Managing a Project by Splitting it into Pieces

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Abstract

I study a dynamic principal-agent problem where there are two routes of completing a project: directly attacking it or splitting it into two subprojects. When the project is split, the principal can better monitor the agent by verifying the completion of the first subproject. However, the inflexible nature of this approach may generate inefficiencies. To mitigate moral hazard, the principal needs to commit to a deadline, which also affects her choice of project management strategy. The optimal contract is determined by the interplay of these three factors: monitoring, efficiency, and an endogenous deadline.

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1 Introduction

In project management, a work breakdown structure (WBS)—a step-by-step approach to complete projects—is widely used (Organ and Bottorff, 2022).[1] The applications of the WBS range from simple projects such as moving an office to complicated projects such as construction, engineering, or software development.[2] Although there are many advantages of employing the WBS (e.g., clarifying the goals, communicating better, etc.), a fundamental benefit is monitoring progress. As a project is broken down into smaller chunks, a manager can better audit a subordinate's progress, which may reduce the moral hazard issue. Nevertheless, decomposing a project into too small pieces may make the project rigid (Golany and Shtub, 2001). It may lead the manager to micromanage the project which in turn slows down project progress, i.e., generates inefficiencies. Thus, when a manager splits a project, she faces a tradeoff between *monitoring* and *efficiency*.

In this article, I introduce a stylized model to study this tension between efficiency and monitoring in breaking a project down.[3] Specifically, I consider a dynamic principal-agent model with two routes to achieving success. One is to attack the project head-on (the direct approach). This method requires a breakthrough which arrives at a low rate. The alternative route is to divide the project into two subprojects and complete them one by one (the sequential approach). This method requires two breakthroughs that each arrive at higher rates. Each breakthrough in this approach can be understood as the completion of a subproject. At the beginning of the game, the principal offers a contract specifying: a schedule dictating which approach to use, how much reward to pay upon success, and a cancellation policy. If the agent accepts the contract, then at each point in time, he chooses whether to work on the specified approach or shirk for private benefit. The principal does

[1] The PMBOK guide provides a formal definition of a work breakdown structure: "A hierarchical decomposition of the total scope of work to be carried out by the project team to accomplish the project objectives and create the required deliverables." (Project Management Institute, 2017)

[2] See Project Management Institute (2006) for further examples.

[3] I refer to splitting the project in two as "breaking it down" to conform with project management nomenclature. My model never involves a project breakdown in the sense of an accident or mishap.

not observe the agent's effort, generating the potential for moral hazard.

To highlight the tension between the direct and sequential approaches, I impose two key assumptions. First, the completion of the first subproject (in the sequential approach) is observable and contractually verifiable. This implies that the project breakdown can help the principal monitor progress. The second assumption is that the direct approach is more efficient than the sequential one. Hence, the sequential approach has an advantage in monitoring the agent but has a disadvantage in efficiency *vis a vis* the direct approach. To be clear, the probability of ultimate project completion is higher under the sequential approach, but the expected running time and concomitant operating cost dominate this effect.

In addition to these elements, there is a third important economic factor that affects the choice of methodology: a *deadline effect*. The principal needs to impose a deadline because, in the absence of a deadline, the agent could (and would) shirk forever without completing the project. Thus, the deadline is essential to mitigate moral hazard. This deadline generates a subtle difference between the incentive schemes implementing each approach. If the principal wants to induce the direct approach, the agent is compensated by an immediate payment upon project success. On the contrary, if the principal wants to induce the sequential approach, the principal slightly extends the deadline after observing the subproject completion and pays the agent only upon the successful completion of the entire project. Thus, the principal chooses between approaches by comparing the expected payoffs from the immediate payment scheme and the deadline extension scheme. When the principal uses the deadline extension scheme, the agent may work for a longer period of time. As noted above, this means that under the sequential approach the probability of ultimate success is higher, but the expected running cost is higher as well. This highlights another motive for the principal to employ the direct approach besides comparing the monitoring loss and efficiency gain. Specifically, when the deadline is close by, i.e., the probability of project completion is low, the principal may prefer saving the expected cost by choosing the direct approach rather than slightly raising the chance of project completion by employing the sequential approach.

To facilitate analysis, I begin by characterizing the optimal contract in the case where both approaches are equally efficient. Here, I can abstract from efficiency considerations and focus on the interaction between monitoring and the deadline. Monitoring gives a generic advantage to the sequential approach while the deadline effect—described in the previous paragraph—gives an advantage to the direct approach when the deadline is imminent. This suggests that the principal would prefer the sequential approach when the deadline is distant and the direct approach when it is near at hand. However, it is possible that the monitoring advantage is so strong that the principal chooses the sequential approach even near the deadline. On the contrary, it is also possible that the optimal deadline is short enough that the principal would want to choose the direct approach even at contract inception. Therefore, depending on the economic environment, different types of contracts may be optimal even in this simplified setting.

The main result of this analysis is that the form of the optimal contract depends crucially on the project return; i.e., the gross value to the principal from completing the project (given its operating cost). As described above, the sequential approach has a higher chance of project completion despite its higher expected cost. Because the project return must be scaled by the probability of successful completion, the sequential approach is preferred when the project return is large and the direct approach is preferred when the project return is small. Based on this observation, I show that the optimal contract is derived as follows:

(a) when the project return is low, the optimal deadline is short and the principal only chooses the direct approach;

(b) when the project return is high, monitoring is highly advantageous and the principal only chooses the sequential approach;

(c) when the project return is intermediate, there is a switching point such that the principal first chooses the sequential approach and then – absent a breakthrough – switches to the direct approach until the deadline is reached.

Next, I introduce the efficiency loss to the sequential approach, representing the idea that requiring milestones may slow down ultimate project development. When the efficiency loss is small enough, I show that a similar result as in the previous case holds: there are three regions of the project return that characterize the form of the optimal contract. In other words, the characterization of the optimal contract when the approaches are equally efficient is robust to a small efficiency loss. This is mainly because efficiency dominates monitoring only if the deadline is distant. I also consider the case where the efficiency loss is large. Here, even if the project return is moderately high, the principal prefers the direct approach over the sequential approach to avoid the efficiency cost. Nevertheless, if the project return is very high, the principal prefers to monitor it to some degree. In fact, there is a cutoff value for the project return such that the principal chooses the direct approach when the return is below the cutoff. Interestingly, if the return is above the threshold, then the principal begins by choosing the direct approach, switches to the sequential approach, then switches back to the direct approach until the deadline is reached.

My results are congruent with the observation that applied scientific research (e.g., development of a new drug, clinical trials) is typically staged. The magnitude of applied research projects is usually large, implying the superiority of the sequential approach. In contrast, the immediate value of basic research (e.g., chemistry, in-vitro experiments) is lower than applied research because "basic research is performed without thought of practical ends" (Bush, 1945).[4] My results suggest that the direct approach should be preferred for basic research because such projects tend to have lower returns than applied ones. For instance, the Research Project designation (R01) grant by the National Institute of Health (NIH) supports "a discrete, specified, circumscribed project" rather than a staged project.[5]

[4]Bush argues that although broad and basic studies seem to be less important than applied ones, they are essential to combat diseases because progress in the treatment "will be made as the result of fundamental discoveries in subjects unrelated to those diseases, and perhaps entirely unexpected by the investigator." However, since this article does not consider externalities, I abstract from this possibility and focus on the principal's return from the completed project.

[5]https://grants.nih.gov/grants/funding/r01.htm

The remainder of this article is organized as follows. Related literature is discussed below. Section 2 introduces the basic setup of the model and analyzes the first-best case. Section 3 provides heuristic arguments on the derivations of the value function and the optimal contract. Then, Section 4 and 5 characterize the optimal contracts for the cases with and without the efficiency losses from the project breakdown. Section 6 concludes. The formal analysis and the proofs are relegated to an Appendix.

Related Literature

There is a growing literature on contracting for multi-stage projects, e.g., Hu (2014); Green and Taylor (2016a); Wolf (2018); Moroni (2022). The most closely related study is Green and Taylor (2016a), who study a model in which multiple breakthroughs are needed to complete a project and in which an agent must be incentivized to exert unobservable effort. The sequential approach considered here comprises the baseline model with the tangible breakthrough in the working paper version of their paper (Green and Taylor, 2016b). However, the option to complete the project directly, which is not considered in their setup, allows the principal to face a choice problem between the two approaches. Moreover, this choice problem arises at every point in time. Therefore, the principal's problem becomes more complex from a dynamic perspective.

Another article that has a similar flavor is Carnehl and Schneider (2021). They consider a two-armed bandit problem where an arm requires one breakthrough (the doing arm) but another arm requires multiple breakthroughs (the thinking arm) to succeed. The arrival rates for the thinking arm are known to the agent whereas the arrival rate for the doing arm is not: the agent needs to infer whether the method is feasible or not by experimenting. The presence of this uncertainty is one key difference between their article and this one. Moreover, their main analysis focuses on a decision problem by a single agent whereas this article considers a principal-agent contracting setting. In addition, they have an exogenous deadline whereas the deadline is endogenously determined in this paper. Despite these differences, we share a common insight in that the chosen approaches may switch up to two times. However, the economic forces that drive choosing the thinking arm (or the sequential approach) are somewhat different. In their paper, as the agent pulls the doing arm and does not achieve any success, the belief that the initial method is feasible goes down. When the belief becomes sufficiently low, the thinking arm would be chosen because it may be more efficient than the doing arm. Thus, experimentation and efficiency are key driving forces for choosing the thinking arm. On the contrary, in this paper, the principal chooses the sequential approach to monitor the agent, not because beliefs about the direct approach have deteriorated.

This article is also somewhat related to the literature on monitoring in dynamic contracts, e.g., Orlov (2015); Piskorski and Westerfield (2016); Dilmé and Garrett (2019); Marinovic and Szydlowski (2019); Varas et al. (2020); Marinovic and Szydlowski (2020); Chen et al. (2020). In most of these papers, a monitoring process provides some information on the agent's current or past action. In this sense, the first breakthrough in the sequential approach can be considered as a monitoring device since it lets the principal know that the agent has worked. However, the completion of the first subproject gives more information than merely the agent's past actions. Before the subproject completion, the success requires one relatively hard breakthrough or two easier breakthroughs. After completing the subproject, it requires only one relatively easy breakthrough. Thus, the subproject completion is distinguished from standard monitoring processes since it also provides information about the subsequent procedure toward success.

The problem of choosing approaches is naturally related to multitasking in the sense that there are multiple options to pursue. In their seminal study, Holmstrom and Milgrom (1991) consider an economic situation where a production worker faces multiple tasks such as producing output and maintaining quality in a static environment. Dewatripont et al. (2000) and Laux (2001) also study multitasking problems in static environments. Several subsequent multitasking problems are also explored in dynamic settings (Manso, 2011; Capponi and Frei, 2015; Varas, 2017; Szydlowski, 2019). A common assumption in these studies is that each task has a different payoff structure.[6] For example, Manso (2011) studies a two-armed bandit problem in a simple agency model with two periods. The main assumption is that if the agent chooses to experiment (pulls the risky arm), the payoff is stochastic, and if the agent chooses to exploit (pulls the safe arm), the payoff is constant. In contrast, the two approaches in this article have the same ultimate payoff. The difference in the approaches is 'how' the main project is completed–via the direct approach or via the sequential approach.

This article is relevant to the literature studying complementary innovations, e.g., Green and Scotchmer (1995); Gilbert and Katz (2011); Bryan and Lemus (2017); Poggi (2021). Two subprojects in the sequential approach can be considered as 'perfect' complements in the sense that completing a subproject does not create any value but completing both of them does. However, to my knowledge, most of the studies in this literature focus on the problems with competing firms or a single decision maker, whereas this article studies an agency problem.

Last, from a technical viewpoint, the current article utilizes Poisson processes which are widely used to address dynamic moral hazard, e.g., Biais et al. (2010); Myerson (2015); Green and Taylor (2016a); Bonatti and Hörner (2017); Varas (2017); Sun and Tian (2017).

2 The Model

2.1 The Setting

A principal (she) hires an agent (he) to complete a (main) project. The project is conducted in continuous time and can be potentially operated over an infinite horizon: $t \in [0, \infty)$. Once the project is completed, the principal realizes a payoff $\Pi > 0$, dubbed the project return, and the game ends. While the project is running, the principal incurs an operating cost of

[6] The only study that does not have this assumption is Varas (2017). He considers a dynamic model with a Poisson process in which the agent chooses between a good project and a bad project. These projects look identical to the principal and yield the same payoff, but differ in the rate of failure.

$c > 0$ per unit of time. The principal is assumed to have an infinite amount of resources to fund the project while the agent is protected by limited liability, i.e., the principal can only transfer nonnegative rewards to the agent.[7] The principal and the agent are both risk-neutral and patient, i.e., they do not discount the future.[8] Both players have outside options of zero.

There are two routes to achieving success. One is to attack the project directly, and I accordingly call this the *direct approach*. Another way is to break the main project into two subprojects and I call this the *sequential approach*. Completing the (first) subproject does not have any independent value for the principal or the agent.[9] However, the completion of the subproject is observable by both players and contractually verifiable by a court. Thus observing the completion of the subproject can be considered a type of monitoring.

At time 0, the principal offers the agent a contract consisting of the deadlines, at which the project is terminated; the reward schedules upon project completion; the approaches to be taken; and the agent's recommended effort. See Section A for the formal definition of the contract. The principal can fully commit to these contractual terms. Note that the contract is contingent on the subproject completion. When the subproject has not been completed, at each point in time t, the contract specifies which approach to take: the direct approach ($a_t = 1$) or the sequential approach ($a_t = 0$). The agent allocates his 1 unit of effort to working ($\tilde{b}_t \in [0, 1]$), and shirking ($1 - \tilde{b}_t$).[10] The allocation of efforts is unobservable to the principal. Then, at time t, the project is completed at the arrival rate $\lambda_D a_t \tilde{b}_t$ (and the agent receives the reward R_t), the subproject is completed at the rate $\lambda_S(1 - a_t)\tilde{b}_t$, and the agent receives $\phi(1 - \tilde{b}_t)$ as a private flow benefit from shirking. I assume that the marginal private benefit ϕ is positive but less than the principal's flow operating cost c. It is easier to complete the subproject than to complete the main project, i.e., λ_S is greater than λ_D. If neither the main project nor the subproject is completed by the deadline, the project is terminated.

[7] See Remark 2 for further discussion of limited liability.

[8] I explore the case of a positive discount factor in Online Appendix Section OA.3.

[9] In Online Appendix Section OA.2, I consider the case where the completion of the first subproject raises outside options for the agent and the principal.

[10] The agent's choice will be denoted with tilde, whereas the principal's choices are not.

When the subproject succeeds, the deadline and the reward schedule are updated. In this case, the agent only needs to complete one more subproject (with the same arrival rate) to make the entire project succeed. Thus, the main project is completed at the rate $\lambda_S \tilde{b}_t$.[11]

2.2 The First-Best Case

As a benchmark, I consider the case where the agent's effort is observable to the principal, namely the first-best situation. Since the benefit from shirking is less than the flow cost, it is optimal for the principal to make the agent work in this case.

To see which approach is more efficient, we need to compare the expected payoffs of both approaches. Suppose that the principal takes the direct approach indefinitely, i.e., until the project is completed. Then, the probability distribution of the date of project completion (τ_m) is given by $\lambda_D e^{-\lambda_D \tau_m}$. From this, we can derive the expected payoff of (indefinitely employing) the direct approach as follows:

$$\int_0^\infty (\Pi - c \cdot \tau_m)\, \lambda_D e^{-\lambda_D \tau_m} d\tau_m = \Pi - \frac{c}{\lambda_D}.$$

Since it does not have a deadline and the agent never shirks, the project will be surely completed and the principal will receive Π (recall that both parties do not discount). The second term comes from the cumulative costs: since the expected duration of the project is $1/\lambda_D$, the expected cost is c/λ_D.

Next, consider the case where the principal takes the sequential approach indefinitely. In this case, in addition to τ_m, we need to consider the date of the first subproject completion– denoted by τ_s. Conditional on the completion of the subproject at the date τ_s, the probability distribution of τ_m is $\lambda_S e^{-\lambda_S(\tau_m - \tau_s)}$ for $\tau_m > \tau_s$ and 0 otherwise. Since the marginal probability distribution of τ_s is $\lambda_S e^{-\lambda_S \tau_s}$, the marginal probability distribution of τ_m can be derived as

[11] I assume that two subprojects in the sequential approach are symmetric. In Online Appendix Section OA.1, I relax this assumption and analyze the case where the two subprojects are asymmetric.

follows:

$$\int_0^{\tau_m} \lambda_S e^{-\lambda_S(\tau_m-\tau_s)} \cdot \lambda_S e^{-\lambda_S\tau_s} d\tau_s = \lambda_S^2 \tau_m e^{-\lambda_S \tau_m}.$$

Then, the expected payoff of (indefinitely employing) the sequential approach is given as follows:

$$\int_0^{\infty} (\Pi - c \cdot \tau_m)\, \lambda_S^2 \tau_m e^{-\lambda_S\tau_m} = \Pi - \frac{2c}{\lambda_S}.$$

Since the expected duration for each step is $1/\lambda_S$, the expected cost is $2c/\lambda_S$ in this case.

I assume that the sequential approach is profitable: $\Pi > 2c/\lambda_S$. I also assume that the sequential approach is not more efficient than the direct approach, i.e., monitoring may harm the efficiency of the project: $\lambda_D \geq \lambda_S/2$. Together with the previous assumption, this implies that the direct approach is also profitable: $\Pi > c/\lambda_D$.

I introduce a parameter $\eta \equiv \lambda_S/\lambda_D - 1$, which measures the efficiency of the sequential approach. If η is equal to one, the sequential approach is equally efficient as the direct approach. As η decreases, the efficiency loss from monitoring increases. In Section 4 and 5, I derive the optimal contract respectively for the case of no efficiency loss from monitoring ($\eta = 1$) and the case of efficiency loss from monitoring ($\eta < 1$), and show that the form of the optimal contract depends crucially on η.

Remark 1. When the sequential approach is less efficient than the direct approach, even if the principal is allowed to switch methods, she will stick to the direct approach in the first-best case. On the other hand, when the sequential approach is equally efficient to the direct approach, she may switch back and forth.

Remark 2. If the agent is not protected by limited liability, the following contract will allow the principal to achieve the first-best outcome. The principal does not use any deadline and always chooses the direct approach. Then, at each point of time, if the project is not completed, she charges ϕ to the agent, i.e., the agent pays ϕ to the principal. If the project is completed, the principal pays $\phi/\lambda_D + \epsilon$ to the agent for some small $\epsilon > 0$. Observe that the agent will always work because the instantaneous payoff from working is $\lambda_D(\phi/\lambda_D + \epsilon) - \phi$

whereas that from shirking is zero. Then, the agent's expected payoff is ϵ, and the principal's expected payoff is $\Pi - c/\lambda_D - \epsilon$. By sending ϵ to zero, the principal is able to achieve the first-best outcome.

3 Minimum-Incentive Contracts and Deadlines

In this section, I provide heuristic arguments on the optimal contract derivation. See Appendix A and B for the formal analysis.

3.1 The Value Function

Following the standard approach in the dynamic contract literature, I consider the agent's promised utility as a state variable and write a contract recursively (see, e.g., Spear and Srivastava, 1987). For a contract Γ, let $P_0(\Gamma)$ and $U_0(\Gamma)$ be the expected payoffs of the principal and the agent at time 0 when the agent adheres to the recommended effort specified in the contract. The core of the analysis is the derivation of the principal's value function, denoted by $V(u)$, which represents her maximized expected utility $P_0(\Gamma)$ subject to the promise-keeping constraint $U_0(\Gamma) = u$ and the incentive compatibility condition—which will be demonstrated in Section 3.2. If a contract Γ satisfies $P_0(\Gamma) = V(u)$ and $U_0(\Gamma) = u$, Γ is said to *implement* a pair of expected payoffs $(u, V(u))$. Once the value function is characterized, the principal solves

$$\bar{u} \equiv \arg\max_{u \geq 0} V(u). \tag{MP}$$

Then, the optimal contract is the contract that implements $(\bar{u}, V(\bar{u}))$. In the rest of this section, I describe how to derive the value function $V(u)$.

I begin by solving the principal's problem given that the subproject is completed at time t. I also consider the agent's promised utility u_S^t as a state variable. Upon the subproject completion at time t, the contract is updated to $\hat{\Gamma}^t$. Define the value function $V_S(u_S^t)$ as the function that maximizes the principal's expected utility $\hat{P}^t(\hat{\Gamma}^t)$ subject to the promise-keeping constraint $\hat{U}^t(\hat{\Gamma}^t) = u_S^t$ and the incentive compatibility condition. Since this case only requires one more breakthrough, it is identical to the single-stage benchmark of Green and Taylor (2016a), thus I can directly use their results. They show that $V_S(u_S^t)$ is the expected profit from running the project for u_S^t/ϕ units of time net of u_S^t:

$$V_S(u_S^t) = \left(\Pi - \frac{c}{\lambda_S}\right)\left(1 - e^{-\frac{\lambda_S}{\phi}u_S^t}\right) - u_S^t. \tag{3.1}$$

In addition, they show that this can be implemented by a contract with a deadline $t + u_S^t/\phi$ and a linearly diminishing reward schedule $\{R_s^t\}_{t \le s \le t+u_S^t/\phi}$ where

$$R_s^t = u_S^t + \phi/\lambda_S - \phi(s-t) \tag{3.2}$$

for all $t \le s \le t + u_S^t/\phi$. The intuition is that when the agent's promised utility is u_S^t, the principal can incentivize the agent to work at most u_S^t/ϕ units of time. This is because if the principal requires him to work more than u_S^t/ϕ units of time, he can achieve higher payoffs than the promised utility by shirking.

3.2 The Agent's Problem

Now consider the agent's problem when the subproject has not been completed. Suppose that the promised utility is u_t at some time t. Under the direct approach, if the agent works for a small interval of time $[t, t+dt)$, the breakthrough occurs and the agent receives the reward R_t with a probability $\lambda_D dt$. However, in this event, he loses the continuation utility, thus, the expected payoff of working is $\lambda_D(R_t - u_t)dt$. On the other hand, if he shirks, his payoff is ϕdt. Therefore, to induce the agent to work, R_t should be greater than or equal to $u_t + \phi/\lambda_D$. Next, under the sequential approach, the agent is compensated in the form of the promised utility upon the subproject completion. Thus, the expected payoff of working for $[t, t+dt)$ is $\lambda_S(u_S^t - u_t)dt$. Then, to induce the agent to work, u_S^t should be greater than or equal to $u_t + \phi/\lambda_S$. To sum up, $R_t \geq u_t + \phi/\lambda_D$ and $u_S^t \geq u_t + \phi/\lambda_S$ serve as incentive compatibility conditions for the direct approach ($a_t = 1$) and the sequential approach ($a_t = 0$) respectively.

3.3 The Principal's Problem

Next, consider the principal's problem. At each point of time t, the principal needs to choose which approach to take (a_t), the reward upon project completion (R_t), the updated contract upon the subproject completion ($\tilde{\Gamma}^t$), and the agent's recommended effort (b_t). If the agent shirks, neither the whole project nor the subproject can be completed, but the flow cost incurs. Thus, the principal would recommend the agent to work unless the contract is terminated. Note that when the subproject is completed, the agent only cares about his updated promised utility, u_S^t, which can eventually determine the updated contract and the principal's expected payoff $V_S(u_S^t)$. Thus, the principal's problem reduces to the choice of a_t, R_t, and u_S^t subject to the incentive compatibility conditions.

We can naturally guess that the incentive compatibility conditions bind in the optimal contract, i.e., the principal provides the minimum incentive for the agent to work.[12] I now show that under the minimum-incentive contract, (i) R_t linearly diminishes over time; (ii) the deadline is extended by $1/\lambda_S$ when the subproject is completed. First, observe that in the absence of any completion, the agent's promised utility is consumed at the same rate with the benefit from shirking: $du/dt = \dot{u}_t = -\phi$. This is because the agent is indifferent between shirking and working under the minimum incentives and it gives the following promised

[12]In the formal analysis, I show that it is indeed optimal for the principal to provide the minimum incentive to the agent. To do this, I use the 'guess and verify' method which is widely used in continuous-time analysis. That is, I first guess the principal's value function by letting her provide the minimum incentives to the agent, then verify that the principal does not have an incentive to deviate from it. The verification parts of the proof are long and tedious, so they are relegated to the latter part of the appendix (Appendix D). Specifically, I provide proof that the principal does not deviate from providing the minimum incentives in D.2.

utility dynamics:

$$u_t = u_{t+dt} + \phi dt = u_t + \dot{u}_t dt + \phi dt \quad \Longrightarrow \quad 0 = \dot{u}_t + \phi.$$

Note that $u_t = u_0 - \phi t$, which means that if any completion has not been made by time t, the agent is promised to the continuation utility $u_0 - \phi t$. Under the direct approach, to make the incentive compatibility condition bind, $R_t = u_t + \phi/\lambda_D = u_0 - \phi t + \phi/\lambda_D$, which linearly diminishes over time. Next, if the completion has not been made by u_0/ϕ, the promised utility is equal to the agent's outside option 0, thus, the contract is terminated, or equivalently, the deadline of the contract is u_0/ϕ. Under the sequential approach, to make incentive compatibility bind, $u_S^t = u_t + \phi/\lambda_S$. Because the updated deadline is $t + u_S^t/\phi = u_0/\phi + 1/\lambda_S$, the deadline is extended by $1/\lambda_S$. In addition, when the original deadline is T ($u_0 = \phi T$), by (3.2) and $u_S^t = u_0 - \phi t + \phi/\lambda_S$, the reward upon project completion at time t is determined as follows:

$$R_s^t = u_0 + 2\phi/\lambda_S - \phi s = \phi\left(T + 2/\lambda_S - s\right),$$

i.e., the reward is linearly diminishing over time.

The remaining step is to determine which approach to take. If the principal chooses the direct approach for $[t, t+dt)$, the expected payoff is

$$\begin{aligned}
&\lambda_D dt(\Pi - R_t) + (1 - \lambda_D dt)V(u_{t+dt}) - cdt \\
&= \lambda_D dt(\Pi - R_t) + (1 - \lambda_D dt)\left(V(u_t) + V'(u_t)\dot{u}_t dt\right) - cdt.
\end{aligned}$$

When dt is small enough, we can take $dt^2 \to 0$, then, the above equation can be rewritten as follows:

$$V(u_t) + \left\{\lambda_D\left(\Pi - R_t - V(u_t)\right) - \phi V'(u_t) - c\right\} dt.$$

Likewise, if the principal chooses the sequential approach for $[t, t+dt)$, the expected payoff can be written as follows:

$$V(u_t) + \left\{\lambda_S(V(u_S^t) - V(u_t)) - \phi V'(u_t) - c\right\} dt.$$

By plugging in $R_t = u_t + \phi/\lambda_D$ and $u_S^t = u_t + \phi/\lambda_S$, the principal determines which approach to take as follows:

$$a_t = \begin{cases} 1, & \text{if } \alpha_D(u_t) > \alpha_S(u_t), \\ 0, & \text{if } \alpha_D(u_t) < \alpha_S(u_t), \end{cases} \tag{3.3}$$ [13]

where

$$\begin{aligned} \alpha_D(u) \equiv\ & \lambda_D \Pi - \lambda_D(V(u) + u) - c - \phi, \\ \alpha_S(u) \equiv\ & \lambda_S \left(V_S \left(u + \frac{\phi}{\lambda_S} \right) + u + \frac{\phi}{\lambda_S} \right) - \lambda_S(V(u) + u) - c - \phi. \end{aligned}$$

3.4 The Optimal Approach at the Deadline

I now explore how the principal chooses the approach at the deadline. Note that $V(0) = 0$ because the contract is terminated when the agent's promised utility is 0. Then, at the deadline, the principal chooses which approach to take by comparing the following two values:

$$\begin{aligned} \alpha_D(0) &= \lambda_D \Pi, \\ \alpha_S(0) &= \lambda_S \left(V_S \left(\frac{\phi}{\lambda_S} \right) + \frac{\phi}{\lambda_S} \right) = \lambda_S \left(\Pi - \frac{c}{\lambda_S} \right) (1 - e^{-1}) \\ &= (1 - e^{-1})(\eta + 1)\lambda_D \Pi - (1 - e^{-1})c. \end{aligned}$$

This can be interpreted as follows. If the direct approach is chosen, the project is completed at rate λ_D, and the contract is terminated otherwise. If the sequential approach is chosen,

[13] If equality holds, the principal is indifferent between the two approaches.

the subproject is completed at rate λ_S. If the subproject is completed, the deadline will be extended by $1/\lambda_S$ and the agent will try to complete the remaining subproject at rate λ_S. This deadline extension has a mixed effect: (i) it may increase the chance of project completion if $(1-e^{-1})(\eta+1)\lambda_D > \lambda_D$, or equivalently, $\eta > 1/(e-1)$; (ii) it will incur running cost ($(1-e^{-1})c$ in expectation) which the principal would not pay under the direct approach. Based on this comparison, the principal chooses which approach to take at the deadline. First, when $\eta \leq 1/(e-1)$, due to high efficiency loss from the project breakdown, the deadline extension does not even deliver a higher chance of project completion, thus, the direct approach will be preferred at the deadline, i.e., $\alpha_D(0) > \alpha_S(0)$ always holds. Second, when $\eta > 1/(e-1)$, the principal needs to take into account the benefit and the cost from the deadline extension. In this case, we have

$$\alpha_D(0) > \alpha_S(0) \quad \Longleftrightarrow \quad \Pi < \frac{e-1}{\eta(e-1)-1} \cdot \frac{c}{\lambda_D} \equiv \Pi_S(\eta). \tag{3.4}$$

Therefore, at the deadline, the principal would split the project if and only if the efficiency loss from project breakdown is low ($\eta > 1/(e-1)$) and the project return is high ($\Pi \geq \Pi_S(\eta)$).

4 Optimal Contracts under No Efficiency Loss

In this section, I characterize the optimal contract under the assumption that there is no efficiency loss from breaking down the project, i.e., $\eta = 1$.

If the principal uses the sequential approach, she can monitor the agent's intermediate progress. On the other hand, if the principal chooses the direct approach, the principal cannot observe progress until the main project is done. Therefore, the sequential approach has a comparative advantage in supervising the agent and can potentially lessen the moral hazard problem. Moreover, in this case, the sequential approach is even more attractive due to no efficiency loss. However, as argued in the previous section, when the deadline is close by, the direct approach can be more appealing despite the monitoring advantage.

Based on this intuition, I construct three types of minimum-incentive contracts which involve at most one switch of the employed approach.

Definition 4.1. (a) A contract is called a *direct-only contract with a deadline* T if (i) the direct approach is employed up to the deadline T, (ii) the reward upon project completion at time t is $R_t = \phi(T - t + 1/\lambda_D)$, and (iii) the contract is terminated if the project is not completed by the deadline T;

(b) A contract is called a *sequential-only contract with a deadline* T if (i) the sequential approach is employed up to the deadline T, (ii) the contract is terminated if the subproject is not completed by the deadline T, (iii) when the subproject is completed before T, the contract is extended to $T + 1/\lambda_S$, i.e., the contract is terminated if the project is not completed by $T + 1/\lambda_S$, and (iv) the reward upon project completion at time t is $R_t = \phi(T - t + 2/\lambda_S)$;

(c) A contract is called a *contract with a switch from the sequential approach to the direct approach at* S *and a deadline* T if (i) the sequential approach is employed up to the intermediate deadline S, (ii) when the subproject is completed before S, the contract is extended to $T + 1/\lambda_S$ and the reward upon project completion at time t is $R_t = \phi(T - t + 2/\lambda_S)$, (iii) if the subproject is not completed by S, the direct approach is employed up to the deadline T and the reward upon project completion at time t is $R_t = \phi(T - t + 1/\lambda_D)$, and (iv) the contract is terminated if the project is not completed by the deadline T.

In the following proposition, I derive the benchmark value functions that can be implemented by the above contracts. The proof is relegated to Appendix B.3.1.

Proposition 4.1. *The following statements hold:*

(a) A direct-only contract with the deadline u/ϕ implements a pair of expected payoffs of the principal and the agent $(V^d(u), u)$ where

$$V^d(u) = \left(\Pi - \frac{c}{\lambda_D}\right)\left(1 - e^{-\frac{\lambda_D}{\phi}u}\right) - u. \tag{4.1}$$

(b) When $0 \leq u_1 \leq u$, a contract with a switch from the sequential approach to the direct approach at $(u - u_1)/\phi$ and the deadline u/ϕ implements $(V^{ds}(u|u_1), u)$ where

$$\begin{aligned} V^{ds}(u|u_1) = &\left(\Pi - \frac{2c}{\lambda_S}\right)\left(1 - e^{\frac{\lambda_S}{\phi}(u_1-u)}\right) + (V^d(u_1) + u_1)e^{\frac{\lambda_S}{\phi}(u_1-u)} \\ &- \left(\Pi - \frac{c}{\lambda_S}\right)\frac{\lambda_S}{\phi}(u - u_1)e^{-\frac{\lambda_S}{\phi}u-1} - u. \end{aligned} \tag{4.2}$$

(c) A sequential-only contract with the deadline u/ϕ implements $(V^{ds}(u|0), u)$.

Next, I show that the principal's value function can be characterized by the above benchmark value functions. From (3.4), the sequential approach is preferred at the deadline if and only if $\Pi \geq \Pi_S(1)$. In this case, we can guess that the principal will use a sequential-only contract, i.e., the principal's value function will take the form $V^{ds}(u|0)$. On the other hand, if Π is less than $\Pi_S(1)$, we can also guess that there exists a cutoff promised utility level $u_1 > 0$ such that the direct approach is employed for $0 \leq u < u_1$ and the sequential approach is employed for $u > u_1$. That is, the value function would take forms of (4.1) when $0 \leq u < u_1$ and (4.2) when $u > u_1$. The following proposition confirms the conjecture. The proof is relegated to Section D.

Proposition 4.2. *Suppose that η is equal to 1. Then, the principal's value function is characterized as follows.*

(a) $\Pi = .8,\ \eta = 1$ (b) $\Pi = .85,\ \eta = 1$ (c) $\Pi = 1.3,\ \eta = 1$

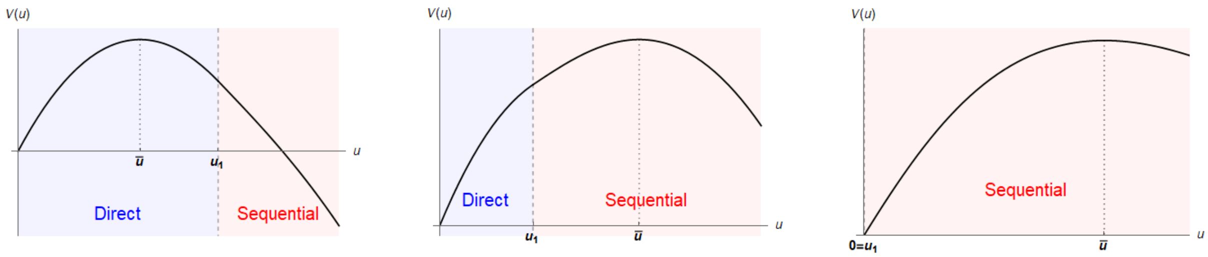

Figure 1: Value functions when there is no efficiency loss
$(\lambda_D = 2,\ \lambda_S = 4,\ c = 1,\ \phi = .5)$

(a) When $\Pi_S(1) > \Pi > \frac{c}{\lambda_D} = \frac{2c}{\lambda_S}$*, there exists* $u_1 > 0$ *such that*

$$V(u) = \begin{cases} V^d(u), & \text{if } 0 \leq u \leq u_1, \\ V^{ds}(u|u_1), & \text{if } u_1 < u. \end{cases} \tag{4.3}$$

(b) When $\Pi \geq \Pi_S(1)$*,* $V(u) = V^{ds}(u|0)$ *for all* $u \geq 0$*.*

This result is illustrated in Figure 1. I set parameters $\lambda_D = 2,\ \lambda_S = 4,\ c = 1,\ \phi = .5$, then plot the graphs of the value functions for $\Pi = .8,\ .85$, and 1.3. The horizontal axis represents the agent's promised utility u and the vertical axis represents the principal's value function $V(u)$. Recall that $\dot{u} = -\phi$ when the principal employs the minimum incentive contract. Thus, time flows along the horizontal axis from the right to the left. Observe that $\Pi_S(1) \approx 1.196$ and $\Pi < \Pi_S(1)$ in Figure 1a and 1b. In these figures, the level u_1 represents the promised utility level at which the approach switches. If u is in the blue region, i.e., below u_1, it is optimal for the principal to employ the direct approach ($a = 1$). If u is in the red region, i.e., above u_1, it is optimal for the principal to use the sequential approach ($a = 0$). Also note that $\Pi > \Pi_S(1)$ in Figure 1c. Thus, the sequential approach is always chosen over the direct approach.

Now that I have characterized the contract that maximizes the principal's expected payoff under the constraint that the agent's promised utility is equal to u, the next step is to pin down the optimal initial promised utility level. Recall that the optimal contract is the one that implements $(\bar{u}, V(\bar{u}))$ where $\bar{u}$ is the solution to (MP). this will determine where the principal will start the contract in Figure 1 and how long the deadline will be.

The feasibility of the project depends on whether $\bar{u}$ is greater than or equal to 0. When $\bar{u}$ is equal to zero, the principal's expected payoff is maximized at $u = 0$. In this case, it is optimal for the principal not to initiate the contract in the first place, i.e., the project is infeasible. This occurs when the project return is very low. In Appendix C.1, for any efficiency level η, I show that the project is feasible if and only if Π is greater than $\Pi_F \equiv (c + \phi)/\lambda_D$. Intuitively, at the deadline, a feasible project has to be profitable at least with the direct approach: $0 < \alpha_D(0) = \lambda_D \Pi - c - \phi$, or equivalently, $\Pi > \Pi_F$.

Next, the form of the optimal contract depends on whether $\bar{u}$ is greater than u_1 or not. For example, the value functions in Figure 1a and 1b both involve a switching point u_1, however, $\bar{u}$ is greater than u_1 in Figure 1a and less than u_1 in Figure 1b. Thus, the optimal contracts are a direct-only contract in Figure 1a and a contract with a switch from the sequential approach to the direct approach in Figure 1b. In Appendix C.2, I show that there exists a threshold $\Pi_D(1)$ such that $\bar{u} > u_1$ if and only if $\Pi > \Pi_D(1)$ (Lemma C.1).

Last but not least, as argued in Proposition 4.2, the form of the optimal contract depends on whether Π is greater than $\Pi_S(1)$ or not. This determines whether the direct approach is ever employed in the optimal contract. Based on the above arguments, the following theorem characterizes the optimal contract and shows that the project return Π determines which case will be applied. The formal proof is presented in Appendix C.3.

Theorem 1. *Suppose that there is no efficiency loss from monitoring ($\eta = 1$). There exists a threshold $\Pi_D(1)$ such that $\Pi_S(1) > \Pi_D(1) > \Pi_F \equiv (c + \phi)/\lambda_D$ and the optimal contract is determined as follows:*

(a) when $\Pi \leq \Pi_F$, the project is infeasible;

(b) when $\Pi_D(1) \geq \Pi > \Pi_F$, $(\bar{u}, V(\bar{u}))$ is implemented by a direct-only contract with a deadline $\bar{u}/\phi$;

(c) *when* $\Pi_S(1) > \Pi > \Pi_D(1)$*, there exists* $u_1 \in (0, \bar{u})$ *such that* $(\bar{u}, V(\bar{u}))$ *is implemented by a contract with a switch from the sequential approach to the direct approach at* $(\bar{u} - u_1)/\phi$ *and a deadline* $\bar{u}/\phi$*;*

(d) *when* $\Pi \geq \Pi_S(1)$*,* $(\bar{u}, V(\bar{u}))$ *is implemented by a sequential-only contract with a deadline* $\bar{u}/\phi$*.*

This theorem provides a clear interpretation of the form of the optimal contract with respect to the project return. When the return is very low ($\Pi \leq \Pi_F$), the project is not feasible and it is optimal not to initiate the contract. As the return grows and Π is now in $(\Pi_F, \Pi_D(1))$, the project becomes feasible and the principal employs the direct approach near the deadline. In addition, the optimal length of the contract is too short to employ the sequential approach, thus, the direct-only contract is optimal. Next, when Π is in $(\Pi_D(1), \Pi_S(1))$, the optimal length of the contract is long enough to begin the contract with the sequential approach. In addition, the return is not too high so the direct approach will be preferred near the deadline. Thus, the optimal contract involves a switch from the sequential approach to the direct approach. Last, when Π is greater than $\Pi_S(1)$, the sequential approach is preferred even at the deadline, thus, the sequential-only contract is optimal.

Remark 3. A mixture of contracts also generates another contract. For example, a contract with a soft deadline—randomly terminating the agent after reaching the soft deadline—as in Green and Taylor (2016a) can be represented by a mixture of two contracts defined here. However, a mixed contract cannot improve the one characterized above. This follows because the value functions derived in Proposition 4.2 are shown to be concave (see Appendix D.3). Consider a set of contracts $\{\Gamma_i\}_{1 \leq i \leq n}$ where the agent's expected utility of Γ_i is u_i and the weight is w_i with $\sum_{i=1}^n w_i = 1$ and $\sum_{i=1}^n u_i w_i = u$. The principal's expected payoff from this mixture is $\sum_{i=1}^n w_i P_0(\Gamma_i)$ and the agent's expected utility is u. By concavity, we have $V(u) \geq \sum_{i=1}^n w_i V(u_i)$. In addition, we have $V(u_i) \geq P_0(\Gamma_i)$ for all $1 \leq i \leq n$ because $V(u_i)$ is the principal's maximized expected profit given that the agent's expected payoff is u_i.

Thus, $V(u)$ is greater than or equal to the expected payoff of the mixed contract. Hence, any mixed contract cannot improve the characterized contract.

5 Optimal Contracts under Efficiency Loss

I now introduce an efficiency loss from breaking down the project, that is, η is less than 1. In this case, we need to consider efficiency as another economic force determining the optimal contract in addition to monitoring and deadlines. I begin by showing that the value function from the previous section cannot be supported as a value function in the efficiency loss case when the promised utility is high enough. Assume the contrary, i.e., $V^{ds}(\cdot|u_1)$ is the principal's value function for all $u \geq u_1$. Note that $\lim_{u\to\infty} V^{ds}(u|u_1) + u = \Pi - \frac{2c}{\lambda_S}$ and $\lim_{u\to\infty} V_S\left(u + \frac{\phi}{\lambda_S}\right) + u + \frac{\phi}{\lambda_S} = \Pi - \frac{c}{\lambda_S}$. Also observe that

$$\begin{aligned}
\lim_{u\to\infty} \alpha_D(u) =& \lambda_D \Pi - \lambda_D \left(\Pi - \frac{2c}{\lambda_S}\right) - (c+\phi) = \frac{1-\eta}{1+\eta} c - \phi, \\
\lim_{u\to\infty} \alpha_S(u) =& \lambda_S \left(\Pi - \frac{c}{\lambda_S}\right) - \lambda_S \left(\Pi - \frac{2c}{\lambda_S}\right) - (c+\phi) = -\phi.
\end{aligned}$$

From $\eta < 1$, $\alpha_D(u) > \alpha_S(u)$ for some high enough u, i.e., the (more efficient) direct approach is always preferred if u is high enough. Thus, $V^{ds}(u|u_1)$ cannot constitute the value function.

Intuitively, for longer time horizons, the sum of expected payoffs for both players from the contract converges to the first-best contract, that is, efficiency determines which approach should be chosen. Since we focus on the case where the sequential approach is less efficient than the direct approach, the principal would choose the direct approach when the deadline is far off. Based on this intuition, I construct a contract that provides a minimum incentive not to shirk and potentially has two switches from the direct approach to the sequential approach, then to the direct approach again.

Definition 5.1. A contract is called a*contract with two switches at S_1 and S_2 and a deadline T* if (i) the direct approach is employed up to the first intermediate deadline S_1 and the reward upon project completion is $R_t = \phi(T - t + 1/\lambda_D)$, (ii) if the project is not completed by S_1, the sequential approach is employed up to the second intermediate deadline S_2, (iii) when the subproject is completed, the deadline is extended to $T + 1/\lambda_S$ and the reward upon project completion at time t is $R_t = \phi(T - t + 2/\lambda_S)$, (iv) if the subproject is not completed by S_2, the direct approach is employed up to the deadline T and the reward upon project completion at time t is $R_t = \phi(T - t + 1/\lambda_D)$, (v) the contract is terminated if the project is not completed by the deadline T.

The following proposition derives another benchmark value function that can be implemented by the above contract. The proof is relegated to Appendix B.3.2.

Proposition 5.1. *When $0 \le u_1 < u_2 \le u$, a contract with two switches at $(u - u_2)/\phi$ and $(u - u_1)/\phi$ and the deadline u/ϕ implements $(V^{dsd}(u|u_1, u_2), u)$ where*

$$V^{dsd}(u|u_1, u_2) = \left(\Pi - \frac{c}{\lambda_D}\right)\left(1 - e^{\frac{\lambda_D}{\phi}(u_2 - u)}\right) + (V^{ds}(u_2|u_1) + u_2)e^{\frac{\lambda_D}{\phi}(u_2 - u)} - u. \tag{5.1}$$

As a next step, I show that the principal's value function can be characterized by the combination of (4.1), (4.2) and (5.1). Recall that the sequential approach is preferred at the deadline if and only if $\eta > 1/(e-1)$ and $\Pi \ge \Pi_S(\eta)$ (Section 3.4). In this case, for a low level of the agent's promised utility u, the principal's value function will take the form of $V^{ds}(u|0)$. However, as argued above, when u is large enough, the direct approach will be employed. Then, we can guess that there exists a cutoff promised utility level $u_2 > 0$ such that the principal's value function would take a form of $V^{ds}(u|0)$ for $0 \le u < u_2$ and $V^{dsd}(u|0, u_2)$ for $u > u_2$.

In the remaining cases, the direct approach is preferred at the deadline. Unlike in the no efficiency loss case, due to the efficiency loss from the project breakdown, it is possible that the sequential approach is never chosen in the value function characterization. Specifically, there exists $\Pi_M(\eta) \ge 2c/\lambda_S$ such that the sequential approach is never employed if and only if $\Pi \leq \Pi_M(\eta)$ (Lemma D.2). Thus, when Π is less than or equal to $\Pi_M(\eta)$, the principal's value function is $V^d(u)$. On the other hand, when Π is greater than $\Pi_M(\eta)$, the sequential approach will be employed at some point, but the direct approach is preferred when u is very low (due to the deadline) or very high (due to efficiency). Therefore, we can guess that there exist $u_2 > u_1 > 0$ such that the value function takes a form of $V^d(u)$ when $u < u_1$, $V^{ds}(u|u_1)$ when $u_1 < u < u_2$, and $V^{dsd}(u|u_1, u_2)$ when $u_2 < u$. The following proposition confirms the conjecture. The proof is relegated to Section D.

Proposition 5.2. *Suppose that η is less than 1. Then, there exists $\Pi_M(\eta)$ such that the following statements hold.*

(a) *When $\Pi_M(\eta) \geq \Pi > \frac{2c}{\lambda_S}$, $V(u) = V^d(u)$ for all $u \geq 0$.*

(b) *Suppose that one of the following hold: (i) $\eta \leq 1/(e-1)$ and $\Pi > \Pi_M(\eta)$; or (ii) $1/(e-1) < \eta < 1$ and $\Pi \in (\Pi_M(\eta), \Pi_S(\eta))$. There exist $u_2 > u_1 > 0$ such that*

$$V(u) = \begin{cases} V^d(u), & \text{if } 0 \leq u \leq u_1, \\ V^{ds}(u|u_1), & \text{if } u_1 < u \leq u_2, \\ V^{dsd}(u|u_1, u_2), & \text{if } u_2 < u. \end{cases} \tag{5.2}$$

(c) *When $1/(e-1) < \eta < 1$ and $\Pi \geq \Pi_S(\eta)$, there exists $u_2 > 0$ such that*

$$V(u) = \begin{cases} V^{ds}(u|0), & \text{if } 0 \leq u \leq u_2, \\ V^{dsd}(u|0, u_2), & \text{if } u_2 < u. \end{cases} \tag{5.3}$$

This result is illustrated in the graphs in Figure 2 and 3. I set parameters $\lambda_D = 2$, $c = 1$, $\phi = .5$, and $\lambda_S = 3.8$ (or equivalently $\eta = .9$) for Figure 2 and $\lambda_S = 3.1$ (or equivalently $\eta = .55$) for Figure 3. Note that $.9 > 1/(e-1) \approx .582$. Then, there are three different cases for Figure 2: (a) when the project return is low, the sequential approach is never employed in

(a) $\Pi = .8,\ \eta = .9$ (b) $\Pi = .95,\ \eta = .9$ (c) $\Pi = 2,\ \eta = .9$

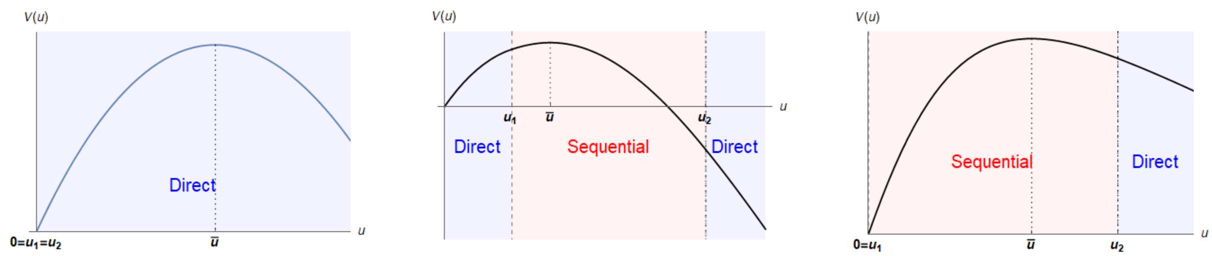

Figure 2: Value functions when the efficiency loss is small
$(\lambda_D = 2,\ \lambda_S = 3.8,\ c = 1,\ \phi = .5)$

the characterized value function (Figure 2a); (b) when the project return is intermediate, the sequential approach is employed for intermediate u ($u \in (u_1, u_2)$) and the direct approach is chosen for high and low u ($u > u_2$ or $u < u_1$) (Figure 2b); (c) when the project return is high, the sequential approach is utilized for low u ($u < u_2$) and the direct approach is employed for high u ($u > u_2$) (Figure 2c). Next, note that $.55 < 1/(e-1)$ and the third case in Proposition 5.2 will not be applied. Thus, there are two cases for Figure 3: (a) when the project return is low, the sequential approach is never employed in the characterized value function (Figure 3a); (b) when the project return is intermediate, the sequential approach is employed for intermediate u ($u \in (u_1, u_2)$) and the direct approach is chosen for high and low u ($u > u_2$ or $u < u_1$) (Figure 3b).

To obtain more precise results, we need to compare two switch points u_1 and u_2 (if they exist) with the profit maximizing promised utility level $\bar{u}$ as we did in the no efficiency loss case. In the efficiency loss case, the type of the optimal contract depends not only on Π but also on η. For the rest of this section, I characterize optimal contracts for two cases: (i) when η is above $\max\{\sqrt{c/(c+\phi)},\ 1/(e-1)\}$, i.e., the efficiency loss is small; (ii) when η is below $\min\{1/(e-1),\ c/(c+\phi)\}$, i.e., the efficiency loss is large.[14]

Theorem 2. *Suppose that η is greater than $\sqrt{c/(c+\phi)}$ and $1/(e-1)$, i.e., the efficiency loss from monitoring is small. There exist thresholds $\Pi_D(\eta)$ and $\Pi_S(\eta)$ with $\Pi_S(\eta) > \Pi_D(\eta) >$*

[14] These cases do not cover the case where the efficiency loss is intermediate. In that case, the form of the optimal contract depends highly on parameter values η and Π and there are many subcases to consider.

(a) $\Pi = .8,\ \eta = .55$ (b) $\Pi = 4,\ \eta = .55$

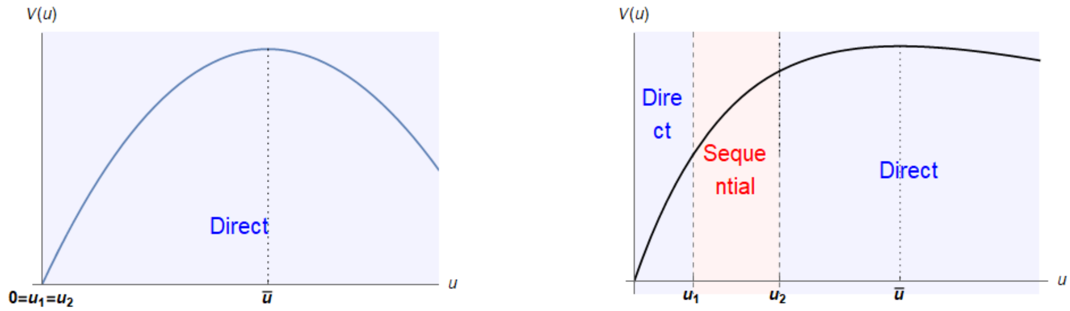

Figure 3: Value functions when the efficiency loss is large
$(\lambda_D = 2,\ \lambda_S = 3.1,\ c = 1,\ \phi = .5)$

$\Pi_F = (c+\phi)/\lambda_D$ *such that the optimal contract is determined as follows:*

(a) when $\Pi \leq \Pi_F$*, the project is infeasible;*

(b) when $\Pi_D(\eta) \geq \Pi > \Pi_F$*,* $(\bar{u}, V(\bar{u}))$ *is implemented by a direct-only contract with a deadline* $\bar{u}/\phi$*;*

(c) when $\Pi_S(\eta) \geq \Pi > \Pi_D(\eta)$*, there exists* $u_1 \in [0, \bar{u}]$ *such that* $(\bar{u}, V(\bar{u}))$ *is implemented by a contract with a switch from the sequential approach to the direct approach at* $(\bar{u} - u_1)/\phi$ *and a deadline* $\bar{u}/\phi$*;*

(d) when $\Pi > \Pi_S(\eta)$*,* $(\bar{u}, V(\bar{u}))$ *is implemented by a sequential-only contract with a deadline* $\bar{u}/\phi$*.*

This result is similar to Theorem 1. Intuitively, when the efficiency loss is small, it only induces the principal to choose the direct approach when the deadline is fairly far away and it may go further than the optimal length of the contract. Thus, the switch from the direct approach to the sequential approach may not occur in the optimal contract. The condition that η is greater than $\sqrt{c/(c+\phi)}$ ensures that the switching point u_2 would be always above the optimal initial promised utility level $\bar{u}$ (Figure 2b and 2c), or the switching point does not exist at all (Figure 2a).[15] Also, the condition that η is greater than $1/(e-1)$ makes the

[15] Refer to Lemma C.3 for details.

principal prefer the sequential approach even at the deadline when Π is large enough (Figure 2c).

These results confirm that the findings from the no efficiency loss case are robust to the introduction of small efficiency costs. However, – as might be expected – the results no longer hold when the efficiency loss is large.

Theorem 3. *Suppose that η is less than $c/(c+\phi)$ and $1/(e-1)$, i.e., the efficiency loss from monitoring is large. There exists a threshold $\Pi_M(\eta)$ with $\Pi_M(\eta) > \Pi_F = (c+\phi)/\lambda_D$ such that the optimal contract is determined as follows:*

(a) when $\Pi \leq \Pi_F$, the project is infeasible;

(b) when $\Pi_M(\eta) \geq \Pi > \Pi_F$, $(\bar{u}, V(\bar{u}))$ is implemented by a direct-only contract with a deadline $\bar{u}/\phi$;

(c) when $\Pi > \Pi_M(\eta)$, there exist u_1 and u_2 such that $0 < u_1 < u_2 < \bar{u}$ and $(\bar{u}, V(\bar{u}))$ is implemented by a contract with two switches at $(\bar{u} - u_2)/\phi$ and $(\bar{u} - u_1)/\phi$ and a deadline $\bar{u}/\phi$.

This result is significantly different from the previous ones in that the optimal contract involves either zero or two switches. Intuitively, when there is a large efficiency loss from monitoring, the principal would choose the direct approach for the majority of the time and choose the sequential approach for only short periods. If Π is not that large, the principal would not employ the sequential approach at all (Figure 3a). However, if Π is large enough, the principal would choose the sequential approach in the middle of the optimal contract (Figure 3b). To get this result, we need to show that $\bar{u} > u_2 > u_1 > 0$. The condition that η is less than $c/(c+\phi)$ ensures that u_2 is always below $\bar{u}$.[16] Also, the condition that η is less than $1/(e-1)$ makes the principal choose the direct approach at the deadline no matter what Π is.[17]

[16] Refer to Lemma C.2 for details.

[17] Refer to Proposition 5.2 for details.

Intuitively, the principal generally prefers the direct approach since there is a large efficiency loss from the sequential one. Nevertheless, when Π is large enough, the principal may take advantage of the monitoring benefit by choosing the sequential approach. If the principal decides to monitor at some point, it is optimal to monitor in the middle of the contract. This is because efficiency outweighs monitoring at the beginning of the contract and the deadline effect outweighs monitoring at the end of the contract. Hence, the optimal contract involves two switches when Π is large.

For a very high-return project, the theorem illustrates that a type of contract involving all three economic forces is optimal. At the beginning of the contract, the principal chooses the direct approach because it is more efficient (i.e., efficiency is initially the dominant concern). When the success is not delivered by a specified time, the principal begins to monitor the agent more closely by switching to the sequential approach (i.e., monitoring becomes the primary concern). She extends the deadline if the agent makes intermediate progress, but if he does not make progress before the deadline is near, the principal switches back to the direct approach in a "last-ditch" attempt at getting the job done (i.e., the deadline effect becomes the preeminent motivation).

6 Conclusion

In this article, I study the economic tradeoffs between a direct approach and a sequential approach for achieving a discrete goal in the context of a principal-agent setting. The optimal contract is determined by the interplay of monitoring, efficiency, and an endogenous deadline. I show that the form of the optimal contract depends on the project return. When the efficiency loss from splitting the project in two does not exist or is small, only the direct approach will be chosen if the project return is low, whereas only the sequential approach will be chosen if the project return is high. If the project return is intermediate, it is optimal to begin with the sequential approach and then switch to the direct approach. When the efficiency loss is large, the principal generally chooses the direct approach. However, if the project return is above a certain cutoff, she may choose the sequential approach for a short period of time in the middle of the contract (i.e., there may be two switches).

There are numerous avenues open for further research. For example, the principal may be able to design the approaches directly. In this article, I assume that the two approaches are exogenously given and the principal chooses between them. However, in practice, a project manager often designs how many milestones to partition the main project into and how difficult each subproject is. We could also introduce 'learning by doing' into the model. If we assume that the agent learns from early errors, the arrival rate of project completion would increase over time.[18] Finally, we might consider competition between firms. Many technology companies are often exposed to competition and this may significantly influence which approach project managers take. For instance, competitive pressure may manifest as increased time sensitivity tipping the choice of approach toward the more efficient direct methodology. I leave these intriguing questions–and others–for future work.

[18] This possibility contrasts with the setting considered by Carnehl and Schneider (2021), where learning causes the expected arrival rate to fall.

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Appendix

A Contracts

At the beginning of the game, the principal offers a contract to the agent and fully commits to all contractual terms. If the agent rejects the offer, the principal and the agent receive zero payoffs. Note that if the agent has not completed either the main project or the subproject, the calendar time is the only relevant variable summarizing the public history.

A (deterministic) contract is denoted by $\Gamma \equiv \left\{T, \{a_t, b_t, R_t, \hat{\Gamma}^t\}_{0\leq t\leq T}\right\}$, where each variable is defined as follows at the calendar time t:[19]

1. $T \in \mathbb{R}_+ \cup \{\infty\}$: the deadline date at which the project is terminated absent the completion of the main project or the subproject. $T = \infty$ means that no deadline is included in the contract;
2. $a_t \in \{0, 1\}$: the principal's choice of an approach at t;
3. $b_t \in [0, 1]$: the agent's recommended effort at t;
4. $R_t \geq 0$: the monetary payment from the principal to the agent for the success of the main project at t;[20]
5. $\hat{\Gamma}^t \equiv \{T^t, \{b^t_s, R^t_s\}_{t\leq s\leq T^t}\}$: an updated contract when the subproject is completed at t;
   (a) $T^t \in \{\check{T} : \check{T} \geq t\} \cup \{\infty\}$: the deadline date at which the project is terminated;
   (b) $b^t_s \in [0, 1]$: the agent's recommended effort at time $s \geq t$;
   (c) $R^t_s \geq 0$: the monetary payment from the principal to the agent for the completion of the main project at time s.

Consider the case where the subproject is completed at time t. Then, the updated contract $\hat{\Gamma}^t$ will be executed. In this case, the agent's admissible action space is $\hat{\mathcal{B}}^t \equiv \{\{\check{b}_s\}_{t\leq s\leq T^t} : \check{b}_s \in [0, 1]\}$. The agent's action profile $\tilde{b}^t \equiv \{\tilde{b}^t_s\}_{t\leq s\leq T^t} \in \hat{\mathcal{B}}^t$ induces a probability distribution $\mathbb{P}^{\tilde{b}^t}$ over a main project completion date τ_m. Let $\mathbb{E}^{\tilde{b}^t}$ denote the corresponding expectation operator. When the agent adheres to the recommended action of

[19] See Remark 3 for discussion on deterministic and mixed contracts.

[20] Since both the principal and the agent are risk neutral and do not discount the future, without loss of generality, all monetary payments to the agent can be backloaded (see, e.g., Ray, 2002). The nonnegativity of R_t is due to limited liability.

$\hat{\Gamma}^t$, the principal's expected utility at time t is given by

$$\hat{P}^t(\hat{\Gamma}^t) = \mathbb{E}^{b^t}\left[\left(\Pi - R^t_{\tau_m}\right)\cdot \mathbf{1}_{\{t\le\tau_m\le T^t\}} - \int_t^{T^t\wedge\tau_m} c\ ds\right],$$ [21]

where the first term in the expectation is the net profit from the success and the second term is the cumulative operating cost. The agent's expected utility is given by

$$\hat{U}^t(\hat{\Gamma}^t) = \mathbb{E}^{b^t}\left[R^t_{\tau_m}\cdot \mathbf{1}_{\{t\le\tau_m\le T^t\}} + \int_t^{T^t\wedge\tau_m} \phi(1-b^t_s)ds\right],$$

where the first term is the payoff from the success and the second term is the benefit from shirking.

Now consider the problem at time 0. The agent's admissible action space (prior to any completion) is $\mathcal{B} \equiv \{\{\check{b}_t\}_{0\le t\le T} \ : \ \check{b}_t \in [0,1]\}$. In this case, any completion depends not only on the agent's effort $(\tilde{b}_t)$ but also the principal's choice of approach (a_t). Then, a pair of actions by the principal and the agent, $(a, \tilde{b})$, induces a probability distribution $\mathbb{P}^{a,\tilde{b}}$ over a pair of completion dates for the main project and the subproject (τ_m, τ_s). Let $\mathbb{E}^{a,\tilde{b}}$ denote the corresponding expectation operator. If the agent adheres to the recommended actions of Γ, the principal's (ex ante) expected utility is given by

$$P_0(\Gamma) = \mathbb{E}^{a,b}\left[(\Pi - R_{\tau_m})\cdot \mathbf{1}_{\{\tau_m<\tau_s\wedge T\}} + \hat{P}_{\tau_s}(\hat{\Gamma}_{\tau_s})\cdot \mathbf{1}_{\{\tau_s<\tau_m\wedge T\}} - \int_0^{T\wedge\tau_m\wedge\tau_s} c\ dt\right], \qquad \text{(A.1)}$$

where the first term is the net profit from the main project completion, the second term is the expected payoff from the subproject completion at time τ_s, and the last term is the cumulative operating cost. The agent's expected utility is given by

$$U_0(\Gamma) = \mathbb{E}^{a,b}\left[R_{\tau_m}\cdot \mathbf{1}_{\{\tau_m\le T\}} + \hat{U}^{\tau_s}(\hat{\Gamma}^{\tau_s})\cdot \mathbf{1}_{\{\tau_s<\tau_m\wedge T\}} + \int_0^{T\wedge\tau_m\wedge\tau_s} \phi(1-b_t)\ dt\right], \qquad \text{(A.2)}$$

where the first term is the payoff from the main project completion, the second term is the expected payoff from the subproject completion at time τ_s, and the last term is the benefit from shirking. By using the agent's expected payoffs, I define incentive compatibility (IC) of contracts as follows.

Definition A.1. A contract $\Gamma = \left\{T, \{a_t, b_t, R_t, \hat{\Gamma}^t\}_{0\le t\le T}\right\}$ is *incentive compatible* if

1. for all $t \le T$, the recommended effort profile $\{b^t_s\}_{t\le s\le T^t}$ in the updated contract $\hat{\Gamma}^t$

[21] For each x and y, let $x \wedge y$ denote the minimum of x and y, and let $x \vee y$ denote the maximum of x and y.

maximizes the agent's expected utility at time t, i.e.,

$$\hat{U}^t(\hat{\Gamma}^t) = \max_{\tilde{b}\in\hat{\mathcal{B}}^t} \mathbb{E}^{\tilde{b}}\left[R^t_{\tau_m}\cdot \mathbf{1}_{\{\tau_m\le T^t\}} + \int_t^{T^t\wedge\tau_m} \phi(1-\tilde{b}_s)ds\right].$$

2. the recommended action profile $\{b_t\}_{0\le t\le T}$ maximizes the agent's expected utility at time 0, i.e.,

$$U_0(\Gamma) = \max_{\tilde{b}\in\mathcal{B}} \mathbb{E}^{a,\tilde{b}}\left[R_{\tau_m}\cdot \mathbf{1}_{\{\tau_m\le T\}} + \hat{U}^{\tau_s}(\hat{\Gamma}^{\tau_s})\cdot \mathbf{1}_{\{\tau_s<\tau_m\wedge T\}} + \int_0^{T\wedge\tau_m\wedge\tau_s} \phi(1-\tilde{b}_t)\ dt\right].$$

The objective of the principal is to find a contract Γ that maximizes her ex ante expected utility $P_0(\Gamma)$ subject to the incentive compatibility constraint and the individual rationality constraint, i.e., $U_0(\Gamma)\ge 0$. Designate such a contract as an *optimal contract*.

B Recursive Formulation

B.1 The Agent's Problem

I now consider the agent's problem when he has not yet completed the first subproject. Given a contract Γ, let $U_t(\Gamma)$ denote the agent's maximized continuation utility at time t, that is,

$$U_t(\Gamma) \equiv \max_{\tilde{b}\in\mathcal{B}_t} \mathbb{E}^{a,\tilde{b}}\left[\begin{array}{c} R_{\tau_m}\cdot \mathbf{1}_{\{\tau_m\le\tau_s\wedge T\}} + \hat{U}^{\tau_s}(\hat{\Gamma}^{\tau_s})\cdot \mathbf{1}_{\{\tau_s<\tau_m\wedge T\}} \\ +\int_t^{T\wedge\tau_m\wedge\tau_s}\phi(1-\tilde{b}_s)\ ds \end{array}\ \middle|\ t\le\tau_m\wedge\tau_s\right], \tag{B.1}$$

where $\mathcal{B}_t \equiv \{\{\tilde{b}_s\}_{t\le s\le T}\ :\ \tilde{b}_s\in[0,1]\}$.

Observe that the agent's continuation utility can be heuristically rewritten as follows:

$$\begin{aligned} u_t = \max_{\tilde{b}_t\in[0,1]}\ & \phi(1-\tilde{b}_t)dt + R_t\cdot\lambda_D a_t\tilde{b}_t dt + u^t_S\cdot\lambda_S(1-a_t)\tilde{b}_t dt \\ & + (1-\lambda_D a_t\tilde{b}_t dt - \lambda_S(1-a_t)\tilde{b}_t dt)\cdot u_{t+dt} \end{aligned}$$

where $u_t = U_t(\Gamma)$ and $u^t_S = \hat{U}^t(\hat{\Gamma}^t)$. Using a Taylor expansion $u_{t+dt} = u_t + \dot{u}_t dt + o(dt)$ where $\dot{u}_t \equiv du_t/dt$, canceling u_t on both sides, and taking the limit as $dt\to 0$, we obtain a Hamilton-Jacobi-Bellman (HJB) equation:

$$0 = \max_{\tilde{b}_t\in[0,1]} \dot{u}_t + \phi(1-\tilde{b}_t) + (R_t-u_t)\lambda_D a_t\tilde{b}_t + (u^t_S-u_t)\lambda_S(1-a_t)\tilde{b}_t. \tag{HJB$_{PK}$}$$

Also note that $U_T(\Gamma) = 0$ since the contract is terminated at time T. The following lemma shows that the HJB equation (HJB_{PK}) with a boundary condition $u_T = 0$ characterizes the evolution of the continuation utility $U_t(\Gamma)$. The proof is relegated to Appendix B.1.1.

Lemma B.1. *Given a contract Γ, suppose that a continuous and differentiable process $\{u_t\}_{0\le t\le T}$ satisfies $u_T = 0$ and (HJB_{PK}). Then, $u_t = U_t(\Gamma)$.*

The HJB equation (HJB_{PK}) provides a clear interpretation of the agent's behavior. The first term is the drift term of the agent's continuation utility from no success and the second term is the benefit from shirking. When the main project is completed at rate $\lambda_D a_t \tilde{b}_t$, the agent receives the immediate payment R_t but he loses the continuation utility since the contract is terminated. When the subproject is completed at rate $\lambda_S(1-a_t)\tilde{b}_t$, the new phase of the contract with the promised utility u_S^t begins and he loses the continuation utility since the current phase of the contract is over.

B.1.1 Proof of Lemma B.1

In this subsection, I formally derive the agent's continuation utility and prove Lemma B.1.

I begin by specifying the probability distribution functions for possible events given a profile of approaches $a = \{a_s\}_{t\le s\le T}$ and an admissible action profile $b \in \mathcal{B}_t$ conditional on no completion has made by time t. The probability that neither the main project nor the subproject is completed by time T is $f(a, b; t, T)$ where

$$f(a, b; x, y) \equiv e^{-\lambda_D \int_x^y a_l b_l dl} \cdot e^{-\lambda_S \int_x^y (1-a_l) b_l dl}.$$

Next, the probability density that the main project is completed at time s and the subproject is not completed by that time ($s = \tau_m < \tau_s$) is $\lambda_D a_s b_s \cdot f(a, b; t, s)$. Similarly, the probability density that the subproject is completed at time s and the main project is not completed by that time ($s = \tau_s < \tau_m$) is $\lambda_S(1-a_s)b_s \cdot f(a, b; t, s)$. Last, the probability density that either the main project or the subproject is completed at time s and the other has not arrived by then, i.e., $\tau_s \wedge \tau_m = s$, is $(\lambda_D a_s + \lambda_S(1-a_s))b_s \cdot f(a, b; t, s)$.

Based on the above results, we can derive that

$$\mathbb{E}^{a,b}\left[R_{\tau_m} \cdot \mathbf{1}_{\{\tau_m \le \tau_s \wedge T\}} \mid t \le \tau_m \wedge \tau_s\right] = \int_t^T R_s \cdot \lambda_D a_s b_s \cdot f(a, b; t, s) ds,$$
$$\mathbb{E}^{a,b}\left[\hat{U}_{\tau_s}(\hat{\Gamma}^{\tau_s}) \cdot \mathbf{1}_{\{\tau_s < \tau_m \wedge T\}} \mid t \le \tau_m \wedge \tau_s\right] = \int_t^T \hat{U}_s(\hat{\Gamma}^s) \cdot f(a, b; t, s) ds.$$

Observe that $\frac{d}{ds} f(a, b; t, s) = -(\lambda_D a_s + \lambda_S(1-a_s))b_s \cdot f(a, b; t, s)$. By using integration by parts, we have

$$
\begin{aligned}
&\mathbb{E}^{a,b}\left[\int_t^{T\wedge\tau_m\wedge\tau_s}\phi(1-b_s)\mid t\le\tau_m\wedge\tau_s\right]\\
&=\int_t^T\left[\int_t^s\phi(1-b_l)dl\right]\cdot(\lambda_D a_s+\lambda_S(1-a_s))b_s\cdot f(a,b;t,s)ds\\
&\quad+\left[\int_t^T\phi(1-b_s)ds\right]\cdot f(a,b;t,T)\\
&=\int_t^T\phi(1-b_s)\cdot f(a,b;t,s)ds.
\end{aligned}
$$

Then, by plugging the above expressions into (B.1), $U_t(\Gamma)$ can be rewritten as follows:

$$
U_t(\Gamma)=\sup_{\tilde{b}\in\mathcal{B}_t}\int_t^T\left[R_s\cdot\lambda_D a_s\tilde{b}_s+\hat{U}_s(\hat{\Gamma}^s)\cdot\lambda_S(1-a_s)\tilde{b}_s+\phi(1-\tilde{b}_s)\right]f(a,\tilde{b};t,s)ds. \tag{B.2}
$$

Now we can prove Lemma B.1 by using the above equation. The proof is inspired by Proposition 3.2.1 in Bertsekas (1995).

Proof of Lemma B.1. Consider an arbitrary admissible action $\tilde{b}\in\mathcal{B}_t$. By rearranging (HJB$_{PK}$), we can derive that

$$
-\dot{u}_s+(\lambda_D a_s+\lambda_S(1-a_s))\tilde{b}_s u_t\ge R_s\lambda_D a_s\tilde{b}_s+u_S^s\lambda_S(1-a_s)\tilde{b}_s+\phi(1-\tilde{b}_s)
$$

and it is equivalent to

$$
\frac{d}{ds}\left[-u_s\cdot f(a,b;t,s)\right]\ge\left[\left(R_s\lambda_D a_s+u_S^s\lambda_S(1-a_s)\right)\tilde{b}_s+\phi(1-\tilde{b}_s)\right]\cdot f(a,b;t,s).
$$

By integrating the above inequality from t to T and using $u_T=0$, we can derive that

$$
u_t\ge\int_t^T\left[\left(R_s\lambda_D a_s+u_S^s\lambda_S(1-a_s)\right)\tilde{b}_s+\phi(1-\tilde{b}_s)\right]\cdot f(a,\tilde{b};t,s)ds
$$

for all $\tilde{b}\in\mathcal{B}_t$.

Suppose that $b^*\in\mathcal{B}_t$ attains the maximum in the equation (HJB$_{PK}$) for all $0\le t\le T$. Then, we have

$$
\begin{aligned}
u_t&=\int_t^T\left[\left(R_s\lambda_D a_s+u_S^s(1-a_s)\right)\lambda_S b_s^*+\phi(1-b_s^*)\right]\cdot f(a,b^*;t,s)ds\\
&\ge\int_t^T\left[\left(R_s\lambda_D a_s+u_S^s\lambda_S(1-a_s)\right)\tilde{b}_s+\phi(1-\tilde{b}_s)\right]\cdot f(a,\tilde{b};t,s)ds
\end{aligned}
$$

for all $\tilde{b} \in \mathcal{B}_t$. Therefore, by (B.2), we have $u_t = U_t(\Gamma)$. $\square$

B.2 The Principal's Problem

I now consider the principal's problem. I begin by considering the incentive compatibility condition. To make a contract incentive compatible, at each point of time, the recommended effort level should coincide with the agent's choice in (HJB_{PK}), that is,

$$b \in \arg\max_{\tilde{b}\in[0,1]} \phi(1-\tilde{b}) + (R-u)\lambda_D a\tilde{b} + (u_S - u)\lambda_S(1-a)\tilde{b}. \tag{IC}$$

In addition, since (HJB_{PK}) is linear in $\tilde{b}$, it can be rewritten as follows:

$$\dot{u}_t = -\left[\phi \vee \left((R_t - u_t)\lambda_D a_t + (u_S^t - u_t)\lambda_S(1-a_t)\right)\right]. \tag{B.3}$$

I now explore how the principal's value function evolves. Note that $V(0) = 0$ since the agent will not participate in the contract when the continuation utility is zero. This will serve as a boundary condition. The value function $V(u_t)$ can be heuristically written as follows:

$$V(u_t) = \max_{\substack{R_t\geq 0,\ u_S^t\geq 0,\\ a_t\in\{0,1\},\ b_t\in[0,1]}} \begin{array}{c} -\,cdt + (\Pi - R_t)\,\lambda_D a_t b_t dt + V_S(u_S^t)\lambda_S(1-a_t)b_t dt \\ +\,(1-\lambda_D a_t b_t dt - \lambda_S(1-a_t)b_t dt)V(u_{t+dt}) \end{array}.$$

By using $V(u_{t+dt}) = V(u_t) + V'(u_t)\dot{u}_t dt + o(dt)$, canceling $V(u_t)$ on both sides, taking the limit as $dt \to 0$ and plugging (B.3) in, we obtain an HJB equation:

$$0 = \max_{\substack{R\geq 0,\ u_S\geq 0,\\ a\in\{0,1\},\ b\in[0,1]}} \begin{array}{c} -\,c + (\Pi - R - V(u))\lambda_D ab + (V_S(u_S) - V(u))\lambda_S(1-a)b \\ -\,[\phi \vee \{(R-u)\lambda_D a + (u_S - u)\lambda_S(1-a)\}]\,V'(u) \end{array}. \tag{HJB_V}$$

Then, the principal's problem is to solve (HJB_V) subject to (IC) with the boundary condition $V(0) = 0$. The following lemma shows that the solution of the problem maximizes the principal's expected payoff subject to a promise keeping constraint $U_0(\Gamma) = u$. The proof is relegated to Appendix B.2.1.

Lemma B.2 (Verification Lemma)**.** *Suppose that a differentiable and concave function $\bar{V}$ solves (HJB_V) subject to (IC) with the boundary condition $\bar{V}(0) = 0$. Then, for any incentive-compatible contract Γ with $U_0(\Gamma) = u$,*

$$\bar{V}(u) \geq P_0(\Gamma).$$

Given this result, I derive the value function by using the 'guess and verify' method. I construct a differentiable and concave value function, $\bar{V} : \mathbb{R}_+ \to \mathbb{R}$, which solves (HJB$_V$) subject to (IC) for all $u \geq 0$. Next, for any $u \geq 0$, I find a deterministic and incentive-compatible contract Γ that implements $(u, \bar{V}(u))$. Then, by the above verification lemma, $\bar{V}(u)$ is the highest expected payoff among incentive-compatible contracts with $U_0(\Gamma) = u$, i.e., $\bar{V}(u) = V(u)$. In the main text, I provide intuition of how I guess the value function. The actual guess appears in Appendix B.4. The formal verification proof is relegated to the Online Appendix.

B.2.1 Proof of Lemma B.2

Proof of Lemma B.2. Consider an arbitrary (deterministic) incentive-compatible contract Γ where the agent's expected payoff is given by u_t. The payoff to the principal under Γ is

$$
\begin{aligned}
P_0(\Gamma) = &\int_0^T (\Pi - R_t - c \cdot t) \cdot \lambda_D a_t b_t f(a,b;0,t)dt \\
&+ \int_0^T (V_S(u_{S,t}) - c \cdot t) \cdot \lambda_S (1-a_t) b_t f(a,b;0,t)dt - c \cdot T \cdot f(a,b;0,T) \\
= &\int_0^T \left((\Pi - R_t)\lambda_D a_t b_t + V_S(u_{S,t})\lambda_S(1-a_t)b_t - c\right) f(a,b;0,t)dt
\end{aligned}
$$

where $u_{S,t} = \hat{U}_t(\hat{\Gamma}^S)$.

Since $\tilde{V}$ solves the HJB equation, we have

$$
\begin{aligned}
0 \geq &- c + (\Pi - R_t - \tilde{V}(u_t))\lambda_D a_t b_t + (V_S(u_{S,t}) - \tilde{V}(u_t))\lambda_S(1-a_t)b_t \\
&- \left[\phi \vee \{(R_t - u_t)\lambda_D a_t + (u_{S,t} - u_t)\lambda_S(1-a_t)\}\right] \tilde{V}'(u_t).
\end{aligned}
$$

By using (B.3), rearranging, and multiplying by $f(a,b;0,t)$, we can obtain that

$$
\begin{aligned}
&(\lambda_D a_t b_t + \lambda_S(1-a_t)b_t) f(a,b;0,t) \cdot \tilde{V}(u_t) - f(a,b;0,t) \cdot \tilde{V}'(u_t)\dot{u}_t \\
\geq &f(a,b;0,t)\left((\Pi - R_t)\lambda_D a_t b_t + V_S(u_{S,t})\lambda_S(1-a_t)b_t - c\right)
\end{aligned}
\tag{B.4}
$$

Note that

$$
\frac{d}{dt}\left(-f(a,b;0,t)\tilde{V}(u_t)\right) = (\lambda_D a_t b_t + \lambda_S(1-a_t)b_t) f(a,b;0,t) \cdot \tilde{V}(u_t) - f(a,b;0,t) \cdot \tilde{V}'(u_t)\dot{u}_t.
$$

Then, by integrating (B.4) over $[0,T]$ and noting that $f(a,b;0,0) = 1$, $u_T = 0$ and $\tilde{V}(0) = 0$, we have

$$
\begin{aligned}
&\tilde{V}(u_0) = \tilde{V}(u_0) - f(a, b; 0, T)\tilde{V}(u_T) \\
&\geq \int_0^T f(a, b; 0, t) \cdot ((\Pi - R_t)\lambda_D a_t b_t + V_S(u_{S,t})\lambda_S(1 - a_t)b_t - c)\, dt = P_0(\Gamma).
\end{aligned}
$$

Therefore, $\tilde{V}(u_0)$ is greater than or equal to any deterministic contract where the agent's expected payoff is equal to u_0. Since $\tilde{V}$ is assumed to be concave, it is greater than or equal to any randomized contract. □

B.3 Implementation

B.3.1 Proof of Proposition 4.1

Proof of Proposition 4.1. (a) Let $\Gamma_d(T)$ denote a direct-only contract with the deadline T.

The agent's expected payoff is

$$
\begin{aligned}
U_0(\Gamma_d(T)) &= \int_0^T R_{\tau_m} \lambda_D e^{-\lambda_D \tau_m} d\tau_m \\
&= \int_0^T \phi \left[T - \tau_m + 1/\lambda_D\right] \lambda_D e^{-\lambda_D \tau_m} d\tau_m \\
&= -\, \phi \left(T - \tau_m\right) e^{-\lambda_D \tau_m} \Big|_0^T \\
&= \phi T.
\end{aligned}
$$

Therefore, $U_0(\Gamma_d(u/\phi)) = u$.

Also note that

$$
\begin{aligned}
P_0(\Gamma_d(T)) + U_0(\Gamma_d(T)) &= \int_0^T \left(\Pi - c\tau_m\right) \lambda_D e^{-\lambda_D \tau_m} d\tau_m - cTe^{-\lambda_D T} \\
&= -\left(\Pi - c\tau_m - c/\lambda_D\right) e^{-\lambda_D \tau_m} \Big|_0^T - cTe^{-\lambda_D T} \\
&= -\left(\Pi - cT - c/\lambda_D\right) e^{-\lambda_D T} + \left(\Pi - c/\lambda_D\right) - cTe^{-\lambda_D T} \\
&= \left(\Pi - c/\lambda_D\right) \left(1 - e^{-\lambda_D T}\right).
\end{aligned}
$$

Therefore,

$$\begin{aligned}
P_0(\Gamma_d(u/\phi)) &= \left(\Pi - \frac{c}{\lambda_D}\right)(1 - e^{-\frac{\lambda_D}{\phi}u}) - U_0(\Gamma_d(u/\phi)) \\
&= \left(\Pi - \frac{c}{\lambda_D}\right)(1 - e^{-\frac{\lambda_D}{\phi}u}) - u = V^d(u).
\end{aligned}$$

(b) Let $\Gamma_{sd}(T_1, T)$ denote a contract with a switch from the sequential approach to the direct approach at T_1 and the deadline T. The subcontract at time $t \leq T_1$ is denoted by $\hat{\Gamma}_{sd}(t|T_1, T)$. Then, the agent's expected payoff for the subcontract $\hat{\Gamma}_{sd}(t|T_1, T)$ at time t is

$$\begin{aligned}
U_t(\hat{\Gamma}_{sd}(t|T_1, T)) = &\int_t^{T+1/\lambda_S} \phi\left(T + 1/\lambda_S - \tau_m + 1/\lambda_S\right)\lambda_S e^{-\lambda_S(\tau_m - t)} d\tau_m \\
= &- \phi\left(T + 1/\lambda_S - \tau_m\right) e^{-\lambda_S(\tau_m - t)}\Big|_t^{T+1/\lambda_S} \\
= &\phi\left(T + 1/\lambda_S - t\right).
\end{aligned}$$

Also note that

$$\begin{aligned}
\int_0^{T_1} U_{\tau_s}(\hat{\Gamma}_s(\tau_s|T_1, T))\lambda_S e^{-\lambda_S \tau_s} d\tau_s = &\int_0^{T_1} \phi(T + 1/\lambda_S - \tau_s)\lambda_S e^{-\lambda_S \tau_s} d\tau_s \\
= &- \phi(T - \tau_s)e^{-\lambda_S \tau_s}\Big|_0^{T_1} \\
= &\phi T - \phi(T - T_1)e^{-\lambda_S T_1}
\end{aligned}$$

Then, the agent's expected payoff at time 0 is

$$\begin{aligned}
U_0(\Gamma_{sd}(T_1, T)) = &\int_0^{T_1} U_{\tau_s}(\hat{\Gamma}_s(\tau_s|T_1, T))\lambda_S e^{-\lambda_S \tau_s} d\tau_s \\
&+ e^{-\lambda_S T_1} \int_{T_1}^{T} \phi(T + 1/\lambda_D - \tau_m)\lambda_D e^{-\lambda_D(\tau_m - T_1)} d\tau_m \\
= &\phi T - \phi(T - T_1)e^{-\lambda_S T_1} - e^{-\lambda_S T_1}\left[\phi\left(T - \tau_m\right) e^{-\lambda_D(\tau_m - T_1)}\Big|_{T_1}^{T}\right] \\
= &\phi T.
\end{aligned}$$

Thus, $U_0(\Gamma_{sd}(T_1, u/\phi)) = u$.

The sum of expected payoffs for the subcontract is

$$
\begin{aligned}
&P_t(\hat{\Gamma}_{sd}(t|T_1,T)) + U_t(\hat{\Gamma}_{sd}(t|T_1,T)) \\
=&\int_t^{T+1/\lambda_S} (\Pi - c(\tau_m - t))\lambda_S e^{-\lambda_S(\tau_m - t)} d\tau_m - c\left(T + \frac{1}{\lambda_S} - t\right) e^{-\lambda_S\left(T+\frac{1}{\lambda_S}-t\right)} \\
=&-\left(\Pi - \frac{c}{\lambda_S} - c(\tau_m - t)\right) e^{-\lambda_S(\tau_m - t)}\Big|_t^{T+1/\lambda_S} - c\left(T + \frac{1}{\lambda_S} - t\right) e^{-\lambda_S\left(T+\frac{1}{\lambda_S}-t\right)} \\
=&-\left(\Pi - \frac{c}{\lambda_S} - c\left(T + \frac{1}{\lambda_S} - t\right)\right) e^{-\lambda_S\left(T+\frac{1}{\lambda_S}-t\right)} + \Pi - \frac{c}{\lambda_S} \\
&- c\left(T + \frac{1}{\lambda_S} - t\right) e^{-\lambda_S\left(T+\frac{1}{\lambda_S}-t\right)} \\
=&\left(\Pi - \frac{c}{\lambda_S}\right)\left(1 - e^{-\lambda_S\left(T+\frac{1}{\lambda_S}-t\right)}\right)
\end{aligned}
$$

Also note that

$$
\begin{aligned}
&\int_0^{T_1} \left[P_{\tau_s}(\hat{\Gamma}_{sd}(\tau_s|T_1,T)) + U_{\tau_s}(\hat{\Gamma}_{sd}(\tau_s|T_1,T)) - c\tau_s\right] \lambda_S e^{-\lambda_S \tau_s} d\tau_s \\
=&-\left(\Pi - 2c/\lambda_S - c\tau_s\right)\Big|_0^{T_1} - \left(\Pi - c/\lambda_S\right) e^{-\lambda_S(T+1/\lambda_S)} \tau_s\Big|_0^{T_1} \\
=&\left(\Pi - \frac{2c}{\lambda_S}\right)\left(1 - e^{-\lambda_S T_1}\right) + cT_1 e^{-\lambda_S T_1} - \left(\Pi - \frac{c}{\lambda_S}\right) T_1 e^{-\lambda_S\left(T+\frac{1}{\lambda_S}\right)}
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{T_1}^{T} \left(\Pi - c(\tau_m - T_1)\right) \lambda_D e^{-\lambda_D(\tau_m - T_1)} d\tau_m - c(T - T_1) e^{-\lambda_D(T - T_1)} \\
=&V^d((T - T_1)/\phi) + (T - T_1)/\phi = V^d(u_1) + u_1.
\end{aligned}
$$

Then, we can derive that

$$
\begin{aligned}
&P_0(\Gamma_{sd}(T_1,T)) + U_0(\Gamma_{sd}(T_1,T)) \\
=& \int_0^{T_1} \left[P_{\tau_s}(\hat{\Gamma}_s(\tau_s|T)) + U_{\tau_s}(\hat{\Gamma}_s(\tau_s|T)) - c\tau_s\right] \lambda_S e^{-\lambda_S \tau_s} d\tau_s - cT_1 e^{-\lambda_S T_1} \\
&+ e^{-\lambda_S T_1}\left[\int_{T_1}^{T} (\Pi - c(\tau_m - T_1))\, \lambda_D e^{-\lambda_D(\tau_m - T_1)} d\tau_m - c(T-T_1)e^{-\lambda_D(T-T_1)}\right] \\
=& \left(\Pi - \frac{2c}{\lambda_S}\right)(1 - e^{-\lambda_S T_1}) - \left(\Pi - \frac{c}{\lambda_S}\right) T_1 e^{-\lambda_S\left(T + \frac{1}{\lambda_S}\right)} + e^{-\lambda_S T_1}\left(V^d(u_1) + u_1\right) \\
=& \left(\Pi - \frac{2c}{\lambda_S}\right)\left(1 - e^{\frac{\lambda_S}{\phi}(u_1 - u)}\right) + (V^d(u_1) + u_1) e^{\frac{\lambda_S}{\phi}(u_1 - u)} \\
&- \left(\Pi - \frac{c}{\lambda_S}\right) \frac{\lambda_S}{\phi}(u - u_1) e^{-\frac{\lambda_S}{\phi}u - 1},
\end{aligned}
$$

thus $P_0(\Gamma_{sd}(T_1,T)) = V^{ds}(u|u_1)$.

(c) Note that a sequential-only contract with a deadline T is equivalent to a contract with a switch from the sequential approach to the direct approach at $T_1 = T$ and a deadline T. Therefore, by the previous result, a sequential-only contract with the deadline u/ϕ implements $(V^{ds}(u|0), u)$.

□

B.3.2 Proof of Proposition 5.1

Proof of Proposition 5.1. Let $\Gamma_{dsd}(T_1, T_2, T)$ denote a contract with two switches at T_1 and T_2 and a deadline T. Note that at time T_1 (if the project has not been successful), the remaining contract is equivalent to $\Gamma_{sd}(T_2 - T_1, T - T_1)$. Then, the agent's expected payoff at time 0 is

$$
\begin{aligned}
U_0(\Gamma_{dsd}(T_1,T_2,T)) &= \int_0^{T_1} \phi(T + 1/\lambda_D - \tau_m)\lambda_D e^{-\lambda_D \tau_m} d\tau_m + e^{-\lambda_S T_1} U_0(\Gamma_{sd}(T_2 - T_1, T - T_1)) \\
&= \phi T - \phi(T - T_1)e^{-\lambda_D T_1} + e^{-\lambda_D T_1}\phi(T - T_1) \\
&= \phi T.
\end{aligned}
$$

Thus, $U_0(\Gamma_{dsd}(T_1, T_2, u/\phi)) = u$.

Also note that

$$
\begin{aligned}
&P_0(\Gamma_{dsd}(T_1,T_2,T)) + U_0(\Gamma_{dsd}(T_1,T_2,T)) \\
=&\int_0^{T_1} (\Pi - c\tau_m)\lambda_D e^{-\lambda_D \tau_m} d\tau_m - cT_1 e^{-\lambda_D T_1} \\
&+ e^{-\lambda_D T_1}(P_0(\Gamma_{sd}(T_2-T_1, T-T_1)) + U_0(\Gamma_{sd}(T_2-T_1,T-T_1))) \\
=&\left(\Pi - \frac{c}{\lambda_D}\right)\left(1-e^{\lambda_D T_1}\right) + \left(V^{ds}(\phi(T-T_1)|\phi(T-T_2)) + \phi(T-T_1)\right)e^{-\lambda_D T_1} \\
=&\left(\Pi - \frac{c}{\lambda_D}\right)\left(1-e^{\frac{\lambda_D}{\phi}(u_2-u)}\right) + \left(V^{ds}(u_2|u_1) + u_2\right)e^{\frac{\lambda_D}{\phi}(u_2-u)},
\end{aligned}
$$

thus $P_0(\Gamma_{dsd}(T_1,T_2,T)) = V^{dsd}(u|u_1,u_2) - u$. □

B.4 Properties of Benchmark Value Functions

Proposition B.1. *The following equations hold:*

$$
\phi V^{d\prime}(u) = -c + \lambda_D\left(\Pi - \frac{\phi}{\lambda_D} - u - V^d(u)\right) = \alpha_D(u), \tag{B.5}
$$

$$
\phi V^{ds\prime}(u|u_1) = -c + \lambda_S\left(V_S\left(u + \phi/\lambda_S\right) - V^{ds}(u|u_1)\right) = \alpha_S(u), \tag{B.6}
$$

$$
\phi V^{dsd\prime}(u|u_1,u_2) = -c + \lambda_D\left(\Pi - \frac{\phi}{\lambda_D} - u - V^{dsd}(u|u_1,u_2)\right) = \alpha_D(u). \tag{B.7}
$$

Proof of Proposition B.1. By taking the derivative of (4.1) and multiplying by ϕ, we have

$$
\begin{aligned}
\phi V^{d\prime}(u) =&\lambda_D\left(\Pi - \frac{c}{\lambda_D}\right)e^{-\frac{\lambda_D}{\phi}u} - \phi = -c + \lambda_D\left(\Pi - \frac{\phi}{\lambda_D} - V^d(u) - u\right), \\
=&\lambda_D\Pi - \lambda_D(V^d(u) + u) - c - \phi = \alpha_D(u),
\end{aligned}
$$

i.e., (B.5) holds.

Next, by taking the derivative of (4.2) and multiplying by ϕ, we have

$$
\begin{aligned}
\phi V^{ds\prime}(u|u_1) =&\lambda_S\left(\Pi - \frac{2c}{\lambda_S}\right)e^{\frac{\lambda_S}{\phi}(u_1-u)} - \lambda_S(V^d(u_1)+u_1)e^{\frac{\lambda_S}{\phi}(u_1-u)} \\
&-\left(\Pi - \frac{c}{\lambda_S}\right)\lambda_S e^{-\frac{\lambda_S}{\phi}u-1} + \lambda_S\left(\Pi - \frac{c}{\lambda_S}\right)\frac{\lambda_S}{\phi}(u-u_1)e^{-\frac{\lambda_S}{\phi}u-1} - \phi.
\end{aligned}
$$

Observe that it can be rewritten as follows:

$$\begin{aligned}\phi V^{ds\prime}(u|u_1) =& -c + \lambda_S \left(\Pi - \frac{c}{\lambda_S}\right)\left(1 - e^{\frac{-\lambda_S}{\phi}u-1}\right) - \lambda_S u - \lambda_S V^{ds}(u|u_1)\\ =& -c + \lambda_S \left(V_S(u+\phi/\lambda_S) - V^{ds}(u|u_1)\right)\\ =& \lambda_S \left(V_S(u+\phi/\lambda_S) + u + \phi/\lambda_S\right) - \lambda_S(V^{ds}(u|u_1) + u) - c - \phi = \alpha_S(u),\end{aligned}$$

i.e., (B.6) holds.

Last, by taking the derivative of (5.1) and multiplying by ϕ, we have

$$\begin{aligned}\phi V^{dsd\prime}(u|u_1,u_2) =& \lambda_D \left(\Pi - \frac{c}{\lambda_D}\right) e^{\frac{\lambda_D}{\phi}(u_2-u)} - \lambda_D \left(V^{ds}(u_2|u_1) + u_2\right) e^{\frac{\lambda_D}{\phi}(u_2-u)} - \phi\\ =& -c + \lambda_D \left(\Pi - \frac{\phi}{\lambda_D} - uV^{dsd}(u|u_1,u_2)\right),\\ =& \lambda_D(\Pi - V^{dsd}(u|u_1,u_2)) - c - \phi = \alpha_D(u),\end{aligned}$$

i.e., (B.7) holds. □

C Optimal Contracts

C.1 Feasibility (Π_F)

Now that we characterized the value function that solves (HJB_V) subject to (IC), the next step is to solve (MP) subject to $u \geq 0$. I begin by checking the feasibility of the project. If the maximum of the value function V is greater than 0, the principal earns positive expected payoff from the contract, thus the project is feasible. If $V'(0) > 0$, there exists $u > 0$ such that $V(u) > 0$. Thus, the project is feasible. On the other hand, if $V'(0) \leq 0$, the maximum of the value function is 0 at $u = 0$ since V is concave (Proposition 4.2 and 5.2). Thus, the project is infeasible. Note that from (HJB_V), $V'(0) > 0$ is equivalent to

$$\max\left[\lambda_D \Pi - \phi, \lambda_S V_S\left(\phi/\lambda_S\right)\right] > c, \tag{C.1}$$

i.e., the project is feasible if at least one of the instantaneous payoff at the deadline covers the operating cost c. Note that

$$\begin{aligned}\lambda_S V_S(\phi/\lambda_S) =& \lambda_S(\Pi - c/\lambda_S)\left(1 - e^{-1}\right) - \phi\\ =& (1+\eta)(1-e^{-1})\lambda_D\Pi - (1-e^{-1})c - \phi.\end{aligned}$$

Then, we can derive that $\lambda_S V_S(\phi/\lambda_S) > c$ is equivalent to

$$\Pi > \frac{(2-e^{-1})c+\phi}{(1+\eta)(1-e^{-1})\lambda_D},$$

whereas $\lambda_D \Pi - \phi > c$ is equivalent to $\Pi > \dfrac{c+\phi}{\lambda_D}$. By simple algebra, we can show that

$$\frac{(2-e^{-1})c+\phi}{(1+\eta)(1-e^{-1})\lambda_D} > \frac{c+\phi}{\lambda_D}.$$

Therefore, $\Pi > \Pi_F \equiv \dfrac{c+\phi}{\lambda_D}$ is equivalent to (C.1) and the project is feasible if this condition is satisfied.

C.2 The Length of the Contract (Π_D)

Given that the project is feasible, there exists $\bar{u} > 0$ that maximizes $V(u)$. Since V is concave and differentiable, $\bar{u}$ is the solution of $V'(\bar{u}) = 0$. To check what type of contract would be utilized to implement $(\bar{u}, V(\bar{u}))$, we need to compare $\bar{u}$ with switch points u_1 and u_2 defined in (4.2) and (5.1).

If Π is less than or equal to $\Pi_M(\eta)$, by Proposition 4.2 and 5.2, the value function is equal to $V^d(u)$, i.e., it does not have any switch point. Therefore, it is enough to restrict attention to the case where Π is greater than $\Pi_M(\eta)$.

If Π is greater than $\Pi_M(\eta)$ and less than $\Pi_S(\eta)$, there exists a switch point $u_1 > 0$ such that $V^d(u_1) = V^{ds'}(u_1|u_1)$. Note that if Π is greater than or equal to $\Pi_S(\eta)$ and η is greater than $1/(e-1)$, the value function near $u = 0$ corresponds to $V^{ds}(u|0)$ by Proposition 4.2 and 5.2. Thus, in this case, we can consider u_1 as 0. The following lemma characterizes the threshold of Π for comparing $\bar{u}$ and u_1.

Lemma C.1. *Suppose that $\Pi > \Pi_M(\eta)$ and $2\lambda_D \geq \lambda_S > \lambda_D$ are satisfied. Then, there exists $\Pi_D(\eta) \geq \Pi_M(\eta)$ such that $u_1 < \bar{u}$ if and only if $\Pi > \Pi_D(\eta)$. Moreover, if $\eta \leq \sqrt{c/(c+\phi)}$, $\Pi_D(\eta)$ is equal to $\Pi_M(\eta)$.*

Now I compare $\bar{u}$ and the second switching point u_2. The following lemma shows that u_2 is less than $\bar{u}$ if η is sufficiently small. Thus, in this case, the optimal contract involves two switches of approaches.

Lemma C.2. *Suppose that $\Pi > \Pi_M(\eta)$ and $\eta \leq c/(c+\phi)$ are satisfied. Then, u_2 is less than $\bar{u}$.*

On the other hand, the following lemma shows that u_2 is greater than $\bar{u}$ if η is close to 1. Thus, in this case, the optimal contract involves at most one switch of approaches.

Lemma C.3. *Suppose that $\Pi \geq \Pi_D(\eta)$ and $\eta \geq \sqrt{c/(c+\phi)}$ are satisfied. Then, u_2 is greater than or equal to $\bar{u}$.*

The proofs for the above lemmas are relegated to Appendix E

C.3 Proofs of Theorems

Proof of Theorem 1. (a) By the argument in Appendix C.1, the project is infeasible when Π is less than Π_F.

(b) When $\Pi_D(1) \geq \Pi > \Pi_F = c/\lambda_D = \Pi_M(1)$, by Lemma C.1, the switching point u_1 is greater than or equal to $\bar{u}$. Then, $V(\bar{u}) = V^d(\bar{u})$ by (a) of Proposition 4.2. In both cases, by (a) of Proposition 4.1, $(\bar{u}, V(\bar{u}))$ is implemented by a direct-only contract with the deadline $\bar{u}/\phi$.

(c) When $\Pi \in (\Pi_D(1), \Pi_S(1))$, u_1 is greater than 0 and less than $\bar{u}$ by Lemma C.1 and Proposition 4.2 (a). Then, $V(\bar{u}) = V^{ds}(\bar{u}|u_1)$ by Proposition 4.2. By (b) of Proposition 4.1, $(\bar{u}, V(\bar{u}))$ is implemented by a contract with a switch from the sequential approach to the direct approach at $(\bar{u} - u_1)/\phi$ and the deadline $\bar{u}/\phi$.

(d) When $\Pi \geq \Pi_S(1)$, $V(u) = V^{ds}(u|0)$ by (b) of Proposition 4.2. By (c) of Proposition 4.1, $(\bar{u}, V(\bar{u}))$ is implemented by a sequential-only contract with the deadline $\bar{u}/\phi$.

□

Proof of Theorem 2. Note that u_2 is greater than $\bar{u}$ by Lemma C.3 since $\eta > \sqrt{c/(c+\phi)}$. When $\Pi \in [\Pi_F, \Pi_M(\eta)]$, $V(\bar{u}) = V^d(\bar{u})$ by (a) of Proposition 5.2. Then, by (a) of Proposition 4.1, $(\bar{u}, V(\bar{u}))$ is implemented by a direct-only contract with the deadline $\bar{u}/\phi$. For other cases, the statements can be similarly proved as in Theorem 1, except that we need to use Proposition 5.2 instead of Proposition 4.2. □

Proof of Theorem 3. The proof for part (a) is same as Theorem 1, thus, it is enough to show (b) and (c).

(b) When $\Pi \in [\Pi_F, \Pi_M(\eta)]$, $V(\bar{u}) = V^d(\bar{u})$ by (a) of Proposition 5.2. By (a) of Proposition 4.1, $(\bar{u}, V(\bar{u}))$ is implemented by a direct-only contract with the deadline $\bar{u}/\phi$.

(c) When $\Pi > \Pi_M(\eta)$, since $\eta < c/(c+\phi)$ is assumed, u_2 is less than $\bar{u}$ by Lemma C.2. Also note that (b) of Proposition 5.2 applies since $\eta < 1/(e-1)$. Therefore, $V(\bar{u})$ is equal to $V^{dsd}(u|u_1,u_2)$. By (a) of Proposition 4.1, $(\bar{u}, V(\bar{u}))$ is implemented by a contract with two switches at $(\bar{u}-u_2)/\phi$ and $(\bar{u}-u_1)/\phi$ and the deadline $\bar{u}/\phi$.

□

D Proofs for Value Function Characterization

In this section, I provide the proofs for value function characterization. In Section D.1, by using the smooth pasting conditions, I identify the thresholds Π_S and Π_M which determine the number of potential switches in the value function. In Section D.2, I define the functions that specify deviations from the given value function and present useful lemmas. By using these results, I prove Proposition 4.2 and 5.2 in Section D.3.

D.1 Conditions for Smooth Pasting

The goal of this analysis is to construct a differentiable and concave function V that solves (HJB$_V$) subject to (IC) with $V(0) = 0$ and apply Lemma B.2. To construct such 'differentiable' function, I need to find a point where one benchmark function has the same first derivative to another benchmark function, i.e., two benchmark functions are 'smoothly pasted.' Depending on the parameter Π, such a switching point may or may not exist. In the following propositions, I formally introduce threshold of Π, Π_S and Π_M, as functions of η $(= \lambda_S/\lambda_D - 1)$ and characterize the conditions for smooth pasting by using those thresholds.

Proposition D.1. *Suppose that η is equal to 1. There exists $\Pi_S(1) > c/\lambda_D$ such that the following statements hold.*

(a) If $\Pi_M(1) = c/\lambda_D \leq \Pi \leq \Pi_S(1)$, there exists $u_1 \geq 0$ such that $V^{d\prime}(u) > V^{ds\prime}(u|u)$ for all $0 \leq u < u_1$, $V^{d\prime}(u_1) = V^{ds\prime}(u_1|u_1)$, $V^{d\prime\prime}(u_1) < V^{ds\prime\prime}(u_1|u_1)$, and $V^{ds\prime}(u|u_1) > V^{dsd\prime}(u|u,u_1)$ for all $u > u_1$.

(b) If $\Pi > \Pi_S(1)$, $V^{ds\prime}(0|0) > V^{d\prime}(0)$ and $V^{ds\prime}(u|0) > V^{dsd\prime}(u|0,u)$ for all $u \geq 0$.

Proposition D.2. *Suppose that η is less than 1. There exists $\Pi_S : (1/(e-1), 1) \to \mathbb{R}_+$ and $\Pi_M : (0,1) \to \mathbb{R}_+$ such that the following statements hold.*

(a) If $c/\lambda_D < \Pi < \Pi_M(\eta)$, $V^{d\prime}(u) > V^{ds\prime}(u|u)$ for all $u \geq 0$.

(b) Suppose that one of the followings hold: (i) $\eta \le 1/(e-1)$ and $\Pi \ge \Pi_M(\eta)$; (ii) $1/(e-1) < \eta < 1$ and $\Pi \in [\Pi_M(\eta), \Pi_S(\eta)]$. Then, there exist a pair (u_1, u_2) with $u_2 \ge u_1 \ge 0$ such that:

(i) $V^{d'}(u) > V^{ds'}(u|u)$ for all $0 \le u < u_1$,

(ii) $V^{d'}(u_1) = V^{ds'}(u_1|u_1)$ and $V^{d''}(u_1) < V^{ds''}(u_1|u_1)$,

(iii) $V^{ds'}(u|u_1) > V^{dsd'}(u|u, u_1)$ for all $u_2 > u > u_1$,

(iv) $V^{ds'}(u_2|u_1) = V^{dsd'}(u_2|u_1, u_2)$ and $V^{ds''}(u_2|u_1) < V^{dsd''}(u_2|u_1, u_2)$,

(v) $V^{dsd'}(u|u_1, u_2) > \frac{1}{\phi}\left[\lambda_S(V_S(u + \phi/\lambda_S) - V^{dsd}(u|u_1, u_2)) - c\right]$ for all $u > u_2$.

(c) If $1/(e-1) < \eta < 1$ and $\Pi > \Pi_S(\eta)$, $V^{ds'}(0|0) > V^{d'}(0)$ and there exists $u_2 > 0$ such that:

(i) $V^{ds'}(u_2|0) = V^{dsd'}(u_2|0, u_2)$, $V^{ds''}(u_2|0) < V^{dsd''}(u_2|0, u_2)$ and $V^{ds'}(u|0) > V^{dsd'}(u|0, u)$ for all $u_2 > u \ge 0$,

(ii) $V^{dsd'}(u|0, u_2) > \frac{1}{\phi}\left[\lambda_S(V_S(u + \phi/\lambda_S) - V^{dsd}(u|0, u_2)) - c\right]$ for all $u > u_2$.

D.1.1 Useful Lemmas

Lemma D.1. *If η is less than or equal to $1/(e-1)$, the inequality $V^{d'}(0) > V^{ds'}(0|0)$ always holds. If η is greater than $1/(e-1)$, $V^{d'}(0) > V^{ds'}(0|0)$ is equivalent to $\Pi < \Pi_S(\eta)$. Moreover, if $\Pi = \Pi_S(\eta)$, then $V^{d'}(0) = V^{ds'}(0|0)$ and $V^{d''}(0) < V^{ds''}(0|0)$.*

Proof of Lemma D.1. By Proposition B.1, $V^{d'}(0) = \alpha_D(0)$ and $V^{ds'}(0|0) = \alpha_S(0)$. By using the arguments in Section 3.4, when $\eta \le 1/(e-1)$, $V^{d'}(0) > V^{ds'}(0|0)$ always hold, and when $\eta > 1/(e-1)$, $V^{d'}(0) > V^{ds'}(0|0)$ is equivalent to $\Pi < \Pi_S(\eta)$.

When $\Pi = \Pi_S(\eta)$, $V^{d'}(0) = V^{ds'}(0|0)$. Also note that

$$\phi V^{d''}(0) = -\lambda_D\left(1 + V^{d'}(0)\right) = -\frac{\lambda_D}{\phi}(\lambda_D\Pi - c),$$

$$\begin{aligned}
\phi V^{ds''}(0|0) &= \lambda_S V_S'(\phi/\lambda_S) - \lambda_S V^{ds'}(0|0) = \frac{\lambda_S}{\phi}\left(\phi V_S'(\phi/\lambda_S) - \phi V^{ds'}(0|0)\right) \\
&= \frac{\lambda_S}{\phi}\left(\phi V_S'(\phi/\lambda_S) - \lambda_S V_S(\phi/\lambda_S) + c\right) \\
&= \frac{\lambda_S}{\phi}\left(\lambda_S\Pi - 2\lambda_S(V_S(\phi/\lambda_S) + \phi/\lambda_S)\right)
\end{aligned}$$

When $\Pi = \Pi_S(\eta)$, $\lambda_D \Pi = \lambda_S \left(V_S(\phi/\lambda_S) + \phi/\lambda_S\right)$ from $V^{d\prime}(0) = V^{ds\prime}(0|0)$. Then,

$$\begin{aligned}
\phi^2 V^{ds\prime\prime}(0|0) - \phi^2 V^{d\prime\prime}(0) &= \lambda_S(\lambda_S \Pi_S(\eta) - 2\lambda_D \Pi_S(\eta)) + \lambda_D(\lambda_D \Pi_S(\eta) - c) \\
&= \lambda_D^2 \left[\left(\frac{\lambda_S}{\lambda_D} - 1 \right)^2 \Pi_S(\eta) - \frac{c}{\lambda_D} \right] \\
&= \lambda_D^2 \left[\eta^2 \Pi_S(\eta) - \frac{c}{\lambda_D} \right] \\
&= \lambda_D c \left[\frac{(e-1)\eta^2}{(e-1)\eta - 1} - 1 \right]
\end{aligned}$$

Since $(e-1)x^2 > (e-1)x - 1$ for all $x > 1/(e-1)$, we can see that $V^{ds\prime\prime}(0|0) > V^{d\prime\prime}(0)$. $\square$

Lemma D.2. *There exists $\Pi_M(\eta) \geq 2c/\lambda_S$ with $\Pi_M(1) = 2c/\lambda_S = c/\lambda_D$ such that the following statements hold.*

(a) If $c/\lambda_D \leq \Pi < \Pi_M(\eta)$, $V^{d\prime}(u) > V^{ds\prime}(u|u)$ for all $u \geq 0$.

(b) Suppose that one of the following statements hold: (i) $\eta \leq 1/(e-1)$ and $\Pi > \Pi_M(\eta)$; (ii) $\eta > 1/(e-1)$ and $\Pi_S(\eta) \geq \Pi > \Pi_M(\eta)$. Then, there exists $u_1 > 0$ such that $V^{d\prime}(u_1) = V^{ds\prime}(u_1|u_1)$, $V^{d\prime\prime}(u_1) < V^{ds\prime\prime}(u_1|u_1)$ and $V^{d\prime}(u) > V^{ds\prime}(u|u)$ for all $u \in [0, u_1)$;

Proof of Lemma D.2. Consider a function $H_1 : \mathbb{R}_+ \to \mathbb{R}$ defined as follows:

$$H_1(u) \equiv \phi V^{ds\prime}(u|u) - \phi V^{d\prime}(u). \tag{D.1}$$

Then, $H_1(u)$ can be rewritten as follows:

$$\begin{aligned}
H_1(u) &= \lambda_S \left((\Pi - c/\lambda_S)(1 - e^{-\frac{\lambda_S}{\phi}u - 1}) - V^d(u) - u \right) - \lambda_D(\Pi - u - V^d(u)) \\
&= (\lambda_S \Pi - c)\left(1 - e^{-\frac{\lambda_S}{\phi}u - 1}\right) - (\lambda_S - \lambda_D)(u + V^d(u)) - \lambda_D \Pi \\
&= (\lambda_S \Pi - c)\left(1 - e^{-\frac{\lambda_S}{\phi}u - 1}\right) - (\lambda_S - \lambda_D)(\Pi - c/\lambda_D)(1 - e^{-\frac{\lambda_D}{\phi}u}) - \lambda_D \Pi \\
&= (\lambda_S/\lambda_D - 1)(\lambda_D \Pi - c)e^{-\frac{\lambda_D}{\phi}u} - (2 - \lambda_S/\lambda_D)c - (\lambda_S \Pi - c)e^{-\frac{\lambda_S}{\phi}u - 1} \\
&= \eta(\lambda_D \Pi - c)e^{-\frac{\lambda_D}{\phi}u} - (\lambda_S \Pi - c)e^{-\frac{\lambda_S}{\phi}u - 1} - (1 - \eta)c \\
&= -\left\{(\eta + 1)\lambda_D \Pi - c\right\} e^{-1} e^{-\frac{(\eta+1)\lambda_D}{\phi}u} + \eta(\lambda_D \Pi - c)e^{-\frac{\lambda_D}{\phi}u} - (1 - \eta)c
\end{aligned}$$

Define $x \equiv e^{-\frac{\lambda_D}{\phi}u}$. Then, by considering $H_1(u)$ as a function of x and Π, it can be rewritten as follows:

$$\tilde{H}_1(x;\Pi) \equiv -\left\{(\eta+1)\lambda_D\Pi - c\right\}e^{-1}x^{\eta+1} + \eta(\lambda_D\Pi - c)x - (1-\eta)c.$$

Observe that

$$\frac{\partial^2 \tilde{H}_1}{\partial x^2}(x;\Pi) = -(\eta+1)\eta\left\{(\eta+1)\lambda_D\Pi - c\right\}e^{-1}x^{\eta-1},$$

thus $\tilde{H}_1$ is a strict concave function in x when $\Pi \geq c/\lambda_D$. Let $x^*(\Pi)$ be the solution of $\max_x H_1(x;\Pi)$ subject to $0 \leq x \leq 1$. Then, when $\Pi \geq c/\lambda_D$, from the first order condition, we can derive that

$$x^*(\Pi) = \left[\frac{\eta(\lambda_D\Pi - c)}{(\eta+1)\{(\eta+1)\lambda_D\Pi - c\}e^{-1}}\right]^{\frac{1}{\eta}}.$$ [22]

Now define

$$h(\Pi) \equiv \tilde{H}_1(x^*(\Pi);\Pi) = K\left(\frac{\lambda_D\Pi - c}{\lambda_S\Pi - c}\right)^{\frac{1}{\eta}}(\lambda_D\Pi - c) - (1-\eta)c$$

where $K = \frac{\eta^2}{\eta+1}\left(\frac{\eta e}{\eta+1}\right)^{\frac{1}{\eta}}$. Observe that

$$h\left(\frac{2c}{\lambda_S}\right) = (1-\eta)c\left[\frac{\eta^2}{(\eta+1)^2}\left(\frac{\eta(1-\eta)e}{(\eta+1)^2}\right)^{\frac{1}{\eta}} - 1\right] < 0$$

from $\eta < 1$ and $\eta(1-\eta)e \leq e/4 < 1 \leq (\eta+1)^2$. In addition, $\lim_{\Pi\to\infty} h(\Pi) = \infty$ and

$$h'(\Pi) = K(\lambda_D\Pi - c)^{1/\eta}(\lambda_S\Pi - c)^{-1/\eta-1}\lambda_D\lambda_S\Pi > 0.$$

Therefore, there exists a unique Π such that $h(\Pi) = 0$ and $\Pi \geq 2c/\lambda_S$. Let the solution of $h(\Pi) = 0$ with $\Pi \geq 2c/\lambda_S$ be $\Pi_M(\eta)$. Also note that when $\eta = 1$, $h(2c/\lambda_S) = 0$ thus $\Pi_M(1) = 2c/\lambda_S = c/\lambda_D$.

(a) Suppose that $c/\lambda_D \leq \Pi < \Pi_M(\eta)$. We have $0 > h(\Pi) = \tilde{H}_1(x^*(\Pi);\Pi) \geq \tilde{H}_1(x;\Pi)$ for all $0 \leq x \leq 1$. It is equivalent to $0 > H_1(u) = \phi(V^{ds\prime}(u|u) - V^{d\prime}(u))$ for all $u \geq 0$, thus $V^{d\prime}(u) > V^{ds\prime}(u|u)$ for all $u \geq 0$ in this case.

(b) First, suppose that $\eta \leq 1/(e-1)$ and $\Pi > \Pi_M(\eta)$. Then, we have $0 < h(\Pi) = \tilde{H}_1(x^*(\Pi);\Pi)$. In addition, by Lemma D.1, we have $\tilde{H}_1(1;\Pi) = \phi(V^{ds\prime}(0|0) - V^{d\prime}(0)) <$

[22] When $\Pi \leq \frac{(\eta+1)e^{-1}-\eta}{(\eta+1)^2e^{-1}-\eta}\cdot\frac{c}{\lambda_D}$, the solution of the maximization problem $\max_{0\leq x\leq 1} H_1(x;\Pi)$ is $x^*(\Pi) = 1$. However, we can show that $\frac{(\eta+1)e^{-1}-\eta}{(\eta+1)^2e^{-1}-\eta} < 1$ for any $0 < \eta$, which implies that we can focus on the interior solution when $\Pi \geq c/\lambda_D$

0. Then, by concavity of $\tilde{H}_1$ w.r.t. x and $\frac{\partial \tilde{H}_1}{\partial x}(x^*(\Pi);\Pi) = 0$, there exists $x_1 \in (x^*(\Pi), 1]$ such that $\tilde{H}_1(x_1;\Pi) = 0$, $\frac{\partial \tilde{H}_1}{\partial x}(x_1;\Pi) < 0$ and $\tilde{H}_1(x;\Pi) < 0$ for all $x \in (x_1, 1]$. By defining $u_1 \equiv -\frac{\phi}{\lambda_D}\log x_1$, the above conditions can be translated into: $V^{d\prime}(u_1) = V^{ds\prime}(u_1|u_1)$, $V^{d\prime\prime}(u_1) < V^{ds\prime\prime}(u_1|u_1)$ and $V^{d\prime}(u) > V^{ds\prime}(u|u)$ for all $u \in [0, u_1)$.

Next, suppose that $\eta > 1/(e-1)$. Note that by the definition of $\Pi_S(\eta)$, if $\Pi \geq \Pi_S(\eta)$, $H_1(0) = \tilde{H}_1(1;\Pi) \geq 0$. It implies that $h(\Pi) \geq \tilde{H}_1(1;\Pi) \geq 0$ and $\Pi \geq \Pi_M(\eta)$. Therefore, we can see that $\Pi_S(\eta) \geq \Pi_M(\eta)$. If $\Pi_S(\eta) \geq \Pi > \Pi_M(\eta)$, we also have $\tilde{H}_1(x^*(\Pi);\Pi) > 0 > \tilde{H}_1(1;\Pi)$. By using the same arguments as above, we can show that there exists $u_1 > 0$ such that $V^{d\prime}(u_1) = V^{ds\prime}(u_1|u_1)$, $V^{d\prime\prime}(u_1) < V^{ds\prime\prime}(u_1|u_1)$ and $V^{d\prime}(u) > V^{ds\prime}(u|u)$ for all $u \in [0, u_1)$.

$\square$

Lemma D.3. *Suppose that $\Pi > c/\lambda_D$ and one of the followings hold: (i) $V^{d\prime}(u_1) < V^{ds\prime}(u_1|u_1)$; (ii) $V^{d\prime}(u_1) = V^{ds\prime}(u_1|u_1)$ and $V^{d\prime\prime}(u_1) < V^{ds\prime\prime}(u_1|u_1)$.*

(a) If $\eta = 1$, $V^{ds\prime}(u|u_1) > V^{dsd\prime}(u|u_1, u)$ for all $u > u_1$.

(b) If $\eta < 1$, there exists $u_2 > u_1$ such that $V^{ds\prime}(u_2|u_1) = V^{dsd\prime}(u_2|u_1, u_2)$ and $V^{ds\prime\prime}(u_2|u_1) < V^{dsd\prime\prime}(u_2|u_1, u_2)$ and such u_2 is unique. Moreover, $V^{ds\prime}(u|u_1) > V^{dsd\prime}(u|u_1, u)$ for all $u \in (u_1, u_2)$.

Proof of Lemma D.3. Define a function $H_2 : [u_1, \infty) \to \mathbb{R}$ as $H_2(u) \equiv \phi V^{dsd\prime}(u|u_1, u) - \phi V^{ds\prime}(u|u_1)$. From $\phi V^{dsd\prime}(u|u_1, u) = -c + \lambda_D \left(\Pi - \phi/\lambda_D - u - V^{dsd}(u|u_1, u)\right)$ (by (B.5)), $V^{dsd}(u|u_1, u) = V^{ds}(u|u_1)$ and (4.2), $H_2(u)$ can be rewritten as follows:

$$
\begin{aligned}
H_2(u) &= \lambda_D \left(\Pi - \frac{\phi}{\lambda_D} - u - V^{ds}(u|u_1)\right) - c - \phi V^{ds\prime}(u|u_1) \\
&= \frac{2\lambda_D - \lambda_S}{\lambda_S} c + (\lambda_S \Pi - c) e^{-1-\frac{\lambda_S}{\phi}u_1} \left[1 + \frac{\lambda_S - \lambda_D}{\lambda_S}\frac{\lambda_S}{\phi}(u_1 - u)\right] e^{\frac{\lambda_S}{\phi}(u_1 - u)} \\
&\quad - (\lambda_S - \lambda_D)\left[\Pi - \frac{2c}{\lambda_S} - (V^d(u_1) + u_1)\right] e^{\frac{\lambda_S}{\phi}(u_1-u)} \\
&= \frac{1-\eta}{1+\eta} c + (\lambda_S \Pi - c) e^{-1-\frac{\lambda_S}{\phi}u_1} \left[1 + \frac{\eta}{1+\eta}\frac{\lambda_S}{\phi}(u_1 - u)\right] e^{\frac{\lambda_S}{\phi}(u_1-u)} \\
&\quad + \eta \left[\frac{1-\eta}{1+\eta} c - (\lambda_D \Pi - c) e^{-\frac{\lambda_D}{\phi}u_1}\right] e^{\frac{\lambda_S}{\phi}(u_1-u)}
\end{aligned}
$$

Define $x \equiv e^{\frac{\lambda_S}{\phi}(u_1-u)}$. Then, $H_2(u)$ can be rewritten as follows:

$$\begin{aligned}\tilde{H}_2(x) \;&\equiv\; \frac{1-\eta}{1+\eta}c + (\lambda_S\Pi - c)e^{-1-\frac{\lambda_S}{\phi}u_1}\left[1+\frac{\eta}{1+\eta}\log x\right]x \\ &\qquad +\eta\left[\frac{1-\eta}{1+\eta}c - (\lambda_D\Pi - c)e^{-\frac{\lambda_D}{\phi}u_1}\right]x.\end{aligned} \tag{D.2}$$

Note that $\tilde{H}_2(1) = H_2(u_1) = \phi V^{dsd'}(u_1|u_1,u_1) - \phi V^{ds'}(u_1|u_1) = \phi V^{d'}(u_1) - \phi V^{ds'}(u_1|u_1) \leq 0$. By differentiating $\tilde{H}$ twice, we have

$$\tilde{H}_2''(x) = \frac{\eta}{1+\eta}(\lambda_S\Pi - c)e^{-1-\frac{\lambda_S}{\phi}u_1}\frac{1}{x} > 0.$$

Since $\Pi > c/\lambda_D > c/\lambda_S$, $\tilde{H}_2$ is strictly convex in x. Also note that

$$\lim_{x\to 0}\tilde{H}_2(x) = \frac{1-\eta}{1+\eta}c$$

and we simply denote by $\tilde{H}_2(0)$.

(a) Suppose that $\eta = 1$. Then, by the strict convexity of $\tilde{H}_2$, for all $x \in (0,1)$,

$$\tilde{H}_2(x) < (1-x)\tilde{H}_2(0) + x\tilde{H}_2(1) \leq 0.$$

Therefore, for all $u > u_1$, $H_2(u) < 0$, i.e., $V^{ds'}(u|u_1) > V^{dsd'}(u|u_1,u)$.

(b) Suppose that $\eta < 1$. Then, $\tilde{H}_2(0) > 0$. If $V^{ds'}(u_1|u_1) < V^{d'}(u_1)$, $\tilde{H}_2(1) < 0$. In this case, there exists $x_2 \in (0,1)$ such that $\tilde{H}_2(x_2) = 0$. Let $u_2 = u_1 - \frac{\phi}{\lambda_S}\log x_2$. Then, $H(u_2) = 0$, i.e., $V^{ds'}(u_2|u_1) = V^{dsd'}(u_2|u_1,u_2)$.

Next, consider the case with $V^{ds'}(u_1|u_1) = V^{d'}(u_1)$ and $V^{ds''}(u_1|u_1) > V^{d''}(u_1)$. By differentiating (B.5) and (B.6) once, we have

$$\begin{aligned}\phi V^{d''}(u_1) &= -\lambda_D(1+V^{d'}(u_1)), \\ \phi V^{ds''}(u_1|u_1) &= \lambda_S\left(V_S'(u+\phi/\lambda_S) - V^{ds'}(u|u_1)\right).\end{aligned}$$

Then, from the above expressions and $V^{ds'}(u_1|u_1) = V^{d'}(u_1)$, $V^{ds''}(u_1|u_1) > V^{d''}(u_1)$ is equivalent to :

$$
\begin{aligned}
& \lambda_S \left(V_S'(u_1 + \phi/\lambda_S) + 1 \right) > (\lambda_S - \lambda_D)(1 + V^{d'}(u_1)) \\
\iff \quad & \frac{\lambda_S}{\phi} \lambda_S \left(\Pi - \frac{c}{\lambda_S} \right) e^{-\frac{\lambda_S}{\phi} u_1 - 1} > (\lambda_S - \lambda_D) \frac{\lambda_D}{\phi} \left(\Pi - \frac{c}{\lambda_D} \right) e^{-\frac{\lambda_D}{\phi} u_1} \\
\iff \quad & (\eta + 1)(\lambda_S \Pi - c) e^{-\frac{\lambda_S}{\phi} u_1 - 1} > \eta(\lambda_D \Pi - c) e^{-\frac{\lambda_D}{\phi} u_1}.
\end{aligned} \tag{D.3}
$$

Note that $\tilde{H}_2(1) = (1-\eta)c + (\lambda_S \Pi - c)e^{-1-\frac{\lambda_S}{\phi}u_1} - \eta(\lambda_D \Pi - c)e^{-\frac{\lambda_S}{\phi}u_1} = 0$ from $V^{ds'}(u_1|u_1) = V^{d'}(u_1)$. Then,

$$
\begin{aligned}
\tilde{H}_2'(1) &= (\lambda_S \Pi - c)e^{-1-\frac{\lambda_S}{\phi}u_1} \left[1 + \frac{\eta}{1+\eta} \right] + \eta \left[\frac{1-\eta}{1+\eta} c - (\lambda_D \Pi - c) e^{-\frac{\lambda_D}{\phi} u_1} \right] \\
&= (\lambda_S \Pi - c)e^{-1-\frac{\lambda_S}{\phi}u_1} - \frac{\eta}{1+\eta}(\lambda_D \Pi - c)e^{-\frac{\lambda_D}{\phi}u_1} \\
&\geq 0.
\end{aligned}
$$

The last inequality is due to (D.3).

$\tilde{H}_2(1) = 0$ and $\tilde{H}_2'(1) > 0$ imply $\tilde{H}_2(1-\epsilon) < 0$ for small enough $\epsilon > 0$. Then, since $\tilde{H}_2(0) > 0$ and $\tilde{H}_2(1-\epsilon) < 0$, there exists $x_2 \in (0, 1-\epsilon)$ such that $\tilde{H}_2(x_2) = 0$ & $\tilde{H}_2'(x_2) < 0$, thus there exists u_2 such that $V^{ds'}(u_2|u_1) = V^{dsd'}(u_2|u_1, u_2)$ & $V^{ds''}(u_2|u_1) < V^{dsd''}(u_2|u_1, u_2)$.

Suppose that there exists another u_2' with $H_2(u_2') = 0$. Consider corresponding x_2', then $\tilde{H}_2(x_2') = 0$. If $x_2' > x_2$,

$$
0 = \tilde{H}_2(x_2'|u_1) < \frac{1-x_2'}{1-x_2}\tilde{H}_2(x_2) + \frac{x_2'-x_2}{1-x_2}\tilde{H}_2(1) \leq 0
$$

from $\tilde{H}_2(x_2) = 0$ and $\tilde{H}_2(1) \leq 0$. Similar logic holds for the case of $x_2' < x_2$. Therefore, there is unique u_2 satisfying $H_2(u_2) = 0$.

From $\tilde{H}_2(x_2) = 0$, $\tilde{H}_2(1) < 0$ and the strict convexity of $\tilde{H}_2$, for all $x \in (x_2, 1)$,

$$
\tilde{H}_2(x) < \frac{1-x}{1-x_2}\tilde{H}_2(x_2) + \frac{x-x_2}{1-x_2}\tilde{H}_2(1) < 0.
$$

Therefore, for all $u_2 > u > u_1$, $H_2(u) < 0$, i.e., $V^{ds'}(u|u_1) > V^{dsd'}(u|u_1, u)$.

□

Lemma D.4. *Suppose that* $\Pi > c/\lambda_D$, $V^{ds\prime}(u_2|u_1) = V^{dsd\prime}(u_2|u_1, u_2)$ *and* $V^{ds\prime\prime}(u_2|u_1) < V^{dsd\prime\prime}(u_2|u_1, u_2)$. *Then,* $\lambda_S \left(V_S(u + \phi/\lambda_S) - V^{dsd}(u|u_1, u_2)\right) - \phi V^{dsd\prime}(u|u_1, u_2) - c < 0$ *for all* $u > u_2$.

Proof of Lemma D.4. By (B.5) and (B.6), we have

$$\phi V^{ds\prime}(u|u_1) = \lambda_S \left(V_S\left(u + \phi/\lambda_S\right) + u + \phi/\lambda_S\right) - \lambda_S \left(V^{ds}(u|u_1) + u\right) - c - \phi,$$

$$\phi V^{dsd\prime}(u|u_1, u_2) = \lambda_D \Pi - \lambda_D \left(V^{dsd}(u|u_1, u_2) + u\right) - c - \phi.$$

By differentiating above equations, we have

$$\phi V^{ds\prime\prime}(u|u_1) = \lambda_S \left(V_S'\left(u + \phi/\lambda_S\right) + 1\right) - \lambda_S \left(V^{ds\prime}(u|u_1) + 1\right),$$

$$\phi V^{dsd\prime\prime}(u|u_1, u_2) = -\lambda_D \left(V^{dsd\prime}(u|u_1, u_2) + 1\right).$$

Then, $V^{ds\prime}(u_2|u_1) = V^{dsd\prime}(u_2|u_1, u_2)$ and $V^{ds\prime\prime}(u_2|u_1) < V^{dsd\prime\prime}(u_2|u_1, u_2)$ imply that

$$\begin{aligned} & (\lambda_S - \lambda_D)(1 + V^{ds\prime}(u_2|u_1)) > \lambda_S(V_S'(u_2 + \phi/\lambda_S) + 1) \\ \iff \quad & \eta(1 + V^{ds\prime}(u_2|u_1)) > (\eta + 1)\left(\frac{\lambda_S \Pi - c}{\phi}\right) e^{-\frac{\lambda_S}{\phi} u_2 - 1}. \end{aligned} \tag{D.4}$$

Define a function $H_3 : [u_2, \infty) \to \mathbb{R}$ as

$$\begin{aligned} H_3(u) &= \lambda_S \left[V_S(u + \phi/\lambda_S) - V^{dsd}(u|u_1, u_2)\right] - \phi V^{dsd\prime}(u|u_1, u_2) - c \\ &= \lambda_S \left[V_S(u + \phi/\lambda_S) + u + \phi/\lambda_S\right] - \lambda_S \left[u + V^{dsd}(u|u_1, u_2)\right] \\ &\quad - \phi(1 + V^{dsd\prime}(u|u_1, u_2)) - c \\ &= \left(\frac{\lambda_S}{\lambda_D} - 2\right) c - (\lambda_S \Pi - c) e^{-\frac{\lambda_S}{\phi} u - 1} \\ &\quad + \left(\frac{\lambda_S}{\lambda_D} - 1\right) \left[\lambda_D \Pi - c - \lambda_D (V^{ds}(u_2|u_1) + u_2)\right] e^{\frac{\lambda_D}{\phi}(u_2 - u)} \\ &= (\eta - 1)\, c - (\lambda_S \Pi - c) e^{-\frac{\lambda_S}{\phi} u_2 - 1} \cdot e^{\frac{\lambda_S}{\phi}(u_2 - u)} \\ &\quad + \eta \left[\lambda_D \Pi - c - \lambda_D \left(V^{ds}(u_2|u_1) + u_2\right)\right] e^{\frac{\lambda_D}{\phi}(u_2 - u)} \\ &= (\eta - 1)\, c - (\lambda_S \Pi - c) e^{-\frac{\lambda_S}{\phi} u_2 - 1} \cdot e^{\frac{\lambda_S}{\phi}(u_2 - u)} + \eta \phi \left(V^{ds\prime}(u_2|u_1) + 1\right) e^{\frac{\lambda_D}{\phi}(u_2 - u)}. \end{aligned}$$

Also note that

$$\begin{aligned}H_3(u_2) &= \lambda_S\left[V_S(u_2+\phi/\lambda_S) - V^{ds}(u_2|u_1)\right] - c - \phi V^{dsd'}(u_2|u_1,u_2)\\ &= \phi V^{ds'}(u_2|u_1) - \phi V^{dsd'}(u_2|u_1,u_2) = 0.\end{aligned}$$

Define $x \equiv e^{\frac{\lambda_D}{\phi}(u_2-u)}$. Then, $H_3(u)$ can be rewritten as follows:

$$\tilde{H}_3(x) = (\eta-1)c - (\lambda_S\Pi - c)e^{-\frac{\lambda_S}{\phi}u_2-1}x^{\eta+1} + \eta\phi\left(V^{ds'}(u_2|u_1)+1\right)x$$

and $\tilde{H}_3(1) = H_3(u_2) = 0$.

Note that

$$\tilde{H}_3'(x) = -(\eta+1)(\lambda_S\Pi - c)e^{-\frac{\lambda_S}{\phi}u_2-1}x^{\eta} + \eta\phi\left(V^{ds'}(u_2|u_1)+1\right).$$

By (D.4), we can derive that

$$\tilde{H}_3'(1) = -(\eta+1)(\lambda_S\Pi - c)e^{-\frac{\lambda_S}{\phi}u_2-1} + \eta\phi\left(V^{ds'}(u_2|u_1)+1\right) > 0.$$

Also note that

$$\tilde{H}_3''(x) = -(\eta+1)\eta(\lambda_S\Pi - c)e^{-\frac{\lambda_S}{\phi}u_2-1}x^{\eta-1} < 0.$$

Therefore, $\tilde{H}_3'(x) > 0$ for all $0 < x < 1$. Since $\tilde{H}_3(1) = 0$, $\tilde{H}_3(x) < 0$ for all $x \in (0,1)$. Thus, $\lambda_S\left(V_S(u+\phi/\lambda_S) - V^{dsd}(u|u_1,u_2)\right) - \phi V^{dsd'}(u|u_1,u_2) - c < 0$ for all $u \geq u_2$. □

Lemma D.5. *Suppose that $V'(u) \geq -1$. The solution of* (HJB_V) *subject to* (IC) *involves $b = 1$.*

Proof of Lemma D.5. Assume that $b^* < 1$ solves (HJB_V) subject to (IC). Observe that $((\Pi - R - V(u))\lambda_D a + (V_S(u_S) - V(u))\lambda_S(1-a))b^* = 0$ from (HJB_V). This is because $(\Pi - R - V(u))\lambda_D a + (V_S(u_S) - V(u))\lambda_S(1-a) = 0$ when $b^* \in (0,1)$.

Also note that $(-\phi + (R-u)\lambda_D a + (u_S - u)\lambda_S(1-a))\, b^* = 0$ from (HJB_{PK}). Then, we have $\dot{u} = -\phi$. By plugging this into (HJB_V), we have $0 = -c - \phi V'(u)$, i.e., $V'(u) = -c/\phi < -1$. It contradicts the assumption of $V'(u) \geq -1$. Therefore, b should be equal to 1 for the solution of (HJB_V) subject to (IC). □

Lemma D.6. *Suppose that $2\lambda_D \geq \lambda_S > \lambda_D$ and $\Pi > c/\lambda_D$. Then, the following statements hold:*

(a) $V^d(u) < \Pi - c/\lambda_D - u$, $V^{d'}(u) > -1$ and $V^{d''}(u) < 0$;

(b) *Suppose that* u_1 *satisfies* $V^{d\prime}(u_1) \le V^{ds\prime}(u_1|u_1)$. *Then, for all* $u \ge u_1$, $V^{ds}(u|u_1) < \Pi - c/\lambda_D - u$, $V^{ds\prime}(u|u_1) > -1$ *and* $V^{ds\prime\prime}(u|u_1) < 0$.

(c) *Suppose that* u_1 *and* u_2 *satisfy* $V^{d\prime}(u_1) \le V^{ds\prime}(u_1|u_1)$ *and* $u_2 > u_1$. *Then, for all* $u \ge u_2$, $V^{dsd}(u|u_1, u_2) < \Pi - c/\lambda_D - u$, $V^{dsd\prime}(u|u_1, u_2) > -1$ *and* $V^{dsd\prime\prime}(u|u_1, u_2) < 0$.

Proof of Lemma D.6. (a) By (4.1) and $e^{-\lambda_D u/\phi} > 0$, $V^d(u) < \Pi - c/\lambda_D - u$. By differentiating (4.1), we have

$$V^{d\prime}(u) = \left(\Pi - \frac{c}{\lambda_D}\right)\frac{\lambda_D}{\phi}e^{-\frac{\lambda_D}{\phi}u} - 1 > -1.$$

By differentiating once again, we have

$$V^{d\prime\prime}(u) = -\left(\Pi - \frac{c}{\lambda_D}\right)\frac{\lambda_D^2}{\phi^2}e^{-\frac{\lambda_D}{\phi}u} < 0.$$

(b) By (4.2), $\Pi - 2c/\lambda_S \le \Pi - c/\lambda_D$, $V^d(u_1) + u_1 < \Pi - c/\lambda_D$ and $u \ge u_1$, we can derive that

$$V^{ds}(u|u_1) < \Pi - \frac{c}{\lambda_D} - u.$$

From (3.1), (B.5) and (B.6), we can derive that

$$\begin{aligned} V^{d\prime}(u_1) &= -\frac{c}{\phi} - 1 + \frac{\lambda_D}{\phi}\left(\Pi - u_1 - V^d(u_1)\right), \\ V^{ds\prime}(u_1|u_1) &= -\frac{c}{\phi} - 1 + \frac{\lambda_S}{\phi}\left(\left(\Pi - \frac{c}{\lambda_S}\right)\left(1 - e^{-\frac{\lambda_S}{\phi}u_1 - 1}\right) - V^d(u_1) - u_1\right). \end{aligned}$$

Then, $V^{d\prime}(u_1) \le V^{ds\prime}(u_1|u_1)$ is equivalent to

$$u_1 + V^d(u_1) \le \Pi - \frac{c}{\lambda_S - \lambda_D} - \frac{\lambda_S}{\lambda_S - \lambda_D}\left(\Pi - \frac{c}{\lambda_S}\right)e^{-\frac{\lambda_S}{\phi}u_1 - 1}. \tag{D.5}$$

By differentiating (4.2) twice, we have

$$
\begin{aligned}
V^{ds''}(u|u_1) = &-\left(\frac{\lambda_S}{\phi}\right)^2 e^{\frac{\lambda_S}{\phi}(u_1-u)}\left[\left(\Pi-\frac{2c}{\lambda_S}\right)-\left(V^d(u_1)+u_1\right)\right] \\
&-\left(\Pi-\frac{c}{\lambda_S}\right)\left(\frac{\lambda_S}{\phi}\right)^2\left[-2+\frac{\lambda_S}{\phi}(u-u_1)\right]e^{-\frac{\lambda_S}{\phi}u-1} \\
= &\left(\frac{\lambda_S}{\phi}\right)^2 e^{\frac{\lambda_S}{\phi}(u_1-u)}\left[\left(V^d(u_1)+u_1\right)-\left(\Pi-\frac{2c}{\lambda_S}\right)+2\left(\Pi-\frac{c}{\lambda_S}\right)e^{-\frac{\lambda_S}{\phi}u_1-1}\right] \\
&-\left(\Pi-\frac{c}{\lambda_S}\right)\left(\frac{\lambda_S}{\phi}\right)^3(u-u_1)e^{-\frac{\lambda_S}{\phi}u-1}.
\end{aligned}
$$

By using (D.5), we can show that

$$
\begin{aligned}
V^{ds''}(u|u_1) \le &\left(\frac{\lambda_S}{\phi}\right)^2 e^{\frac{\lambda_S}{\phi}(u_1-u)}\left[\frac{\lambda_S-2\lambda_D}{\lambda_S(\lambda_S-\lambda_D)}c+\frac{\lambda_S-2\lambda_D}{\lambda_S-\lambda_D}\left(\Pi-\frac{c}{\lambda_S}\right)e^{-\frac{\lambda_S}{\phi}u_1-1}\right] \\
&-\left(\Pi-\frac{c}{\lambda_S}\right)\left(\frac{\lambda_S}{\phi}\right)^3(u-u_1)e^{-\frac{\lambda_S}{\phi}u-1}.
\end{aligned}
$$

Then, from $2\lambda_D \ge \lambda_S > \lambda_D$ and $\Pi > c/\lambda_S$, we can derive that $V^{ds''}(u|u_1) \le 0$.

Note that

$$
\begin{aligned}
V^{ds'}(u|u_1) = &\frac{\lambda_S}{\phi}e^{\frac{\lambda_S}{\phi}(u_1-u)}\left[\left(\Pi-\frac{2c}{\lambda_S}\right)-(V^d(u_1)+u_1)\right] \\
&-\left(\Pi-\frac{c}{\lambda_S}\right)\left(\frac{\lambda_S}{\phi}\right)\left[1-\frac{\lambda_S}{\phi}(u-u_1)\right]e^{-\frac{\lambda_S}{\phi}u-1}-1, \\
\lim_{u\to\infty} V^{ds'}(u|u_1) = &-1.
\end{aligned}
$$

Then, by the concavity of $V^{ds}(u|u_1)$, $V^{ds'}(u|u_1) > -1$.

(c) By (5.1), $V^{ds}(u_2|u_1)+u_2 < \Pi - c/\lambda_D$ and $u \ge u_2$, we can derive that

$$
V^{dsd}(u|u_1,u_2) < \Pi - \frac{c}{\lambda_D} - u.
$$

By differentiating (5.1) once, we have

$$
V^{dsd'}(u|u_1,u_2) = \frac{\lambda_D}{\phi}\left(\Pi-\frac{c}{\lambda_D}-(V^{ds}(u_2|u_1,u_2)+u_2)\right)e^{\frac{\lambda_D}{\phi}(u_2-u)}-1.
$$

By the previous result ($V^{ds}(u_2|u_1,u_2)+u_2 < \Pi - c/\lambda_D$), we can derive that $V^{dsd'}(u|u_1,u_2) >$

-1.

By differentiating (5.1) twice, we can derive that

$$V^{dsd''}(u|u_1,u_2) = -\left(\frac{\lambda_D}{\phi}\right)^2 \left(\Pi - \frac{c}{\lambda_D} - (V^{ds}(u_2|u_1,u_2)+u_2)\right) e^{\frac{\lambda_D}{\phi}(u_2-u)} < 0.$$

□

D.1.2 Proofs for Proposition D.1 and D.2

Proof of Proposition D.1. (a) Observe that $\Pi_M(1) = c/\lambda_D < \Pi_S(1)$ from Lemma D.1 and D.2. Suppose that $\Pi_M(1) = c/\lambda_D < \Pi \leq \Pi_S(1)$. Then, by Lemma D.2, there exists $u_1 \geq 0$ such that $V^{d'}(u_1) = V^{ds'}(u_1|u_1)$, $V^{d''}(u_1) < V^{ds''}(u_1|u_1)$ and $V^{d'}(u) > V^{ds'}(u|u)$ for all $u \in [0, u_1)$. By Lemma D.3 (a), we have $V^{ds'}(u|u_1) > V^{dsd'}(u|u_1,u)$ for all $u > u_1$.

(b) Suppose that $\Pi > \Pi_S(1)$. By Lemma D.1, $V^{d'}(0) < V^{ds'}(0|0)$. Set $u_1 = 0$. Note that $V^{ds'}(0|0) > V^{d'}(0) = V^{dsd'}(0|0,0)$. Next, by Lemma D.3 (a), we have $V^{ds'}(u|0) > V^{dsd'}(u|0,u)$ for all $u > 0$.

□

Proof of Proposition D.2. (a) If $c/\lambda_D < \Pi < \Pi_M(\eta)$, by Lemma D.2, $V^{d'}(u) > V^{ds'}(u|u)$ for all $u \geq 0$.

(b) (i-ii) By Lemma D.2, there exists $u_1 > 0$ such that $V^{d'}(u_1) = V^{ds'}(u_1|u_1)$, $V^{d''}(u_1) < V^{ds''}(u_1|u_1)$ and $V^{d'}(u) > V^{ds'}(u|u)$ for all $u \in [0, u_1)$.

(iii-iv) By Lemma D.3, there exists $u_2 > u_1$ such that $V^{ds'}(u_2|u_1) = V^{dsd'}(u_2|u_1,u_2)$, $V^{ds''}(u_2|u_1) < V^{dsd''}(u_2|u_1,u_2)$ and $V^{ds'}(u|u_1) > V^{dsd'}(u|u_1,u)$ for all $u \in (u_1,u_2)$.

(v) By Lemma D.4, for all $u > u_2$, we have

$$V^{dsd'}(u|u_1,u_2) > \frac{1}{\phi}\left[\lambda_S\left(V_S(u+\phi/\lambda_S) - V^{dsd}(u|u_1,u_2)\right) - c\right].$$

(c) Suppose that $1/(e-1) < \eta < 1$ and $\Pi > \Pi_S(\eta)$. By Lemma D.1, we have $V^{d'}(0) < V^{ds'}(0|0)$. Now set $u_1 = 0$, then $V^{d'}(u_1) < V^{d''}(u_1|u_1)$.

(i) By Lemma D.3, there exists $u_2 > u_1 = 0$ such that $V^{ds'}(u_2|0) = V^{dsd'}(u_2|0,u_2)$, $V^{ds''}(u_2|0) < V^{dsd''}(u_2|0,u_2)$ and $V^{ds'}(u|0) > V^{dsd'}(u|0,u)$ for all $u \in (0,u_2)$.

(ii) By Lemma D.4, for all $u > u_2$, we have

$$V^{dsd'}(u|u_1, u_2) > \frac{1}{\phi}\left[\lambda_S\left(V_S(u+\phi/\lambda_S) - V^{dsd}(u|u_1,u_2)\right) - c\right].$$

□

D.2 Functions for Deviation

In this subsection, I introduce functions that specify deviations from the given value functions. These functions will be used when I verify that the constructed value functions solve (HJB_V) subject to (IC). Then I present some properties of these functions.[23]

1. Functions for deviation given V^d

(a) Define

$$L_1^D(u, R) \equiv \lambda_D(\Pi - R - V^d(u)) - c - \lambda_D(R-u)V^{d'}(u).$$

Given u, maximizing this function with respect to $R \geq u + \phi/\lambda_D$ is equivalent to maximizing the right hand side of (HJB_V) under the condition that $b = 1$ solves (HJB_{PK}) with $a = 1$.

(b) Define

$$\begin{aligned} L_1^S(u, w) \equiv& \lambda_S(V_S(w) - V^d(u)) - c - \lambda_S(w-u)V^{d'}(u) \\ =& \lambda_S\left[\left(\Pi - \frac{c}{\lambda_S}\right)\left(1 - e^{-\frac{\lambda_S}{\phi}u - \frac{\lambda_S}{\phi}(w-u)}\right)\right. \\ & \left. - \left(\Pi - \frac{c}{\lambda_D}\right)\left(1 - e^{-\frac{\lambda_D}{\phi}u}\right) - (w-u)\left(\frac{\lambda_D\Pi - c}{\phi}\right)e^{-\frac{\lambda_D}{\phi}u}\right] - c. \end{aligned} \tag{D.6}$$

Given u, maximizing this function with respect to $w \geq u + \phi/\lambda_S$ is equivalent to maximize the right hand side of (HJB_V) under the condition that $b = 1$ solves (HJB_{PK}) with $a = 0$.

2. Functions for deviation given V^{ds}

(a) Define

$$L_2^D(u, R|u_1) \equiv \lambda_D\left(\Pi - R - V^{ds}(u|u_1)\right) - c - \lambda_D(R-u)V^{ds'}(u|u_1).$$

[23] This approach is inspired by the tangible first breakthrough case of Green and Taylor (2016b). In their paper, they only need to consider the deviation from working to shirking. In this paper, we also need to consider the deviation from an approach to another approach. Thus, we need to define two functions for each case.

(b) Define

$$L_2^S(u, w|u_1) \equiv \lambda_S \left(V_S(w) - V^{ds}(u|u_1)\right) - c - \lambda_S(w-u)V^{ds\prime}(u|u_1).$$

3. Functions for deviation given V^{dsd}

(a) Define

$$L_3^D(u, R|u_1, u_2) \equiv \lambda_D \left(\Pi - R - V^{dsd}(u|u_1, u_2)\right) - c - \lambda_D(R-u)V^{dsd\prime}(u|u_1, u_2).$$

(b) Define

$$L_3^S(u, w|u_1, u_2) \equiv \lambda_S \left(V_S(w) - V^{dsd}(u|u_1, u_2)\right) - c - \lambda_S(w-u)V^{dsd\prime}(u|u_1, u_2).$$

D.2.1 Useful Lemmas

Lemma D.7. *Suppose that $\Pi > c/\lambda_D$ and $2\lambda_D \geq \lambda_S > \lambda_D$ are satisfied. Then, L_1^D and L_1^S satisfy the following properties:*

(a) $L_1^D(u, R) \leq 0$ for all $u \geq 0$ and $R \geq u + \phi/\lambda_D$.

(b) If $\Pi \leq \Pi_M(\eta)$, $L_1^S(u, w) \leq 0$ for all $u \geq 0$ and $w \geq u + \phi/\lambda_S$.

(c) If $\Pi_M(\eta) < \Pi < \Pi_S(\eta)$, $L_1^S(u, w) \leq 0$ for all $u \in [0, u_1]$ and $w \geq u + \phi/\lambda_S$ where u_1 is the threshold defined on Proposition D.1 or D.2.

Proof of Lemma D.7. (a) Note that $\frac{\partial}{\partial R}L_1^D = -\lambda_D(1 + V^{d\prime}(u)) < 0$ by Lemma D.6. Therefore, for a fixed u, L_1^D is maximized at $R = u + \phi/\lambda_D$. Note that by the definition of V^d, $L_1^D(u, u+\phi/\lambda_D) = 0$, thus, $L_1^D(u, R) \leq 0$ for all $R \geq u + \phi/\lambda_D$.

(b) Define $x \equiv e^{-\frac{\lambda_D}{\phi}u}$ and $y \equiv w - u$. Note that $u \geq 0$, $w \geq u + \phi/\lambda_S$ and $2\lambda_D \geq \lambda_S > \lambda_D$ imply that $1 \geq x > 0$, $y \geq \phi/\lambda_S$ and $1 \geq \eta > 0$. Then, L_1^S can be rewritten as follows:

$$\tilde{L}_1^b(x, y) \equiv -\lambda_S \left[\left(\Pi - \frac{c}{\lambda_S}\right) e^{-\frac{\lambda_S}{\phi}y} \cdot x^\eta - \left(1 - \frac{\lambda_D y}{\phi}\right)\left(\Pi - \frac{c}{\lambda_D}\right)\right] x + (\eta - 1)c.$$

By $\Pi > c/\lambda_D > c/\lambda_S$, $\eta > 0$ and $x > 0$,

$$\frac{\partial^2 \tilde{L}_1^S}{\partial x^2} = -\lambda_S \left(\Pi - \frac{c}{\lambda_S}\right) e^{-\frac{\lambda_S}{\phi}y}(\eta+1)\eta x^{\eta-1} < 0,$$

thus, $\tilde{L}_1^S$ is strictly concave in x.

By differentiating $\tilde{L}_1^S$ once by x,

$$\frac{\partial \tilde{L}_1^S}{\partial x} = -\lambda_S \left[(\eta+1) \left(\Pi - \frac{c}{\lambda_S} \right) e^{-\frac{\lambda_S}{\phi} y} x^{\eta} - \left(1 - \frac{\lambda_D y}{\phi} \right) \left(\Pi - \frac{c}{\lambda_D} \right) \right].$$

If $y \geq \phi/\lambda_D$ and $x \geq 0$, $\tilde{L}_1^S$ is decreasing in x, thus, $\tilde{L}_1^S$ is maximized at $x = 0$. Then, for all $1 \geq x > 0$ and $y \geq \phi/\lambda_D$, the following inequalities hold:

$$\tilde{L}_1^S(x, y) \leq \tilde{L}_1^S(0, y) = (\eta - 1)c \leq 0,$$

thus, $L_1^S(u, w) \leq 0$ for all $u \geq 0$ and $w \geq u + \phi/\lambda_D$.

When $\phi/\lambda_D > y \geq \phi/\lambda_S$ and y is fixed, since $\tilde{L}_1^S(x, y)$ is concave in x, $\tilde{L}_1^S(\cdot, y)$ is maximized at

$$x^*(y) \equiv \left[\frac{(\lambda_D \Pi - c)\,(1 - \lambda_D y/\phi)\, e^{\frac{\lambda_S}{\phi} y}}{(\lambda_S \Pi - c)} \right]^{\frac{1}{\eta}}.$$

Define $g(y) \equiv (1 - \lambda_D y/\phi)\, e^{(\lambda_S/\phi) y}$. Then, differentiating $g(y)$ gives

$$\begin{aligned} g'(y) =& -\frac{\lambda_D}{\phi} e^{\frac{\lambda_S}{\phi} y} + \frac{\lambda_S}{\phi} \left(1 - \frac{\lambda_D y}{\phi} \right) e^{\frac{\lambda_S}{\phi} y} \\ =& \frac{\lambda_D \lambda_S}{\phi} e^{\frac{\lambda_S}{\phi} y} \left(-\frac{1}{\lambda_S} + \frac{1}{\lambda_D} - \frac{y}{\phi} \right). \end{aligned}$$

Note that since $y \geq \phi/\lambda_S$ and $2\lambda_D \geq \lambda_S$, $g(y)$ is decreasing in y, hence, $x^*(y)$ is also decreasing in y.

Now, restrict attention to $1 \geq x > 0$. If $x^*(y) < 1$, the maximum value of $\tilde{L}_1^S(\cdot, y)$ is

$$\begin{aligned} \tilde{L}_1^S(x^*(y), y) =& \eta \lambda_S \left(\Pi - \frac{c}{\lambda_S} \right) e^{-\frac{\lambda_S}{\phi} y} x^*(y)^{\eta+1} + (\eta - 1)c \\ =& \eta\, (\lambda_S \Pi - c) \left[\frac{\lambda_D \Pi - c}{\lambda_S \Pi - c} \left(1 - \frac{\lambda_D y}{\phi} \right) e^{\frac{\lambda_D y}{\phi}} \right]^{\frac{1+\eta}{\eta}} + (\eta - 1)c \end{aligned}$$

Note that $(1 - \lambda_D y/\phi) e^{\lambda_D y/\phi}$ is decreasing in y,[24] thus, $\tilde{L}_1^S(x^*(y), y)$ is also decreasing in y.

[24] Differentiating the term by y gives $-(\lambda_D^2 y/\phi^2) e^{\lambda_D y/\phi} < 0$.

If $x^*(y) \geq 1$, since $\dfrac{\partial \tilde{L}_1^b}{\partial x}$ is negative for all $0 \leq x \leq 1$, the maximum value of $\tilde{L}_1^S(\cdot, y)$ is

$$\tilde{L}_1^S(1, y) = -\lambda_S \left[\left(\Pi - \frac{c}{\lambda_S} \right) e^{-\frac{\lambda_S}{\phi} y} - \left(1 - \frac{\lambda_D y}{\phi} \right) \left(\Pi - \frac{c}{\lambda_D} \right) \right] + (\eta - 1)c.$$

Note that $x^*(y) \geq 1$ implies that $0 > -\lambda_D y/\phi \geq (\lambda_S \Pi - c) e^{-\frac{\lambda_S}{\phi} y} - (\lambda_D \Pi - c)$. Also note that

$$\frac{\partial\, \tilde{L}_1^S(1, y)}{\partial y} = \frac{\lambda_S}{\phi} \left[(\lambda_S \Pi - c)\, e^{-\frac{\lambda_S}{\phi} y} - (\lambda_D \Pi - c) \right] < 0.$$

Therefore, $\tilde{L}_1^S(1, y)$ is decreasing in y.

When $x^*(\phi/\lambda_S) \leq 1$, $x^*(y) \leq 1$ holds for all $\phi/\lambda_D > y \geq \phi/\lambda_S$ since $x^*(y)$ is decreasing in y. Then,

$$\tilde{L}_1^S(x, y) \leq \tilde{L}_1^S(x^*(y), y) \leq \tilde{L}_1^S\left(x^*\left(\phi/\lambda_S\right), \phi/\lambda_S\right)$$

since $x^*(y)$ maximizes $\tilde{L}_1^S(x, y)$ and $\tilde{L}_1^S(x^*(y), y)$ is decreasing in y.

When $x^*(\phi/\lambda_S) > 1$, there exists $y^* \in (\phi/\lambda_S, \phi/\lambda_D)$ such that $x^*(y^*) = 1$ since $x^*(\phi/\lambda_D) = 0$. Then, $x^*(y) < 1$ for $y > y^*$ and $x^*(y) > 1$ for $y < y^*$. When $y < y^*$, by using the decreasingness of $\tilde{L}_1^S(1, y)$ for $x^*(y) > 1$,

$$\tilde{L}_1^S(x, y) \leq \tilde{L}_1^S(1, y) \leq \tilde{L}_1^S(1, \phi/\lambda_S).$$

When $y > y^*$,

$$\tilde{L}_1^S(x, y) \leq \tilde{L}_1^S(x^*(y), y) \leq \tilde{L}_1^S(x^*(y^*), y^*) = \tilde{L}_1^S(1, y^*) \leq \tilde{L}_1^S(1, \phi/\lambda_S).$$

By combining the above results, we can show that

$$\max_{\substack{1 \geq x > 0, \\ \phi/\lambda_D > y \geq \phi/\lambda_S}} \tilde{L}_1^S(x, y) = \begin{cases} \tilde{L}_1^S\left(x^*\left(\phi/\lambda_S\right),\ \phi/\lambda_S\right) & \text{if } \ x^*\left(\phi/\lambda_S\right) \leq 1, \\ \tilde{L}_1^S\left(1,\ \phi/\lambda_S\right) & \text{if } \ x^*\left(\phi/\lambda_S\right) > 1. \end{cases}$$

Note that

$$\tilde{L}_1^S(x, \phi/\lambda_S) = L_1^S(u, u + \phi/\lambda_S) = \phi V^{ds'}(u|u) - \phi V^{d'}(u)$$

where $u = -\frac{\phi}{\lambda_D} \log x$. Then, by Lemma D.2, if $c/\lambda_D < \Pi \leq \Pi_M(\eta)$, $\tilde{L}_1^S(x, \phi/\lambda_S) \leq 0$ for all $x \in (0, 1]$. Therefore, if $c/\lambda_D < \Pi \leq \Pi_M(\eta)$, for all $x \in (0, 1]$, $y \geq \phi/\lambda_S$,

$$\tilde{L}_1^S(x, y) \leq \tilde{L}_1^S(x^*(\phi/\lambda_S) \wedge 1, \phi/\lambda_S) \leq 0,$$

thus, $L_1^S(u,w) \le 0$ for all $u \ge 0$ and $u + \phi/\lambda_D > w \ge u + \phi/\lambda_S$.

(c) By Lemma D.1, if $\Pi_S(\eta) \ge \Pi$,

$$\tilde{L}_1^S(1, \phi/\lambda_S) = \phi V^{ds'}(u|u) - \phi V^{d'}(u) < 0.$$

In the previous case, we show that $\frac{\partial}{\partial y}\tilde{L}_1^S(1,y) < 0$, thus $\tilde{L}_1^S(1,y) < 0$ for all $y \ge \phi/\lambda_S$. On the other hand, by the definition of u_1, $V^{d'}(u_1) = V^{ds'}(u_1|u_1)$ and $V^{d''}(u_1) < V^{ds''}(u_1|u_1)$. Let $x_1 = e^{-\frac{\lambda_S}{\phi}u_1}$. Then,

$$\tilde{L}_1^S(x_1, \phi/\lambda_S) = L_1^S(u_1, \phi/\lambda_S) = \phi V^{ds'}(u_1|u_1) - \phi V^{d'}(u_1) = 0.$$

Then, we can derive that

$$-(\lambda_S\Pi - c)e^{-1}x_1^{\eta+1} + \eta(\lambda_D\Pi - c)x_1 + (\eta-1)c = 0.$$

Note that

$$\begin{aligned}
\frac{\partial\,\tilde{L}_1^S}{\partial y}(x_1,y) =& \frac{\lambda_S}{\phi}\left[(\lambda_S\Pi - c)e^{-\frac{\lambda_S}{\phi}y}x_1^{\eta} - (\lambda_D\Pi - c)\right]x_1\\
=& \frac{\lambda_S}{\phi}\left[(\lambda_S\Pi - c)x_1^{\eta}e^{-1}\cdot e^{-\frac{\lambda_S}{\phi}\left(y-\frac{\phi}{\lambda_S}\right)} - (\lambda_D\Pi - c)\right]x_1\\
=& \frac{\lambda_S}{\phi}\left[(\eta(\lambda_D\Pi - c)x_1 + (\eta-1)c)\cdot e^{-\frac{\lambda_S}{\phi}\left(y-\frac{\phi}{\lambda_S}\right)} - (\lambda_D\Pi - c)x_1\right]\\
=& \frac{\lambda_S}{\phi}\left[(\lambda_D\Pi - c)\cdot\left(\eta e^{-\frac{\lambda_S}{\phi}\left(y-\frac{\phi}{\lambda_S}\right)} - 1\right)x_1 + (\eta-1)c\cdot e^{-\frac{\lambda_S}{\phi}\left(y-\frac{\phi}{\lambda_S}\right)}\right].
\end{aligned}$$

Since $\eta \le 1$, $y \ge \phi/\lambda_S$ and $\lambda_D\Pi > c$, $\dfrac{\partial\,\tilde{L}_1^S}{\partial y}(x_1,y) < 0$. Therefore, $\tilde{L}_1^S(x_1,y) < 0$ for all $y \ge \phi/\lambda_S$.

In the previous case, we show that $\tilde{L}_1^S$ is strictly concave, thus, for all $x_1 \le x \le 1$ and $y \ge \phi/\lambda_S$,

$$\tilde{L}_1^S(x,y) \le \frac{x - x_1}{1 - x_1}\,\tilde{L}_1^S(1,y) + \frac{1-x}{1-x_1}\,\tilde{L}_1^S(x_1,y) \le 0.$$

Therefore, $L_1^S(u,w) \le 0$ for all $u \in [0,u_1]$ and $w \ge u + \phi/\lambda_S$.

□

Lemma D.8. *Suppose that $2\lambda_D \ge \lambda_S > \lambda_D$, $\Pi > \Pi_M(\eta)$ and $V^{d'}(u_1) < V^{ds'}(u_1|u_1)$ or $V^{d'}(u_1) = V^{ds'}(u_1|u_1)$ & $V^{d''}(u_1) < V^{ds''}(u_1|u_1)$ are satisfied. Then, L_2^D and L_2^S satisfy the following properties:*

(a) $L_2^S(u, w|u_1) \leq 0$ *for all* $u \geq u_1$ *and* $w \geq u + \phi/\lambda_S$.

(b) When $\eta = 1$, *for all* $u > u_1$ *and* $R \geq u + \phi/\lambda_D$, $L_2^D(u, R|u_1) < 0$ *is satisfied.*

(c) When $\eta < 1$, $L_2^D(u_2, u_2 + \phi/\lambda_D|u_1) = 0$, $L_2^D(u, R|u_1) < 0$ *for all* $u_1 < u < u_2$ *and* $R \geq u + \phi/\lambda_D$, *where* u_2 *is the threshold defined on D.2.*

Proof of Lemma D.8. (a) Note that $\frac{\partial}{\partial w}L_2^S = \lambda_S(V_S'(w) - V^{ds'}(u|u_1))$ and $\frac{\partial^2}{\partial w^2}L_2^S = \lambda_S V_S''(w) < 0$. By differentiating (B.6) once, we can derive that

$$\phi V^{ds''}(u|u_1) = \lambda_S \left(V_S'\left(u + \frac{\phi}{\lambda_S}\right) - V^{ds'}(u|u_1) \right).$$

By (b) of Lemma D.6, $V^{ds''}(u|u_1) < 0$. Thus, the following inequality holds:

$$0 > \frac{\partial}{\partial w} L_2^S(u, u + \phi/\lambda_S|u_1) = \lambda_S \left(V_S'\left(u + \frac{\phi}{\lambda_S}\right) - V^{ds'}(u|u_1) \right).$$

Then, $L_2^S(u, w|u_1)$ subject to $w \geq u + \phi/\lambda_S$ is maximized at $w = u + \phi/\lambda_S$ for a given u. Also note that $L_2^S(u, u + \phi/\lambda_S|u_1) = 0$ holds by (B.6). Therefore, $L_2^S(u, w|u_1) \leq 0$ for all $u \geq u_1$ and $w \geq u + \phi/\lambda_S$.

(b) Note that if $V^{d'}(u_1) \leq V^{ds'}(u_1|u_1)$, $\frac{\partial}{\partial R}L_2^D = -\lambda_D(1 + V^{ds'}(u|u_1)) < 0$ by Lemma D.6. Therefore, for all $u > u_1$ and $R \geq u + \phi/\lambda_D$,

$$L_2^D(u, R|u_1) \leq L_2^D(u, u + \phi/\lambda_D|u_1).$$

Note that

$$L_2^D(u, u + \phi/\lambda_D|u_1) = \phi V^{dsd'}(u|u_1, u) - \phi V^{ds'}(u|u_1)$$

By (a) of Lemma D.3, when $\eta = 1$, $V^{dsd'}(u|u_1, u) < V^{ds'}(u|u_1)$ for all $u > u_1$, thus, $L_2^D(u, R|u_1) < 0$ for all $u \geq u_1$ and $R \geq u + \phi/\lambda_D$.

(c) By the definition of u_2, $V^{ds'}(u_2|u_1) = V^{dsd'}(u_2|u_1, u_2)$ and $V^{ds''}(u_2|u_1) < V^{dsd''}(u_2|u_1, u_2)$, thus $L_2^D(u_2, u_2 + \phi/\lambda_D|u_1) = \phi V^{dsd'}(u_2|u_1, u) - \phi V^{ds'}(u_2|u_1) = 0$. Moreover, $V^{dsd'}(u|u_1, u) < V^{ds'}(u|u_1)$ for all $u \in (u_1, u_2)$, thus, $L_2^D(u, R|u_1) < 0$ for all $u \in (u_1, u_2)$ and $R \geq u + \phi/\lambda_D$.

$\square$

Lemma D.9. *Suppose that* $2\lambda_D \geq \lambda_S > \lambda_D$, $\Pi > \Pi_M(\eta)$, $V^{d\prime}(u_1) < V^{ds\prime}(u_1|u_1)$ *or* $V^{d\prime}(u_1) = V^{ds\prime}(u_1|u_1)$ *&* $V^{d\prime\prime}(u_1) < V^{ds\prime\prime}(u_1|u_1)$, $V^{ds\prime}(u_2|u_1) = V^{dsd\prime}(u_2|u_1,u_2)$ *and* $V^{ds\prime\prime}(u_2|u_1) < V^{dsd\prime\prime}(u_2|u_1,u_2)$ *are satisfied. Then,* L_3^D *and* L_3^S *satisfy the following properties:*

(a) $L_3^D(u,R|u_1,u_2) \leq 0$ *for all* $u \geq u_2$ *and* $R \geq u + \phi/\lambda_D$.

(b) $L_3^S(u,w|u_1,u_2) \leq 0$ *for all* $u \geq u_2$ *and* $w \geq u + \phi/\lambda_S$.

Proof of Lemma D.9. (a) Note that $\frac{\partial}{\partial R} L_3^D(u,R) = -\lambda_D\left(1 + V^{dsd\prime}(u|u_1,u_2)\right)$. By (c) of Lemma D.6, $V^{dsd\prime}(u|u_1,u_2) > -1$, thus $\frac{\partial}{\partial R} L_3^D(u,R)$ and $L_3^D(u,R) \leq L_3^D(u, u+\phi/\lambda_D)$ for all $u \geq u_2$ and $R \geq u + \phi/\lambda_D$.

By (B.5),

$$L_3^D\left(u, u+\frac{\phi}{\lambda_D} \mid u_1,u_2\right) = \lambda_D(\Pi - u - \phi/\lambda_D - V^{dsd}(u|u_1,u_2)) - c - \phi V^{dsd\prime}(u|u_1,u_2) = 0.$$

Therefore, $L_3^D(u,R) \leq 0$ for all $u \geq u_2$ and $R \geq u + \phi/\lambda_D$.

(b) By differentiating L_3^S by w, we have

$$\begin{aligned}\frac{\partial}{\partial w} L_3^S(u,w|u_1,u_2) =& \lambda_S\left[V_S'(w) - V^{dsd\prime}(u|u_1,u_2)\right] \\ =& \lambda_S\left[\frac{\lambda_S}{\phi}\left(\Pi - \frac{c}{\lambda_S}\right)e^{-\frac{\lambda_S}{\phi}w} - \frac{1}{\phi}\left(-c + \lambda_D(\Pi - u - V^{dsd}(u|u_1,u_2))\right)\right] \\ =& \frac{\lambda_S}{\phi}\left[(\lambda_S\Pi - c)e^{-\frac{\lambda_S}{\phi}w} - \lambda_D\left(\Pi - \frac{c}{\lambda_D} - (V^{ds}(u_2|u_1) + u_2)\right)e^{\frac{\lambda_D}{\phi}(u_2-u)}\right].\end{aligned}$$

By $w \geq u + \phi/\lambda_S$,

$$\begin{aligned}\frac{\partial}{\partial w} L_3^S(u,w|u_1,u_2) <& \frac{\lambda_S}{\phi}\left[(\lambda_S\Pi - c)e^{-\frac{\lambda_S}{\phi}u-1} - \lambda_D\left(\Pi - \frac{c}{\lambda_D} - (V^{ds}(u_2|u_1) + u_2)\right)e^{\frac{\lambda_D}{\phi}(u_2-u)}\right] \\ =& \frac{\lambda_S}{\phi}e^{\frac{\lambda_D}{\phi}(u_2-u)}\Bigg[(\lambda_S\Pi - c)e^{-\frac{\lambda_S}{\phi}u_2-1}e^{\frac{\lambda_S-\lambda_D}{\phi}(u_2-u)} \\ & - \lambda_D\left(\Pi - \frac{c}{\lambda_D} - (V^{ds}(u_2|u_1) + u_2)\right)\Bigg].\end{aligned}$$

Since $u \geq u_2$ and $\lambda_S > \lambda_D$, we have

$$\frac{\partial}{\partial w} L_3^S(u,w|u_1,u_2) < \frac{\lambda_S}{\phi}e^{\frac{\lambda_D}{\phi}(u_2-u)}\left[(\lambda_S\Pi - c)e^{-\frac{\lambda_S}{\phi}u_2-1} - \lambda_D\left(\Pi - \frac{c}{\lambda_D} - (V^{ds}(u_2|u_1) + u_2)\right)\right].$$

By (B.5), (B.6), $V^{ds}(u_2|u_1) = V^{dsd}(u_2|u_1, u_2)$ and $V^{ds'}(u_2|u_1) = V^{dsd'}(u_2|u_1, u_2)$, we can derive that

$$(\lambda_S \Pi - c)e^{-\frac{\lambda_S}{\phi}u_2 - 1} = (\lambda_S - \lambda_D)(\Pi - (V^{ds}(u_2|u_1) + u_2)) - c.$$

By plugging this into the above inequality, we have

$$\frac{\partial}{\partial w} L_3^S(u, w|u_1, u_2) < \frac{\lambda_S}{\phi} e^{\frac{\lambda_D}{\phi}(u_2 - u)} (\lambda_S - 2\lambda_D) \left(\Pi - (V^{ds}(u_2|u_1) + u_2) \right),$$

thus, since $2\lambda_D \geq \lambda_S$ and $\Pi - c/\lambda_D > V^{ds}(u_2|u_1) + u_2$ ((b) of Lemma D.6), $L_3^S(u, w|u_1, u_2)$ is decreasing in w. Therefore, $L_3^S(u, w|u_1, u_2) \leq L_3^S(u, u + \phi/\lambda_S \mid u_1, u_2)$ for all $w \geq u + \phi/\lambda_S$.

Note that $L_3^S(u, u + \phi/\lambda_S|u_1, u_2) = H_3(u)$ and $H_3(u) \leq 0$ for all $u \geq u_2$ by Lemma D.4. Therefore, $L_3^S(u, w|u_1, u_2) \leq 0$ for all $u \geq u_2$ and $w \geq u + \phi/\lambda_S$.

□

D.3 Proof of Proposition 4.2 and 5.2

Proof of Proposition 4.2. (a) From Proposition D.1 (a), there exists $u_1 \geq 0$ such that $V^{d'}(u_1) = V^{ds'}(u_1|u_1)$ and $V^{d''}(u_1) < V^{ds''}(u_1|u_1)$. Therefore, the function defined in (4.3) is differentiable.

First, consider the case where $u \in [0, u_1]$. Then, by Lemma D.6 (a), we have $V^{d'}(u) > -1$. By Lemma D.5, we can set $b = 1$. When $a = 0$, (IC) with $b = 1$ is equivalent to $u_S \geq u + \phi/\lambda_S$. By (c) of Lemma D.7, $L_1^S(u, u_S) \leq 0$ for all $u_S \geq u + \phi/\lambda_S$. When $a = 1$, (IC) with $b = 1$ is equivalent to $R \geq u + \phi/\lambda_D$. By (a) of Lemma D.7, $0 \geq L_1^D(u, R)$ for all $R \geq u + \phi/\lambda_D$. In addition, from the definition of V^d, we have $L_1^D(u, u + \phi/\lambda_D) = 0$. Therefore, when $u \in [0, u_1]$, $V^d(u)$ solves (HJB_V) subject to (IC).

Second, consider the case where $u > u_1$. Then, by Lemma D.6 (b), we have $V^{ds'}(u|u_1) > -1$. By Lemma D.5, we can set $b = 1$. When $a = 1$, (IC) with $b = 1$ is equivalent to $R \geq u + \phi/\lambda_D$. By (b) of Lemma D.8, $0 \geq L_2^D(u, R|u_1)$ for all $R \geq u + \phi/\lambda_D$. When $a = 0$, (IC) with $b = 1$ is equivalent to $u_S \geq u + \phi/\lambda_S$. By (a) of Lemma D.8, $L_2^S(u, u_S|u_1) \leq 0$ for all $u_S \geq u + \phi/\lambda_S$. In addition, from the definition of V^{ds}, we have $L_2^S(u, u + \phi/\lambda_D|u_1) = 0$. Thus, for all $u > u_1$, $V^{ds}(u|u_1)$ solves (HJB_V) subject to (IC). Therefore, the value function specified in (4.3) solves (HJB_V) subject to (IC).

Next, by (a) of Lemma D.6, $V^{d''}(u) < 0$ for all $0 < u < u_1$. By (b) of Lemma D.6 and $V^{d'}(u_1) = V^{ds'}(u_1|u_1)$, we have $V^{ds''}(u|u_1) < 0$ for all $u > u_1$. It remains to show that the function defined in (4.3) is concave at u_1. Observe that

$$V^d(u) < V^d(u_1) - V^{d'}(u_1)(u_1 - u) \tag{D.7}$$

for all $0 < u < u_1$, and

$$V^{ds}(u|u_1) < V^{ds}(u_1|u_1) + V^{ds'}(u_1|u_1)(u - u_1) \tag{D.8}$$

for all $u_1 < u$. Therefore, the function defined in (4.3) is concave for all $u > 0$.

By Lemma B.2, any incentive compatible contract delivers the principal's expected payoff less than or equal to $V^d(u)$ for $u \le u_1$ and $V^{ds}(u|u_1)$ for $u > u_1$. In addition, by Proposition 4.1 (a) and (b), these values can be implemented by some incentive compatible contracts, i.e., they are maximized expected payoff of the principal given the agent's promised utility. Thus, the function defined in (4.3) is the principal's value function.

(b) By Proposition D.1 (b), we have $V^{ds'}(0|0) > V^{d'}(0)$ ($\Pi > \Pi_S(1)$) or $V^{ds'}(0|0) = V^{d'}(0)$ & $V^{ds''}(0|0) > V^{d''}(0)$ ($\Pi = \Pi_S(1)$). Consider any $u \ge 0$. By using $V^{ds'}(0|0) \ge V^{d'}(0)$ and Lemma D.6 (b), we have $V^{ds'}(u|0) > -1$. By Lemma D.5, we can set $b = 1$. When $a = 1$, (IC) with $b = 1$ is equivalent to $R \ge u + \phi/\lambda_D$. By (b) of Lemma D.8, $0 \ge L_2^D(u, R|0)$ for all $R \ge u + \phi/\lambda_D$. When $a = 0$, (IC) with $b = 1$ is equivalent to $u_S \ge u + \phi/\lambda_S$. By (a) of Lemma D.8, $L_2^S(u, u_S|0) \le 0$ for all $u_S \ge u + \phi/\lambda_S$. In addition, from the definition of V^{ds}, we have $L_2^S(u, u + \phi/\lambda_D|0) = 0$. Therefore, the value function $V^{ds}(u|0)$ solves (HJB$_V$) subject to (IC).

By (b) of Lemma D.6 and $V^{d'}(0) < V^{ds'}(0|0)$, we have $V^{ds''}(u|0) < 0$ for all $u > 0$, i.e., $V^d(\cdot|0)$ is concave. By Lemma B.2 and Proposition 4.1 (c), $V^{ds}(u|0)$ is the principal's value function.

□

Proof of Proposition 5.2. (a) By Lemma D.6 (a), we have $V^{d'}(u) > -1$. By Lemma D.5, we can set $b = 1$. When $a = 0$, (IC) with $b = 1$ is equivalent to $u_S \ge u + \phi/\lambda_S$. By (b) of Lemma D.7, $L_1^S(u, u_S) \le 0$ for all $u_S \ge u + \phi/\lambda_S$. When $a = 1$, (IC) with $b = 1$ is equivalent to $R \ge u + \phi/\lambda_D$. By (a) of Lemma D.7, $0 \ge L_1^D(u, R)$ for all $R \ge u + \phi/\lambda_D$. In addition, from the definition of V^d, we have $L_1^D(u, u + \phi/\lambda_D) = 0$. Therefore, $V^d(u)$ solves (HJB$_V$) subject to (IC). By (a) of Lemma D.6, V^d is concave. By Lemma B.2 and Proposition 4.1 (a), V^d is the principal's value function.

(b) Set the thresholds u_1 and u_2 as in Proposition D.2 (b).

First, when $u \in [0, u_1]$, by following the same steps as in Proposition 4.2 (a), we can show that $V^d(u)$ solves (HJB$_V$) subject to (IC).

Second, consider the case where $u_1 < u \leq u_2$. Then, by Lemma D.6 (b), we have $V^{ds\prime}(u|u_1) > -1$. By Lemma D.5, we can set $b = 1$. When $a = 1$, (IC) with $b = 1$ is equivalent to $R \geq u + \phi/\lambda_D$. By (c) of Lemma D.8, $0 \geq L_2^D(u, R|u_1)$ for all $R \geq u + \phi/\lambda_D$. When $a = 0$, (IC) with $b = 1$ is equivalent to $u_S \geq u + \phi/\lambda_S$. By (a) of Lemma D.8, $L_2^S(u, u_S|u_1) \leq 0$ for all $u_S \geq u + \phi/\lambda_S$. In addition, from the definition of V^{ds}, we have $L_2^S(u, u+\phi/\lambda_D|u_1) = 0$. Thus, for all $u > u_1$, $V^{ds}(u|u_1)$ solves (HJB$_V$) subject to (IC).

Last, consider the case where $u > u_2$. Then, by Lemma D.6 (c), we have $V^{dsd\prime}(u|u_1, u_2) > -1$. By Lemma D.5, we can set $b = 1$. When $a = 0$, (IC) with $b = 1$ is equivalent to $u_S \geq u + \phi/\lambda_S$. By (b) of Lemma D.9, $L_3^S(u, u_S|u_1, u_2) \leq 0$ for all $u_S \geq u + \phi/\lambda_S$. When $a = 1$, (IC) with $b = 1$ is equivalent to $R \geq u + \phi/\lambda_D$. By (a) of Lemma D.9, $L_3^S(u, u_S|u_1, u_2) \leq 0$ for all $u_S \geq u + \phi/\lambda_S$. In addition, from the definition of V^{dsd}, we have $L_3^D(u, u+\phi/\lambda_D|u_1, u_2) = 0$. Therefore, the value function specified in (5.2) solves (HJB$_V$) subject to (IC).

By using Lemma D.6, $V^{d\prime}(u_1) = V^{ds\prime}(u_1|u_1)$ and $V^{ds\prime}(u_2|u_1) = V^{dsd\prime}(u_2|u_1, u_2)$, the function defined in (5.2) is concave.

(c) Set the thresholds $u_1 = 0$ and $u_2 > 0$ as in Proposition D.2 (c).

First, consider the case where $u \in [0, u_2]$, so $V(u) = V^{ds}(u|0)$. By using $V^{ds\prime}(0|0) > V^{d\prime}(0)$ and Lemma D.6 (b), we have $V'(u) = V^{ds\prime}(u|u_1) > -1$. By Lemma D.5, we can set $b = 1$. When $a = 1$, (IC) with $b = 1$ is equivalent to $R \geq u + \phi/\lambda_D$. By (b) of Lemma D.8, $0 \geq L_2^D(u, R|0)$ for all $R \geq u + \phi/\lambda_D$. When $a = 0$, (IC) with $b = 1$ is equivalent to $u_S \geq u + \phi/\lambda_S$. By (a) of Lemma D.8, $L_2^S(u, u_S|0) \leq 0$ for all $u_S \geq u+\phi/\lambda_S$. In addition, from the definition of V^{ds}, we have $L_2^S(u, u+\phi/\lambda_D|0) = 0$. Therefore, for all $u \in [0, u_2]$, the value function $V^{ds}(u|0)$ solves (HJB$_V$) subject to (IC).

Next, consider the case where $u > u_2$, so $V(u) = V^{dsd}(u|0, u_2)$. By using $V^{ds\prime}(0|0) > V^{d\prime}(0)$ and Lemma D.6 (c), we have $V'(u) = V^{dsd\prime}(u|0, u_2) > -1$. By Lemma D.5, we can set $b = 1$. When $a = 0$, (IC) with $b = 1$ is equivalent to $u_S \geq u + \phi/\lambda_S$. By (b) of Lemma D.9, $L_3^S(u, u_S|0, u_2) \leq 0$ for all $u_S \geq u + \phi/\lambda_S$. When $a = 1$, (IC) with $b = 1$ is equivalent to $R \geq u + \phi/\lambda_D$. By (a) of Lemma D.9, $L_3^S(u, u_S|0, u_2) \leq 0$ for all $u_S \geq u + \phi/\lambda_S$. In addition, from the definition of V^{dsd}, we have $L_3^D(u, u+\phi/\lambda_D|0, u_2) = 0$. Thus, for all $u > u_2$, the value function $V^{dsd}(u|0, u_2)$ solves (HJB$_V$) subject to (IC). Therefore, the value function specified in (5.3) solves (HJB$_V$) subject to (IC).

By using Lemma D.6, $V^{d'}(0) < V^{ds'}(0|0)$ and $V^{ds'}(u_2|0) = V^{dsd'}(u_2|0, u_2)$, we can show that V is concave.

□

E Proofs of Lemmas in Section C

E.1 Proof of Lemma C.1

Proof of Lemma C.1. If Π is greater than or equal to $\Pi_S(\eta)$ and η is greater than $1/(e-1)$, u_1 is equal to zero by Proposition D.1 and D.2. Note that

$$\Pi_S(\eta) = \frac{e-1}{(e-1)\eta - 1} \cdot \frac{c}{\lambda_D} \geq \frac{e-1}{e-2} \cdot \frac{c}{\lambda_D} > \frac{2c}{\lambda_D} \geq \frac{c+\phi}{\lambda_D} = \Pi_F.$$

Then, the project is feasible and $\bar{u}$ is greater than 0, thus, $\bar{u}$ is always greater than u_1.

Now suppose that $\Pi_D(\eta) < \Pi < \Pi_S(\eta)$. Since V is strictly concave, $u_1 < \bar{u}$ is equivalent to $0 < V'(u_1) = V^{d'}(u_1)$. Note that $0 < V^{d'}(u_1)$ is equivalent to:

$$\frac{\phi}{\lambda_D \Pi - c} < e^{-\frac{\lambda_D}{\phi} u_1}.$$

Recall that u_1 is the solution of

$$-H_1(u) = ((\eta+1)\lambda_D \Pi - c)e^{-1} e^{-\frac{(\eta+1)\lambda_D}{\phi} u} - \eta(\lambda_D \Pi - c) e^{-\frac{\lambda_D}{\phi} u} + (1-\eta)c = 0.$$ [25]

Define $x_1 \equiv e^{-\frac{\lambda_D}{\phi} u_1}$. Then, x_1 is the solution of

$$\tilde{H}_1(x) = ((\eta+1)\lambda_D \Pi - c)e^{-1} x^{\eta+1} - \eta(\lambda_D \Pi - c)x + (1-\eta)c = 0,$$

and we need to identify a condition for $x_1 > \phi/(\lambda_D \Pi - c)$.

Note that

$$\frac{\partial^2 \tilde{H}_1}{\partial x^2}(x) = (\eta+1)\eta((\eta+1)\lambda_D \Pi - c)e^{-1} x^{\eta-1} > 0,$$

thus there exists $\underline{x} \in (0,1)$ that minimizes $\tilde{H}_1$. Also note that $\Pi_S(\eta) > \Pi > \Pi_M(\eta)$ imply that $\tilde{H}_1(1) > 0 > \tilde{H}_1(\underline{x})$ and $x_1 \in (\underline{x}, 1)$.

There are two possible cases that satisfy $x_1 > \phi/(\lambda_D \Pi - c)$: (i) $\underline{x} \geq \phi/(\lambda_D \Pi - c)$; (ii) $\phi/(\lambda_D \Pi - c) > \underline{x}$ and $\tilde{H}_1(\phi/(\lambda_D \Pi - c)) < 0$.

[25] See the proof of Lemma D.2 for the definition of H_1.

The first case is equivalent to $\tilde{H}_1'(\phi/(\lambda_D\Pi - c)) < 0$. By algebra, we can show that it is equivalent to

$$\frac{(\eta+1)\lambda_D\Pi - c}{(\lambda_D\Pi - c)^{\eta+1}} < \frac{\eta e}{\eta+1}\phi^{-\eta}. \tag{E.1}$$

The second case is equivalent to $\tilde{H}_1'(\phi/(\lambda_D\Pi - c)) \geq 0$ and $\tilde{H}_1(\phi/(\lambda_D\Pi - c)) < 0$. By algebra, we can show that it is equivalent to

$$\frac{\eta e}{\eta+1}\phi^{-\eta} \leq \frac{(\eta+1)\lambda_D\Pi - c}{(\lambda_D\Pi - c)^{\eta+1}} < (\eta(c+\phi) - c)\, e\phi^{-\eta-1}. \tag{E.2}$$

Last, by the proof of Lemma D.2, we can show that $\Pi > \Pi_M(\eta)$ is equivalent to

$$\frac{(\eta+1)\lambda_D\Pi - c}{(\lambda_D\Pi - c)^{\eta+1}} < \left(\frac{\eta^2}{1-\eta^2}\right)^{\eta} \frac{\eta e}{1+\eta}c^{-\eta}. \tag{E.3}$$

Now I compare the above three conditions. When $\eta > \sqrt{c/(c+\phi)}$, by simple algebra, we can show that

$$\frac{\eta e}{\eta+1}\phi^{-\eta} < (\eta(c+\phi) - c)e\phi^{-\eta-1} < \left(\frac{\eta^2}{1-\eta^2}\right)^{\eta} \frac{\eta e}{1+\eta}c^{-\eta}.$$

Therefore, the inequality

$$\frac{(\eta+1)\lambda_D\Pi - c}{(\lambda_D\Pi - c)^{\eta+1}} < (\eta(c+\phi) - c)\, e\phi^{-\eta-1}$$

imply that (E.1), (E.2) and (E.3). Define $\Pi_D(\eta)$ be the value of Π that makes both sides of the above inequality equal. Then, $\Pi_D(\eta) > \Pi_M(\eta)$ since $\Pi < \Pi_M(\eta)$ implies $\Pi < \Pi_D(\eta)$. Therefore, there exists $\Pi_D(\eta) > \Pi_M(\eta)$ such that $u_1 < \bar{u}$ if and only if $\Pi > \Pi_D(\eta)$.

When $\eta \leq \sqrt{c/(c+\phi)}$, by simple algebra, we can show that

$$\frac{\eta e}{\eta+1}\phi^{-\eta} \geq (\eta(c+\phi) - c)e\phi^{-\eta-1} \quad \& \quad \frac{\eta e}{\eta+1}\phi^{-\eta} \geq \left(\frac{\eta^2}{1-\eta^2}\right)^{\eta} \frac{\eta e}{1+\eta}c^{-\eta}.$$

Therefore, (E.2) cannot hold in this case and (E.3) implies (E.1). It means that $\Pi > \Pi_M(\eta)$ implies $\tilde{H}_1'(\phi/(\lambda_D\Pi - c)) < 0$. Moreover, $\Pi > \Pi_M(\eta)$ is necessary for the existence of u_1. Hence, $u_1 < \bar{u}$ holds if and only if $\Pi > \Pi_M(\eta)$. □

E.2 Proof of Lemma C.2

Proof of Lemma C.2. Since V is strictly concave, $u_2 < \bar{u}$ is equivalent to $0 < V'(u_2)$. By (b) of Proposition 5.2, we have $V'(u_2) = V^{ds'}(u_2|u_1) = V^{dsd'}(u_2|u_1, u_2)$. By (B.5) and

$V^{dsd}(u_2|u_1, u_2) = V^{ds}(u_2|u_1)$, $0 < V^{dsd\prime}(u_2|u_1, u_2)$ is equivalent to:

$$\lambda_D(u_2 + V^{ds}(u_2|u_1)) < \lambda_D \Pi - c - \phi. \tag{E.4}$$

Also note that $V^{dsd\prime}(u_2|u_1, u_2) = \phi V^{ds\prime}(u_2|u_1)$ and $V^{dsd}(u_2|u_1, u_2) = V^{ds}(u_2|u_1)$ imply that

$$\lambda_D(\Pi - u_2 - V^{ds}(u_2|u_1)) = \lambda_S \left(V_S \left(u_2 + \frac{\phi}{\lambda_S} \right) + u_2 + \frac{\phi}{\lambda_S} \right) - \lambda_S \left(V^{ds}(u_2|u_1) + u_2 \right)$$

by (B.5) and (B.6). By plugging (3.1) into the above equation, we can derive that

$$\begin{aligned} (\lambda_S - \lambda_D)(V^{ds}(u_2|u_1) + u_2) =& \lambda_S \left(\Pi - \frac{c}{\lambda_S} \right) \left(1 - e^{-\frac{\lambda_S}{\phi} u_2 - 1} \right) - \lambda_D \Pi \\ \iff \qquad \eta \lambda_D (V^{ds}(u_2|u_1) + u_2) =& \eta \lambda_D \Pi - c - (\lambda_S \Pi - c) e^{-\frac{\lambda_S}{\phi} u_2 - 1}. \end{aligned}$$

Then, by plugging this into (E.4), $0 < V^{dsd\prime}(u_2|u_1, u_2)$ is equivalent to

$$(\eta - 1)c + \eta \phi < (\lambda_S \Pi - c) e^{-\frac{\lambda_S}{\phi} u_2 - 1}.$$

Since $\Pi > c/\lambda_D > c/\lambda_S$, the right hand side of the above inequality is always greater than 0. Since it is assumed that $\eta > \dfrac{c}{c + \phi}$, the left hand side of the above inequality is always less than 0. Therefore, the above inequality always holds, i.e., u_2 is less than $\bar{u}$.

$\square$

E.3 Proof of Lemma C.3

Proof of Lemma C.3. By following the proof of Lemma C.2, $u_2 \geq \bar{u}$ is equivalent to

$$\hat{y} \equiv \frac{(\eta - 1)c + \eta \phi}{(\lambda_S \Pi - c) e^{-\frac{\lambda_S}{\phi} u_1 - 1}} \geq e^{\frac{\lambda_S}{\phi}(u_1 - u_2)} \tag{E.5}$$

By the proof of Lemma D.3, $x_2 \equiv e^{\frac{\lambda_S}{\phi}(u_1 - u_2)}$ is the solution, which is not equal to 1, of $\tilde{H}_2(x) = 0$.[26] Since $u_2 \geq u_1$, if $\hat{y} \geq 1$, the above inequality holds, thus, I restrict attention to the case of $\hat{y} < 1$. Observe that the inequality $\tilde{H}_2(\hat{y}) \leq 0$ implies (E.5) because $\tilde{H}_2$ is strictly convex in x and $\tilde{H}_2(1) \leq 0$.

[26] The function $\tilde{H}_2$ is defined in (D.2)

Note that $\tilde{H}_2(x)$ can be rewritten as follows:

$$\tilde{H}_2(x) = \frac{1-\eta}{1+\eta}c - H_1(u_1)x + \left[-\frac{1-\eta}{1+\eta}c + \frac{\eta}{1+\eta}(\lambda_S\Pi - c)e^{-\frac{\lambda_D}{\phi}u_1 - 1}\log x\right]x$$

where H_1 is a function defined in (D.1). Also note that $H_1(u_1) = \phi V^{ds''}(u_1|u_1) - \phi V^{d'}(u_1) \geq 0$.

By plugging the definition of $\hat{y}$ into the above equation, we can derive that

$$\tilde{H}_2(\hat{y}) = \frac{1-\eta}{1+\eta}c(1-\hat{y}) - H_1(u_1)\hat{y} + \frac{\eta}{1+\eta}\left((\eta-1)c + \eta\phi\right)\log\hat{y}.$$

Now define a new function G as follows:

$$G(y) \equiv \frac{1-\eta}{1+\eta}c(1-y) - H_1(u_1)y + \frac{\eta}{1+\eta}\left((\eta-1)c + \eta\phi\right)\log y,$$

and it is enough to show that $G(y) \leq 0$ for all $y < 1$.

Note that

$$G''(y) = -\frac{\eta}{1+\eta}\left(\frac{(\eta-1)c + \eta\phi}{y^2}\right) < 0$$

from $\eta \geq \sqrt{c/(c+\phi)} > c/(c+\phi)$. Also note that

$$G'(1) = -H_1(u_1) + \frac{1}{1+\eta}\left((\eta^2-1)c + \eta^2\phi\right) < 0.$$

from $\eta \geq \sqrt{c/(c+\phi)}$ and $H_1(u_1) \geq 0$. Lastly, note that $G(1) = -H_1(u_1) \leq 0$. Therefore, for all $y < 1$, $G(y) \leq G(1) + G'(1)(1-y) \leq 0$. Therefore, $\tilde{H}_2(\hat{y}) \leq 0$ and $u_2 \geq \bar{u}$. $\square$

Online Appendix for “Managing a Project by Splitting it into Pieces”

I also explore three variations of the model that reflect relevant features in some economic applications. First, I analyze the case of asymmetric arrival rates of subprojects and ask whether this can change the form of the optimal contract (Section OA.1). Next, I consider outside options for both players that emerge after the completion of the subproject under the sequential approach and investigate how they affect the optimal contract (Section OA.2). Last, I introduce discounting and discuss how it affects the desirability of each approach (Section OA.3).

OA.1 Asymmetric Arrival Rates for Subprojects

Here I investigate a setting where the arrival rates for the subprojects in the sequential approach are no longer the same. Let $\lambda_{S,1}$ and $\lambda_{S,2}$ denote the arrival rates for the first and second subprojects. The ratio between these two arrival rates, $\lambda_{S,2}/\lambda_{S,1}$, is denoted by κ. In Theorem 4, I show that the form of the optimal contract depends crucially on κ.

To simplify the discussion, I restrict attention to the case where there is no efficiency loss from monitoring. I begin by considering the first-best scenario to derive the condition that fulfills this restriction. Under the assumption that the agent's allocation of effort is observable to the principal, the principal's expected profit from the sequential-only contract without a deadline is derived as follows:

$$\int_0^\infty \int_{\tau_s}^\infty \left(\Pi - \int_0^{\tau_m} c \ dt \right) \lambda_{S,2} e^{-\lambda_{S,2}(\tau_m - \tau_s)} d\tau_m \ \lambda_{S,1} e^{-\lambda_{S,1}\tau_s} d\tau_s = \Pi - \frac{c}{\lambda_{S,1}} - \frac{c}{\lambda_{S,2}}.$$

Note that the principal's expected profit from the direct-only contract without a deadline is $\Pi - c/\lambda_D$. By using similar steps to those in Section 2.2, we can show that there is no efficiency loss from monitoring if and only if $1/\lambda_D = 1/\lambda_{S,1} + 1/\lambda_{S,2}$.

Under this condition, by using the definition of κ, we can derive that $\lambda_{S,1} = (1 + 1/\kappa)\,\lambda_D$ and $\lambda_{S,2} = (1+\kappa)\lambda_D$. From these equations, we observe that higher κ implies that it is harder to achieve the first subproject (lower $\lambda_{S,1}$) but it is easier to complete the second subproject (higher $\lambda_{S,2}$). Note that the completion of the first subproject is monitored by the principal. If the first subproject becomes more difficult and the second subproject becomes easier, we can think of monitoring as relatively more effective. Thus, κ can be interpreted as a parameter that measures the *effectiveness* of monitoring. The following theorem characterizes the optimal contract under an arbitrary κ.

Theorem 4. *Suppose that there is no efficiency loss from monitoring and the arrival rates of the subprojects are not necessarily symmetric. There is a threshold $\kappa^* > 0$ such that the optimal contract is implemented as follows.*

(a) If $\kappa < \kappa^*$, there exist $\Pi_S^A(\kappa) > \Pi_D^A(\kappa) > \Pi_F^A(\kappa) = \Pi_F = (c+\phi)/\lambda_D$ such that the optimal contract is determined as follows:

(i) when $\Pi \leq \Pi_F^A(\kappa) = \Pi_F$, the project is infeasible;

(ii) when $\Pi_D^A(\kappa) \geq \Pi > \Pi_F^A = \Pi_F$, $(\bar{u}, V(\bar{u}))$ is implemented by a direct-only contract with a deadline $\bar{u}/\phi$;

(iii) when $\Pi_S^A(\kappa) > \Pi > \Pi_D^A(\kappa)$, there exists $u_1 \in (0, \bar{u})$ such that $(\bar{u}, V(\bar{u}))$ is implemented by a contract with a switch from the sequential approach to the direct approach at $(\bar{u} - u_1)/\phi$ and a deadline $\bar{u}/\phi$;

(iv) when $\Pi \geq \Pi_S^A(\kappa)$, $(\bar{u}, V(\bar{u}))$ is implemented by a sequential-only contract with a deadline $\bar{u}/\phi$.

(b) If $\kappa \geq \kappa^*$, there exists $\Pi_F^A(\kappa) \leq \Pi_F$ such that the optimal contract is determined as follows:

(i) when $\Pi \leq \Pi_F^A(\kappa)$, the project is infeasible;

(ii) when $\Pi > \Pi_F^A(\kappa)$, $(\bar{u}, V(\bar{u}))$ is implemented by a sequential-only contract with a deadline $\bar{u}/\phi$.

This theorem is illustrated in Figure 4. When monitoring is relatively ineffective, i.e., κ is below κ^*, optimal contracts are characterized in the same way as in Theorem 1: (i) a sequential-only contract when Π is high; (ii) a contract with one switch when Π is intermediate; (iii) a direct-only contract when Π is relatively low; (iv) infeasible when Π is very low. However, when monitoring is relatively effective, i.e., κ is above κ^*, optimal contracts are characterized in a different manner. In this case, the optimal contract is in the form of a sequential-only contract when Π is not low, and the project is infeasible when Π is low.

From this exercise, we can conclude that the form of the optimal contract is determined not only by the return of the project (Π) but also by the effectiveness of monitoring (κ). In particular, when monitoring is effective enough, monitoring is a more important factor than time pressure. Thus, even when the return of the project is low, the principal always chooses the sequential approach so long as the project is feasible.

The rest of this section is devoted to the derivation of Theorem 4.

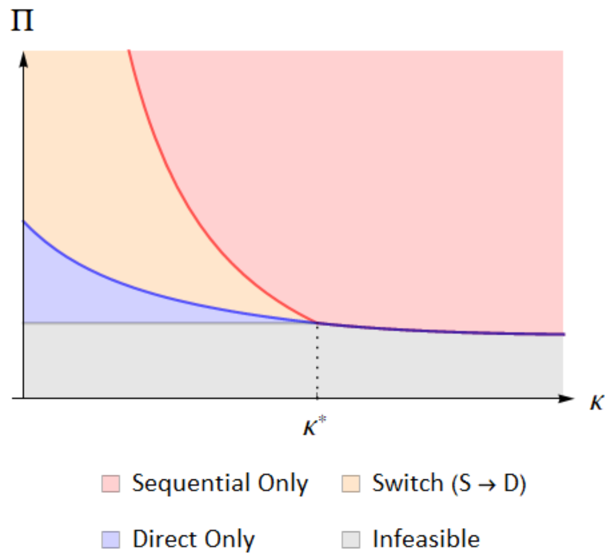


Figure 4: Optimal contracts under asymmetric arrival rates and no efficiency loss

OA.1.1 Benchmark Value Functions

Under the assumption that the arrival rates for the subprojects are no longer symmetric, the agent's HJB equation (HJB$_{PK}$) can be rewritten as follows:

$$0 = \max_{\substack{a_t \in \{0,1\}, \\ b_t \in [0,1]}} \dot{u}_t + \phi(1-b_t) + (R_t - u_t)\lambda_D a_t b_t + (u_S^t - u_t)\lambda_{S,1}(1-a_t)b_t. \tag{PK$_A$}$$

Then, we can observe that the sequential approach can be induced with a minimum incentive by setting $u_S^t = u_t + \phi/\lambda_{S,1}$.

Also note that the principal's value function given that the subproject, V_S^A, is already completed can be written as follows:

$$V_S^A(u_S) = \left(\Pi - \frac{c}{\lambda_{S,2}} - u_S\right) - \left(\Pi - \frac{c}{\lambda_{S,2}}\right) e^{-\frac{\lambda_{S,2}}{\phi} u_S}. \tag{OA.1.1}$$

Let the principal's value function prior to the completion of the subproject be V^A. The HJB equation for $V^A(u)$ is

$$0 = \max_{\substack{R \geq 0,\ u_S \geq 0, \\ a \in \{0,1\},\ b \in [0,1]}} -c + (\Pi - R - V^A(u))\lambda_D ab + (V_S^A(u_S) - V^A(u))\lambda_{S,1}(1-a)b + V^{A\prime}(u)\ \dot{u}. \tag{HJB$_A$}$$

Then, the principal's problem is to solve (HJB$_A$) subject to (PK$_A$). We can derive two benchmark value functions as follows:[27]

1. When the agent's promised utility is lower than a switching point u_1, the principal would recommend the direct approach and the benchmark value function, V_A^d, would be the same as (4.1):

$$V_A^d(u) = \left(\Pi - \frac{c}{\lambda_D}\right)\left(1 - e^{-\frac{\lambda_D}{\phi}u}\right) - u.$$

2. When the agent's promised utility is higher than a switching point u_1, the principal would recommend the sequential approach. By following the similar steps in Section B.4, the benchmark value function, V_A^{ds}, is derived as follows:

$$\begin{aligned} V_A^{ds}(u|u_1) = &\left(\Pi - \frac{c}{\lambda_{S,1}} - \frac{c}{\lambda_{S,2}}\right)\left(1 - e^{\frac{\lambda_{S,1}}{\phi}(u_1-u)}\right) + (V_A^d(u_1) + u_1)e^{\frac{\lambda_{S,1}}{\phi}(u_1-u)} \\ &- \left(\Pi - \frac{c}{\lambda_{S,2}}\right)\frac{e^{\frac{\lambda_{S,2}}{\phi}(u_1-u)} - e^{\frac{\lambda_{S,1}}{\phi}(u_1-u)}}{\kappa - 1}e^{-\frac{\lambda_{S,1}}{\phi}u_1-\kappa} - u.^{28} \end{aligned} \tag{OA.1.2}$$

OA.1.2 Value Function Derivation

To derive the value function, we need to characterize a threshold of Π that determines the recommended approach at the deadline as in Lemma D.1.

Lemma OA.1.1. *Suppose that there is no efficiency loss from splitting the project and $\kappa = \lambda_{S,2}/\lambda_{S,1}$. The inequality $V_A^{d\prime}(0) > V_A^{ds\prime}(0|0)$ holds if and only if*

$$\Pi < \Pi_S^A(\kappa) \equiv \frac{1 - e^{-\kappa}}{1 - (\kappa + 1)e^{-\kappa}} \cdot \frac{c}{\lambda_D}.^{29} \tag{OA.1.3}$$

In addition, $\Pi_S^A(\kappa)$ is decreasing in κ.

[27] Since I focus on the case of no efficiency loss from splitting the project, a benchmark value function involving two switches such as (5.1) is not needed.

[28] Note that

$$\lim_{\kappa\to 1} \frac{e^{\frac{\lambda_{S,2}}{\phi}(u_1-u)} - e^{\frac{\lambda_{S,1}}{\phi}(u_1-u)}}{\kappa - 1} = \frac{(u - u_1)e^{\frac{\lambda_{S,2}}{\phi}(u_1-u)}}{\phi}.$$

Hence, when $\kappa = 1$, i.e., $\lambda_{S,1} = \lambda_{S,2}$, (OA.1.2) is equivalent to (4.2).

[29] Note that $1 - e^{-\kappa} > 1 - (\kappa + 1)e^{-\kappa} > 0$ for all $\kappa > 0$, thus, $\Pi_S^A(\kappa) > c/\lambda_D$.

Proof of Lemma OA.1.1. Note that

$$V_A^{d\prime}(0) = \frac{1}{\phi}\left(\lambda_D \Pi - \phi - c\right),$$

$$\begin{aligned} V_A^{ds\prime}(0|0) &= \frac{1}{\phi}\left(\lambda_{S,1} V_S^A\left(\frac{\phi}{\lambda_{S,1}}\right) - c\right) \\ &= \frac{1}{\phi}\left(\left(1 + \frac{1}{\kappa}\right)(1-e^{-\kappa})\lambda_D \Pi - \frac{1}{\kappa}(1-e^{-\kappa})c - \phi - c\right). \end{aligned}$$

Therefore, $V_A^{d\prime}(0) > V_A^{ds\prime}(0|0)$ is equivalent to

$$(1-e^{-\kappa})c > (1-(\kappa+1)e^{-\kappa})\lambda_D \Pi.$$

Since $1-(\kappa+1)e^{-\kappa} > 0$ for all $\kappa > 0$, the above inequality is equivalent to $\Pi_S^A(\kappa) > \Pi$.

By differentiating Π_S^A with respect to κ, we have

$$\Pi_S^{A\prime}(\kappa) = -\frac{e^{\kappa}\left(e^{-\kappa} + \kappa - 1\right)}{\left(1-(\kappa+1)e^{-\kappa}\right)^2} \cdot \frac{c}{\lambda_D}.$$

Since $e^{-\kappa} > -\kappa + 1$, we have $\Pi_S^{A\prime}(\kappa) < 0$. Thus, Π_S^A is decreasing in κ. □

Proposition OA.1.1. *Assume that both technologies are equally efficient and $\lambda_{S,2}/\lambda_{S,1}$ is equal to κ. Then, the following statements hold:*

(a) When $\Pi \in (c/\lambda_D, \Pi_S^A(\kappa))$, for some $u_1 > 0$, $V_A^{d\prime}(u_1) = V_A^{ds\prime}(u_1|u_1)$ and

$$V^A(u) = \begin{cases} V_A^d(u) & \text{if } 0 \le u \le u_1, \\ V_A^{ds}(u|u_1) & \text{if } u_1 < u \end{cases} \tag{OA.1.4}$$

solves (HJB$_A$) subject to (PK$_A$).

(b) When $\Pi \ge \Pi_S^A(\kappa)$, $V^A(u) = V_A^{ds}(u|0)$ solves (HJB$_A$) subject to (PK$_A$).

OA.1.3 Thresholds

OA.1.3.1 Feasibility (Π_F^A)

Now I derive a threshold that determines the feasibility of the project. By using the same logic in Appendix C.1, the project is feasible if and only if $V'(0) > 0$, and it is equivalent to

$$\max\left[\lambda_D \Pi - \phi,\ \lambda_{S,1} V_S^A(\phi/\lambda_{S,1})\right] > c.$$

Note that

$$\begin{aligned}\lambda_{S,1}V_S^A(\phi/\lambda_{S,1}) =& \lambda_{S,1}\left(\Pi - \frac{c}{\lambda_{S,2}}\right)(1-e^{-\kappa}) - \phi \\ =& \left(1+\frac{1}{\kappa}\right)\lambda_D(1-e^{-\kappa})\Pi - \frac{1-e^{-\kappa}}{\kappa}c - \phi.\end{aligned}$$

Therefore, $\lambda_{S,1}V_S^A(\phi/\lambda_{S,1}) > c$ is equivalent to

$$\Pi > \frac{1}{(\kappa+1)\lambda_D}\left[c + \frac{\kappa}{1-e^{-\kappa}}(c+\phi)\right].$$

Then, the project is feasible if and only if $\Pi > \min\left[\frac{c+\phi}{\lambda_D}, \frac{1}{(\kappa+1)\lambda_D}\left[c+\frac{\kappa}{1-e^{-\kappa}}(c+\phi)\right]\right] \equiv \Pi_F^A(\kappa)$.

OA.1.3.2 The Length of the Contract (Π_D^A)

Next, I derive another threshold that determines whether there is a switch in the optimal contract. Let $\bar{u}$ denote the value that maximizes $V^A(u)$. Then, we need to compare $\bar{u}$ with a switch point u_1. The following lemma characterizes the threshold Π_D^A.

Lemma OA.1.2. *Suppose that there is no efficiency loss from splitting the project and $\kappa = \lambda_{S,2}/\lambda_{S,1}$. Then, there exists $\Pi_D^A(\kappa)$ such that $u_1 < \bar{u}$ if and only if $\Pi > \Pi_D^A(\kappa)$.*

Proof of Lemma OA.1.2. I begin by deriving the closed form of u_1. Define a function H_1^A as in (D.1):

$$H_1^A(u) \equiv \phi V_A^{ds\prime}(u|u) - \phi V_A^{d\prime}(u). \tag{OA.1.5}$$

Then, $H_1^A(u)$ can be rewritten as follows:

$$\begin{aligned}H_1^A(u) &= \lambda_{S,1}\left((\Pi - c/\lambda_{S,2})(1-e^{-\frac{\lambda_{S,2}}{\phi}u-\kappa}) - V_A^d(u) - u\right) - \lambda_D(\Pi - u - V_A^d(u)) \\ &= (\lambda_{S,1}\Pi - c/\kappa)\left(1-e^{-\frac{\lambda_{S,2}}{\phi}u-\kappa}\right) - (\lambda_{S,1}-\lambda_D)(u+V_A^d(u)) - \lambda_D\Pi \\ &= (\lambda_{S,1}\Pi - c/\kappa)\left(1-e^{-\frac{\lambda_{S,2}}{\phi}u-\kappa}\right) - (\lambda_{S,1}-\lambda_D)(\Pi - c/\lambda_D)(1-e^{-\frac{\lambda_D}{\phi}u}) - \lambda_D\Pi \\ &= (\lambda_{S,1}/\lambda_D - 1)(\lambda_D\Pi - c)e^{-\frac{\lambda_D}{\phi}u} - (1/\kappa - \lambda_{S,1}/\lambda_D + 1)c - (\lambda_{S,1}\Pi - c/\kappa)e^{-\frac{\lambda_{S,1}}{\phi}u-\kappa} \\ &= (\lambda_D\Pi - c)e^{-\frac{\lambda_D}{\phi}u}/\kappa - (\lambda_{S,2}\Pi - c)e^{-\frac{\lambda_{S,1}}{\phi}u-\kappa}/\kappa.\end{aligned}$$

By the definition of u_1, it is the solution of $H_1^A(u) = 0$. Then, by using the above equation, we can derive that

$$u_1 = \frac{\phi}{\lambda_D}\left[\frac{1}{\kappa}\log\left(\frac{\lambda_{S,2}\Pi - c}{\lambda_D\Pi - c}\right) - 1\right].$$

The next step is to compare u_1 with $\bar{u}$. Since V^A is strictly concave, $u_1 < \bar{u}$ is equivalent to $0 < V^{A\prime}(u_1) = V_A^{d\prime}(u_1)$. By using the above equation, we can derive that $u_1 = \bar{u}$ is equivalent to

$$\frac{(\lambda_D\Pi - c)^{1+\kappa}}{(1+\kappa)\lambda_D\Pi - c}\left(\frac{e}{\phi}\right)^{\kappa} = 1. \tag{OA.1.6}$$

Note that the left hand side is increasing in Π, is equal to zero when Π is equal to c/λ_D, and diverges as Π goes to infinity. Therefore, there exists a unique solution that satisfies (OA.1.6) and I denote the solution as $\Pi_D^A(\kappa)$. Then, $\Pi > \Pi_D^A(\kappa)$ is equivalent to $\bar{u} > u_1$. $\square$

OA.1.3.3 Comparison of Thresholds

Lemma OA.1.3. *Let κ^* be the positive solution of the equation*

$$0 = \phi + (c+\phi)\kappa - \phi e^{\kappa}. \tag{OA.1.7}$$

Then,

1. *if $\kappa > \kappa^*$,*

$$\Pi_F^A(\kappa) = \frac{1}{(\kappa+1)\lambda_D}\left(c + \frac{\kappa}{1-e^{-\kappa}}(c+\phi)\right) > \max\left[\Pi_D^A(\kappa),\ \Pi_S^A(\kappa)\right],$$

2. *if $\kappa = \kappa^*$,*

$$\Pi_F^A(\kappa^*) = \frac{c+\phi}{\lambda_D} = \frac{1}{(\kappa^*+1)\lambda_D}\left(c + \frac{\kappa^*}{1-e^{-\kappa^*}}(c+\phi)\right) = \Pi_D^A(\kappa^*) = \Pi_S^A(\kappa^*);$$

3. *if $\kappa < \kappa^*$,*

$$\Pi_F^A(\kappa) = \frac{c+\phi}{\lambda_D} < \Pi_D^A(\kappa) < \Pi_S^A(\kappa);$$

4. *as $\kappa \to 0$,*

$$\lim_{\kappa\to 0}\Pi_D^A(\kappa) = \frac{c + \phi\cdot\psi\left(c/\phi\right)}{\lambda_D} \quad \textit{and} \quad \lim_{\kappa\to 0}\Pi_S^A(\kappa) = \infty,$$

where $\psi : \mathbb{R}_+ \to [1,\infty)$ is the inverse function of $x\log(x)$ for $x \geq 1$.

Proof of Lemma OA.1.3. Note that for all $\kappa > 0$ and $\lambda_D > 0$,

$$\begin{aligned}
& \frac{c+\phi}{\lambda_D} > \frac{1}{(\kappa+1)\lambda_D}\left(c + \frac{\kappa}{1-e^{-\kappa}}(c+\phi)\right) \\
\Leftrightarrow \quad & (c+\phi)(\kappa+1)(e^{\kappa}-1) > c(e^{\kappa}-1) + (c+\phi)(\kappa+1-e^{\kappa}) \\
\Leftrightarrow \quad & 0 > g(\kappa) \equiv \phi + (c+\phi)\kappa - \phi e^{\kappa}.
\end{aligned}$$

Also note that $g(\kappa)$ is concave in κ, $\lim_{\kappa\to 0} g(\kappa) = 0$, $\lim_{\kappa\to\infty} g(\kappa) = -\infty$ and $\lim_{\kappa\to 0} g'(\kappa) = c > 0$. Then, there exists a unique positive solution to $g(\kappa) = 0$, which is κ^*. Then, $g(\kappa) < 0$ is equivalent to $\kappa > \kappa^*$. Therefore,

$$\Pi_F^A(\kappa) = \begin{cases} \dfrac{c+\phi}{\lambda_D}, & \text{if } \kappa < \kappa^*, \\ \dfrac{c+\phi}{\lambda_D} = \dfrac{1}{(\kappa^*+1)\lambda_D}\left(c + \dfrac{\kappa^*}{1-e^{-\kappa^*}}(c+\phi)\right), & \text{if } \kappa = \kappa^*, \\ \dfrac{1}{(\kappa+1)\lambda_D}\left(c + \dfrac{\kappa}{1-e^{-\kappa}}(c+\phi)\right), & \text{if } \kappa > \kappa^*. \end{cases}$$

For $i \in \{F, W, S\}$, note that $\Pi_i(\kappa)$ can be considered as a unique solution (greater than c/λ) to the equation

$$L(\Pi) = R_i(\Pi|\kappa),$$

where

$$\begin{aligned}
L(\Pi) &= (\kappa+1)\lambda_D \Pi - c, \\
R_F(\Pi|\kappa) &= \begin{cases} \dfrac{\kappa}{1-e^{-\kappa}}(c+\phi) & \text{if } \kappa \geq \kappa^* \\ \kappa(c+\phi)+\phi & \text{if } \kappa \leq \kappa^* \end{cases}, \text{[30]} \\
R_W(\Pi|\kappa) &= \phi \cdot e^{\kappa} \cdot \left(\frac{\lambda_D \Pi - c}{\phi}\right)^{\kappa+1}, \\
R_S(\Pi|\kappa) &= \phi \cdot e^{\kappa} \cdot \left(\frac{\lambda_D \Pi - c}{\phi}\right).
\end{aligned}$$

Note that $L(c/\lambda_D) < R_i(c/\lambda_D|\kappa)$, $\lim_{\Pi\to\infty} L(\Pi) > \lim_{\Pi\to\infty} R_i(\Pi|\kappa)$ and L and $R_i(\cdot|\kappa)$ cross only once for all $i \in \{F, W, S\}$ and $\kappa > 0$.

[30] Note that $\kappa^*(c+\phi)/(1-e^{-\kappa^*}) = \kappa^*(c+\phi)+\phi$ by the definition of κ^*.

If $R_i(\Pi_i(\kappa)|\kappa) > R_j(\Pi_i(\kappa)|\kappa)$,

$$L(\Pi_i(\kappa)) = R_i(\Pi_i(\kappa)|\kappa) > R_j(\Pi_i(\kappa)|\kappa),$$

and it implies that $\Pi_j(\kappa)$ is lower than $\Pi_i(\kappa)$. Similarly, $R_i(\Pi_i(\kappa)|\kappa) = R_j(\Pi_i(\kappa)|\kappa)$ implies that $\Pi_j(\kappa)$ is equal to $\Pi_i(\kappa)$ and $R_i(\Pi_i(\kappa)|\kappa) < R_j(\Pi_i(\kappa)|\kappa)$ implies that $\Pi_j(\kappa)$ is greater than $\Pi_i(\kappa)$.

1. When $\kappa > \kappa^*$, to prove that $\Pi_F^A(\kappa) > \max\left[\Pi_D^A(\kappa), \Pi_S^A(\kappa)\right]$, it is enough to show that $R_F(\Pi_F^A(\kappa)|\kappa) < R_W(\Pi_F^A(\kappa)|\kappa)$ and $R_F(\Pi_F^A(\kappa)|\kappa) < R_S(\Pi_F^A(\kappa)|\kappa)$.

   Define $x(\kappa)$ as follows:
   $$x(\kappa) = \frac{\kappa}{e^\kappa - 1}\left(\frac{c+\phi}{\phi}\right).$$
   Then, $x(\kappa) < 1$ is equivalent to $g(\kappa) < 0$, i.e., $\kappa > \kappa^*$. Also note that
   $$\frac{\lambda_D \Pi_F^A(\kappa) - c}{\phi} = \frac{\kappa}{\kappa+1}\left(\frac{c + e^\kappa \phi}{e^\kappa - 1}\right) = \frac{x(\kappa)+\kappa}{\kappa+1}.$$

   By using the definition of $x(\kappa)$ and the above equation, we can see that
   $$\begin{aligned} R_F(\Pi_F^A(\kappa)|\kappa) &= \phi \cdot e^\kappa \cdot x(\kappa), \\ R_W(\Pi_F^A(\kappa)|\kappa) &= \phi \cdot e^\kappa \cdot \left(\frac{x(\kappa)+\kappa}{\kappa+1}\right)^{\kappa+1}, \\ R_S(\Pi_F^A(\kappa)|\kappa) &= \phi \cdot e^\kappa \cdot \left(\frac{x(\kappa)+\kappa}{\kappa+1}\right). \end{aligned} \tag{OA.1.8}$$

   Consider a function $h(x) = \left(\frac{x+\kappa}{1+\kappa}\right)^{\kappa+1}$. Note that $h'(x) = \left(\frac{x+\kappa}{1+\kappa}\right)^{\kappa}$ and $h''(x) = \frac{\kappa}{1+\kappa}\left(\frac{x+\kappa}{1+\kappa}\right)^{\kappa-1} > 0$. Then, $h(x) > h(1) + h'(1)(x-1) = x$ for $x < 1$. Hence, $R_W(\Pi_F^A(\kappa)|\kappa) > R_F(\Pi_F^A(\kappa)|\kappa)$. Also, we can easily see that $\dfrac{x+\kappa}{\kappa+1} > x$ is equivalent to $x < 1$, i.e., $R_S(\Pi_F^A(\kappa)|\kappa) > R_F(\Pi_F^A(\kappa)|\kappa)$.

2. When $\kappa = \kappa^*$, to prove that $\Pi_F^A(\kappa) = \Pi_D^A(\kappa) = \Pi_S^A(\kappa)$, it is enough to show that $R_F(\Pi_F^A(\kappa)|\kappa) = R_W(\Pi_F^A(\kappa)|\kappa) = R_S(\Pi_F^A(\kappa)|\kappa)$.

   Note that $x(\kappa^*) = 1$. Hence, by (OA.1.8), $R_F(\Pi_F^A(\kappa)|\kappa) = R_W(\Pi_F^A(\kappa)|\kappa) = R_S(\Pi_F^A(\kappa)|\kappa)$.

3. When $\kappa < \kappa^*$, to prove that $\Pi_S^A(\kappa) > \Pi_D^A(\kappa) > \Pi_F^A(\kappa)$, it is enough to show that $R_F(\Pi_F^A(\kappa)|\kappa) > R_W(\Pi_F^A(\kappa)|\kappa)$ and $R_W(\Pi_S^A(\kappa)|\kappa) > R_S(\Pi_S^A(\kappa)|\kappa)$.

In this case, $\Pi_F^A(\kappa) = (c+\phi)/\lambda_D$. Then, by the definition of R_F and R_W, $R_F(\Pi_F^A(\kappa)|\kappa) = \kappa(c+\phi)+\phi$ and $R_W(\Pi_F^A(\kappa)|\kappa) = \phi \cdot e^{\kappa}$. Since $\kappa < \kappa^*$ is equivalent to $\kappa(c+\phi)+\phi > \phi e^{\kappa}$, $R_F(\Pi_F^A(\kappa)|\kappa) > R_W(\Pi_F^A(\kappa)|\kappa)$.

Also note that

$$\frac{\lambda_D \Pi_S^A(\kappa) - c}{\phi} = \frac{\frac{1-e^{-\kappa}}{1-(\kappa+1)e^{-\kappa}}c - c}{\phi} = \frac{\kappa \cdot c}{(e^{\kappa} - (\kappa+1))\phi} > 1.$$

Then, since $R_W(\Pi_S^A(\kappa)|\kappa) = R_S(\Pi_S^A(\kappa)|\kappa) \cdot \left(\frac{\lambda_D \Pi - c}{\phi}\right)^{\kappa}$, $R_W(\Pi_S^A(\kappa)|\kappa) > R_S(\Pi_S^A(\kappa)|\kappa)$.

4. When $\kappa \to 0$, by L'Hôpital's Rule,

$$\lim_{\kappa \to 0} \Pi_S^A(\kappa) = \lim_{\kappa \to 0} \frac{1 - e^{-\kappa}}{1 - (\kappa+1)e^{-\kappa}} \cdot \frac{c}{\lambda_D} = \lim_{\kappa \to 0} \frac{e^{-\kappa}}{\kappa \; e^{-\kappa}} \cdot \frac{c}{\lambda_D} = \infty.$$

Define $y(\kappa) \equiv \left(\lambda_D \Pi_D^A(\kappa) - c\right)/\phi > 0$. Then, from (OA.1.6), $y(\kappa)$ satisfies the following equations for all $\kappa > 0$:

$$y(\kappa)^{1+\kappa} \cdot e^{\kappa} = (1+\kappa)y(\kappa) + \frac{c}{\phi}\kappa$$

$$\Rightarrow \qquad (1+\kappa)\log\left[y(\kappa)\right] + \kappa = \log\left[(1+\kappa)y(\kappa) + \frac{c}{\phi}\kappa\right].$$

By differentiating the above equation by κ, we have

$$\log\left[y(\kappa)\right] + 1 + \frac{1+\kappa}{y(\kappa)}y'(\kappa) = \frac{y(\kappa) + \frac{c}{\phi}}{(1+\kappa)y(\kappa) + \frac{c}{\phi}\kappa} + \frac{1+\kappa}{(1+\kappa)y(\kappa) + \frac{c}{\phi}\kappa}y'(\kappa).$$

By sending $\kappa \to 0$, we have

$$y(0) \cdot \log\left[y(0)\right] = \frac{c}{\phi},$$

i.e., $y(0) = \psi(c/\phi)$. Then, we have

$$\lim_{\kappa \to 0} \Pi_D^A(\kappa) = \frac{c + \phi \cdot y(0)}{\lambda_D} = \frac{c + \phi \cdot \psi\left(\frac{c}{\phi}\right)}{\lambda_D}.$$

□

OA.1.4 Proof of Theorem 4

Proof of Theorem 4. (a) By Lemma OA.1.3, when $\kappa < \kappa^*$, $\Pi_S^A(\kappa) > \Pi_D^A(\kappa) > \Pi_F^A(\kappa) = (c+\phi)/\lambda_D$.

(i) By the argument in Section OA.1.3.1, the project is infeasible if $\Pi \leq \Pi_F^A(\kappa)$.

(ii) When $\Pi_D^A(\kappa) \geq \Pi > \Pi_F^A(\kappa)$, u_1 is greater than or equal to $\bar{u}$ by Lemma OA.1.2. By Proposition OA.1.1, the value function is $V^A(u) = V_A^g(u)$ for $u \leq \bar{u} \leq u_1$. Thus, the optimal contract is to execute the direct approach for all $u \leq \bar{u}$. Therefore, the direct-only contract with $T = \bar{u}/\phi$ implements $(\bar{u}, V(\bar{u}))$.

(iii) When $\Pi_S^A(\kappa) > \Pi > \Pi_D^A(\kappa)$, u_1 is less than $\bar{u}$ by Lemma OA.1.2 and greater than zero by Lemma OA.1.1. By Proposition OA.1.1, the value function is $V(u) = V_A^{ds}(u|u_1)$ for $u_1 < u < \bar{u}$ and $V(u) = V_A^g(u)$ for $0 \leq u \leq u_1$. Thus, the optimal contract is to execute the sequential approach for $u_1 < u \leq \bar{u}$ and the direct approach for $0 \leq u \leq u_1$. Therefore, the contract with a switch from the sequential approach to the direct approach at $(\bar{u} - u_1)/\phi$ and a deadline $\bar{u}/\phi$.

(iv) When $\Pi \geq \Pi_S^A(\kappa)$, $V^A(u) = V_A^{ds}(u|0)$ by Proposition OA.1.1. Thus, the optimal contract is to execute the sequential approach for $0 \leq u \leq \bar{u}$. Therefore, the sequential-only contract with $T = \bar{u}/\phi$ implements $(\bar{u}, V(\bar{u}))$.

(b) By Lemma OA.1.3, when $\kappa \geq \kappa^*$, $\Pi_F \geq \Pi_F^A(\kappa) \geq \Pi_S^A(\kappa)$.

(i) By the argument in Section OA.1.3.1, the project is infeasible if $\Pi \leq \Pi_F^A(\kappa)$.

(ii) When $\Pi > \Pi_F^A(\kappa)$, Π is greater than $\Pi_S^A(\kappa)$. Thus, $V^A(u) = V_A^{ds}(u|0)$ by Proposition OA.1.1 and the sequential-only contract with $T = \bar{u}/\phi$ implements $(\bar{u}, V(\bar{u}))$.

□

OA.2 Independent Values from a Subproject

In many relevant applications, both players may benefit from the completion of the subproject. In particular, their outside options will differ before and after the intermediate breakthrough. For example, by adding experience to his resume, the agent can get a better offer from other firms. For the principal, even if the agent does not finish the main project, she may be able to license out the current progress to another company.

The goal of this section is to investigate how these outside options affect the optimal contract. I address this question by exploring a numerical example. I fix parameter values as follows: $\Pi = 5$, $c = 1$, $\phi = .5$, $\lambda_D = 1$, $\lambda_S = 2$. Note that λ_S is equal to $2\lambda_D$ which means that two approaches are equally efficient. Moreover, Π is greater than $\Pi_S(1)$ implying that the optimal contract takes a form of a sequential-only contract with a deadline when there are no intermediate values from the subproject.

Let $B_P \geq 0$ and $B_A \geq 0$ denote the outside options for the principal and the agent when the subproject is completed. We can interpret a high B_P involving an active buyout market for projects and a high B_A involving an active labor market for experienced workers. The goal is to provide comparative statics on the deadlines and the principal's expected payoffs from the optimal contracts with respect to B_A and B_P. To derive the optimal contracts, we need to reformulate the principal's problem by adding B_A and B_P.

I begin by rewriting the promise-keeping constraint (HJB_{PK}). Once I introduce the outside option B_A for the agent, the effective promised utility for the agent at time t would be $u_S^{B,t} \equiv u_S^t - B_A$ where u_S^t is the agent's promised utility for the success of the subproject at t. Then, (HJB_{PK}) can be rewritten as follows:

$$0 = \max_{\substack{a_t \in \{0,1\}, \\ b_t \in [0,1]}} \dot{u}_t + \phi(1 - b_t) + (R_t - u_t)\lambda_D a_t b_t + (u_S^{B,t} - u_t + B_A)\lambda_S(1 - a_t)b_t. \qquad (\text{HJB}^B_{PK})$$

From this equation, we can infer that to induce the sequential approach, $u_S^{B,t}$ has to be greater than or equal to $u_t + \phi/\lambda_S - B_A$. To simplify discussion, I focus on the case where the principal "extends" the deadline after the completion of the subproject, i.e., $B_A < \phi/\lambda_S = .25$.

Next, we need to reconsider the principal's expected payoff after the completion of the subproject. Note that the additional value for the second subproject would be $\Pi - B_P$ since the principal has the outside option of B_P. In addition, the principal needs to work with the agent's effective promised utility rather than the original promised utility. Then, given the subproject completion, the principal's value function can be written as in (3.1):

$$V_S^B(u_S^B) = \left(\Pi - B_P - \frac{c}{\lambda_S}\right)\left(1 - e^{-\frac{\lambda_S}{\phi}u_S^B}\right) - u_S^B. \qquad \text{(OA.2.1)}$$

This value stands for the principal's expected payoff additional to the outside option B_P when the agent's effective promised utility is u_S^B. Thus, $V_S^B(u_S^B) + B_P$ is the principal's expected payoff after the completion of the subproject.

Note that the introduction of B_A and B_P does not affect the direct approach, thus V^d remains the same. Since (HJB_{PK}) is replaced by (HJB^B_{PK}), when the sequential approach is chosen, we need to substitute $u_S^B = u + \phi/\lambda_S - B_A$ for u_S. The principal's value function after the intermediate breakthrough ($V_S(u_S)$) also needs to be replaced by $V_S^B(u_S^B) + B_P$.

Then, the HJB equation (B.6) can be rewritten as follows:

$$0 = -c + \lambda_S \left(V_S^B \left(u + \phi/\lambda_S - B_A \right) + B_P - V_B^{ds}(u|u_1) \right) - \phi V_B^{ds\prime}(u|u_1) \tag{OA.2.2}$$

where V_B^{ds} is the new value function.

By plugging (OA.2.1) into (OA.2.2) and following the same steps in Appendix B.4, we can derive the closed form of V_B^{ds} as follows:

$$\begin{aligned} V_B^{ds}(u|u_1) = & \left(\Pi - \frac{2c}{\lambda_S} + B_A \right) \left(1 - e^{\frac{\lambda_S}{\phi}(u_1 - u)} \right) + (V^d(u_1) + u_1) e^{\frac{\lambda_S}{\phi}(u_1 - u)} \\ & - \left(\Pi - B_P - \frac{c}{\lambda_S} \right) \frac{\lambda_S}{\phi} (u - u_1) e^{\frac{\lambda_S}{\phi}(B_A - u) - 1} - u. \end{aligned}$$

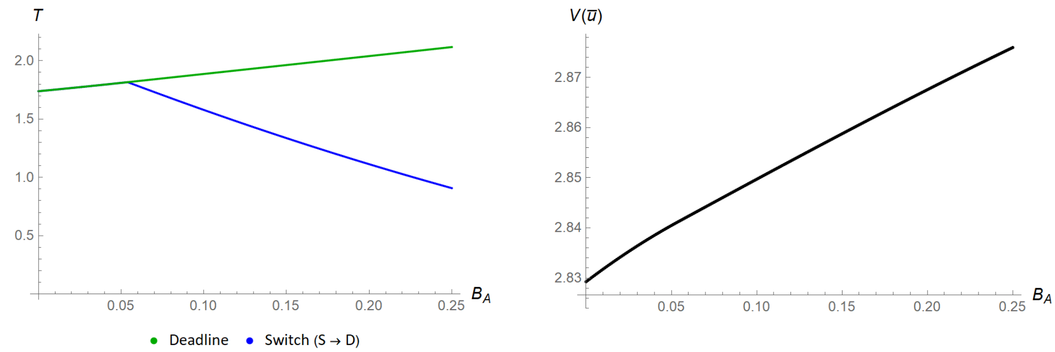
If there is no switching point, $V_B^{ds}(u|0)$ would serve as the value function. If there exists a switching point, we can pin down the switching point by following the same steps in Appendix D.1. When the switching point is u_1, the value function for $u \leq u_1$ is $V^d(u)$ and that for $u > u_1$ is $V^{ds}(u|u_1)$.

In Figure 5, I illustrate the comparative statics on the optimal deadlines and the principal's expected payoffs under four different scenarios. In Figure 5a and 5b, I demonstrate the comparative statics with respect to B_A when B_P is 0 or 3. In Figure 5c and 5d, I display the comparative statics with respect to B_P when B_A is 0 or .2. In the left panels, the green curves demonstrate the optimal deadlines and the blue curves display the switching time from the sequential approach to the direct approach. If the blue curves are not present, it means that the sequential approach is chosen until the deadline. The principal's expected payoffs (before the intermediate breakthrough) are shown in the right panels.

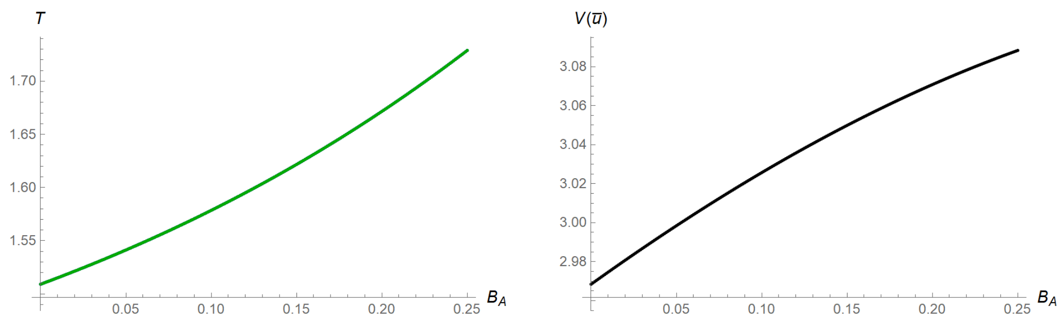
First, we can observe that the principal's expected payoffs are increasing in B_A or B_P in every panel in Figure 5. This means that the principal benefits from both the active buyout market for projects and the labor market for experienced workers. Since the active buyout market allows the principal to liquidate the project easier, it directly helps the principal. When the labor market for experienced workers becomes more active, it is easier for the principal to incentivize the agent to split the project. Given the intermediate breakthrough, while the principal needs to extend the deadline by $1/\lambda_S$ without the labor market, she only needs to extend the deadline by $1/\lambda_S - B_A/\phi$ with the labor market. This numerical example shows that the principal benefits from this decreased incentive under the restriction that B_A is less than ϕ/λ_S.[31]

[31] The case where B_A is greater than ϕ/λ_S is problematic for the following reason. In this case, if the principal provides the minimum incentive not to shirk, the deadline would be shortened after the completion of the subproject. This lowers the probability for the success of the project implying that it may not be optimal to employ "the minimum incentive contract" in the first place. This possibility significantly

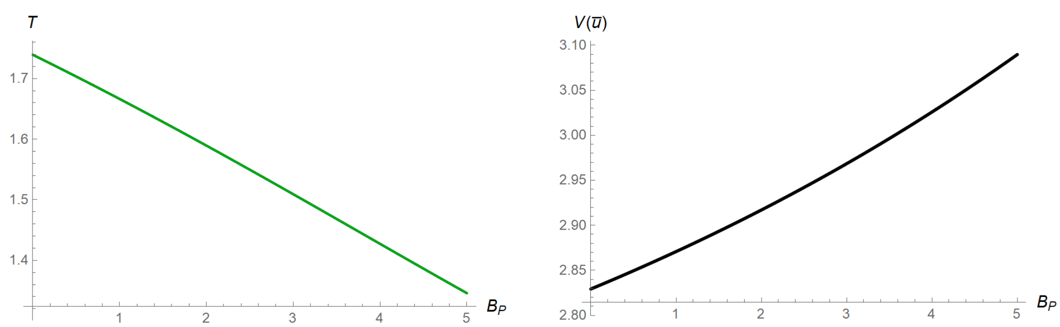
(a) Deadlines and the principal expected payoffs when $B_P = 0$



(b) Deadlines and the principal expected payoffs when $B_P = 3$



(c) Deadlines and the principal expected payoffs when $B_A = 0$



(d) Deadlines and the principal expected payoffs when $B_A = .2$

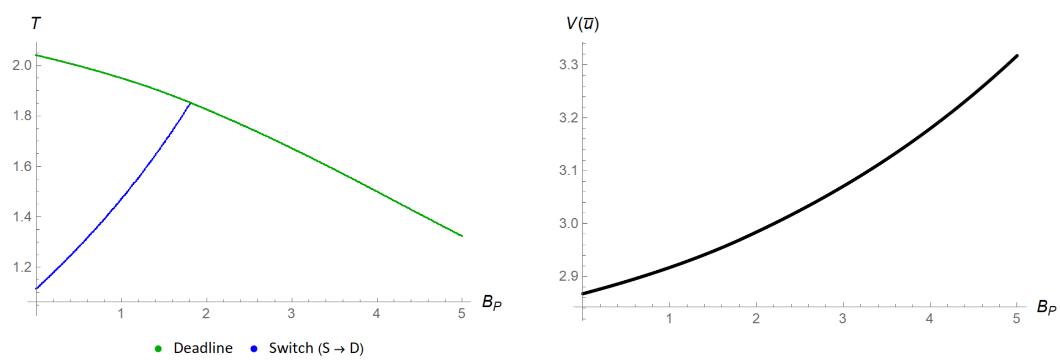


Figure 5: Comparative Statics with respect to B_A and B_P

Next, in the left panels of Figure 5a and 5b, we can see that the optimal deadline increases as B_A increases. The principal expects that the deadline extension would be shortened as B_A increases, thus she preemptively prolonds the deadline. Another interesting feature is that switching to the direct approach occurs when B_A is large and B_P is small as in the left panel of Figure 5a. As B_A rises, the sequential approach becomes less appealing to the principal since the probability of success is lowered by the shortened deadline extension. Therefore, it may be more profitable to switch to the direct approach near the deadline. When B_P is large, even though the sequential approach becomes less desirable, it may still be better than the direct approach. Thus, there may not exist a switch at all as in 5b.

Last, the left panels of Figure 5c and 5d show that the optimal deadline decreases as B_P rises. This is simply because an "early buyout" becomes more attractive as the buyout market grows. Moreover, as B_P increases, the sequential approach becomes more desirable. The left panel of Figure 5d shows that the direct approach is optimal when B_P is small, but the sequential approach eventually dominates as B_P grows.

OA.3 The Role of the Discount Rate

Finally, I introduce discounting to the model and investigate how it distorts the efficiency of each approach. Let $r \geq 0$ denote the common discount rate for the principal and the agent. The expected payoffs for indefinitely employing the direct approach and the sequential approach can be rewritten respectively as follows:

$$
\begin{aligned}
F_D(r) \equiv & \int_0^\infty \left(e^{-r\tau_m}\Pi - \int_0^{\tau_m} e^{-rt} c dt \right) \lambda_D e^{-\lambda_D \tau_m} d\tau_m \\
= & \int_0^\infty \left(\frac{\lambda_D}{\lambda_D + r} \cdot \left(\Pi + \frac{c}{r} \right) \cdot (\lambda_D + r) e^{-(\lambda_D + r)\tau_m} - \frac{c}{r} \cdot \lambda_D e^{-\lambda_D \tau_m} \right) d\tau_m \\
= & \frac{\lambda_D}{\lambda_D + r} \left(\Pi - \frac{c}{\lambda_D} \right) = \frac{\lambda_D}{\lambda_D + r}\Pi - \frac{1}{\lambda_D + r} c,
\end{aligned}
$$

$$
\begin{aligned}
F_S(r) \equiv & \int_0^\infty \int_{\tau_s}^\infty \left(e^{-r\tau_m}\Pi - \int_0^{\tau_m} e^{-rt} c dt \right) \lambda_S e^{-\lambda_S(\tau_m - \tau_s)} d\tau_m \; \lambda_S e^{-\lambda_S \tau_s} d\tau_s \\
= & \int_0^\infty \left(\frac{\lambda_S}{\lambda_S + r} \cdot \left(\Pi + \frac{c}{r} \right) \cdot e^{-r\tau_s} - \frac{c}{r} \right) \lambda_S e^{-\lambda_S \tau_s} d\tau_s \\
= & \frac{\lambda_S^2}{(\lambda_S + r)^2} \left(\Pi + \frac{c}{r} \right) - \frac{c}{r} = \frac{\lambda_S^2}{(\lambda_S + r)^2}\Pi - \frac{2\lambda_S + r}{(\lambda_S + r)^2} c.
\end{aligned}
$$

From these expressions, when $r > 0$, we observe that the efficiency relationship depends not only on the arrival rates but also on the return of the project (Π) and the flow cost (c).

complicates the analysis of the model even numerically.

This is significantly different from the no-discounting case. When $r = 0$, the efficiency relationship is simply determined by comparing the expected durations under the two approaches ($1/\lambda_D$ and $2/\lambda_S$). Thus, the presence of the discount factor complicates the analysis.

To simplify the argument, I focus on the case where there is no efficiency loss from monitoring; i.e., $2\lambda_D = \lambda_S$. The following proposition shows how the discount rate distorts the efficiency of each approach.

Proposition OA.3.1. *Suppose that $2\lambda_D = \lambda_S$ and $\Pi > c/\lambda_D$. Then, for all $r > 0$, the following inequality holds: $F_D(0) = F_S(0) > F_D(r) > F_S(r)$.*

Proof of Proposition OA.3.1. Note that

$$F_D(0) = \Pi - \frac{c}{\lambda_D} = \Pi - \frac{2c}{\lambda_S} = F_S(0),$$

and

$$F_D(0) = \frac{\lambda_D \Pi - c}{\lambda_D} > \frac{\lambda_D \Pi - c}{\lambda_D + r} = F_D(r)$$

since $\Pi > c/\lambda_D$ and $r > 0$. Also note that

$$F_D(r) - F_S(r) = \frac{\lambda_D r}{(\lambda_D + r)(\lambda_S + r)^2}(r\Pi + c) > 0,$$

thus $F_D(r) > F_S(r)$. □

This proposition says that the introduction of the discount rate harms the efficiency of the sequential approach more than that of the direct approach. In other words, if players begin to discount the future, the sequential approach becomes less appealing in terms of efficiency. Recall that the sequential approach is less advantageous in the short run. Thus, a high discount rate distorts the efficiency of the sequential approach more than that of the direct approach.