Strategic Concealment in Innovation Races *

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Abstract

We investigate firms' incentives to conceal intermediate research discoveries in innovation races. To study this, we introduce an innovation game where two racing firms dynamically allocate their resources between two distinct research and development (R&D) paths towards a final innovation: (i) developing it with the currently available but slower technology; (ii) conducting research to discover a faster new technology for developing it. We fully characterize the equilibrium behavior of the firms in the cases where their research progress is public and private information. Then, we extend the private information setting by allowing firms to conceal or license their intermediate discoveries. Firms may conceal their interim discoveries during innovation races, which can lead to a slower pace of innovation that is inefficient, particularly when the reward for winning the race is high.

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1 Introduction

In the course of research and development (R&D), firms often discover interim knowledge that brings them closer to successfully producing a final innovation. When multiple firms race towards such innovation, a firm's optimal R&D strategy is likely to be influenced by the information about whether its rivals have made intermediate breakthroughs. Thus, a firm may want to conceal intermediate discoveries in order to hinder its rivals from adjusting their R&D strategies. On the other hand, it may prefer to disclose an intermediate discovery because this can open the opportunity for monetization via licensing the technological breakthrough. In this paper, we introduce and study an innovation race model that captures the tradeoffs between licensing and concealing interim discoveries and characterize firms' equilibrium behavior.

We consider a situation where two firms race towards developing an innovative product, such as a COVID-19 vaccine or a full self-driving (FSD) vehicle. The first firm to develop the product receives a reward (e.g., a transitory flow of monopoly profit) and the other firm does not. At each point in time, the firms allocate their limited resources between two routes for developing the product and incur constant flow costs. One route is to conduct basic *research* to discover a new technology that does not directly deliver the product but makes developing it faster, e.g., messenger RNA (mRNA) or light detection and ranging (LIDAR) technology.[1][2] This route requires two breakthroughs: discovering the new technology and developing the product with it. The other route is to *develop* the product with a currently available but slow technology, namely the old technology. For example, the viral vector method for developing

[1]The mRNA technology was not utilized in practice before the COVID-19 outbreak. Thus, pharmaceutical firms had to first acquire basic knowledge in order to employ this new methodology. The advantage of possessing this intermediate technology is that firms can develop vaccines in a laboratory by using readily available materials. Hence vaccines can be developed faster with mRNA technology than with older methods. Moderna and Pfizer-BioNTech utilize mRNA technology to develop COVID-19 vaccines. For more information, see the web page of the Centers for Disease Control and Prevention (CDC): https://www.cdc.gov/coronavirus/2019-ncov/vaccines/different-vaccines/mrna.html.

[2] LIDAR is a laser radar that can provide extensive and reliable information surrounding a vehicle including an object's distance, size, position, and velocity if it is moving. Most FSD vehicle developers including Waymo—formerly the Google self-driving car project—use LIDAR combined with cameras. The main drawback of LIDAR is its current high cost. Thus, to develop a commercializable FSD vehicle, firms first need to discover a way to make LIDAR less expensive. Once LIDAR becomes affordable, it will be relatively easy to develop a commercializable FSD vehicle. In this sense, successfully developing an FSD vehicle with the LIDAR technology can be understood as a route requiring two breakthroughs.

a COVID vaccine and the camera-based vision technology for developing an FSD vehicle can be considered old technologies.[3][4] This path requires a single breakthrough but the arrival rate is relatively low. We assume that the path with the new technology is more efficient: the total expected completion time of doing research for the new technology and developing the product with it is shorter than that of developing with the old strategy. Thus, the socially efficient policy is to have both firms allocate all their resources to research, and once one of them discovers the new technology, have it share the breakthrough with the other firm to prevent duplication of research costs.

We investigate three different settings in the context of this framework. First, we consider the case where it is public information whether a firm has discovered the new technology or not. In this setting, a firm can condition its strategy not only on its own technological breakthrough but also on its rival's progress. We show that there exists a unique equilibrium and its form is determined by the relative efficiency of the new technology. The efficiency measure is defined to be inversely proportional to the expected total completion time of the path with the new technology, i.e., doing research is more attractive when efficiency is high. It is shown that when efficiency is extreme (high or low), a firm's equilibrium strategy does not depend on its rival's progress. Specifically, when the new technology is highly efficient, both firms allocate all their resources to research (i.e., perform research only); and when the new technology is not much more efficient, both firms allocate all their resources to development (i.e., develop with the old technology only) regardless of their rival's status. On the contrary, when efficiency is intermediate, the equilibrium strategy of each firm does depend on its rival's progress. In this case, both firms begin by conducting research, but once one firm makes the intermediate technological breakthrough, the other switches to developing with the old technology, namely it pursues a *fall-back* strategy.

[3]The viral vector technology was used during recent disease outbreaks including the 2014-2016 Ebola outbreak in West Africa. Many pharmaceutical firms had access to this methodology when the COVID-19 outbreak began. Indeed, this technology was utilized to develop COVID-19 vaccines by Oxford-AstraZeneca and Janssen (Johnson&Johnson). For more information, see the web page of the CDC: https://www.cdc.gov/coronavirus/2019-ncov/vaccines/different-vaccines/viralvector.html.

[4] Unlike other companies, Tesla's approach towards developing an FSD vehicle is to use only cameras without LIDAR (Templeton, 2019). Since camera technology is already very cheap, no cost-saving breakthrough is needed to implement it. However, the quality of information attained from cameras is inferior to that attained from LIDAR, thus it will take more time to develop an FSD vehicle utilizing only cameras.

Next, we analyze the setting where technological discoveries are private information, i.e., a firm cannot observe its rivals' technological progress. As in the public information setting, when efficiency is high, each firm conducts research until it succeeds or its rival produces the final innovation. Similarly, when efficiency is low, both firms endeavor to develop with the old technology. This invariance occurs because, in the extreme cases of very high and very low efficiency, firms do not use the information about their rival's progress even when it is observable. However, in the case of intermediate efficiency, the firms cannot use the fall-back strategy as in the public information setting since they are no longer able to make their resource allocations contingent on their rivals' state of technology. Instead, their resource allocations must depend on their 'beliefs' about their rivals' progress. We characterize the unique symmetric equilibrium that is Markov with respect to these beliefs. The equilibrium strategy has a cutoff structure: firms conduct research exclusively up to a certain date (belief), then they start allocating their resources between developing with the old technology and researching the new one, namely they employ a *stationary fall-back* strategy. The most intriguing feature of this equilibrium is that beliefs remain constant once the allocation of resources to development begins. This stationarity derives from two conflicting effects in the belief evolution. First, as time passes, it becomes more likely that one's rival has found the new technology (the *duration effect*). On the other hand, the lack of one's rival producing the final innovation (which is publically observable) implies that it is less likely that the new technology has been discovered (the *still-in-the-race effect*).

Last, we extend the private information setting by allowing firms to protect their discoveries by using either a *patent* or a *trade secret*. First, when a firm treats the new technology as a trade secret, it conceals the discovery, i.e., its rival still cannot observe its progress. However, this does not prohibit the firm's rival from discovering the new technology independently. Second, when a firm files a patent, it discloses the discovery of the new technology. On the one hand, if its rival has not yet made the technological breakthrough, then the exclusive right to use the new technology is bestowed on the patenting firm. In addition, the patenting firm may *license* the new technology, i.e., it may permit its rival to use the new technology for a fee. Once the licensee pays the fee, both firms race for the final innovation employing the new technology. On the other hand, if the rival firm has already discovered the new technology, i.e., it was protected as a trade secret. Then, the patenting firm cannot claim the exclusive right—rather, the new technology is now considered common property—and both firms can use it without making transfers.[5][6]

We first show that if a firm files a patent and the rival firm does not possess the new technology, the patenting firm always licenses. Thus, both firms develop the final innovation with the new technology, which is socially efficient. Once a firm files a patent, its rival can only try to develop the product with the old slow technology. Given this, the patenting firm can extract rent from its rival by allowing it to use the new technology for a fee. This is an application of the classical result of Coase (1960) in the sense that the socially efficient outcome can be achieved when the property right of the new technology is given to a firm and trade involves no transaction costs. Therefore, disclosing the new technology implies licensing it.

Finally, we explore whether a firm prefers to disclose or conceal the new technology. We show that this decision crucially depends on the size of the reward of winning the race: when the reward is high, firms may prefer to conceal their discoveries, whereas when the reward is low, they disclose and license them. Intuitively, this is because concealment involves a higher chance of winning the race, which is more attractive when the reward is high. Whereas, disclosure delivers an immediate payment from licensing, which is more appealing when the reward is low. More specifically, when a firm conceals a discovery, its rival does not know whether it possesses the new technology. Thus, per the results from the private information setting, the rival firm continues allocating some of its resources to researching the new technology. This is not desirable for the rival, especially when efficiency of the new technology is intermediate, because if it knew that the other firm already possessed the new technology, then its best response would be to allocate all its resources to development with the old technology (i.e., to employ the fall-back strategy). In this sense, concealing the new technology hinders the rival firm from strategically responding to its discovery.

Concealment is detrimental not only to the rival firm but also to social surplus because

[5] When a firm files a patent, the firm with the trade secret can dispute the patent based on 35 U.S. Code §273 - Defense to infringement based on prior commercial use.

[6] For more information about trade secrets and patents, see the web page of the World Intellectual Property Organization: `https://www.wipo.int/about-ip/en/`. Also, see Lobel (2013) for examples.

it generates duplicate research efforts. This slows down the pace of innovation. On the contrary, the socially efficient outcome could be achieved by disclosing and licensing the new technology. These results on firms' incentives for concealment imply a simple policy intervention. Reducing the reward of winning the race (e.g., weakening the transitory monopoly power in the innovative product market by imposing a tax,) reduces incentives to conceal and promotes licensing, thus speeding up the pace of innovation.

Related Literature

This paper primarily contributes to the literature on patent vs. secrecy by introducing a novel incentive to conceal a firm's discovery: hindering its rival's strategic response. Previous studies mainly focused on the limited protection power of patents. For example, the seminal article by Horstmann et al. (1985) posits that "patent coverage may not exclude profitable imitation." Thus, in their framework, the main reason why a firm may choose secrecy over a patent is not to be imitated.[7] Another limitation of a patent is that it expires in a finite time. For instance, Denicolò and Franzoni (2004) consider a framework where a patent gives the patenting firm monopoly power only for a certain period of time (and no profit after expiration), whereas secrecy can give indefinite monopoly power to a firm but it can be leaked or duplicated by a rival with some probability. On the contrary, in this paper, we abstract from the restrictions of patents and focus analysis on the potential advantages of concealment.

Another hallmark of this paper is its consideration of 'interim' discoveries. Therefore, it is naturally related to the literature on licensing of interim R&D knowledge, e.g., Bhattacharya et al. (1992); d'Aspremont et al. (2000); Bhattacharya and Guriev (2006); Spiegel (2008). In these papers it is assumed that firms already know which of them has superior knowledge, i.e., the firm that will license the technology is exogenously given. Unlike in those studies, we allow firms to choose when to license (and even allow them not to license), i.e., the licensing decision is endogenous.

We also contribute to the innovation literature by introducing a model with two char-

[7]Many subsequent papers study the imitation threat and potential patent infringement, e.g., Gallini (1992); Takalo (1998); Anton and Yao (2004); Kultti et al. (2007); Kwon (2012); Zhang (2012).

acteristics. First, there are different avenues towards innovation: developing with the old technology and doing research for the new technology. Second, one of the paths involves multiple stages: once a firm discovers the new technology, then the firm develops the innovative product with it.

With respect to the first characteristic, there is a recent branch of the literature that studies races where there are different routes to achieve a final objective. Das and Klein (2020) and Akcigit and Liu (2016) study a patent race where two firms compete for a breakthrough and there are two methods to get the breakthrough: a safe method and a risky method. In Das and Klein (2020) the safe method has a known constant arrival intensity while the risky method has an unknown constant arrival intensity. In Akcigit and Liu (2016), instead, the safe method has a known payoff associated with breakthrough arrival, while there is uncertainty about the payoff if the risky method is used. In this paper, firms face no uncertainty about whether the innovation is feasible. Instead, they are uncertain whether their rival possesses the new and faster technology.

The second characteristic, multi-stage innovation, is also widely studied in the literature, e.g., Scotchmer and Green (1990); Denicolò (2000); Green and Taylor (2016); Song and Zhao (2021). Our paper shares the framework with these in that we use two sequential Poisson discovery processes and ask whether a firm would patent the first discovery or not. A feature setting apart from their works is that there is another path that only requires one but slower breakthrough toward innovation. This feature connects our model to Carnehl and Schneider (2022) and Kim (2022) in the sense that players can choose between a sequential approach—which requires two breakthroughs—and a direct approach, which requires only one breakthrough, but its riskier or slower.[8] Our model mainly differs from theirs in that multiple players compete by choosing between these approaches, whereas Carnehl and Schneider (2022) considers a problem by a single decision maker and Kim (2022) studies a contracting setup between a principal and an agent. In their studies, a key factor for a player to choose the direct approach is a deadline that is either exogenously given or endogenously determined

[8]In Carnehl and Schneider (2022), an agent is uncertain whether the direct approach is feasible or not, i.e., this approach is risky. On the other hand, in Kim (2022), there is no uncertainty on the feasibility of the direct approach, but its completion rate is slower than the ones for the sequential approach. In this sense, our framework is closer to Kim (2022).

to reduce moral hazard. In contrast to these, a deadline is not involved in our model. Rather, the race with the rival firm may induce a firm to develop with the old technology, which can be considered as a direct approach.

Last, this paper is related to the recent literature on information disclosure in priority races, e.g., Hopenhayn and Squintani (2016); Bobtcheff et al. (2017). In those papers, once a firm makes a breakthrough, the innovation value grows as time passes until one of the firms files a patent. Thus, firms face a tradeoff between disclosing to claim the priority and delaying in order to grow the innovation value. On the contrary, in this paper, the value of innovation is fixed and the discovery of the new technology only allows the firm to develop the innovative product faster. Therefore, a firm may delay the disclosure purely to confound the rival's R&D decisions.

Roadmap

We introduce the model in the next section, then characterize equilibria in the private and the public information settings in Section 4 and 5. In Section 6, we extend the private information setting by allowing firms to disclose their discoveries. We conclude in Section 7. All proofs appear in the appendix.

2 Model

We consider a race between two firms, A and B, to develop an *innovative product*. Time is continuous and infinite: $t \in [0, \infty)$. The innovative product can be developed using either *old* or *new* technology, each with a different development speed. At the outset of the race, both firms have access to an old technology, but they can gain access to a new technology by conducting research.

Each firm owns one unit of resources per unit of time, which can be allocated for either conducting research to discover the new technology or developing the innovative product. When a firm gains access to the new technology, it directs all its resources towards product development, resulting in a development rate of λ_H. When a firm does not yet possess the new technology, let $\sigma_t^i \in [0,1]$ denote the resources that Firm i allocates to 'research' at time t. Then, $1-\sigma_t^i$ is the amount of resources that Firm i allocates to 'develop' the innovative product, and the product can be stochastically developed at rate $\lambda_L \cdot (1-\sigma_t^i)$. In addition, Firm i stochastically discovers the new technology at rate $\sigma_t^i \cdot \mu$, where μ is a constant parameter. We call that the research progress is made once the firm discovers the new technology, and this progress is irreversible. The parameters μ, λ_L, and λ_H are positive.

The race ends once one of the firms develops the innovative product. During the race, firms pay a flow cost $c > 0$. The first firm to develop the innovative product receives a lump-sum reward worth Π.[9] Firms do not discount the future and maximize their expected total payoff.[10] The successful development of the innovative product is publicly observable. Thus, firms know at all times if they are still on the race. However, firms do not observe their opponents' resource allocations over time. Regarding the research progress, we explore different setups where they are publicly or privately observed by rival firms.

For the rest of the paper, we make the following two parametric assumptions:

$$\Pi - \frac{c}{\mu} - \frac{c}{\lambda_H} > \Pi - \frac{c}{\lambda_L} > 0. \tag{2.1}$$

The first inequality states that when there is only one firm, conducting research and developing with the new technology is more efficient than developing with the old technology. Note that this condition is equivalent to $\frac{1}{\mu} + \frac{1}{\lambda_H} < \frac{1}{\lambda_L}$, implying that in expectation, the product can be developed faster by conducting research and developing with the new technology. Then, the second inequality implies that developing with the old technology is profitable.

3 Benchmark: Constant Development Rate

As a benchmark, imagine a scenario where Firm j does not engage in the resource allocation problem and the rate of development is held constant at λ. Solving this benchmark will provide valuable insights for the main analysis of the paper.

[9] We model the race as winner-takes-all competition. This payoff structure has been commonly used in the innovation race literature, e.g., Loury (1979); Lee and Wilde (1980); Denicolò and Franzoni (2010).

[10] With discounting the firms are not risk-neutral over the duration of the race conditional on the outcome. This complicates the closed-form solutions without affecting the qualitative results of the paper.

Suppose that Firm i has already discovered the new technology. Then, Firm i develops with the rate λ_H and Firm j develops with the rate λ. Then, Firm i's probability of winning the race is $\frac{\lambda_H}{\lambda_H+\lambda}$ and the expected duration of the remaining race is $\frac{1}{\lambda_H+\lambda}$. Therefore, Firm i's expected payoff is given by

$$\mathcal{V}_\lambda^1 \equiv \frac{\lambda_H}{\lambda_H+\lambda}\cdot\Pi - \frac{1}{\lambda_H+\lambda}\cdot c = \frac{\lambda_H\Pi - c}{\lambda_H+\lambda}. \tag{3.1}$$

Now suppose that Firm i has yet to discover the new technology. Consider research allocation strategies, which allocate a fixed amount of resources to research until either the new technology is discovered or the race ends, i.e., for some $x \in [0,1]$, $\sigma_t^i = x$ for all $t \geq 0$. In Lemma 6, we show that it is without loss to focus on these strategies. When Firm i allocates x amount of resources towards research, there are three potential outcomes: (i) Firm i develops the product with the old technology at rate $\lambda_L(1-x)$; (ii) Firm i discovers the new technology at rate μx; (iii) Firm j develops the product at rate λ. In the first scenario, Firm i wins the race and receives Π, and the probability of this event happening is $\frac{\lambda_L(1-x)}{\lambda_L(1-x)+\mu x+\lambda}$. In the second scenario, Firm i enters the post-research phase, and its expected payoff is $\mathcal{V}_\lambda^1$. The probability of this event occurring is $\frac{\mu x}{\lambda_L(1-x)+\mu x+\lambda}$. In the third scenario, Firm i receives nothing, and the probability of this event happening is $\frac{\lambda}{\lambda_L(1-x)+\mu x+\lambda}$. The expected duration of the game is $\frac{1}{\lambda_L(1-x)+\mu x+\lambda}$. Therefore, Firm i's expected payoff is given by

$$u(x) \equiv \frac{\lambda_L(1-x)\cdot\Pi + \mu x\cdot\mathcal{V}_\lambda^1 - c}{\lambda_L(1-x)+\mu x+\lambda}. \tag{3.2}$$

After taking the first derivative of u, with some algebra, we can derive that

$$u'(x) = \frac{\lambda_L(\lambda\Pi + c)(\lambda_\star - \lambda)}{(\lambda+\lambda_H)(\lambda+(1-x)\lambda_L + x\mu)^2} \tag{3.3}$$

where

$$\lambda_\star \equiv \mu\lambda_H\left(\frac{1}{\lambda_L} - \frac{1}{\mu} - \frac{1}{\lambda_H}\right) > 0.^{11} \tag{3.4}$$

Therefore, from $x \in [0,1]$, we have that $x = 1$ is optimal if $\lambda < \lambda_\star$, and $x = 0$ is optimal if $\lambda > \lambda_\star$. The following proposition formally states this result.

[11] Note that $\lambda_\star$ is a function of λ_L, μ and λ_H, but we suppress it to ease the notation.

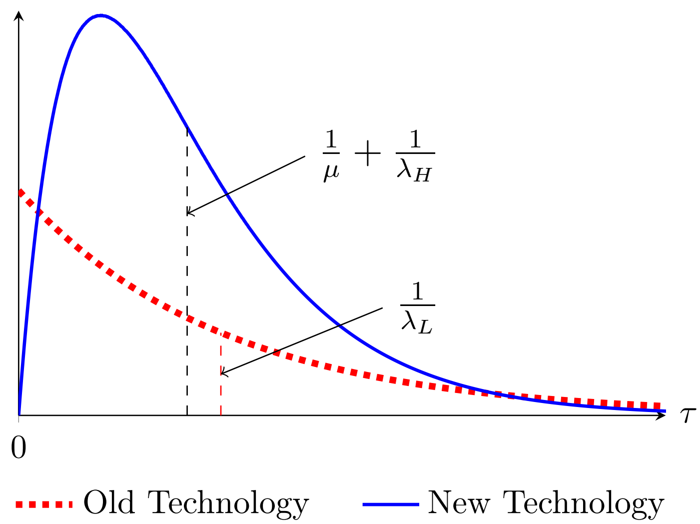


Figure 1: Probability distribution functions of a firm's development time

Proposition 1. *Suppose that Firm j has a constant development rate λ.*

(a) When $\lambda < \lambda_\star$, Firm i's optimal resource allocation strategy is to conduct research: $\sigma_t^i = 1$ for all $t \in \mathbb{R}_+$;

(b) When $\lambda > \lambda_\star$, Firm i's optimal resource allocation strategy is to develop with the old technology: $\sigma_t^i = 0$ for all $t \in \mathbb{R}_+$.

To illustrate the intuition behind this proposition, it is useful to understand the difference between the probability distributions of development times for conducting research and developing with the old technology. In Figure 1, the red dotted curve represents the probability distribution of Firm i's development time when it develops with the old technology. The blue solid curve is the probability distribution of the development time when Firm i conducts research and then develops with the new technology. By the parametric assumption in the previous section, the expected development time under research is shorter than that under development with the old technology. However, as illustrated in the figure, within a short time horizon, a firm may have a higher probability of successfully developing the product when using the old technology. This is because the firm only needs one breakthrough to develop with the old technology, whereas to develop the product with the new technology, it requires two breakthroughs: discovering the new technology and developing the product with it. Therefore, when a firm faces an opponent who has a high development rate, the firm may choose to develop with the old technology since it would give a higher chance of winning the race.

4 Public Information Setting

We begin our analysis by exploring a setting where the research progress of firms is publicly available information, i.e., each firm observes the research progress made by its competitor. In this case, the set of firms that have successfully obtained the new technology is common knowledge, and we represent it as a state variable denoted by $\omega \in \Omega \equiv \{\{A, B\}, \{A\}, \{B\}, \emptyset\}$.

We focus on firms' Markov strategies. Specifically, Firm i's Markov strategy is defined as $\mathbf{s}^i : \Omega \to [0, 1]$, where $\mathbf{s}^i(\omega)$ denotes the amount of resources allocated by Firm i to research in state ω. Recall that a firm allocates all its resources to development once it has the new technology, i.e., $\mathbf{s}^i(\omega) = 0$ for all $i \in \omega$. A pair of Markov strategies $(\mathbf{s}^A, \mathbf{s}^B)$ constitutes a Markov perfect equilibrium (MPE) if, for any given state, each firm's strategy is the best response to the opponent's strategy.

Next, we introduce three benchmark Markov strategies.

Definition 1. (a) The *research strategy* $\mathbf{s}^i_R$ for Firm i fully allocates resources to research regardless of the opponent's research progress ($\mathbf{s}^i_R \equiv \mathbb{1}_{\{\omega | i \notin \omega\}}$).[12]

(b) The *fall-back strategy* $\mathbf{s}^i_F$ fully allocates resources to research if neither firm has the new technology. If one of the firms has obtained the new technology, it fully allocates resources to development ($\mathbf{s}^i_F \equiv \mathbb{1}_{\{\emptyset\}}$).

(c) The *direct-development strategy* $\mathbf{s}^i_D$ fully allocates the resources to development regardless of the state ($\mathbf{s}^i_D \equiv 0$).

Now, we demonstrate that one of these strategies constitutes an MPE depending on the parameters. First, assume that $\lambda_\star > \lambda_H$. Note that the development rate cannot exceed λ_H, thus, it is always lower than $\lambda_\star$. By (a) of Proposition 1, we can guess that Firms would

[12]The function $\mathbb{1}_X$ is an indicator function: $\mathbb{1}_X(\omega) = 1$ if $\omega \in X$ and $\mathbb{1}_X(\omega) = 0$ if $\omega \notin X$.

conduct research regardless of the rival's strategy, thus, we can guess that the research strategy would constitute an equilibrium.

Next, suppose that $\lambda_L > \lambda_\star$. If a firm develops with the old technology, its development rate is λ_L, which is greater than $\lambda_\star$. Then, by (b) of Proposition 1, the rival firm would also develop with the old technology. Therefore, we can guess that the direct-development strategy would constitute an equilibrium.

Last, assume that $\lambda_H > \lambda_\star > \lambda_L$. Consider the case where only Firm j has discovered the new technology, i.e., $\omega = \{j\}$. Then, Firm j will develop with the new technology, i.e., the development rate of Firm j is λ_H, which is higher than $\lambda_\star$. Then, by (b) of Proposition 1, Firm i develops with the old technology. Since $\lambda_\star > \lambda_L$, the direct-development strategy cannot constitute an equilibrium. Thus, among the benchmark strategies, the fall-back strategy is the only candidate for an equilibrium strategy under this parametric region.

The following theorem shows that the above benchmark Markov strategies are unique MPE strategies within their respective parametric regions.

Theorem 1. *Suppose that firms' research progress is public information. Then, the Markov perfect equilibrium is uniquely characterized as follows:*

(a) if $\lambda_\star > \lambda_H$, both firms play their respective research strategies $(\mathbf{s}_R^A, \mathbf{s}_R^B)$;

(b) if $\lambda_H > \lambda_\star > \lambda_L$, both firms play the fall-back strategies $(\mathbf{s}_F^A, \mathbf{s}_F^B)$;

(c) if $\lambda_L > \lambda_\star$, both firms play the direct-development strategies $(\mathbf{s}_D^A, \mathbf{s}_D^B)$.[13]

Figure 2 illustrates the relevant parametric regions in the above theorem. Note that $\lambda_\star > 0$ is equivalent to $\frac{\lambda_L}{\mu} + \frac{\lambda_L}{\lambda_H} < 1$, which confines the parametric region to the triangular area depicted in Figure 2. With some algebra, we can show that $\lambda_\star > \lambda_H$ is equivalent to $1 > \frac{\lambda_L}{\lambda_H} + 2 \cdot \frac{\lambda_L}{\mu}$. This gives the transparent triangle-shaped region where firms employ the research strategy. Next, we can also show that $\lambda_\star < \lambda_L$ is equivalent to

$$\frac{\lambda_L}{\mu} > \frac{1 - \frac{\lambda_L}{\lambda_H}}{1 + \frac{\lambda_L}{\lambda_H}}.$$

[13] We do not confine our analysis solely to symmetric equilibria; rather, symmetry emerges as a result of our analysis.

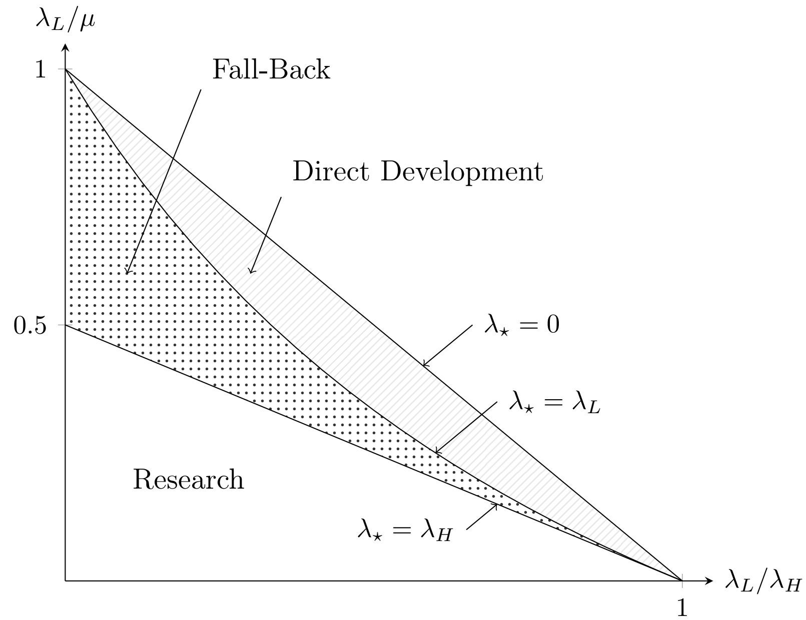


Figure 2: Markov Perfect Equilibrium in the Public Information Setting

With $\lambda_\star > 0$, it gives the shaded region where firms employ the direct-development strategy. In the remaining dotted region, $\lambda_H > \lambda_\star > \lambda_L$ holds and firms use the fall-back strategy.

5 Private Information Setting

In this section, we consider the private information framework, in which firms don't observe whether their opponents have the new technology. In this setting, as before, a firm with the new technology fully allocates the resources to development. A firm without the new technology, however, can only condition its resource allocation on the calendar time t. An *allocation policy* is a right-continuous function $\boldsymbol{\sigma} : \mathbb{R}_+ \to [0, 1]$ that represents the research allocation at a given time, conditional on the new technology not being obtained. We denote $\mathcal{S}$ the set of allocation policies.

New technology access and rate of development

Before analyzing the strategic aspects of this problem, it is useful to note that for each allocation policy $\boldsymbol{\sigma} \in \mathcal{S}$ there is an associated distribution of research breakthroughs and development times.[14] Let τ_D be the development time and τ_R be the research breakthrough time. For example, a firm that follows a policy $\boldsymbol{\sigma} = 0$ will never have a research breakthrough in finite time, and the development time τ_D is exponentially distributed with parameter λ_L. For a firm that follows a policy $\boldsymbol{\sigma} = 1$, τ_R is exponentially distributed with parameter μ, and τ_D is the sum of two exponentially distributed variables with parameters μ and λ_H.

Since the research breakthrough affects the rate of development, the random variables τ_R and τ_D are not independent. In this section, we derive for any policy $\boldsymbol{\sigma}$, two important objects: the probability of access to the new technology $\mathbf{p}_{\boldsymbol{\sigma}}$ and the rate of development $\mathbf{h}_{\boldsymbol{\sigma}}$. Formally, $\mathbf{p}_{\boldsymbol{\sigma}}$ represents the probability that a firm that follows allocation $\boldsymbol{\sigma}$ obtained the new technology by time t, conditional that it has not developed the product yet. $\mathbf{p}_{\boldsymbol{\sigma}}$ can be expressed in terms of τ_R and τ_D as follows:

$$\mathbf{p}_{\boldsymbol{\sigma}}(t) := \Pr[\tau_R < t < \tau_D \mid \tau_D > t]. \tag{5.1}$$

The following proposition characterizes for any $\boldsymbol{\sigma} \in \mathcal{S}$, the evolution of $\mathbf{p}_{\boldsymbol{\sigma}}$ over time.

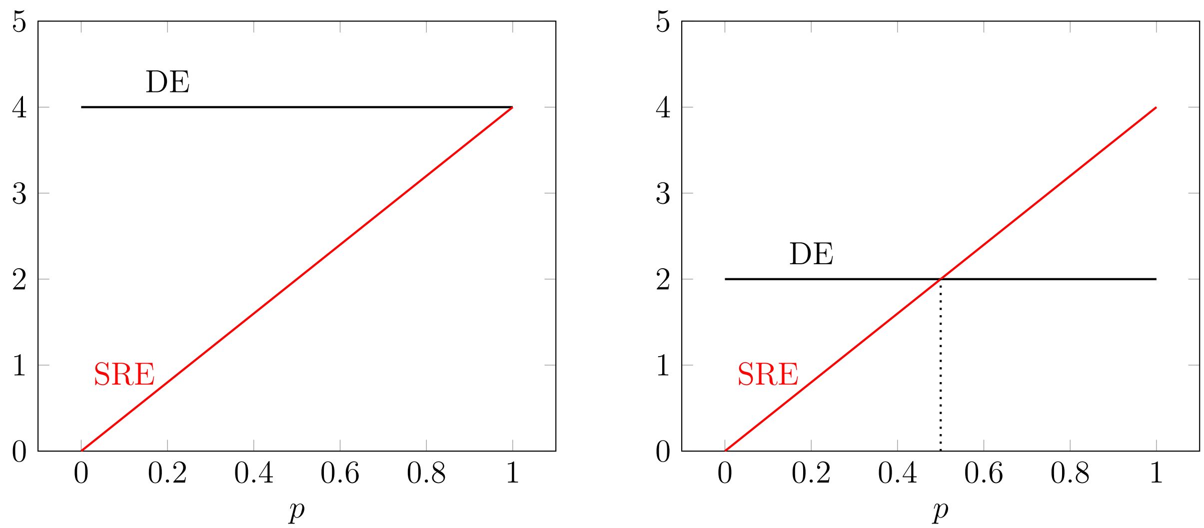
Proposition 2. *For any allocation policy $\boldsymbol{\sigma} \in \mathcal{S}$, the conditional probability $\mathbf{p}_{\boldsymbol{\sigma}}(t)$ satisfies the initial condition $\mathbf{p}_{\boldsymbol{\sigma}}(0) = 0$ and evolves according to the differential equation $\dot{\mathbf{p}}_{\boldsymbol{\sigma}}(t) = \delta(\mathbf{p}_{\boldsymbol{\sigma}}(t), \boldsymbol{\sigma}(t))$, where*

$$\delta(p, \sigma) := \mu \cdot \sigma \cdot (1 - p) - (\lambda_H - (1 - \sigma) \cdot \lambda_L) \cdot p \cdot (1 - p) \tag{5.2}$$

The proof of Proposition 2 is provided in Appendix D.1. The function δ highlights two distinct effects of the resource allocation $\sigma(t)$ on the evolution of $\mathbf{p}_{\boldsymbol{\sigma}}$, captured by the two terms in Eq. 5.2. First, if the firm doesn't have the new technology—which happens with probability $(1 - p)$—there is a research success arrival at rate $\mu \cdot \sigma$. We dub the effect of this arrival rate the *duration effect* (DE). On the other hand, the lack of development success indicates that it is less likely that the firm has the new technology. This second effect, which we dub the still-in-the-race effect (SRE), is reflected in the second term.[15] Notice that the

[14] We provide mathematical details about these arrival times including survival function $S^D_\sigma(t) \equiv \Pr[t < \tau_D]$ in Appendix D.1.

[15] Similar types of belief updating can be found in the strategic experimentation literature, e.g., Keller



(a) $\sigma = 1$, $\gamma = \rho = 1/2$ and $\lambda_L = 2$.

(b) $\sigma = 1$, $\gamma = 1$, $\rho = 1/2$ and $\lambda_L = 2$.

Figure 3: Duration Effect (Black) and Still-in-the-Race Effect (Red)

SRE is proportional to $\lambda_H - (1 - \sigma)\lambda_L$, which is the difference in the rate of development of the firm with and without the new technology.

In Figure 3, we illustrate the duration and still-in-the-race effects for $\sigma = 1$.[16] Specifically, we provide the graphs of each of the effects effect divided by $(1 - p)$: μ (DE), $\lambda_H p$ (SRE). In Figure 3a, we depict the case where $\mu = \lambda_H$. Observe that, in this case, the duration effect is larger than the still-in-the-race effect for every p. If we fix λ_H and increase μ, we observe that the duration effect continues to dominate the still-in-the-race effect. Hence, when $\mu > \lambda_H$, the conditional probability of having the new technology by time t when the firm follows policy $\boldsymbol{\sigma} = 1$, $\mathbf{p}_1(t)$, converges to 1. On the other hand, in Figure 3b, we illustrate the case where $\mu < \lambda_H$. In this case, the duration effect is greater than the still-in-the-race effect only when $p < \mu/\lambda_H$. Thus, $\mathbf{p}_{\boldsymbol{\sigma}}(t)$ does not exceed μ/λ_H for any $\boldsymbol{\sigma} \in \mathcal{S}$.

The access to the new technology and the allocation of resources determine the development rate of the firm. We can define the development rate of a policy as follows.

Lemma 1. *Given a policy $\boldsymbol{\sigma} \in \mathcal{S}$, the associated* development rate function $\mathbf{h}_{\boldsymbol{\sigma}}$ *is given by*

et al. (2005); Bonatti and Hörner (2011). The main difference is that, in that literature, the agents form beliefs about whether a project is good or bad. In this paper, on the other hand, firms only form beliefs about the technology access of the rival.

[16] The function $\mathbf{0} : \mathbb{R}_+ \to \{0, 1\}$ is $\mathbf{0}(t) = 0$ for all $t \in \mathbb{R}_+$, and the function $\mathbf{1} : \mathbb{R}_+ \to \{0, 1\}$ is $\mathbf{1}(t) = 1$ for all $t \in \mathbb{R}_+$,

$\mathbf{h}_{\boldsymbol{\sigma}}(t) = \xi(\mathbf{p}_{\boldsymbol{\sigma}}(t), \boldsymbol{\sigma}(t))$ *where*

$$\xi(p, \sigma) := p \cdot \lambda_H + (1 - p) \cdot (1 - \sigma) \cdot \lambda_L. \tag{5.3}$$

The first term of Eq. 5.3 captures that a firm with the new technology develops at rate λ_H. If the firm doesn't have the new technology, they only develop at a rate $(1 - \sigma)\lambda_L$.

Expected Payoffs and Solution Concept

Fixing the firm and opponent's allocation policies as $\boldsymbol{\sigma}$ and $\hat{\boldsymbol{\sigma}}$ respectively, let τ_D and $\hat{\tau}_D$ be the random variables that represent the arrival of successful developments. The expected payoff of the firm is therefore:

$$\mathcal{U}(\boldsymbol{\sigma}, \hat{\boldsymbol{\sigma}}) = \Pr[\tau_D < \hat{\tau}_D] \cdot \Pi - c \cdot \mathbb{E}[\tau_D \wedge \hat{\tau}_D].$$

In Appendix D.1, we show that we can write this expected payoff in terms of the associated development rates, $h_{\boldsymbol{\sigma}}$ and $h_{\hat{\boldsymbol{\sigma}}}$, as follows:

$$\mathcal{U}(\boldsymbol{\sigma}, \hat{\boldsymbol{\sigma}}) = \int_0^{\infty} e^{-\int_0^t h_{\boldsymbol{\sigma}}(s) + h_{\hat{\boldsymbol{\sigma}}}(s)\ ds} \left[h_{\boldsymbol{\sigma}}(t) \cdot \Pi - c\right] dt. \tag{5.4}$$

Intuitively, the exponential term captures the probability that no firm has developed the innovative product by time t, i.e. the probability that the race is still ongoing. In that case, the firm captures an expected flow payoff equal to $h_{\boldsymbol{\sigma}}(t) \cdot \Pi$, because it might develop the innovative product, and pays the flow cost c.

As in the literature on dynamic games with unobservable actions (e.g., Bonatti and Hörner, 2011), we aim to characterize the Nash equilibria (NE) in this game. Especially, we focus on the NE with the following property.

Definition 2. An allocation policy $\boldsymbol{\sigma} \in \mathcal{S}$ exhibits the *monotone development rate (MDR) property* if $h_{\boldsymbol{\sigma}}$ is weakly increasing. An allocation policy profile $(\boldsymbol{\sigma}_A, \boldsymbol{\sigma}_B)$ is a *Nash equilibrium with monotone development* (MDNE) if (i) $(\boldsymbol{\sigma}_A, \boldsymbol{\sigma}_B)$ is a Nash equilibrium and (ii) $\boldsymbol{\sigma}_A$ and $\boldsymbol{\sigma}_B$ are MDR.

When the opponent uses an allocation policy that satisfies the MDR property, the likelihood that the opponent wins the race in the next instant is larger as time passes. This makes the firm more eager to allocate resources in a way that increases the chance of obtaining a quick development success. In the next section, we prove the uniqueness of Nash equilibrium with monotone development rates.

Note that the MDR property is weaker than jointy requiring an increasing conditional probability $\mathbf{p}_{\boldsymbol{\sigma}}$ and decreasing allocation $\boldsymbol{\sigma}$, but it is stronger than any of the two separate requirements.

Equilibrium Characterization

Before the equilibrium characterization, we define, for intermediate parameters, an important probability and allocation level.

Lemma 2. *If $\lambda_\star \in (\lambda_L, \lambda_H)$, there is a unique combination of probability and allocation $(p_\star, \sigma_\star) \in (0,1)^2$ such that (i) $\xi(p_\star, \sigma_\star) = \lambda_\star$ and (ii) $\delta(p_\star, \sigma_\star) = 0$. Moreover,*

$$p_\star = \frac{\mu(\lambda_\star - \lambda_L)}{2\lambda_L\lambda_\star} = 1 - \frac{(\mu - \lambda_L)(\lambda_H - \lambda_\star)}{2\lambda_L\lambda_\star} \tag{5.5}$$

$$\sigma_\star = \frac{\lambda_\star - \lambda_L}{\mu - \lambda_L} \tag{5.6}$$

We can think of $(p_\star, \sigma_\star)$ as a special *steady state*. On one hand, $p_\star$ is the unique interior stationary probability, given a constant allocation $\sigma_\star$. On the other hand, $\sigma_\star$ is the unique allocation that induces a development rate $\lambda_\star$ when the probability is constant at $p_\star$. The following theorem characterizes the unique MDNE of the innovation race under private information.

Theorem 2. *Suppose that firms' research progress is private information. For almost all parameters there is a unique MDNE $(\boldsymbol{\sigma}_A, \boldsymbol{\sigma}_B)$, which is characterized as follows.*

*(i) If $\lambda_\star < \lambda_L$, firms play a **direct-development policy** $\boldsymbol{\sigma}_A = \boldsymbol{\sigma}_B = 0$.*

*(ii) If $\lambda_\star > \min\{\lambda_H, \mu\}$, firms play a **research policy**: $\boldsymbol{\sigma}_A = \boldsymbol{\sigma}_B = 1$.*

(iii) If $\lambda_\star \in (\lambda_L, \min\{\lambda_H, \mu\})$, firms play a ***stationary fall-back policy****: $\boldsymbol{\sigma}_A = \boldsymbol{\sigma}_B = \boldsymbol{\sigma}^{SF}$, which is defined as follows:*

$$\boldsymbol{\sigma}^{SF}(t) = \begin{cases} 1, & \text{if } t < T_\star, \\ \sigma_\star & \text{if } t \geq T_\star. \end{cases}$$

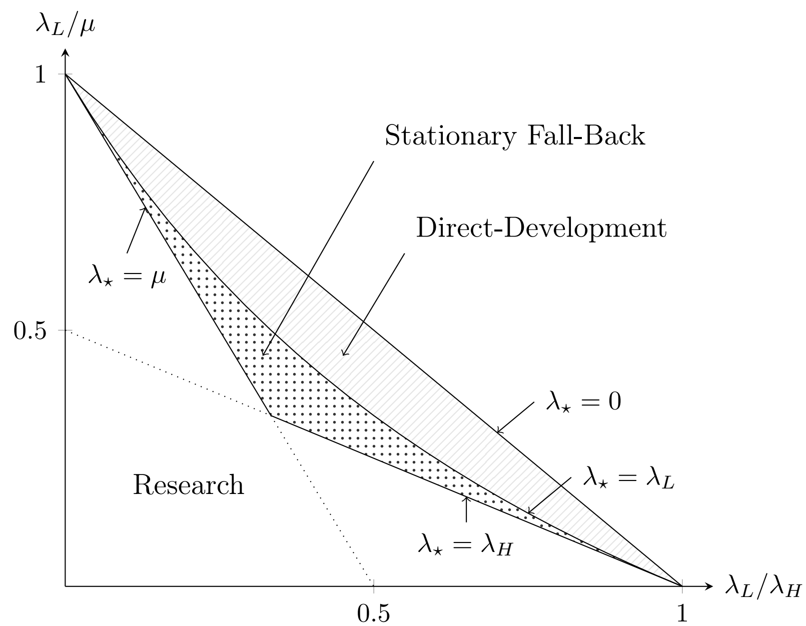
where $T_\star$ is the unique time such that $\mathbf{p}_1(T_\star) = p_\star$.

We provide intuition for this Theorem in Section 5.1 and prove it formally in Appendix D.4. When the parameters are such that $\lambda_\star > \lambda_H$ or $\lambda_\star < \lambda_L$, we know from Theorem 1, specifically from points (a) and (c), that firms do not tailor their allocation to the status of the opponent even when this information is publicly available. Thus, under the private information setting, it is intuitive that the firms adopt the same equilibrium allocations as in the public information setting for those regions.

The more interesting case occurs when $\lambda_\star \in (\lambda_L, \lambda_H)$. For these parameters, firms would like to adjust their allocation according to their rival's access to the new technology. If a firm believes that the rival has the new technology with enough likelihood, the firm may consider allocating more resources to develop using the old technology. In particular, if beliefs were ever to reach the stationary point $p_\star$, both firms implementing the interior allocation $\sigma_\star$ thereafter would constitute an equilibrium of the continuation game: the development rate remains constant at $\lambda_\star$, and therefore the firms are indifferent between any allocation.

Before reaching the belief $p_\star$, however, the development rates of the firms must be lower than $\lambda_\star$ by the monotonicity assumption. We show that firms have incentives to do research before this point. Moreover, if $\lambda_\star \in (\mu, \lambda_H)$, even when firms allocate all resources to research, the beliefs $p_{\boldsymbol{\sigma}}(t)$ remain bounded away from $p_\star$. Thus, firms never reach the indifference point and continue to conduct research until the research breakthrough.

In the following subsection, we shed some light on the details of Theorem 2.



5.1 Proof of Theorem 2

Recursive Formulation Let $V_1(t; \mathbf{h})$ and $V_0(t; \mathbf{h})$ be the continuation payoffs of a firm with and without the new technology at time t, respectively, when the opponent employs an allocation policy with associated development rate $\mathbf{h}$, and no firm has succeeded in development so far. Formally, we define V_1 as follows:

$$V_1(t; \mathbf{h}) := \int_t^\infty \{\lambda_H \Pi - c\} \cdot e^{-\int_t^s (\mathbf{h}(u) + \lambda_H) du} \, ds \tag{5.7}$$

The exponential term captures the probability that the race is still on by time s, given that the race is on by time by time t. The term $\lambda_H \Pi - c$ captures the flow expected payoff of the firm with the new technology. On top of fixing the opponent's development rate $\mathbf{h}$, we can fix the firm's policy $\boldsymbol{\sigma} \in \mathcal{S}$ to compute the continuation value v_0 of the firm without the new technology as follows:

$$v_0(t; \boldsymbol{\sigma}, \mathbf{h}) := \int_t^\infty \left\{ \boldsymbol{\sigma}(s) \mu V^1(s; \mathbf{h}) + (1 - \boldsymbol{\sigma}(s)) \lambda_L \Pi - c \right\} \cdot e^{-\int_t^s \{\mathbf{h}(u) + \boldsymbol{\sigma}(u)\mu + (1 - \boldsymbol{\sigma}(u))\lambda_H\} du} \, ds \tag{5.8}$$

In this expression, as before, the exponential term captures the probability that race is on and the firm doesn't have the new technology by time s, given that both hold at time t.

Conditional on this event, the firm enjoys an expected flow payoff captured by the expression in brackets: the firm pays the cost c and, at rate $\boldsymbol{\sigma}(s)\mu$, the firm obtains the new technology which induces a continuation payoff $V_1(s, \mathbf{h})$. At rate $(1 - \boldsymbol{\sigma}(s))\lambda_L$ the firm successfully develops, which induces a lump-sum payoff Π. By maximizing over all the allocation policies in $\mathcal{S}$, we obtain the continuation value of a firm without the new technology V_0.

$$V_0(t; \mathbf{h}) \ := \ \max_{\boldsymbol{\sigma} \in \mathcal{S}} \quad v_0(t; \boldsymbol{\sigma}, \mathbf{h})$$

Best responses

The aim is to characterize the optimal policy $\boldsymbol{\sigma}$ given the hazard rate $\mathbf{h}$ of the opponent. For any development rate function $\mathbf{h}$, let $R(t; \mathbf{h})$ be define as:

$$R(t; \mathbf{h}) := \mu(V_1(t; \mathbf{h}) - V_0(t; \mathbf{h})) - \lambda_L(\Pi - V_0(t; \mathbf{h}))$$

We interpret R as capturing the relative incentives to do research: by allocating resources to research, the firm obtains the new technology at rate μ, and this induces a jump in the continuation payoff equal to $V_1 - V_0$. By allocating resources to the old technology, the firm develops at rate λ_L, which generates a lump-sum payoff Π and ends the race $(-V_0)$. The following verification result identifies the relationship between the continuation values of the game and the optimal allocation of resources.

Proposition 3. *An allocation policy $\boldsymbol{\sigma}^*$ is a best-response to $\hat{\boldsymbol{\sigma}}$, i.e. $\mathcal{U}(\boldsymbol{\sigma}^*, \hat{\boldsymbol{\sigma}}) \geq \mathcal{U}(\boldsymbol{\sigma}, \hat{\boldsymbol{\sigma}})$ for all $\boldsymbol{\sigma} \in \mathcal{S}$, if and only if for every time $t \geq 0$ the following two conditions hold:*
(i) $v_0(t; \boldsymbol{\sigma}^*, \mathbf{h}_{\hat{\boldsymbol{\sigma}}}) > 0$ and (ii)

$$\boldsymbol{\sigma}^*(t) \quad \in \quad \arg\max_{\sigma \in [0,1]} \quad \sigma \cdot R(t; \mathbf{h}_{\hat{\boldsymbol{\sigma}}})$$

The proof of this proposition can be found in Appendix D.3. Intuitively, since allocations of resources are unobservable, the firms' allocation does not affect the future allocations of their opponent. Therefore, firms take the development rate of the opponent as given and optimally choose the research and development. Such allocation must, at each point in time, maximize the net flow of expected gains. Otherwise, small local deviations would be profitable.

Nash Equilibria with Monotone Development

Being equipped with the verification result from Proposition 3, we move to explain the intuition behind the proof of Theorem 2 for the different parameters we consider.

Theorem 2 (i) When $\lambda_\star < \lambda_L$, $(\boldsymbol{\sigma}^A, \boldsymbol{\sigma}^B) = (\mathbf{0}, \mathbf{0})$ is the unique MDNE.

In Theorem 1, we obtained that when $\lambda_\star < \lambda_L$ and firms observe the technology of their opponents, the unique MPE involves both firms developing with the old technology. Intuitively, the equilibrium allocations from this MPE survive as an equilibrium in the unobservable case because the information about the opponent's technology was not used anyway. To show that this is the unique MDNE, first note that any optimal policy has to eventually generate development rates higher than λ_L. Otherwise, the policy would be dominated by developing with the old technology. Thus, the development rates must converge to a rate higher than $\lambda_\star$. We show that the incentives to do research R must therefore converge to a negative number. By Proposition 3, there must be a time after which the firms stop allocating resources to research. However, we show in Lemma 17 that if a firm ever allocates resources to research, stopping induces a decreasing development rate. Thus, the only possibility is that firms don't research at all.

Theorem 2 (ii) When $\lambda_\star > \min\{\mu, \lambda_H\}$, $(\boldsymbol{\sigma}^A, \boldsymbol{\sigma}^B) = (\mathbf{1}, \mathbf{1})$ is the unique MDNE.

First, we show that for any policy that satisfies monotone development rate, the development rates are bounded above by $\min\{\mu, \lambda_H\}$. This is true because maintaining a development rate higher than $\min\{\mu, \lambda_H\}$ requires a strictly decreasing $\boldsymbol{\sigma}$ to compensate for the decrease in beliefs $\mathbf{p}_{\boldsymbol{\sigma}}$. At some time, $\boldsymbol{\sigma}(t)$ must reach zero, and such development rate cannot be maintained anymore. Thus, the development rates of the firms must converge to some rate weakly lower than $\min\{\mu, \lambda_H\}$, which is lower than $\lambda_\star$. Therefore, for any $\boldsymbol{\sigma}$ satisfying MDR, there is a time T for which $R(t; \mathbf{h}_{\boldsymbol{\sigma}}) > 0$ for all $t > T$. However, we also

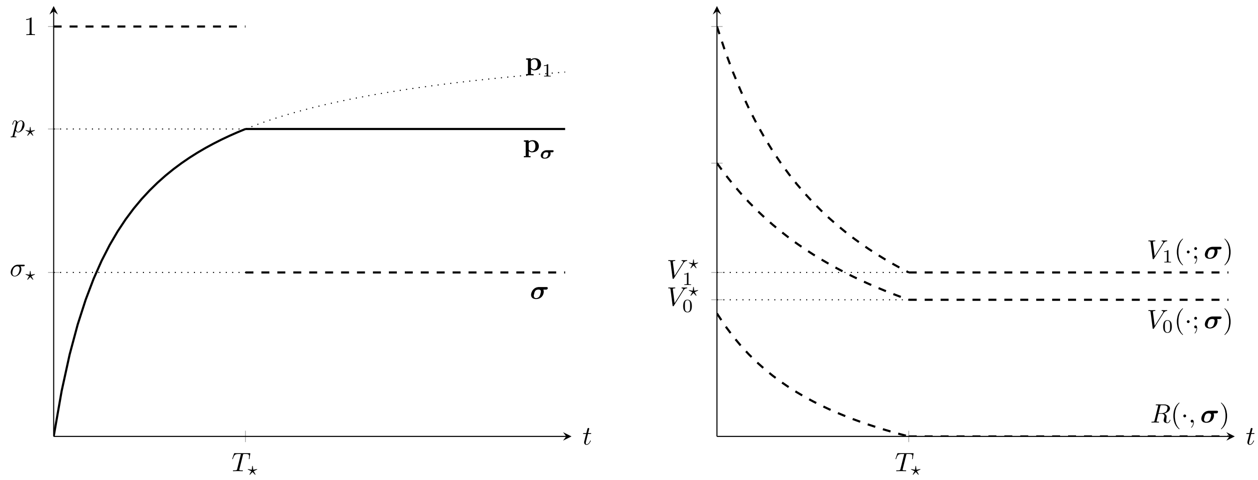


Figure 4: Stationary fall-back equilibrium.

show in a corollary of Lemma 15 that if a firm finds it optimal to allocate resources to development with the old technology ($R(t, \mathbf{h}) < 0$) then it must be that this is always optimal ($R(t, \mathbf{h}) < 0$ for all $s > t$). The only equilibrium candidate is therefore $(1, 1)$.

Theorem 2 (iii) When $\lambda_\star \in (\lambda_L, \min\{\lambda_H, \mu\})$, the stationary fall-back policy profile is the unique MDNE.

First, we establish that in any MDNE both firms' development must converge precisely to $\lambda_\star$ (Lemma 22). Essentially, we show that any other converging limits lead to a contradiction. Next, we prove that, in any MDNE, the two firms must reach the development rate $\lambda_\star$ simultaneously. If one firm reaches $\lambda_\star$ first, we show using Lemma 20 that the firm has incentives to allocate all resources to research until the opponent reaches $\lambda_\star$. However, this allocation would necessarily elevate the development rate, pushing it beyond $\lambda_\star$.

Let's define $T_\star$ as the time when both firms reach the development rate $\lambda_\star$. From T onward, the firms develop at the rate $\lambda_\star$. We show that there is a unique constant probability and allocation, $p_\star$ and $\sigma_\star$, that can maintain the development rate at $\lambda_\star$, as any deviation from these levels would induce the development rates to diverge. To obtain the allocations before time $T_\star$, we apply Lemma 20 again to show that firms must strictly allocate all resources to research before $T_\star$. The continuity of the probability function pins down the time $T_\star$, since it must therefore be that $\mathbf{p}_1(T_\star) = p_\star$.

Figure 4 depicts the dynamics of the key variables in the equilibrium for the parameters in which $\lambda_\star \in (\lambda_L, \min\{\mu, \lambda_H\})$. The equilibrium presents two distinct stages, demarcated by the time $T_\star$. During the first stage, the firms allocate all resources to research. Consequently, $\mathbf{p}$ and $\mathbf{h}$ are strictly increasing. Additionally, the continuation values for the firm with and without the new technology decline over time.

When $\mathbf{p}_1$ reaches the level $p_\star$, the equilibrium transitions to a second stage where the research allocation shifts to $\sigma_\star$ and the functions $\mathbf{p}$ and $\mathbf{h}$ remain constant at the levels $p_\star$ and $\lambda_\star$, respectively. The continuation values for the firms with and without the new technology remain constant at the levels

$$V_1^\star = \frac{\lambda_H \Pi - c}{\lambda_H + \lambda_\star}, \qquad \text{and} \qquad V_0^\star = \frac{\mu V_1^\star - \lambda_L \Pi}{\mu - \lambda_L}. \tag{5.9}$$

6 Patent, License and Trade Secret

6.1 Setup

In this section, we extend the model by allowing the firms to patent and license the new technology. The main components of the model remain the same as in Section 2, with a key difference: once a firm discovers the new technology, it has the option to apply for a patent. If the firm decides to not apply for a patent for the new technology, its discovery remains secret: in this case, we say that the firm uses *trade secret protection*.

We assume that firms cannot fraudulently claim the possessions of the new technology, and the patent applications are publicly available information. Thus, when a firm applies for a patent, the rival firm becomes aware of its possession of the new technology. On the other hand, if a firm decides to protect the new technology by using the trade secret, it does not release the information about this firm's possession of the new technology. However, this firm may face a risk of losing the right to use the new technology, as described in the paragraphs to follow. For simplicity, we also assume that the entire patent process is instantly completed and free of cost, and patents never expire: once a firm acquires the right to use the new technology, it retains ownership indefinitely. We intentionally impose these strong assumptions in favor of patents to isolate the pure effect of information in firms' patenting decisions regarding research progress. The only incentive for a firm to not apply for a patent is to conceal its research progress from the rival firm.

The timing of the game is as follows. When a firm discovers the new technology, it chooses to apply for a patent or not.[17] If neither firm applies for a patent, the game is the same as in our baseline model. When one of the firms applies for a patent, firms enter the subgame described in Figure 5. Suppose that Firm i has just applied for a patent. First, when the rival firm (Firm j) does not possess the new technology yet, Firm i owns the patent for the new technology. Then, Firm i offers a take-it-or-leave-it (TIOLI) offer of the license fee l to Firm j. If the offer is rejected, Firm j cannot use the new technology even if it independently discovers the new technology. Therefore, Firm j allocates all its resources toward developing with the old technology—since it is useless to discover the new technology—whereas Firm i develops with the new technology. If the offer is accepted, Firm j pays l to Firm i, and Firm i allows Firm j to use the new technology, thus, firms race developing the product by using the new technology.

Next, consider the case where Firm j already possesses the new technology but has not applied for a patent, i.e., Firm j has protected the right to use the new technology by using the trade secret. Let $\alpha \in [0, 1]$ denote the level of trade secret protection. Firm j (the challenger) can contest the patent based on the evidence demonstrating their prior possession of the new technology. With the probability α, Firm j's discovery is protected by trade secret, thereby both firms have the right to use the new technology. With the probability $1 - \alpha$, Firm i retains sole ownership of the patent. The subsequent game involving Firm i's patent is the same as above.

First-Best Outcome Consider the social planner whose goal is to maximize the joint expected profit of the firms. Assume that the planner can observe the research progress, and make the patent and license decisions. Suppose that one of the firms has discovered the new technology. Then, it is socially efficient to direct this firm to license the new technology to

[17]Here, to simplify the discussion, we assume that a firm can only apply for a patent right after the discovery. In practice, it is possible to delay the patent application, e.g., a firm can protect the new technology by trade secret for six months, then apply for a patent.

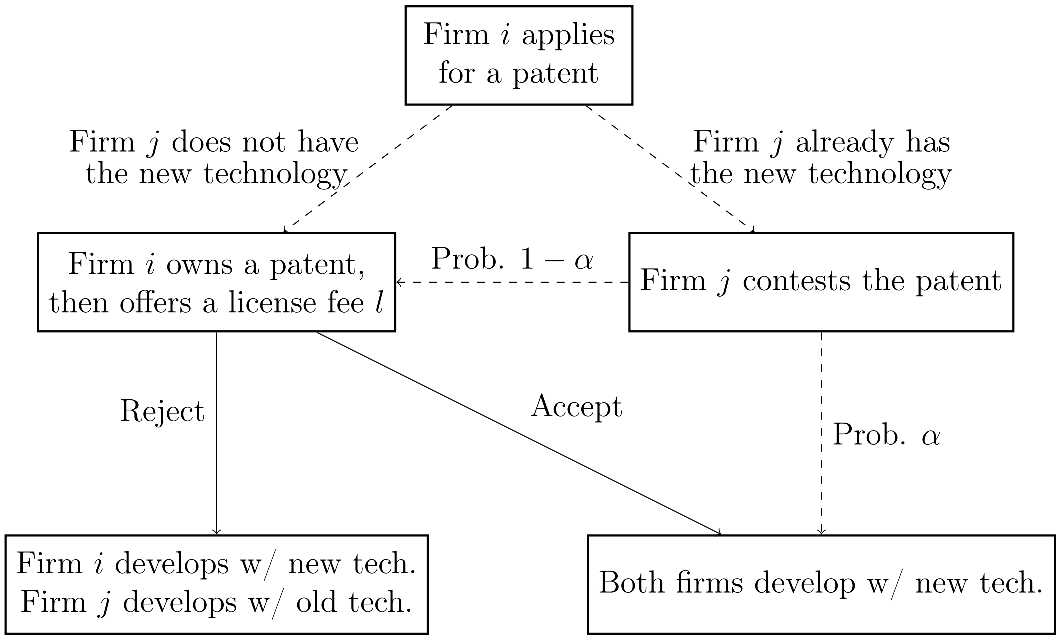


Figure 5: Timing of the patent game after the patent application

the other firm, enabling both firms to develop with the new technology. Given this, consider the planner's problem in allocating firms' resources when neither firm has yet discovered the new technology. Note that the expected cost is constant with respect to the number of firms: even if the flow cost is doubled, the expected completion time will be halved. Then, by (2.1), it is socially efficient when both firms conduct research. Therefore, the first-best resource allocation is to allocate all the resources toward research, and once one of the firms discovers the new technology, it applies for a patent and license it, then both firms develop with the new technology.

Optimal License Fee Consider the subgame that Firm i has obtained the patent for the new technology. If the license offer is accepted, both firms develop with the new technology, thus, the social welfare is $\Pi - \frac{2c}{2\lambda_H}$. When it is rejected, since Firm i develops with the new technology and Firm j develops with the old technology, the social expected cost is $\frac{2c}{\lambda_L+\lambda_H}$, thus, the social welfare is

$$\Pi - \frac{2c}{\lambda_L + \lambda_H} < \Pi - \frac{2c}{2\lambda_H}.$$

Since the licensing firm has an exclusive bargaining power, Firm i would capture the entire surplus from licensing:

$$\left(\Pi - \frac{2c}{2\lambda_H}\right) - \left(\Pi - \frac{2c}{\lambda_L + \lambda_H}\right) = \frac{\lambda_H - \lambda_L}{2\lambda_H(\lambda_H + \lambda_L)}c. \tag{6.1}$$

By using these, we can derive the optimal license fee.

Proposition 4. *Suppose that Firm i has obtained the patent for the new technology. Then, Firm i offers a license fee*

$$l^* = \frac{\lambda_H - \lambda_L}{\lambda_H + \lambda_L} \cdot \frac{\lambda_H \Pi + c}{2\lambda_H}. \tag{6.2}$$

to Firm j. Then, Firm i's expected payoff is $U_{Licensor} = V_C + l^*$ and Firm j's expected payoff is $U_{Licensee} = V_C - l^*$ where $V_C = \frac{\lambda_H \Pi - c}{2\lambda_H}$.

Proof of Proposition 4. When the offer is rejected, Firm j's expected payoff is $\frac{\lambda_L \Pi - c}{\lambda_H + \lambda_L}$. Note that V_C is the expected payoff when both firms race with the new technology. thus, when the license offer with the fee l is accepted, Firm j's expected payoff is $V_C - l$. Then, Firm i's optimal offer is

$$l^* = V_C - \frac{\lambda_L \Pi - c}{\lambda_H + \lambda_L}, \tag{6.3}$$

and we can derive (6.2) with simple algebra. Then, once the offer is accepted, Firm i's expected payoff is $V_C + l^*$ and Firm j's expected payoff is $V_C - l^*$. □

With some algebra, we can also show that

$$U_{Licensor} = \frac{\lambda_H \Pi - c}{\lambda_H + \lambda_L} + \frac{\lambda_H - \lambda_L}{2\lambda_H(\lambda_H + \lambda_L)}c, \qquad U_{Licensee} = \frac{\lambda_L \Pi - c}{\lambda_H + \lambda_L}. \tag{6.4}$$

Intuitively, l^* was deliberately chosen to make $U_{Licensee}$ equal to Firm j's outside option, and Firm i is capturing the surplus from licensing derived in (6.1).

Now consider the case when Firm i applies a patent, Firm j already has the new technology. With probability α, both firms have the right to use the new technology, and their expected payoffs are V_C. With probability $1 - \alpha$, Firm i retains the patent, and Firm i receives $U_{Licensor}$ and Firm j receives $U_{Licensee}$.

Corollary 1. *When a patent is contested, the expected continuation payoffs of the applicant* ($U^{\alpha}_{Applicant}$) *and the challenger are* ($U^{\alpha}_{Challenger}$) *are:*

$$U^{\alpha}_{Applicant} = V_C + (1-\alpha) \cdot l^*, \qquad U^{\alpha}_{Challenger} = V_C - (1-\alpha) \cdot l^*. \tag{6.5}$$

6.2 Patents under Public Information

We start by considering a setting where, as in Section 4, research progress is public information. In this setting, a firm deciding on patent application is aware of its opponent's available technologies. For the first firm to obtain the new technology, the only incentive to forego patenting would be to conceal the discovery from its rival. In public information settings, concealment is not possible, hence the first firm to obtain the new technology patents it. We formally state this result in the following proposition.

Proposition 5. *Suppose that research progress is public information. In any SPNE, the first firm to discover the new technology applies for a patent.*

The proof is provided in Appendix E.1.1. Note that the patent application of the first firm to obtain the new technology can not be challenged. With this result and the equilibrium license fee from Proposition 4, we pin down the continuation payoffs of both firms after the new technology is first discovered. We use these continuation payoffs to analyze the resource allocation of the firms before the new technology is first discovered. Note that, on the equilibrium path of any SPNE, research is only conducted before the first discovery. Let $\mathbf{s}^i_P$ be the research allocation of the firm i before the new technology is first discovered.[18]

Proposition 6. *Suppose that research progress is public information. In any MPE, the resource allocations before the new technology is first discovered are characterized as follows:*

(a) if $\lambda_\star > \lambda_L$*, both firms do research:* $\mathbf{s}^A_P = \mathbf{s}^B_P = 1$*;*

(b) if $\frac{\lambda_H \lambda_L}{2\lambda_H + \lambda_L} > \lambda_\star$*, both firms develop with old technology:* $\mathbf{s}^A_P = \mathbf{s}^B_P = 0$*;*

(c) if $\lambda_L > \lambda_\star > \frac{\lambda_H \lambda_L}{2\lambda_H + \lambda_L}$*, there exist thresholds* $\tilde{\pi}_0 > \tilde{\pi}_1 > 1$ *such that*

[18] We focus, as in Section 4, on Markov strategies, i.e. allocations that are independent of calendar time.

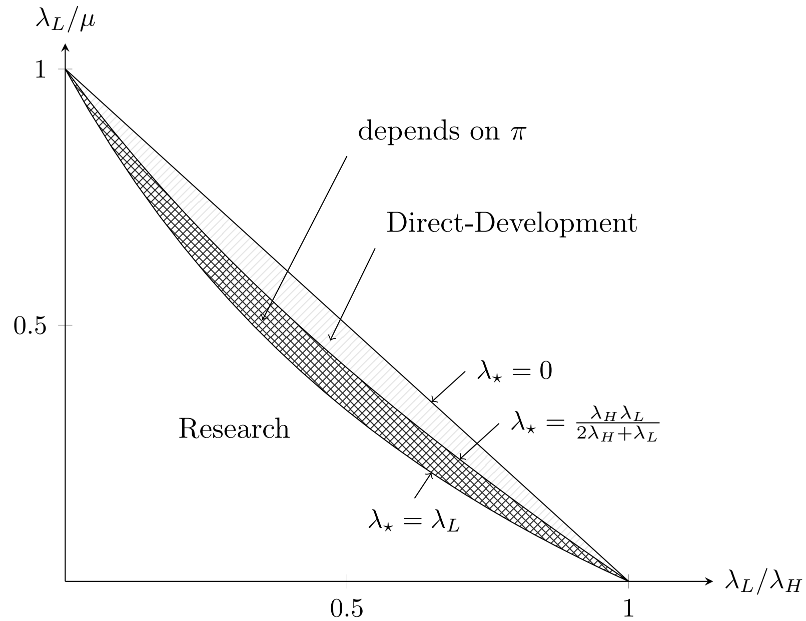


Figure 6: Equilibrium Resource Allocations in the Patent Game under Public Information

(i) *when* $\pi \equiv \lambda_L \Pi / c > \tilde{\pi}_0$*, both firms develop with old technology:* $\mathbf{s}_P^A = \mathbf{s}_P^B = 0$*;*

(ii) *when* $\tilde{\pi}_0 > \pi > \tilde{\pi}_1$*, there are three equilibrium allocations: one firm does research and the other firm develops with the old technology, i.e.,* $(\mathbf{s}_P^A, \mathbf{s}_P^B) = (1, 0)$ *or* $(0, 1)$*; both firms allocate some amount* $z^* \in (0, 1)$ *resources to research:* $\mathbf{s}_P^A = \mathbf{s}_P^B = z^*$*;*

(iii) *when* $\tilde{\pi}_1 > \pi$*, both firms do research:* $\mathbf{s}_P^A = \mathbf{s}_P^B = 1$*;*

The proof of this Proposition is in Appendix E.1.2. Part (i) reflects the fact that when the new technology represents a sufficiently important improvement compared to the old technology and relative to the difficulty of obtaining it, firms will race to obtain the new technology. On the other hand, Part (ii) says that when the improvement is sufficiently small, firms prefer to develop with the old technology. Part (iii) analyzes the case of intermediate rate parameters, for which there can be multiple equilibria. These multiplicity arise, for intermediate levels of the stakes π, as a consequence of strategic substitutability in research, that is induced by the possibility of patenting.[19] It is worth highlighting that, since patents are never challenged on the equilibrium path, the research allocations are independent of α.

[19]In Proposition 8, we study the general problem of first-stage research with arbitrary continuation payoffs and both research complementarities and substitutabilities.

We can compare the equilibrium allocations characterized in Proposition 6 and the on-path patenting decisions characterized in 4 to the first-best obtained in Section 6.1.

Corollary 2. *Suppose that research progress is public information. The first-best can be sustained in a MPE if and only if (i)* $\lambda_\star > \lambda_L$*; or (ii)* $\lambda_L > \lambda_\star > \frac{\lambda_H \lambda_L}{2\lambda_H + \lambda_L}$ *and* $\tilde{\pi}_1 > \pi$*.*

6.3 Patents under Private Information

Now we explore the case where research progress is private information. To simplify our discussion, we focus on a case in which $\lambda_H > \lambda_\star > \mu$: both firms employ the fall-back strategy in the public information setting, but conduct research in the private information setting. Then, we identify conditions under which (i) both firms apply for patents, namely *Efficient Patent Equilibrium*; or (ii) both firms do not apply for patents, namely *Concealment Equilibrium.*

6.3.1 Efficient Patent Equilibrium

First, we explore parametric conditions where an efficient patent equilibrium exists. Assume that Firm j's resource allocation strategy is to do research indefinitely and apply for a patent once the new technology is discovered. Suppose that Firm i has discovered the new technology, and Firm j has not applied for a patent yet. Given Firm j's patent application strategy, the fact that Firm j has not applied for a patent implies that Firm j does not have the new technology yet. Therefore, if Firm i applies for a patent, it will attain the patent and the expected payoff is $U_{Licensor} = V_C + l^*$. Suppose that Firm i does not apply for a patent. Firm j keeps conducting research and applies for a patent when it discovers the new technology. Firm i's payoff in this case is $U^{\alpha}_{Challenger} = V_C - (1-\alpha) \cdot l^*$. Therefore, Firm i's expected payoff of not applying for a patent is

$$\frac{\lambda_H \Pi + \mu \cdot U^{\alpha}_{Challenger} - c}{\lambda_H + \mu} = \frac{(\mu + 2\lambda_H)V_C - \mu(1-\alpha)l^*}{\lambda_H + \mu}. \tag{6.6}$$

Lemma 3. *Suppose that research progress is private information, and Firm* j*'s resource allocation strategy is to do research indefinitely (*$\sigma_t = 1$ *for all* $t \geq 0$*) and apply for a patent* *once the new technology is discovered. When Firm i discovers the new technology, it applies for a patent if and only if*

$$\frac{l^*}{V_C} > \frac{\lambda_H}{\lambda_H + \mu(2-\alpha)}. \tag{6.7}$$

Proof of Proposition 3. Firm i applies for a patent when $U_{Licensor}$ is greater than (6.6), which is equivalent to:

$$\begin{aligned} & (\lambda_H + \mu)V_C + (\lambda_H + \mu)l^* > (\mu + 2\lambda_H)V_C - \mu(1-\alpha)l^* \\ \iff \quad & \{\lambda_H + \mu(2-\alpha)\}\, l^* > \lambda_H V_C. \end{aligned}$$

Since $1 > \alpha$, $\lambda_H, \mu > 0$ and $V_C > 0$, it is equivalent to (6.7). □

Observe that the left hand side of (6.7) remains constant with respect to α, while the right hand side increases with α. Therefore, as α increases, (6.7) becomes more difficult to hold. This result aligns with intuition: as the trade secret protection level increases, firms are less inclined to apply for patents. Also note that the license fee l^* is endogenously determined. The following lemma provides the parametric conditions under which (6.7) holds.

Lemma 4. *The inequality (6.7) holds if and only if (i) $\lambda_\star \geq \frac{\alpha}{2-\alpha}\lambda_H$; or (ii) $\lambda_\star < \frac{\alpha}{2-\alpha}\lambda_H$ and*

$$\hat{\pi}(\alpha) \equiv 1 + \frac{\lambda_L + \lambda_H}{\lambda_H} \cdot \frac{\lambda_\star + \lambda_H}{\frac{\alpha}{2-\alpha}\lambda_H - \lambda_\star} > \pi. \tag{6.8}$$

By using this lemma, we can pin down the conditions under which the efficient patent equilibrium exists.

Proposition 7. *Suppose that research progress is private information and $\lambda_H > \lambda_\star > \mu$. The efficient patent equilibrium exists if and only if one of the following conditions holds: (i) $\alpha < \frac{2\mu}{\lambda_H+\mu}$; or (ii) $\alpha > \frac{2\mu}{\lambda_H+\mu}$ and $\lambda_\star \geq \frac{\alpha}{2-\alpha}\lambda_H$ or $\hat{\pi}(\alpha) > \pi$.*

6.3.2 Concealment Equilibrium

Now we explore parametric conditions for which an equilibrium without patents exists. Conditional on no patents, the equilibrium policies have to be the unique policies that form an equilibrium in Theorem 2. Let $\boldsymbol{\sigma}^*$ denote such policy.

Observation There is a concealment equilibrium if and only if, for all $t \geq 0$,

$$V_1(t; \mathbf{h}_{\boldsymbol{\sigma}^*}) \geq V_C + (1 - \alpha\, \mathbf{p}_{\boldsymbol{\sigma}^*}(t)) \cdot l^*. \tag{6.9}$$

To understand the observation, notice that Eq. 6.9 captures the trade-off in the patenting decision of a firm that discovers the new technology at time t, when the opponent follows policy $\boldsymbol{\sigma}^*$ and never patents. The left-hand-side denotes the payoff obtained by not patenting, i.e by keeping the discovery secret. The right-hand-side is equivalent to $p_t \cdot V^{\alpha}_{Applicant} + (1 - p_t) \cdot V_{Licensor}$ and captures the expected payoff if the firm decides to patent at time t. If the equation 6.9 holds for all t, then it is a best response to never patent.

Suppose that $\lambda_H > \lambda_\star > \mu$. In this case, firms indefinitely conduct research in equilibrium in the private information setting. If a firm applies for a patent upon discovery, its expected payoff is $V_C + (1 - \alpha \cdot p_t) \cdot l^*$. On the other hand, if the firm keeps it secret, its expected payoff is $V_1(t; \mathbf{h}_{\boldsymbol{\sigma}^*})$ is $\left\{1 + \frac{\lambda_H}{\lambda_H + \mu}(1 - p_t)\right\} V_C$. Then, (6.9) is equivalent to:

$$\frac{l^*}{V_C} < \frac{\lambda_H}{\lambda_H + \mu} \cdot \frac{1 - p_t}{1 - \alpha \cdot p_t}. \tag{6.10}$$

The right hand side is decreasing in p_t. Under $\lambda_H > \lambda_\star > \mu$, p_t converges to μ/λ_H, thus, we can plug this into (6.10):

$$\frac{l^*}{V_C} < \frac{\lambda_H(\lambda_H - \mu)}{(\lambda_H + \mu)(\lambda_H - \alpha\mu)}. \tag{6.11}$$

With simple algebra, we can show that $\frac{\lambda_H(\lambda_H - \mu)}{(\lambda_H + \mu)(\lambda_H - \alpha\mu)} \leq \frac{\lambda_H}{\lambda_H + \mu(2 - \alpha)}$. Therefore, the threshold for the concealment equilibrium is below the one for the efficient patent equilibrium, i.e., there is no parameter such that both the efficient patent equilibrium and the concealment equilibrium exist.

7 Conclusion

In this article, we study the long-lasting question of patent vs. secrecy by highlighting the firm's incentives to conceal breakthroughs to hinder the rival's strategic response. To do so, we introduce an innovation race model with multiple paths and show that firms' disclosing decisions depend on the reward for winning the race.

We show that when interim breakthroughs are public, patent protection is effective in inducing a more efficient allocation of R&D resources. However, when interim breakthroughs are private and stakes are high, patent protection has a limited effect. Based on this result, we can argue that, in some situations, higher stakes may reduce patenting and licensing which would decrease the pace of innovation.

There are many avenues open for further research. For example, we assume that there are exogenously given two paths towards innovation, and one of the paths requires two breakthroughs. However, in practice, there are numerous ways to make an innovation, and it often requires more than two breakthroughs. We also assume that a firm's R&D resources are fixed over time, but we could also allow firms to endogenously choose how much effort to put into each point in time. Finally, we assume the contest structure is given by the winner-takes-all competition, but we might consider a contest designing problem. We leave these intriguing questions and others for future work.

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Appendix

A Preliminaries

A.1 Useful Observations

Let τ be a random variable on $\mathbb{R}_+$. Suppose that it has a continuous and differentiable cumulative distribution function $F : \mathbb{R}_+ \to [0, 1]$. Let $S(t)$ denote the survival function of τ, i.e., $S(t) = 1 - F(t)$. If $\lim_{t\to\infty} t \cdot S(t) = 0$, the following equation holds:

$$\mathbb{E}[\tau] = \int_0^\infty t \cdot F'(t)dt = -t \cdot S(t)\Big|_0^\infty + \int_0^\infty S(t)dt = \int_0^\infty S(t)dt. \tag{A.1}$$

Let h be a development rate function of τ: $h(t) = -S'(t)/S(t)$.[20] Then, under the assumption that $F(0) = 0$, we can derive that $S(t) = e^{-\int_0^t h(s)ds}$. Then, (A.1) can be rewritten as follows:

$$\mathbb{E}[\tau] = \int_0^\infty e^{-\int_0^t h(s)ds} dt. \tag{A.2}$$

Consider another random variable $\hat{\tau}$ independent to τ. Let $\hat{S}$ and $\hat{h}$ be its survival and development rate functions. Observe that

$$\Pr[\tau < \hat{\tau}] = \int_0^\infty \hat{S}(t)\ dF(t) = -\int_0^\infty S'(t) \cdot \hat{S}(t)\ dt. \tag{A.3}$$

Then, (A.3) can be rewritten as follows:

$$\Pr[\tau < \hat{\tau}] = \int_0^\infty h(t) \cdot S(t) \cdot \hat{S}(t)\ dt = \int_0^\infty h(t) \cdot e^{-\int_0^t (h(s)+\hat{h}(s))ds}\ dt. \tag{A.4}$$

Now consider another random variable which is a minimum of τ and $\hat{\tau}$, denoted by $(\tau \wedge \hat{\tau})$. Then, the survival function of $(\tau \wedge \hat{\tau})$ is $S(t) \cdot \hat{S}(t)$, and the development function of $(\tau \wedge \hat{\tau})$

[20]In the literature, the function $h(t)$ is often referred to as a 'hazard rate' function. The term hazard rate originated from the tradition of describing arrivals as negative events such as failures. In our context, where we are analyzing the timing of product developments, we use the term 'development rate' instead of hazard rate.

is $h(t)+\hat{h}(t)$. By applying (A.2), when $\lim_{t\to\infty} t\cdot S(t)\cdot \hat{S}(t)=0$, we have

$$\mathbb{E}[\tau \wedge \hat{\tau}] = \int_0^\infty e^{-\int_0^t (h(s)+\hat{h}(s))ds}\, dt. \tag{A.5}$$

A.2 Formal Definitions of Arrival Times

Given an allocation policy $\sigma : \mathbb{R}_+ \to [0,1]$, we define the following random variables:

1. τ_L: the arrival time of successful development with the old technology;

2. τ_R: the arrival time of the new technology discovery.

Let $\Sigma_t \equiv \int_0^t \sigma_s ds$. Then, the survival functions of τ_L and τ_R are given as follows: for all $t \geq 0$,

$$S^L_\sigma(t) = e^{-\lambda_L(t-\Sigma_t)} \quad \text{and} \quad S^R_\sigma(t) = e^{-\mu\Sigma_t}. \tag{A.6}$$

In addition, the development rate functions can be derived as follows:

$$h^L_\sigma(t) = \lambda_L(1-\sigma_t) \quad \text{and} \quad h^R_\sigma(t) = \mu\sigma_t. \tag{A.7}$$

Intuitively, the product is developed with the old technology at the rate $h^L_\sigma(t) = \lambda_L(1-\sigma_t)$ and the new technology is discovered at the rate $h^R_\sigma(t) = \mu\sigma_t$.

B Proofs for the Constant Development Rate Case

Lemma 5. *Suppose that Firm j has a constant development rate λ. When Firm i employs an allocation policy σ, its expected payoff is given as follows:*

$$V^0_\lambda(\sigma) = \int_0^\infty \big(\lambda_L(1-\sigma_t)\cdot\Pi + \mu\ \sigma_t\cdot\mathcal{V}^1_\lambda - c\big)\cdot e^{-\lambda_L(t-\Sigma_t)-\mu\Sigma_t-\lambda t}\, dt, \tag{B.1}$$

where $\Sigma_t \equiv \int_0^t \sigma_s ds$.

Proof of Lemma 5. Let τ_λ be the arrival time of Firm j. When any of the arrival times τ_L, τ_R and τ_λ occurs, we can regard that Firm i's payoff is realized. Furthermore, it incurs a flow cost c until one of these arrival times takes place. Thus, Firm i's expected payoff can be written as follows:

$$V_\lambda^0(\sigma) = \Pr[\tau_L < (\tau_R \wedge \tau_\lambda)] \cdot \Pi + \Pr[\tau_R < (\tau_L \wedge \tau_\lambda)] \cdot \mathcal{V}_\lambda^1 - \mathbb{E}[(\tau_L \wedge \tau_R \wedge \tau_\lambda)] \cdot c. \tag{B.2}$$

Note that the survival function of $(\tau_R \wedge \tau_\lambda)$ is $e^{-\int_0^t (\mu\sigma_s+\lambda)ds} = e^{-\mu\Sigma_t - \lambda t}$. By using (A.4) and (A.7), we have

$$\Pr[\tau_L < (\tau_R \wedge \tau_\lambda)] = \int_0^\infty \lambda_L(1-\sigma_t) \cdot e^{-\lambda_L(t-\Sigma_t)-\mu\Sigma_t-\lambda t}\, dt.$$

Likewise, we can derive that

$$\Pr[\tau_R < (\tau_L \wedge \tau_\lambda)] = \int_0^\infty \mu\ \sigma_t \cdot e^{-\lambda_L(t-\Sigma_t)-\mu\Sigma_t-\lambda t}\, dt.$$

Next, observe that the survival function of $(\tau_L \wedge \tau_R \wedge \tau_\lambda)$ is

$$e^{-\lambda_L(t-\Sigma_t)-\mu\Sigma_t-\lambda t} = e^{-(\lambda_L+\lambda)t-(\mu-\lambda_L)\Sigma_t}.$$

Then, from $\mu \geq \lambda_L$ and $\Sigma_t + \hat{\Sigma}_t \geq 0$, we have $\lim_{t\to\infty} t \cdot e^{-\lambda_L(t-\Sigma_t)-\mu\Sigma_t-\lambda t} = 0$. By applying (A.1), we have

$$\mathbb{E}[(\tau_L \wedge \tau_R \wedge \tau_\lambda)] = \int_0^\infty e^{-\lambda_L(t-\Sigma_t)-\mu\Sigma_t-\lambda t}\, dt.$$

By plugging the above equations into (B.2), we obtain (B.1). □

Lemma 6. *Suppose that $x_0 \in \arg\max_{x\in[0,1]} u(x)$ where u is a function defined in (3.2). Let $\sigma^* : \mathbb{R}_+ \to [0,1]$ be $\sigma_t^* = x_0$ for all $t \geq 0$. Then, $\sigma^* \in \arg\max_\sigma V_\lambda^0(\sigma)$.*

Proof of Lemma 6. Let r_t denote $e^{-\lambda_L(t-\Sigma_t)-\mu\Sigma t-\lambda t}$. By taking a derivative, we have

$$\dot{r}_t = -\left\{\lambda_L(1-\sigma_t) + \mu\sigma_t + \lambda\right\} \cdot r_t. \tag{B.3}$$

By Lemma 5, Firm i's problem is

$$\max_{\sigma} \int_0^\infty \left\{\lambda_L(1-\sigma_t)\cdot \Pi + \mu\sigma_t \cdot \mathcal{V}_\lambda^1 - c\right\} \cdot r_t \; dt \tag{B.4}$$

subject to (B.3).

Observe that the Hamiltonian of this optimal control problem is

$$\begin{aligned} H(\sigma_t, r_t, \eta_t) &= \left\{\lambda_L(1-\sigma_t)\cdot \Pi + \mu\sigma_t \cdot \mathcal{V}_\lambda^1 - c\right\} \cdot r_t \\ &\quad - \eta_t \left\{\lambda_L(1-\sigma_t) + \mu\sigma_t + \lambda\right\} \cdot r_t \\ &= \left\{u(\sigma_t) - \eta_t\right\} \cdot \left\{\lambda_L(1-\sigma_t) + \mu\sigma_t + \lambda\right\} \cdot r_t, \end{aligned} \tag{B.5}$$

where η_t is a co-state variable.

To show that σ^* is a solution of (B.4) subject to (B.3) by using the Arrow sufficiency condition (Seierstad and Sydsaeter, 1987, Theorem 3.14), we consider (η^*, r^*) defined as follows: for all $t \geq 0$, $\eta_t^* = u(x_0)$ and $r_t^* = e^{-\{\mu x_0 + \lambda_L(1-x_0)+\lambda\}\cdot t}$.

Then, we need to check following four primitive conditions:

1. Maximum principle: for all $t \geq 0$,

$$\sigma_t^* = x_0 \in \arg\max_{\sigma_t \in [0,1]} H(\sigma_t, r_t^*, \eta_t^*). \tag{B.6}$$

2. Evolution of the co-state variable:

$$\dot{\eta}_t^* = -\frac{\partial H}{\partial r_t} = -\left\{u(\sigma_t^*) - \eta_t^*\right\} \cdot \left\{\lambda_L(1-\sigma_t^*) + \mu\sigma_t^* + \lambda\right\}. \tag{B.7}$$

3. Transversality condition: If r^* is the optimal trajectory, i.e., $r_t^* = e^{-\{\mu x_0 + \lambda_L(1-x_0)+\lambda\}\cdot t}$, $\lim_{t\to\infty} \eta_t^*(r_t^* - r_t) \leq 0$ for all feasible trajectories r_t.

4. $\hat{H}(r_t, \eta_t) = \max_{\sigma_t \in [0,1]} H(\sigma_t, r_t, \eta_t)$ is concave in r_t.

First, by plugging r_t^* and η_t^* into (B.5), we have

$$H(\sigma_t, r_t^*, \eta_t^*) = \left\{u(\sigma_t) - u(x_0)\right\} \cdot \left\{\lambda_L(1-\sigma_t^*) + \mu\sigma_t^* + \lambda\right\} \cdot r_t \tag{B.8}$$

Recall that $x_0 \in \arg\max_{x\in[0,1]} u(x)$. Thus, $H(\sigma_t, r_t^*, \eta_t^*) \leq 0$ for all $\sigma_t \in [0,1]$. In addition, $H(x_0, r_t^*, \eta_t^*) = 0$. Therefore, $x_0 \in \arg\max_{\sigma_t\in[0,1]} H(\sigma_t, r_t, \eta_t)$, i.e., (B.6) holds.

Second, by the definition of η^*, (B.7) holds.

Third, note that for any admissible allocation σ,

$$r_t = e^{-\{\mu\Sigma_t+\lambda_L(t-\Sigma_t)+\lambda t\}} = r_t^* \cdot e^{(\mu-\lambda_L)\cdot(x_0 t-\Sigma_t)}.$$

Then, we have

$$\lim_{t\to\infty} \eta_t^* \cdot (r_t^* - r_t) = \lim_{t\to\infty} u(x_0)\cdot r_t^* \cdot \left(1 - e^{(\mu-\lambda_L)\cdot(x_0 t-\Sigma_t)}\right) = 0.$$

Last, we can see that $\hat{H}$ is linear in r_t, thus, the fourth condition holds. Hence, by the Arrow sufficiency condition, σ^* is the best response to $\hat{\sigma}^*$. □

C Proofs for the Public Information Setting

C.1 Proof Sketches of Theorem 1

A MPE consists of a profile of Markov strategies such that each of the players is best responding to the strategy of their opponent. By using the similar argument as in Lemma 6, we only need to consider Markov deviations to construct the set of Markov Perfect Equilibria.[21]

Given a Markov strategy profile of the firms, we can define U_ω^i as the continuation payoff of Firm i in state ω. Next, we provide some intuition for the proof of Theorem 1 by splitting the problem of the firms in two: On one hand, we solve the problem of the firms before any research progress has been made (and fixing the continuation payoffs). On the other hand, we compute the best responses of the firms after one of them obtains the new technology, and therefore the equilibrium continuation payoffs. Finally, by plugging these continuation payoffs into the problem of the firms at the initial state, we prove the theorem.

[21] Intuitively, when the opponent is using a Markov strategy, the problem of a firm is independent of calendar time. Thus, there exists a best response that is Markov and, therefore, the best Markov response is also a best response. For general treatment on the existence of Markov equilibria in a larger class of continuous-time stochastic games with finite states and actions, see Neyman (2017).

Best Responses under no Research Progress We first consider the case where neither firm has made research progress, i.e., $\omega = \emptyset$. The conventional approach is to solve the problem with backward induction. However, in order to facilitate the analysis in various extensions, we present the problem under the state $\omega = \emptyset$ in a general manner by treating the continuation payoffs $U^i_{\{i\}}$ and $U^i_{\{j\}}$ as exogenous values.

When Firm i and j play $\mathbf{s}(\emptyset) = x$ and $\hat{\mathbf{s}}(\emptyset) = y$, Firm i's expected payoff at the state $\emptyset$ is

$$u_0(x, y) \equiv \frac{x\mu U^i_{\{i\}} + (1-x)\lambda_L \Pi + y\mu U^i_{\{j\}} - c}{x\mu + (1-x)\lambda_L + y\mu + (1-y)\lambda_L}. \tag{C.1}$$

Define $\Delta_y := u_0(1, y) - u_0(0, y)$. The following proposition characterizes the equilibrium allocations at state $\emptyset$ in any MPE.

Proposition 8. *The equilibrium allocations at state $\emptyset$ are characterized as follows:*

(a) when $\Delta_0, \Delta_1 > 0$, both firms do research, i.e., $(\mathbf{s}^A(\emptyset), \mathbf{s}^B(\emptyset)) = (1, 1)$;

(b) when $\Delta_0, \Delta_1 < 0$, both firms develop with the old technology, i.e., $(\mathbf{s}^A(\emptyset), \mathbf{s}^B(\emptyset)) = (0, 0)$;

(c) when $\Delta_0 > 0 > \Delta_1$, there are three possible equilibrium allocations:

- *one firm does research and the other firm develops with the old technology, i.e., $(\mathbf{s}^A(\emptyset), \mathbf{s}^B(\emptyset)) = (1, 0)$ or $(0, 1)$,*
- *both firms allocate $z^* = \Delta_0/(\Delta_0 - \Delta_1)$ amount of resources to research and the remainder to the development with the old technology, i.e., $(\mathbf{s}^A(\emptyset), \mathbf{s}^B(\emptyset)) = (z^*, z^*)$;*

(d) when $\Delta_1 > 0 > \Delta_0$, there are three possible equilibrium allocations:

- *both firms do research, i.e., $(\mathbf{s}^A(\emptyset), \mathbf{s}^B(\emptyset)) = (1, 1)$,*
- *both firms develop with the old technology, i.e., $(\mathbf{s}^A(\emptyset), \mathbf{s}^B(\emptyset)) = (0, 0)$*
- *both firms allocate $z^* = -\Delta_0/(\Delta_1 - \Delta_0)$ amount of resources to research and the remainder to the development with the old technology, i.e., $(\mathbf{s}^A(\emptyset), \mathbf{s}^B(\emptyset)) = (z^*, z^*)$.*

The set of equilibria as a function of Δ_1 and Δ_0 characterized in Proposition 8 is summarized in Figure 7. The proof, in Appendix, boils down to showing that the derivative of u_0 with respect to x shares the same sign with $\lambda_L \cdot \Delta_0 \cdot (1-y) + \mu \cdot \Delta_1 \cdot y$ (Lemma 7). As this sign is independent of x, the best response function exhibits a 'bang-bang' characteristic: the optimal response to the allocation y of the opponent is either 0, 1, or any value in $[0, 1]$.

In scenarios where Δ_0 and Δ_1 share the same sign, the best response is independent of the opponent's resource allocation. Specifically, when both Δ_0 and Δ_1 are positive, it is optimal to assign all resources to research. Conversely, when both Δ_0 and Δ_1 are negative, it is optimal to develop with the old technology.

When Δ_0 and Δ_1 have different signs, the optimal response depends on the resource allocation y of the opponent. When Δ_1 is positive and Δ_0 is negative, the function u_0 satisfies the single-crossing property from Milgrom and Shannon (1994) and the best-response functions of firms are increasing with respect to the opponent's resource allocation y. Thus, we can interpret the firms' allocations as strategic complements. This complementarity explains the two symmetric equilibria at the extreme allocations and the equilibrium with interior allocation.

When Δ_0 is positive and Δ_1 is negative, the negative of u_0 satisfies the single-crossing property and the best response functions are decreasing. In this context, we can interpret the firms' allocations as strategic substitutes. This substitutability explains why we obtain two asymmetric extreme equilibria and one symmetric equilibrium with interior allocations.

Best Responses under Research Progress We now consider the cases where at least one of the firms has made the research progress.

When both firms have made the research progress ($\omega = \{i, j\}$), they will develop with the new technology and their expected payoffs are $U^i_{\{i,j\}} = U^j_{\{i,j\}} = V_C \equiv \frac{\lambda_H \Pi - c}{2\lambda_H}$. Next, suppose that only one of the firms, say Firm i, has made the research progress, i.e., $\omega = \{i\}$. In this case, Firm i develops the product at rate λ_H with the new technology. Then, we can derive the continuation values by applying Proposition 1:

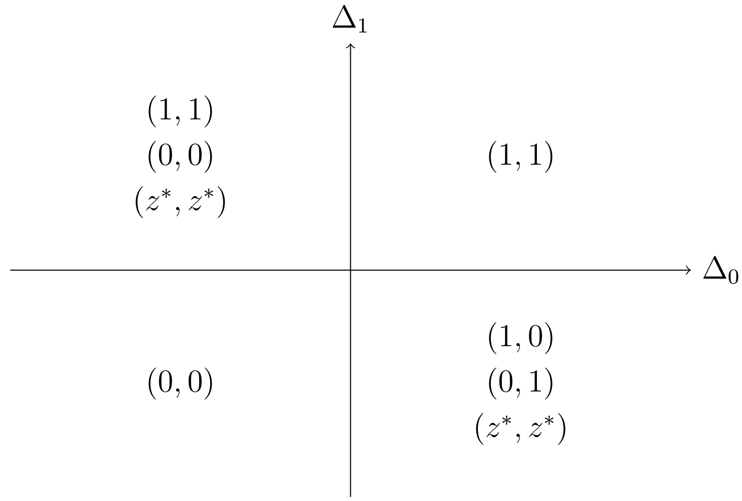

Figure 7: Equilibrium Allocations under No Research Progress

(i) if $\lambda_\star > \lambda_H$, Firm j keeps conducting research:

$$U^i_{\{i\}} = U^j_{\{j\}} = \frac{\lambda_H \Pi + \mu V_C - c}{\mu + \lambda_H} = \frac{\mu + 2\lambda_H}{\mu + \lambda_H} V_C, \qquad U^i_{\{j\}} = U^j_{\{i\}} = \frac{\mu V_C - c}{\mu + \lambda_H}, \tag{C.2}$$

(ii) if $\lambda_\star < \lambda_H$, Firm j develops with the old technology:

$$U^i_{\{i\}} = U^j_{\{j\}} = V_H \equiv \frac{\lambda_H \Pi - c}{\lambda_L + \lambda_H}, \qquad U^i_{\{j\}} = U^j_{\{i\}} = V_L \equiv \frac{\lambda_L \Pi - c}{\lambda_L + \lambda_H}. \tag{C.3}$$

Equilibrium Characterization Now that we have derived the continuation values, we can finalize the proof of Theorem 1 by plugging these values into Proposition 8.

First, when $\lambda_\star > \lambda_H$, in Lemma 8, we show that the following equations hold:

$$\Delta_0 = \frac{\lambda_H \cdot \lambda_\star \cdot (\lambda_L \Pi + c) + \mu \cdot (\lambda_\star - \lambda_H) \cdot c}{2\lambda_H(\lambda_H + \mu)(\lambda_L + \mu)}, \tag{C.4}$$

$$\Delta_1 = \frac{\lambda_L \cdot \{\lambda_H \cdot \lambda_\star \cdot (\mu \Pi + c) + \mu \cdot (\lambda_\star - \lambda_H) \cdot c\}}{2\mu\lambda_H(\lambda_H + \mu)(\lambda_L + \mu)}. \tag{C.5}$$

From the above equations and $\lambda_\star > \lambda_H$, we can see that $\Delta_0, \Delta_1 > 0$. By applying Proposition 8 (a), both firms do research at the state $\emptyset$. Then, when one of the firms, say Firm j, succeeds in research, by Proposition 1 (a), Firm i will keep doing research. Therefore, the unique MPE is for firms to follow the research strategy (Theorem 1 (a)).

Next, when $\lambda_\star < \lambda_H$, in Lemma 9, we show that the following equations hold:

$$\Delta_0 = \frac{(\lambda_L \Pi + c) \cdot (\lambda_\star - \lambda_L)}{2(\lambda_L + \mu)(\lambda_L + \lambda_H)}, \tag{C.6}$$

$$\Delta_1 = \frac{(\mu \Pi + c) \cdot \lambda_L \cdot (\lambda_\star - \lambda_L)}{2\mu(\lambda_L + \mu)(\lambda_L + \lambda_H)}. \tag{C.7}$$

When $\lambda_\star \in (\lambda_L, \lambda_H)$, (C.6) and (C.7) imply that Δ_0 and Δ_1 are positive. Thus, by Proposition 8 (a), both firms do research at the state $\emptyset$. Then, when one of the firms, say Firm j, succeeds in research, by Proposition 1 (b), Firm i will switch to develop with the old technology. Therefore, the unique MPE is for firms to follow the fall-back strategy (Theorem 1 (b)).

Last, when $\lambda_\star < \lambda_L$, we can see that Δ_0 and Δ_1 are negative. Then, by Proposition 8 (b), both firms develop with the old technology at the state $\emptyset$. Additionally, even if a firm happens to succeed in research, the other firm will keep developing with the old technology due to Proposition 1 (b). Thus, the unique MPE is for firms to employ the direct-development strategy (Theorem 1 (c)).

C.2 Lemmas

Lemma 7. *The following equation holds:*

$$\frac{\partial u_0}{\partial x} = \mathcal{C}(x, y) \cdot \{\lambda_L \cdot \Delta_0 \cdot (1 - y) + \mu \cdot \Delta_1 \cdot y\}, \tag{C.8}$$

where

$$\mathcal{C}(x, y) = \frac{2(\lambda_L + \mu)}{\{\mu x + \lambda_L(1 - x) + \mu y + \lambda_L(1 - y)\}^2} > 0.$$

Proof of Lemma 7. Observe that

$$\Delta_0 = \frac{\mu U^i_{\{i\}} - c}{\mu + \lambda_L} - \frac{\lambda_L \Pi - c}{2\lambda_L},$$

$$\Delta_1 = \frac{\mu(U^i_{\{i\}} + U^i_{\{j\}}) - c}{2\mu} - \frac{\lambda_L \Pi + \mu U^i_{\{j\}} - c}{\lambda_L + \mu}.$$

Thus, we have

$$2(\lambda_L+\mu)\lambda_L\cdot\Delta_0 = 2\lambda_L\mu U^i_{\{i\}} - \lambda_L(\lambda_L+\mu)\Pi + (\mu-\lambda_L)c, \tag{C.9}$$

$$2(\lambda_L+\mu)\mu\cdot\Delta_1 = (\lambda_L+\mu)\mu U^i_{\{i\}} - (\mu-\lambda_L)\mu U^i_{\{j\}} - 2\lambda_L\mu\Pi + (\mu-\lambda_L)c, \tag{C.10}$$

and

$$2(\lambda_L+\mu)\mu\cdot\Delta_1 - 2(\lambda_L+\mu)\lambda_L\cdot\Delta_0 = (\mu-\lambda_L)\left(\mu U^i_{\{i\}} - \mu U^i_{\{j\}} - \lambda_L\Pi\right).$$

Also note that

$$\frac{\partial u_0}{\partial x} = \frac{NUM_0}{\{\mu x + \lambda_L(1-x) + \mu y + \lambda_L(1-y)\}^2}$$

where

$$\begin{aligned} NUM_0 =& (\mu U^i_{\{i\}} - \lambda_L\Pi)\cdot(\mu x + \lambda_L(1-x) + \mu y + \lambda_L(1-y)) \\ &- (x\mu U^i_{\{i\}} + (1-x)\lambda_L\Pi + y\mu U^i_{\{j\}} - c)\cdot(\mu-\lambda_L). \end{aligned}$$

With some algebra, we can show that

$$\begin{aligned} NUM_0 =& \left\{2\lambda_L\mu U^i_{\{i\}} - \lambda_L(\lambda_L+\mu)\Pi + (\mu-\lambda_L)c\right\} \\ &+ (\mu-\lambda_L)(\mu U^i_{\{i\}} - \mu U^i_{\{j\}} - \lambda_L\Pi)y \\ =& 2(\lambda_L+\mu)\lambda_L\cdot\Delta_0 + (2(\lambda_L+\mu)\mu\cdot\Delta_1 - 2(\lambda_L+\mu)\lambda_L\cdot\Delta_0)\cdot y. \end{aligned}$$

By plugging this in, we can show that (C.8) holds. □

Lemma 8. *When $\lambda_\star > \lambda_H$, the equations (C.4) and (C.5) hold.*

Proof of Lemma 8. By plugging (C.2) into (C.9),

$$
\begin{aligned}
2(\lambda_L+\mu)\lambda_L\cdot\Delta_0 =& 2\lambda_L\mu\cdot U^i_{\{i\}} - \lambda_L(\lambda_L+\mu)\cdot\Pi + (\mu-\lambda_L)\cdot c \\
=& \frac{\lambda_L\mu}{\lambda_H}\cdot\frac{2\lambda_H+\mu}{\lambda_H+\mu}\cdot(\lambda_H\Pi - c) - \lambda_L(\lambda_L+\mu)\cdot\Pi + (\mu-\lambda_L)c \\
=& \frac{\lambda_H\mu - \lambda_L\lambda_H - \mu\lambda_L}{\lambda_H+\mu}\cdot(\lambda_L\Pi + c) + \frac{\mu\left(\lambda_H\mu - \lambda_L\lambda_H - \mu\lambda_L - \lambda_L\lambda_H\right)}{\lambda_H(\lambda_H+\mu)}\cdot c.
\end{aligned}
$$

By using (3.4), we have

$$
2(\lambda_L+\mu)\lambda_L\cdot\Delta_0 = \frac{\lambda_L}{\lambda_H(\lambda_H+\mu)}\cdot\left[\lambda_H\cdot\lambda_\star\cdot(\lambda_L\Pi+c) + \mu\cdot(\lambda_\star-\lambda_H)\cdot c\right].
$$

Then, by dividing both sides by $2(\lambda_L+\mu)\lambda_L$, we can show that (C.4) holds.

Next, by plugging (C.2) into (C.10),

$$
\begin{aligned}
2(\lambda_L+\mu)\mu\cdot\Delta_1 =& (\lambda_L+\mu)\mu\cdot U^i_{\{i\}} - (\mu-\lambda_L)\mu\cdot U^i_{\{j\}} - 2\lambda_L\mu\cdot\Pi + (\mu-\lambda_L)\cdot c \\
=& \mu\frac{2(\lambda_H\lambda_L + \lambda_H\mu + \lambda_L\mu)}{\mu+\lambda_H}\cdot\frac{\lambda_H\Pi - c}{2\lambda_H} - 2\lambda_L\mu\cdot\Pi + (\mu-\lambda_L)\frac{2\mu+\lambda_H}{\mu+\lambda_H}\cdot c \\
=& \frac{\lambda_H\mu - \lambda_L\lambda_H - \mu\lambda_L}{\lambda_H+\mu}\cdot(\mu\Pi + c) + \frac{\mu(\lambda_H\mu - \lambda_L\lambda_H - \mu\lambda_L - \lambda_L\lambda_H)}{\lambda_H(\lambda_H+\mu)}\cdot c.
\end{aligned}
$$

By using (3.4), we have

$$
2(\lambda_L+\mu)\mu\cdot\Delta_1 = \frac{\lambda_L}{\lambda_H(\lambda_H+\mu)}\cdot\left[\lambda_H\cdot\lambda_\star\cdot(\mu\Pi+c) + \mu\cdot(\lambda_\star-\lambda_H)\cdot c\right]
$$

Then, by dividing both sides by $2(\lambda_L+\mu)\mu$, we can show that (C.5) holds. □

Lemma 9. *When $\lambda_\star < \lambda_H$, the equations* (C.6) *and* (C.7) *hold.*

Proof of Lemma 9. By plugging (C.3) into (C.9),

$$
\begin{aligned}
2(\lambda_L+\mu)\lambda_L\cdot\Delta_0 =& 2\lambda_L\mu\cdot U^i_{\{i\}} - \lambda_L(\lambda_L+\mu)\cdot\Pi + (\mu-\lambda_L)\cdot c \\
=& \frac{2\lambda_L\mu}{\lambda_L+\lambda_H}(\lambda_H\Pi - c) - \lambda_L(\lambda_L+\mu)\cdot\Pi + (\mu-\lambda_L)c \\
=& \frac{\lambda_L\Pi + c}{\lambda_L+\lambda_H}\cdot\left\{\lambda_H\mu - \lambda_L\lambda_H - \mu\lambda_L - \lambda_L^2\right\}
\end{aligned}
$$

By using (3.4), we have

$$2(\lambda_L + \mu)\lambda_L \cdot \Delta_0 = \frac{(\lambda_L \Pi + c) \cdot \lambda_L \cdot (\lambda_\star - \lambda_L)}{\lambda_L + \lambda_H}.$$

Then, by dividing both sides by $2(\lambda_L + \mu)\lambda_L$, we can show that (C.6) holds.

Next, by plugging (C.3) into (C.10),

$$\begin{aligned}
2(\lambda_L + \mu)\mu \cdot \Delta_1 =& (\lambda_L + \mu)\mu \cdot U^i_{\{i\}} - (\mu - \lambda_L)\mu \cdot U^i_{\{j\}} - 2\lambda_L\mu \cdot \Pi + (\mu - \lambda_L) \cdot c \\
=& (\lambda_L + \mu)\mu \cdot \frac{\lambda_H \Pi - c}{\lambda_L + \lambda_H} - (\mu - \lambda_L)\mu \cdot \frac{\lambda_L \Pi - c}{\lambda_L + \lambda_H} - 2\lambda_L\mu \cdot \Pi + (\mu - \lambda_L) \cdot c \\
=& \frac{\mu\Pi + c}{\lambda_L + \lambda_H} \cdot \left\{ \lambda_H \mu - \lambda_L \lambda_H - \mu \lambda_L - \lambda_L^2 \right\}.
\end{aligned}$$

By using (3.4), we have

$$2(\lambda_L + \mu)\mu \cdot \Delta_1 = \frac{(\mu\Pi + c) \cdot \lambda_L \cdot (\lambda_\star - \lambda_L)}{\lambda_L + \lambda_H}.$$

Then, by dividing both sides by $2(\lambda_L + \mu)\mu$, we can show that (C.5) holds. □

C.3 Proof of Proposition 8

Proof of Proposition 8. (a) When $\Delta_0, \Delta_1 > 0$, from (C.8), $\frac{\partial u_0}{\partial x} > 0$ for all $y \in [0, 1]$, i.e., $x = 1$ is optimal. Thus, both firms play $\mathbf{s}(\emptyset) = 1$ in any MPE.

(b) When $\Delta_0, \Delta_1 < 0$, from (C.8), $\frac{\partial u_0}{\partial x} < 0$ for all $y \in [0, 1]$, i.e., $x = 0$ is optimal. Thus, both firms play $\mathbf{s}(\emptyset) = 0$ in any MPE.

(c) From $\Delta_0 > 0$ and (C.8), we have $\frac{\partial u_0}{\partial x}|_{y=0} > 0$, i.e., $x = 1$ is the best response for $y = 0$. In addition, from $0 > \Delta_1$ and (C.8), we have $\frac{\partial u_0}{\partial x}|_{y=1} < 0$, i.e., $x = 0$ is the best response for $y = 1$. Therefore, $(1, 0)$ and $(0, 1)$ can be supported equilibrium allocations at $\omega = \emptyset$.

Next, note that $z^* \in (0, 1)$ and $\frac{\partial u_0}{\partial x}|_{y=z^*} = 0$, i.e., any $x \in [0, 1]$ is the best response for $y = z^*$. Thus, (z^*, z^*) can be supported as an equilibrium allocation.

Last, consider any $\tilde{y} \in (0,1)$ with $\tilde{y} \neq z^*$. Then, $\frac{\partial u_0}{\partial x}|_{y=\tilde{y}} \neq 0$, i.e., the best response is $x = 1$ or $x = 0$. Recall that the best response of $x = 1$ ($x = 0$) is $y = 0$ ($y = 1$), thus, $y = \tilde{y}$ cannot be a part of an equilibrium allocation.

(d) From $\Delta_0 < 0$ and (C.8), we have $\frac{\partial u_0}{\partial x}|_{y=0} < 0$, i.e., $x = 0$ is the best response for $y = 0$. Thus, $(0,0)$ can be supported as an equilibrium allocation.

Similarly, from $0 < \Delta_1$ and (C.8), we have $\frac{\partial u_0}{\partial x}|_{y=1} > 0$, i.e., $x = 1$ is the best response for $y = 1$. Therefore, $(1,1)$ can also be supported as an equilibrium allocation.

Next, note that $z^* \in (0,1)$ and $\frac{\partial u_0}{\partial x}|_{y=z^*} = 0$, i.e., any $x \in [0,1]$ is the best response for $y = z^*$. Thus, (z^*, z^*) can be supported as an equilibrium allocation.

Last, by using the similar argument as in the previous case, $\tilde{y} \in (0,1)$ with $\tilde{y} \neq z^*$ cannot be a part of an equilibrium allocation.

□

D Proofs for the Private Information Setting

D.1 Preliminaries

In this section, we derive, the probability $\mathbf{p}_{\boldsymbol{\sigma}}$ of having the new technology under an allocation policy $\boldsymbol{\sigma}$ given no development. Using this probability, we then compute the development rate $\mathbf{h}_{\boldsymbol{\sigma}}$. To achieve this, we fix the allocation policy $\boldsymbol{\sigma} \in \mathcal{S}$ and define two arrival times: τ_M represents the time at which the first breakthrough occurs. Meanwhile, τ_D is the time of successful development. Observe that, by definition, τ_M must be weakly lower than τ_D. This inequality is strict if and only if the first breakthrough is a research breakthrough. Therefore, we use $(\tau_M = \tau_D)$ to indicate the event that the first breakthrough was the development using the old technology and $(\tau_M < \tau_D)$ the event that the first breakthrough was the new technology discovery.

Probability of new technology given no development $\mathbf{p}_{\boldsymbol{\sigma}}$

Let $\Sigma_t \equiv \int_0^t \boldsymbol{\sigma}(s)\, ds$ represent the cumulative research. We begin by observing that the probability that there is no breakthrough by time t is given by

$$S_{\boldsymbol{\sigma}}^M(t) := \Pr[\tau_M > t] = e^{-\lambda_L(t-\Sigma_t)-\mu\Sigma_t} \tag{D.1}$$

If the new technology is obtained at time s, the probability that development is not attained by time $t > s$ is given by:

$$\Pr[\tau_D > t \mid \tau_D > \tau_M = s] = e^{-\lambda_H(t-s)} \tag{D.2}$$

If instead there is a breakthrough at some $s < t$, with some probability the breakthrough is a research breakthrough. This probability is given by:

$$\Pr(\tau_D > \tau_M \mid \tau_M = s) = \frac{\boldsymbol{\sigma}(s)\mu}{\boldsymbol{\sigma}(s)\mu + (1-\boldsymbol{\sigma}(s))\lambda_L} \tag{D.3}$$

This result is standard in competing risks models. Thus, we can derive the probability that a research breakthrough is obtained before t, but no development was obtained by time t.

$$L_{\boldsymbol{\sigma}}(t) := \Pr(\tau_M < t < \tau_D) = \int_0^t f_M(s) \cdot \Pr(\tau_D > t \mid \tau_M = s)\, ds \tag{D.4}$$

Where f_M is the marginal density function of τ_M. Observe that, for any $s < t$,

$$\begin{aligned}
f_M(s) \cdot \Pr(\tau_D > t \mid \tau_M = s) &= f_M(s) \cdot \Pr(\tau_D > \tau_M \mid \tau_M = s) \cdot \Pr[\tau_D > t \mid \tau_D > \tau_M = s] \\
&= f_M(s) \cdot \Pr(\tau_D > \tau_M \mid \tau_M = s) \cdot e^{-\lambda_H(t-s)} \\
&= [\boldsymbol{\sigma}(s)\mu + (1-\boldsymbol{\sigma}(s))\lambda_L]\, S_{\boldsymbol{\sigma}}^M(s) \cdot \frac{\boldsymbol{\sigma}(s)\mu}{\boldsymbol{\sigma}(s)\mu + (1-\boldsymbol{\sigma}(s))\lambda_L} \cdot e^{-\lambda_H(t-s)} \\
&= \mu\, \boldsymbol{\sigma}(s) e^{-\lambda_L(s-\Sigma_s)-\mu\Sigma_s} e^{-\lambda_H(t-s)}
\end{aligned}$$

Where the first equality uses the law of total probability. For the second equality, we invoke Eq. D.2. The third equality holds by expressing the density function f_M as the product of the hazard rate $[\boldsymbol{\sigma}\,\mu + (1-\boldsymbol{\sigma})\lambda_L]$ the survival function $S^M_{\boldsymbol{\sigma}}$. We also invoke Eq. D.3. The last equality applies Eq. D.1. Using the previous derivation in Eq. D.4, we obtain:

$$L_{\boldsymbol{\sigma}}(t) = \int_0^t \mu\,\boldsymbol{\sigma}(s) e^{-\lambda_L(s-\Sigma_s)-\mu\Sigma_s} e^{-\lambda_H(t-s)}\; ds \tag{D.5}$$

The probability $S^D_{\boldsymbol{\sigma}}$ that no development was made by time t can be written as:

$$\begin{aligned} S^D_{\boldsymbol{\sigma}}(t) := \Pr(\tau_D > t) &= \Pr(\tau_M > t) + \Pr(\tau_M < t < \tau_D) \\ &= S^M_{\boldsymbol{\sigma}}(t) + L_{\boldsymbol{\sigma}}(t) \end{aligned} \tag{D.6}$$

Finally, we obtain an expression for our conditional probability $\mathbf{p}_{\boldsymbol{\sigma}}$ in terms of $L_{\boldsymbol{\sigma}}$ and $S^M_{\boldsymbol{\sigma}}$:

$$\mathbf{p}_{\boldsymbol{\sigma}}(t) = \Pr(\tau_M < t \mid \tau_D > t) = \frac{\Pr(\tau_M < t < \tau_D)}{S^D_{\boldsymbol{\sigma}}(t)} = \frac{L_{\boldsymbol{\sigma}}(t)}{S^M_{\boldsymbol{\sigma}}(t) + L_{\boldsymbol{\sigma}}(t)} \tag{D.7}$$

Proof of Proposition 2. From (D.7), we can derive that

$$\frac{\mathbf{p}_{\boldsymbol{\sigma}}(t)}{1-\mathbf{p}_{\boldsymbol{\sigma}}(t)} = \frac{L_{\boldsymbol{\sigma}}(t)}{S^M_{\boldsymbol{\sigma}}(t)}$$

By differentiating this equation side-by-side, we have

$$\frac{\dot{\mathbf{p}}_{\boldsymbol{\sigma}}(t)}{(1-\mathbf{p}_{\boldsymbol{\sigma}}(t))^2} = \frac{L_{\boldsymbol{\sigma}}(t)}{S^M_{\boldsymbol{\sigma}}(t)}\left[\frac{L'_{\boldsymbol{\sigma}}(t)}{L_{\boldsymbol{\sigma}}(t)} - \frac{S^{M\prime}_{\boldsymbol{\sigma}}(t)}{S^M_{\boldsymbol{\sigma}}(t)}\right] = \frac{\mathbf{p}_{\boldsymbol{\sigma}}(t)}{1-\mathbf{p}_{\boldsymbol{\sigma}}(t)}\left[\frac{L'_{\boldsymbol{\sigma}}(t)}{L_{\boldsymbol{\sigma}}(t)} - \frac{S^{M\prime}_{\boldsymbol{\sigma}}(t)}{S^M_{\boldsymbol{\sigma}}(t)}\right] \tag{D.8}$$

From deriving (D.1) and (D.5), we obtain that

$$S^{M\prime}_{\boldsymbol{\sigma}}(t) = -\left\{\lambda_L(1-\boldsymbol{\sigma}(t)) + \mu\,\boldsymbol{\sigma}(t)\right\}\cdot S^M_{\boldsymbol{\sigma}}(t), \tag{D.9}$$

$$L'_{\boldsymbol{\sigma}}(t) = \mu\cdot\boldsymbol{\sigma}(t)\cdot S^M_{\boldsymbol{\sigma}}(t) - \lambda_H\cdot L_{\boldsymbol{\sigma}}(t) \tag{D.10}$$

Using these expressions in (D.8) and multiplying side by side by $(1-\mathbf{p}_{\boldsymbol{\sigma}}(t))^2$, we obtain the desired result. □

D.1.1 Development rates

For any continuous random variable, the hazard rate can be expressed as the negative of the log of the survival function. The development rate of a firm that follows policy $\boldsymbol{\sigma} \in \mathcal{S}$ is the hazard rate associated with the random variable τ_D. Therefore, it can be derived as follows:

$$\begin{aligned}\mathbf{h}_{\boldsymbol{\sigma}}(t) = -\frac{\partial \log\left[S^D_{\boldsymbol{\sigma}}(t)\right]}{\partial t} = -\frac{S^{D\prime}_{\boldsymbol{\sigma}}(t)}{S^D_{\boldsymbol{\sigma}}(t)} =& \frac{\lambda_L(1-\boldsymbol{\sigma}(t)) \cdot S^M_{\boldsymbol{\sigma}}(t) + \lambda_H \cdot L_{\boldsymbol{\sigma}}(t)}{S^M_{\boldsymbol{\sigma}}(t) + L_{\boldsymbol{\sigma}}(t)} \\ =& \lambda_L(1-\boldsymbol{\sigma}(t)) \cdot (1-\mathbf{p}_{\boldsymbol{\sigma}}(t)) + \lambda_H \cdot \mathbf{p}_{\boldsymbol{\sigma}}(t).\end{aligned} \tag{D.11}$$

D.1.2 Steady State

Proof of Lemma 2. From $\xi(p_\star, \sigma_\star) = \lambda_\star$, $\delta(p_\star, \sigma_\star) = 0$ and $p_\star < 1$, we have

$$\lambda_\star = p_\star \lambda_H + (1-p_\star)(1-\sigma_\star)\lambda_L, \tag{D.12}$$

$$0 = \mu\sigma_\star - \{\lambda_H - (1-\sigma_\star)\lambda_L\} p_\star. \tag{D.13}$$

By rearranging (D.13), we have

$$\begin{aligned}\mu\sigma_\star =& \{\lambda_H - (1-\sigma_\star)\lambda_L\} p_\star \\ =& \lambda_H p_\star + (1-\sigma_\star)\lambda_L(1-p_\star) - \lambda_L(1-\sigma_\star) = \lambda_\star - \lambda_L(1-\sigma_\star).\end{aligned}$$

By solving this, we can derive (5.6).

Next, from (D.13) and (5.6), we have

$$p_\star = \frac{\mu\sigma_\star}{\lambda_H - (1-\sigma_\star)\lambda_L} = \frac{\mu(\lambda_\star - \lambda_L)}{(\mu-\lambda_L)\lambda_H - (\mu-\lambda_\star)\lambda_L}.$$

Note that $\lambda_L\lambda_\star = (\mu-\lambda_L)\lambda_H - \mu\lambda_L$. By plugging this into the above equation, we have the first equality of (5.5). Observe that

$$1 - p_\star = \frac{2\lambda_L\lambda_\star - \mu\lambda_\star + \mu\lambda_L}{2\lambda_L\lambda_\star} = \frac{\lambda_L(\mu+\lambda_\star) - (\mu-\lambda_L)\lambda_\star}{2\lambda_L\lambda_\star} = \frac{(\mu-\lambda_L)(\lambda_H-\lambda_L)}{2\lambda_L\lambda_\star},$$

which confirms the second equality of (5.5).

From $\lambda_L < \lambda_\star < \min\{\mu, \lambda_H\}$, we can see that $p_\star, \sigma_\star \in (0,1)$. □

D.2 Recursive Formulation

The opponent's allocation policy is only payoff-relevant for a firm through the distribution of development times. Thus, in this section, we focus on characterizing the continuation payoffs of firms fixing the development rate function $\mathbf{h}$ of the opponent.

Lemma 10. *Let $V_1(t;\mathbf{h})$ be the continuation payoff of a firm at time t when the firm has the new technology, neither firm had succeeded in development by time t, and the opponent employs an allocation policy with development rate $\mathbf{h}$. $V_1(t;\mathbf{h})$ can be written as follows:*

$$V_1(t;\mathbf{h}) = (\lambda_H \Pi - c) \cdot \int_t^\infty e^{-\int_t^s (\lambda_H + \mathbf{h}(u))du} ds. \tag{D.14}$$

In addition, the following differential equation holds:

$$0 = V_1'(t;\mathbf{h}) + (\lambda_H \Pi - c) - (\lambda_H + \mathbf{h}(t)) \cdot V_1(t;\mathbf{h}). \tag{HJB$_1$}$$

Proof. The proof of this lemma is relegated to the online appendix. □

Lemma 11. *Let v_0 be the continuation payoff at time t of a firm that doesn't have the new technology and employs allocation policy $\boldsymbol{\sigma} \in \mathcal{S}$ when the opponent has a development rate $\mathbf{h} \in \mathcal{H}$. Then,*

$$v_0(t;\boldsymbol{\sigma},\mathbf{h}) = \int_t^\infty e^{-\int_t^s \mathbf{h}(u)\ du} \cdot \frac{S^M_{\boldsymbol{\sigma}}(s)}{S^M_{\boldsymbol{\sigma}}(t)} \cdot [\lambda_L(1-\boldsymbol{\sigma}(s)) \cdot \Pi + \mu\, \boldsymbol{\sigma}(s) \cdot V_1(s;\mathbf{h}) - c]\ ds. \tag{D.15}$$

In addition, the following differential equation holds:

$$\begin{aligned} 0 = V_0'(t;\boldsymbol{\sigma},\mathbf{h}) + \lambda_L(1-\boldsymbol{\sigma}(t)) \cdot \Pi + \mu\boldsymbol{\sigma}(t) \cdot V_1(t;\mathbf{h}) - c \\ - \{\lambda_L(1-\boldsymbol{\sigma}(t)) + \mu\boldsymbol{\sigma}(t) + \mathbf{h}(t)\} \cdot V_0(t;\sigma,\hat{\sigma}). \end{aligned} \tag{HJB$_0$}$$

Proof. The proof of this lemma is relegated to the online appendix. □

Observation. Let $\mathbf{h}, \hat{\mathbf{h}} \in \mathcal{H}$ be two development functions such that $\mathbf{h}(s) = \hat{\mathbf{h}}(s)$ for all $s > t$. Then $V_0(s; \mathbf{h}) = V_0(s; \hat{\mathbf{h}})$ and $V_1(s; \mathbf{h}) = V_1(s; \hat{\mathbf{h}})$.

D.3 Verification

In this subsection, we prove the verification result (Proposition 3). To prove the verification result, it is useful to first introduce two convergence results.

Lemma 12. *For any $\boldsymbol{\sigma} \in \mathcal{S}$, the following holds:*

$$\lim_{t\to\infty} S^D_{\boldsymbol{\sigma}}(t) \cdot V_1(t; \mathbf{h}_{\boldsymbol{\sigma}}) = 0.$$

Proof. Let $\Sigma_t := \int_0^t \boldsymbol{\sigma}(s)\ ds$. From $\lambda_H > \lambda_L$ and $\mu > \lambda_L$, we have

$$e^{-\mu t} \le S^M_{\boldsymbol{\sigma}}(t) = e^{-\lambda_L(t-\Sigma_t)-\mu\Sigma_t} \le e^{-\lambda_L t}, \tag{D.16}$$

$$\begin{aligned} 0 \le L_{\boldsymbol{\sigma}}(t) = & \int_0^t \mu\, \boldsymbol{\sigma}(s) \cdot S^M_{\boldsymbol{\sigma}}(s) \cdot e^{-\lambda_H(t-s)}\ ds \\ & < e^{-(\lambda_L+\lambda_H)t} \cdot \int_0^t \mu \cdot e^{\lambda_H s}\ ds < \frac{\mu}{\lambda_H} e^{-\lambda_L t}. \end{aligned} \tag{D.17}$$

Note that the left inequality of (D.16) binds when $\Sigma_t = t$, and the left inequality of (D.17) binds when $\Sigma_t = 0$. By (D.6), we have

$$e^{-\mu t} < S^D_{\boldsymbol{\sigma}}(t) = S^M_{\boldsymbol{\sigma}}(t) + L_{\boldsymbol{\sigma}}(t) < e^{-\lambda_L t} \cdot \left(\frac{\mu+\lambda_H}{\lambda_H}\right). \tag{D.18}$$

From (D.14), we have

$$S^D_{\boldsymbol{\sigma}}(t) \cdot V_1(t; \boldsymbol{\sigma}) = (\lambda_H \Pi - c) \cdot \int_t^\infty e^{-\lambda_H(s-t)} \cdot S^D_{\boldsymbol{\sigma}}(s)\ ds.$$

By applying (D.18) and since $\lambda_H \Pi > \lambda_L \Pi > c$, we have

$$\begin{aligned} (\lambda_H \Pi - c) \cdot \int_t^\infty e^{-\lambda_H(s-t)} \cdot S^D_{\boldsymbol{\sigma}}(s)\ ds \ > \ & (\lambda_H \Pi - c) \cdot \int_t^\infty e^{-\lambda_H(s-t)} \cdot e^{-\mu s}\ ds \\ = & \frac{\lambda_H}{\mu+\lambda_H}\left(\Pi - \frac{c}{\lambda_H}\right) \cdot e^{-\mu t} \end{aligned}$$

and

$$
\begin{aligned}
(\lambda_H \Pi - c) \cdot \int_t^\infty e^{-\lambda_H(s-t)} \cdot S^D_{\boldsymbol{\sigma}}(s)\ ds \ &< \ (\lambda_H \Pi - c) \cdot \int_t^\infty e^{-\lambda_H(s-t)} \cdot \frac{\mu + \lambda_H}{\lambda_H} e^{-\lambda_L s}\ ds \\
&= \frac{\mu + \lambda_H}{\lambda_L + \lambda_H} \left(\Pi - \frac{c}{\lambda_H} \right) \cdot e^{-\lambda_L t}.
\end{aligned}
$$

Therefore, we have that

$$
\frac{\lambda_H}{\mu + \lambda_H} \left(\Pi - \frac{c}{\lambda_H} \right) \cdot e^{-\mu t} < S^D_{\boldsymbol{\sigma}}(t) \cdot V_1(t; \boldsymbol{\sigma}) < \frac{\mu + \lambda_H}{\lambda_L + \lambda_H} \left(\Pi - \frac{c}{\lambda_H} \right) \cdot e^{-\lambda_L t}, \tag{D.19}
$$

Since the lower bound and the upper bound converge to 0 as t goes to infinity, we obtain the desired result. □

Lemma 13. *For any $\boldsymbol{\sigma}, \hat{\boldsymbol{\sigma}} \in \mathcal{S}$,*

$$
\lim_{t \to \infty} v_0(t; \boldsymbol{\sigma}, \mathbf{h}_{\hat{\boldsymbol{\sigma}}}) \cdot S^M_{\boldsymbol{\sigma}}(t) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(t) = 0. \tag{D.20}
$$

Proof. Note that for any time $s \in \mathbb{R}_+$, $-c < \lambda_L(1 - \boldsymbol{\sigma}(s))\Pi - c < \lambda_L \Pi$. Since $\lambda_L \Pi > c$, we have $|\lambda_L(1 - \boldsymbol{\sigma}(s))\Pi - c| < \lambda_L \Pi$.

From (D.15), we have

$$
\begin{aligned}
\left| v_0(t; \boldsymbol{\sigma}, \mathbf{h}_{\hat{\boldsymbol{\sigma}}}) \cdot S^M_{\boldsymbol{\sigma}}(t) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(t) \right| < \ & \lambda_L \Pi \cdot \int_t^\infty S^M_{\boldsymbol{\sigma}}(s) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(s)\ ds \\
& + \mu \cdot \int_t^\infty V_1(s; \hat{\boldsymbol{\sigma}}) \cdot S^M_{\boldsymbol{\sigma}}(s) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(s)\ ds.
\end{aligned}
$$

Observe that from (D.16) and (D.18), we have

$$
\int_t^\infty S^M_{\boldsymbol{\sigma}}(s) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(s)\ ds < \frac{\mu + \lambda_H}{\lambda_H} \cdot \int_t^\infty e^{-2\lambda_L s} ds = \frac{\mu + \lambda_H}{2\lambda_L \lambda_H} \cdot e^{-2\lambda_L t}.
$$

In addition, from (D.19) and (D.18), we have

$$\begin{aligned}\int_t^\infty V_1(s;\mathbf{h}_{\hat{\boldsymbol\sigma}})\cdot S^M_{\boldsymbol\sigma}(s)\cdot S^D_{\hat{\boldsymbol\sigma}}(s)\;ds <&\frac{(\mu+\lambda_H)^2}{\lambda_H(\lambda_L+\lambda_H)}\cdot\left(\Pi-\frac{c}{\lambda_H}\right)\cdot\int_t^\infty e^{-2\lambda_L s}ds\\ =&\frac{(\mu+\lambda_H)^2}{2\lambda_L\lambda_H(\lambda_L+\lambda_H)}\cdot\left(\Pi-\frac{c}{\lambda_H}\right)\cdot e^{-2\lambda_L t}.\end{aligned}$$

Then, we have

$$\left|v_0(t;\boldsymbol\sigma,\mathbf{h}_{\hat\sigma})\cdot S^M_{\boldsymbol\sigma}(t)\cdot S^D_{\hat{\boldsymbol\sigma}}(t)\right| < \frac{\mu+\lambda_H}{2\lambda_L\lambda_H}\left[\lambda_L\Pi+\frac{\mu(\mu+\lambda_H)}{\lambda_L+\lambda_H}\left(\Pi-\frac{c}{\lambda_H}\right)\right]\cdot e^{-2\lambda_L t}.$$

Since the right-hand side of the above inequality converges to 0 as $t\to\infty$, (D.20) holds. □

D.3.1 Proof of Proposition 3

In this proof, we fix the policy of the opponent at $\hat{\boldsymbol\sigma}$. To save on notation, we will drop the dependency of the value and survival functions on $\hat{\boldsymbol\sigma}$ and the opponent's development rate $\mathbf{h}_{\hat{\boldsymbol\sigma}}$. Specifically, we will abuse notation and use $V_1(t)\equiv V_1(t;\mathbf{h}_{\hat{\boldsymbol\sigma}})$, $v_0(t;\boldsymbol\sigma)\equiv v_0(t;\boldsymbol\sigma,\mathbf{h}_{\hat{\boldsymbol\sigma}})$, $\hat S(t)\equiv S^D_{\hat{\boldsymbol\sigma}}(t)$.

Proof of Proposition 3. **(** $\Longleftarrow$ **)**

From $\boldsymbol\sigma^*$, we have that for all $\boldsymbol\sigma\in\mathcal{S}$ and $t\in R_+$

$$(\boldsymbol\sigma^*(t)-\boldsymbol\sigma(t))\cdot[\mu\cdot(V_1(t)-v_0(t;\boldsymbol\sigma^*))-\lambda_L\cdot(\Pi-v_0(t;\boldsymbol\sigma^*))]\geq 0 \tag{D.21}$$

Suppose that $v_0(t;\boldsymbol\sigma^*)>0$. From ($\mathrm{HJB}_0$), we have

$$\begin{aligned}0=&v_0'(t;\boldsymbol\sigma^*)-c-\mathbf{h}_{\hat{\boldsymbol\sigma}}(t)\cdot v_0(t;\boldsymbol\sigma^*)+\lambda_L\cdot(\Pi-v_0(t;\boldsymbol\sigma^*))\\ &+\boldsymbol\sigma^*(t)\cdot\left[\mu\cdot(V_1(t)-v_0(t;\boldsymbol\sigma^*))-\lambda_L\cdot(\Pi-v_0(t;\boldsymbol\sigma^*))\right].\end{aligned}$$

Then, (D.21) implies that, for any $\boldsymbol\sigma\in\mathcal{S}$ and $t\geq 0$,

$$\left\{h^D_{\hat{\boldsymbol\sigma}}(t)+h^M_{\boldsymbol\sigma}(t)\right\}\cdot v_0(t;\boldsymbol\sigma^*)-v_0'(t;\boldsymbol\sigma^*)\geq\lambda_L(1-\boldsymbol\sigma(t))\cdot\Pi+\mu\,\boldsymbol\sigma(t)\cdot V_1(t)-c.$$

Multiplying side-by-side by $S^M_{\boldsymbol{\sigma}}(t) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(t)$, we have

$$-\frac{d}{dt}\left[v_0(t;\boldsymbol{\sigma}^*) \cdot S^M_{\boldsymbol{\sigma}}(t) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(t)\right] \geq [\lambda_L(1-\boldsymbol{\sigma}(t)) \cdot \Pi + \mu\, \boldsymbol{\sigma}(t) \cdot V_1(t) - c] \cdot S^M_{\boldsymbol{\sigma}}(t) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(t)$$

for all $t \geq 0$. Integrating this inequality from 0 to ∞ and using Lemma 11, we have

$$\begin{aligned}
&v_0(0;\boldsymbol{\sigma}^*) \cdot S^M_{\boldsymbol{\sigma}}(0) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(0) - \lim_{t\to\infty} v_0(t;\boldsymbol{\sigma}^*) \cdot S^M_{\boldsymbol{\sigma}}(t) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(t) \\
\geq &\int_0^\infty [\lambda_L(1-\boldsymbol{\sigma}(t)) \cdot \Pi + \mu\, \boldsymbol{\sigma}(t) \cdot V_1(t) - c] \cdot S^M_{\boldsymbol{\sigma}}(t) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(t) dt = \mathcal{U}(\boldsymbol{\sigma}, \hat{\boldsymbol{\sigma}}).
\end{aligned}$$

Since $v_0(t;\boldsymbol{\sigma}^*)$, $S^M_{\boldsymbol{\sigma}}(t)$ and $S^D_{\hat{\boldsymbol{\sigma}}}(t)$ are strictly positive, we have

$$\lim_{t\to\infty} v_0(t;\boldsymbol{\sigma}^*) \cdot S^M_{\boldsymbol{\sigma}}(t) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(t) \geq 0.$$

By using this, $\mathcal{U}(\boldsymbol{\sigma}^*, \hat{\boldsymbol{\sigma}}) = v_0(0;\boldsymbol{\sigma}^*)$, and $S^M_{\boldsymbol{\sigma}}(0) = S^D_{\hat{\boldsymbol{\sigma}}}(0) = 1$, we obtain $\mathcal{U}(\boldsymbol{\sigma}^*, \hat{\boldsymbol{\sigma}}) \geq \mathcal{U}(\boldsymbol{\sigma}, \hat{\boldsymbol{\sigma}})$. $\square$

Proof. ($\Longrightarrow$) Suppose that $\boldsymbol{\sigma}^* \in \arg\max_{\boldsymbol{\sigma}\in\mathcal{S}} \mathcal{U}(\boldsymbol{\sigma}, \hat{\boldsymbol{\sigma}})$. From Lemma 11, observe that for any $t \geq 0$, a firm's expected payoff can be rewritten as follows:

$$\begin{aligned}
\mathcal{U}(\boldsymbol{\sigma}, \hat{\boldsymbol{\sigma}}) = &\int_0^t [\lambda_L(1-\boldsymbol{\sigma}(s)) \cdot \Pi + \mu\, \boldsymbol{\sigma}(s) \cdot V_1(s) - c] \cdot S^M_{\boldsymbol{\sigma}}(s) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(s)\, ds \\
&+ S^M_{\boldsymbol{\sigma}}(t) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(t) \cdot v_0(t;\boldsymbol{\sigma}).
\end{aligned}$$

Now consider the following allocation policy $\tilde{\boldsymbol{\sigma}}(s) := \boldsymbol{\sigma}^*(s)1_{s<t}$. Then, $S^M_{\boldsymbol{\sigma}^*}(s) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(s) = S^M_{\tilde{\boldsymbol{\sigma}}}(s) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(s)$ for all $s \leq t$.[22] In addition, by using $\sigma^*(s) = \tilde{\boldsymbol{\sigma}}(s)$ for all $s < t$ and $\mathcal{U}(\boldsymbol{\sigma}^*, \hat{\boldsymbol{\sigma}}) \geq \mathcal{U}(\tilde{\boldsymbol{\sigma}}, \hat{\boldsymbol{\sigma}})$, we have $v_0(t;\boldsymbol{\sigma}^*) \geq v_0(t;\tilde{\boldsymbol{\sigma}})$.

Note that

$$v_0(t;\tilde{\boldsymbol{\sigma}}) = \int_t^\infty (\lambda_L \Pi - c) \cdot \frac{S^M_{\tilde{\boldsymbol{\sigma}}}(s)}{S^M_{\tilde{\boldsymbol{\sigma}}}(t)} \cdot \frac{S^D_{\hat{\boldsymbol{\sigma}}}(s)}{S^D_{\hat{\boldsymbol{\sigma}}}(t)}\, ds > 0$$

from $\lambda_L \Pi > c$, $S^M_{\tilde{\boldsymbol{\sigma}}}(s) > 0$, and $S^D_{\hat{\boldsymbol{\sigma}}}(s) > 0$. Therefore, $v_0(t;\boldsymbol{\sigma}^*) > 0$ for all $t \geq 0$.

Now assume that there exists $\boldsymbol{\sigma} \in \mathcal{S}$ such that (D.21) does not hold for some $t \geq 0$.

[22] Note that the equality also holds at $s = t$, since $\boldsymbol{\sigma}^*$ and $\tilde{\boldsymbol{\sigma}}$ differ only at $\{t\}$, which is negligible after integration.

Observe that $V_1(\cdot;\mathbf{h})$ and $v_0(\cdot;\boldsymbol{\sigma},\mathbf{h})$ are continuous. Since $\boldsymbol{\sigma}^*$ and $\boldsymbol{\sigma}$ are right-continuous, there exists $\epsilon > 0$ such that for all $s \in [t, t+\epsilon)$,

$$(\boldsymbol{\sigma}^*(s) - \boldsymbol{\sigma}(s)) \cdot [\mu \cdot (V_1(s) - v_0(s;\boldsymbol{\sigma}^*) - \lambda_L \cdot (\Pi - v_0(s;\boldsymbol{\sigma}^*))] < 0. \tag{D.22}$$

Consider the following allocation policy $\boldsymbol{\sigma}^{**}$ defined by:

$$\boldsymbol{\sigma}^{**}(s) := \begin{cases} \boldsymbol{\sigma}^*(s), & \text{if } s \notin [t, t+\epsilon), \\ \boldsymbol{\sigma}(s), & \text{if } s \in [t, t+\epsilon). \end{cases}$$

By using a similar reformulation as in the previous case, we have

$$\begin{aligned} &-\frac{d}{ds}\left[v_0(s;\boldsymbol{\sigma}^*) \cdot S^M_{\boldsymbol{\sigma}^{**}}(s) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(s)\right] \\ &\le [\lambda_L(1 - \boldsymbol{\sigma}^{**}(s)) \cdot \Pi + \mu\sigma^{**}(s) \cdot V_1(s) - c] \cdot S^M_{\boldsymbol{\sigma}^{**}}(s) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(s) \end{aligned} \tag{D.23}$$

for all $s \ge 0$, and the inequality strictly holds for $s \in [t, t+\epsilon)$. Also note that by Lemma 13,

$$\lim_{s\to\infty} v_0(s;\boldsymbol{\sigma}^*) \cdot S^M_{\boldsymbol{\sigma}^{**}}(s) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(s) = \lim_{s\to\infty} v_0(s;\boldsymbol{\sigma}^*) \cdot S^M_{\boldsymbol{\sigma}^{*}}(s) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(s) = 0.$$

By integrating (D.23) from 0 to ∞, we have

$$\begin{aligned} \mathcal{U}(\boldsymbol{\sigma}^*, \hat{\boldsymbol{\sigma}}) &= v_0(0;\boldsymbol{\sigma}^*) \\ &< \int_0^\infty [\lambda_L(1 - \boldsymbol{\sigma}^{**}(s)) \cdot \Pi + \mu\, \boldsymbol{\sigma}^{**}(s) \cdot V_1(s) - c] \cdot S^M_{\boldsymbol{\sigma}^{**}}(s) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(s)\; ds = \mathcal{U}(\boldsymbol{\sigma}^{**}, \hat{\boldsymbol{\sigma}}), \end{aligned}$$

which contradicts $\boldsymbol{\sigma}^* \in \arg\max_{\boldsymbol{\sigma}\in\mathcal{S}} \mathcal{U}(\boldsymbol{\sigma}, \hat{\boldsymbol{\sigma}})$. Therefore, (D.21) holds for all $t \ge 0$. □

D.4 Equilibrium Characterization

We start by considering an opponent with a constant development rate. The following lemma characterizes the best response in this case, in line with the best responses described for the case of public information in equations C.2 and C.3.

Lemma 14. *For any constant development rate $\lambda \in \mathbb{R}_+$, $V_1(\cdot;\lambda)$, $V_0(\cdot;\lambda)$, and $R(\cdot;\lambda)$ are* *constant. Moreover, sgn*$(R(0;\lambda)) =$ *sgn*$(\lambda_\star - \lambda)$.

Proof. Since the allocation problem of a firm when the opponent develops at a constant rate λ is memoryless, there must be a constant research rate $\sigma^* \in [0,1]$ that is optimal. Then,

$$V_1(t;\lambda) = \frac{\lambda_H \Pi - c}{\lambda_H + \lambda} \qquad \text{and} \qquad V_0(t;\lambda) = \frac{(1-\sigma^*)\lambda_L \Pi + \sigma^* \mu V_1(t;\lambda) - c}{(1-\sigma^*)\lambda_L + \sigma^*\mu + \lambda}$$

Observe that these two value functions are constant in t. Thus, using these expressions, we obtain:

$$\begin{aligned} R(t;\lambda) &= \mu(V_1(0;\lambda) - V_0(0;\lambda) - \lambda_L(\Pi - V_0(0;\lambda)) \\ &= \frac{(\lambda\Pi + c)\lambda_L(\lambda_\star - \lambda)}{(\lambda + \lambda_H)(\lambda + (1-\sigma^*)\lambda_L + \sigma^*\mu)} \end{aligned}$$

Which is also constant in t and shares the sign of $\lambda_\star - \lambda$. □

D.4.1 Increasing development rate

We now consider an opponent with a weakly increasing development rate. In the next lemma, we show that the opponent having an increasing development rate implies that the firm is worst-off as the race continues and, moreover, if the development rate is always below $\lambda_\star$, there is at most one time at which $R_h(t)$ crosses zero from above, and thus research stops being profitable.

Lemma 15 (Increasing hazard)**.** *Let* $\mathbf{h} \in \mathcal{H}$ *be weakly increasing. Then,* $V_1(\cdot;\mathbf{h})$ *and* $V_0(\cdot,\mathbf{h})$ *are weakly decreasing. Moreover,* $R(t;\mathbf{h}) \leq 0$ *implies sgn*$[\dot{V}_1(t;\mathbf{h}) - \dot{V}_0(t;\mathbf{h})] =$ *sgn*$(\mathbf{h}(t) - \lambda_\star)$.

Proof. Note that

$$V_1(t;\mathbf{h}) = (\lambda_H\Pi - c)\cdot \int_t^\infty e^{-\int_t^s (\mathbf{h}(u)+\lambda_H)\ du}\, ds \leq (\lambda_H\Pi - c)\cdot \int_t^\infty e^{-(\mathbf{h}(t)+\lambda_H)(s-t)} ds = \frac{\lambda_H\Pi - c}{\mathbf{h}(t) + \lambda_H}$$

From HJB$_1$, we have

$$V_1'(t;\mathbf{h}) = -(\lambda_H\Pi - c) + (\lambda_H + \mathbf{h}(t)) \cdot V_1(t;\mathbf{h}) \leq 0. \tag{D.24}$$

Next, let $\boldsymbol{\sigma}^*$ be a policy satisfying $V_0(t;\mathbf{h}) = v_0(t;\boldsymbol{\sigma}^*,\mathbf{h})$. Note that for all $s \geq t$, $V_0(s;\mathbf{h}) \geq v_0(s;\mathbf{0},\mathbf{h}) > 0$ from $\Pi > c/\lambda_L$. Additionally, from HJB_0,

$$\begin{aligned}
&(1-\boldsymbol{\sigma}^*(s))\lambda_L(\Pi - V_0(s;\mathbf{h})) + \boldsymbol{\sigma}^*(s)\mu(V_1(s;\mathbf{h}) - V_0(s;\mathbf{h})) \geq \lambda_L(\Pi - V_0(s;\mathbf{h})) \\
\Rightarrow \quad & \boldsymbol{\sigma}^*(s)\cdot\mu\cdot V_1(s;\mathbf{h}) + (1-\boldsymbol{\sigma}^*(s))\cdot\lambda_L\Pi - c \\
& \geq \boldsymbol{\sigma}^*(s)\cdot\mu\cdot V_0(s;\mathbf{h}) + (1-\boldsymbol{\sigma}^*(s))\cdot\lambda_L\cdot V_0(s;\mathbf{h}) + \lambda_L(\Pi - V_0(s;\mathbf{h})) - c \\
& = (\lambda_L\Pi - c) + \boldsymbol{\sigma}^*(s)\cdot(\mu - \lambda_L)\cdot V_0(s;\mathbf{h}) \geq 0.
\end{aligned}$$

$$\begin{aligned}
v_0(t;\sigma^*,\mathbf{h}) &= \int_t^\infty \{\boldsymbol{\sigma}^*(s)\cdot\mu\cdot V_1(s;\mathbf{h}) + (1-\boldsymbol{\sigma}^*(s))\cdot\lambda_L\Pi - c\}\cdot e^{-\int_t^s\{\mathbf{h}(u)+\boldsymbol{\sigma}^*(u)\mu+(1-\boldsymbol{\sigma}^*(u))\cdot\lambda_H\}du}ds \\
&\leq \int_t^\infty \left\{\boldsymbol{\sigma}^*(s)\cdot\mu\cdot V^1(t;\mathbf{h}) + (1-\boldsymbol{\sigma}^*(s))\cdot\lambda_L\Pi - c\right\}\cdot e^{-\int_t^s\{\mathbf{h}(t)+\boldsymbol{\sigma}^*(u)\mu+(1-\boldsymbol{\sigma}^*(u))\cdot\lambda_H\}du}ds \\
&\leq \max_{\sigma\in[0,1]} \frac{\sigma\cdot\mu\cdot V_1(t;\mathbf{h}) + (1-\sigma)\cdot\lambda_L\Pi - c}{(1-\sigma)\lambda_L + \sigma\mu + \mathbf{h}(t)}
\end{aligned}$$

Let the solution of the maximization problem of the right hand side is $\hat{\sigma}$. Then, we have

$$\begin{aligned}
0 \geq & -\{(1-\hat{\sigma})\lambda_L(\Pi - V_0(t;\mathbf{h})) + \hat{\sigma}\mu(V_1(t;\mathbf{h}) - V_0(t;\mathbf{h}))\} + c + \mathbf{h}(t)\cdot V_0(t;\mathbf{h}) && \text{(D.25)} \\
\geq & -\max_{\sigma\in[0,1]}\{(1-\hat{\sigma})\lambda_L(\Pi - V_0(t;\mathbf{h})) + \hat{\sigma}\mu(V_1(t;\mathbf{h}) - V_0(t;\mathbf{h}))\} + c + \mathbf{h}(t)\cdot V_0(t;\mathbf{h}) && \text{(D.26)} \\
= & V_0'(t;\mathbf{h}). && \text{(D.27)}
\end{aligned}$$

For the second part of the lemma, we start by subtracting HJB_1 and HJB_0 to obtain, for all t, the following:

$$
\begin{aligned}
V_1'(t;\mathbf{h}) - V_0'(t;\mathbf{h}) &= -\lambda_H \Pi + c + (\lambda_H + \mathbf{h}(t))V_1(t;\mathbf{h}) + \lambda_L(\Pi - V_0(t;\mathbf{h})) - c - \mathbf{h}(t)\cdot V_0(t;\mathbf{h}) \\
&= -\lambda_H(\Pi - V_1(t;\mathbf{h})) + \lambda_L(\Pi - V_h^0) + \mathbf{h}(t)(V_1(t;\mathbf{h}) - V_h^0) \\
&= -\lambda_H(\Pi - V_0(t;\mathbf{h})) + \lambda_H(V_1(t;\mathbf{h}) - V_0(t;\mathbf{h})) \\
&\qquad\qquad + \lambda_L(\Pi - V_0(t;\mathbf{h})) + \mathbf{h}(t)(V_1(t;\mathbf{h}) - V_0(t;\mathbf{h})) \\
&= (\lambda_H + \mathbf{h}(t))(V_1(t;\mathbf{h}) - V_0(t;\mathbf{h})) - (\lambda_H - \lambda_L)(\Pi - V_0(t;\mathbf{h})) \\
&\leq (\lambda_H + \mathbf{h}(t))\frac{\lambda_L}{\mu}[\Pi - V_0(t;\mathbf{h})] - (\lambda_H - \lambda_L)(\Pi - V_0(t;\mathbf{h})) \\
&= \frac{\lambda_L}{\mu}\left[\mathbf{h}(t) - \frac{\mu(\lambda_H - \lambda_L) - \lambda_H\lambda_L}{\lambda_L}\right](\Pi - V_0(t;\mathbf{h})) \\
&= \frac{\lambda_L}{\mu}\left[\mathbf{h}(t) - \lambda_\star\right](\Pi - V_0(t;\mathbf{h})).
\end{aligned}
$$

□

Corollary. If $\mathbf{h}$ is increasing with $\mathbf{h}(t) < \lambda_\star$ for all t, $R(t,\mathbf{h}) < 0$ implies $R(s,\mathbf{h}) < 0$ for all $s > t$.

Lemma 16 (Limit incentives)**.** *Let $\mathbf{h} \in \mathcal{H}$ be increasing with $\mathbf{h}(t) \to \bar{h}$. Then $R(t;\mathbf{h}) \to R(t;\bar{h})$.*

Proof. First, we show that $V_1(t;\mathbf{h})$ converges to $V_1(0;\bar{h})$. Since $\mathbf{h}(t) \leq \mathbf{h}(s) \leq \bar{h}$ for all $s > t$, we can bound V_1 by the value when the opponent has constant hazard rates $\mathbf{h}(t)$ and $\bar{h}$.

$$V_1(0;\bar{h}) = V_1(t;\bar{h}) \leq V_1(t;\mathbf{h}) \leq V_1(t;\mathbf{h}(t)) = V_1(0;\mathbf{h}(t))$$

By continuity of $V_1(0;h)$ in h, and the fact that the upper-bound $V_1(0;\mathbf{h}(t))$ converges to the lower-bound $V_1(0;\bar{h})$, we can apply the squeeze theorem to get $V_1(t;\mathbf{h}) \to V_1(0;\bar{h})$. We can obtain bounds for $V_0(t;\mathbf{h})$ using a similar logic. Since $h(t) \leq h(s) \leq \bar{h}$,

$$V_0(0;\bar{h}) = V_0(t;\bar{h}) \leq V_0(t;\mathbf{h}) \leq V_0(t;\mathbf{h}(t)) = V_0(0;\mathbf{h}(t))$$

Using the continuity of $V_0(0;h)$ in h, as V_0 is the maximum of continuous functions, and applying the squeeze theorem, we obtain that $V_0(t;\mathbf{h}) \to V_0(0;\bar{h})$. $\square$

Lemma 17. *Let $\boldsymbol{\sigma} \in \mathcal{S}$. If $\mathbf{h}_{\boldsymbol{\sigma}}$ is increasing and $\boldsymbol{\sigma}(s) = 0$ then $\boldsymbol{\sigma}(t) = 0$ for all $t < s$.*

Proof. Since $\boldsymbol{\sigma}(s) = 0$ and $\boldsymbol{\sigma}$ is right-continuous, it must be that $\mathbf{h}(\tilde{s}) = \lambda_L \cdot \mathbf{p}_{\boldsymbol{\sigma}}(\tilde{s})$ for $\tilde{s}$ slightly above s. This means that

$$0 \leq \mathbf{h}'(s) = \lambda_L \cdot \dot{\mathbf{p}}_{\boldsymbol{\sigma}}(s) = -\lambda_L \, \mathbf{p}_{\boldsymbol{\sigma}}(s)(1 - \mathbf{p}_{\boldsymbol{\sigma}}(s))[\lambda_H - \lambda_L]$$

Where the inequality holds since $\mathbf{h}$ is weakly increasing. Since $\mathbf{p}_{\boldsymbol{\sigma}}(s) < 1$, it must be the case that $\mathbf{p}_{\boldsymbol{\sigma}}(s) = 0$. This only holds if $\boldsymbol{\sigma}(t) = 0$ for all $t < s$. $\square$

Lemma 18. *Let T be a finite time and consider a policy $\boldsymbol{\sigma}$ such that $\boldsymbol{\sigma}(t) = 1$ for all $t > T$. Then, $\mathbf{h}_{\boldsymbol{\sigma}}(t) \to \min\{\lambda_H, \mu\}$.*

Proof. From the evolution of beliefs, using that $\boldsymbol{\sigma}(t) = 1$, we get that, for all $t > T$,

$$\dot{\mathbf{p}}_{\boldsymbol{\sigma}}(t) = (1 - \mathbf{p}_{\boldsymbol{\sigma}}(t) \, [\mu - \lambda_H \, \mathbf{p}_{\boldsymbol{\sigma}}(t)]$$

This evolution of beliefs gives us that $\mathbf{p}_{\boldsymbol{\sigma}}(t)$ converges to 1 when $\mu > \lambda_H$ and to μ/λ_H when $\mu \leq \lambda_H$. Using this, together with $\boldsymbol{\sigma}_t = 0$, in the hazard rate function and taking limits, we obtain that

$$\lim_{t\to\infty} \mathbf{h}_{\boldsymbol{\sigma}}(t) = \lim_{t\to\infty} \lambda_H \, \mathbf{p}_{\boldsymbol{\sigma}}(t) = \lambda_H \cdot \min\{1, \mu/\lambda_H\} = \min\{\lambda_H, \mu\}$$

$\square$

D.4.2 Case (i): $\lambda_\star < \lambda_L$.

In this subsection we prove Theorem 2 (i).

Proof. Let $(\boldsymbol{\sigma}_A, \boldsymbol{\sigma}_B)$ be a MDNE. Then, by MDR of $\boldsymbol{\sigma}_j$, it must be that $\mathbf{h}_{\boldsymbol{\sigma}_j}$ is increasing. $h_{\boldsymbol{\sigma}_j}$ is also bounded by λ_H, and therefore it converges. We denote $\bar{h}$ the limit of $h_{\boldsymbol{\sigma}_j}(t)$ when $t \to \infty$.

Note that $\bar{h} \geq \lambda_L$: otherwise $\mathbf{h}_{\boldsymbol{\sigma}_j}(t) < \bar{h} < \lambda_L$ for all t and, thus, it would be more profitable for the firm to choose $\boldsymbol{\sigma} = 0$, which induces a constant rate of development equal to λ_L. By continuity, the relative attractiveness of research $R(t; \mathbf{h}_{\boldsymbol{\sigma}_j})$ converges to $R_{\bar{h}} < 0$, where the inequality holds since $\bar{h} \geq \lambda_L > \lambda_\star$. This implies that there is a time T such that $R(t; \mathbf{h}_{\boldsymbol{\sigma}_j}) < 0$ for all $t \geq T$. By Proposition 3, it must be that $\boldsymbol{\sigma}_i(t) = 0$ for all $t \geq T$. It remains to show that $\boldsymbol{\sigma}_i(t) = 0$ for all $t \leq T$, which follows immediately from applying Lemma 17.

Summarizing, $(0, 0)$ is the unique candidate for MDNE. First notice that the policy 0 satisfies MDR since $\mathbf{h}_0$ is constant and equal to λ_L. Moreover, to check that $(0, 0)$ is a Nash equilibrium, notice that $\lambda_L > \lambda_\star$, which implies by Proposition 1 that developing with the old technology is a best response. $\square$

D.4.3 Case (ii): $\lambda_\star > \min\{\mu, \lambda_H\}$.

We begin the proof of Theorem 2 part (ii) by obtaining an upper bound for the development rate for any policy with monotone development rates.

Lemma 19. *Let $\boldsymbol{\sigma} \in \mathcal{S}$ be MDR. Then, $\mathbf{h}_{\boldsymbol{\sigma}} < \min\{\mu, \lambda_H\}$.*

Proof. First, observe that for any $\boldsymbol{\sigma} \in \mathcal{S}$ and $t \geq 0$, $\mathbf{p}_{\boldsymbol{\sigma}}(t) \leq \min\{\mu/\lambda_H, 1\}$. Suppose toward a contradiction that there is a T such that $\mathbf{p}_{\boldsymbol{\sigma}}(T) > \min\{\mu/\lambda_H, 1\}$. Then, by continuity of $\mathbf{p}_{\boldsymbol{\sigma}}$, there must be a $t < T$ such that $\mathbf{p}_{\boldsymbol{\sigma}}(t) \in (\min\{\mu/\lambda_H, 1\}, \mathbf{p}_{\boldsymbol{\sigma}}(T))$ and $\dot{\mathbf{p}}_{\boldsymbol{\sigma}}(t) > 0$. However,

$$
\begin{aligned}
\dot{\mathbf{p}}_{\boldsymbol{\sigma}}(t) &= \mu(1 - \mathbf{p}_{\boldsymbol{\sigma}}(t))\, \boldsymbol{\sigma}(t) - (\lambda_H - (1 - \boldsymbol{\sigma}(t))\lambda_L)\, \mathbf{p}_{\boldsymbol{\sigma}}(t)(1 - \mathbf{p}_{\boldsymbol{\sigma}}(t)) \\
&\leq [\mu - \lambda_H\, \mathbf{p}_{\boldsymbol{\sigma}}(t)]\, (1 - \mathbf{p}_{\boldsymbol{\sigma}}(t)) < 0
\end{aligned}
$$

Where the first inequality holds because the $\delta(\sigma, p)$, as defined in Eq. 2, is increasing in σ and the second inequality holds because if $p_{\boldsymbol{\sigma}}(t) > \min\{\mu/\lambda_H, 1\}$ is only possible if $\mu < \lambda_H$ and $p_{\boldsymbol{\sigma}}(t) > \mu/\lambda_H$.

Next we prove that for any policy $\boldsymbol{\sigma}$ satisfying MDR, the hazard rate $\mathbf{h}_{\boldsymbol{\sigma}}$ never exceeds

$\min\{\mu, \lambda_H\}$. First,

$$\mathbf{h}_{\boldsymbol{\sigma}}(t) = \mathbf{p}_{\boldsymbol{\sigma}}(t) \cdot \lambda_H + (1 - \mathbf{p}_{\boldsymbol{\sigma}}(t)) \underbrace{(1 - \boldsymbol{\sigma}(t))\lambda_L}_{<\lambda_H} < \lambda_H$$

It remains to show that, when $\mu < \lambda_H$, $\mathbf{h}_{\boldsymbol{\sigma}}(t) < \mu$. First, we can see that $\dot{\mathbf{p}}_{\boldsymbol{\sigma}}(t) \geq 0$ implies $\mathbf{h}_{\boldsymbol{\sigma}}(t) \leq \mu$.

$$\dot{\mathbf{p}}_{\boldsymbol{\sigma}}(t) = [\mu\, \boldsymbol{\sigma}(t) - (\lambda_H - (1 - \boldsymbol{\sigma}(t))\lambda_L)\, \mathbf{p}_{\boldsymbol{\sigma}}(t)](1 - \mathbf{p}_{\boldsymbol{\sigma}}(t)) \geq 0$$

Since $\mathbf{p}_{\boldsymbol{\sigma}}(t) < 1$, this holds if and only if

$$\mu\, \boldsymbol{\sigma}(t) \geq (\lambda_H - (1 - \boldsymbol{\sigma}(t))\lambda_L)\, \mathbf{p}_{\boldsymbol{\sigma}}(t)$$

In this case,

$$\mathbf{h}_{\boldsymbol{\sigma}}(t) = (1 - \boldsymbol{\sigma}(t))\lambda_L + p(\lambda_H - (1 - \boldsymbol{\sigma}(t))\lambda_L) \leq (1 - \boldsymbol{\sigma}(t))\lambda_L + \boldsymbol{\sigma}(t)\mu \leq \mu$$

Thus, $\mathbf{h}_{\boldsymbol{\sigma}}(T) > \mu$ implies $\dot{\mathbf{p}}_{\boldsymbol{\sigma}}(t) < 0$ for all $t > T$. $\mathbf{p}_{\boldsymbol{\sigma}}$ is bounded below by 0, thus it must converge. Let $\bar{p}$ be the limit of $\mathbf{p}_{\boldsymbol{\sigma}}(t)$ when $t \to \infty$. Moreover, $\mathbf{h}_{\boldsymbol{\sigma}}$ increasing with decreasing $\mathbf{p}_{\boldsymbol{\sigma}}$ implies that $\boldsymbol{\sigma}$ has to be decreasing as well. Since $\boldsymbol{\sigma}$ is bounded, it must converge as well. Let $\bar{\sigma}$ be the limit of $\boldsymbol{\sigma}(t)$ when $t \to \infty$. However, δ is continuous at $(\bar{p}, \bar{\sigma})$ and $\delta(\bar{p}, \bar{\sigma})$ is bounded away from zero, which contradicts the limit of $\mathbf{p}_{\boldsymbol{\sigma}}$. $\square$

Proof of Theorem 2 (ii). Let $\mathbf{h}$ be the opponent's equilibrium hazard rate. Since $\mathbf{h}$ is increasing and bounded, it must be that it converges. Let $\bar{h}$ be the limit of $\mathbf{h}(t)$ when $t \to \infty$. Note that, for all t, $\mathbf{h}(t) < \bar{h} < \min\{\mu, \lambda_H\} < \lambda_\star$, where the first inequality holds by monotonicity of $\mathbf{h}$, the second inequality by Lemma 17, and the third inequality by assumption. By applying Lemma 14, we obtain that $R(t; \bar{h}) > 0$. Thus, by Lemma 16 there is a time T such that $R(t; \mathbf{h}) > 0$ for all $t > T$. Suppose toward a contradiction that $R(s; \mathbf{h}) < 0$ for some $s \in \mathbb{R}_+$. Then, by the Corollary of Lemma 15, it must be that $R(s; \mathbf{h}) < 0$ for all $s > T$. Thus, there is no such s and $R(t; \mathbf{h}) \geq 0$ for all t. This is true for both firms, so using Proposition 3, we have that $(1, 1)$ is the only equilibrium candidate.

It remains to check that $(1, 1)$ is a MDNE. First, observe that $\mathbf{h}_1$ is increasing, since

$\dot{\mathbf{h}}_1(t) = \lambda_H \dot{\mathbf{p}}_1(t) = \lambda_H(\mu - \lambda_H \mathbf{p}_1(t))(1 - \mathbf{p}_1(t))$ and $(\mu - \lambda_H \mathbf{p}_1(t)) > 0$ by Lemma 19. By Lemma 18, $\mathbf{h}_1$ converges to $\min\{\mu, \lambda_H\}$, which is lower than $\lambda_\star$. Therefore, there is a time T such that $R(t; \mathbf{h}_1) > 0$ for all $t > T$. Moreover, suppose toward a contradiction that $R(s; \mathbf{h}) < 0$ for some $s \in \mathbb{R}_+$. Then, by the Corollary of Lemma 15, it must be that $R(s; \mathbf{h}) < 0$ for all $s > T$. Thus, there is no such s and $R(t; \mathbf{h}) \geq 0$ for all t. Thus, by the verification result, $\boldsymbol{\sigma} = 1$ is a best response to $\mathbf{h}_1$ and $(1, 1)$ is a NE. □

D.4.4 Case (iii): $\lambda_\star \in (\lambda_L, \min\{\mu, \lambda_H\})$.

Lemma 20. *Let $\lambda_\star \in (\lambda_L, \lambda_H)$, and let $\mathbf{h}$ be increasing with $\mathbf{h}(t) \to \lambda_\star$. Let T be the first time at which $\mathbf{h}(T) = \lambda_\star$. Then $R(t; \mathbf{h}) > 0$ for all $t < T$ and $R(t; \mathbf{h}) = 0$ for all $t \geq T$.*

Proof. First, note that $\mathbf{h}(s) = \lambda_\star$ for all $s \geq T$. Therefore, by the observation in Section D.2, $R(t; \mathbf{h}) = R(0; \lambda_\star) = 0$ for all $t \geq T$. Let $\hat{T}$ be the first time it is profitable to use the old technology, i.e. $\hat{T} := \inf\{t \in [0, \infty] : R(t; \mathbf{h}) \leq 0\}$. Observe that, since $R(T; \mathbf{h}) = 0$, it must be that $\hat{T} \leq T$. Next, we show that $\hat{T} < T$ leads to a contradiction.

Suppose towards a contradiction that $\hat{T} < T$. Note that,

$$R'(t; \mathbf{h}) = \mu(V_1'(t; \mathbf{h}) - V_0'(t; \mathbf{h})) + \lambda_L V_0'(t; \mathbf{h})$$

The second term is weakly negative by the first part of Lemma 15. The first term is strictly negative when evaluated at $\hat{T}$, since $R(\hat{T}; \mathbf{h}) \leq 0$ and applying the last part of Lemma 15. Thus, we have that $R(\hat{T}; \mathbf{h}) = R(T; \mathbf{h}) = 0$ and $R'(\hat{T}; \mathbf{h}) < 0$. By continuity of $R(\cdot; \mathbf{h})$, there must exist a time $t \in (\hat{T}, T)$ such that $R(t; \mathbf{h}) \leq 0$ and $R'(t; \mathbf{h}) > 0$. This enters in contradiction with the corollary of Lemma 15. Therefore $\hat{T} = T$, which means that $R(t; \mathbf{h}) > 0$ for all $t < T$. □

The next lemma shows that if the opponent does research first ($\boldsymbol{\sigma}_j(t) = 1$ for all t) it is not a best-response to do direct development.

Lemma 21. *Let $\lambda_\star \in (\lambda_L, \min\{\lambda_H, \mu\})$. Then $R(0, \mathbf{h}_1) > 0$.*

Proof. $\mathbf{h}_1$ is the development rate associated with the research policy ($\boldsymbol{\sigma} = 1$). We can compute the continuation value, at time zero, of doing direct development $v_0(0; 0, \mathbf{h}_1)$.

$$v_0(0;0,\mathbf{h}_1) = \Pi\left[\frac{\lambda_L}{\lambda_L+\mu} + \frac{\mu}{\lambda_L+\mu}\cdot\frac{\lambda_L}{\lambda_L+\lambda_H}\right] - c\left[\frac{1}{\lambda_L+\mu} + \frac{\mu}{\lambda_L+\mu}\cdot\frac{1}{\lambda_L+\lambda_H}\right]$$

The first bracket captures the probability of the firm winning the race. The firm can win by developing before the opponent finds the new technology—which happens with probability $\lambda_L/(\lambda_L+\mu)$—or the opponent can find the new technology first, in which case the firm wins with probability $\lambda_L/\lambda_L+\lambda_H$. The second bracket captures the expected duration of the race. The expected time before the first breakthrough in the race is $1/(\lambda_L+\mu)$. If the opponent finds the new technology—which happens with probability $\mu/(\lambda_L+\mu)$—the race is extended by $1/(\lambda_L+\lambda_H)$ in expectation. By doing some algebra, we obtain that:

$$v_0(0;0,\mathbf{h}_1) = \frac{\lambda_L\Pi - c}{\lambda_L+\mu}\cdot\frac{\lambda_L+\lambda_H+\mu}{\lambda_L+\lambda_H}$$

We can obtain $V_1(0,h_1)$ by using the same logic, but replacing the development rate of the incumbent technology λ_L with the development rate of the new technology λ_H.

$$V_1(0;\mathbf{h}_1) = \frac{\lambda_H\Pi - c}{\lambda_H+\mu}\cdot\frac{\lambda_H+\lambda_H+\mu}{\lambda_H+\lambda_H}$$

Suppose toward a contradiction that direct development ($\sigma=0$) is a best response toward research first ($\sigma=1$). This implies that $V_0(t;\mathbf{h}_1) = v_0(t;0,\mathbf{h}_1)$ and that $R(t;h_1)\leq 0$ for all t. However,

$$\begin{aligned}
R(0;\mathbf{h}_1) &= \mu(V_1(0;\mathbf{h}_1) - V_0(0;\mathbf{h}_1)) - \lambda_L(\Pi - V_0(0,\mathbf{h}_1)) \\
&= \frac{c\left(\left(\lambda_\star-\lambda_L\right)\left(2\lambda_H+\mu\right)+\lambda_L\left(\lambda_H-\lambda_\star\right)\right)}{2\lambda_H\left(\lambda_H+\mu\right)} \\
&\qquad + \mu\cdot(\lambda_L\Pi - c)\cdot\frac{\left(\lambda_\star-\lambda_L\right)\left(2\lambda_H+\mu+\lambda_L\right)+\lambda_L\left(2\lambda_H+\lambda_L-\lambda_\star\right)}{2\left(\lambda_H+\mu\right)\left(\lambda_H+\lambda_L\right)\left(\mu+\lambda_L\right)} \\
&> 0
\end{aligned}$$

Where the inequality uses that $\lambda_L\Pi - c > 0$ and that $\lambda_\star \in (\lambda_L,\lambda_H)$. $\square$

Lemma 22. *Let $\lambda_\star \in (\lambda_L, \min\{\lambda_H,\mu\})$ and let $(\boldsymbol{\sigma}_A,\boldsymbol{\sigma}_B)$ be a MDNE. Then $\mathbf{h}_{\boldsymbol{\sigma}_A}(t)$ and*

$\mathbf{h}_{\boldsymbol{\sigma}_B}(t)$ *converge to* $\lambda_\star$.

Proof. First, note that $\mathbf{h}_{\boldsymbol{\sigma}_i}$ is weakly increasing and bounded above by λ_H. Thus, $\mathbf{h}_{\boldsymbol{\sigma}_i}(t)$ must converge. Let $\bar{h}_i$ be the limit of $\mathbf{h}_{\boldsymbol{\sigma}_i}(t)$ when t goes to infinity.

Suppose towards a contradiction that $\bar{h}_i > \lambda_\star$. Then, by Lemma 16, $R(t; \mathbf{h}_{\boldsymbol{\sigma}_i})$ converges $R(0; \bar{h}_i)$. Since $\bar{h} > \lambda_\star$, applying Lemma 14, we get that $R(0; \bar{h}_i) < 0$. Thus, there is a time T for which $R(t; \mathbf{h}_{\boldsymbol{\sigma}_i}) < 0$ for all $t > T$. This implies that $\boldsymbol{\sigma}_j = 0$ for all $t > T$ and moreover, by Lemma 17, $\boldsymbol{\sigma}_j = 0$. Therefore, $\mathbf{h}_{\boldsymbol{\sigma}_j} = \mathbf{h}_0 = \lambda_L$. Since $\lambda_L < \lambda_\star$, it must be, by 14, that $R(t; \lambda_L) > 0$. Thus, since $\boldsymbol{\sigma}_i$ is a best-response, $\boldsymbol{\sigma}_i = 1$. However, $(\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_j) = (0, 1)$ is ruled out as an equilibrium by Lemma 21. Therefore, there cannot be an equilibrium in which one of the development rates converges to a rate greater than $\lambda_\star$.

Suppose towards a contradiction that $\bar{h}_i < \lambda_\star$. Then, by Lemma 16, $R(t; \mathbf{h}_{\boldsymbol{\sigma}_i})$ converges to $R(0; \bar{h}) > 0$. Thus, there is a time T such that $R(t, \mathbf{h}_{\boldsymbol{\sigma}_i}) > 0$ for all $t > T$. Since $\boldsymbol{\sigma}_j$ is a best-response, it must be that $\boldsymbol{\sigma}_j(t) = 1$ for all $t > T$. By Lemma 18, $\mathbf{h}_j$ converges to $\min\{\mu, \lambda_H\} > \lambda_\star$. However, we showed that this was not possible. $\square$

Proposition 9. *Let* $\lambda_\star \in (\lambda_L, \min\{\lambda_H, \mu\})$ *and* $(\boldsymbol{\sigma}_A, \boldsymbol{\sigma}_B)$ *be a MDNE. Then* $\boldsymbol{\sigma}_A = \boldsymbol{\sigma}_B = \boldsymbol{\sigma}_\star$*, where* $\boldsymbol{\sigma}_\star(t) = 1$ *for every* t *such that* $\mathbf{p}_{\boldsymbol{\sigma}_\star}(t) < p_\star$ *and* $\boldsymbol{\sigma}_\star(t) = \sigma_\star$ *when* $p_{\boldsymbol{\sigma}_\star}(t) = p_\star$.

Proof. By Lemma 22, it must be that $\mathbf{h}_{\boldsymbol{\sigma}_A}$ and $\mathbf{h}_{\boldsymbol{\sigma}_B}$ converge to $\lambda_\star$. For $i = A, B$, let $T_i = \sup\{t \ : \ \mathbf{h}_{\boldsymbol{\sigma}_i}(t) < \lambda_\star\}$ and let $T = \min\{T_A, T_B\}$.

Suppose towards a contradiction that $T_A < T_B$. By Lemma 20, we know that $R(t; \mathbf{h}_B) > 0$ for all $t < T_B$. This means that $\boldsymbol{\sigma}_A(t) = 1$ for $t \in (T_A, T_B)$. This, however, contradicts the fact that $\mathbf{h}_{\boldsymbol{\sigma}_A}$ is constant and equal to $\lambda_\star$ on that interval:

$$\dot{\mathbf{h}}_{\boldsymbol{\sigma}_A} = \dot{\mathbf{p}}_{\boldsymbol{\sigma}_A} = (\mu - \lambda_H \, \mathbf{p}_{\boldsymbol{\sigma}_A}(t))(1 - \mathbf{p}_{\boldsymbol{\sigma}_A}(t))\lambda_H > 0 \qquad \forall t \in (T_A, T_B)$$

Thus, $T_A = T_B = T$ with $\boldsymbol{\sigma}_i(t) = 1$ for $t < T$.

For $s > T$ and $i \in \{A, B\}$, we have $\mathbf{h}_{\boldsymbol{\sigma}_i}(s) = \lambda_\star$. Using the definition of $\mathbf{h}_{\boldsymbol{\sigma}_i}$, we have that

$$\lambda_\star = \lambda_H \, \mathbf{p}_{\boldsymbol{\sigma}_i}(s) + \lambda_L(1 - \mathbf{p}_{\boldsymbol{\sigma}_i}(s))(1 - \boldsymbol{\sigma}^*(s)),$$

or equivalently,

$$1 - \sigma^*(s) = \frac{\lambda_\star - \lambda_H \, \mathbf{p}_{\boldsymbol{\sigma}_i}(s)}{\lambda_L(1 - \mathbf{p}_{\boldsymbol{\sigma}_i}(s))}. \tag{D.28}$$

From the evolution of beliefs, we have that for every $s > T$

$$\begin{aligned}
\dot{\mathbf{p}}_{\boldsymbol{\sigma}_i}(s) =&(1 - \mathbf{p}_{\boldsymbol{\sigma}_i}(s)) \left[\mu - \lambda_H \, \mathbf{p}_{\boldsymbol{\sigma}_i}(s) - (1 - \boldsymbol{\sigma}_i(s))(\mu - \lambda_L \, \mathbf{p}_{\boldsymbol{\sigma}_i}(s))\right] \\
=&(1 - \mathbf{p}_{\boldsymbol{\sigma}_i}(s))(\mu - \lambda_H \, \mathbf{p}_{\boldsymbol{\sigma}_i}(s)) - \left\{\lambda_\star - \lambda_H \, \mathbf{p}_{\boldsymbol{\sigma}_i}(s)\right\} \left(\frac{\mu}{\lambda_L} - \mathbf{p}_{\boldsymbol{\sigma}_i}(s)\right) \\
=&\mu - \frac{\mu}{\lambda_L}\lambda_\star + \left(\lambda_\star + \frac{\lambda_H \mu}{\lambda_L} - \mu - \lambda_H\right) \mathbf{p}_{\boldsymbol{\sigma}_i}(s) \\
=&-\frac{\mu}{\lambda_L}(\lambda_\star - \lambda_L) + 2\lambda_\star \, \mathbf{p}_{\boldsymbol{\sigma}_i}(s) = 2\lambda_\star(\mathbf{p}_{\boldsymbol{\sigma}_i}(s) - p_\star).
\end{aligned}$$

If there is an $s > T$ such that $\mathbf{p}_{\boldsymbol{\sigma}_i}(s) \neq p_\star$, then the solution of the above differential equation diverges. Therefore, $\mathbf{p}_{\boldsymbol{\sigma}_i}(s) = p_\star$ for all $s \geq T$. Using this, in conjunction with $\dot{\mathbf{p}}_{\boldsymbol{\sigma}_i}(s) = 0$, we obtain

$$\boldsymbol{\sigma}_i(s) = \frac{(\lambda_H - \lambda_L) \, \mathbf{p}_{\boldsymbol{\sigma}_i}(s)}{\mu - \lambda_L \, \mathbf{p}_{\boldsymbol{\sigma}_i}(s)} = \frac{(\lambda_H - \lambda_L)p_\star}{\mu - \lambda_L p_\star} = \sigma_\star \qquad \text{for all } s \geq T.$$

By 20, we have that $\boldsymbol{\sigma}_A(s) = \boldsymbol{\sigma}_B(s) = 1$ for all $s < T$. Finally, the fact that $\mathbf{p}_1(T) = p_\star$ is given by the continuity of the probability function $\mathbf{p}_{\boldsymbol{\sigma}}$. □

E Proofs for Patent, License and Trade Secret

E.1 Public Information Setting

E.1.1 Proof of Proposition 5

Proof of Proposition 5. Suppose that Firm i has just discovered the new technology and Firm j does not have the patent for the new technology. If Firm j already has the patent, Firm i cannot apply for a patent in the first place.

First, consider the case where Firm j already has the new technology (not the patent). If Firm i does not apply for a patent, both firms race toward development with the new technology. Thus, Firm i's expected payoff is $\frac{\lambda_H \Pi - c}{2\lambda_H}$. If Firm i applies for a patent, with probability α, Firm j's right to use the new technology is protected, and with probability $1-\alpha$, Firm i acquires the patent. In either case, Firm i's expected payoff is at least $\frac{\lambda_H \Pi - c}{2\lambda_H}$, thus, Firm i prefers to apply for a patent.

Next, consider the case where Firm j does not have the new technology. Suppose that in equilibrium, Firm j allocates $x \in [0,1]$ to research and $1-x$ to development with old technology, when it observes the research progress by Firm i (without a patent). To maximize Firm j's expected payoff, we have

$$\frac{\mu x \cdot \tilde{U}^j + \lambda_L(1-x)\cdot \Pi - c}{\lambda_H + \mu x + \lambda_L(1-x)} \geq \frac{\lambda_L \Pi - c}{\lambda_H + \lambda_L}, \tag{E.1}$$

where $\tilde{U}^j$ is Firm j's expected payoff when it also discovers the new technology. To constitute an equilibrium, Firm i's expected payoff under this Firm j's strategy should be greater than or equal to Firm i's expected payoff from applying for a patent:

$$\frac{\lambda_H \cdot \Pi + \mu x \cdot \tilde{U}^i - c}{\lambda_H + \mu x + \lambda_L(1-x)} \geq U_{Licensor}, \tag{E.2}$$

where $\tilde{U}^i$ is Firm i's expected payoff when Firm j discovers the new technology.

Note that $\tilde{U}^i + \tilde{U}^j \leq \Pi - \frac{2c}{2\lambda_H}$ since the social welfare is maximized when both firms use the new technology, and $U_{Licensor} + \frac{\lambda_L \Pi - c}{\lambda_H + \lambda_L} = \Pi - \frac{c}{\lambda_H}$ from (6.4). By using these and summing (E.1) and (E.2) up, we have

$$\begin{aligned}\Pi - \frac{c}{\lambda_H} &\leq \frac{\lambda_H \Pi + \mu x \cdot (\tilde{U}^j + \tilde{U}^i) + \lambda_L(1-x)\Pi - 2c}{\lambda_H + \mu x + \lambda_L(1-x)} \\ &\leq \Pi - \frac{\frac{\mu x}{\lambda_H} + 2}{\lambda_H + \mu x + \lambda_L(1-x)} c.\end{aligned}$$

However, this inequality is equivalent to $\lambda_H + \mu x + \lambda_L(1-x) \geq 2\lambda_H + \mu x$, which contradicts $\lambda_H > \lambda_L$ and $x \leq 1$. Therefore, in equilibrium, Firm i applies for a patent. $\square$

E.1.2 Proof of Proposition 6

Proof of Proposition 6. Given these continuation payoffs, we now consider the equilibrium allocations when neither firm possesses the new technology yet. To apply Proposition 8, we first compute $\hat{\Delta}_0$ and $\hat{\Delta}_1$ by replacing $(U^i_{\{i\}}, U^j_{\{i\}})$ to $(U_{Licensor}, U_{Licensee})$ in (C.1):

$$\begin{aligned}
\hat{\Delta}_0 &= \frac{\mu U_{Licensor} - c}{\mu + \lambda_L} - \frac{\lambda_L \Pi - c}{2\lambda_L}, \\
\hat{\Delta}_1 &= \frac{\mu U_{Licensor} + \mu U_{Licensee} - c}{2\mu} - \frac{\lambda_L \Pi + \mu U_{Licensee} - c}{\mu + \lambda_L}.
\end{aligned}$$

By plugging (6.4) in, with some algebra, we can derive that

$$\begin{aligned}
\hat{\Delta}_0 =& \frac{\lambda_H \lambda_L (\lambda_\star - \lambda_L)\Pi + (\lambda_H + \lambda_L)\lambda_\star c}{2\lambda_H(\lambda_H + \lambda_L)(\lambda_L + \mu)}, \\
\hat{\Delta}_1 =& \frac{\lambda_H \lambda_L (\lambda_\star - \lambda_L)\Pi + \frac{\lambda_L}{2\mu}\left\{(2\lambda_H + \mu + \lambda_L)\lambda_\star + (\mu - \lambda_L)\lambda_H\right\} c}{2\lambda_H(\lambda_H + \lambda_L)(\lambda_L + \mu)}.
\end{aligned}$$

First, observe that $\lambda_\star \geq \lambda_L$ implies $\hat{\Delta}_0,\ \hat{\Delta}_1 > 0$. Then, by Proposition 8 (a), both firms do research, thus, Proposition 6 (a) holds. Next, when $\lambda_L > \lambda_\star$, we have

$$\begin{aligned}
\hat{\Delta}_0 > 0 \qquad &\Longleftrightarrow \qquad \tilde{\pi}_0 \equiv \frac{\lambda_\star(\lambda_H + \lambda_L)}{\lambda_H(\lambda_L - \lambda_\star)} > \frac{\lambda_L \Pi}{c} = \pi, \\
\hat{\Delta}_1 > 0 \qquad &\Longleftrightarrow \qquad \tilde{\pi}_1 \equiv \frac{\frac{\lambda_L}{2\mu}\left\{(2\lambda_H + \mu + \lambda_L)\lambda_\star + (\mu - \lambda_L)\lambda_H\right\}}{\lambda_H(\lambda_L - \lambda_\star)} > \pi.
\end{aligned}$$

Suppose that $\lambda_\star \in \left(\frac{\lambda_H \lambda_L}{2\lambda_H + \lambda_L}, \lambda_L\right)$. By using $\mu > \lambda_L$, we can show that $\tilde{\pi}_0 > \tilde{\pi}_1 > 1$.

(i) if $\pi > \tilde{\pi}_0 > \tilde{\pi}_1$, we have $\hat{\Delta}_0,\ \hat{\Delta}_1 < 0$, then, by Proposition 1 (b), both firms develop with old technology;

(ii) if $\tilde{\pi}_0 > \pi > \tilde{\pi}_1$, we have $\hat{\Delta}_0 > 0 > \hat{\Delta}_1$, then, by Proposition 1 (c), there are three equilibria including the asymmetric one;

(iii) if $\tilde{\pi}_1 > \pi > 1$, we have $\hat{\Delta}_0, \hat{\Delta}_1 > 0$, then, by Proposition 1 (a), both firms do research.

Thus, Proposition 6 (b) holds.

Now suppose that $\lambda_\star \leq \frac{\lambda_H \lambda_L}{2\lambda_H + \lambda_L}$. With some algebra, we have $1 \geq \tilde{\pi}_1 \geq \tilde{\pi}_0$. From $\pi > 1$, we have $\hat{\Delta}_0, \hat{\Delta}_1 < 0$, then, by Proposition 1 (b), both firms develop with old technology. Thus, Proposition 6 (c) holds. □

E.2 Private Information Setting

E.2.1 Proof of Lemma 4

Proof of Lemma 4. By plugging (6.2) in, we have that (6.7) is equivalent to:

$$
\begin{aligned}
& \frac{\lambda_H - \lambda_L}{\lambda_H + \lambda_L} \cdot \frac{\lambda_H \Pi + c}{\lambda_H \Pi - c} > \frac{\lambda_H}{\lambda_H + \mu(2-\alpha)} \\
\iff \quad & \{\lambda_H(\lambda_H - \lambda_L) + \mu(\lambda_H - \lambda_L)(2-\alpha)\}\,(\lambda_H \Pi + c) > \lambda_H(\lambda_H + \lambda_L)(\lambda_H \Pi - c) \\
\iff \quad & \{\mu(\lambda_H - \lambda_L)(2-\alpha) - 2\lambda_L \lambda_H\} \cdot \lambda_H \Pi + \left\{\mu(\lambda_H - \lambda_L)(2-\alpha) + 2\lambda_H^2\right\} \cdot c > 0.
\end{aligned}
$$

Note that $\mu(\lambda_H - \lambda_L) = \lambda_L(\lambda_\star + \lambda_H)$ from (3.4). By plugging this in, the above inequality is equivalent to:

$$
\begin{aligned}
& \{(2-\alpha)\lambda_\star - \alpha\lambda_H\} \cdot \lambda_H \lambda_L \Pi + \left\{(2-\alpha)\lambda_L(\lambda_\star + \lambda_H) + 2\lambda_H^2\right\} \cdot c > 0 \\
\iff \quad & \{(2-\alpha)\lambda_\star - \alpha\lambda_H\} \cdot \lambda_H \left(\frac{\lambda_L \Pi}{c} - 1\right) + (2-\alpha)(\lambda_L + \lambda_H)(\lambda_\star + \lambda_H) > 0.
\end{aligned}
$$

If $\lambda_\star \geq \frac{\alpha}{2-\alpha}\lambda_H$, the first term in the above inequality is nonnegative and the second term is positive from $\alpha < 1$ and $\lambda_L,\ \lambda_H,\ \lambda_\star > 0$. If $\lambda_\star < \frac{\alpha}{2-\alpha}\lambda_H$, by rearranging it and using $\pi = \frac{\lambda_L \Pi}{c}$, we can show that the above inequality is equivalent to (6.8). □