- Absolute continuity ensures that a derivative exists a.e. and is integrable
- In the talk, assume that  $C$  is differentiable and the gradient exists

- **Blackwell Monotonicity**

- An information cost function  $C \in \mathcal{C}_m$  is **Blackwell monotone** if for all  $f, g \in \mathcal{E}_m$ ,  $C(f) \geq C(g)$  whenever  $f \succeq_B g$ .

- **Permutation Invariance**

- Any Blackwell-monotone information cost function is **permutation invariant**, i.e.,  $C(f) = C(f P)$  for any permutation matrix  $P$

# Binary Experiments

---

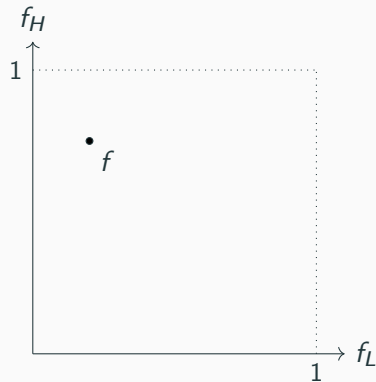


# Binary Experiments

- Focus on the case where  $n = m = 2$
- Any experiment can be represented by  $f \equiv (f_L, f_H)^\top \in [0, 1]^2$ :

$$[1 - f, f] = \begin{array}{c|cc} & s_L & s_H \\ \hline \omega_L & 1 - f_L & f_L \\ \omega_H & 1 - f_H & f_H \end{array}$$

- $1 - f$  is a permutation of  $f$
- When  $f_L = f_H$ , it is completely uninformative

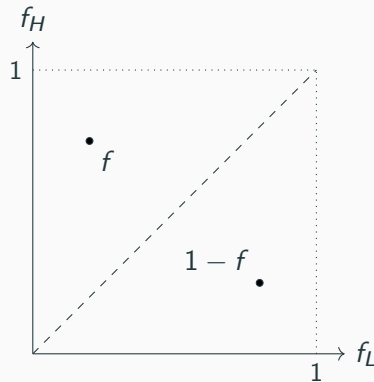


# Binary Experiments

- Focus on the case where  $n = m = 2$
- Any experiment can be represented by  $f \equiv (f_L, f_H)^\top \in [0, 1]^2$ :

$$[1 - f, f] = \begin{array}{c|cc} & s_L & s_H \\ \hline \omega_L & 1 - f_L & f_L \\ \omega_H & 1 - f_H & f_H \end{array}$$

- $1 - f$  is a permutation of  $f$
- When  $f_L = f_H$ , it is completely uninformative

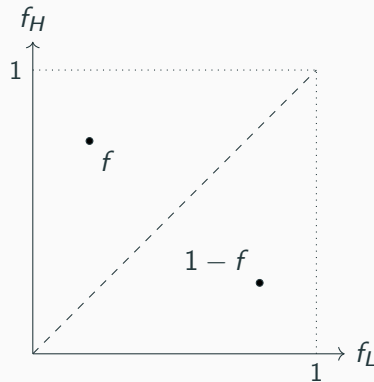


# Binary Experiments

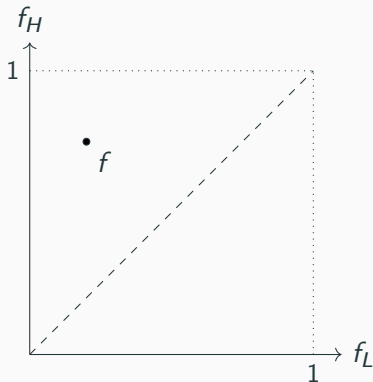
- Focus on the case where  $n = m = 2$
- Any experiment can be represented by  $f \equiv (f_L, f_H)^\top \in [0, 1]^2$ :

$$[1 - f, f] = \begin{array}{c|cc} & s_L & s_H \\ \hline \omega_L & 1 - f_L & f_L \\ \omega_H & 1 - f_H & f_H \end{array}$$

- $1 - f$  is a permutation of  $f$
- When  $f_L = f_H$ , it is completely uninformative



# Blackwell Informativeness under Binary Experiments



- Recall that  $f \succeq_B g$  iff

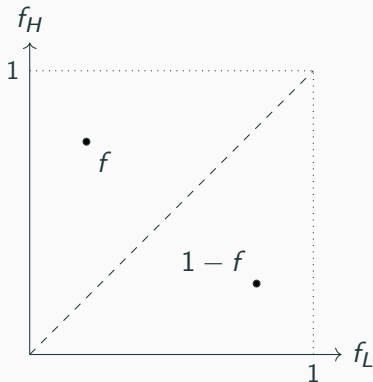
$$[1 - g, g] = [1 - f, f] M$$

for some stochastic matrix  $M$

- Extreme points of  $M$ :

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad M_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$M_3 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \quad M_4 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

# Blackwell Informativeness under Binary Experiments



- Recall that  $f \succeq_B g$  iff

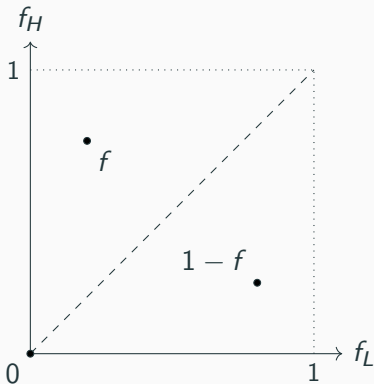
$$[1 - g, g] = [1 - f, f] M$$

for some stochastic matrix  $M$

- Extreme points of  $M$ :

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad M_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$M_3 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \quad M_4 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

# Blackwell Informativeness under Binary Experiments



- Recall that  $f \succeq_B g$  iff

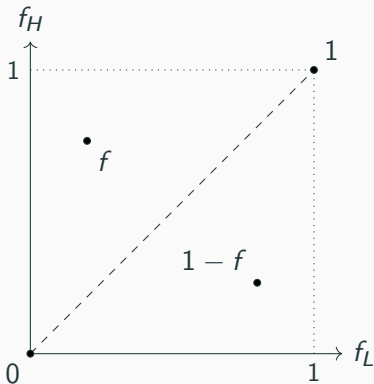
$$[1 - g, g] = [1 - f, f] M$$

for some stochastic matrix  $M$

- Extreme points of  $M$ :

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad M_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$M_3 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \quad M_4 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

# Blackwell Informativeness under Binary Experiments



- Recall that  $f \succeq_B g$  iff

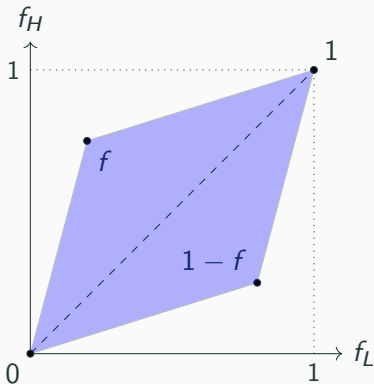
$$[1 - g, g] = [1 - f, f] M$$

for some stochastic matrix  $M$

- Extreme points of  $M$ :

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad M_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$M_3 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \quad M_4 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

# Blackwell Informativeness under Binary Experiments



- Recall that  $f \succeq_B g$  iff

$$[1 - g, g] = [1 - f, f] M$$

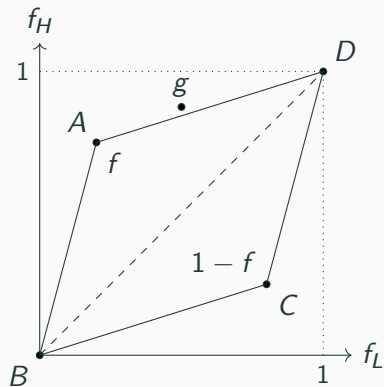
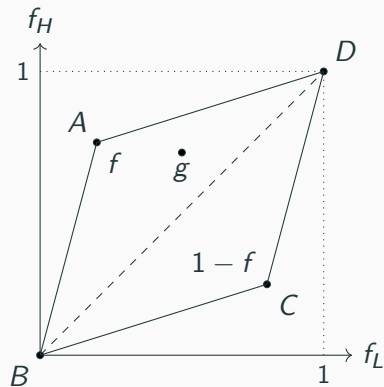
for some stochastic matrix  $M$

- Extreme points of  $M$ :

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad M_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$M_3 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \quad M_4 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

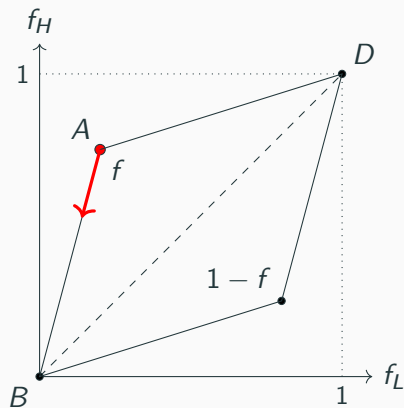


# Blackwell Informativeness under Binary Experiments

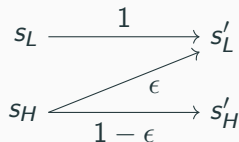


# Necessary Conditions for Blackwell Monotonicity

When an information cost  $C$  is Blackwell monotone,



$$1. \langle \nabla C(f), -f \rangle \leq 0$$

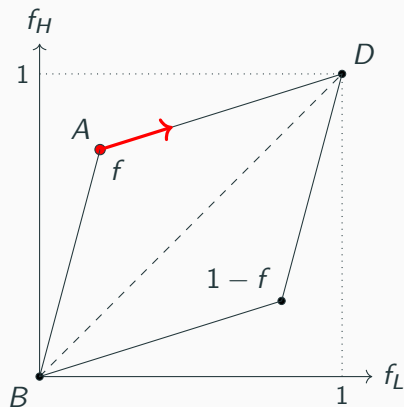


$$2. \langle \nabla C(f), 1 - f \rangle \leq 0$$

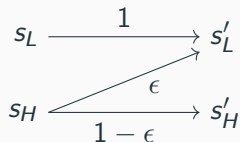


# Necessary Conditions for Blackwell Monotonicity

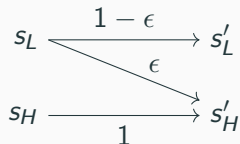
When an information cost  $C$  is Blackwell monotone,



$$1. \langle \nabla C(f), -f \rangle \leq 0$$



$$2. \langle \nabla C(f), 1 - f \rangle \leq 0$$



# Theorem for Binary Experiments

## Theorem 1

$C \in \mathcal{C}_2$  is Blackwell monotone if and only if it is

1. permutation invariant;
2. for all  $f \in \mathcal{E}_2$ ,

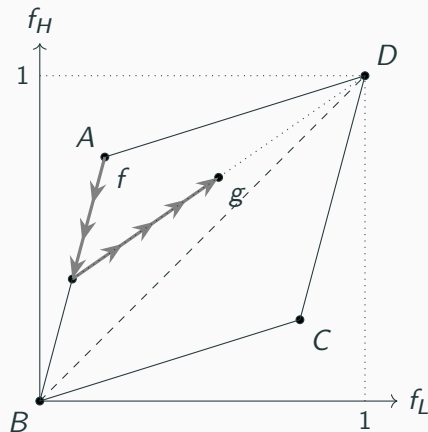
$$\langle \nabla C(f), f \rangle \geq 0 \geq \langle \nabla C(f), 1 - f \rangle. \quad (1)$$

- This theorem holds for the cases with more than two states, but the binary signal assumption is crucial.

► Quiz

## Proof for Sufficiency

For any  $f \succeq_B g$ , we can find a path from  $f$  to  $g$  (or the permutation of it) along which Blackwell informativeness decreases



## Further Characterizations with Binary States

---

## Quiz

Which of the followings (defined over  $f_H > f_L$ ) are Blackwell-monotone information cost functions?

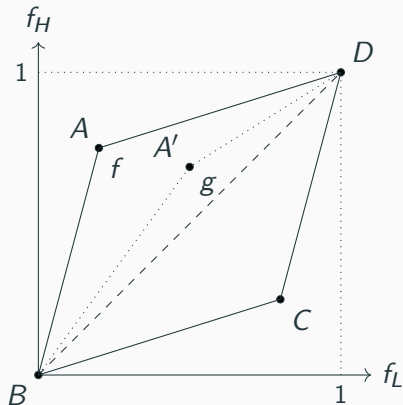
1.  $C(f_L, f_H) = \frac{f_H(1 - f_H)}{f_L(1 - f_L)} - 1$

2.  $C(f_L, f_H) = \frac{f_H}{f_L} + \frac{1 - f_L}{1 - f_H} - 2$

3.  $C(f_L, f_H) = (f_H - f_L)^2$

4.  $C(f_L, f_H) = f_H - 2f_L$

## Further Characterizations with Binary States



$f \succeq_B g$  is equivalent to:

1.  $AB$  steeper than  $A'B$ :

$$\alpha \equiv \frac{f_H}{f_L} \geq \frac{g_H}{g_L} \equiv \alpha'$$

 $\alpha$ : likelihood ratio (LR) of receiving  $s_H$ 

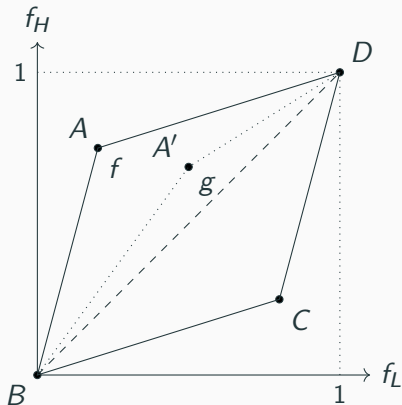
2.  $AD$  shallower than  $A'D$ :

$$\beta \equiv \frac{1 - f_L}{1 - f_H} \geq \frac{1 - g_L}{1 - g_H} \equiv \beta'$$

$\beta$ : the inverse of LR of receiving  $s_L$



## Further Characterizations with Binary States



$f \succeq_B g$  is equivalent to:

1.  $AB$  steeper than  $A'B$ :

$$\alpha \equiv \frac{f_H}{f_L} \geq \frac{g_H}{g_L} \equiv \alpha'$$

 $\alpha$ : likelihood ratio (LR) of receiving  $s_H$ 

2.  $AD$  shallower than  $A'D$ :

$$\beta \equiv \frac{1 - f_L}{1 - f_H} \geq \frac{1 - g_L}{1 - g_H} \equiv \beta'$$

$\beta$ : the inverse of LR of receiving  $s_L$

- $C$  is Blackwell monotone iff it is increasing in  $\alpha$  and  $\beta$  after reparametrization

## Further Characterizations with Binary States

1.  $C(f_L, f_H) = \frac{f_H(1 - f_H)}{f_L(1 - f_L)} - 1$  with  $1 > f_H > f_L > 0$

$$\tilde{C}(\alpha, \beta) = \frac{\alpha}{\beta} - 1$$

- $\tilde{C}$  is increasing in  $\alpha$  but not in  $\beta$ , thus,  $\tilde{C}$  is not Blackwell monotone.

2.  $C(f_L, f_H) = \frac{f_H}{f_L} + \frac{1 - f_L}{1 - f_H} - 2$  with  $1 > f_H > f_L > 0$

$$\tilde{C}(\alpha, \beta) = \alpha + \beta - 2$$

- $\tilde{C}$  is increasing in both  $\alpha$  and  $\beta$ , thus,  $\tilde{C}$  is **Blackwell monotone**.

## Further Characterizations with Binary States

1.  $C(f_L, f_H) = \frac{f_H(1 - f_H)}{f_L(1 - f_L)} - 1$  with  $1 > f_H > f_L > 0$

$$\tilde{C}(\alpha, \beta) = \frac{\alpha}{\beta} - 1$$

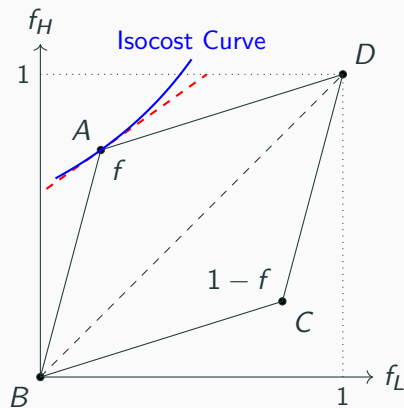
- $\tilde{C}$  is increasing in  $\alpha$  but not in  $\beta$ , thus,  $\tilde{C}$  is not Blackwell monotone.

2.  $C(f_L, f_H) = \frac{f_H}{f_L} + \frac{1 - f_L}{1 - f_H} - 2$  with  $1 > f_H > f_L > 0$

$$\tilde{C}(\alpha, \beta) = \alpha + \beta - 2$$

- $\tilde{C}$  is increasing in both  $\alpha$  and  $\beta$ , thus,  $\tilde{C}$  is **Blackwell monotone**.

## Further Characterizations with Binary States

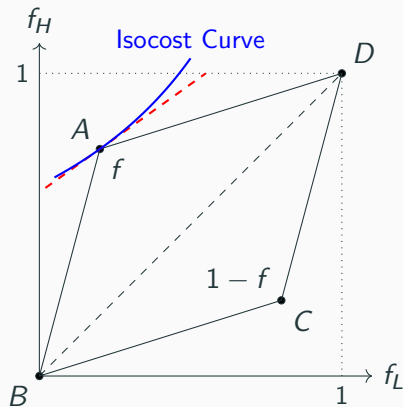


$\langle \nabla C(f), f \rangle \geq 0 \geq \langle \nabla C(f), 1 - f \rangle$   
 is equivalent to:

$$\underbrace{\frac{f_H}{f_L}}_{\text{the slope of } \overline{AB}} \geq \underbrace{-\frac{\partial C / \partial f_L}{\partial C / \partial f_H}}_{\text{the slope of the isocost curve}} \geq \underbrace{\frac{1 - f_H}{1 - f_L}}_{\text{the slope of } \overline{AD}}$$

- **Interpretation:** a *marginal rate of information transformation* (MRIT) lies between the two likelihood ratios provided by the experiment.

## Further Characterizations with Binary States



$\langle \nabla C(f), f \rangle \geq 0 \geq \langle \nabla C(f), 1 - f \rangle$   
 is equivalent to:

$$\underbrace{\frac{f_H}{f_L}}_{\text{the slope of } \overline{AB}} \geq \underbrace{-\frac{\partial C / \partial f_L}{\partial C / \partial f_H}}_{\text{the slope of the isocost curve}} \geq \underbrace{\frac{1 - f_H}{1 - f_L}}_{\text{the slope of } \overline{AD}}$$

- **Interpretation:** a *marginal rate of information transformation* (MRIT) lies between the two likelihood ratios provided by the experiment.

## Further Characterizations with Binary States

3.  $C(f_L, f_H) = (f_H - f_L)^2$  with  $1 > f_H > f_L > 0$

$$\frac{f_H}{f_L} \geq -\frac{\partial C / \partial f_L}{\partial C / \partial f_H} = 1 \geq \frac{1 - f_H}{1 - f_L}$$

- The above inequalities hold for all  $1 > f_H > f_L > 0$ , thus, it is **Blackwell monotone**.

4.  $C(f_L, f_H) = f_H - 2f_L$  with  $1 > f_H > f_L > 0$

$$\frac{f_H}{f_L} \geq -\frac{\partial C / \partial f_L}{\partial C / \partial f_H} = 2 \geq \frac{1 - f_H}{1 - f_L}$$

- The above inequalities does not always hold, e.g.,  $f_L = .5$  and  $f_H = .6$ , thus, it is not Blackwell monotone.

## Further Characterizations with Binary States

3.  $C(f_L, f_H) = (f_H - f_L)^2$  with  $1 > f_H > f_L > 0$

$$\frac{f_H}{f_L} \geq -\frac{\partial C / \partial f_L}{\partial C / \partial f_H} = 1 \geq \frac{1 - f_H}{1 - f_L}$$

- The above inequalities hold for all  $1 > f_H > f_L > 0$ , thus, it is **Blackwell monotone**.

4.  $C(f_L, f_H) = f_H - 2f_L$  with  $1 > f_H > f_L > 0$

$$\frac{f_H}{f_L} \geq -\frac{\partial C / \partial f_L}{\partial C / \partial f_H} = 2 \geq \frac{1 - f_H}{1 - f_L}$$

- The above inequalities does not always hold, e.g.,  $f_L = .5$  and  $f_H = .6$ , thus, it is not Blackwell monotone.

## Answer for the Quiz

Which of the followings are Blackwell-monotone information cost functions?

1.  $C(f_L, f_H) = \frac{f_H(1 - f_H)}{f_L(1 - f_L)} - 1$

2.  $C(f_L, f_H) = \frac{f_H}{f_L} + \frac{1 - f_L}{1 - f_H} - 2$

3.  $C(f_L, f_H) = (f_H - f_L)^2$

4.  $C(f_L, f_H) = f_H - 2f_L$



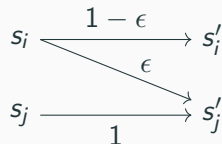
## Finite Experiments: more than two signals

---

# Necessary Conditions for Blackwell Monotonicity

Now assume that there are more than two signals.

- Permutation invariance is still necessary
- For any pair  $(i, j)$ , the following garbling worsens the informativeness:



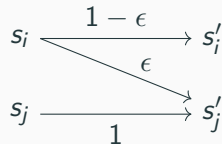
- This gives us  $\langle \nabla^j C(f) - \nabla^i C(f), f^i \rangle \leq 0$ , where

$$\langle \nabla^j C(f) - \nabla^i C(f), f^i \rangle = \sum_{s=1}^n \frac{\partial C}{\partial f_s^j} \cdot f_s^i - \sum_{s=1}^n \frac{\partial C}{\partial f_s^i} \cdot f_s^i$$

# Necessary Conditions for Blackwell Monotonicity

Now assume that there are more than two signals.

- Permutation invariance is still necessary
- For any pair  $(i, j)$ , the following garbling worsens the informativeness:



- This gives us  $\langle \nabla^j C(f) - \nabla^i C(f), f^i \rangle \leq 0$ , where

$$\langle \nabla^j C(f) - \nabla^i C(f), f^i \rangle = \sum_{s=1}^n \frac{\partial C}{\partial f_s^j} \cdot f_s^i - \sum_{s=1}^n \frac{\partial C}{\partial f_s^i} \cdot f_s^i$$

# Sufficient Conditions for Blackwell Monotonicity

- For binary experiments, sufficiency was established by finding a path between two experiments along which informativeness decreases
- However, when  $m \geq 3$ , there may not exist such path

► Illustrations

- To overcome this issue, we impose quasiconvexity on  $C$ :

$$C(\lambda f + (1 - \lambda)g) \leq \max\{C(f), C(g)\}.$$

With quasiconvexity, the first-order condition serves as a sufficient condition for Blackwell monotonicity

- **Remarks**

- Quasiconvexity is not a necessary condition for Blackwell Monotonicity
- We found a weaker (but less standard) version of Quasiconvexity serving as a necessary condition for Blackwell monotonicity

► Details

► Details 21/32

# Sufficient Conditions for Blackwell Monotonicity

- For binary experiments, sufficiency was established by finding a path between two experiments along which informativeness decreases
- However, when  $m \geq 3$ , there may not exist such path [▶ Illustrations](#)
- To overcome this issue, we impose quasiconvexity on  $C$ :

$$C(\lambda f + (1 - \lambda)g) \leq \max\{C(f), C(g)\}.$$

With quasiconvexity, the first-order condition serves as a sufficient condition for Blackwell monotonicity

## • Remarks

- Quasiconvexity is not a necessary condition for Blackwell Monotonicity [▶ Details](#)
- We found a weaker (but less standard) version of Quasiconvexity serving as a necessary condition for Blackwell monotonicity [▶ Details](#)

# Sufficient Conditions for Blackwell Monotonicity

- For binary experiments, sufficiency was established by finding a path between two experiments along which informativeness decreases
- However, when  $m \geq 3$ , there may not exist such path [▶ Illustrations](#)
- To overcome this issue, we impose quasiconvexity on  $C$ :

$$C(\lambda f + (1 - \lambda)g) \leq \max\{C(f), C(g)\}.$$

With quasiconvexity, the first-order condition serves as a sufficient condition for Blackwell monotonicity

- **Remarks**

- Quasiconvexity is not a necessary condition for Blackwell Monotonicity [▶ Details](#)
- We found a weaker (but less standard) version of Quasiconvexity serving as a necessary condition for Blackwell monotonicity [▶ Details](#)

# Sufficient Conditions for Blackwell Monotonicity

- For binary experiments, sufficiency was established by finding a path between two experiments along which informativeness decreases
- However, when  $m \geq 3$ , there may not exist such path [▶ Illustrations](#)
- To overcome this issue, we impose quasiconvexity on  $C$ :

$$C(\lambda f + (1 - \lambda)g) \leq \max\{C(f), C(g)\}.$$

With quasiconvexity, the first-order condition serves as a sufficient condition for Blackwell monotonicity

- **Remarks**

- Quasiconvexity is not a necessary condition for Blackwell Monotonicity [▶ Details](#)
- We found a weaker (but less standard) version of Quasiconvexity serving as a necessary condition for Blackwell monotonicity [▶ Details](#)

# Theorem for Finite Experiments

## Theorem 2

Suppose that  $C \in \mathcal{C}_m$  is absolutely continuous and quasiconvex. Then,  $C$  is Blackwell monotone if and only if it is

1. permutation invariant;
2. for all  $f \in \mathcal{E}_m$  and  $i \neq j$ ,

$$\langle \nabla^j C(f) - \nabla^i C(f), f \rangle \leq 0. \quad (2)$$

- $S_B(f)$ : the set of experiments that are less Blackwell informative than  $f$
- Two conditions ensure that extreme points of  $S_B(f)$  are not more costly than  $f$
- Then, we can apply quasiconvexity



# Theorem for Finite Experiments

## Theorem 2

Suppose that  $C \in \mathcal{C}_m$  is absolutely continuous and quasiconvex. Then,  $C$  is Blackwell monotone if and only if it is

1. permutation invariant;
2. for all  $f \in \mathcal{E}_m$  and  $i \neq j$ ,

$$\langle \nabla^j C(f) - \nabla^i C(f), f \rangle \leq 0. \quad (2)$$

- $S_B(f)$ : the set of experiments that are less Blackwell informative than  $f$
- Two conditions ensure that extreme points of  $S_B(f)$  are not more costly than  $f$
- Then, we can apply quasiconvexity

## Likelihood Separable Costs

---

# Likelihood Separable Costs

## Likelihood Separable Costs

$C$  is *likelihood separable* if there exist a constant  $a$  and an absolutely continuous function  $\psi : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$  such that, for all  $m$  and  $f \in \mathcal{E}_m$ ,

$$C(f) = \sum_{j=1}^m \psi(f^j) + a.$$

Let  $\mathcal{C}^{LS}$  be the class of likelihood separable costs

### Theorem 3

When  $C \in \mathcal{C}^{LS}$ ,  $C$  is Blackwell monotone if and only if  $\psi$  is sublinear:

► proof

1. positive homogeneity:  $\psi(\alpha h) = \alpha \psi(h)$ ;
2. subadditivity:  $\psi(k) + \psi(l) \geq \psi(k + l)$

# Likelihood Separable Costs

## Likelihood Separable Costs

$C$  is *likelihood separable* if there exist a constant  $a$  and an absolutely continuous function  $\psi : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$  such that, for all  $m$  and  $f \in \mathcal{E}_m$ ,

$$C(f) = \sum_{j=1}^m \psi(f^j) + a.$$

Let  $\mathcal{C}^{LS}$  be the class of likelihood separable costs

## Theorem 3

When  $C \in \mathcal{C}^{LS}$ ,  $C$  is Blackwell monotone if and only if  $\psi$  is sublinear:

► proof

1. positive homogeneity:  $\psi(\alpha h) = \alpha \psi(h)$ ;
2. subadditivity:  $\psi(k) + \psi(l) \geq \psi(k + l)$

## Groundedness

$C$  is *grounded* if it assigns zero cost to uninformative experiments.

Let  $\mathcal{C}^G$  be the class of grounded costs.

## GSLS costs

$C$  is called *grounded sublinear likelihood separable (GSLS)* if there exists a sublinear and absolutely continuous function  $\psi$  such that

$$C(f) = \sum_{j=1}^m \psi(f^j) - \psi(1).$$

Then,

$$\mathcal{C}^{GSLS} = \mathcal{C}^{LS} \cap \mathcal{C}^G \cap \mathcal{C}^{BM}$$

## Groundedness

$C$  is *grounded* if it assigns zero cost to uninformative experiments.

Let  $\mathcal{C}^G$  be the class of grounded costs.

## GSLS costs

$C$  is called *grounded sublinear likelihood separable (GSLS)* if there exists a sublinear and absolutely continuous function  $\psi$  such that

$$C(f) = \sum_{j=1}^m \psi(f^j) - \psi(1).$$

Then,

$$\mathcal{C}^{GSLS} = \mathcal{C}^{LS} \cap \mathcal{C}^G \cap \mathcal{C}^{BM}$$

## Examples: GSLS Costs

### 1. Supnorm Costs

$$C(f) = \sum_{j=1}^m \max_i f_i^j - 1,$$

### 2. Absolute-Linear Costs

$$C(f) = \sum_{j=1}^m |\langle a, f^j \rangle| - |\langle a, 1 \rangle| = \sum_{j=1}^m \left| \sum_{i=1}^n a_i f_i^j \right| - \left| \sum_{i=1}^n a_i \right|.$$

### 3. Linear $\phi$ -divergence Costs (including LLR costs of Pomatto, Strack, Tamuz (2023))

$$C(f) = \sum_{j=1}^m \sum_{i,i'} \beta_{ii'} f_{i'}^j \phi_{ii'} \left( \frac{f_i^j}{f_{i'}^j} \right) = \sum_{i,i'} \beta_{ii'} \sum_{j=1}^m f_{i'}^j \phi_{ii'} \left( \frac{f_i^j}{f_{i'}^j} \right), \quad (3)$$

where  $\phi_{ii'} : [0, \infty] \rightarrow \mathbb{R} \cup \{+\infty\}$  is a convex function with  $\phi_{ii'}(1) = 0$  and  $\beta_{ii'} \geq 0$

## Posterior Separability

$C$  has a *posterior separable (PS)* representation at a prior belief  $\mu \in \Delta(\Omega)$  if there exists a concave and absolutely continuous function  $H : \Delta(\Omega) \rightarrow \mathbb{R}$  such that

$$C(f) = H(\mu) - \sum_{j=1}^m \tau_{\mu}(f^j) \cdot H(q_{\mu}(f^j))$$

where  $q_{\mu}(f^j)$  is the posterior belief upon receiving  $s_j$  and  $\tau_{\mu}(f^j)$  is the probability of receiving  $s_j$ .

Let  $C_{\mu}^{PS}$  denote the class of cost functions that have PS representations at  $\mu$ .

## Proposition

For any full support prior  $\mu \in \Delta(\Omega)$ ,  $C^{GSLS} = C_{\mu}^{PS}$ .



## Posterior Separability

$C$  has a *posterior separable (PS)* representation at a prior belief  $\mu \in \Delta(\Omega)$  if there exists a concave and absolutely continuous function  $H : \Delta(\Omega) \rightarrow \mathbb{R}$  such that

$$C(f) = H(\mu) - \sum_{j=1}^m \tau_{\mu}(f^j) \cdot H(q_{\mu}(f^j))$$

where  $q_{\mu}(f^j)$  is the posterior belief upon receiving  $s_j$  and  $\tau_{\mu}(f^j)$  is the probability of receiving  $s_j$ .

Let  $C_{\mu}^{PS}$  denote the class of cost functions that have PS representations at  $\mu$ .

## Proposition

For any full support prior  $\mu \in \Delta(\Omega)$ ,  $C^{GSLS} = C_{\mu}^{PS}$ .

## **Application: Costly Persuasion**

---

## Gentzkow, Kamenica (2014) Revisited

- Consider a costly persuasion problem with the standard example
  - State:  $\{innocent, guilty\}$
  - Receiver's action: **A**cquit or **C**onvict
  - Sender's payoff:  $u_S(C) = 1, u_S(A) = 0$
  - Receiver's payoff:  $u_R(A, innocent) = u_R(C, guilty) = 1$   
 $u_R(C, innocent) = u_R(A, guilty) = 0$
  - Sender commits to an experiment at some cost
- GK focuses on posterior separable costs (e.g., entropy cost) to utilize concavification technique
- Can we solve this problem with any Blackwell-monotone information cost function?

## Gentzkow, Kamenica (2014) Revisited

- Consider a costly persuasion problem with the standard example
  - State:  $\{innocent, guilty\}$
  - Receiver's action: **A**cquit or **C**onvict
  - Sender's payoff:  $u_S(C) = 1, u_S(A) = 0$
  - Receiver's payoff:  $u_R(A, innocent) = u_R(C, guilty) = 1$   
 $u_R(C, innocent) = u_R(A, guilty) = 0$
  - Sender commits to an experiment at some cost
- GK focuses on posterior separable costs (e.g., entropy cost) to utilize concavification technique
- Can we solve this problem with any Blackwell-monotone information cost function?

## Gentzkow, Kamenica (2014) Revisited

- Consider a costly persuasion problem with the standard example
  - State:  $\{innocent, guilty\}$
  - Receiver's action: **A**cquit or **C**onvict
  - Sender's payoff:  $u_S(C) = 1, u_S(A) = 0$
  - Receiver's payoff:  $u_R(A, innocent) = u_R(C, guilty) = 1$   
 $u_R(C, innocent) = u_R(A, guilty) = 0$
  - Sender commits to an experiment at some cost
- GK focuses on posterior separable costs (e.g., entropy cost) to utilize concavification technique
- Can we solve this problem with any Blackwell-monotone information cost function?

# Costly Persuasion with Blackwell-Monotone Information Cost

- It is without loss to consider binary experiments since  $\mathbf{R}$ 's action is binary
  - $f_2 = \Pr(C|guilty)$  and  $f_1 = \Pr(C|innocent)$
- When the prior is  $p$ , the sender's problem is

$$\max_{0 \leq f_1 \leq f_2 \leq 1} pf_2 + (1 - p)f_1 - C(f_1, f_2)$$

subject to

$$\frac{pf_2}{pf_2 + (1 - p)f_1} \geq \frac{1}{2}.$$

- When  $p \geq 1/2$ , the solution is  $f_1 = f_2 = 1$ : always convict costlessly

# Costly Persuasion with Blackwell-Monotone Information Cost

- It is without loss to consider binary experiments since  $\mathbf{R}$ 's action is binary
  - $f_2 = \Pr(C|guilty)$  and  $f_1 = \Pr(C|innocent)$
- When the prior is  $p$ , the sender's problem is

$$\max_{0 \leq f_1 \leq f_2 \leq 1} pf_2 + (1 - p)f_1 - C(f_1, f_2)$$

subject to

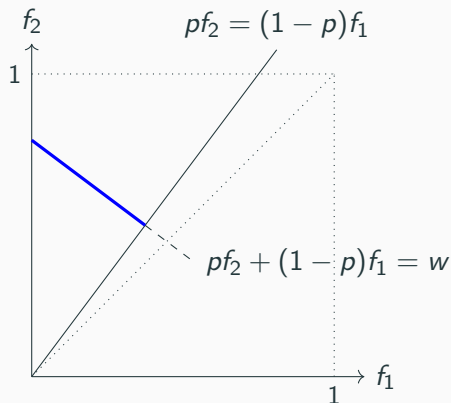
$$\frac{pf_2}{pf_2 + (1 - p)f_1} \geq \frac{1}{2}.$$

- When  $p \geq 1/2$ , the solution is  $f_1 = f_2 = 1$ : always convict costlessly

# Cost Minimization

- Suppose  $p < 1/2$ .
- Cost minimization problem under  $pf_2 + (1 - p)f_1 = w$ :

$$\min C(f_1, f_2) \quad \text{s.t.} \quad \begin{aligned} pf_2 + (1 - p)f_1 &= w, \\ pf_2 &\geq (1 - p)f_1 \end{aligned}$$



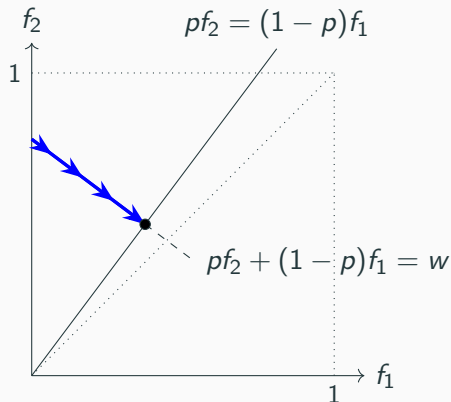
- **Proposition:** for any Blackwell-monotone information cost function, the cost is minimized when  $pf_2 = (1 - p)f_1$



# Cost Minimization

- Suppose  $p < 1/2$ .
- Cost minimization problem under  $pf_2 + (1 - p)f_1 = w$ :

$$\min C(f_1, f_2) \quad \text{s.t.} \quad \begin{aligned} pf_2 + (1 - p)f_1 &= w, \\ pf_2 &\geq (1 - p)f_1 \end{aligned}$$



- **Proposition:** for any Blackwell-monotone information cost function, the cost is minimized when  $pf_2 = (1 - p)f_1$

## Sender's Problem

- When  $pf_2 + (1 - p)f_1 = w$ , the cost is minimized at

$$f_2 = \frac{w}{2p} \quad \text{and} \quad f_1 = \frac{w}{2(1-p)}.$$

- Now the sender's problem is

$$\max_{0 \leq w \leq 2p} w - C\left(\frac{w}{2(1-p)}, \frac{w}{2p}\right) \quad (4)$$

- From here on, a specific cost function is needed

## Sender's Problem

- When  $pf_2 + (1 - p)f_1 = w$ , the cost is minimized at

$$f_2 = \frac{w}{2p} \quad \text{and} \quad f_1 = \frac{w}{2(1-p)}.$$

- Now the sender's problem is

$$\max_{0 \leq w \leq 2p} w - C\left(\frac{w}{2(1-p)}, \frac{w}{2p}\right) \quad (4)$$

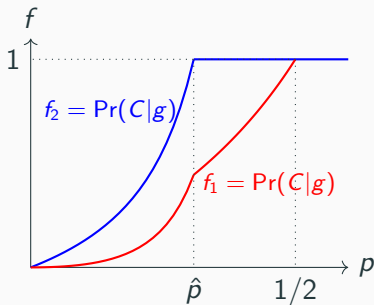
- From here on, a specific cost function is needed

# Costly Persuasion with Non-Posterior-Separable Cost

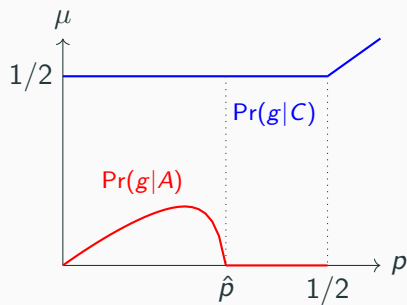
- When  $C(f_1, f_2) = (f_2 - f_1)^2$ , the solution for  $p < 1/2$  is

$$f_2(p) = \min \left\{ 1, \frac{(1-p)^2 p}{(1-2p)^2} \right\} \quad \text{and} \quad f_1(p) = \frac{p}{1-p} \cdot f_2(p).$$

► Entropy



Optimal Experiments



Posteriors

## Conclusion

---

# Conclusion

- We identify necessary and sufficient conditions for Blackwell Monotonicity.
- Under likelihood separability, we show that the sublinearity of the component function is equivalent to Blackwell Monotonicity.
- **Applications:** we apply our results to extend
  1. Costly Persuasion (Gentzkow, Kamenica, 2014)
  2. Bargaining and Information Acquisition (Chatterjee, Dong, Hoshino, 2024) ▶ Bargaining
- Future Research: Lehmann-Monotone Information Costs based on Kim [2023]

# Conclusion

- We identify necessary and sufficient conditions for Blackwell Monotonicity.
- Under likelihood separability, we show that the sublinearity of the component function is equivalent to Blackwell Monotonicity.
- **Applications:** we apply our results to extend
  1. Costly Persuasion (Gentzkow, Kamenica, 2014)
  2. Bargaining and Information Acquisition (Chatterjee, Dong, Hoshino, 2024) ▶ Bargaining
- Future Research: Lehmann-Monotone Information Costs based on Kim [2023]

# Conclusion

- We identify necessary and sufficient conditions for Blackwell Monotonicity.
- Under likelihood separability, we show that the sublinearity of the component function is equivalent to Blackwell Monotonicity.
- **Applications:** we apply our results to extend
  1. Costly Persuasion (Gentzkow, Kamenica, 2014)
  2. Bargaining and Information Acquisition (Chatterjee, Dong, Hoshino, 2024) ▶ Bargaining
- Future Research: Lehmann-Monotone Information Costs based on Kim [2023]



# Conclusion

- We identify necessary and sufficient conditions for Blackwell Monotonicity.
- Under likelihood separability, we show that the sublinearity of the component function is equivalent to Blackwell Monotonicity.
- **Applications:** we apply our results to extend
  1. Costly Persuasion (Gentzkow, Kamenica, 2014)
  2. Bargaining and Information Acquisition (Chatterjee, Dong, Hoshino, 2024) ▶ Bargaining
- Future Research: Lehmann-Monotone Information Costs based on Kim [2023]

# Thank You!

- **Posterior-based information costs**

- Entropy cost: Sims [2003]; Matějka, McKay [2015]
- Decision theory: Caplin, Dean [2015]; Caplin, Dean, Leahy [2022]; Chambers, Liu, Rehbeck [2020]; Denti [2022]
- Applications: Ravid [2020]; Zhong [2022]; Gentzkow, Kamenica [2014]

- **Experiment-based information costs**

- LLR cost: Pomatto, Strack, Tamuz [2023];
- Applications: Denti, Marinacci, Rustichini [2022]; Ramos-Mercado [2023]

# Sufficient Conditions for Blackwell Monotonicity

When  $m \geq 3$ , there may not exist a path along which informativeness decreases

## Proposition

Let

$$g = \begin{bmatrix} 4/5 & 1/5 & 0 \\ 0 & 4/5 & 1/5 \\ 1/5 & 0 & 4/5 \end{bmatrix} \in \mathcal{E}_3.$$

If  $f \succeq_B g$  and  $f \in \mathcal{E}_3$ , then  $f$  is a permutation of  $I_3$  or  $g$ .

- $I_3$  is Blackwell more informative than  $g$ , but we cannot find a path from  $I_3$  to  $g$  along which Blackwell informativeness decreases

# Sufficient Conditions for Blackwell Monotonicity

When  $m \geq 3$ , there may not exist a path along which informativeness decreases

## Proposition

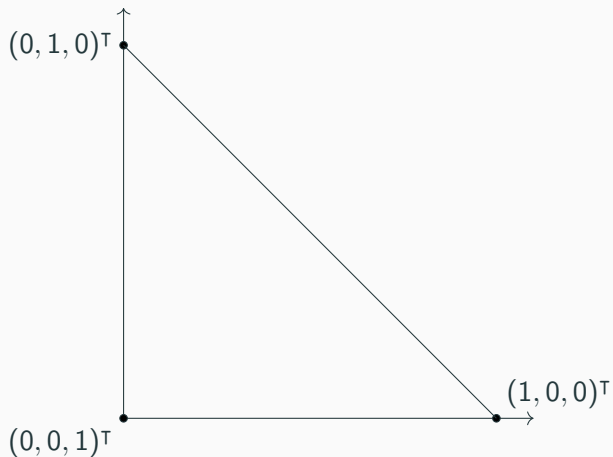
Let

$$g = \begin{bmatrix} 4/5 & 1/5 & 0 \\ 0 & 4/5 & 1/5 \\ 1/5 & 0 & 4/5 \end{bmatrix} \in \mathcal{E}_3.$$

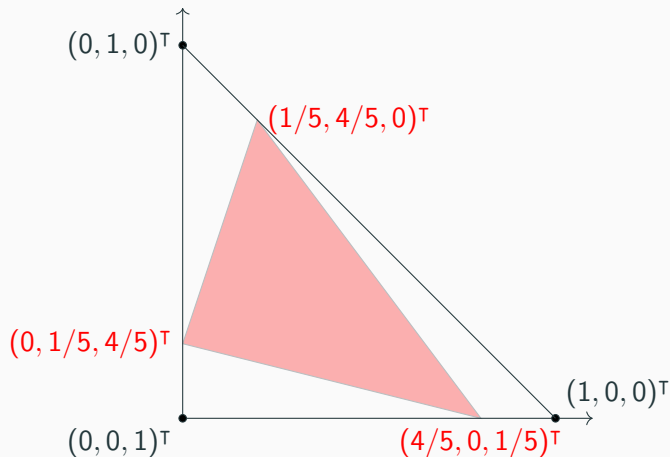
If  $f \succeq_B g$  and  $f \in \mathcal{E}_3$ , then  $f$  is a permutation of  $I_3$  or  $g$ .

- $I_3$  is Blackwell more informative than  $g$ , but we cannot find a path from  $I_3$  to  $g$  along which Blackwell informativeness decreases

- When  $n = m = 3$ ,  $f \succeq_B g$  iff the triangle generated by  $f^1, f^2, f^3$  includes the one generated by  $g^1, g^2, g^3$



- When  $n = m = 3$ ,  $f \succeq_B g$  iff the triangle generated by  $f^1, f^2, f^3$  includes the one generated by  $g^1, g^2, g^3$



# Quasiconvexity

- Observe that there is a permutation of  $l_3$  such that

$$g = \frac{4}{5} \cdot l_3 + \frac{1}{5} \cdot (l_3 \cdot P).$$

- If we impose **quasiconvexity**, with permutation invariance, we have

$$C(l_3) = C(l_3 \cdot P) \geq C\left(\frac{4}{5} \cdot l_3 + \frac{1}{5} \cdot l_3 \cdot P\right) = C(g).$$

# Quasiconvexity

- Observe that there is a permutation of  $l_3$  such that

$$g = \frac{4}{5} \cdot l_3 + \frac{1}{5} \cdot (l_3 \cdot P).$$

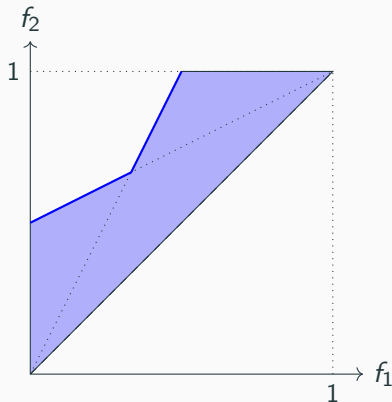
- If we impose **quasiconvexity**, with permutation invariance, we have

$$C(l_3) = C(l_3 \cdot P) \geq C\left(\frac{4}{5} \cdot l_3 + \frac{1}{5} \cdot l_3 \cdot P\right) = C(g).$$



# Quasiconvexity

- The following information cost function for binary experiments is not quasiconvex



$$\begin{aligned} C(f_1, f_2) &= \min \left\{ \frac{f_2}{f_1}, \frac{1-f_1}{1-f_2} \right\} \\ &= \min\{\alpha, \beta\} \end{aligned}$$

# Garbling Quasiconvexity

## Garbling Quasiconvexity

$C \in \mathcal{C}_m$  is *garbling-quasiconvex* if for all  $f \in \mathcal{E}_m$ , any finite collection of its garblings,  $\{g_1, \dots, g_n\}$ , and  $\lambda_0, \dots, \lambda_n \in [0, 1]$  with  $\sum_{i=0}^n \lambda_i = 1$ ,

$$C\left(\lambda_0 f + \sum_{i=1}^n \lambda_i g_i\right) \leq \max\{C(f), C(g_1), \dots, C(g_n)\}$$

## Theorem 4

$C \in \mathcal{C}_m$  is Blackwell monotone if and only if (i)  $C$  is permutation invariant; (ii)  $C$  is garbling-quasiconvex; and (iii) for all  $f \in \mathcal{E}_m$ ,

► Go Back

$$\langle \nabla^j C(f) - \nabla^i C(f), f \rangle \leq 0.$$

# Garbling Quasiconvexity

## Garbling Quasiconvexity

$C \in \mathcal{C}_m$  is *garbling-quasiconvex* if for all  $f \in \mathcal{E}_m$ , any finite collection of its garblings,  $\{g_1, \dots, g_n\}$ , and  $\lambda_0, \dots, \lambda_n \in [0, 1]$  with  $\sum_{i=0}^n \lambda_i = 1$ ,

$$C\left(\lambda_0 f + \sum_{i=1}^n \lambda_i g_i\right) \leq \max\{C(f), C(g_1), \dots, C(g_n)\}$$

## Theorem 4

$C \in \mathcal{C}_m$  is Blackwell monotone if and only if (i)  $C$  is permutation invariant; (ii)  $C$  is garbling-quasiconvex; and (iii) for all  $f \in \mathcal{E}_m$ ,

[► Go Back](#)

$$\langle \nabla^j C(f) - \nabla^i C(f), f \rangle \leq 0.$$

## Proof of Theorem 3

### [Sublinearity $\Rightarrow$ Blackwell Monotonicity]

- From sublinearity, we can show that  $C$  is convex.
- Consider the garbling of replacing  $s_j$  to  $s_k$  with prob.  $\epsilon$ :

$$\begin{aligned}\Delta C &= \psi(f^k + \epsilon \cdot f^j) + \psi((1 - \epsilon)f^j) - [\psi(f^k) + \psi(f^j)] \\ &= \psi(f^k + \epsilon \cdot f^j) + (1 - \epsilon) \cdot \psi(f^j) - \psi(f^k) - \psi(f^j) \\ &= \psi(f^k + \epsilon \cdot f^j) - \psi(f^k) -\end{aligned}$$

## Proof of Theorem 3

### [Sublinearity $\Rightarrow$ Blackwell Monotonicity]

- From sublinearity, we can show that  $C$  is convex.
- Consider the garbling of replacing  $s_j$  to  $s_k$  with prob.  $\epsilon$ :

$$\begin{aligned}\Delta C &= \psi(f^k + \epsilon \cdot f^j) + \psi((1 - \epsilon)f^j) - [\psi(f^k) + \psi(f^j)] \\ &= \psi(f^k + \epsilon \cdot f^j) + (1 - \epsilon) \cdot \psi(f^j) - \psi(f^k) - \psi(f^j) \\ &= \psi(f^k + \epsilon \cdot f^j) - \psi(f^k) -\end{aligned}$$

## Proof of Theorem 3

### [Sublinearity $\Rightarrow$ Blackwell Monotonicity]

- From sublinearity, we can show that  $C$  is convex.
- Consider the garbling of replacing  $s_j$  to  $s_k$  with prob.  $\epsilon$ :

$$\begin{aligned}\Delta C &= \psi(f^k + \epsilon \cdot f^j) + \psi((1 - \epsilon)f^j) - [\psi(f^k) + \psi(f^j)] \\ &= \psi(f^k + \epsilon \cdot f^j) + (1 - \epsilon) \cdot \psi(f^j) - \psi(f^k) - \psi(f^j) \\ &= \psi(f^k + \epsilon \cdot f^j) - \psi(f^k) - \epsilon \cdot \psi(f^j)\end{aligned}$$

## Proof of Theorem 3

### [Sublinearity $\Rightarrow$ Blackwell Monotonicity]

- From sublinearity, we can show that  $C$  is convex.
- Consider the garbling of replacing  $s_j$  to  $s_k$  with prob.  $\epsilon$ :

$$\begin{aligned}\Delta C &= \psi(f^k + \epsilon \cdot f^j) + \psi((1 - \epsilon)f^j) - [\psi(f^k) + \psi(f^j)] \\ &= \psi(f^k + \epsilon \cdot f^j) + (1 - \epsilon) \cdot \psi(f^j) - \psi(f^k) - \psi(f^j) \\ &= \psi(f^k + \epsilon \cdot f^j) - \psi(f^k) - \psi(\epsilon \cdot f^j) \leq 0\end{aligned}$$

## Proof of Theorem 3

### [Blackwell Monotonicity $\Rightarrow$ Sublinearity]

1. **Positive homogeneity:** Note that  $\psi(0) = 0$ . For any  $k \in \mathbb{N}$ ,

$$[\hat{f}, 0, \dots, 0, 1 - \hat{f}] \sim_B [\hat{f}/k, \hat{f}/k, \dots, \hat{f}/k, 1 - \hat{f}] \Rightarrow \psi(\hat{f}) = k \psi(\hat{f}/k).$$

Then, for any  $(k, l) \in \mathbb{N}^2$ , we also have

$$\frac{l}{k} \psi(\hat{f}) = l \psi\left(\frac{\hat{f}}{k}\right) = \psi\left(\frac{l}{k} \hat{f}\right)$$

By density of  $\mathbb{Q}$  in  $\mathbb{R}$  and the continuity of  $\psi$ ,  $\psi(\alpha \hat{f}) = \alpha \psi(\hat{f})$  for all  $\alpha \in \mathbb{R}_+$

2. **Subadditivity:**

$$[\hat{f}, \hat{g}, 1 - \hat{f} - \hat{g}] \succeq_B [\hat{f} + \hat{g}, 0, 1 - \hat{f} - \hat{g}] \Rightarrow \psi(\hat{f}) + \psi(\hat{g}) \geq \psi(\hat{f} + \hat{g})$$



## **Application II: Bargaining and Information Acquisition**

---

# Chatterjee, Dong, Hoshino (2023)

- Consider a bargaining problem with information acquisition
  - Players: **S**eller and **B**uyer
  - State (**B**'s valuation):  $v \in \{L, H\}$  with  $H > L > 0$ 
    - Prior belief:  $\pi \equiv \Pr(v = H) \in (0, 1)$
  - Timing of the game
    1. Nature draws  $v$  and **S** observes  $v$
    2. **S** offers  $p$
    3. **B** costly acquires information about  $v$  and then accepts or rejects
- Chatterjee et al. focus on specific types of information acquisition
- We extend their analysis by allowing **B** to choose information flexibly

# Chatterjee, Dong, Hoshino (2023)

- Consider a bargaining problem with information acquisition
  - Players: **S**eller and **B**uyer
  - State (**B**'s valuation):  $v \in \{L, H\}$  with  $H > L > 0$ 
    - Prior belief:  $\pi \equiv \Pr(v = H) \in (0, 1)$
  - Timing of the game
    1. Nature draws  $v$  and **S** observes  $v$
    2. **S** offers  $p$
    3. **B** costly acquires information about  $v$  and then accepts or rejects
- Chatterjee et al. focus on specific types of information acquisition
- We extend their analysis by allowing **B** to choose information flexibly

# Chatterjee, Dong, Hoshino (2023): H-focused information

**B's cost:**  $\lambda \cdot c(f_H)$

**Result 1:** pooling eq'm

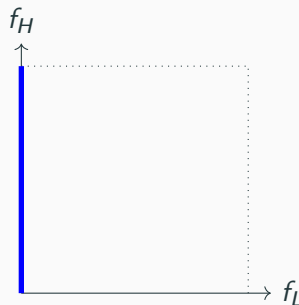
under H-focused signal structure, for any  $\lambda$ , there exists  $\epsilon > 0$  such that every equilibrium is a pooling equilibrium where

1. both types of **S** offer  $p^* \in [L, L + \epsilon)$ ;
2. **B** accepts without information acquisition.

Moreover,  $\epsilon \rightarrow 0$  as  $\lambda \rightarrow 0$ , thus, **B** extracts full surplus as  $\lambda \rightarrow 0$

## H-focused Information

	$s_L$	$s_H$
L	1	0
H	$1 - f_H$	$f_H$



# Chatterjee, Dong, Hoshino (2023): L-focused information

**B's cost:**  $\lambda \cdot c(1 - f_L)$

**Result 2:** almost-separating eq'm

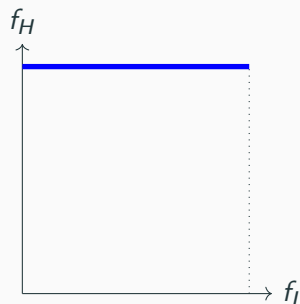
under L-focused signal structure, for any small enough  $\lambda$ , there exists an equilibrium such that

1. type H **S** offers  $p^* \approx H$ ;
2. type L **S** offers  $L$  with prob.  $1 - \epsilon$ ,  
 $p^*$  with prob.  $\epsilon$ ;
3. **B** acquires information and conditions her purchase decision on the signal realization

Moreover, **S's** payoff is close to  $v$  and **B's** payoff is close to zero

## L-focused Information

	$s_L$	$s_H$
L	$1 - f_L$	$f_L$
H	0	1



# Flexible Information Acquisition

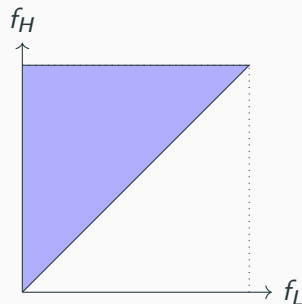
- We extend to the full domain and consider  $\lambda|f_2 - f_1|$  and  $\lambda(f_2 - f_1)^2$

**Result 1'**: when  $C(f_1, f_2) = \lambda|f_2 - f_1|$ , the unique equilibrium is the pooling equilibrium, and as  $\lambda \rightarrow 0$ , **B** extracts full surplus

**Result 2'**: when  $C(f_1, f_2) = \lambda(f_2 - f_1)^2$ , there exists an almost-separating equilibrium, and **S**'s payoff is close to  $v$  and **B**'s payoff is close to zero

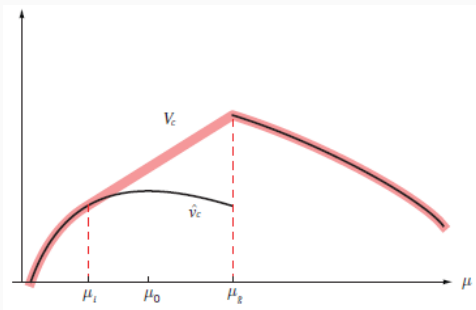
## Flexible Information

	$s_L$	$s_H$
L	$1 - f_L$	$f_L$
H	$1 - f_H$	$f_H$

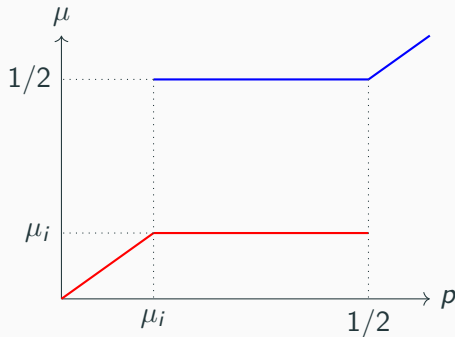


# Gentzkow, Kamenica (2014) Revisited

- Entropy cost:  $k \cdot \mathbb{E}_{\pi|p}[H(p) - H(\mu_s)]$  where  $H(\mu) \equiv -\sum_{\omega} \mu(\omega) \log(\mu(\omega))$ 
  - $p$  is prior, and  $\mu_i$  and  $\mu_g$  are posteriors from an experiment  $\pi$



Concavification



Posteriors