Strategic Concealment in Innovation Races*

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Abstract

Do firms in innovation races benefit from concealing intermediate discoveries? To study this, we introduce an innovation game where two firms dynamically allocate their resources across two distinct research and development (R&D) paths: (i) developing an innovative product with the currently available technology; (ii) conducting research to discover a faster technology for developing it. In equilibrium, firms may adjust their innovation paths if they observe rivals' progress. This creates an incentive for firms to conceal their interim discoveries, thereby slowing down the overall pace of social innovation.

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1 Introduction

The advantages of being the first firm to market an innovative product serve as a driving force behind technological progress. Even when the ultimate innovation goal is clear, firms often reach the marketing stage through different innovation paths. For instance, the development of COVID-19 vaccines has been accomplished through the messenger RNA (mRNA) technology by Moderna and Pfizer-BioNTech, and viral vector technology by Oxford-AstraZeneca and Janssen (Johnson&Johnson). Each of these routes may require the completion of a series of critical breakthroughs. Thus, firms' innovation strategies are often intertwined with the progress of their competitors. This interplay raises two key questions. First, if a firm observes its rivals' progress, will it adjust its innovation path? Second, if a firm makes progress that is not observable by its rivals, does the firm benefit from keeping this progress secret?

To answer these questions, we consider a model in which two firms race to develop an innovative product. The first firm to successfully develop the product obtains a fixed reward, e.g., a transitory flow of monopoly profit. The innovative product can be developed using two alternative technologies. The first path is to develop the product with an *old* technology, which is available from the beginning of the game. This path to innovation requires a single breakthrough.[1][2] Alternatively, firms can conduct *research* to discover a *new* technology, which is not available at the beginning of the game, but increases the rate of development rate once discovered.[3][4] This route of innovation requires two breakthroughs:

[1]In the Covid-19 vaccine example, the viral vector technology can be considered as an old technology. It was used during recent disease outbreaks including the 2014-2016 Ebola outbreak in West Africa. Many pharmaceutical firms had access to this methodology when the COVID-19 outbreak began. For more information, see the web page of the Centers for Disease Control and Prevention (CDC): https://www.cdc.gov/coronavirus/2019-ncov/vaccines/different-vaccines/viralvector.html.

[2]Another example of an innovation product is a full self-driving (FSD) vehicle. Tesla's approach towards developing an FSD vehicle is to use only cameras (Templeton, 2019). Since the camera technology is already on the cutting edge, the remaining step is to develop the FSD vehicle based on it.

[3]The mRNA technology can be considered as a new technology in the COVID-19 vaccine example. It was not in practical use before the COVID-19 outbreak. Thus, pharmaceutical firms had to acquire basic knowledge in order to employ this new methodology. The advantage of possessing this intermediate technology is that firms can develop vaccines in a laboratory by using readily available materials. Hence vaccines can be developed faster with mRNA technology than with older methods like viral vector. For more information, see the web page of the Centers for Disease Control and Prevention (CDC): https://www.cdc.gov/coronavirus/2019-ncov/vaccines/different-vaccines/mrna.html.

[4] In the FSD vehicle example, the affordable light detection and ranging (LIDAR) technology can be considered as a new technology. LIDAR is a laser radar that can provide extensive and reliable information surrounding a vehicle including an object's distance, size, position, and velocity if it is moving. Most FSD

first, the discovery of the new technology, and, second, the development of the product with it.

We consider two distinct informational settings: a firm's research progress—whether the firm has discovered the new technology or not—is publicly or privately available information. In the case where firms' research progress is public, a firm can base its choice of an innovation path not only on its own research progress but also on that of its competitors. We characterize the unique Markov perfect equilibrium across various parametric regions. When the development rate with the new technology is sufficiently higher than that with the old technology, firms engage in research independently of their rivals' progress until obtaining the new technology, namely a *research* strategy. In contrast, when the development rate of the new technology falls below a certain threshold, both firms always develop the product using the old technology, namely a *direct-development* strategy. In both of these cases, firms do not utilize information about their rivals' research progress. However, for intermediate development rates of the new technology, both firms initially conduct research; but once a firm observes that its rival has attained the new technology, it switches to developing with the old technology. We call this a *fall-back* strategy. This switch occurs due to two paths having distinct distributions of development times: while the research and development path may have a shorter expected development time, the old technology path is more likely to be completed within a short period. Under the fall-back strategy, a firm's choice of innovation paths depends on its rival's research progress. Thus, we can interpret this as a firm leveraging information about its competitor's progress to adjust its innovation strategy.

When a firm's research progress is private information, firms cannot base their innovation paths on their rivals' advancements. Consequently, the fallback strategy becomes unfeasible. However, firms can opt for a research strategy or a direct-development strategy, as these approaches do not require any information about rivals' progress. If a pair of research or direct-development strategies forms an equilibrium under the public information setting,

vehicle developers including Waymo—formerly the Google self-driving car project—use LIDAR combined with cameras. The main drawback of LIDAR is its current high cost. Thus, to develop a commercially viable FSD vehicle, firms first need to discover a way to make LIDAR more affordable. Once LIDAR becomes cost-effective, it will be relatively easy to develop a commercializable FSD vehicle. In this sense, successfully developing an FSD vehicle with the LIDAR technology can be understood as a route requiring two breakthroughs.

it will also be an equilibrium under the private information setting. This is because firms disregard any information about rivals' progress, even if it were publicly available. Therefore, when the development rate of the new technology is sufficiently high or low, firms will follow the same path as in the public information setting. In the intermediate case, however, firms have incentives to adjust their R&D strategies according to their rivals' progress. Although a firm does not observe its rival's progress, they form beliefs about whether its opponent has access to the new technology. We characterize the evolution of these beliefs as a function of the firms' R&D strategies. On one hand, as time passes and more research is conducted, it becomes more likely that a firm has obtained the new technology, namely the *duration effect*. On the other hand, since a firm with the new technology has a higher rate of development, the absence of an innovation indicates that it is less likely that the new technology has been discovered yet, namely the *still-in-the-race effect*. These two conflicting effects result in an equilibrium with two phases. In the first phase, firms without the new technology conduct research and the beliefs about rival have discovered the new technology increase over time. In the second stage, that starts at a deterministic time, firms without the new technology partially allocate their resources to developing with the old technology and conducting research, and beliefs remain constant.

The main challenge of equilibrium characterization lies in expressing a firm's problem as an optimal control problem, given a strategy of the rival. The solution is not solely dependent on beliefs regarding whether the rival possesses the new technology; it also hinges on the rival's future allocations, influencing the probability of winning the race. This interdependence introduces complexity in characterizing the set of all fixed points of the game's best response correspondence within the infinite-dimensional space of strategies. However, by concentrating on monotone development rates, we can derive structural properties of the best response correspondence. Specifically, we establish a single-crossing property of the relative incentives for research, utilizing this insight to characterize the unique equilibrium featuring monotone development rates.

Last, we extend our model by allowing firms to patent and license the new technology. In this extension, firms with the new technology can publicly apply for a patent and, when the patent is granted, the patent holder obtains the exclusive right to use the new technology. The patent holder can also license the technology to the rival for a fee. This enables us to investigate the trade-off between patenting and concealing the discovery of the new technology. The former analysis suggests that a firm, say Firm A, may adjust its innovation path once it notices that the rival firm, say Firm B, has discovered the new technology. This creates an incentive for Firm B to conceal its research progress. When Firm B discovers the new technology and decides not to patent it, i.e., keeps the discovery secret, it bears the risk of losing the right to use it in the event that Firm A independently discovers the new technology and files a patent. In this case, however, Firm B can contest the Firm A's patent application using a trade secret protection argument. We parametrize the probability of the patent challenge being successful when the new technology was already obtained by the challenger, namely the level of trade secret protection.[5][6]

We first show that if a firm obtains a patent, there is always a license fee such that both licensor and licensee are willing to accept. Thus, in the spirit of Coase (1960), both firms obtain access to the new technology, which is socially efficient. Moreover, we find that when firms' research progress is public information, firms benefit from patenting the new technology independently of the trade secret protection level.

When the firms' research progress is private information, however, applying for a patent would reveal information to competitors. Under these circumstances, firms may forgo patenting in order to keep their advancements concealed. We show that this decision crucially depends on the level of trade secret protection and the *stake*, namely the size of the reward of winning the race relative to the cost associated with the duration of the race. Focusing on parameters for which the still-in-race effect is sufficiently strong, we show that when the stake and trade secret protection are sufficiently high, there is an equilibrium in which firms do not patent the new technology and there is no licensing, even when the patent holders have all the bargaining power in the licensing negotiations. Intuitively, this failure of the Coase Theorem is explained because the efficiency gains associated with licensing of the new technology, and that can be captured by the patent holder through the license fee, are

[5]We focus on first-to-file patent systems with some level of trade secret protection, such as the protection given by the defense to infringement based on prior commercial use (US Code §273). However, it is possible to extend our framework to capture *first to invent* patent systems as well.

[6]For more information about trade secrets and patents, see the World Intellectual Property Organization's webpage: `https://www.wipo.int/about-ip/en/`. Also, see Lobel (2013) for examples.

associated to a reduction in the expected duration of the race. When the reward of winning the race is large compared with the costs of a longer race, firms prefer to conceal their advancements, which maximizes their chances of winning the race to the detriment of slowing down the overall speed of innovation.

Related Literature

This paper primarily contributes to the literature on patent vs. secrecy by introducing a novel incentive to conceal a firm's discovery: hindering its rival's strategic response.[7] Previous studies mainly focused on the limited protection power of patents. For example, the seminal article by Horstmann et al. (1985) posits that "patent coverage may not exclude profitable imitation." Thus, in their framework, the main reason why a firm may choose secrecy over a patent is not to be imitated.[8] Another limitation of a patent is that it expires in a finite time. For instance, Denicolò and Franzoni (2004) consider a framework where a patent gives the patenting firm monopoly power only for a certain period of time (and no profit after expiration), whereas secrecy can give indefinite monopoly power to a firm but it can be leaked or duplicated by a rival with some probability. On the contrary, in this paper, we abstract from the restrictions of patents and focus analysis on the potential advantages of concealment.

Another hallmark of this paper is its consideration of 'interim' discoveries. Therefore, it is naturally related to the literature on licensing of interim R&D knowledge, e.g., Bhattacharya et al. (1992); d'Aspremont et al. (2000); Bhattacharya and Guriev (2006); Spiegel (2008). In these papers it is assumed that firms already know which of them has superior knowledge, i.e., the firm that will license the technology is exogenously given. Unlike in those studies, we allow firms to choose when to license (and even allow them not to license), i.e., the licensing decision is endogenous.

We also contribute to the innovation literature by introducing a model with two char-

[7] There exists an extensive body of literature addressing both the empirical and theoretical aspects of the patent vs. secrecy discussion. A comprehensive overview of this literature can be found in the excellent survey paper authored by Hall et al. (2014).

[8] Many subsequent papers study the imitation threat and potential patent infringement, e.g., Gallini (1992); Takalo (1998); Anton and Yao (2004); Kultti et al. (2007); Kwon (2012); Zhang (2012); Krasteva (2014); Krasteva et al. (2020).

acteristics. First, there are different avenues towards innovation: developing with the old technology and doing research for the new technology. Second, one of the paths involves multiple stages: once a firm discovers the new technology, then the firm develops the innovative product with it.

With respect to the first characteristic, there is a recent branch of the literature that studies races where there are different routes to achieve a final objective. Das and Klein (2020) and Akcigit and Liu (2016) study a patent race where two firms compete for a breakthrough and there are two methods to get the breakthrough: a safe method and a risky method. In Das and Klein (2020) the safe method has a known constant arrival intensity while the risky method has an unknown constant arrival intensity. In Akcigit and Liu (2016), instead, the safe method has a known payoff associated with breakthrough arrival, while there is uncertainty about the payoff if the risky method is used. In this paper, firms face no uncertainty about whether the innovation is feasible. Instead, they are uncertain whether their rivals possess the new and faster technology. Another related paper concerning this characteristic is the study by Bryan and Lemus (2017). They introduce a general model of direction of innovation using acyclic graphs, where a node denotes a set of available inventions in society, and an edge represents a feasible innovation path. They assume that whenever a new invention is discovered, the first firm to invent it receives the prize, and the access to the invention is given to all the other firms. In contrast, in our model, interim discoveries can remain private.

The second characteristic, multi-stage innovation, is also widely studied in the literature, e.g., Scotchmer and Green (1990); Denicolò (2000); Green and Taylor (2016); Song and Zhao (2021). Our paper shares the framework with these in that we use two sequential Poisson discovery processes and ask whether a firm would patent the first discovery or not. A feature setting apart from their works is that there is another path that only requires one but slower breakthrough toward innovation. This feature connects our model to Carnehl and Schneider (2022) and Kim (2022) in the sense that players can choose between a sequential approach—which requires two breakthroughs—and a direct approach, which requires only one breakthrough, but its riskier or slower.[9] Our model mainly differs from theirs in that multiple

[9] In Carnehl and Schneider (2022), an agent is uncertain whether the direct approach is feasible or not,

players compete by choosing between these approaches, whereas Carnehl and Schneider (2022) considers a problem by a single decision maker and Kim (2022) studies a contracting setup between a principal and an agent. In their studies, a key factor for a player to choose the direct approach is a deadline that is either exogenously given or endogenously determined to reduce moral hazard. In contrast to these, a deadline is not involved in our model. Rather, the race with the rival firm may induce a firm to develop with the old technology, which can be considered as a direct approach.

This paper is related to the recent studies on information disclosure in priority races, e.g., Hopenhayn and Squintani (2016); Bobtcheff et al. (2017).[10] In those papers, once a firm makes a breakthrough, the innovation value grows as time passes until one of the firms files a patent. Thus, firms face a tradeoff between disclosing to claim the priority and delaying in order to grow the innovation value. On the contrary, in this paper, the value of innovation is fixed and the discovery of the new technology only allows the firm to develop the innovative product faster. Therefore, a firm may delay the disclosure purely to confound the rival's R&D decisions.

Lastly, a closely related study is the recent paper by Chatterjee et al. (2023). They also explore a disclosure problem concerning an intermediate research finding in a two-step project. The key distinction lies in their assumption of an exogenous payoff from disclosing the intermediate discovery, whereas in our paper, the payoff is endogenously determined, considering the option to develop with the old technology. As in our paper, they also find that a high reward of the final discovery may induce firms to conceal their intermediate discoveries, resulting in socially inefficiency.

i.e., this approach is risky. On the other hand, in Kim (2022), there is no uncertainty on the feasibility of the direct approach, but its completion rate is slower than the ones for the sequential approach. In this sense, our framework is closer to Kim (2022).

[10] There is a strand of literature on strategic disclosure, e.g., Lichtman et al. (2000); Baker and Mezzetti (2005); Gill (2008); Baker et al. (2011); Ponce (2011). These works are well summarized in Section 3.3 of Hall et al. (2014).

2 Model

We consider a race between two firms, A and B, trying to develop an *innovative product*. Time is continuous and infinite: $t \in [0, \infty)$. Firms can develop the innovative product using either *old* or *new* technology, each with a different development speed. At the outset of the race, both firms have access to an old technology, but they can gain access to a new technology by conducting research.

Each firm owns one unit of resources per unit of time, which can be allocated for either conducting research to discover the new technology or developing the innovative product. When a firm gains access to the new technology, it directs all its resources towards product development, resulting in a development rate of λ_H. When Firm i does not yet possess the new technology, it allocates a fraction $\sigma_t^i \in [0, 1]$ to 'research' at time t. Then, $1 - \sigma_t^i$ is the amount of resources that Firm i allocates to 'develop' the innovative product using the old technology, and the product can be stochastically developed at rate $\lambda_L \cdot (1 - \sigma_t^i)$. In addition, Firm i stochastically discovers the new technology at rate $\sigma_t^i \cdot \mu$, where μ is a constant parameter. Firm i can observe its own discovery of the new technology. We consider two different settings regarding whether Firm i can observe Firm j's research progress, whether it has discovered the new technology or not. The parameters μ, λ_L, and λ_H are positive.

The race ends once one of the firms develops the innovative product. During the race, firms pay a flow cost $c > 0$. The first firm to develop the innovative product receives a lump-sum reward worth Π.[11] Firms do not discount the future and maximize their expected total payoff.[12] The successful development of the innovative product is publicly observable. Thus, firms always know whether they are still on the race. However, firms do not observe their opponents' resource allocations over time.

For the rest of the paper, we make the following two parametric assumptions:

$$\Pi - \frac{c}{\mu} - \frac{c}{\lambda_H} > \Pi - \frac{c}{\lambda_L} > 0. \tag{2.1}$$

[11] We model the race as winner-takes-all competition. This payoff structure has been commonly used in the innovation race literature, e.g., Loury (1979); Lee and Wilde (1980); Denicolò and Franzoni (2010).

[12] With discounting the firms are not risk-neutral over the duration of the race conditional on the outcome. This complicates the closed-form solutions without affecting the qualitative results of the paper.

The first inequality states that when there is only one firm, conducting research and developing with the new technology is more efficient than developing with the old technology. Note that this condition is equivalent to $\frac{1}{\mu} + \frac{1}{\lambda_H} < \frac{1}{\lambda_L}$, implying that in expectation, the product can be developed faster by conducting research and developing with the new technology. Then, the second inequality implies that developing with the old technology is profitable.[13]

3 Benchmark: Constant Development Rate

As a benchmark, imagine a scenario where Firm j does not engage in the resource allocation problem and the rate of development is held constant at λ. Solving this benchmark will provide valuable insights for the main analysis of the paper.

Suppose that Firm i has already discovered the new technology. Then, Firm i develops with the rate λ_H and Firm j develops with the rate λ. Firm i's probability of winning the race is $\frac{\lambda_H}{\lambda_H+\lambda}$ and the expected duration of the remaining race is $\frac{1}{\lambda_H+\lambda}$. Therefore, Firm i's expected payoff is given by

$$\mathcal{V}^1_\lambda \equiv \frac{\lambda_H}{\lambda_H + \lambda} \cdot \Pi - \frac{1}{\lambda_H + \lambda} \cdot c = \frac{\lambda_H \Pi - c}{\lambda_H + \lambda}. \tag{3.1}$$

Now suppose that Firm i has yet to discover the new technology. Consider research allocation strategies, which allocate a fixed amount of resources to research until either the new technology is discovered or the race ends, i.e., for some $x \in [0,1]$, $\sigma^i_t = x$ for all $t \geq 0$. In Appendix B, we show that it is without loss to focus on these strategies (Lemma B.2). When Firm i allocates x amount of resources towards research, there are three potential outcomes: (i) Firm i develops the product with the old technology at rate $\lambda_L(1-x)$; (ii) Firm i discovers the new technology at rate μx; (iii) Firm j develops the product at rate λ. In the first scenario, Firm i wins the race and receives Π, and the probability of this event happening is $\frac{\lambda_L(1-x)}{\lambda_L(1-x)+\mu x+\lambda}$. In the second scenario, Firm i enters the post-research phase, and its expected payoff is $\mathcal{V}^1_\lambda$. The probability of this event occurring is $\frac{\mu x}{\lambda_L(1-x)+\mu x+\lambda}$.

[13] This assumption leads us to abstract away from firms' exit decisions: the flow expected payoff of staying in the race is at least $\lambda_L \Pi$, which is greater than the flow cost (c). If this assumption is violated, firms completely disregard the old technology and, therefore, there is no strategic choice of innovation path.

In the third scenario, Firm i receives nothing, and the probability of this event happening is $\frac{\lambda}{\lambda_L(1-x)+\mu x+\lambda}$. The expected duration of the game is $\frac{1}{\lambda_L(1-x)+\mu x+\lambda}$. Therefore, Firm i's expected payoff is given by

$$u(x) \equiv \frac{\lambda_L(1-x)\cdot\Pi + \mu x \cdot \mathcal{V}^1_\lambda - c}{\lambda_L(1-x) + \mu x + \lambda}. \tag{3.2}$$

After taking the first derivative of u, with some algebra, we can derive that

$$u'(x) = \frac{\lambda_L(\lambda\Pi + c)(\lambda_\star - \lambda)}{(\lambda+\lambda_H)(\lambda + (1-x)\lambda_L + x\mu)^2} \tag{3.3}$$

where

$$\lambda_\star \equiv \mu\lambda_H\left(\frac{1}{\lambda_L} - \frac{1}{\mu} - \frac{1}{\lambda_H}\right) > 0.\text{[14]} \tag{3.4}$$

Therefore, from $x \in [0,1]$, we have that $x = 1$ is optimal if $\lambda < \lambda_\star$, and $x = 0$ is optimal if $\lambda > \lambda_\star$. If $\lambda = \lambda_\star$, any $x \in [0,1]$ is optimal. The following proposition formally states this result.

Proposition 3.1. *Suppose that Firm j has a constant development rate λ.*

(a) When $\lambda < \lambda_\star$, Firm i's optimal resource allocation strategy is to conduct research: $\sigma^i_t = 1$ for all $t \in \mathbb{R}_+$;

(b) When $\lambda > \lambda_\star$, Firm i's optimal resource allocation strategy is to develop with the old technology: $\sigma^i_t = 0$ for all $t \in \mathbb{R}_+$;

(c) when $\lambda = \lambda_\star$, Firm i is indifferent between conducting research and developing with the old technology.

To illustrate the intuition behind this proposition, it is useful to understand the difference between the probability distributions of development times for conducting research and developing with the old technology. In Figure 1, the red dotted curve represents the probability distribution of Firm i's development time when it develops with the old technology. The blue solid curve is the probability distribution of the development time when Firm i conducts

[14] Note that $\lambda_\star$ is a function of λ_L, μ and λ_H, but we suppress it to ease the notation.

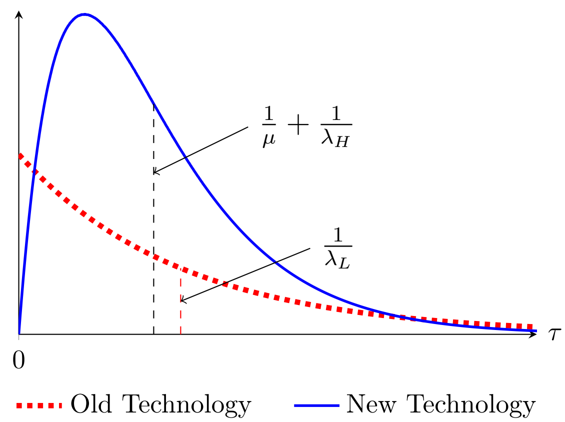


Figure 1: Probability distribution functions of a firm's development time

research and then develops with the new technology. By the parametric assumption in the previous section, the expected development time under research is shorter than that under development with the old technology. However, as illustrated in the figure, within a short time horizon, a firm may have a higher probability of successfully developing the product when using the old technology. This is because the firm only needs one breakthrough to develop with the old technology, whereas to develop the product with the new technology, it requires two breakthroughs: discovering the new technology and developing the product with it. Therefore, when a firm faces an opponent who has a high development rate, the firm may choose to develop with the old technology since it would give a higher chance of winning the race.

4 Public Information Setting

We explore a setting where firms' research progress is publicly available information. In this case, the set of firms that have successfully obtained the new technology is common knowledge, and we represent it as a state variable denoted by $\omega \in \Omega \equiv \{\{A, B\}, \{A\}, \{B\}, \emptyset\}$.

We focus on firms' Markov strategies. Specifically, Firm i's Markov strategy is defined as $\mathbf{s}^i : \Omega \to [0, 1]$, where $\mathbf{s}^i(\omega)$ denotes the amount of resources allocated by Firm i to research in state ω. Recall that a firm allocates all its resources to development once it has the new technology, i.e., $\mathbf{s}^i(\omega) = 0$ for all $i \in \omega$. A pair of Markov strategies $(\mathbf{s}^A, \mathbf{s}^B)$ constitutes a

Markov perfect equilibrium (MPE) if, for any given state, each firm's strategy is the best response to the opponent's strategy.

Next, we introduce three benchmark Markov strategies.

Definition 1. (a) The *research strategy* $\mathbf{s}_R^i$ for Firm i fully allocates resources to research regardless of the opponent's progress ($\mathbf{s}_R^i \equiv \mathbb{1}_{\{\omega | i \notin \omega\}}$).[15]

(b) The *fall-back strategy* $\mathbf{s}_F^i$ fully allocates resources to research if neither firm has the new technology. If one of the firms has obtained the new technology, it fully allocates resources to development ($\mathbf{s}_F^i \equiv \mathbb{1}_{\{\emptyset\}}$).

(c) The *direct-development strategy* $\mathbf{s}_D^i$ fully allocates the resources to development regardless of the state ($\mathbf{s}_D^i \equiv 0$).

Now, we demonstrate that one of these strategies constitutes an MPE depending on the parameters. First, assume that $\lambda_\star > \lambda_H$. Note that the development rate cannot exceed λ_H, thus, it is always lower than $\lambda_\star$. Referring to (a) in Proposition 3.1, we can infer that firms would conduct research regardless of the rival's strategy. Therefore, both firms employing the research strategy would constitute an equilibrium.

Next, suppose that $\lambda_L > \lambda_\star$. If a firm develops with the old technology, its development rate is λ_L, which is greater than $\lambda_\star$. Then, by (b) of Proposition 3.1, the rival firm would also develop with the old technology. Therefore, we can guess that both firms adopting the direct-development strategy would constitute an equilibrium.

Last, assume that $\lambda_H > \lambda_\star > \lambda_L$. Consider the case where only Firm j has discovered the new technology, i.e., $\omega = \{j\}$. Then, Firm j will develop with the new technology, i.e., the development rate of Firm j is λ_H, which is higher than $\lambda_\star$. Then, by (b) of Proposition 3.1, Firm i develops with the old technology. Since $\lambda_\star > \lambda_L$, the direct-development strategy cannot constitute an equilibrium. Thus, among the benchmark strategies, the fall-back strategy is the only candidate for an equilibrium strategy under this parametric region.

The following theorem shows that the above benchmark Markov strategies are unique MPE strategies within their respective parametric regions.

[15] The function $\mathbb{1}_X$ is an indicator function: $\mathbb{1}_X(\omega) = 1$ if $\omega \in X$ and $\mathbb{1}_X(\omega) = 0$ if $\omega \notin X$.

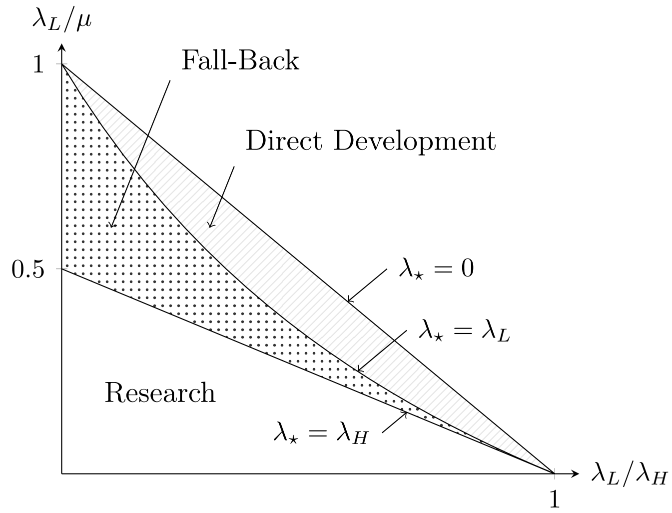


Figure 2: Markov Perfect Equilibrium in the Public Information Setting

Theorem 1. *Suppose that firms' research progress is public information. Then, the Markov perfect equilibrium is uniquely characterized as follows:*

(a) if $\lambda_\star > \lambda_H$, both firms play their respective research strategies $(\mathbf{s}_R^A, \mathbf{s}_R^B)$;

(b) if $\lambda_H > \lambda_\star > \lambda_L$, both firms play the fall-back strategies $(\mathbf{s}_F^A, \mathbf{s}_F^B)$;

(c) if $\lambda_L > \lambda_\star$, both firms play the direct-development strategies $(\mathbf{s}_D^A, \mathbf{s}_D^B)$.

It is worth noting that we do not limit our analysis solely to symmetric equilibrium; instead, symmetry emerges as a result of our analysis. Figure 2 illustrates the relevant parametric regions in the above theorem. Note that $\lambda_\star > 0$ is equivalent to $\frac{\lambda_L}{\mu} + \frac{\lambda_L}{\lambda_H} < 1$, which confines the parametric region to the triangular area depicted in Figure 2. With some algebra, we can show that $\lambda_\star > \lambda_H$ is equivalent to $1 > \frac{\lambda_L}{\lambda_H} + 2 \cdot \frac{\lambda_L}{\mu}$. This gives the transparent triangle-shaped region where firms employ the research strategy. Next, we can also show that $\lambda_\star < \lambda_L$ is equivalent to

$$\frac{\lambda_L}{\mu} > \frac{1 - \frac{\lambda_L}{\lambda_H}}{1 + \frac{\lambda_L}{\lambda_H}}.$$

With $\lambda_\star > 0$, it gives the shaded region where firms employ the direct-development strategy. In the remaining dotted region, $\lambda_H > \lambda_\star > \lambda_L$ holds and firms use the fall-back strategy.

5 Private Information Setting

In this section, we consider the private information framework, in which firms do not observe whether their opponents have the new technology. In this setting, as before, a firm with the new technology fully allocates the resources to development. However, a firm without the new technology can only condition its resource allocation on the calendar time t. An *allocation policy* is a right-continuous function $\boldsymbol{\sigma} : \mathbb{R}_+ \to [0, 1]$ that represents the research allocation at a given time, conditional on not having obtained the new technology. We denote $\mathcal{S}$ as the set of allocation policies.

New Technology Access and Development Rate

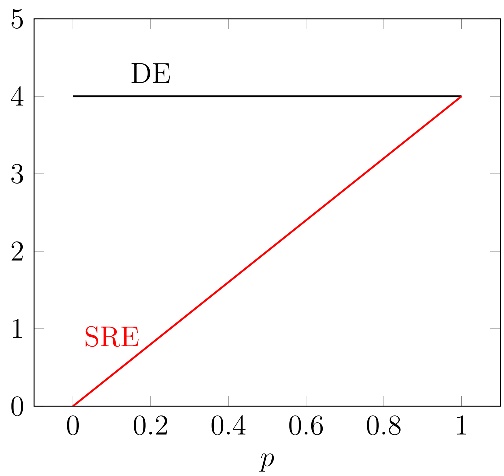
Before analyzing the strategic aspects of this problem, it is useful to note that, for each allocation policy $\boldsymbol{\sigma} \in \mathcal{S}$, there is an associated distribution of the time of new technology discovery and the time of product development.[16] Let τ_D be the (arrival) time of product development and τ_R be the (arrival) time of new technology discovery. For example, a firm that follows a policy $\boldsymbol{\sigma} = 0$ will never discover the new technology in finite time, and the development time τ_D is exponentially distributed with parameter λ_L. For a firm that follows a policy $\boldsymbol{\sigma} = 1$, τ_R is exponentially distributed with parameter μ, and τ_D is the sum of two exponentially distributed variables with parameters μ and λ_H.

Since new technology discovery affects the rate of product development, the random variables τ_R and τ_D are interdependent. In this section, we derive two important objects for any policy $\boldsymbol{\sigma}$: the probability of access to the new technology $\mathbf{p}_{\boldsymbol{\sigma}}$, and the rate of development $\mathbf{h}_{\boldsymbol{\sigma}}$. Formally, $\mathbf{p}_{\boldsymbol{\sigma}}$ represents the probability that a firm following allocation $\boldsymbol{\sigma}$ obtains the new technology by time t, conditional on not having developed the product yet. $\mathbf{p}_{\boldsymbol{\sigma}}$ can be expressed in terms of τ_R and τ_D as follows:

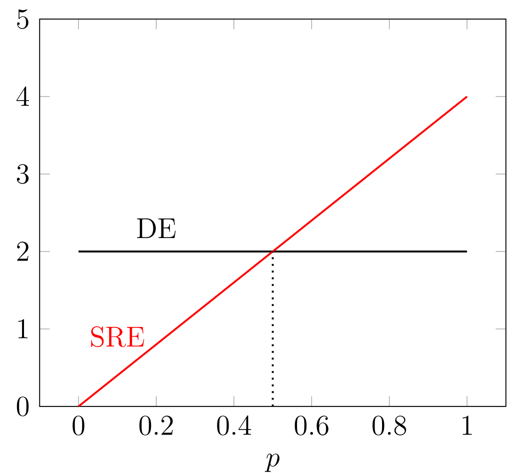
$$\mathbf{p}_{\boldsymbol{\sigma}}(t) := \Pr[\tau_R < t < \tau_D \mid \tau_D > t]. \tag{5.1}$$

The following proposition characterizes for any $\boldsymbol{\sigma} \in \mathcal{S}$, the evolution of $\mathbf{p}_{\boldsymbol{\sigma}}$ over time.

[16] We provide mathematical details about these arrival times including survival function $S^D_{\sigma}(t) \equiv \Pr[t < \tau_D]$ in Appendix D.2.



(a) $\sigma = 1$, $\gamma = \rho = 1/2$ and $\lambda_L = 2$.



(b) $\sigma = 1$, $\gamma = 1$, $\rho = 1/2$ and $\lambda_L = 2$.

Figure 3: Duration Effect (Black) and Still-in-the-Race Effect (Red)

Proposition 5.1. *For any allocation policy $\boldsymbol{\sigma} \in \mathcal{S}$, the conditional probability $\mathbf{p}_{\boldsymbol{\sigma}}(t)$ satisfies the initial condition $\mathbf{p}_{\boldsymbol{\sigma}}(0) = 0$ and evolves according to the differential equation $\dot{\mathbf{p}}_{\boldsymbol{\sigma}}(t) = \delta(\mathbf{p}_{\boldsymbol{\sigma}}(t), \boldsymbol{\sigma}(t))$, where*

$$\delta(p, \sigma) := \mu \cdot \sigma \cdot (1-p) - (\lambda_H - (1-\sigma) \cdot \lambda_L) \cdot p \cdot (1-p). \tag{5.2}$$

The proof of Proposition 5.1 is provided in Appendix D.3.1. The function δ highlights two distinct effects of the resource allocation $\sigma(t)$ on the evolution of $\mathbf{p}_{\boldsymbol{\sigma}}$, captured by the two terms in (5.2). First, if the firm does not have the new technology—which happens with probability $(1-p)$—the new technology is discovered at rate $\mu \cdot \sigma$. We dub the effect of this arrival rate the *duration effect* (DE). On the other hand, the lack of development success indicates that it is less likely that the firm has the new technology. This second effect, which we dub the still-in-the-race effect (SRE), is reflected in the second term.[17] Notice that the SRE is proportional to $\lambda_H - (1-\sigma)\lambda_L$, which is the difference in the rate of development of the firm with and without the new technology.

In Figure 3, we illustrate the duration and still-in-the-race effects for $\boldsymbol{\sigma} = \mathbf{1}$.[18] Specifically,

[17]Similar types of belief updating can be found in the strategic experimentation literature, e.g., Keller et al. (2005); Bonatti and Hörner (2011). The main difference is that, in that literature, the agents form beliefs about whether a project is good or bad. In this paper, on the other hand, firms only form beliefs about the technology access of the rival.

[18]The function $\mathbf{0} : \mathbb{R}_+ \to \{0,1\}$ is $\mathbf{0}(t) = 0$ for all $t \in \mathbb{R}_+$, and the function $\mathbf{1} : \mathbb{R}_+ \to \{0,1\}$ is $\mathbf{1}(t) = 1$ for

we provide the graphs of each of the effects effect divided by $(1-p)$: μ (DE), $\lambda_H p$ (SRE). In Figure 3a, we depict the case where $\mu = \lambda_H$. Observe that, in this case, the duration effect is larger than the still-in-the-race effect for every p. If we fix λ_H and increase μ, we observe that the duration effect continues to dominate the still-in-the-race effect. Hence, when $\mu > \lambda_H$, the conditional probability of having the new technology by time t when the firm follows policy $\boldsymbol{\sigma} = \mathbf{1}$, $\mathbf{p_1}(t)$, converges to 1.[19] On the other hand, in Figure 3b, we illustrate the case where $\mu < \lambda_H$. In this case, the duration effect is greater than the still-in-the-race effect only when $p < \mu/\lambda_H$. Thus, $\mathbf{p}_\sigma(t)$ does not exceed μ/λ_H for any $\boldsymbol{\sigma} \in \mathcal{S}$.

The access to the new technology and the allocation of resources determine the development rate of the firm. We can define the development rate of a policy as follows.

Lemma 5.1. *Given a policy $\boldsymbol{\sigma} \in \mathcal{S}$, the associated* development rate function $\mathbf{h}_{\boldsymbol{\sigma}}$ *is given by $\mathbf{h}_{\boldsymbol{\sigma}}(t) = \xi(\mathbf{p}_{\boldsymbol{\sigma}}(t), \boldsymbol{\sigma}(t))$ where*

$$\xi(p, \sigma) := p \cdot \lambda_H + (1-p) \cdot (1-\sigma) \cdot \lambda_L. \tag{5.3}$$

The first term of (5.3) captures that a firm with the new technology develops at rate λ_H. If the firm does not have the new technology, they only develop at a rate $(1-\sigma)\lambda_L$.

Expected Payoffs and Solution Concept

Fixing the firm and opponent's allocation policies as $\boldsymbol{\sigma}$ and $\hat{\boldsymbol{\sigma}}$ respectively, let τ_D and $\hat{\tau}_D$ be the random variables that represent the arrival of successful developments. The expected payoff of the firm is therefore:

$$\mathcal{U}(\boldsymbol{\sigma}, \hat{\boldsymbol{\sigma}}) = \Pr[\tau_D < \hat{\tau}_D] \cdot \Pi - c \cdot \mathbb{E}[\tau_D \wedge \hat{\tau}_D].$$

In Appendix D.2, we show that we can write this expected payoff in terms of the associated development rates, $h_{\boldsymbol{\sigma}}$ and $h_{\hat{\boldsymbol{\sigma}}}$, as follows:

$$\mathcal{U}(\boldsymbol{\sigma}, \hat{\boldsymbol{\sigma}}) = \int_0^\infty e^{-\int_0^t \{h_{\boldsymbol{\sigma}}(s) + h_{\hat{\boldsymbol{\sigma}}}(s)\}\, ds} \left[h_{\boldsymbol{\sigma}}(t) \cdot \Pi - c\right] dt. \tag{5.4}$$

all $t \in \mathbb{R}_+$,

[19] For the closed form expression of $\mathbf{p_1}$, see Lemma D.1 in Appendix D.3

Intuitively, the exponential term captures the probability that no firm has developed the innovative product by time t, i.e. the probability that the race is still ongoing. In that case, the firm captures an expected flow payoff equal to $h_{\boldsymbol{\sigma}}(t) \cdot \Pi$, because it might develop the innovative product, and pays the flow cost c.

As in the literature on dynamic games with unobservable actions (e.g., Bonatti and Hörner, 2011), we aim to characterize the Nash equilibria (NE) in this game. The main challenge in solving for Nash equilibrium arises from the fact that Firm A's best response at time t is contingent not only on Firm B's past resource allocations—determining Firm A's belief regarding whether Firm B possesses the new technology—but also on Firm B's future allocations, which determine the continuation payoffs. To overcome this challenge, we focus on the NE with the following property.

Definition 2. An allocation policy $\boldsymbol{\sigma} \in \mathcal{S}$ exhibits the *monotone development rate (MDR) property* if $h_{\boldsymbol{\sigma}}$ is weakly increasing. An allocation policy profile $(\boldsymbol{\sigma}_A, \boldsymbol{\sigma}_B)$ is a *Nash equilibrium with monotone development* (MDNE) if (i) $(\boldsymbol{\sigma}_A, \boldsymbol{\sigma}_B)$ is a Nash equilibrium; and (ii) $\boldsymbol{\sigma}_A$ and $\boldsymbol{\sigma}_B$ are MDR.[20]

Focusing on Firm A satisfying the MDR property is convenient for two reasons. First, it restricts the feasible set of allocation policies for Firm A. In Appendix D.1, we show that with the MDR property, if Firm A fully allocates its resources to the development with the old technology at time s ($\boldsymbol{\sigma}_A(s) = 0$), it implies that Firm A has fully allocated its resources to development with the old technology ($\boldsymbol{\sigma}_A(t) = 0$ for all $t < s$) (Proposition D.2). Additionally, it provides a simple characterization of the set of best response allocation policies for Firm B. Proposition D.3 shows that when Firm B faces an opponent with the MDR property, under certain conditions, the objective of the optimal control problem has the single-crossing property in resource allocation and time (from negative to positive). Based on the usual results of the monotone comparative statics, we can infer that Firm B's best response would be to (weakly) decrease the amount of resource allocated to research.

[20]Another way to refine Nash equilibrium in this context is to restrict attention to strategy profiles that adhere to Markov property concerning the belief about rival's progress. In other words, given the rival's resource allocation policy $\hat{\boldsymbol{\sigma}}$, $\boldsymbol{\sigma}(t) = \boldsymbol{\sigma}(s)$ when $\mathbf{p}_{\hat{\boldsymbol{\sigma}}}(t) = \mathbf{p}_{\hat{\boldsymbol{\sigma}}}(s)$. This differs from a standard Markov perfect equilibrium, as a firm cannot observe $\hat{\boldsymbol{\sigma}}$, which is essential for defining the state variable $\mathbf{p}_{\hat{\boldsymbol{\sigma}}}$. Under this equilibrium concept, we can obtain the same set of equilibria as under the MDR property. The proof can be provided upon request.

Steady State

We now define a pair consisting of a probability and a resource allocation that can emerge in a MDNE.

Definition 3. A pair of a probability and an allocation, $(p_\star, \sigma_\star) \in (0,1)^2$, is called a *steady state* if (i) $\xi(p_\star, \sigma_\star) = \lambda_\star$; and (ii) $\delta(p_\star, \sigma_\star) = 0$.

Suppose that a steady state exists, and for some $\boldsymbol{\sigma} \in \mathcal{S}$ and $T \in \mathbb{R}_+$, $\mathbf{p}_{\boldsymbol{\sigma}}(T) = p_\star$ and $\boldsymbol{\sigma}_t = \sigma_\star$ for all $t \geq T$. Then, given $\delta(p_\star, \sigma_\star) = 0$, we have $\mathbf{p}_{\boldsymbol{\sigma}}(t) = p_\star$ for all $t \geq T$. Moreover, the development rate is $\mathbf{h}_{\boldsymbol{\sigma}}(t) = \xi(p_\star, \sigma_\star) = \lambda_\star$ for all $t \geq T$. Therefore, if Firm A employs $\boldsymbol{\sigma}$, from time T, Firm B perceives that Firm A's development is fixed at $\lambda_\star$. Since Firm B faces the opponent with the development rate $\lambda_\star$, it is indifferent between conducting research and developing with the old technology by Proposition 3.1. Thus, for $t \geq T$, $\boldsymbol{\sigma}$ can be a best response for Firm B against Firm A playing $\boldsymbol{\sigma}$. Therefore, if a steady state exists, this can be part of a Nash equilibrium.

The following lemma provides a condition for the existence of the steady state. The proof is in Appendix D.3.4.

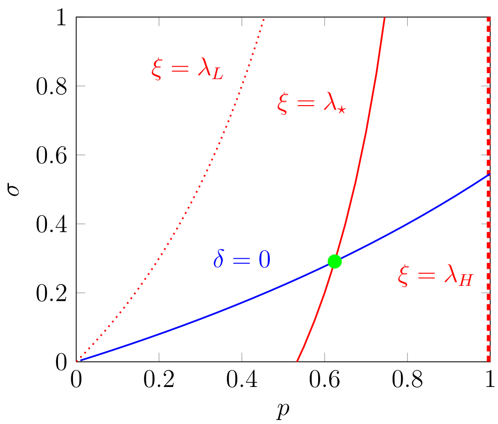
Proposition 5.2. *If $\lambda_L < \lambda_\star < \min\{\lambda_H, \mu\}$, there exists a steady state $(p_\star, \sigma_\star)$, and*

$$p_\star = \frac{\mu(\lambda_\star - \lambda_L)}{2\lambda_L \lambda_\star} = 1 - \frac{(\mu - \lambda_L)(\lambda_H - \lambda_\star)}{2\lambda_L \lambda_\star}, \tag{5.5}$$

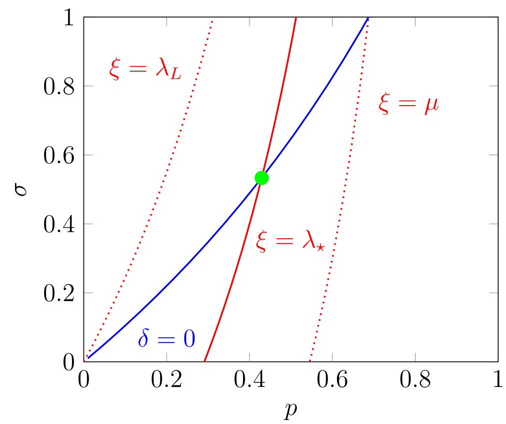
$$\sigma_\star = \frac{\lambda_\star - \lambda_L}{\mu - \lambda_L}. \tag{5.6}$$

Figure 4 illustrates this result. The red curves demonstrate the iso-development-rate curve, showing pairs of probability and allocation that yield the same value of $\xi(p, \sigma)$. Note that $\xi(0,0) = \lambda_L$, thus, the iso-development-rate curve with the value λ_L passes through the origin. Additionally, for all $\sigma \in [0,1]$, $\xi(1, \sigma) = \lambda_H$. Therefore, as depicted in the left panel of Figure 4, the iso-development-rate curve with the value λ_H is represented by the vertical line passing through $(1,0)$ and $(1,1)$.

The blue curves in Figure 4 illustrate the stationary-belief curve, depicting pairs of probability and allocation that result in no drift in belief, i.e., $\delta(p, \sigma) = 0$. This curve shows



(a) $\mu > \lambda_H$



(b) $\lambda_H > \mu$

Figure 4: Iso-Development-Rate Curves, Stationary-Belief Curves, and Steady State

differences depending on whether $\mu < \lambda_H$ or $\mu > \lambda_H$. When $\mu > \lambda_H$, as depicted in the left panel, the stationary-belief curve touches the vertical line passing through $(1, 0)$. Thus, in this case, we can find an interior solution $(p_\star, \sigma_\star)$ satisfying $\xi(p_\star, \sigma_\star) = \lambda_\star$ and $\delta(p_\star, \sigma_\star) = 0$ if and only if $\lambda_\star \in (\lambda_L, \lambda_H)$. Next, when $\mu < \lambda_H$, as depicted in the right panel, the stationary-belief curve touches the horizontal line passing through $(0, 1)$. Observe that $\delta(\mu/\lambda_H, 1) = 0$ and $\xi(\mu/\lambda_H, 1) = \mu$. Thus, in this case, the interior solution exists if and only if $\lambda_\star \in (\lambda_L, \mu)$. In sum, we can see that a steady state exists if and only if $\lambda_\star \in (\lambda_L, \min\{\mu, \lambda_H\})$.

Equilibrium Characterization

Now we provide the characterization of the MDNE.

Theorem 2. *Suppose that firms' research progress is private information. Then, the MDNE is uniquely characterized as follows:*

(i) if $\lambda_\star < \lambda_L$, firms play ***direct-development policies****: $\boldsymbol{\sigma}_A = \boldsymbol{\sigma}_B = 0$;*

(ii) if $\lambda_\star > \min\{\lambda_H, \mu\}$, firms play ***research policies****: $\boldsymbol{\sigma}_A = \boldsymbol{\sigma}_B = 1$;*

(iii) if $\lambda_\star \in (\lambda_L, \min\{\lambda_H, \mu\})$, firms play ***stationary fall-back policies****: $\boldsymbol{\sigma}_A = \boldsymbol{\sigma}_B = \boldsymbol{\sigma}^{SF}$,*

which is defined as follows:

$$\boldsymbol{\sigma}^{SF}(t) = \begin{cases} 1, & \text{if } t < T_\star, \\ \sigma_\star & \text{if } t \geq T_\star, \end{cases}$$

where $T_\star$ is the unique time such that $\mathbf{p_1}(T_\star) = p_\star$.

We provide the proof sketch in Appendix D.1, and the formal proof follows. As in the public information setting, symmetry is achieved as a result.

When the parameters are such that $\lambda_\star > \lambda_H$ or $\lambda_\star < \lambda_L$, we know from Theorem 1, specifically from points (a) and (c), that firms do not tailor their allocation to the opponent's progress even when this information is publicly available. Thus, under the private information setting, it is intuitive that the firms adopt the same equilibrium allocations as in the public information setting for those regions.

The more interesting case occurs when $\lambda_\star \in (\lambda_L, \lambda_H)$. For these parameters, firms would like to adjust their allocation according to their rival's access to the new technology. If a firm believes that the rival has the new technology with enough likelihood, the firm may consider allocating more resources to develop using the old technology. In particular, if beliefs were ever to reach the steady state probability $p_\star$, both firms implementing the steady state allocation $\sigma_\star$ thereafter would constitute an equilibrium in the continuation game. Thus, when a steady state exists, i.e., $\lambda_\star \in (\lambda_L, \min\{\mu, \lambda_H\})$, both firms playing the stationary fall-back policies can constitute an equilibrium. Last, when $\mu < \lambda_\star < \lambda_H$, as depicted in the right panel of Figure 4, a firm's development rate cannot exceed μ, thus, it is always lower than $\lambda_\star$. Therefore, it is optimal for both firms to indefinitely conduct research.

Figure 5 illustrates the relevant parametric regions in the above theorem. Note that $\lambda_\star < \mu$ is equivalent to $\frac{\lambda_L}{\mu} + 2 \cdot \frac{\lambda_L}{\lambda_H} > 1$. Thus, the dotted region represents the parametric region where firms employ the stationary fall-back policies. In addition, observe that there is a triangular region ($\lambda_H > \lambda_\star > \mu$) where firms adjust their resource allocations when research progress is public, but conduct research when research progress is private.

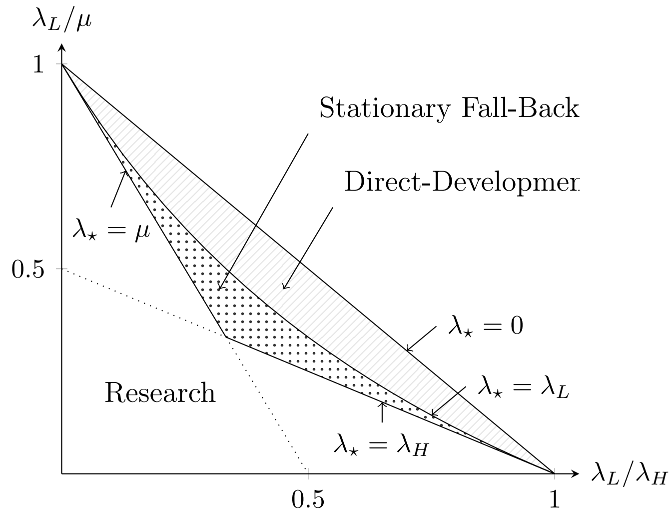


Figure 5: Nash Equilibrium with Monotone Development in the Private Information Setting

6 Patent, License and Trade Secret

In this section, we extend the model by allowing the firms to patent and license the new technology. The main components of the model remain the same as in Section 2, with a key difference: once a firm discovers the new technology, it has an option to apply for a patent. If the firm decides not to apply for a patent for the new technology, its discovery remains secret: in this case, we say that the firm uses *trade secret protection*.

We assume that firms cannot fraudulently claim the possessions of the new technology, and the patent applications are publicly available information. Thus, when a firm applies for a patent, the rival firm becomes aware of its possession of the new technology. On the other hand, if a firm decides to protect the new technology by using the trade secret, it does not release the information about this firm's possession of the new technology. However, this firm may face a risk of losing the right to use the new technology, as described in the paragraphs to follow. For simplicity, we also assume that the entire patent process is instantly completed and free of cost, and patents never expire: once a firm acquires the right to use the new technology, it retains ownership indefinitely. We intentionally impose these strong assumptions in favor of patents to isolate the pure effect of information in firms' patenting decisions regarding their intermediate discoveries. The only incentive for a firm to not apply for a patent is to conceal its research progress from the rival firm.

The timing of the game is as follows. When a firm discovers the new technology, it chooses to apply for a patent or not.[21] If neither firm applies for a patent, the game is the same as in our baseline model. When one of the firms applies for a patent, firms enter the subgame described in Figure 6. Suppose that Firm i has just applied for a patent. First, when the rival firm (Firm j) does not possess the new technology yet, Firm i owns the patent for the new technology. Then, Firm i offers a take-it-or-leave-it (TIOLI) offer of the license fee l to Firm j. If the offer is rejected, Firm j cannot use the new technology even if it independently discovers the new technology. Therefore, Firm j allocates all its resources toward developing with the old technology—since it is useless to discover the new technology—whereas Firm i develops with the new technology. If the offer is accepted, Firm j pays l to Firm i, and Firm i allows Firm j to use the new technology, thus, firms race developing the product by using the new technology.

Next, consider the case where Firm j already possesses the new technology but has not applied for a patent, i.e., Firm j has protected the right to use the new technology by using the trade secret. Let $\alpha \in [0, 1]$ denote the level of trade secret protection. Firm j (the challenger) can contest the patent based on the evidence demonstrating their prior possession of the new technology. With the probability α, Firm j's discovery is protected by trade secret, thereby both firms have the right to use the new technology. With the probability $1 - \alpha$, Firm i retains sole ownership of the patent. The subsequent game involving Firm i's patent is the same as above.

6.1 Preliminary Results

First-Best Outcome Consider the social planner whose goal is to maximize the joint expected profit of the firms. Assume that the planner can observe both firms' research progress, and make the patent and license decisions. Suppose that one of the firms has discovered the new technology. Then, it is socially efficient to direct this firm to license the new technology to the other firm, enabling both firms to develop with the new technology.

[21] Here, to simplify the discussion, we assume that a firm can only apply for a patent right after the discovery. In practice, it is possible to delay the patent application, e.g., a firm can protect the new technology by trade secret for six months, then apply for a patent.

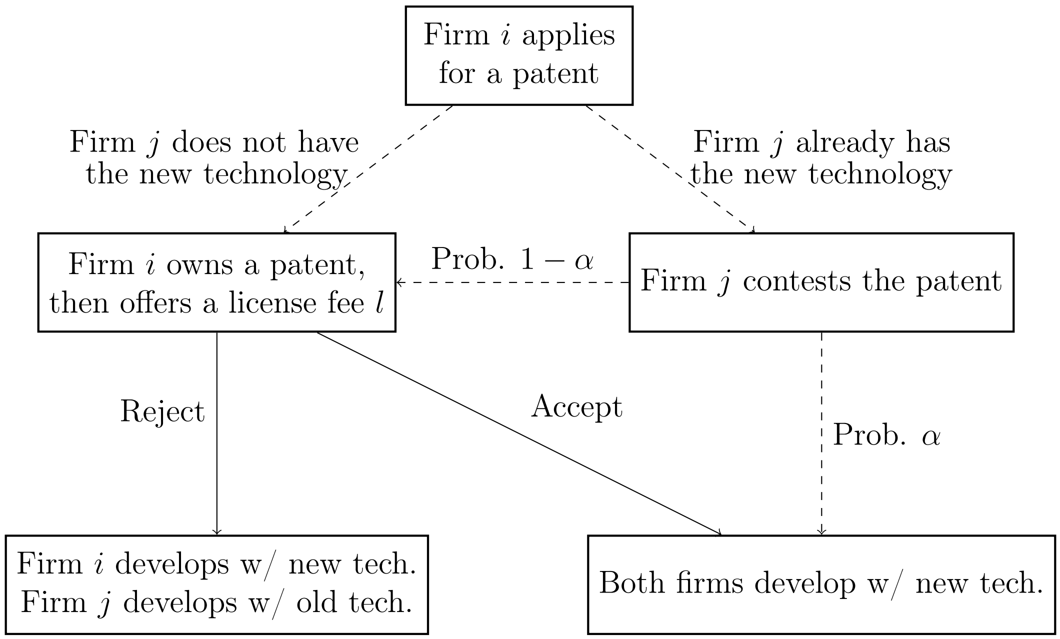


Figure 6: Timing of the patent game after the patent application

Given this, consider the planner's problem in allocating firms' resources when neither firm has yet discovered the new technology. Note that the expected cost is constant with respect to the number of firms: even if the flow cost is doubled, the expected completion time will be halved. Then, by (2.1), it is socially efficient when both firms conduct research. Therefore, the first-best resource allocation is to allocate all the resources toward research, and once one of the firms discovers the new technology, it applies for a patent and license it, then both firms develop with the new technology.

Optimal License Fee Consider the subgame that Firm i has obtained the patent for the new technology. If the license offer is accepted, both firms develop with the new technology, thus, the social welfare is $\Pi - \frac{2c}{2\lambda_H}$. When it is rejected, since Firm i develops with the new technology and Firm j develops with the old technology, the social expected cost is $\frac{2c}{\lambda_L+\lambda_H}$, thus, the social welfare is

$$\Pi - \frac{2c}{\lambda_L + \lambda_H} < \Pi - \frac{2c}{2\lambda_H}.$$

Since the licensing firm has an exclusive bargaining power, Firm i would capture the entire surplus from licensing:

$$\left(\Pi - \frac{2c}{2\lambda_H}\right) - \left(\Pi - \frac{2c}{\lambda_L + \lambda_H}\right) = \frac{\lambda_H - \lambda_L}{2\lambda_H(\lambda_H + \lambda_L)}c. \tag{6.1}$$

This essentially represents the savings in social cost achieved by allowing Firm j to use the new technology rather than the old technology. By using these, we can derive the optimal license fee.

Proposition 6.1. *Suppose that Firm i has obtained the patent for the new technology. Then, Firm i offers a license fee*

$$l^* = \frac{\lambda_H - \lambda_L}{\lambda_H + \lambda_L} \cdot \frac{\lambda_H \Pi + c}{2\lambda_H}. \tag{6.2}$$

to Firm j. Then, Firm i's expected payoff is $U_{Licensor} = V_C + l^*$ and Firm j's expected payoff is $U_{Licensee} = V_C - l^*$ where $V_C = \frac{\lambda_H \Pi - c}{2\lambda_H}$.

Proof of Proposition 6.1. When the offer is rejected, Firm j's expected payoff is $\frac{\lambda_L \Pi - c}{\lambda_H + \lambda_L}$. Note that V_C is the expected payoff when both firms race with the new technology. thus, when the license offer with the fee l is accepted, Firm j's expected payoff is $V_C - l$. Then, Firm i's optimal offer is

$$l^* = V_C - \frac{\lambda_L \Pi - c}{\lambda_H + \lambda_L}, \tag{6.3}$$

and we can derive (6.2) with simple algebra. Then, once the offer is accepted, Firm i's expected payoff is $V_C + l^*$ and Firm j's expected payoff is $V_C - l^*$. □

With some algebra, we can also show that

$$U_{Licensor} = \frac{\lambda_H \Pi - c}{\lambda_H + \lambda_L} + \frac{\lambda_H - \lambda_L}{2\lambda_H(\lambda_H + \lambda_L)}c, \qquad U_{Licensee} = \frac{\lambda_L \Pi - c}{\lambda_H + \lambda_L}. \tag{6.4}$$

Intuitively, l^* was deliberately chosen to make $U_{Licensee}$ equal to Firm j's outside option, and Firm i is capturing the surplus from licensing derived in (6.1).

Now consider the case when Firm i applies a patent, Firm j already has the new technology. With probability α, both firms have the right to use the new technology, and their expected payoffs are V_C. With probability $1-\alpha$, Firm i retains the patent, and Firm i receives $U_{Licensor}$ and Firm j receives $U_{Licensee}$.

Corollary 1. *When a patent is contested, the expected continuation payoffs of the applicant ($U^{\alpha}_{Applicant}$) and the challenger are ($U^{\alpha}_{Challenger}$) are:*

$$U^{\alpha}_{Applicant} = V_C + (1-\alpha)\cdot l^*, \qquad U^{\alpha}_{Challenger} = V_C - (1-\alpha)\cdot l^*. \tag{6.5}$$

6.2 Patents under Public Information

We start by considering a setting where, as in Section 4, the research progress is public information. In this setting, a firm deciding on patent application is aware of its opponent's available technologies. For the first firm to obtain the new technology, the only incentive to forego patenting would be to conceal the discovery from its rival. In public information settings, concealment is not possible, hence the first firm to obtain the new technology patents it. We formally state this result in the following proposition.

Proposition 6.2. *Suppose that firms' research progress is public information. In any subgame perfect Nash Equilibrium (SPNE), the first firm to discover the new technology applies for a patent.*

The proof is provided in Appendix E.1.1. Note that the patent application of the first firm to obtain the new technology can not be challenged. With this result and the equilibrium license fee from Proposition 6.1, we pin down the continuation payoffs of both firms after the new technology is first discovered. We use these continuation payoffs to analyze the resource allocation of the firms before the new technology is first discovered. Note that, on the equilibrium path of any SPNE, research is only conducted before the first discovery.

Next, we need to find equilibrium resource allocations when neither firm has discovered the new technology. As in Section 4, we focus on Markov strategies, i.e., allocations that are independent of calendar time. Let $\mathbf{s}^i_P$ denote the research allocation of Firm i in the absence of the new technology discovery by either firm.

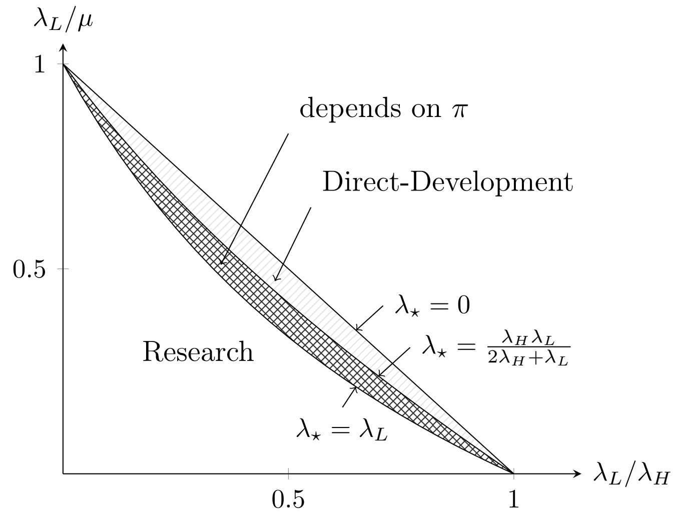


Figure 7: Equilibrium Resource Allocations in the Patent Game under Public Information

Proposition 6.3. *Suppose that firms' research progress is public information. In any MPE, the resource allocations before the new technology is first discovered are characterized as follows:*

(a) if $\lambda_\star > \lambda_L$, both firms conduct research: $\mathbf{s}_P^A = \mathbf{s}_P^B = 1$;

(b) if $\frac{\lambda_H\lambda_L}{2\lambda_H+\lambda_L} > \lambda_\star$, both firms develop with the old technology: $\mathbf{s}_P^A = \mathbf{s}_P^B = 0$;

(c) if $\lambda_L > \lambda_\star > \frac{\lambda_H\lambda_L}{2\lambda_H+\lambda_L}$, there exist thresholds $\tilde{\pi}_0 > \tilde{\pi}_1 > 1$ such that

(i) when $\pi \equiv \lambda_L \Pi/c > \tilde{\pi}_0$, both firms develop with the old technology: $\mathbf{s}_P^A = \mathbf{s}_P^B = 0$;

(ii) when $\tilde{\pi}_0 > \pi > \tilde{\pi}_1$, there are three equilibrium allocations: one firm does research and the other firm develops with the old technology, i.e., $(\mathbf{s}_P^A, \mathbf{s}_P^B) = (1, 0)$ or $(0, 1)$; both firms allocate some amount $z^* \in (0, 1)$ resources to research: $\mathbf{s}_P^A = \mathbf{s}_P^B = z^*$;

(iii) when $\tilde{\pi}_1 > \pi$, both firms do research: $\mathbf{s}_P^A = \mathbf{s}_P^B = 1$;

The proof of this Proposition is in Appendix E.1.2. Figure 7 summarizes the result. Intuitively, the option of patenting increases the value of conducting research. Recall that when $\lambda_\star > \lambda_L$, firms begin by conducting research in both public and private information settings without patenting. Therefore, firms will continue conducting research in this parametric region when patenting the new technology is possible (Part (a)). When $\lambda_\star < \lambda_L$, the equilibrium without patents involves both first doing direct development. In this region, the extra incentive to conduct research due to the option of patenting is insufficient to induce firms to conduct any research when π is high, or the expected completion time with the new technology path is close to that with the old technology path (Parts (b) and (c)). It is worth highlighting that, since patents are never challenged on the equilibrium path, the research allocations are independent of α.

We can compare the equilibrium allocations characterized in Proposition 6.3 and the on-path patenting decisions characterized in 6.1 to the first-best obtained in Section 6.1.

Corollary 2. *Suppose that firms' research progress is public information. The first-best outcome can be sustained in a MPE if and only if (i) $\lambda_\star > \lambda_L$; or (ii) $\lambda_L > \lambda_\star > \frac{\lambda_H \lambda_L}{2\lambda_H + \lambda_L}$ and $\tilde{\pi}_1 > \pi$.*

6.3 Patents under Private Information

Now we explore the case where firms' research progress is private information. We identify conditions under which (i) both firms apply for patents, namely *Efficient Patent Equilibrium*; or (ii) both firms do not apply for patents, namely *Concealment Equilibrium*.

6.3.1 Efficient Patent Equilibrium

First, we explore parametric conditions where an efficient patent equilibrium exists. Assume that Firm j's resource allocation strategy is to do research indefinitely and apply for a patent once the new technology is discovered. Suppose that Firm i has discovered the new technology, and Firm j has not applied for a patent yet. Given Firm j's patent application strategy, the fact that Firm j has not applied for a patent implies that Firm j does not have the new technology yet. Therefore, if Firm i applies for a patent, it will attain the patent and the expected payoff is $U_{Licensor} = V_C + l^*$. Suppose that Firm i does not apply for a patent. Firm j keeps conducting research and applies for a patent when it discovers the new technology. Firm i's payoff in this case is $U^{\alpha}_{Challenger} = V_C - (1 - \alpha) \cdot l^*$. Therefore, Firm i's expected payoff of not applying for a patent is

$$\frac{\lambda_H \Pi + \mu \cdot U^{\alpha}_{Challenger} - c}{\lambda_H + \mu} = \frac{(\mu + 2\lambda_H)V_C - \mu(1-\alpha)l^*}{\lambda_H + \mu}. \tag{6.6}$$

Lemma 6.1. *Suppose that firms' research progress is private information, and Firm j's resource allocation strategy is to do research indefinitely ($\sigma_t = 1$ for all $t \geq 0$) and apply for a patent once the new technology is discovered. When Firm i discovers the new technology, it applies for a patent if and only if*

$$\frac{l^*}{V_C} > \frac{\lambda_H}{\lambda_H + \mu(2-\alpha)}. \tag{6.7}$$

Proof of Proposition 6.1. Firm i applies for a patent when $U_{Licensor}$ is greater than (6.6), which is equivalent to:

$$\begin{aligned} & (\lambda_H + \mu)V_C + (\lambda_H + \mu)l^* > (\mu + 2\lambda_H)V_C - \mu(1-\alpha)l^* \\ \iff \quad & \{\lambda_H + \mu(2-\alpha)\}\, l^* > \lambda_H V_C. \end{aligned}$$

Since $1 > \alpha$, $\lambda_H, \mu > 0$ and $V_C > 0$, it is equivalent to (6.7). □

This result is intuitive in that a firm is willing to apply for a patent if and only if the license fee is attractive enough compare to the firm's expected payoff after licensing. Observe that the left hand side of (6.7) remains constant with respect to α, while the right hand side increases with α. Therefore, as α increases, (6.7) becomes more difficult to hold. This result aligns with intuition: as the trade secret protection level increases, firms are less inclined to apply for patents.

Also note that from (6.2) and $\pi = \lambda_L \Pi / c$, we have

$$\frac{l^*}{V_C} = \frac{\lambda_H - \lambda_L}{\lambda_H + \lambda_L} \cdot \frac{\lambda_H \Pi + c}{\lambda_H \Pi - c} = \frac{\lambda_H - \lambda_L}{\lambda_H + \lambda_L} \cdot \frac{\lambda_H \pi + \lambda_L}{\lambda_H \pi - \lambda_L}.$$

Therefore, the left hand side of (6.7) is decreasing in π, i.e., as π increases, (6.7) becomes more difficult to hold. Intuitively, since a part of the license fee comes from the saving of the cost, it does not increase proportionally with V_C. By solving (6.7), we can pin down the parametric conditions under which the efficient patent equilibrium exists.

Proposition 6.4. *Suppose that firms' research progress is private information. The efficient patent equilibrium exists if and only if one of the following conditions holds: (i)* $\alpha \leq \hat{\alpha} \equiv \frac{2\lambda_\star}{\lambda_H + \lambda_\star}$*; or (ii)* $\alpha > \hat{\alpha}$ *and*

$$\hat{\pi}(\alpha) \equiv 1 + \frac{\lambda_L + \lambda_H}{\lambda_H} \cdot \frac{2 - \alpha}{\alpha - \hat{\alpha}} > \pi. \tag{6.8}$$

Note that when $\lambda_\star > \lambda_H$, the efficient patent equilibrium exists, since $\hat{\alpha} > 1$. In this case, firms conduct research regardless of their rivals' progress. Therefore, when a firm discovers the new technology, there is no informational advantage to concealing it. Instead, firms can benefit from licensing the new technology to the rival firms, allowing the efficient patent equilibrium to be attained. On the other hand, when $\lambda_\star < \lambda_H$, it is possible that the efficient patent equilibrium does not exist. To illustrate this, consider a scenario where Firm A discovers the new technology. If Firm A patents and licenses the new technology, the license fee is determined based on the assumption that, if the offer is rejected, Firm B will develop with the old technology. Recall that, in the case of $\lambda_\star < \lambda_H$, developing with the old technology is the best response for Firm B when it knows that the rival has the new technology (Proposition 3.1). Therefore, by applying for a patent, Firm A provides an opportunity for Firm B to exercise its best response. In contrast, if Firm A keeps the discovery secret, it may induce Firm B to make suboptimal choices in R&D strategies, e.g., Firm B may squander its time in conducting research for the new technology, which Firm A already possesses. This trade-off creates the possibility that the efficient patent equilibrium does not exist.

6.3.2 Concealment Equilibrium

Now we explore parametric conditions for which a concealment equilibrium exists. To simplify our discussion, we focus on a case in which $\lambda_H > \lambda_\star > \mu$: both firms employ the fall-back strategy in the public information setting, but conduct research in the private information setting. Conditional on no patent applications, the equilibrium policies have to be the unique policies that form an equilibrium in Theorem 2. Let $\boldsymbol{\sigma}^*$ denote such policy.

Observation There is a concealment equilibrium if and only if, for all $t \geq 0$,

$$V_1(t; \mathbf{h}_{\boldsymbol{\sigma}^*}) \geq V_C + (1 - \alpha\, \mathbf{p}_{\boldsymbol{\sigma}^*}(t)) \cdot l^*. \tag{6.9}$$

To understand the observation, notice that (6.9) captures the trade-off in the patenting decision of a firm that discovers the new technology at time t, when the opponent follows policy $\boldsymbol{\sigma}^*$ and never patents. The left-hand-side denotes the payoff obtained by not patenting, i.e by keeping the discovery secret. The right-hand-side is equivalent to $\mathbf{p}_{\boldsymbol{\sigma}^*}(t) \cdot V^{\alpha}_{Applicant} + (1 - \mathbf{p}_{\boldsymbol{\sigma}^*}(t)) \cdot V_{Licensor}$ and captures the expected payoff if the firm decides to patent at time t. If (6.9) holds for all t, then it is a best response to never patent.

Under $\lambda_H > \lambda_\star > \mu$, by Theorem 2 (ii), firms employ the research policy in the private information setting, i.e., $\boldsymbol{\sigma}^* = \mathbf{1}$. The following lemma provides the closed form solution of $V_1(t; \mathbf{h_1})$.

Lemma 6.2. *The following equation holds:*

$$V_1(t; \mathbf{h_1}) = \left\{1 + \frac{\lambda_H}{\lambda_H + \mu}(1 - \mathbf{p_1}(t))\right\} \cdot V_C. \tag{6.10}$$

By using this lemma, (6.9) is equivalent to:

$$\frac{l^*}{V_C} < \frac{\lambda_H}{\lambda_H + \mu} \cdot \frac{1 - \mathbf{p_1}(t)}{1 - \alpha \cdot \mathbf{p_1}(t)}. \tag{6.11}$$

The right hand side is decreasing in $\mathbf{p_1}(t)$. Under $\lambda_H > \lambda_\star > \mu$, $\mathbf{p_1}(t)$ converges to μ/λ_H, thus, we can plug this into (6.11):

$$\frac{l^*}{V_C} < \frac{\lambda_H(\lambda_H - \mu)}{(\lambda_H + \mu)(\lambda_H - \alpha\mu)}. \tag{6.12}$$

With simple algebra, we can show that $\frac{\lambda_H(\lambda_H-\mu)}{(\lambda_H+\mu)(\lambda_H-\alpha\mu)} \leq \frac{\lambda_H}{\lambda_H+\mu(2-\alpha)}$. Therefore, the threshold for the concealment equilibrium is below the one for the efficient patent equilibrium, i.e., there is no parameter such that both the efficient patent equilibrium and the concealment equilibrium exist. By solving (6.12), we can pin down the parametric conditions under which the concealment equilibrium exists.

Proposition 6.5. *Suppose that firms' research progress is private information and $\lambda_H > \lambda_\star > \mu$. The concealment equilibrium exists if and only if*

$$\alpha > \tilde{\alpha} \equiv \frac{2\lambda_H(\mu + \lambda_\star)}{(\lambda_H + \mu)(\lambda_H + \lambda_\star)} \tag{6.13}$$

and

$$\pi > \tilde{\pi}(\alpha) \equiv 1 + \frac{\lambda_H + \lambda_L}{\lambda_H + \mu} \cdot \frac{2\lambda_H - (\lambda_H + \mu)\alpha}{\lambda_H(\alpha - \tilde{\alpha})}. \tag{6.14}$$

In addition, $\tilde{\alpha} > \hat{\alpha}$, $\tilde{\pi}(\alpha) \geq \hat{\pi}(\alpha)$, and the equality holds if and only if $\alpha = 1$.

6.4 Discussion

The results in this section enlighten us about firms' incentives to conceal their research progress and their impact on the social speed of innovation. When firms' research progress is publicly observable, firms voluntarily patent and license the new technology, which is the socially optimal outcome, unless the direct development with the old technology is appealing enough (Corollary 2 (i)). With private information about their progress, firms may choose to conceal their discovery of the new technology if the right to use the technology is well protected by trade secret laws and the stake from winning the race is high (Proposition 6.5). This slows down the social speed of innovation not only because the discovered new technology cannot be used by the other firm, but also because the other firm has to invest time in independently discovering the new technology. Proposition 6.4 suggests two potential policies to resolve this inefficiency. One approach is to lower the trade secret protection level, denoted as α, as this would discourage firms from concealing their discoveries. Another is to decrease the stake of winning the race, represented by π, as this would make licensing more appealing. A caveat of this policy is that lowering π too much would discourage firms from participating in the innovation races in the first place.

We can also modify the model to reflect the first-to-invest patent system. For instance, when Firm i applies for a patent and Firm j contests it, with probability α, the firm that discovered the new technology earlier obtains the patent, and with probability $1 - \alpha$, Firm i obtains the patent. Then, the first-to-invent system can be represented by $\alpha = 0$, while the first-to-file system can be represented by $\alpha = 1$. With this modification, firms have more incentives to conceal their discoveries as they now have a chance of becoming a patentee by contesting the other firms' patent application. Nevertheless, when α is low enough, these incentives cannot outweigh the advantage of licensing the new technology. Therefore, the socially efficient outcome can also be attained by decreasing α.

7 Conclusion

In this article, we study the long-lasting question of patent vs. secrecy by highlighting the firm's incentives to conceal interim breakthroughs to hinder the rival's strategic response. To do so, we introduce an innovation race model with multiple paths and show that firms' disclosing decisions depend on the stake for winning the race and the trade secret protection level.

We show that when firms' research progress is public information, the patent protection is effective in inducing a more efficient allocation of R&D resources. However, when the research progress is private information, patent protection has a limited effect. Based on this result, we can argue that, in some situations, a higher stake may reduce patenting and licensing which slows down the pace of innovation.

There are many avenues open for further research. For example, we assume that there are exogenously given two paths towards innovation, and one of the paths requires two breakthroughs. However, in practice, there are numerous ways to make an innovation, and it often requires more than two breakthroughs. We also assume that a firm's R&D resources are fixed over time, but we could also allow firms to endogenously choose how much effort to put into each point in time. Finally, we assume the contest structure is given by the winner-takes-all competition, but we might consider a contest designing problem. We leave these intriguing questions and others for future work.

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Appendix

A Preliminaries

A.1 Useful Observations

Let τ be a random variable on $\mathbb{R}_+$. Suppose that it has a continuous and differentiable cumulative distribution function $F : \mathbb{R}_+ \to [0,1]$. Let $S(t)$ denote the survival function of τ, i.e., $S(t) = 1 - F(t)$. If $\lim_{t\to\infty} t \cdot S(t) = 0$, the following equation holds:

$$\mathbb{E}[\tau] = \int_0^\infty t \cdot F'(t)dt = -t \cdot S(t)\Big|_0^\infty + \int_0^\infty S(t)dt = \int_0^\infty S(t)dt. \tag{A.1}$$

Let h be a development rate function of τ: $h(t) = -S'(t)/S(t)$.[22] Then, under the assumption that $F(0) = 0$, we can derive that $S(t) = e^{-\int_0^t h(s)ds}$. Then, (A.1) can be rewritten as follows:

$$\mathbb{E}[\tau] = \int_0^\infty e^{-\int_0^t h(s)ds}dt. \tag{A.2}$$

Consider another random variable $\hat{\tau}$ independent to τ. Let $\hat{S}$ and $\hat{h}$ be its survival and development rate functions. Observe that

$$\Pr[\tau < \hat{\tau}] = \int_0^\infty \hat{S}(t)\ dF(t) = -\int_0^\infty S'(t) \cdot \hat{S}(t)\ dt. \tag{A.3}$$

Then, (A.3) can be rewritten as follows:

$$\Pr[\tau < \hat{\tau}] = \int_0^\infty h(t) \cdot S(t) \cdot \hat{S}(t)\ dt = \int_0^\infty h(t) \cdot e^{-\int_0^t (h(s)+\hat{h}(s))ds}\ dt. \tag{A.4}$$

Now consider another random variable which is a minimum of τ and $\hat{\tau}$, denoted by $(\tau \wedge \hat{\tau})$. Then, the survival function of $(\tau \wedge \hat{\tau})$ is $S(t) \cdot \hat{S}(t)$, and the development function of $(\tau \wedge \hat{\tau})$

[22] In the literature, the function $h(t)$ is often referred to as a 'hazard rate' function. The term hazard rate originated from the tradition of describing arrivals as negative events such as failures. In our context, where we are analyzing the timing of product developments, we use the term 'development rate' instead of hazard rate.

is $h(t) + \hat{h}(t)$. By applying (A.2), when $\lim_{t\to\infty} t \cdot S(t) \cdot \hat{S}(t) = 0$, we have

$$\mathbb{E}[\tau \wedge \hat{\tau}] = \int_0^\infty e^{-\int_0^t (h(s)+\hat{h}(s))ds}\, dt. \tag{A.5}$$

A.2 Formal Definitions of Arrival Times

Given an allocation policy $\sigma : \mathbb{R}_+ \to [0,1]$, we define the following random variables:

1. τ_L: the arrival time of successful development with the old technology;

2. τ_R: the arrival time of the new technology discovery.

Let $\Sigma_t \equiv \int_0^t \sigma_s ds$. Then, the survival functions of τ_L and τ_R are given as follows: for all $t \geq 0$,

$$S^L_\sigma(t) = e^{-\lambda_L(t-\Sigma_t)} \quad \text{and} \quad S^R_\sigma(t) = e^{-\mu\Sigma_t}. \tag{A.6}$$

In addition, the development rate functions can be derived as follows:

$$h^L_\sigma(t) = \lambda_L(1-\sigma_t) \quad \text{and} \quad h^R_\sigma(t) = \mu\sigma_t. \tag{A.7}$$

Intuitively, the product is developed with the old technology at the rate $h^L_\sigma(t) = \lambda_L(1-\sigma_t)$ and the new technology is discovered at the rate $h^R_\sigma(t) = \mu\sigma_t$.

B Proofs for the Constant Development Rate Case

Lemma B.1. *Suppose that Firm j has a constant development rate λ. When Firm i employs an allocation policy σ, its expected payoff is given as follows:*

$$V^0_\lambda(\sigma) = \int_0^\infty \big(\lambda_L(1-\sigma_t)\cdot\Pi + \mu\ \sigma_t \cdot \mathcal{V}^1_\lambda - c\big)\cdot e^{-\lambda_L(t-\Sigma_t)-\mu\Sigma_t-\lambda t}\, dt, \tag{B.1}$$

where $\Sigma_t \equiv \int_0^t \sigma_s ds$.

Proof of Lemma B.1. Let τ_λ be the arrival time of Firm j. When any of the arrival times τ_L, τ_R and τ_λ occurs, we can regard that Firm i's payoff is realized. Furthermore, it incurs a flow cost c until one of these arrival times takes place. Thus, Firm i's expected payoff can be written as follows:

$$V_\lambda^0(\sigma) = \Pr[\tau_L < (\tau_R \wedge \tau_\lambda)] \cdot \Pi + \Pr[\tau_R < (\tau_L \wedge \tau_\lambda)] \cdot \mathcal{V}_\lambda^1 - \mathbb{E}[(\tau_L \wedge \tau_R \wedge \tau_\lambda)] \cdot c. \tag{B.2}$$

Note that the survival function of $(\tau_R \wedge \tau_\lambda)$ is $e^{-\int_0^t(\mu\sigma_s+\lambda)ds} = e^{-\mu\Sigma_t - \lambda t}$. By using (A.4) and (A.7), we have

$$\Pr[\tau_L < (\tau_R \wedge \tau_\lambda)] = \int_0^\infty \lambda_L(1-\sigma_t) \cdot e^{-\lambda_L(t-\Sigma_t)-\mu\Sigma_t-\lambda t}\, dt.$$

Likewise, we can derive that

$$\Pr[\tau_R < (\tau_L \wedge \tau_\lambda)] = \int_0^\infty \mu\ \sigma_t \cdot e^{-\lambda_L(t-\Sigma_t)-\mu\Sigma_t-\lambda t}\, dt.$$

Next, observe that the survival function of $(\tau_L \wedge \tau_R \wedge \tau_\lambda)$ is

$$e^{-\lambda_L(t-\Sigma_t)-\mu\Sigma_t-\lambda t} = e^{-(\lambda_L+\lambda)t-(\mu-\lambda_L)\Sigma_t}.$$

Then, from $\mu \geq \lambda_L$ and $\Sigma_t + \hat{\Sigma}_t \geq 0$, we have $\lim_{t\to\infty} t \cdot e^{-\lambda_L(t-\Sigma_t)-\mu\Sigma_t-\lambda t} = 0$. By applying (A.1), we have

$$\mathbb{E}[(\tau_L \wedge \tau_R \wedge \tau_\lambda)] = \int_0^\infty e^{-\lambda_L(t-\Sigma_t)-\mu\Sigma_t-\lambda t}\, dt.$$

By plugging the above equations into (B.2), we obtain (B.1). □

Lemma B.2. *Suppose that $x_0 \in \arg\max_{x\in[0,1]} u(x)$ where u is a function defined in (3.2). Let $\sigma^* : \mathbb{R}_+ \to [0,1]$ be $\sigma_t^* = x_0$ for all $t \geq 0$. Then, $\sigma^* \in \arg\max_\sigma V_\lambda^0(\sigma)$.*

Proof of Lemma B.2. Let r_t denote $e^{-\lambda_L(t-\Sigma_t)-\mu\Sigma t-\lambda t}$. By taking a derivative, we have

$$\dot{r}_t = -\left\{\lambda_L(1-\sigma_t) + \mu\sigma_t + \lambda\right\} \cdot r_t. \tag{B.3}$$

By Lemma B.1, Firm i's problem is

$$\max_{\sigma} \int_0^{\infty} \left\{\lambda_L(1-\sigma_t)\cdot \Pi + \mu\sigma_t \cdot \mathcal{V}_\lambda^1 - c\right\} \cdot r_t \; dt \tag{B.4}$$

subject to (B.3).

Observe that the Hamiltonian of this optimal control problem is

$$\begin{aligned} H(\sigma_t, r_t, \eta_t) &= \left\{\lambda_L(1-\sigma_t)\cdot \Pi + \mu\sigma_t \cdot \mathcal{V}_\lambda^1 - c\right\} \cdot r_t \\ &\quad - \eta_t \left\{\lambda_L(1-\sigma_t) + \mu\sigma_t + \lambda\right\} \cdot r_t \\ &= \left\{u(\sigma_t) - \eta_t\right\} \cdot \left\{\lambda_L(1-\sigma_t) + \mu\sigma_t + \lambda\right\} \cdot r_t, \end{aligned} \tag{B.5}$$

where η_t is a co-state variable.

To show that σ^* is a solution of (B.4) subject to (B.3) by using the Arrow sufficiency condition (Seierstad and Sydsaeter, 1987, Theorem 3.14), we consider (η^*, r^*) defined as follows: for all $t \geq 0$, $\eta_t^* = u(x_0)$ and $r_t^* = e^{-\{\mu x_0 + \lambda_L(1-x_0)+\lambda\}\cdot t}$.

Then, we need to check following four primitive conditions:

1. Maximum principle: for all $t \geq 0$,

$$\sigma_t^* = x_0 \in \operatorname*{arg\,max}_{\sigma_t \in [0,1]} H(\sigma_t, r_t^*, \eta_t^*). \tag{B.6}$$

2. Evolution of the co-state variable:

$$\dot{\eta}_t^* = -\frac{\partial H}{\partial r_t} = -\left\{u(\sigma_t^*) - \eta_t^*\right\} \cdot \left\{\lambda_L(1-\sigma_t^*) + \mu\sigma_t^* + \lambda\right\}. \tag{B.7}$$

3. Transversality condition: If r^* is the optimal trajectory, i.e., $r_t^* = e^{-\{\mu x_0 + \lambda_L(1-x_0)+\lambda\}\cdot t}$, $\lim_{t\to\infty} \eta_t^*(r_t^* - r_t) \leq 0$ for all feasible trajectories r_t.

4. $\hat{H}(r_t, \eta_t) = \max_{\sigma_t \in [0,1]} H(\sigma_t, r_t, \eta_t)$ is concave in r_t.

First, by plugging r_t^* and η_t^* into (B.5), we have

$$H(\sigma_t, r_t^*, \eta_t^*) = \left\{u(\sigma_t) - u(x_0)\right\} \cdot \left\{\lambda_L(1-\sigma_t^*) + \mu\sigma_t^* + \lambda\right\} \cdot r_t \tag{B.8}$$

Recall that $x_0 \in \arg\max_{x\in[0,1]} u(x)$. Thus, $H(\sigma_t, r_t^*, \eta_t^*) \leq 0$ for all $\sigma_t \in [0,1]$. In addition, $H(x_0, r_t^*, \eta_t^*) = 0$. Therefore, $x_0 \in \arg\max_{\sigma_t\in[0,1]} H(\sigma_t, r_t, \eta_t)$, i.e., (B.6) holds.

Second, by the definition of η^*, (B.7) holds.

Third, note that for any admissible allocation σ,

$$r_t = e^{-\{\mu\Sigma_t+\lambda_L(t-\Sigma_t)+\lambda t\}} = r_t^* \cdot e^{(\mu-\lambda_L)\cdot(x_0 t-\Sigma_t)}.$$

Then, we have

$$\lim_{t\to\infty} \eta_t^* \cdot (r_t^* - r_t) = \lim_{t\to\infty} u(x_0) \cdot r_t^* \cdot \left(1 - e^{(\mu-\lambda_L)\cdot(x_0 t-\Sigma_t)}\right) = 0.$$

Last, we can see that $\hat{H}$ is linear in r_t, thus, the fourth condition holds. Hence, by the Arrow sufficiency condition, σ^* is the best response to $\hat{\sigma}^*$. □

C Proofs for the Public Information Setting

C.1 Proof Sketches of Theorem 1

A MPE consists of a profile of Markov strategies such that each of the players is best responding to the strategy of their opponent. By using the similar argument as in Lemma B.2, we only need to consider Markov deviations to construct the set of Markov Perfect Equilibria.[23]

Given a Markov strategy profile of the firms, we can define U_ω^i as the continuation payoff of Firm i in state ω. Next, we provide some intuition for the proof of Theorem 1 by splitting the problem of the firms in two: On one hand, we solve the problem of the firms before any research progress has been made (and fixing the continuation payoffs). On the other hand, we compute the best responses of the firms after one of them obtains the new technology,

[23]Intuitively, when the opponent is using a Markov strategy, the problem of a firm is independent of calendar time. Thus, there exists a best response that is Markov and, therefore, the best Markov response is also a best response. For general treatment on the existence of Markov equilibria in a larger class of continuous-time stochastic games with finite states and actions, see Neyman (2017).

and therefore the equilibrium continuation payoffs. Finally, by plugging these continuation payoffs into the problem of the firms at the initial state, we prove the theorem.

Best Responses under no New Technology Discovery We first consider the case where neither firm has discovered the new technology, i.e., $\omega = \emptyset$. The conventional approach is to solve the problem with backward induction. However, in order to facilitate the analysis in various extensions, we present the problem under the state $\omega = \emptyset$ in a general manner by treating the continuation payoffs $U^i_{\{i\}}$ and $U^i_{\{j\}}$ as exogenous values.

When Firm i and j play $\mathbf{s}(\emptyset) = x$ and $\hat{\mathbf{s}}(\emptyset) = y$, Firm i's expected payoff at the state $\emptyset$ is

$$u_0(x, y) \equiv \frac{x\mu U^i_{\{i\}} + (1-x)\lambda_L \Pi + y\mu U^i_{\{j\}} - c}{x\mu + (1-x)\lambda_L + y\mu + (1-y)\lambda_L}. \tag{C.1}$$

Define $\Delta_y := u_0(1, y) - u_0(0, y)$. The following proposition characterizes the equilibrium allocations at state $\emptyset$ in any MPE.

Proposition C.1. *The equilibrium allocations at state $\emptyset$ are characterized as follows:*

(a) when $\Delta_0, \Delta_1 > 0$, both firms do research, i.e., $(\mathbf{s}^A(\emptyset), \mathbf{s}^B(\emptyset)) = (1, 1)$;

(b) when $\Delta_0, \Delta_1 < 0$, both firms develop with the old technology, i.e., $(\mathbf{s}^A(\emptyset), \mathbf{s}^B(\emptyset)) = (0, 0)$;

(c) when $\Delta_0 > 0 > \Delta_1$, there are three possible equilibrium allocations:

- *one firm does research and the other firm develops with the old technology, i.e., $(\mathbf{s}^A(\emptyset), \mathbf{s}^B(\emptyset)) = (1, 0)$ or $(0, 1)$,*
- *both firms allocate $z^* = \Delta_0/(\Delta_0 - \Delta_1)$ amount of resources to research and the remainder to the development with the old technology, i.e., $(\mathbf{s}^A(\emptyset), \mathbf{s}^B(\emptyset)) = (z^*, z^*)$;*

(d) when $\Delta_1 > 0 > \Delta_0$, there are three possible equilibrium allocations:

- *both firms do research, i.e., $(\mathbf{s}^A(\emptyset), \mathbf{s}^B(\emptyset)) = (1, 1)$,*
- *both firms develop with the old technology, i.e., $(\mathbf{s}^A(\emptyset), \mathbf{s}^B(\emptyset)) = (0, 0)$*

- *both firms allocate $z^* = -\Delta_0/(\Delta_1 - \Delta_0)$ amount of resources to research and the remainder to the development with the old technology, i.e.,* $(\mathbf{s}^A(\emptyset), \mathbf{s}^B(\emptyset)) = (z^*, z^*)$.

The set of equilibria as a function of Δ_1 and Δ_0 characterized in Proposition C.1 is summarized in Figure 8. The proof, in Appendix, boils down to showing that the derivative of u_0 with respect to x shares the same sign with $\lambda_L \cdot \Delta_0 \cdot (1-y) + \mu \cdot \Delta_1 \cdot y$ (Lemma C.1). As this sign is independent of x, the best response function exhibits a 'bang-bang' characteristic: the optimal response to the allocation y of the opponent is either 0, 1, or any value in $[0, 1]$.

In scenarios where Δ_0 and Δ_1 share the same sign, the best response is independent of the opponent's resource allocation. Specifically, when both Δ_0 and Δ_1 are positive, it is optimal to assign all resources to research. Conversely, when both Δ_0 and Δ_1 are negative, it is optimal to develop with the old technology.

When Δ_0 and Δ_1 have different signs, the optimal response depends on the resource allocation y of the opponent. When Δ_1 is positive and Δ_0 is negative, the function u_0 satisfies the single-crossing property from Milgrom and Shannon (1994) and the best-response functions of firms are increasing with respect to the opponent's resource allocation y. Thus, we can interpret the firms' allocations as strategic complements. This complementarity explains the two symmetric equilibria at the extreme allocations and the equilibrium with interior allocation.

When Δ_0 is positive and Δ_1 is negative, the negative of u_0 satisfies the single-crossing property and the best response functions are decreasing. In this context, we can interpret the firms' allocations as strategic substitutes. This substitutability explains why we obtain two asymmetric extreme equilibria and one symmetric equilibrium with interior allocations.

Best Responses under New Technology Discovery We now consider the cases where at least one of the firms has discovered the new technology.

When both firms have discovered the new technology ($\omega = \{i, j\}$), they will develop with the new technology and their expected payoffs are $U^i_{\{i,j\}} = U^j_{\{i,j\}} = V_C \equiv \frac{\lambda_H \Pi - c}{2\lambda_H}$. Next, suppose that only one of the firms, say Firm i, has discovered the new technology, i.e.,

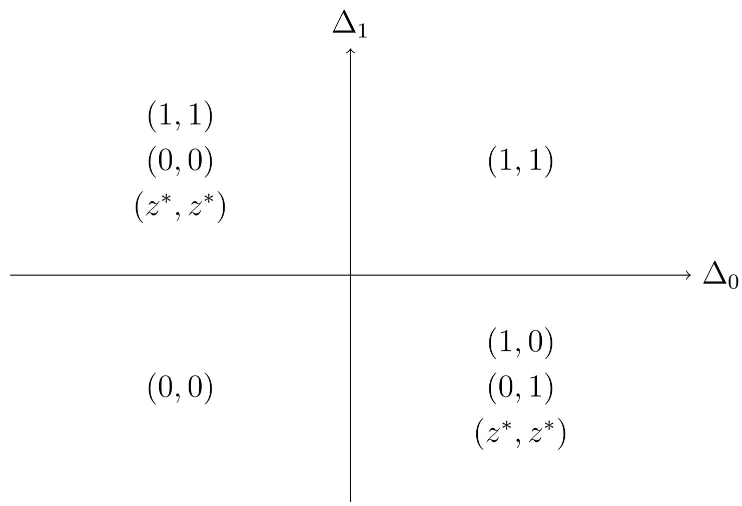

Figure 8: Equilibrium Allocations under No Discovery

$\omega = \{i\}$. In this case, Firm i develops the product at rate λ_H with the new technology. Then, we can derive the continuation values by applying Proposition 3.1:

(i) if $\lambda_\star > \lambda_H$, Firm j keeps conducting research:

$$U^i_{\{i\}} = U^j_{\{j\}} = \frac{\lambda_H \Pi + \mu V_C - c}{\mu + \lambda_H} = \frac{\mu + 2\lambda_H}{\mu + \lambda_H} V_C, \qquad U^i_{\{j\}} = U^j_{\{i\}} = \frac{\mu V_C - c}{\mu + \lambda_H}, \tag{C.2}$$

(ii) if $\lambda_\star < \lambda_H$, Firm j develops with the old technology:

$$U^i_{\{i\}} = U^j_{\{j\}} = V_H \equiv \frac{\lambda_H \Pi - c}{\lambda_L + \lambda_H}, \qquad U^i_{\{j\}} = U^j_{\{i\}} = V_L \equiv \frac{\lambda_L \Pi - c}{\lambda_L + \lambda_H}. \tag{C.3}$$

Equilibrium Characterization Now that we have derived the continuation values, we can finalize the proof of Theorem 1 by plugging these values into Proposition C.1.

First, when $\lambda_\star > \lambda_H$, in Lemma C.2, we show that the following equations hold:

$$\Delta_0 = \frac{\lambda_H \cdot \lambda_\star \cdot (\lambda_L \Pi + c) + \mu \cdot (\lambda_\star - \lambda_H) \cdot c}{2\lambda_H(\lambda_H + \mu)(\lambda_L + \mu)}, \tag{C.4}$$

$$\Delta_1 = \frac{\lambda_L \cdot \{\lambda_H \cdot \lambda_\star \cdot (\mu \Pi + c) + \mu \cdot (\lambda_\star - \lambda_H) \cdot c\}}{2\mu\lambda_H(\lambda_H + \mu)(\lambda_L + \mu)}. \tag{C.5}$$

From the above equations and $\lambda_\star > \lambda_H$, we can see that $\Delta_0, \Delta_1 > 0$. By applying Proposition

C.1 (a), both firms do research at the state $\emptyset$. Then, when one of the firms, say Firm j, succeeds in research, by Proposition 3.1 (a), Firm i will keep doing research. Therefore, the unique MPE is for firms to follow the research strategy (Theorem 1 (a)).

Next, when $\lambda_\star < \lambda_H$, in Lemma C.3, we show that the following equations hold:

$$\Delta_0 = \frac{(\lambda_L \Pi + c) \cdot (\lambda_\star - \lambda_L)}{2(\lambda_L + \mu)(\lambda_L + \lambda_H)}, \tag{C.6}$$

$$\Delta_1 = \frac{(\mu \Pi + c) \cdot \lambda_L \cdot (\lambda_\star - \lambda_L)}{2\mu(\lambda_L + \mu)(\lambda_L + \lambda_H)}. \tag{C.7}$$

When $\lambda_\star \in (\lambda_L, \lambda_H)$, (C.6) and (C.7) imply that Δ_0 and Δ_1 are positive. Thus, by Proposition C.1 (a), both firms do research at the state $\emptyset$. Then, when one of the firms, say Firm j, succeeds in research, by Proposition 3.1 (b), Firm i will switch to develop with the old technology. Therefore, the unique MPE is for firms to follow the fall-back strategy (Theorem 1 (b)).

Last, when $\lambda_\star < \lambda_L$, we can see that Δ_0 and Δ_1 are negative. Then, by Proposition C.1 (b), both firms develop with the old technology at the state $\emptyset$. Additionally, even if a firm happens to succeed in research, the other firm will keep developing with the old technology due to Proposition 3.1 (b). Thus, the unique MPE is for firms to employ the direct-development strategy (Theorem 1 (c)).

C.2 Lemmas

Lemma C.1. *The following equation holds:*

$$\frac{\partial u_0}{\partial x} = \mathcal{C}(x, y) \cdot \{\lambda_L \cdot \Delta_0 \cdot (1 - y) + \mu \cdot \Delta_1 \cdot y\}, \tag{C.8}$$

where

$$\mathcal{C}(x, y) = \frac{2(\lambda_L + \mu)}{\{\mu x + \lambda_L(1 - x) + \mu y + \lambda_L(1 - y)\}^2} > 0.$$

Proof of Lemma C.1. Observe that

$$\Delta_0 = \frac{\mu U^i_{\{i\}} - c}{\mu + \lambda_L} - \frac{\lambda_L \Pi - c}{2\lambda_L},$$
$$\Delta_1 = \frac{\mu(U^i_{\{i\}} + U^i_{\{j\}}) - c}{2\mu} - \frac{\lambda_L \Pi + \mu U^i_{\{j\}} - c}{\lambda_L + \mu}.$$

Thus, we have

$$2(\lambda_L + \mu)\lambda_L \cdot \Delta_0 = 2\lambda_L \mu U^i_{\{i\}} - \lambda_L(\lambda_L + \mu)\Pi + (\mu - \lambda_L)c, \tag{C.9}$$
$$2(\lambda_L + \mu)\mu \cdot \Delta_1 = (\lambda_L + \mu)\mu U^i_{\{i\}} - (\mu - \lambda_L)\mu U^i_{\{j\}} - 2\lambda_L \mu \Pi + (\mu - \lambda_L)c, \tag{C.10}$$

and

$$2(\lambda_L + \mu)\mu \cdot \Delta_1 - 2(\lambda_L + \mu)\lambda_L \cdot \Delta_0 = (\mu - \lambda_L)\left(\mu U^i_{\{i\}} - \mu U^i_{\{j\}} - \lambda_L \Pi\right).$$

Also note that

$$\frac{\partial u_0}{\partial x} = \frac{NUM_0}{\{\mu x + \lambda_L(1-x) + \mu y + \lambda_L(1-y)\}^2}$$

where

$$NUM_0 = (\mu U^i_{\{i\}} - \lambda_L \Pi) \cdot (\mu x + \lambda_L(1-x) + \mu y + \lambda_L(1-y))$$
$$- (x\mu U^i_{\{i\}} + (1-x)\lambda_L \Pi + y\mu U^i_{\{j\}} - c) \cdot (\mu - \lambda_L).$$

With some algebra, we can show that

$$NUM_0 = \left\{2\lambda_L \mu U^i_{\{i\}} - \lambda_L(\lambda_L + \mu)\Pi + (\mu - \lambda_L)c\right\}$$
$$+ (\mu - \lambda_L)(\mu U^i_{\{i\}} - \mu U^i_{\{j\}} - \lambda_L \Pi)y$$
$$= 2(\lambda_L + \mu)\lambda_L \cdot \Delta_0 + (2(\lambda_L + \mu)\mu \cdot \Delta_1 - 2(\lambda_L + \mu)\lambda_L \cdot \Delta_0) \cdot y.$$

By plugging this in, we can show that (C.8) holds. □

Lemma C.2. *When $\lambda_\star > \lambda_H$, the equations (C.4) and (C.5) hold.*

Proof of Lemma C.2. By plugging (C.2) into (C.9),

$$\begin{aligned}2(\lambda_L+\mu)\lambda_L\cdot\Delta_0 =& 2\lambda_L\mu\cdot U^i_{\{i\}}-\lambda_L(\lambda_L+\mu)\cdot\Pi+(\mu-\lambda_L)\cdot c\\ =&\frac{\lambda_L\mu}{\lambda_H}\cdot\frac{2\lambda_H+\mu}{\lambda_H+\mu}\cdot(\lambda_H\Pi-c)-\lambda_L(\lambda_L+\mu)\cdot\Pi+(\mu-\lambda_L)c\\ =&\frac{\lambda_H\mu-\lambda_L\lambda_H-\mu\lambda_L}{\lambda_H+\mu}\cdot(\lambda_L\Pi+c)\\ &+\frac{\mu\left(\lambda_H\mu-\lambda_L\lambda_H-\mu\lambda_L-\lambda_L\lambda_H\right)}{\lambda_H(\lambda_H+\mu)}\cdot c.\end{aligned}$$

By using (3.4), we have

$$2(\lambda_L+\mu)\lambda_L\cdot\Delta_0 =\frac{\lambda_L}{\lambda_H(\lambda_H+\mu)}\cdot\left[\lambda_H\cdot\lambda_\star\cdot(\lambda_L\Pi+c)+\mu\cdot(\lambda_\star-\lambda_H)\cdot c\right].$$

Then, by dividing both sides by $2(\lambda_L+\mu)\lambda_L$, we can show that (C.4) holds.

Next, by plugging (C.2) into (C.10),

$$\begin{aligned}2(\lambda_L+\mu)\mu\cdot\Delta_1 =&(\lambda_L+\mu)\mu\cdot U^i_{\{i\}}-(\mu-\lambda_L)\mu\cdot U^i_{\{j\}}-2\lambda_L\mu\cdot\Pi+(\mu-\lambda_L)\cdot c\\ =&\mu\frac{2(\lambda_H\lambda_L+\lambda_H\mu+\lambda_L\mu)}{\mu+\lambda_H}\cdot\frac{\lambda_H\Pi-c}{2\lambda_H}-2\lambda_L\mu\cdot\Pi+(\mu-\lambda_L)\frac{2\mu+\lambda_H}{\mu+\lambda_H}\cdot c\\ =&\frac{\lambda_H\mu-\lambda_L\lambda_H-\mu\lambda_L}{\lambda_H+\mu}\cdot(\mu\Pi+c)+\frac{\mu(\lambda_H\mu-\lambda_L\lambda_H-\mu\lambda_L-\lambda_L\lambda_H)}{\lambda_H(\lambda_H+\mu)}\cdot c.\end{aligned}$$

By using (3.4), we have

$$2(\lambda_L+\mu)\mu\cdot\Delta_1 =\frac{\lambda_L}{\lambda_H(\lambda_H+\mu)}\cdot\left[\lambda_H\cdot\lambda_\star\cdot(\mu\Pi+c)+\mu\cdot(\lambda_\star-\lambda_H)\cdot c\right]$$

Then, by dividing both sides by $2(\lambda_L+\mu)\mu$, we can show that (C.5) holds. □

Lemma C.3. *When $\lambda_\star<\lambda_H$, the equations (C.6) and (C.7) hold.*

Proof of Lemma C.3. By plugging (C.3) into (C.9),

$$
\begin{aligned}
2(\lambda_L+\mu)\lambda_L\cdot\Delta_0 =& 2\lambda_L\mu\cdot U^i_{\{i\}} - \lambda_L(\lambda_L+\mu)\cdot\Pi + (\mu-\lambda_L)\cdot c\\
=& \frac{2\lambda_L\mu}{\lambda_L+\lambda_H}(\lambda_H\Pi - c) - \lambda_L(\lambda_L+\mu)\cdot\Pi + (\mu-\lambda_L)c\\
=& \frac{\lambda_L\Pi + c}{\lambda_L+\lambda_H}\cdot\left\{\lambda_H\mu - \lambda_L\lambda_H - \mu\lambda_L - \lambda_L^2\right\}
\end{aligned}
$$

By using (3.4), we have

$$
2(\lambda_L+\mu)\lambda_L\cdot\Delta_0 = \frac{(\lambda_L\Pi+c)\cdot\lambda_L\cdot(\lambda_\star-\lambda_L)}{\lambda_L+\lambda_H}.
$$

Then, by dividing both sides by $2(\lambda_L+\mu)\lambda_L$, we can show that (C.6) holds.

Next, by plugging (C.3) into (C.10),

$$
\begin{aligned}
2(\lambda_L+\mu)\mu\cdot\Delta_1 =& (\lambda_L+\mu)\mu\cdot U^i_{\{i\}} - (\mu-\lambda_L)\mu\cdot U^i_{\{j\}} - 2\lambda_L\mu\cdot\Pi + (\mu-\lambda_L)\cdot c\\
=& (\lambda_L+\mu)\mu\cdot\frac{\lambda_H\Pi - c}{\lambda_L+\lambda_H} - (\mu-\lambda_L)\mu\cdot\frac{\lambda_L\Pi - c}{\lambda_L+\lambda_H}\\
& - 2\lambda_L\mu\cdot\Pi + (\mu-\lambda_L)\cdot c\\
=& \frac{\mu\Pi + c}{\lambda_L+\lambda_H}\cdot\left\{\lambda_H\mu - \lambda_L\lambda_H - \mu\lambda_L - \lambda_L^2\right\}.
\end{aligned}
$$

By using (3.4), we have

$$
2(\lambda_L+\mu)\mu\cdot\Delta_1 = \frac{(\mu\Pi+c)\cdot\lambda_L\cdot(\lambda_\star-\lambda_L)}{\lambda_L+\lambda_H}.
$$

Then, by dividing both sides by $2(\lambda_L+\mu)\mu$, we can show that (C.5) holds. □

C.3 Proof of Proposition C.1

Proof of Proposition C.1. (a) When $\Delta_0,\Delta_1>0$, from (C.8), $\frac{\partial u_0}{\partial x}>0$ for all $y\in[0,1]$, i.e., $x=1$ is optimal. Thus, both firms play $\mathbf{s}(\emptyset)=1$ in any MPE.

(b) When $\Delta_0,\Delta_1<0$, from (C.8), $\frac{\partial u_0}{\partial x}<0$ for all $y\in[0,1]$, i.e., $x=0$ is optimal. Thus, both firms play $\mathbf{s}(\emptyset) = 0$ in any MPE.

(c) From $\Delta_0 > 0$ and (C.8), we have $\frac{\partial u_0}{\partial x}|_{y=0} > 0$, i.e., $x = 1$ is the best response for $y = 0$. In addition, from $0 > \Delta_1$ and (C.8), we have $\frac{\partial u_0}{\partial x}|_{y=1} < 0$, i.e., $x = 0$ is the best response for $y = 1$. Therefore, $(1, 0)$ and $(0, 1)$ can be supported equilibrium allocations at $\omega = \emptyset$.

Next, note that $z^* \in (0, 1)$ and $\frac{\partial u_0}{\partial x}|_{y=z^*} = 0$, i.e., any $x \in [0, 1]$ is the best response for $y = z^*$. Thus, (z^*, z^*) can be supported as an equilibrium allocation.

Last, consider any $\tilde{y} \in (0, 1)$ with $\tilde{y} \neq z^*$. Then, $\frac{\partial u_0}{\partial x}|_{y=\tilde{y}} \neq 0$, i.e., the best response is $x = 1$ or $x = 0$. Recall that the best response of $x = 1$ ($x = 0$) is $y = 0$ ($y = 1$), thus, $y = \tilde{y}$ cannot be a part of an equilibrium allocation.

(d) From $\Delta_0 < 0$ and (C.8), we have $\frac{\partial u_0}{\partial x}|_{y=0} < 0$, i.e., $x = 0$ is the best response for $y = 0$. Thus, $(0, 0)$ can be supported as an equilibrium allocation.

Similarly, from $0 < \Delta_1$ and (C.8), we have $\frac{\partial u_0}{\partial x}|_{y=1} > 0$, i.e., $x = 1$ is the best response for $y = 1$. Therefore, $(1, 1)$ can also be supported as an equilibrium allocation.

Next, note that $z^* \in (0, 1)$ and $\frac{\partial u_0}{\partial x}|_{y=z^*} = 0$, i.e., any $x \in [0, 1]$ is the best response for $y = z^*$. Thus, (z^*, z^*) can be supported as an equilibrium allocation.

Last, by using the similar argument as in the previous case, $\tilde{y} \in (0, 1)$ with $\tilde{y} \neq z^*$ cannot be a part of an equilibrium allocation.

□

D Proofs for the Private Information Setting

D.1 Proof Sketch of Theorem 2

Recursive Formulation Let $V_1(t; \mathbf{h})$ and $V_0(t; \mathbf{h})$ be the continuation payoffs of a firm with and without the new technology at time t, respectively, when the opponent employs an allocation policy with associated development rate $\mathbf{h}$, and no firm has succeeded in development so far. Formally, we define V_1 as follows:

$$V_1(t;\mathbf{h}) \equiv \int_t^\infty \{\lambda_H \Pi - c\} \cdot e^{-\int_t^s (\mathbf{h}(u)+\lambda_H) du} \; ds \tag{D.1}$$

The exponential term captures the probability that the race is still on by time s, given that the race is on by time by time t. The term $\lambda_H \Pi - c$ captures the flow expected payoff of the firm with the new technology. On top of fixing the opponent's development rate $\mathbf{h}$, we can fix the firm's policy $\boldsymbol{\sigma} \in \mathcal{S}$ to compute the continuation value v_0 of the firm without the new technology as follows:

$$v_0(t;\boldsymbol{\sigma},\mathbf{h}) \equiv \int_t^\infty \left\{\boldsymbol{\sigma}(s)\mu V^1(s;\mathbf{h}) + (1-\boldsymbol{\sigma}(s))\lambda_L \Pi - c\right\} \cdot \mathbf{r}_{\mathbf{h},\boldsymbol{\sigma}}(s;t) \; ds, \tag{D.2}$$

$$\mathbf{r}_{\mathbf{h},\boldsymbol{\sigma}}(s;t) \equiv e^{-\int_t^s \{\mathbf{h}(u)+\boldsymbol{\sigma}(u)\mu+(1-\boldsymbol{\sigma}(u))\lambda_H\} du}.$$

In this expression, as before, the exponential term captures the probability that race is on and the firm does not have the new technology by time s, given that both hold at time t. Conditional on this event, the firm enjoys an expected flow payoff captured by the expression in brackets: the firm pays the cost c and, at rate $\boldsymbol{\sigma}(s)\mu$, the firm obtains the new technology which induces a continuation payoff $V_1(s,\mathbf{h})$. At rate $(1-\boldsymbol{\sigma}(s))\lambda_L$ the firm successfully develops, which induces a lump-sum payoff Π. By maximizing over all the allocation policies in $\mathcal{S}$, we obtain the continuation value of a firm without the new technology V_0.

$$V_0(t;\mathbf{h}) \; := \; \max_{\boldsymbol{\sigma}\in\mathcal{S}} \quad v_0(t;\boldsymbol{\sigma},\mathbf{h}).$$

Best responses To characterize the optimal policy $\boldsymbol{\sigma}$ given the opponent's development rate $\mathbf{h}$, define $\mathcal{R}(x,t;\mathbf{h})$ and $R(t;\mathbf{h})$ as follows:

$$\mathcal{R}(x,t;\mathbf{h}) \equiv \mu x(V_1(t;\mathbf{h}) - V_0(t;\mathbf{h})) + \lambda_L(1-x)(\Pi - V_0(t;\mathbf{h})), \tag{D.3}$$

$$R(t;\mathbf{h}) \equiv \frac{\partial \mathcal{R}}{\partial x}(x,t;\mathbf{h}) = \mu(V_1(t;\mathbf{h}) - V_0(t;\mathbf{h})) - \lambda_L(\Pi - V_0(t;\mathbf{h})).$$

We can interpret $\mathcal{R}$ as the instantaneous payoff at time t by allocating x to research and $1-x$ to development with the old technology. The new technology is discovered at the rate μx, yielding the new continuation payoff $V_1(x;\mathbf{h})$ but losing the present continuation payoff $V_0(x;\mathbf{h})$. Similarly, the product is developed with the old technology at the rate $\lambda_L(1-x)$, resulting in the reward Π but losing $V_0(x;\mathbf{h})$. At each time t, the firm chooses a resource allocation to maximize $\mathcal{R}$. Therefore, we interpret R as capturing the relative incentives to conduct research: when R is positive, conducting research is preferred over developing with the old technology, conversely, when R is negative, developing with the old technology is preferred. The following proposition formalizes this verification arguments given the opponent's resource allocation policy $\hat{\sigma}$. The proof is in Appendix D.5.

Proposition D.1. *An allocation policy $\boldsymbol{\sigma}^*$ is a best-response to $\hat{\boldsymbol{\sigma}}$, i.e. $\mathcal{U}(\boldsymbol{\sigma}^*, \hat{\boldsymbol{\sigma}}) \geq \mathcal{U}(\boldsymbol{\sigma}, \hat{\boldsymbol{\sigma}})$ for all $\boldsymbol{\sigma} \in \mathcal{S}$, if and only if the following two conditions hold for every time $t \geq 0$:*

(i) $v_0(t; \boldsymbol{\sigma}^*, \mathbf{h}_{\hat{\boldsymbol{\sigma}}}) > 0$*; and*

(ii) $\boldsymbol{\sigma}^*(t) \in \arg\max_{x\in[0,1]} \mathcal{R}(x, t; \mathbf{h}_{\hat{\boldsymbol{\sigma}}})$*, or equivalently,*

$$\boldsymbol{\sigma}^*(t) = \begin{cases} 1, & \text{if } R(t; \mathbf{h}_{\hat{\boldsymbol{\sigma}}}) > 0, \\ \in [0,1], & \text{if } R(t; \mathbf{h}_{\hat{\boldsymbol{\sigma}}}) = 0, \\ 0, & \text{if } R(t; \mathbf{h}_{\hat{\boldsymbol{\sigma}}}) < 0. \end{cases}$$

Properties with Monotone Development Rates We now highlight two features of MDNE. The first feature arises when an allocation policy satisfies MDR property. The proof is in Section D.3.3.

Proposition D.2. *Suppose that $\boldsymbol{\sigma} \in \mathcal{S}$ satisfies the MDR property. If $\boldsymbol{\sigma}(s) = 0$, then $\boldsymbol{\sigma}(t) = 0$ for all $t < s$.*

The intuition for this result is as follows. Suppose that $\boldsymbol{\sigma}(s) = 0$ and $\mathbf{p}_{\boldsymbol{\sigma}}(s) > 0$. Then, the probability that this firm has discovered the new technology decreases as it does not conduct research at time s, which in turn decreases the development rate. This violates the MDR property. To eliminate this effect, we need to ensure $\mathbf{p}_{\boldsymbol{\sigma}} = 0$, which can only be achieved by $\boldsymbol{\sigma}(t) = 0$ for all $t < s$.

The next feature emerges when a firm faces a rival employing an allocation policy with the MDR property. The proof is in Section D.6.2.

Proposition D.3 (Single-Crossing Property)**.** *Suppose that* $\mathbf{h}$ *is increasing with* $\mathbf{h}(t) < \lambda_\star$ *for all* t*. Then,* $-\mathcal{R}(x, t; \mathbf{h})$ *satisfies the single-crossing property: for all* $x' > x$ *and* $t' > t$*,*

$$-\mathcal{R}(x', t; \mathbf{h}) \geq -\mathcal{R}(x, t; \mathbf{h}) \quad \Rightarrow \quad -\mathcal{R}(x', t'; \mathbf{h}) \geq -\mathcal{R}(x, t'; \mathbf{h}),$$
$$-\mathcal{R}(x', t; \mathbf{h}) > -\mathcal{R}(x, t; \mathbf{h}) \quad \Rightarrow \quad -\mathcal{R}(x', t'; \mathbf{h}) > -\mathcal{R}(x, t'; \mathbf{h}),$$

or equivalently, $0 \;\geq (>)\; R(t; \mathbf{h})$ *implies* $0 \;\geq (>)\; R(t'; \mathbf{h})$.

Roughly speaking, from the single-crossing property of $-\mathcal{R}$, we obtain that, under the best response policy, the resource allocation to research weakly decreases over time.[24]

Equilbirium Characterization Equipped with the verification result (Proposition D.1) and the properties derived from the MDR property (Proposition D.2 and D.3), we proceed to explain the intuition behind the proof of Theorem 2 concerning the different parameters.

Theorem 2 (i) When $\lambda_\star < \lambda_L$, $(\boldsymbol{\sigma}^A, \boldsymbol{\sigma}^B) = (\mathbf{0}, \mathbf{0})$ is the unique MDNE.

In Theorem 1, we obtained that when $\lambda_\star < \lambda_L$ and firms observe their rivals' research progress, the unique MPE involves both firms developing with the old technology. Intuitively, the equilibrium allocations from this MPE survive as an equilibrium in the unobservable case because the information about the opponent's technology was not used anyway. To show that this is the unique MDNE, first note that any optimal policy has to eventually generate development rates higher than λ_L. Otherwise, the policy would be dominated by developing with the old technology. Thus, the development rates must converge to a rate higher than $\lambda_\star$. In D.8, we show that the incentives to do research R must therefore converge to a negative number. By Proposition D.1, there must be a time after which the firms stop allocating

[24] To be precise, the single-crossing property only guarantees monotonicity in the strong set of order (Milgrom and Shannon, 1994). We present additional arguments to substantiate this claim.

resources to research. However, Proposition D.2 implies that if a firm ever allocates resources to research, stopping would induce a decreasing development rate. Thus, the only possibility is that firms do not conduct research at all.

Theorem 2 (ii) When $\lambda_\star > \min\{\mu, \lambda_H\}$, $(\boldsymbol{\sigma}^A, \boldsymbol{\sigma}^B) = (\mathbf{1}, \mathbf{1})$ is the unique MDNE.

First, we show that for any policy that satisfies monotone development rate, the development rates are bounded above by $\min\{\mu, \lambda_H\}$. This is true because maintaining a development rate higher than $\min\{\mu, \lambda_H\}$ requires a strictly decreasing $\boldsymbol{\sigma}$ to compensate for the decrease in beliefs $\mathbf{p}_{\boldsymbol{\sigma}}$. At some time, $\boldsymbol{\sigma}(t)$ must reach zero, and such development rate cannot be maintained anymore. Thus, the development rates of the firms must converge to some rate weakly lower than $\min\{\mu, \lambda_H\}$, which is lower than $\lambda_\star$. Therefore, for any $\boldsymbol{\sigma}$ satisfying MDR, there is a time T for which $R(t; \mathbf{h}_{\boldsymbol{\sigma}}) > 0$ for all $t > T$. However, we also show in Proposition D.3 that if a firm finds it optimal to allocate resources to development with the old technology ($R(t, \mathbf{h}) < 0$) then it must be that this is always optimal ($R(t, \mathbf{h}) < 0$ for all $s > t$). The only equilibrium candidate is therefore $(1, 1)$.

Theorem 2 (iii) When $\lambda_\star \in (\lambda_L, \min\{\lambda_H, \mu\})$, the stationary fall-back policy profile is the unique MDNE.

First, we establish that in any MDNE both firms' development must converge precisely to $\lambda_\star$ (Lemma D.13). Essentially, we show that any other converging limits lead to a contradiction. Next, we prove that, in any MDNE, the two firms must reach the development rate $\lambda_\star$ simultaneously. If one firm reaches $\lambda_\star$ first, we show using Lemma D.11 that the firm has incentives to allocate all resources to research until the opponent reaches $\lambda_\star$. However, this allocation would necessarily elevate the development rate, pushing it beyond $\lambda_\star$.

Let's define $T_\star$ as the time when both firms reach the development rate $\lambda_\star$. From T onward, the firms develop at the rate $\lambda_\star$. We show that there is a unique constant probability and allocation, $p_\star$ and $\sigma_\star$, that can maintain the development rate at $\lambda_\star$, as any deviation from these levels would induce the development rates to diverge. To obtain the allocations before time $T_\star$, we apply Lemma D.11 again to show that firms must strictly allocate all resources to research before $T_\star$. The continuity of the probability function pins down the

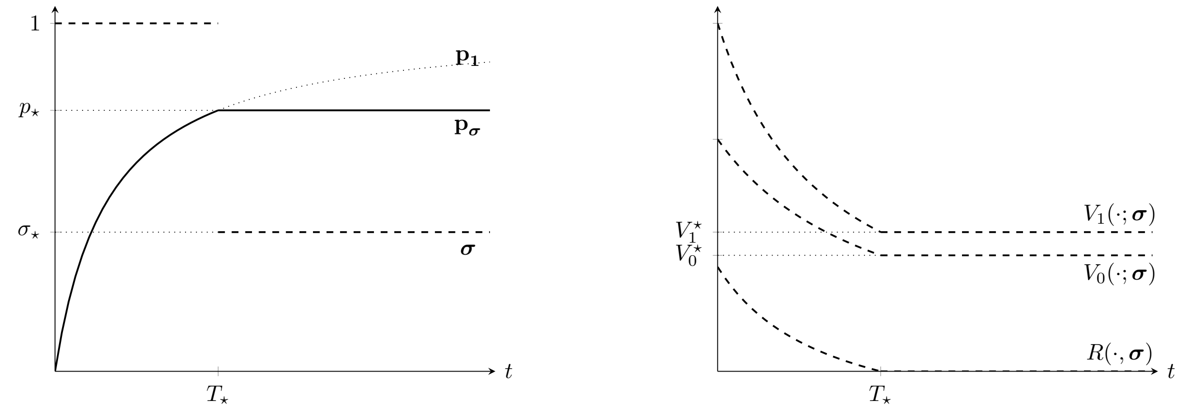

Figure 9: Stationary fall-back equilibrium.

time $T_\star$, since it must therefore be that $\mathbf{p_1}(T_\star) = p_\star$.

Figure 9 depicts the dynamics of the key variables in the equilibrium for the parameters in which $\lambda_\star \in (\lambda_L, \min\{\mu, \lambda_H\})$. The equilibrium presents two distinct stages, demarcated by the time $T_\star$. During the first stage, the firms allocate all resources to research. Consequently, $\mathbf{p}$ and $\mathbf{h}$ are strictly increasing. Additionally, the continuation values for the firm with and without the new technology decline over time.

When $\mathbf{p_1}$ reaches the level $p_\star$, the equilibrium transitions to a second stage where the research allocation shifts to $\sigma_\star$ and the functions $\mathbf{p}$ and $\mathbf{h}$ remain constant at the levels $p_\star$ and $\lambda_\star$, respectively. The continuation values for the firms with and without the new technology remain constant at the levels

$$V_1^\star = \frac{\lambda_H \Pi - c}{\lambda_H + \lambda_\star}, \qquad \text{and} \qquad V_0^\star = \frac{\mu V_1^\star - \lambda_L \Pi}{\mu - \lambda_L}. \tag{D.4}$$

D.2 Preliminaries

Given an allocation policy $\boldsymbol{\sigma} \in \mathcal{S}$, we define two arrival times: (i) τ_M represents the time at which either the new technology is discovered or the product is developed by the old technology; (ii) τ_D represents the time of the product development. Observe that, τ_M must be less than or equal to τ_D by definition. This inequality is strict if and only if the new technology is discovered prior to the product development. Therefore, we use $(\tau_M = \tau_D)$ to indicate the event that the new technology is discovered before the product is developed using the old technology and $(\tau_M < \tau_D)$ to indicate the event that the product is developed before the new technology discovery.

D.3 Evolution of $\mathbf{p}_{\boldsymbol{\sigma}}$

In this section, we derive, the probability $\mathbf{p}_{\boldsymbol{\sigma}}$ of having the new technology under an allocation policy $\boldsymbol{\sigma}$ given no development. Using this probability, we then compute the development rate $\mathbf{h}_{\boldsymbol{\sigma}}$. Let $\Sigma_t \equiv \int_0^t \boldsymbol{\sigma}(s)\, ds$ represent the cumulative research. We begin by observing that the probability that there is no breakthrough by time t is given by

$$S_{\boldsymbol{\sigma}}^M(t) := \Pr[\tau_M > t] = e^{-\lambda_L(t-\Sigma_t)-\mu\Sigma_t} \tag{D.5}$$

If the new technology is obtained at time s, the probability that development is not attained by time $t > s$ is given by:

$$\Pr[\tau_D > t \mid \tau_D > \tau_M = s] = e^{-\lambda_H(t-s)} \tag{D.6}$$

If instead there is a breakthrough at some $s < t$, with some probability the breakthrough is a research breakthrough. This probability is given by:

$$\Pr(\tau_D > \tau_M \mid \tau_M = s) = \frac{\boldsymbol{\sigma}(s)\mu}{\boldsymbol{\sigma}(s)\mu + (1-\boldsymbol{\sigma}(s))\lambda_L} \tag{D.7}$$

This result is standard in competing risks models. Thus, we can derive the probability that a research breakthrough is obtained before t, but no development was obtained by time t.

$$L_{\boldsymbol{\sigma}}(t) := \Pr(\tau_M < t < \tau_D) = \int_0^t f_M(s) \cdot \Pr(\tau_D > t \mid \tau_M = s)\, ds, \tag{D.8}$$

where f_M is the marginal density function of τ_M. Observe that, for any $s < t$,

$$
\begin{aligned}
& f_M(s) \cdot \Pr(\tau_D > t \mid \tau_M = s) \\
&= f_M(s) \cdot \Pr(\tau_D > \tau_M \mid \tau_M = s) \cdot \Pr[\tau_D > t \mid \tau_D > \tau_M = s] \\
&= f_M(s) \cdot \Pr(\tau_D > \tau_M \mid \tau_M = s) \cdot e^{-\lambda_H(t-s)} \\
&= [\boldsymbol{\sigma}(s)\mu + (1-\boldsymbol{\sigma}(s))\lambda_L]\, S^M_{\boldsymbol{\sigma}}(s) \cdot \frac{\boldsymbol{\sigma}(s)\mu}{\boldsymbol{\sigma}(s)\mu + (1-\boldsymbol{\sigma}(s))\lambda_L} \cdot e^{-\lambda_H(t-s)} \\
&= \mu\, \boldsymbol{\sigma}(s) e^{-\lambda_L(s-\Sigma_s)-\mu\Sigma_s} e^{-\lambda_H(t-s)}
\end{aligned}
$$

Where the first equality uses the law of total probability. For the second equality, we invoke (D.6). The third equality holds by expressing the density function f_M as the product of the hazard rate $[\boldsymbol{\sigma}\,\mu + (1-\boldsymbol{\sigma})\lambda_L]$ the survival function $S^M_{\boldsymbol{\sigma}}$. We also invoke (D.7). The last equality applies (D.5). Using the previous derivation in (D.8), we obtain:

$$
L_{\boldsymbol{\sigma}}(t) = \int_0^t \mu\, \boldsymbol{\sigma}(s) e^{-\lambda_L(s-\Sigma_s)-\mu\Sigma_s} e^{-\lambda_H(t-s)}\, ds \tag{D.9}
$$

The probability $S^D_{\boldsymbol{\sigma}}$ that no development was made by time t can be written as:

$$
\begin{aligned}
S^D_{\boldsymbol{\sigma}}(t) := \Pr(\tau_D > t) &= \Pr(\tau_M > t) + \Pr(\tau_M < t < \tau_D) \\
&= S^M_{\boldsymbol{\sigma}}(t) + L_{\boldsymbol{\sigma}}(t)
\end{aligned}
\tag{D.10}
$$

Finally, we obtain an expression for our conditional probability $\mathbf{p}_{\boldsymbol{\sigma}}$ in terms of $L_{\boldsymbol{\sigma}}$ and $S^M_{\boldsymbol{\sigma}}$:

$$
\mathbf{p}_{\boldsymbol{\sigma}}(t) = \Pr(\tau_M < t \mid \tau_D > t) = \frac{\Pr(\tau_M < t < \tau_D)}{S^D_{\boldsymbol{\sigma}}(t)} = \frac{L_{\boldsymbol{\sigma}}(t)}{S^M_{\boldsymbol{\sigma}}(t) + L_{\boldsymbol{\sigma}}(t)}. \tag{D.11}
$$

Based on the above observations, we provide the proof of Proposition 5.1.

D.3.1 Proof of Proposition 5.1

Proof of Proposition 5.1. From (D.11), we can derive that

$$
\frac{\mathbf{p}_{\boldsymbol{\sigma}}(t)}{1-\mathbf{p}_{\boldsymbol{\sigma}}(t)} = \frac{L_{\boldsymbol{\sigma}}(t)}{S^M_{\boldsymbol{\sigma}}(t)}
$$

By differentiating this equation side-by-side, we have

$$\frac{\dot{\mathbf{p}}_{\boldsymbol{\sigma}}(t)}{(1-\mathbf{p}_{\boldsymbol{\sigma}}(t))^2}=\frac{L_{\boldsymbol{\sigma}}(t)}{S^M_{\boldsymbol{\sigma}}(t)}\left[\frac{L'_{\boldsymbol{\sigma}}(t)}{L_{\boldsymbol{\sigma}}(t)}-\frac{S^{M\prime}_{\boldsymbol{\sigma}}(t)}{S^M_{\boldsymbol{\sigma}}(t)}\right]=\frac{\mathbf{p}_{\boldsymbol{\sigma}}(t)}{1-\mathbf{p}_{\boldsymbol{\sigma}}(t)}\left[\frac{L'_{\boldsymbol{\sigma}}(t)}{L_{\boldsymbol{\sigma}}(t)}-\frac{S^{M\prime}_{\boldsymbol{\sigma}}(t)}{S^M_{\boldsymbol{\sigma}}(t)}\right] \tag{D.12}$$

From deriving (D.5) and (D.9), we obtain that

$$S^{M\prime}_{\boldsymbol{\sigma}}(t)=-\left\{\lambda_L(1-\boldsymbol{\sigma}(t))+\mu\,\boldsymbol{\sigma}(t)\right\}\cdot S^M_{\boldsymbol{\sigma}}(t), \tag{D.13}$$

$$L'_{\boldsymbol{\sigma}}(t)=\mu\cdot\boldsymbol{\sigma}(t)\cdot S^M_{\boldsymbol{\sigma}}(t)-\lambda_H\cdot L_{\boldsymbol{\sigma}}(t) \tag{D.14}$$

Using these expressions in (D.12) and multiplying side by side by $(1-\mathbf{p}_{\boldsymbol{\sigma}}(t))^2$, we obtain the desired result. □

Next, we provide the closed form solution of $\mathbf{p}_{\mathbf{1}}(t)$.

Lemma D.1. *Suppose that a firm follows an allocation policy $\boldsymbol{\sigma}$, with $\boldsymbol{\sigma}(s)=1$ for $s\in[0,t)$. Then, the conditional probability $\mathbf{p}_{\boldsymbol{\sigma}}(t)$ of having access to the new technology by time t given that the race is ongoing is the same as that under the research policy ($\mathbf{p}_{\mathbf{1}}(t)$), which is given as follows:*

$$\mathbf{p}_{\boldsymbol{\sigma}}(t)=\mathbf{p}_{\mathbf{1}}(t)\equiv\begin{cases}\dfrac{\frac{1}{\lambda_H}\left(e^{-\mu t}-e^{-\lambda_H t}\right)}{\frac{1}{\mu}e^{-\mu t}-\frac{1}{\lambda_H}e^{-\lambda_H t}}, & \text{if } \mu\neq\lambda_H,\\[2ex] \dfrac{\mu t}{1+\mu t}, & \text{if } \mu=\lambda_H.\end{cases} \tag{D.15}$$

In addition, $\mathbf{p}_{\mathbf{1}}(t)$ is increasing in t, with $\lim_{t\to\infty}\mathbf{p}_{\mathbf{1}}(t)=\min\{1,\mu/\lambda_H\}$.

Proof of Lemma D.1. Note that the conditional probability of having access to the new technology by time t only depends on the resource allocations prior to time t. Thus, since $\boldsymbol{\sigma}$ and $\mathbf{1}$ have the same resource allocation by time t, $\mathbf{p}_{\boldsymbol{\sigma}}(t)$ and $\mathbf{p}_{\mathbf{1}}(t)$ are equal. By plugging $\boldsymbol{\sigma}(t)=1$ to the result of Proposition 5.1, we have $\mathbf{p}'_{\boldsymbol{\sigma}}(t)=(\mu-\lambda_H\,\mathbf{p}_{\boldsymbol{\sigma}}(t))(1-\mathbf{p}_{\boldsymbol{\sigma}}(t))$. By rearranging the differential equation, we can derive that

$$\begin{cases}\lambda_H-\mu=\dfrac{d}{dt}\log\left(\dfrac{\lambda_H-\lambda_H\,\mathbf{p}_{\boldsymbol{\sigma}}(t)}{\mu-\lambda_H\,\mathbf{p}_{\boldsymbol{\sigma}}(t)}\right), & \text{if } \mu\neq\lambda_H,\\[2ex] \mu=\dfrac{d}{dt}\dfrac{1}{1-\mathbf{p}_{\boldsymbol{\sigma}}(t)}, & \text{if } \mu=\lambda_H.\end{cases}$$

Then, from $\mathbf{p}_{\boldsymbol{\sigma}}(0) = 0$, we can derive that

$$\begin{cases} \dfrac{\lambda_H(1-\mathbf{p}_{\boldsymbol{\sigma}}(t))}{\mu - \lambda_H\,\mathbf{p}_{\boldsymbol{\sigma}}(t)} = \dfrac{\lambda_H}{\mu} e^{(\lambda_H-\mu)t}, & \text{if } \mu \neq \lambda_H, \\ \dfrac{1}{1-\mathbf{p}_{\boldsymbol{\sigma}}(t)} - 1 = \mu t, & \text{if } \mu = \lambda_H. \end{cases}$$

By rearranging the above equation, we have (D.15).

Observe that

$$\mathbf{p}_{\mathbf{1}}'(t) = \begin{cases} \dfrac{\mu(\lambda_H-\mu)^2 e^{(\lambda_H+\mu)t}}{(\lambda_H e^{\lambda_H t} - \mu e^{\mu t})^2} > 0, & \text{if } \mu \neq \lambda_H, \\ \dfrac{\mu}{(1+\mu t)^2} > 0, & \text{if } \mu = \lambda_H. \end{cases}$$

Thus, $\mathbf{p}_{\mathbf{1}}(t)$ is increasing in t.

When $\mu > \lambda_H$,

$$\lim_{t\to\infty} \mathbf{p}_{\mathbf{1}}(t) = \lim_{t\to\infty} \frac{\frac{1}{\lambda_H}\left(e^{(\lambda_H-\mu)t}-1\right)}{\frac{1}{\mu}e^{(\lambda_H-\mu)t} - \frac{1}{\lambda_H}} = 1.$$

When $\mu < \lambda_H$,

$$\lim_{t\to\infty} \mathbf{p}_{\mathbf{1}}(t) = \lim_{t\to\infty} \frac{\frac{1}{\lambda_H}\left(1-e^{(\mu-\lambda_H)t}\right)}{\frac{1}{\mu} - \frac{1}{\lambda_H}e^{(\mu-\lambda_H)t}} = \frac{\mu}{\lambda_H}.$$

When $\mu = \lambda_H$,

$$\lim_{t\to\infty} \mathbf{p}_{\mathbf{1}}(t) = \lim_{t\to\infty} \frac{\mu t}{1+\mu t} = 1 = \frac{\mu}{\lambda_H}.$$

□

D.3.2 Development rates

For any continuous random variable, the hazard rate can be expressed as the negative of the log of the survival function. The development rate of a firm that follows policy $\boldsymbol{\sigma} \in \mathcal{S}$ is the hazard rate associated with the random variable τ_D. Therefore, it can be derived as follows:

$$\begin{aligned} \mathbf{h}_{\boldsymbol{\sigma}}(t) = -\frac{\partial \log\left[S^D_{\boldsymbol{\sigma}}(t)\right]}{\partial t} = -\frac{S^{D\prime}_{\boldsymbol{\sigma}}(t)}{S^D_{\boldsymbol{\sigma}}(t)} &= \frac{\lambda_L(1-\boldsymbol{\sigma}(t))\cdot S^M_{\boldsymbol{\sigma}}(t) + \lambda_H \cdot L_{\boldsymbol{\sigma}}(t)}{S^M_{\boldsymbol{\sigma}}(t) + L_{\boldsymbol{\sigma}}(t)} \\ &= \lambda_L(1-\boldsymbol{\sigma}(t))\cdot(1-\mathbf{p}_{\boldsymbol{\sigma}}(t)) + \lambda_H \cdot \mathbf{p}_{\boldsymbol{\sigma}}(t). \end{aligned} \tag{D.16}$$

Also note that from $S^D_{\boldsymbol{\sigma}}(0) = 1$, $S^D_{\boldsymbol{\sigma}}(t)$ can be rewritten as follows:

$$S^D_{\boldsymbol{\sigma}}(t) = e^{-\int_0^t \mathbf{h}_{\boldsymbol{\sigma}}(s)ds} \tag{D.17}$$

D.3.3 Proof of Proposition D.2

Proof of Proposition D.2. Since $\boldsymbol{\sigma}(s) = 0$ and $\boldsymbol{\sigma}$ is right-continuous, it must be that $\mathbf{h}_{\boldsymbol{\sigma}}(\tilde{s}) = \lambda_L \cdot (1 - \mathbf{p}_{\boldsymbol{\sigma}}(\tilde{s})) + \lambda_H \cdot \mathbf{p}_{\boldsymbol{\sigma}}(\tilde{s})$ for $\tilde{s}$ slightly above s. This means that

$$0 \leq \mathbf{h}'_{\boldsymbol{\sigma}}(s) = (\lambda_H - \lambda_L) \cdot \dot{\mathbf{p}}_{\boldsymbol{\sigma}}(s) = -(\lambda_H - \lambda_L)^2 \cdot \mathbf{p}_{\boldsymbol{\sigma}}(s)(1 - \mathbf{p}_{\boldsymbol{\sigma}}(s))$$

where the inequality holds since $\mathbf{h}_{\boldsymbol{\sigma}}$ is weakly increasing. Since $\mathbf{p}_{\boldsymbol{\sigma}}(s) < 1$, it must be the case that $\mathbf{p}_{\boldsymbol{\sigma}}(s) = 0$. This holds only if $\boldsymbol{\sigma}(t) = 0$ for all $t < s$. □

D.3.4 Proof of Proposition 5.2

Proof of Proposition 5.2. From $\xi(p_\star, \sigma_\star) = \lambda_\star$, $\delta(p_\star, \sigma_\star) = 0$ and $p_\star < 1$, we have

$$\lambda_\star = p_\star \lambda_H + (1 - p_\star)(1 - \sigma_\star)\lambda_L, \tag{D.18}$$

$$0 = \mu\sigma_\star - \{\lambda_H - (1 - \sigma_\star)\lambda_L\} \, p_\star. \tag{D.19}$$

By rearranging (D.19), we have

$$\begin{aligned} \mu\sigma_\star &= \{\lambda_H - (1 - \sigma_\star)\lambda_L\} \, p_\star \\ &= \lambda_H p_\star + (1 - \sigma_\star)\lambda_L(1 - p_\star) - \lambda_L(1 - \sigma_\star) = \lambda_\star - \lambda_L(1 - \sigma_\star). \end{aligned}$$

By solving this, we can derive (5.6).

Next, from (D.19) and (5.6), we have

$$p_\star = \frac{\mu\sigma_\star}{\lambda_H - (1 - \sigma_\star)\lambda_L} = \frac{\mu(\lambda_\star - \lambda_L)}{(\mu - \lambda_L)\lambda_H - (\mu - \lambda_\star)\lambda_L}.$$

Note that $\lambda_L \lambda_\star = (\mu - \lambda_L)\lambda_H - \mu\lambda_L$. By plugging this into the above equation, we have the first equality of (5.5). Observe that

$$1 - p_\star = \frac{2\lambda_L\lambda_\star - \mu\lambda_\star + \mu\lambda_L}{2\lambda_L\lambda_\star} = \frac{\lambda_L(\mu + \lambda_\star) - (\mu - \lambda_L)\lambda_\star}{2\lambda_L\lambda_\star} = \frac{(\mu - \lambda_L)(\lambda_H - \lambda_L)}{2\lambda_L\lambda_\star},$$

which confirms the second equality of (5.5).

From $\lambda_L < \lambda_\star < \min\{\mu, \lambda_H\}$, we can see that $p_\star, \sigma_\star \in (0, 1)$. □

D.4 Recursive Formulation

The opponent's allocation policy is only payoff-relevant for a firm through the distribution of development times. Thus, in this section, we focus on characterizing the continuation payoffs of firms fixing the development rate function $\mathbf{h}$ of the opponent.

Lemma D.2. *Let $V_1(t; \mathbf{h})$ be the continuation payoff of a firm at time t when the firm has the new technology, neither firm had succeeded in development by time t, and the opponent employs an allocation policy with development rate $\mathbf{h}$. Then, $V_1(t; \mathbf{h})$ takes a form of (D.1). In addition, the following differential equation holds:*

$$0 = V_1'(t; \mathbf{h}) + (\lambda_H \Pi - c) - (\lambda_H + \mathbf{h}(t)) \cdot V_1(t; \mathbf{h}). \tag{HJB$_1$}$$

Proof of Lemma D.2. Let $\hat{\tau}_D$ be the arrival time of the product development by the opponent whose development rate is $\mathbf{h}$. Note that the continuation payoffs can be written as follows.

$$\begin{aligned} V_1(t; \mathbf{h}) = & \Pr[\tau_D < \hat{\tau}_D \mid \tau_M = t < (\tau_D \wedge \hat{\tau}_D)] \cdot \Pi \\ & - c \cdot \mathbb{E}[\tau_D \wedge \hat{\tau}_D - t \mid \tau_M = t < (\tau_D \wedge \hat{\tau}_D)]. \end{aligned} \tag{D.20}$$

Note that (conditional) survival functions of $\hat{\tau}_D$ and τ_D can be written as follows:

$$\begin{aligned} \Pr[\hat{\tau}_D > s \mid \tau_M = t < (\tau_D \wedge \hat{\tau}_D)] = & e^{-\int_t^s \mathbf{h}(u) du}, \\ \Pr[\tau_D = \tau_H > s \mid \tau_R = t < (\tau_L \wedge \hat{\tau}_D)] = & e^{-\lambda_H (s-t)}. \end{aligned}$$

By applying (A.4) and (A.5), we have

$$
\begin{aligned}
\Pr[\tau_D < \hat{\tau}_D \mid \tau_R = t < (\tau_L \wedge \hat{\tau}_D)] &= \int_t^{\infty} \lambda_H e^{-\int_t^s (\lambda_H + \mathbf{h}(u))du} ds, \\
\mathbb{E}[\tau_D \wedge \hat{\tau}_D - t \mid \tau_R = t < (\tau_L \wedge \hat{\tau}_D)] &= \int_t^{\infty} e^{-\int_t^s (\lambda_H + \mathbf{h}(u))du} ds.
\end{aligned}
$$

By plugging these equations into (D.20), we can derive that (D.1) holds.

By taking a derivative of (D.20), we have

$$
\begin{aligned}
V_1'(t; \mathbf{h}) = &- (\lambda_H \Pi - c) \cdot e^{-\int_t^t (\lambda_H + \mathbf{h}(u))du} \\
&+ (\lambda_H + \mathbf{h}(t)) \cdot (\lambda_H \Pi - c) \cdot \int_t^{\infty} e^{-\int_t^s (\lambda_H + \mathbf{h}(u))du} ds \\
= &- (\lambda_H \Pi - c) + (\lambda_H + \mathbf{h}(t)) \cdot V_1(t; \hat{\sigma}),
\end{aligned}
$$

which is equivalent to (HJB_1). □

Lemma D.3. *Let v_0 be the continuation payoff at time t of a firm that does not have the new technology and employs allocation policy $\boldsymbol{\sigma} \in \mathcal{S}$ when the opponent has a development rate $\mathbf{h} \in \mathcal{H}$. Then, v_0 takes a form of (D.2). In addition, the following differential equation holds:*

$$
\begin{aligned}
0 = v_0'(t; \boldsymbol{\sigma}, \mathbf{h}) + \lambda_L (1 - \boldsymbol{\sigma}(t)) \cdot \Pi + \mu \boldsymbol{\sigma}(t) \cdot V_1(t; \mathbf{h}) - c \\
- \{\lambda_L (1 - \boldsymbol{\sigma}(t)) + \mu \boldsymbol{\sigma}(t) + \mathbf{h}(t)\} \cdot v_0(t; \sigma, \hat{\sigma}).
\end{aligned}
\tag{HJB_0}
$$

Proof of Lemma D.3. We focus on the event such that $(\tau_M \wedge \hat{\tau}_D) > t$. The continuation payoff can be written as follows:

$$
\begin{aligned}
v_0(t; \boldsymbol{\sigma}, \mathbf{h}) = &\Pr[\tau_D < \hat{\tau}_D \mid (\tau_M \wedge \hat{\tau}_D) > t] \cdot \Pi \\
&- c \cdot \mathbb{E}[\tau_D \wedge \hat{\tau}_D - t \mid (\tau_M \wedge \hat{\tau}_D) > t].
\end{aligned}
\tag{D.21}
$$

Note that

$$\Pr[\tau_M > s \mid \tau_M > t] = \frac{S^M_{\boldsymbol{\sigma}}(s)}{S^M_{\boldsymbol{\sigma}}(t)},$$

$$\Pr[\tau_D > s > \tau_M > t \mid \tau_M > t] = \int_t^s e^{-\lambda_H(s-u)} \cdot \mu\, \boldsymbol{\sigma}(u) \cdot \frac{S^M_{\boldsymbol{\sigma}}(u)}{S^M_{\boldsymbol{\sigma}}(t)} du = \frac{L_{\boldsymbol{\sigma}}(s|t)}{S^M_{\boldsymbol{\sigma}}(t)},$$

where $L_{\boldsymbol{\sigma}}(s|t) \equiv \int_t^s e^{-\lambda_H(s-u)} \cdot \mu\, \boldsymbol{\sigma}(u) \cdot S^M_{\boldsymbol{\sigma}}(u) du$. Then, the survival function of τ_D conditional on $\tau_M > t$ can be written as follows:

$$S^D_{\boldsymbol{\sigma}|_t}(s) \equiv \Pr\left[\tau_D > s \mid \tau_M > t\right] = \frac{S^M_{\boldsymbol{\sigma}}(s) + L_{\boldsymbol{\sigma}}(s|t)}{S^M_{\boldsymbol{\sigma}}(t)}$$

Also note that $\Pr[\hat{\tau}_D > s \mid \hat{\tau}_D > t] = e^{-\int_t^s \mathbf{h}(u)du}$.

Observe that

$$L'_{\boldsymbol{\sigma}}(s|t) = \mu\, \boldsymbol{\sigma}(s) \cdot S^M_{\boldsymbol{\sigma}}(s) - \lambda_H \cdot L_{\boldsymbol{\sigma}}(s|t). \tag{D.22}$$

Since τ_D and $\hat{\tau}_D$ are independent, we can apply (A.3) and (A.5) by resetting the initial time to t. Then, by using (D.13) and (D.22), we have

$$\begin{aligned}
\Pr[\tau_D < \hat{\tau}_D \mid (\tau_M \wedge \hat{\tau}_D) > t] &= -\int_t^\infty {S^D_{\boldsymbol{\sigma}|_t}}'(s) \cdot e^{-\int_t^s \mathbf{h}(u)du} ds \\
&= -\int_t^\infty \frac{S^{M\prime}_{\boldsymbol{\sigma}}(s) + L'_{\boldsymbol{\sigma}}(s|t)}{S^M_{\boldsymbol{\sigma}}(t)} \cdot e^{-\int_t^s \mathbf{h}(u)du} ds \\
&= \int_t^\infty \frac{\lambda_L(1-\boldsymbol{\sigma}(s)) \cdot S^M_{\boldsymbol{\sigma}}(s) + \lambda_H \cdot L_{\boldsymbol{\sigma}}(s|t)}{S^M_{\boldsymbol{\sigma}}(t)} \cdot e^{-\int_t^s \mathbf{h}(u)du} ds,
\end{aligned}$$

$$\mathbb{E}[\tau_D \wedge \hat{\tau}_D - t \mid (\tau_M \wedge \hat{\tau}_D) > t] = \int_t^\infty \frac{S^M_{\boldsymbol{\sigma}}(s) + L_{\boldsymbol{\sigma}}(s|t)}{S^M_{\boldsymbol{\sigma}}(t)} \cdot e^{-\int_t^s \mathbf{h}(u)du} ds.$$

By plugging these into (D.21) and using (D.16), we can derive that

$$v_0(t; \boldsymbol{\sigma}, \mathbf{h}) = \int_t^\infty \left[\{\lambda_L(1-\boldsymbol{\sigma}(s))\Pi - c\} \cdot S^M_{\boldsymbol{\sigma}}(s) + (\lambda_H \Pi - c) \cdot L_{\boldsymbol{\sigma}}(s|t)\right] \cdot \frac{e^{-\int_t^s \mathbf{h}(u)du}}{S^M_{\boldsymbol{\sigma}}(t)} ds.$$

Thus, it remains to show that

$$\int_t^\infty \mu\, \boldsymbol{\sigma}(s) \cdot V_1(s; \mathbf{h}) \cdot S^M_{\boldsymbol{\sigma}}(s) \cdot e^{-\int_t^s \mathbf{h}(u)du}\, ds = (\lambda_H \Pi - c) \cdot \int_0^\infty L_{\boldsymbol{\sigma}}(s|t) \cdot e^{-\int_t^s \mathbf{h}(u)du}\, ds. \tag{D.23}$$

By plugging (D.1) into the left hand side of (D.23), we have

$$
\begin{aligned}
&\int_t^\infty \mu\,\boldsymbol{\sigma}(s)\cdot(\lambda_H\Pi-c)\cdot\left[\int_s^\infty e^{-\int_s^u(\lambda_H+\mathbf{h}(v))dv}du\right]\cdot S_{\boldsymbol{\sigma}}^M(s)\cdot e^{-\int_t^s\mathbf{h}(v)dv}\ ds\\
=&(\lambda_H\Pi-c)\cdot\int_t^\infty\left[\int_s^\infty \mu\,\boldsymbol{\sigma}(s)\cdot e^{-\lambda_H(u-s)}\cdot e^{-\int_t^u\mathbf{h}(v)dv}\cdot S_{\boldsymbol{\sigma}}^M(u)\ du\right]\ ds\\
=&(\lambda_H\Pi-c)\cdot\int_t^\infty\left[\int_t^u \mu\,\boldsymbol{\sigma}(u)\cdot e^{-\lambda_H(u-s)}\cdot S_{\boldsymbol{\sigma}}^M(s)\ ds\right]\cdot e^{-\int_t^s\mathbf{h}(v)dv}\ du\\
=&(\lambda_H\Pi-c)\cdot\int_t^\infty L_{\boldsymbol{\sigma}}(u|t)\cdot e^{-\int_t^u\mathbf{h}(v)dv}\ du.
\end{aligned}
$$

Thus, (D.2) holds.

Last, to show that (HJB_0) holds, we multiply $S_{\boldsymbol{\sigma}}^M(t)\cdot e^{-\int_0^t\mathbf{h}(u)du}$ to (D.2) and take a derivative:

$$
\begin{aligned}
&-\left[\lambda_L(1-\boldsymbol{\sigma}(t))\cdot\Pi+\mu\,\boldsymbol{\sigma}(t)\cdot V_1(t;\mathbf{h})-c\right]\cdot S_{\boldsymbol{\sigma}}^M(t)\cdot e^{-\int_0^t\mathbf{h}(u)du}\\
&=\frac{d}{dt}\left[v_0(t;\sigma,\mathbf{h})\cdot S_{\boldsymbol{\sigma}}^M(t)\cdot e^{-\int_0^t\mathbf{h}(u)du}\right]\\
&=\left[v_0'(t;\sigma,\mathbf{h})-\left(-\frac{S_{\boldsymbol{\sigma}}^{M\prime}(t)}{S_{\boldsymbol{\sigma}}^M(t)}+\mathbf{h}(t)\right)\cdot v_0(t;\boldsymbol{\sigma},\mathbf{h})\right]\cdot S_{\boldsymbol{\sigma}}^M(t)\cdot e^{-\int_0^t\mathbf{h}(u)du}.
\end{aligned}
$$

By using (D.13) and $S_\sigma^M(t)\cdot e^{-\int_0^t\mathbf{h}(u)du}>0$, we can see that ($\mathrm{HJB}_0$) holds. □

Corollary 3. *Let $\mathbf{h},\hat{\mathbf{h}}\in\mathcal{H}$ be two development functions such that $\mathbf{h}(s)=\hat{\mathbf{h}}(s)$ for all $s>t$. Then $V_0(t;\mathbf{h})=V_0(t;\hat{\mathbf{h}})$ and $V_1(t;\mathbf{h})=V_1(t;\hat{\mathbf{h}})$.*

D.5 Verification

In this subsection, we prove the verification result (Proposition D.1). To prove the verification result, it is useful to first introduce two convergence results.

Lemma D.4. *For any $\boldsymbol{\sigma}\in\mathcal{S}$, the following holds:*

$$\lim_{t\to\infty}S_{\boldsymbol{\sigma}}^D(t)\cdot V_1(t;\mathbf{h}_{\boldsymbol{\sigma}})=0.$$

Proof. Let $\Sigma_t := \int_0^t \boldsymbol{\sigma}(s)\, ds$. From $\lambda_H > \lambda_L$ and $\mu > \lambda_L$, we have

$$e^{-\mu t} \le S^M_{\boldsymbol{\sigma}}(t) = e^{-\lambda_L(t-\Sigma_t)-\mu\Sigma_t} \le e^{-\lambda_L t}, \tag{D.24}$$

$$\begin{aligned} 0 \le L_{\boldsymbol{\sigma}}(t) &= \int_0^t \mu\, \boldsymbol{\sigma}(s) \cdot S^M_{\boldsymbol{\sigma}}(s) \cdot e^{-\lambda_H(t-s)}\, ds \\ &< e^{-(\lambda_L+\lambda_H)t} \cdot \int_0^t \mu \cdot e^{\lambda_H s}\, ds < \frac{\mu}{\lambda_H} e^{-\lambda_L t}. \end{aligned} \tag{D.25}$$

Note that the left inequality of (D.24) binds when $\Sigma_t = t$, and the left inequality of (D.25) binds when $\Sigma_t = 0$. By (D.10), we have

$$e^{-\mu t} < S^D_{\boldsymbol{\sigma}}(t) = S^M_{\boldsymbol{\sigma}}(t) + L_{\boldsymbol{\sigma}}(t) < e^{-\lambda_L t} \cdot \left(\frac{\mu + \lambda_H}{\lambda_H}\right). \tag{D.26}$$

From (D.17) and (D.1), we have

$$S^D_{\boldsymbol{\sigma}}(t) \cdot V_1(t; \mathbf{h}_{\boldsymbol{\sigma}}) = (\lambda_H \Pi - c) \cdot \int_t^\infty e^{-\lambda_H(s-t)} \cdot S^D_{\boldsymbol{\sigma}}(s)\, ds.$$

By applying (D.26) and since $\lambda_H \Pi > \lambda_L \Pi > c$, we have

$$\begin{aligned} (\lambda_H \Pi - c) \cdot \int_t^\infty e^{-\lambda_H(s-t)} \cdot S^D_{\boldsymbol{\sigma}}(s)\, ds \ &> \ (\lambda_H \Pi - c) \cdot \int_t^\infty e^{-\lambda_H(s-t)} \cdot e^{-\mu s}\, ds \\ &= \frac{\lambda_H}{\mu + \lambda_H}\left(\Pi - \frac{c}{\lambda_H}\right) \cdot e^{-\mu t} \end{aligned}$$

and

$$\begin{aligned} (\lambda_H \Pi - c) \cdot \int_t^\infty e^{-\lambda_H(s-t)} \cdot S^D_{\boldsymbol{\sigma}}(s)\, ds \ &< \ (\lambda_H \Pi - c) \cdot \int_t^\infty e^{-\lambda_H(s-t)} \cdot \frac{\mu + \lambda_H}{\lambda_H} e^{-\lambda_L s}\, ds \\ &= \frac{\mu + \lambda_H}{\lambda_L + \lambda_H}\left(\Pi - \frac{c}{\lambda_H}\right) \cdot e^{-\lambda_L t}. \end{aligned}$$

Therefore, we have that

$$\frac{\lambda_H}{\mu + \lambda_H}\left(\Pi - \frac{c}{\lambda_H}\right) \cdot e^{-\mu t} < S^D_{\boldsymbol{\sigma}}(t) \cdot V_1(t; \mathbf{h}_{\boldsymbol{\sigma}}) < \frac{\mu + \lambda_H}{\lambda_L + \lambda_H}\left(\Pi - \frac{c}{\lambda_H}\right) \cdot e^{-\lambda_L t}. \tag{D.27}$$

Since the lower bound and the upper bound converge to 0 as t goes to infinity, we obtain the desired result. $\square$

Lemma D.5. *For any $\boldsymbol{\sigma}, \hat{\boldsymbol{\sigma}} \in \mathcal{S}$,*

$$\lim_{t\to\infty} v_0(t; \boldsymbol{\sigma}, \mathbf{h}_{\hat{\boldsymbol{\sigma}}}) \cdot S^M_{\boldsymbol{\sigma}}(t) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(t) = 0. \tag{D.28}$$

Proof. Note that for any time $s \in \mathbb{R}_+$, $-c < \lambda_L(1-\boldsymbol{\sigma}(s))\Pi - c < \lambda_L\Pi$. Since $\lambda_L\Pi > c$, we have $|\lambda_L(1-\boldsymbol{\sigma}(s))\Pi - c| < \lambda_L\Pi$.

From (D.2), we have

$$\begin{aligned} \left|v_0(t; \boldsymbol{\sigma}, \mathbf{h}_{\hat{\boldsymbol{\sigma}}}) \cdot S^M_{\boldsymbol{\sigma}}(t) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(t)\right| <\; & \lambda_L\Pi \cdot \int_t^\infty S^M_{\boldsymbol{\sigma}}(s) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(s)\ ds \\ & + \mu \cdot \int_t^\infty V_1(s; \mathbf{h}_{\hat{\boldsymbol{\sigma}}}) \cdot S^M_{\boldsymbol{\sigma}}(s) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(s)\ ds. \end{aligned}$$

Observe that from (D.24) and (D.26), we have

$$\int_t^\infty S^M_{\boldsymbol{\sigma}}(s) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(s)\ ds < \frac{\mu+\lambda_H}{\lambda_H} \cdot \int_t^\infty e^{-2\lambda_L s} ds = \frac{\mu+\lambda_H}{2\lambda_L\lambda_H} \cdot e^{-2\lambda_L t}.$$

In addition, from (D.27) and (D.26), we have

$$\begin{aligned} \int_t^\infty V_1(s; \mathbf{h}_{\hat{\boldsymbol{\sigma}}}) \cdot S^M_{\boldsymbol{\sigma}}(s) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(s)\ ds <& \frac{(\mu+\lambda_H)^2}{\lambda_H(\lambda_L+\lambda_H)} \cdot \left(\Pi - \frac{c}{\lambda_H}\right) \cdot \int_t^\infty e^{-2\lambda_L s} ds \\ =& \frac{(\mu+\lambda_H)^2}{2\lambda_L\lambda_H(\lambda_L+\lambda_H)} \cdot \left(\Pi - \frac{c}{\lambda_H}\right) \cdot e^{-2\lambda_L t}. \end{aligned}$$

Then, we have

$$\left|v_0(t; \boldsymbol{\sigma}, \mathbf{h}_{\hat{\sigma}}) \cdot S^M_{\boldsymbol{\sigma}}(t) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(t)\right| < \frac{\mu+\lambda_H}{2\lambda_L\lambda_H}\left[\lambda_L\Pi + \frac{\mu(\mu+\lambda_H)}{\lambda_L+\lambda_H}\left(\Pi - \frac{c}{\lambda_H}\right)\right] \cdot e^{-2\lambda_L t}.$$

Since the right-hand side of the above inequality converges to 0 as $t \to \infty$, (D.28) holds. $\square$

D.5.1 Proof of Proposition D.1

In this proof, we fix the policy of the opponent at $\hat{\boldsymbol{\sigma}}$. To save on notation, we will drop the dependency of the value and survival functions on $\hat{\boldsymbol{\sigma}}$ and the opponent's development rate

$\mathbf{h}_{\hat{\boldsymbol{\sigma}}}$. Specifically, we will abuse notation and use $V_1(t) \equiv V_1(t; \mathbf{h}_{\hat{\boldsymbol{\sigma}}})$, $v_0(t; \boldsymbol{\sigma}) \equiv v_0(t; \boldsymbol{\sigma}, \mathbf{h}_{\hat{\boldsymbol{\sigma}}})$, $\hat{S}(t) \equiv S^D_{\hat{\boldsymbol{\sigma}}}(t)$.

Proof of Proposition D.1 ($\Longleftarrow$). From $\boldsymbol{\sigma}^*$, we have that for all $\boldsymbol{\sigma} \in \mathcal{S}$ and $t \in R_+$

$$(\boldsymbol{\sigma}^*(t) - \boldsymbol{\sigma}(t)) \cdot [\mu \cdot (V_1(t) - v_0(t; \boldsymbol{\sigma}^*)) - \lambda_L \cdot (\Pi - v_0(t; \boldsymbol{\sigma}^*))] \geq 0 \tag{D.29}$$

Suppose that $v_0(t; \boldsymbol{\sigma}^*) > 0$. From (HJB$_0$), we have

$$\begin{aligned} 0 =& v_0'(t; \boldsymbol{\sigma}^*) - c - \mathbf{h}_{\hat{\boldsymbol{\sigma}}}(t) \cdot v_0(t; \boldsymbol{\sigma}^*) + \lambda_L \cdot (\Pi - v_0(t; \boldsymbol{\sigma}^*)) \\ &+ \boldsymbol{\sigma}^*(t) \cdot [\mu \cdot (V_1(t) - v_0(t; \boldsymbol{\sigma}^*)) - \lambda_L \cdot (\Pi - v_0(t; \boldsymbol{\sigma}^*))] \, . \end{aligned}$$

Then, (D.29) implies that, for any $\boldsymbol{\sigma} \in \mathcal{S}$ and $t \geq 0$,

$$\left\{ h^D_{\hat{\boldsymbol{\sigma}}}(t) + h^M_{\boldsymbol{\sigma}}(t) \right\} \cdot v_0(t; \boldsymbol{\sigma}^*) - v_0'(t; \boldsymbol{\sigma}^*) \geq \lambda_L (1 - \boldsymbol{\sigma}(t)) \cdot \Pi + \mu \, \boldsymbol{\sigma}(t) \cdot V_1(t) - c.$$

Multiplying side-by-side by $S^M_{\boldsymbol{\sigma}}(t) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(t)$, we have

$$-\frac{d}{dt} \left[v_0(t; \boldsymbol{\sigma}^*) \cdot S^M_{\boldsymbol{\sigma}}(t) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(t) \right] \geq \left[\lambda_L (1 - \boldsymbol{\sigma}(t)) \cdot \Pi + \mu \, \boldsymbol{\sigma}(t) \cdot V_1(t) - c \right] \cdot S^M_{\boldsymbol{\sigma}}(t) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(t)$$

for all $t \geq 0$. Integrating this inequality from 0 to ∞ and using Lemma D.3, we have

$$\begin{aligned} & v_0(0; \boldsymbol{\sigma}^*) \cdot S^M_{\boldsymbol{\sigma}}(0) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(0) - \lim_{t \to \infty} v_0(t; \boldsymbol{\sigma}^*) \cdot S^M_{\boldsymbol{\sigma}}(t) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(t) \\ \geq & \int_0^\infty \left[\lambda_L (1 - \boldsymbol{\sigma}(t)) \cdot \Pi + \mu \, \boldsymbol{\sigma}(t) \cdot V_1(t) - c \right] \cdot S^M_{\boldsymbol{\sigma}}(t) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(t) dt = \mathcal{U}(\boldsymbol{\sigma}, \hat{\boldsymbol{\sigma}}). \end{aligned}$$

Since $v_0(t; \boldsymbol{\sigma}^*)$, $S^M_{\boldsymbol{\sigma}}(t)$ and $S^D_{\hat{\boldsymbol{\sigma}}}(t)$ are strictly positive, we have

$$\lim_{t \to \infty} v_0(t; \boldsymbol{\sigma}^*) \cdot S^M_{\boldsymbol{\sigma}}(t) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(t) \geq 0.$$

By using this, $\mathcal{U}(\boldsymbol{\sigma}^*, \hat{\boldsymbol{\sigma}}) = v_0(0; \boldsymbol{\sigma}^*)$, and $S^M_{\boldsymbol{\sigma}}(0) = S^D_{\hat{\boldsymbol{\sigma}}}(0) = 1$, we obtain $\mathcal{U}(\boldsymbol{\sigma}^*, \hat{\boldsymbol{\sigma}}) \geq \mathcal{U}(\boldsymbol{\sigma}, \hat{\boldsymbol{\sigma}})$. $\square$

Proof of Proposition D.1 ($\Longrightarrow$). Suppose that $\boldsymbol{\sigma}^* \in \arg\max_{\boldsymbol{\sigma}\in\mathcal{S}} \mathcal{U}(\boldsymbol{\sigma}, \hat{\boldsymbol{\sigma}})$. From Lemma D.3, observe that for any $t \geq 0$, a firm's expected payoff can be rewritten as follows:

$$\begin{aligned}\mathcal{U}(\boldsymbol{\sigma}, \hat{\boldsymbol{\sigma}}) = &\int_0^t [\lambda_L(1-\boldsymbol{\sigma}(s)) \cdot \Pi + \mu\, \boldsymbol{\sigma}(s) \cdot V_1(s) - c] \cdot S^M_{\boldsymbol{\sigma}}(s) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(s)\ ds \\ &+ S^M_{\boldsymbol{\sigma}}(t) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(t) \cdot v_0(t; \boldsymbol{\sigma}).\end{aligned}$$

Now consider the following allocation policy $\tilde{\boldsymbol{\sigma}}(s) := \boldsymbol{\sigma}^*(s)1_{s<t}$. Then, $S^M_{\boldsymbol{\sigma}^*}(s) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(s) = S^M_{\tilde{\boldsymbol{\sigma}}}(s) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(s)$ for all $s \leq t$.[25] In addition, by using $\sigma^*(s) = \tilde{\boldsymbol{\sigma}}(s)$ for all $s < t$ and $\mathcal{U}(\boldsymbol{\sigma}^*, \hat{\boldsymbol{\sigma}}) \geq \mathcal{U}(\tilde{\boldsymbol{\sigma}}, \hat{\boldsymbol{\sigma}})$, we have $v_0(t; \boldsymbol{\sigma}^*) \geq v_0(t; \tilde{\boldsymbol{\sigma}})$.

Note that

$$v_0(t; \tilde{\boldsymbol{\sigma}}) = \int_t^\infty (\lambda_L \Pi - c) \cdot \frac{S^M_{\tilde{\boldsymbol{\sigma}}}(s)}{S^M_{\tilde{\boldsymbol{\sigma}}}(t)} \cdot \frac{S^D_{\hat{\boldsymbol{\sigma}}}(s)}{S^D_{\hat{\boldsymbol{\sigma}}}(t)}\ ds > 0$$

from $\lambda_L \Pi > c$, $S^M_{\tilde{\boldsymbol{\sigma}}}(s) > 0$, and $S^D_{\hat{\boldsymbol{\sigma}}}(s) > 0$. Therefore, $v_0(t; \boldsymbol{\sigma}^*) > 0$ for all $t \geq 0$.

Now assume that there exists $\boldsymbol{\sigma} \in \mathcal{S}$ such that (D.29) does not hold for some $t \geq 0$. Observe that $V_1(\cdot; \mathbf{h})$ and $v_0(\cdot; \boldsymbol{\sigma}, \mathbf{h})$ are continuous. Since $\boldsymbol{\sigma}^*$ and $\boldsymbol{\sigma}$ are right-continuous, there exists $\epsilon > 0$ such that for all $s \in [t, t+\epsilon)$,

$$(\boldsymbol{\sigma}^*(s) - \boldsymbol{\sigma}(s)) \cdot [\mu \cdot (V_1(s) - v_0(s; \boldsymbol{\sigma}^*) - \lambda_L \cdot (\Pi - v_0(s; \boldsymbol{\sigma}^*))] < 0. \tag{D.30}$$

Consider the following allocation policy $\boldsymbol{\sigma}^{**}$ defined by:

$$\boldsymbol{\sigma}^{**}(s) := \begin{cases} \boldsymbol{\sigma}^*(s), & \text{if } s \notin [t, t+\epsilon), \\ \boldsymbol{\sigma}(s), & \text{if } s \in [t, t+\epsilon). \end{cases}$$

By using a similar reformulation as in the previous case, we have

$$\begin{aligned} &-\frac{d}{ds}\left[v_0(s; \boldsymbol{\sigma}^*) \cdot S^M_{\boldsymbol{\sigma}^{**}}(s) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(s)\right] \\ &\leq [\lambda_L(1-\boldsymbol{\sigma}^{**}(s)) \cdot \Pi + \mu\sigma^{**}(s) \cdot V_1(s) - c] \cdot S^M_{\boldsymbol{\sigma}^{**}}(s) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(s) \end{aligned} \tag{D.31}$$

for all $s \geq 0$, and the inequality strictly holds for $s \in [t, t+\epsilon)$. Also note that by Lemma

[25] Note that the equality also holds at $s = t$, since $\boldsymbol{\sigma}^*$ and $\tilde{\boldsymbol{\sigma}}$ differ only at $\{t\}$, which is negligible after integration.

D.5,

$$\lim_{s\to\infty} v_0(s;\boldsymbol{\sigma}^*)\cdot S^M_{\boldsymbol{\sigma}^{**}}(s)\cdot S^D_{\hat{\boldsymbol{\sigma}}}(s) = \lim_{s\to\infty} v_0(s;\boldsymbol{\sigma}^*)\cdot S^M_{\boldsymbol{\sigma}^{*}}(s)\cdot S^D_{\hat{\boldsymbol{\sigma}}}(s) = 0.$$

By integrating (D.31) from 0 to ∞, we have

$$\begin{aligned}\mathcal{U}(\boldsymbol{\sigma}^*,\hat{\boldsymbol{\sigma}}) &= v_0(0;\boldsymbol{\sigma}^*)\\ &< \int_0^\infty [\lambda_L(1-\boldsymbol{\sigma}^{**}(s))\cdot\Pi + \mu\,\boldsymbol{\sigma}^{**}(s)\cdot V_1(s) - c]\cdot S^M_{\boldsymbol{\sigma}^{**}}(s)\cdot S^D_{\hat{\boldsymbol{\sigma}}}(s)\; ds\\ &= \mathcal{U}(\boldsymbol{\sigma}^{**},\hat{\boldsymbol{\sigma}}),\end{aligned}$$

which contradicts $\boldsymbol{\sigma}^* \in \arg\max_{\boldsymbol{\sigma}\in\mathcal{S}} \mathcal{U}(\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}})$. Therefore, (D.29) holds for all $t \geq 0$. □

D.6 The Single-Crossing Property

D.6.1 Monotonicity of V_1 and V_0

We start by considering an opponent with a constant development rate. The following lemma characterizes the best response in this case, in line with the best responses described for the case of public information in equations (C.2) and (C.3).

Lemma D.6. *For any constant development rate $\lambda \in \mathbb{R}_+$, $V_1(t;\lambda)$, $V_0(t;\lambda)$, and $R(t;\lambda)$ are constant over time. Moreover,*

$$sgn(R(t;\lambda)) = sgn(\lambda_\star - \lambda).$$

Proof. Since the allocation problem of a firm when the opponent develops at a constant rate λ is memoryless, there must be a constant research rate $\sigma^* \in [0,1]$ that is optimal. Then,

$$V_1(t;\lambda) = \frac{\lambda_H \Pi - c}{\lambda_H + \lambda} \qquad \text{and} \qquad V_0(t;\lambda) = \frac{(1-\sigma^*)\lambda_L\Pi + \sigma^*\mu V_1(t;\lambda) - c}{(1-\sigma^*)\lambda_L + \sigma^*\mu + \lambda}.$$

Observe that these two value functions are constant in t. Thus, using these expressions, we obtain:

$$
\begin{aligned}
R(t;\lambda) &= \mu(V_1(0;\lambda) - V_0(0;\lambda) - \lambda_L(\Pi - V_0(0;\lambda)) \\
&= \frac{(\lambda\Pi + c)\lambda_L(\lambda_\star - \lambda)}{(\lambda+\lambda_H)(\lambda + (1-\sigma^*)\lambda_L + \sigma^*\mu)}
\end{aligned}
$$

Which is also constant in t and shares the sign of $(\lambda_\star - \lambda)$. □

We now consider an opponent with a weakly increasing development rate.

Lemma D.7. *Let $\mathbf{h} \in \mathcal{H}$ be weakly increasing. Then, $V_1(t;\mathbf{h})$ and $V_0(t,\mathbf{h})$ are weakly decreasing in t.*

Proof. Note that

$$
\begin{aligned}
V_1(t;\mathbf{h}) =& (\lambda_H\Pi - c) \cdot \int_t^\infty e^{-\int_t^s (\mathbf{h}(u)+\lambda_H)\ du}\ ds \\
&\leq (\lambda_H\Pi - c) \cdot \int_t^\infty e^{-(\mathbf{h}(t)+\lambda_H)(s-t)} ds = \frac{\lambda_H\Pi - c}{\mathbf{h}(t) + \lambda_H}
\end{aligned}
$$

From (HJB$_1$), we have

$$
V_1'(t;\mathbf{h}) = -(\lambda_H\Pi - c) + (\lambda_H + \mathbf{h}(t)) \cdot V_1(t;\mathbf{h}) \leq 0. \tag{D.32}
$$

Therefore, $V_1(t;\mathbf{h})$ is decreasing in t.

Next, let $\boldsymbol{\sigma}^*$ be a policy satisfying $V_0(t;\mathbf{h}) = v_0(t;\boldsymbol{\sigma}^*, \mathbf{h})$. Note that for all $s \geq t$, $V_0(s;\mathbf{h}) \geq v_0(s;\mathbf{0},\mathbf{h}) > 0$ from $\Pi > c/\lambda_L$. Additionally, from (HJB$_0$),

$$
\begin{aligned}
& (1 - \boldsymbol{\sigma}^*(s))\lambda_L(\Pi - V_0(s;\mathbf{h})) + \boldsymbol{\sigma}^*(s)\mu(V_1(s;\mathbf{h}) - V_0(s;\mathbf{h})) \geq \lambda_L(\Pi - V_0(s;\mathbf{h})) \\
\Rightarrow \quad & \boldsymbol{\sigma}^*(s) \cdot \mu \cdot V_1(s;\mathbf{h}) + (1 - \boldsymbol{\sigma}^*(s)) \cdot \lambda_L\Pi - c \\
& \geq \boldsymbol{\sigma}^*(s) \cdot \mu \cdot V_0(s;\mathbf{h}) + (1 - \boldsymbol{\sigma}^*(s)) \cdot \lambda_L \cdot V_0(s;\mathbf{h}) + \lambda_L(\Pi - V_0(s;\mathbf{h})) - c \\
& \quad = (\lambda_L\Pi - c) + \boldsymbol{\sigma}^*(s) \cdot (\mu - \lambda_L) \cdot V_0(s;\mathbf{h}) \geq 0.
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
v_0(t;\sigma^*,\mathbf{h}) &= \int_t^\infty \{\boldsymbol{\sigma}^*(s)\cdot\mu\cdot V_1(s;\mathbf{h}) + (1-\boldsymbol{\sigma}^*(s))\cdot\lambda_L\Pi - c\}\cdot \mathbf{r}_{\mathbf{h},\boldsymbol{\sigma}^*}(s;t)ds \\
&\le \int_t^\infty \left\{\boldsymbol{\sigma}^*(s)\cdot\mu\cdot V^1(t;\mathbf{h}) + (1-\boldsymbol{\sigma}^*(s))\cdot\lambda_L\Pi - c\right\}\cdot \mathbf{r}_{\mathbf{h},\boldsymbol{\sigma}^*}(s;t)ds \\
&\le \max_{\sigma\in[0,1]} \frac{\sigma\cdot\mu\cdot V_1(t;\mathbf{h}) + (1-\sigma)\cdot\lambda_L\Pi - c}{(1-\sigma)\lambda_L + \sigma\mu + \mathbf{h}(t)}
\end{aligned}
$$

Let the solution of the maximization problem of the right hand side is $\hat{\sigma}$. Then, we have

$$
\begin{aligned}
0 \ge& - \{(1-\hat{\sigma})\lambda_L(\Pi - V_0(t;\mathbf{h})) + \hat{\sigma}\mu(V_1(t;\mathbf{h}) - V_0(t;\mathbf{h}))\} \\
&+ c + \mathbf{h}(t)\cdot V_0(t;\mathbf{h}) \\
\ge& - \max_{\sigma\in[0,1]} \{(1-\sigma)\lambda_L(\Pi - V_0(t;\mathbf{h})) + \sigma\mu(V_1(t;\mathbf{h}) - V_0(t;\mathbf{h}))\} \\
&+ c + \mathbf{h}(t)\cdot V_0(t;\mathbf{h}) \\
&= V_0'(t;\mathbf{h}).
\end{aligned}
$$

Therefore, $V_0(t;\mathbf{h})$ is decreasing in t. □

D.6.2 Proof of Proposition D.3

Proof of Proposition D.3. It is sufficient to show that $R(t;\mathbf{h}) \le 0$ implies $\frac{\partial R}{\partial t}(t;\mathbf{h}) < 0$. Note that

$$
\frac{\partial R}{\partial t}(t;\mathbf{h}) = \mu\cdot(V_1'(t;\mathbf{h}) - V_0'(t;\mathbf{h})) + \lambda_L\cdot V_0'(t;\mathbf{h}).
$$

By Lemma D.7, we have $V_0'(t;\mathbf{h}) \le 0$.

By subtracting (HJB$_1$) and (HJB$_0$), we have

$$
\begin{aligned}
&V_1'(t;\mathbf{h}) - V_0'(t;\mathbf{h}) \\
&= -\lambda_H \Pi + c + (\lambda_H + \mathbf{h}(t))V_1(t;\mathbf{h}) + \lambda_L(\Pi - V_0(t;\mathbf{h})) \\
&\quad - c - \mathbf{h}(t) \cdot V_0(t;\mathbf{h}) \\
&= -\lambda_H(\Pi - V_1(t;\mathbf{h})) + \lambda_L(\Pi - V_h^0) + \mathbf{h}(t)(V_1(t;\mathbf{h}) - V_h^0) \\
&= -\lambda_H(\Pi - V_0(t;\mathbf{h})) + \lambda_H(V_1(t;\mathbf{h}) - V_0(t;\mathbf{h})) \\
&\qquad\qquad + \lambda_L(\Pi - V_0(t;\mathbf{h})) + \mathbf{h}(t)(V_1(t;\mathbf{h}) - V_0(t;\mathbf{h})) \\
&= (\lambda_H + \mathbf{h}(t))(V_1(t;\mathbf{h}) - V_0(t;\mathbf{h})) - (\lambda_H - \lambda_L)(\Pi - V_0(t;\mathbf{h})).
\end{aligned}
$$

If $R(t;\mathbf{h}) \leq 0$, we have

$$
V_1(t;\mathbf{h}) - V_0(t;\mathbf{h}) \leq \frac{\lambda_L}{\mu}(\Pi - V_0(t;\mathbf{h})).
$$

By plugging this in, we have

$$
\begin{aligned}
V_1'(t;\mathbf{h}) - V_0'(t;\mathbf{h}) \leq& (\lambda_H + \mathbf{h}(t))\frac{\lambda_L}{\mu}[\Pi - V_0(t;\mathbf{h})] - (\lambda_H - \lambda_L)(\Pi - V_0(t;\mathbf{h})) \\
=& \frac{\lambda_L}{\mu}\left[\mathbf{h}(t) - \frac{\mu(\lambda_H - \lambda_L) - \lambda_H\lambda_L}{\lambda_L}\right](\Pi - V_0(t;\mathbf{h})) \\
=& \frac{\lambda_L}{\mu}\left[\mathbf{h}(t) - \lambda_\star\right](\Pi - V_0(t;\mathbf{h})).
\end{aligned}
$$

From $\mathbf{h}(t) < \lambda_\star$, we have $V_1'(t;\mathbf{h}) - V_0'(t;\mathbf{h}) < 0$. Then, by (D.6.2), we have $\frac{\partial R}{\partial t}(t;\mathbf{h}) < 0$. □

D.7 Equilibrium Characterization

D.7.1 Useful Properties

Lemma D.8 (Limit incentives). *Let $\mathbf{h} \in \mathcal{H}$ be increasing with $\mathbf{h}(t) \to \bar{h}$. Then $R(t;\mathbf{h}) \to R(t;\bar{h})$.*

Proof. First, we show that $V_1(t;\mathbf{h})$ converges to $V_1(0;\bar{h})$. Since $\mathbf{h}(t) \leq \mathbf{h}(s) \leq \bar{h}$ for all $s > t$, we can bound V_1 by the value when the opponent has constant hazard rates $\mathbf{h}(t)$ and $\bar{h}$.

$$V_1(0;\bar{h}) = V_1(t;\bar{h}) \leq V_1(t;\mathbf{h}) \leq V_1(t;\mathbf{h}(t)) = V_1(0;\mathbf{h}(t))$$

By continuity of $V_1(0;h)$ in h, and the fact that the upper-bound $V_1(0;\mathbf{h}(t))$ converges to the lower-bound $V_1(0;\bar{h})$, we can apply the squeeze theorem to get $V_1(t;\mathbf{h}) \to V_1(0;\bar{h})$. We can obtain bounds for $V_0(t;\mathbf{h})$ using a similar logic. Since $h(t) \leq h(s) \leq \bar{h}$,

$$V_0(0;\bar{h}) = V_0(t;\bar{h}) \leq V_0(t;\mathbf{h}) \leq V_0(t;\mathbf{h}(t)) = V_0(0;\mathbf{h}(t))$$

Using the continuity of $V_0(0;h)$ in h, as V_0 is the maximum of continuous functions, and applying the squeeze theorem, we obtain that $V_0(t;\mathbf{h}) \to V_0(0;\bar{h})$. □

Lemma D.9. *Let T be a finite time and consider a policy $\boldsymbol{\sigma}$ such that $\boldsymbol{\sigma}(t) = 1$ for all $t > T$. Then, $\mathbf{h}_{\boldsymbol{\sigma}}(t) \to \min\{\lambda_H, \mu\}$.*

Proof. From the evolution of beliefs, using that $\boldsymbol{\sigma}(t) = 1$, we get that, for all $t > T$,

$$\dot{\mathbf{p}}_{\boldsymbol{\sigma}}(t) = (1 - \mathbf{p}_{\boldsymbol{\sigma}}(t))\,[\mu - \lambda_H\,\mathbf{p}_{\boldsymbol{\sigma}}(t)]$$

This evolution of beliefs gives us that $\mathbf{p}_{\boldsymbol{\sigma}}(t)$ converges to 1 when $\mu > \lambda_H$ and to μ/λ_H when $\mu \leq \lambda_H$. Using this, together with $\boldsymbol{\sigma}_t = 0$, in the hazard rate function and taking limits, we obtain that

$$\lim_{t\to\infty} \mathbf{h}_{\boldsymbol{\sigma}}(t) = \lim_{t\to\infty} \lambda_H\,\mathbf{p}_{\boldsymbol{\sigma}}(t) = \lambda_H \cdot \min\{1, \mu/\lambda_H\} = \min\{\lambda_H, \mu\}.$$

□

D.7.2 Case (i): $\lambda_\star < \lambda_L$.

In this subsection we prove Theorem 2 (i).

Proof. Let $(\boldsymbol{\sigma}_A, \boldsymbol{\sigma}_B)$ be a MDNE. Then, by MDR of $\boldsymbol{\sigma}_j$, it must be that $\mathbf{h}_{\boldsymbol{\sigma}_j}$ is increasing.

$h_{\boldsymbol{\sigma}_j}$ is also bounded by λ_H, and therefore it converges. We denote $\bar{h}$ the limit of $h_{\boldsymbol{\sigma}_j}(t)$ when $t \to \infty$.

Note that $\bar{h} \geq \lambda_L$: otherwise $\mathbf{h}_{\boldsymbol{\sigma}_j}(t) < \bar{h} < \lambda_L$ for all t and, thus, it would be more profitable for the firm to choose $\boldsymbol{\sigma} = 0$, which induces a constant rate of development equal to λ_L. By continuity, the relative attractiveness of research $R(t; \mathbf{h}_{\boldsymbol{\sigma}_j})$ converges to $R_{\bar{h}} < 0$, where the inequality holds since $\bar{h} \geq \lambda_L > \lambda_\star$. This implies that there is a time T such that $R(t; \mathbf{h}_{\boldsymbol{\sigma}_j}) < 0$ for all $t \geq T$. By Proposition D.1, it must be that $\boldsymbol{\sigma}_i(t) = 0$ for all $t \geq T$. It remains to show that $\boldsymbol{\sigma}_i(t) = 0$ for all $t \leq T$, which follows immediately from applying Proposition D.2.

Summarizing, $(0, 0)$ is the unique candidate for MDNE. First notice that the policy 0 satisfies MDR since $\mathbf{h}_0$ is constant and equal to λ_L. Moreover, to check that $(0, 0)$ is a Nash equilibrium, notice that $\lambda_L > \lambda_\star$, which implies by Proposition 3.1 that developing with the old technology is a best response. □

D.7.3 Case (ii): $\lambda_\star > \min\{\mu, \lambda_H\}$.

We begin the proof of Theorem 2 part (ii) by obtaining an upper bound for the development rate for any policy with monotone development rates.

Lemma D.10. *Let $\boldsymbol{\sigma} \in \mathcal{S}$ be MDR. Then, $\mathbf{h}_{\boldsymbol{\sigma}} < \min\{\mu, \lambda_H\}$.*

Proof. First, observe that for any $\boldsymbol{\sigma} \in \mathcal{S}$ and $t \geq 0$, $\mathbf{p}_{\boldsymbol{\sigma}}(t) \leq \min\{\mu/\lambda_H, 1\}$. Suppose toward a contradiction that there is a T such that $\mathbf{p}_{\boldsymbol{\sigma}}(T) > \min\{\mu/\lambda_H, 1\}$. Then, by continuity of $\mathbf{p}_{\boldsymbol{\sigma}}$, there must be a $t < T$ such that $\mathbf{p}_{\boldsymbol{\sigma}}(t) \in (\min\{\mu/\lambda_H, 1\}, \mathbf{p}_{\boldsymbol{\sigma}}(T))$ and $\dot{\mathbf{p}}_{\boldsymbol{\sigma}}(t) > 0$. However,

$$
\begin{aligned}
\dot{\mathbf{p}}_{\boldsymbol{\sigma}}(t) &= \mu(1 - \mathbf{p}_{\boldsymbol{\sigma}}(t))\,\boldsymbol{\sigma}(t) - (\lambda_H - (1 - \boldsymbol{\sigma}(t))\lambda_L)\,\mathbf{p}_{\boldsymbol{\sigma}}(t)(1 - \mathbf{p}_{\boldsymbol{\sigma}}(t)) \\
&\leq [\mu - \lambda_H\,\mathbf{p}_{\boldsymbol{\sigma}}(t)]\,(1 - \mathbf{p}_{\boldsymbol{\sigma}}(t)) < 0
\end{aligned}
$$

Where the first inequality holds because the $\delta(\sigma, p)$, as defined in (5.2), is increasing in σ and the second inequality holds because if $p_{\boldsymbol{\sigma}}(t) > \min\{\mu/\lambda_H, 1\}$ is only possible if $\mu < \lambda_H$ and $p_{\boldsymbol{\sigma}}(t) > \mu/\lambda_H$.

Next we prove that for any policy $\boldsymbol{\sigma}$ satisfying MDR, the hazard rate $\mathbf{h}_{\boldsymbol{\sigma}}$ never exceeds $\min\{\mu, \lambda_H\}$. First,

$$\mathbf{h}_{\boldsymbol{\sigma}}(t) = \mathbf{p}_{\boldsymbol{\sigma}}(t) \cdot \lambda_H + (1 - \mathbf{p}_{\boldsymbol{\sigma}}(t)) \underbrace{(1 - \boldsymbol{\sigma}(t))\lambda_L}_{<\lambda_H} < \lambda_H$$

It remains to show that, when $\mu < \lambda_H$, $\mathbf{h}_{\boldsymbol{\sigma}}(t) < \mu$. First, we can see that $\dot{\mathbf{p}}_{\boldsymbol{\sigma}}(t) \geq 0$ implies $\mathbf{h}_{\boldsymbol{\sigma}}(t) \leq \mu$.

$$\dot{\mathbf{p}}_{\boldsymbol{\sigma}}(t) = [\mu\, \boldsymbol{\sigma}(t) - (\lambda_H - (1 - \boldsymbol{\sigma}(t))\lambda_L)\, \mathbf{p}_{\boldsymbol{\sigma}}(t)](1 - \mathbf{p}_{\boldsymbol{\sigma}}(t)) \geq 0$$

Since $\mathbf{p}_{\boldsymbol{\sigma}}(t) < 1$, this holds if and only if

$$\mu\, \boldsymbol{\sigma}(t) \geq (\lambda_H - (1 - \boldsymbol{\sigma}(t))\lambda_L)\, \mathbf{p}_{\boldsymbol{\sigma}}(t)$$

In this case,

$$\mathbf{h}_{\boldsymbol{\sigma}}(t) = (1 - \boldsymbol{\sigma}(t))\lambda_L + p(\lambda_H - (1 - \boldsymbol{\sigma}(t))\lambda_L) \leq (1 - \boldsymbol{\sigma}(t))\lambda_L + \boldsymbol{\sigma}(t)\mu \leq \mu$$

Thus, $\mathbf{h}_{\boldsymbol{\sigma}}(T) > \mu$ implies $\dot{\mathbf{p}}_{\boldsymbol{\sigma}}(t) < 0$ for all $t > T$. $\mathbf{p}_{\boldsymbol{\sigma}}$ is bounded below by 0, thus it must converge. Let $\bar{p}$ be the limit of $\mathbf{p}_{\boldsymbol{\sigma}}(t)$ when $t \to \infty$. Moreover, $\mathbf{h}_{\boldsymbol{\sigma}}$ increasing with decreasing $\mathbf{p}_{\boldsymbol{\sigma}}$ implies that $\boldsymbol{\sigma}$ has to be decreasing as well. Since $\boldsymbol{\sigma}$ is bounded, it must converge as well. Let $\bar{\sigma}$ be the limit of $\boldsymbol{\sigma}(t)$ when $t \to \infty$. However, δ is continuous at $(\bar{p}, \bar{\sigma})$ and $\delta(\bar{p}, \bar{\sigma})$ is bounded away from zero, which contradicts the limit of $\mathbf{p}_{\boldsymbol{\sigma}}$. □

Proof of Theorem 2 (ii). Let $\mathbf{h}$ be the opponent's equilibrium hazard rate. Since $\mathbf{h}$ is increasing and bounded, it must be that it converges. Let $\bar{h}$ be the limit of $\mathbf{h}(t)$ when $t \to \infty$. Note that, for all t, $\mathbf{h}(t) < \bar{h} < \min\{\mu, \lambda_H\} < \lambda_\star$, where the first inequality holds by monotonicity of $\mathbf{h}$, the second inequality by Proposition D.2, and the third inequality by assumption. By applying Lemma D.6, we obtain that $R(t; \bar{h}) > 0$. Thus, by Lemma D.8 there is a time T such that $R(t; \mathbf{h}) > 0$ for all $t > T$. Suppose toward a contradiction that $R(s; \mathbf{h}) < 0$ for some $s \in \mathbb{R}_+$. Then, by Proposition D.3, it must be that $R(s; \mathbf{h}) < 0$ for all $s > T$. Thus, there is no such s and $R(t; \mathbf{h}) \geq 0$ for all t. This is true for both firms, so using Proposition D.1, we have that $(1, 1)$ is the only equilibrium candidate.

It remains to check that $(1,1)$ is a MDNE. First, observe that $\mathbf{h}_1$ is increasing, since $\dot{\mathbf{h}}_1(t) = \lambda_H \dot{\mathbf{p}}_1(t) = \lambda_H(\mu - \lambda_H \, \mathbf{p_1}(t))(1 - \mathbf{p_1}(t))$ and $(\mu - \lambda_H \, \mathbf{p_1}(t)) > 0$ by Lemma D.10. By Lemma D.9, $\mathbf{h}_1$ converges to $\min\{\mu, \lambda_H\}$, which is lower than $\lambda_\star$. Therefore, there is a time T such that $R(t; \mathbf{h}_1) > 0$ for all $t > T$. Moreover, suppose toward a contradiction that $R(s; \mathbf{h}) < 0$ for some $s \in \mathbb{R}_+$. Then, by Proposition D.3, it must be that $R(s; \mathbf{h}) < 0$ for all $s > T$. Thus, there is no such s and $R(t; \mathbf{h}) \geq 0$ for all t. Thus, by the verification result, $\boldsymbol{\sigma} = 1$ is a best response to $\mathbf{h}_1$ and $(1,1)$ is a NE. □

D.7.4 Case (iii): $\lambda_\star \in (\lambda_L, \min\{\mu, \lambda_H\})$.

Lemma D.11. *Let $\lambda_\star \in (\lambda_L, \lambda_H)$, and let $\mathbf{h}$ be increasing with $\mathbf{h}(t) \to \lambda_\star$. Let T be the first time at which $\mathbf{h}(T) = \lambda_\star$. Then $R(t; \mathbf{h}) > 0$ for all $t < T$ and $R(t; \mathbf{h}) = 0$ for all $t \geq T$.*

Proof. First, note that $\mathbf{h}(s) = \lambda_\star$ for all $s \geq T$. Therefore, by Corollary 3, $R(t; \mathbf{h}) = R(0; \lambda_\star) = 0$ for all $t \geq T$. Let $\hat{T}$ be the first time it is profitable to use the old technology, i.e. $\hat{T} := \inf\{t \in [0, \infty] : R(t; \mathbf{h}) \leq 0\}$. Observe that, since $R(T; \mathbf{h}) = 0$, it must be that $\hat{T} \leq T$. Next, we show that $\hat{T} < T$ leads to a contradiction.

Suppose towards a contradiction that $\hat{T} < T$. By Proposition D.3, $R(t, \mathbf{h}) \leq 0$ for all $t \geq \hat{T}$. Additionally, in the proof of Proposition D.3, we show that $R(t, \mathbf{h}) \leq 0$ implies $R'(t, \mathbf{h}) < 0$, which gives $R(T, \mathbf{h}) < 0$ which contradicts $R(T, \mathbf{h}) = 0$. □

The next lemma shows that if the opponent does research first ($\boldsymbol{\sigma}_j(t) = 1$ for all t) it is not a best-response to do direct development.

Lemma D.12. *Let $\lambda_\star \in (\lambda_L, \min\{\lambda_H, \mu\})$. Then $R(0, \mathbf{h}_1) > 0$.*

Proof. $\mathbf{h}_1$ is the development rate associated with the research policy ($\boldsymbol{\sigma} = 1$). We can compute the continuation value, at time zero, of doing direct development $v_0(0; 0, \mathbf{h}_1)$.

$$v_0(0; 0, \mathbf{h}_1) = \Pi \left[\frac{\lambda_L}{\lambda_L + \mu} + \frac{\mu}{\lambda_L + \mu} \cdot \frac{\lambda_L}{\lambda_L + \lambda_H} \right] - c \left[\frac{1}{\lambda_L + \mu} + \frac{\mu}{\lambda_L + \mu} \cdot \frac{1}{\lambda_L + \lambda_H} \right]$$

The first bracket captures the probability of the firm winning the race. The firm can win by developing before the opponent finds the new technology—which happens with probability

$\lambda_L/(\lambda_L+\mu)$—or the opponent can find the new technology first, in which case the firm wins with probability $\lambda_L/\lambda_L+\lambda_H$. The second bracket captures the expected duration of the race. The expected time before the first breakthrough in the race is $1/(\lambda_L+\mu)$. If the opponent finds the new technology—which happens with probability $\mu/(\lambda_L+\mu)$—the race is extended by $1/(\lambda_L+\lambda_H)$ in expectation. By doing some algebra, we obtain that:

$$v_0(0;0,\mathbf{h}_1)=\frac{\lambda_L\Pi-c}{\lambda_L+\mu}\cdot\frac{\lambda_L+\lambda_H+\mu}{\lambda_L+\lambda_H}$$

We can obtain $V_1(0,h_1)$ by using the same logic, but replacing the development rate of the incumbent technology λ_L with the development rate of the new technology λ_H.

$$V_1(0;\mathbf{h}_1)=\frac{\lambda_H\Pi-c}{\lambda_H+\mu}\cdot\frac{\lambda_H+\lambda_H+\mu}{\lambda_H+\lambda_H}$$

Suppose toward a contradiction that direct development ($\sigma=0$) is a best response toward research first ($\sigma=1$). This implies that $V_0(t;\mathbf{h}_1)=v_0(t;0,\mathbf{h}_1)$ and that $R(t;h_1)\leq 0$ for all t. However,

$$\begin{aligned}
R(0;\mathbf{h}_1) &= \mu(V_1(0;\mathbf{h}_1)-V_0(0;\mathbf{h}_1))-\lambda_L(\Pi-V_0(0,\mathbf{h}_1))\\
&= \frac{c\left(\left(\lambda_\star-\lambda_L\right)\left(2\lambda_H+\mu\right)+\lambda_L\left(\lambda_H-\lambda_\star\right)\right)}{2\lambda_H\left(\lambda_H+\mu\right)}\\
&\qquad +\mu\cdot(\lambda_L\Pi-c)\cdot\frac{\left(\lambda_\star-\lambda_L\right)\left(2\lambda_H+\mu+\lambda_L\right)+\lambda_L\left(2\lambda_H+\lambda_L-\lambda_\star\right)}{2\left(\lambda_H+\mu\right)\left(\lambda_H+\lambda_L\right)\left(\mu+\lambda_L\right)}\\
&> 0
\end{aligned}$$

Where the inequality uses that $\lambda_L\Pi-c>0$ and that $\lambda_\star\in(\lambda_L,\lambda_H)$. $\square$

Lemma D.13. *Let $\lambda_\star\in(\lambda_L,\min\{\lambda_H,\mu\})$ and let $(\boldsymbol{\sigma}_A,\boldsymbol{\sigma}_B)$ be a MDNE. Then $\mathbf{h}_{\boldsymbol{\sigma}_A}(t)$ and $\mathbf{h}_{\boldsymbol{\sigma}_B}(t)$ converge to $\lambda_\star$.*

Proof. First, note that $\mathbf{h}_{\boldsymbol{\sigma}_i}$ is weakly increasing and bounded above by λ_H. Thus, $\mathbf{h}_{\boldsymbol{\sigma}_i}(t)$ must converge. Let $\bar{h}_i$ be the limit of $\mathbf{h}_{\boldsymbol{\sigma}_i}(t)$ when t goes to infinity.

Suppose towards a contradiction that $\bar{h}_i>\lambda_\star$. Then, by Lemma D.8, $R(t;\mathbf{h}_{\boldsymbol{\sigma}_i})$ converges $R(0;\bar{h}_i)$. Since $\bar{h}>\lambda_\star$, applying Lemma D.6, we get that $R(0;\bar{h}_i)<0$. Thus, there is a time

T for which $R(t;\mathbf{h}_{\boldsymbol{\sigma}_i}) < 0$ for all $t > T$. This implies that $\boldsymbol{\sigma}_j = 0$ for all $t > T$ and moreover, by Proposition D.2, $\boldsymbol{\sigma}_j = 0$. Therefore, $\mathbf{h}_{\boldsymbol{\sigma}_j} = \mathbf{h}_0 = \lambda_L$. Since $\lambda_L < \lambda_\star$, it must be, by D.6, that $R(t;\lambda_L) > 0$. Thus, since $\boldsymbol{\sigma}_i$ is a best-response, $\boldsymbol{\sigma}_i = 1$. However, $(\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_j) = (0,1)$ is ruled out as an equilibrium by Lemma D.12. Therefore, there cannot be an equilibrium in which one of the development rates converges to a rate greater than $\lambda_\star$.

Suppose towards a contradiction that $\bar{h}_i < \lambda_\star$. Then, by Lemma D.8, $R(t;\mathbf{h}_{\boldsymbol{\sigma}_i})$ converges to $R(0;\bar{h}) > 0$. Thus, there is a time T such that $R(t,\mathbf{h}_{\boldsymbol{\sigma}_i}) > 0$ for all $t > T$. Since $\boldsymbol{\sigma}_j$ is a best-response, it must be that $\boldsymbol{\sigma}_j(t) = 1$ for all $t > T$. By Lemma D.9, $\mathbf{h}_j$ converges to $\min\{\mu, \lambda_H\} > \lambda_\star$. However, we showed that this was not possible. □

Proposition D.4. *Let $\lambda_\star \in (\lambda_L, \min\{\lambda_H, \mu\})$ and $(\boldsymbol{\sigma}_A, \boldsymbol{\sigma}_B)$ be a MDNE. Then $\boldsymbol{\sigma}_A = \boldsymbol{\sigma}_B = \boldsymbol{\sigma}_\star$, where $\boldsymbol{\sigma}_\star(t) = 1$ for every t such that $\mathbf{p}_{\boldsymbol{\sigma}_\star}(t) < p_\star$ and $\boldsymbol{\sigma}_\star(t) = \sigma_\star$ when $p_{\boldsymbol{\sigma}_\star}(t) = p_\star$.*

Proof. By Lemma D.13, it must be that $\mathbf{h}_{\boldsymbol{\sigma}_A}$ and $\mathbf{h}_{\boldsymbol{\sigma}_B}$ converge to $\lambda_\star$. For $i = A, B$, let $T_i = \sup\{t \;:\; \mathbf{h}_{\boldsymbol{\sigma}_i}(t) < \lambda_\star\}$ and let $T = \min\{T_A, T_B\}$.

Suppose towards a contradiction that $T_A < T_B$. By Lemma D.11, we know that $R(t;\mathbf{h}_B) > 0$ for all $t < T_B$. This means that $\boldsymbol{\sigma}_A(t) = 1$ for $t \in (T_A, T_B)$. This, however, contradicts the fact that $\mathbf{h}_{\boldsymbol{\sigma}_A}$ is constant and equal to $\lambda_\star$ on that interval:

$$\dot{\mathbf{h}}_{\boldsymbol{\sigma}_A} = \dot{\mathbf{p}}_{\boldsymbol{\sigma}_A} = (\mu - \lambda_H\, \mathbf{p}_{\boldsymbol{\sigma}_A}(t))(1 - \mathbf{p}_{\boldsymbol{\sigma}_A}(t))\lambda_H > 0 \qquad \forall t \in (T_A, T_B)$$

Thus, $T_A = T_B = T$ with $\boldsymbol{\sigma}_i(t) = 1$ for $t < T$.

For $s > T$ and $i \in \{A, B\}$, we have $\mathbf{h}_{\boldsymbol{\sigma}_i}(s) = \lambda_\star$. Using the definition of $\mathbf{h}_{\boldsymbol{\sigma}_i}$, we have that

$$\lambda_\star = \lambda_H\, \mathbf{p}_{\boldsymbol{\sigma}_i}(s) + \lambda_L(1 - \mathbf{p}_{\boldsymbol{\sigma}_i}(s))(1 - \boldsymbol{\sigma}^*(s)),$$

or equivalently,

$$1 - \sigma^*(s) = \frac{\lambda_\star - \lambda_H\, \mathbf{p}_{\boldsymbol{\sigma}_i}(s)}{\lambda_L(1 - \mathbf{p}_{\boldsymbol{\sigma}_i}(s))}. \tag{D.33}$$

From the evolution of beliefs, we have that for every $s > T$

$$\begin{aligned}
\dot{\mathbf{p}}_{\boldsymbol{\sigma}_i}(s) =& (1-\mathbf{p}_{\boldsymbol{\sigma}_i}(s))\left[\mu - \lambda_H\,\mathbf{p}_{\boldsymbol{\sigma}_i}(s) - (1-\boldsymbol{\sigma}_i(s))(\mu - \lambda_L\,\mathbf{p}_{\boldsymbol{\sigma}_i}(s))\right] \\
=& (1-\mathbf{p}_{\boldsymbol{\sigma}_i}(s))(\mu - \lambda_H\,\mathbf{p}_{\boldsymbol{\sigma}_i}(s)) - \left\{\lambda_\star - \lambda_H\,\mathbf{p}_{\boldsymbol{\sigma}_i}(s)\right\}\left(\frac{\mu}{\lambda_L} - \mathbf{p}_{\boldsymbol{\sigma}_i}(s)\right) \\
=& \mu - \frac{\mu}{\lambda_L}\lambda_\star + \left(\lambda_\star + \frac{\lambda_H\mu}{\lambda_L} - \mu - \lambda_H\right)\mathbf{p}_{\boldsymbol{\sigma}_i}(s) \\
=& -\frac{\mu}{\lambda_L}(\lambda_\star - \lambda_L) + 2\lambda_\star\,\mathbf{p}_{\boldsymbol{\sigma}_i}(s) = 2\lambda_\star(\mathbf{p}_{\boldsymbol{\sigma}_i}(s) - p_\star).
\end{aligned}$$

If there is an $s > T$ such that $\mathbf{p}_{\boldsymbol{\sigma}_i}(s) \neq p_\star$, then the solution of the above differential equation diverges. Therefore, $\mathbf{p}_{\boldsymbol{\sigma}_i}(s) = p_\star$ for all $s \geq T$. Using this, in conjunction with $\dot{\mathbf{p}}_{\boldsymbol{\sigma}_i}(s) = 0$, we obtain

$$\boldsymbol{\sigma}_i(s) = \frac{(\lambda_H - \lambda_L)\,\mathbf{p}_{\boldsymbol{\sigma}_i}(s)}{\mu - \lambda_L\,\mathbf{p}_{\boldsymbol{\sigma}_i}(s)} = \frac{(\lambda_H - \lambda_L)p_\star}{\mu - \lambda_L p_\star} = \sigma_\star \qquad \text{for all } s \geq T.$$

By Lemma D.11, we have that $\boldsymbol{\sigma}_A(s) = \boldsymbol{\sigma}_B(s) = 1$ for all $s < T$. Finally, the fact that $\mathbf{p}_{\mathbf{1}}(T) = p_\star$ is given by the continuity of the probability function $\mathbf{p}_{\boldsymbol{\sigma}}$. □

E Proofs for Patent, License and Trade Secret

E.1 Public Information Setting

E.1.1 Proof of Proposition 6.2

Proof of Proposition 6.2. Suppose that Firm i has just discovered the new technology and Firm j does not have the patent for the new technology. If Firm j already has the patent, Firm i cannot apply for a patent in the first place.

First, consider the case where Firm j already has the new technology (not the patent). If Firm i does not apply for a patent, both firms race toward development with the new technology. Thus, Firm i's expected payoff is $\frac{\lambda_H \Pi - c}{2\lambda_H}$. If Firm i applies for a patent, with probability α, Firm j's right to use the new technology is protected, and with probability

$1-\alpha$, Firm i acquires the patent. In either case, Firm i's expected payoff is at least $\frac{\lambda_H \Pi - c}{2\lambda_H}$, thus, Firm i prefers to apply for a patent.

Next, consider the case where Firm j does not have the new technology. Suppose that in equilibrium, Firm j allocates $x \in [0,1]$ to research and $1-x$ to development with old technology, when it observes the new technology discovery by Firm i (without a patent). To maximize Firm j's expected payoff, we have

$$\frac{\mu x \cdot \tilde{U}^j + \lambda_L(1-x)\cdot \Pi - c}{\lambda_H + \mu x + \lambda_L(1-x)} \geq \frac{\lambda_L \Pi - c}{\lambda_H + \lambda_L}, \tag{E.1}$$

where $\tilde{U}^j$ is Firm j's expected payoff when it also discovers the new technology. To constitute an equilibrium, Firm i's expected payoff under this Firm j's strategy should be greater than or equal to Firm i's expected payoff from applying for a patent:

$$\frac{\lambda_H \cdot \Pi + \mu x \cdot \tilde{U}^i - c}{\lambda_H + \mu x + \lambda_L(1-x)} \geq U_{Licensor}, \tag{E.2}$$

where $\tilde{U}^i$ is Firm i's expected payoff when Firm j discovers the new technology.

Note that $\tilde{U}^i + \tilde{U}^j \leq \Pi - \frac{2c}{2\lambda_H}$ since the social welfare is maximized when both firms use the new technology, and $U_{Licensor} + \frac{\lambda_L \Pi - c}{\lambda_H + \lambda_L} = \Pi - \frac{c}{\lambda_H}$ from (6.4). By using these and summing (E.1) and (E.2) up, we have

$$\begin{aligned}\Pi - \frac{c}{\lambda_H} &\leq \frac{\lambda_H \Pi + \mu x \cdot (\tilde{U}^j + \tilde{U}^i) + \lambda_L(1-x)\Pi - 2c}{\lambda_H + \mu x + \lambda_L(1-x)} \\ &\leq \Pi - \frac{\frac{\mu x}{\lambda_H} + 2}{\lambda_H + \mu x + \lambda_L(1-x)} c.\end{aligned}$$

However, this inequality is equivalent to $\lambda_H + \mu x + \lambda_L(1-x) \geq 2\lambda_H + \mu x$, which contradicts $\lambda_H > \lambda_L$ and $x \leq 1$. Therefore, in equilibrium, Firm i applies for a patent. □

E.1.2 Proof of Proposition 6.3

Proof of Proposition 6.3. Given these continuation payoffs, we now consider the equilibrium allocations when neither firm possesses the new technology yet. To apply Proposition C.1, we first compute $\hat{\Delta}_0$ and $\hat{\Delta}_1$ by replacing $(U^i_{\{i\}}, U^j_{\{i\}})$ to $(U_{Licensor}, U_{Licensee})$ in (C.1):

$$\hat{\Delta}_0 = \frac{\mu U_{Licensor} - c}{\mu + \lambda_L} - \frac{\lambda_L \Pi - c}{2\lambda_L},$$
$$\hat{\Delta}_1 = \frac{\mu U_{Licensor} + \mu U_{Licensee} - c}{2\mu} - \frac{\lambda_L \Pi + \mu U_{Licensee} - c}{\mu + \lambda_L}.$$

By plugging (6.4) in, with some algebra, we can derive that

$$\hat{\Delta}_0 = \frac{\lambda_H \lambda_L (\lambda_\star - \lambda_L)\Pi + (\lambda_H + \lambda_L)\lambda_\star c}{2\lambda_H(\lambda_H + \lambda_L)(\lambda_L + \mu)},$$
$$\hat{\Delta}_1 = \frac{\lambda_H \lambda_L (\lambda_\star - \lambda_L)\Pi + \frac{\lambda_L}{2\mu}\left\{(2\lambda_H + \mu + \lambda_L)\lambda_\star + (\mu - \lambda_L)\lambda_H\right\} c}{2\lambda_H(\lambda_H + \lambda_L)(\lambda_L + \mu)}.$$

First, observe that $\lambda_\star \geq \lambda_L$ implies $\hat{\Delta}_0,\ \hat{\Delta}_1 > 0$. Then, by Proposition C.1 (a), both firms do research, thus, Proposition 6.3 (a) holds. Next, when $\lambda_L > \lambda_\star$, we have

$$\hat{\Delta}_0 > 0 \quad \iff \quad \tilde{\pi}_0 \equiv \frac{\lambda_\star(\lambda_H + \lambda_L)}{\lambda_H(\lambda_L - \lambda_\star)} > \frac{\lambda_L \Pi}{c} = \pi,$$

$$\hat{\Delta}_1 > 0 \quad \iff \quad \tilde{\pi}_1 \equiv \frac{\frac{\lambda_L}{2\mu}\left\{(2\lambda_H + \mu + \lambda_L)\lambda_\star + (\mu - \lambda_L)\lambda_H\right\}}{\lambda_H(\lambda_L - \lambda_\star)} > \pi.$$

Suppose that $\lambda_\star \in \left(\frac{\lambda_H \lambda_L}{2\lambda_H + \lambda_L}, \lambda_L\right)$. By using $\mu > \lambda_L$, we can show that $\tilde{\pi}_0 > \tilde{\pi}_1 > 1$.

(i) if $\pi > \tilde{\pi}_0 > \tilde{\pi}_1$, we have $\hat{\Delta}_0,\ \hat{\Delta}_1 < 0$, then, by Proposition 1 (b), both firms develop with old technology;

(ii) if $\tilde{\pi}_0 > \pi > \tilde{\pi}_1$, we have $\hat{\Delta}_0 > 0 > \hat{\Delta}_1$, then, by Proposition 1 (c), there are three equilibria including the asymmetric one;

(iii) if $\tilde{\pi}_1 > \pi > 1$, we have $\hat{\Delta}_0, \hat{\Delta}_1 > 0$, then, by Proposition 1 (a), both firms do research.

Thus, Proposition 6.3 (b) holds.

Now suppose that $\lambda_\star \leq \frac{\lambda_H \lambda_L}{2\lambda_H + \lambda_L}$. With some algebra, we have $1 \geq \tilde{\pi}_1 \geq \tilde{\pi}_0$. From $\pi > 1$, we have $\hat{\Delta}_0, \hat{\Delta}_1 < 0$, then, by Proposition 1 (b), both firms develop with old technology. Thus, Proposition 6.3 (c) holds. $\square$

E.2 Private Information Setting

E.2.1 Proof of Proposition 6.4

Proof of Proposition 6.4. By plugging (6.2) in, we have that (6.7) is equivalent to:

$$
\begin{aligned}
& \frac{\lambda_H-\lambda_L}{\lambda_H+\lambda_L}\cdot\frac{\lambda_H\Pi+c}{\lambda_H\Pi-c} > \frac{\lambda_H}{\lambda_H+\mu(2-\alpha)} \\
\iff\quad & \{\lambda_H(\lambda_H-\lambda_L)+\mu(\lambda_H-\lambda_L)(2-\alpha)\}\,(\lambda_H\Pi+c) > \lambda_H(\lambda_H+\lambda_L)(\lambda_H\Pi-c) \\
\iff\quad & \{\mu(\lambda_H-\lambda_L)(2-\alpha)-2\lambda_L\lambda_H\}\cdot\lambda_H\Pi+\left\{\mu(\lambda_H-\lambda_L)(2-\alpha)+2\lambda_H^2\right\}\cdot c>0.
\end{aligned}
$$

Note that $\mu(\lambda_H-\lambda_L)=\lambda_L(\lambda_\star+\lambda_H)$ from (3.4). By plugging this in, the above inequality is equivalent to:

$$
\begin{aligned}
& \{(2-\alpha)\lambda_\star-\alpha\lambda_H\}\cdot\lambda_H\lambda_L\Pi+\left\{(2-\alpha)\lambda_L(\lambda_\star+\lambda_H)+2\lambda_H^2\right\}\cdot c>0 \\
\iff\quad & (\lambda_\star+\lambda_H)\,(\hat{\alpha}-\alpha)\cdot\lambda_H\left(\frac{\lambda_L\Pi}{c}-1\right)+(2-\alpha)(\lambda_L+\lambda_H)(\lambda_\star+\lambda_H)>0.
\end{aligned}
$$

If $\alpha\le\hat{\alpha}$, the first term in the above inequality is nonnegative and the second term is positive from $\alpha<1$ and $\lambda_L,\ \lambda_H,\ \lambda_\star>0$. If $\alpha>\hat{\alpha}$, by rearranging it and using $\pi=\frac{\lambda_L\Pi}{c}$, we can show that the above inequality is equivalent to (6.8). □

E.3 Proof of Lemma 6.2

Proof of Lemma 6.2. By Lemma D.1, $\mathbf{p_1}(t)$ is increasing in t. Then, $V_1(t;\mathbf{h_1})$ can be written as a function of $\mathbf{p_1}(t)$: $V_1(t;\mathbf{h_1})=v_1(\mathbf{p_1}(t))$. Observe that

$$V_1'(t;\mathbf{h_1})=v_1'(\mathbf{p_1'}(t))\cdot\mathbf{p_1'}(t)=v_1'(\mathbf{p_1'}(t)))(\mu-\lambda_H\,\mathbf{p_1}(t))(1-\mathbf{p_1}(t)).$$

By plugging this into (HJB_1), we have

$$0=v_1'(p)(\mu-\lambda_Hp)(1-p)-\lambda_H(1+p)v_1(p)+\lambda_H\Pi-c. \tag{E.3}$$

Define two function $g(p)$ and $k(p)$ as follows:

$$g(p) \equiv \frac{(\mu - \lambda_H p)^{\frac{\mu+\lambda_H}{\lambda_H-\mu}}}{(1-p)^{\frac{2\lambda_H}{\lambda_H-\mu}}} \quad \text{and} \quad k(p) \equiv 1 + \frac{\lambda_H}{\lambda_H+\mu}(1-p). \tag{E.4}$$

Observe that

$$\frac{g'(p)}{g(p)} = \frac{d\log(g(p))}{dp} = -\frac{\mu+\lambda_H}{\lambda_H-\mu}\cdot\frac{\lambda_H}{\mu-\lambda_H p} + \frac{2\lambda_H}{\lambda_H-\mu}\cdot\frac{1}{1-p} = -\frac{\lambda_H(1+p)}{(1-p)(\mu-\lambda_H p)} \tag{E.5}$$

and

$$\frac{d}{dp}(g(p)\cdot k(p)) = -\frac{\lambda_H(1+p)k(p)}{(1-p)(\mu-\lambda_H p)}g(p) - \frac{\lambda_H}{\lambda_H+\mu}g(p) = -\frac{2\lambda_H}{(1-p)(\mu-\lambda_H p)}g(p) \tag{E.6}$$

By multiplying (E.3) by $\frac{g(p)}{(\mu-\lambda_H p)(1-p)}$ and using above two equations, we have

$$\begin{aligned} 0 =& v_1'(p)\cdot g(p) + g'(p)\cdot v_1(p) + \frac{\lambda_H\Pi - c}{2\lambda_H}\cdot\frac{g(p)}{(1-p)(\mu-\lambda_H p)} \\ =& \frac{d}{dp}\left[(v_1(p) - V_C\cdot k(p))\cdot g(p)\right]. \end{aligned}$$

Therefore, there exists $C \in \mathbb{R}$ such that

$$v_1(p) = V_C\cdot k(p) + \frac{C}{g(p)}. \tag{E.7}$$

Recall that by Lemma D.1, if $\mu \geq \lambda_H$, $\lim_{t\to\infty}\mathbf{p_1}(t) = 1$, and if $\mu < \lambda_H$, $\lim_{t\to\infty}\mathbf{p_1}(t) = \mu/\lambda_H$. By using these, we have that $\lim_{t\to\infty} g(\mathbf{p_1}(t)) = 0$. Then, to satisfy $V_1(t;\mathbf{h_1}) = v_1(\mathbf{p_1}(t))$ and (E.7), the constant C has to be zero, and (6.10) holds. $\square$

E.4 Proof of Proposition 6.5

Proof of Proposition 6.5. By plugging (6.2) in, we have that (6.12) is equivalent to:

$$
\begin{aligned}
& \frac{\lambda_H - \lambda_L}{\lambda_H + \lambda_L} \cdot \frac{\lambda_H \Pi + c}{\lambda_H \Pi - c} < \frac{\lambda_H}{\lambda_H + \mu} \cdot \frac{\lambda_H - \mu}{\lambda_H - \alpha\mu} \\
\iff \quad & (\lambda_H + \mu)(\lambda_H - \alpha\mu)(\lambda_H - \lambda_L)(\lambda_H \Pi + c) < \lambda_H(\lambda_H - \mu)(\lambda_H + \lambda_L)(\lambda_H \Pi - c) \\
\iff \quad & \left[2(\lambda_H^2 - \mu\lambda_L)\lambda_H - \alpha\mu(\lambda_H - \lambda_L)(\lambda_H + \mu)\right] \cdot c \\
& < \left[-2\lambda_H^2(\mu - \lambda_L) + \alpha\mu(\lambda_H - \lambda_L)(\lambda_H + \mu)\right] \cdot \lambda_H \Pi.
\end{aligned}
$$

By using $\mu(\lambda_H - \lambda_L) = \lambda_L(\lambda_\star + \lambda_H)$ and $\lambda_H(\mu - \lambda_L) = \lambda_L(\lambda_\star + \mu)$,

$$
\begin{aligned}
& \left[2(\lambda_H^2 - \mu\lambda_L)\lambda_H - \alpha\lambda_L(\lambda_\star + \lambda_H)(\mu + \lambda_H)\right] c \\
& < [(\lambda_H + \mu)(\lambda_\star + \lambda_H)\alpha - 2\lambda_H(\lambda_\star + \mu)] \cdot \lambda_H \cdot \lambda_L \Pi \\
\iff \quad & (\lambda_H + \lambda_L)(\lambda_\star + \lambda_H)(2\lambda_H - \alpha(\lambda_H + \mu)) \\
& < (\lambda_H + \mu)(\lambda_\star + \lambda_H)(\alpha - \tilde{\alpha})\lambda_H \cdot \left(\frac{\lambda_L \Pi}{c} - 1\right).
\end{aligned}
$$

Note that $2\lambda_H - \alpha(\lambda_H + \mu) > 0$ from $\lambda_H > \mu$ and $1 \geq \alpha$. Therefore, if $\alpha \leq \tilde{\alpha}$, the above inequality cannot hold. When $\alpha > \tilde{\alpha}$, by rearranging the above inequality, we have (6.14).

Observe that $\tilde{\alpha} > \hat{\alpha}$ is equivalent to:

$$
2\lambda_H(\mu + \lambda_\star) > 2\lambda_\star(\lambda_H + \mu)
$$

and it holds from the assumption that $\lambda_H > \lambda_\star$.

Next, observe that $\tilde{\pi}(\alpha) \geq \hat{\pi}(\alpha)$ is equivalent to:

$$
\frac{\frac{2\lambda_H}{\lambda_H + \mu} - \alpha}{\alpha - \tilde{\alpha}} \geq \frac{2 - \alpha}{\alpha - \hat{\alpha}} \qquad \iff \qquad \frac{\frac{2\lambda_H}{\lambda_H + \mu} - \tilde{\alpha}}{\alpha - \tilde{\alpha}} \geq \frac{2 - \hat{\alpha}}{\alpha - \hat{\alpha}}. \tag{E.8}
$$

Also note that

$$
\frac{2\lambda_H}{\lambda_H + \mu} - \tilde{\alpha} = \frac{2\lambda_H}{\lambda_H + \mu} \cdot \frac{\lambda_H - \mu}{\lambda_H + \lambda_\star} \qquad \text{and} \qquad 2 - \hat{\alpha} = \frac{2\lambda_H}{\lambda_H + \lambda_\star}.
$$

By plugging these in, (E.8) is equivalent to:

$$\tilde{\alpha} - \frac{\lambda_H - \mu}{\lambda_H + \mu}\hat{\alpha} \geq \frac{2\mu}{\lambda_H + \mu}\alpha. \tag{E.9}$$

Note that

$$\tilde{\alpha} - \frac{\lambda_H - \mu}{\lambda_H + \mu}\hat{\alpha} = \frac{2\lambda_H(\mu + \lambda_\star)}{(\lambda_H + \mu)(\lambda_H + \lambda_\star)} - \frac{\lambda_H - \mu}{\lambda_H + \mu} \cdot \frac{2\lambda_\star}{\lambda_H + \lambda_\star} = \frac{2\mu}{\lambda_H + \mu}.$$

Therefore, (E.9) is equivalent to $1 \geq \alpha$. Therefore, $\tilde{\pi}(\alpha) \geq \hat{\pi}(\alpha)$ holds for all $1 \geq \alpha > \tilde{\alpha}$ and the equality holds if and only if $\alpha = 1$. □

Online Appendix

F Omitted Proofs for the Public Information Setting

Lemma F.1. *Suppose that Firm i and j employ allocation policies σ and $\hat{\sigma}$ at the state $\emptyset$. Let $U^i_{\{i\}}$ and $U^i_{\{j\}}$ be Firm i's continuation payoffs at the states $\{i\}$ and $\{j\}$. Then, Firm i's expected payoffs are given as follows:*

$$U_0(\sigma,\hat{\sigma}) = \int_0^\infty \left(\lambda_L(1-\sigma_t)\cdot\Pi + \mu\ \sigma_t \cdot U^i_{\{i\}} + \mu\ \hat{\sigma}_t \cdot U^i_{\{j\}} - c\right)\cdot e^{-\lambda_L(2t-\Sigma_t-\hat{\Sigma}_t)-\mu(\Sigma_t+\hat{\Sigma}_t)}\ dt, \tag{F.1}$$

where $\Sigma_t = \int_0^t \sigma_s ds$ and $\hat{\Sigma}_t = \int_0^t \hat{\sigma}_s ds$.

Proof. When any of the arrival times $\tau_L,\ \tau_R,\ \hat{\tau}_L$ and $\hat{\tau}_R$ occurs, the Firm i's payoff is realized. Furthermore, it incurs a flow cost c until one of these arrival times takes place. Thus, Firm i's expected payoff can be written as follows:

$$\begin{aligned} U_0(\sigma,\hat{\sigma}) =& \Pr[\tau_L < (\tau_R\wedge\hat{\tau}_L\wedge\hat{\tau}_R)]\cdot\Pi + \Pr[\tau_R < (\tau_L\wedge\hat{\tau}_L\wedge\hat{\tau}_R)]\cdot U^i_{\{i\}} \\ &+ \Pr[\hat{\tau}_R < (\tau_L\wedge\tau_R\wedge\hat{\tau}_L)]\cdot U^i_{\{j\}} - \mathbb{E}[(\tau_L\wedge\tau_R\wedge\hat{\tau}_L\wedge\hat{\tau}_R)]\cdot c. \end{aligned} \tag{F.2}$$

Note that the survival function of $(\tau_R\wedge\hat{\tau}_L\wedge\hat{\tau}_R)$ is $e^{-\lambda_L(t-\hat{\Sigma}_t)-\mu(\Sigma_t+\hat{\Sigma}_t)}$. By using (A.4) and (A.7), we have

$$\Pr[\tau_L < (\tau_R\wedge\hat{\tau}_L\wedge\hat{\tau}_R)] = \int_0^\infty \lambda_L(1-\sigma_t)\cdot e^{-\lambda_L(2t-\Sigma_t-\hat{\Sigma}_t)-\mu(\Sigma_t+\hat{\Sigma}_t)}\ dt.$$

Likewise, we can derive that

$$\begin{aligned} \Pr[\tau_R < (\tau_L\wedge\hat{\tau}_L\wedge\hat{\tau}_R)] &= \int_0^\infty \mu\ \sigma_t\cdot e^{-\lambda_L(2t-\Sigma_t-\hat{\Sigma}_t)-\mu(\Sigma_t+\hat{\Sigma}_t)}\ dt, \\ \Pr[\hat{\tau}_R < (\hat{\tau}_L\wedge\tau_L\wedge\tau_R)] &= \int_0^\infty \mu\ \hat{\sigma}_t\cdot e^{-\lambda_L(2t-\Sigma_t-\hat{\Sigma}_t)-\mu(\Sigma_t+\hat{\Sigma}_t)}\ dt. \end{aligned}$$

Next, observe that the survival function of $(\tau_L \wedge \tau_R \wedge \hat{\tau}_L \wedge \hat{\tau}_R)$ is

$$e^{-\lambda_L(2t-\Sigma_t-\hat{\Sigma}_t)-\mu(\Sigma_t+\hat{\Sigma}_t)} = e^{-2\lambda_L t-(\mu-\lambda_L)(\Sigma_t+\hat{\Sigma}_t)}.$$

Then, from $\mu \geq \lambda_L$ and $\Sigma_t + \hat{\Sigma}_t \geq 0$, we have $\lim_{t\to\infty} t \cdot e^{-\lambda_L(2t-\Sigma_t-\hat{\Sigma}_t)-\mu(\Sigma_t+\hat{\Sigma}_t)} = 0$. By applying (A.1), we have

$$\mathbb{E}[(\tau_L \wedge \tau_R \wedge \hat{\tau}_L \wedge \hat{\tau}_R)] = \int_0^\infty e^{-\lambda_L(2t-\Sigma_t-\hat{\Sigma}_t)-\mu(\Sigma_t+\hat{\Sigma}_t)}\, dt.$$

By plugging the above equations into (F.2), we obtain (F.1). □

Lemma F.2. *Suppose that $(x_0, y_0) \in [0,1]^2$ satisfies $x_0 \in \arg\max_{x\in[0,1]} u_0(x, y_0)$. Let $\sigma^*, \hat{\sigma}^* : \mathbb{R}_+ \to [0,1]$ be $\sigma^*_t = x_0$ and $\hat{\sigma}^*_t = y_0$ for all $t \geq 0$. Then, σ^* is a best response to $\hat{\sigma}^*$.*

Proof of Lemma F.2. Let r_t denote $S^M_{\sigma,\hat{\sigma}^*}(t)$. By taking a derivative, we have

$$\dot{r}_t = -\left\{\mu(\sigma_t + y_0) + \lambda_L(2 - \sigma_t - y_0)\right\} \cdot r_t. \tag{F.3}$$

By Lemma F.1, given Firm j's allocation profile $\hat{\sigma}^*$, Firm i's problem is

$$\max_{\sigma} \int_0^\infty \left\{\lambda_L(1-\sigma_t)\cdot\Pi + \mu\sigma_t \cdot U^i_{\{i\}} + \mu y_0 \cdot U^i_{\{j\}} - c\right\} \cdot r_t\, dt \tag{F.4}$$

subject to (F.3).

Observe that the Hamiltonian of this optimal control problem is

$$\begin{aligned} H(\sigma_t, r_t, \eta_t) &= \left\{\lambda_L(1-\sigma_t)\cdot\Pi + \mu\sigma_t \cdot U^i_{\{i\}} + \mu y_0 \cdot U^i_{\{j\}} - c\right\} \cdot r_t \\ &\qquad - \eta_t \left\{\mu(\sigma_t + y_0) + \lambda_L(2-\sigma_t-y_0)\right\} \cdot r_t \\ &= \left\{u_0(\sigma_t, y_0) - \eta_t\right\} \cdot \left\{\mu(\sigma_t + y_0) + \lambda_L(2-\sigma_t-y_0)\right\} \cdot r_t, \end{aligned} \tag{F.5}$$

where η_t is a co-state variable.

To show that σ^* is a solution of (F.4) subject to (F.3) by using the Arrow sufficiency condition (Seierstad and Sydsaeter, 1987, Theorem 3.14), we consider (η^*, r^*) defined as follows: for all $t \geq 0$, $\eta^*_t = u_0(x_0, y_0)$ and $r^*_t = e^{-\{\mu(x_0+y_0)+\lambda_L(2-x_0-y_0)\}\cdot t}$.

Then, we need to check following four primitive conditions:

1. Maximum principle: for all $t \geq 0$,

$$\sigma_t^* = x_0 \in \arg\max_{\sigma_t \in [0,1]} H(\sigma_t, r_t^*, \eta_t^*). \tag{F.6}$$

2. Evolution of the co-state variable:

$$\dot{\eta}_t^* = -\frac{\partial H}{\partial r_t} = -\{u_0(\sigma_t^*, y_0) - \eta_t^*\} \cdot \{\mu(\sigma_t^* + y_0) + \lambda_L(2 - \sigma_t^* - y_0)\}. \tag{F.7}$$

3. Transversality condition: If r^* is the optimal trajectory, i.e., $r_t^* = S^M_{\sigma^*,\hat{\sigma}^*}(t)$, $\lim_{t\to\infty} \eta_t^*(r_t^* - r_t) \leq 0$ for all feasible trajectories r_t.

4. $\hat{H}(r_t, \eta_t) = \max_{\sigma_t \in [0,1]} H(\sigma_t, r_t, \eta_t)$ is concave in r_t.

First, by plugging r_t^* and η_t^* into (F.5), we have

$$H(\sigma_t, r_t^*, \eta_t^*) = \{u_0(\sigma_t, y_0) - u_0(x_0, y_0)\} \cdot \{\mu(\sigma_t + y_0) + \lambda_L(2 - \sigma_t - y_0)\} \cdot r_t \tag{F.8}$$

Recall that $x_0 \in \arg\max_{x \in [0,1]} u_0(x, y_0)$. Thus, $H(\sigma_t, r_t^*, \eta_t^*) \leq 0$ for all $\sigma_t \in [0,1]$. In addition, $H(x_0, r_t^*, \eta_t^*) = 0$. Therefore, $x_0 \in \arg\max_{\sigma_t \in [0,1]} H(\sigma_t, r_t, \eta_t)$, i.e., (F.6) holds.

Second, by the definition of η^*, (F.7) holds.

Third, note that for any admissible allocation σ,

$$r_t = e^{-\{\mu(\Sigma_t + y_0 t) + \lambda_L(2t - \Sigma_t - y_0 t)\}} = r_t^* \cdot e^{(\mu - \lambda_L)\cdot(x_0 t - \Sigma_t)}.$$

Then, we have

$$\lim_{t\to\infty} \eta_t^* \cdot (r_t^* - r_t) = \lim_{t\to\infty} u_0(x_0, y_0) \cdot r_t^* \cdot \left(1 - e^{(\mu - \lambda_L)\cdot(x_0 t - \Sigma_t)}\right) = 0.$$

Last, we can see that $\hat{H}$ is linear in r_t, thus, the fourth condition holds. Hence, by the Arrow sufficiency condition, σ^* is the best response to $\hat{\sigma}^*$. $\square$