Comparing Information in General Monotone Decision Problems

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Abstract

I study the value of information in monotone decision problems with potentially multidimensional action spaces. As a criterion for comparing information structures, I develop a condition called *monotone quasi-garbling*, which involves adding *reversely monotone noise* to an existing information structure. Specifically, this noise is more likely to return a higher signal in a lower state and a lower signal in a higher state. I show that monotone quasi-garbling is a necessary and sufficient condition for decision makers to obtain a higher ex-ante expected payoff. This new criterion refines the garbling condition by Blackwell (1951, 1953) and is equivalent to the accuracy condition by Lehmann (1988) under the monotone likelihood ratio property. To illustrate, I apply the result to problems in nonlinear monopoly pricing and optimal insurance.

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1 Introduction

Consider a pair of information structures that provide signals about uncertain states. When can we say that one is superior to the other? This fundamental question has numerous economic applications including investment, monopoly pricing, and auctions. A common feature in such settings is that the decision maker would like to take a higher action when a higher signal is realized, that is, the decision problem is often *monotone*.

The classical way of comparing information structures is to use the *garbling condition* developed by Blackwell (1951, 1953). By this criterion an information structure (G) is worse than another (F) if G can be obtained from F by adding some noise—in other words, G is a garbling of F. Intuitively, the added noise reduces the value of information. Indeed, Blackwell's condition implies that for *any* preferences that satisfy the von Neumann-Morgenstern axioms, the expected payoff under F is higher than under G. This is a powerful result because once a pair of information structures are ranked by the garbling order, the rank is preserved in every decision problem. Although the garbling order is an important criterion for comparing two information structures, its usefulness is restricted because it is difficult to meet its requirements.

If we restrict attention to monotone decision problems, it is sometimes possible to compare information structures that are unrankable by Blackwell's condition. One well-known criterion for doing so is the *accuracy condition* by Lehmann (1988). Lehmann considers a specific class of monotone decision problems and shows that his criterion refines Blackwell's garbling condition. Although the accuracy condition has been applied widely in economic settings, its precise meaning remains underexplored. Specifically, the accuracy criterion does not deliver as clear an economic interpretation as Blackwell's garbling condition does. This leads to the first main question of this paper: can we understand Lehmann's condition by using the garbling notion?

To answer this question, I introduce a novel concept called *monotone quasi-garbling* by modifying Blackwell's garbling order. Specifically, I relax the assumption of the garbling condition that the added noise is independent of the state. I define the monotone quasi-garbling order by (i) allowing noise to be state-dependent; (ii) but restricting noise to be *reversely monotone* in the sense that it is more likely to return a higher signal in a lower state and a lower signal in a higher state. In other words, G is a monotone quasi-garbling of F if G can be obtained from F by adding reversely monotone noise. I show that if the monotone likelihood ratio property (MLRP) holds, then Lehmann's order and the monotone quasi-garbling order are equivalent (Theorem 1). This result provides a fresh view for interpreting Lehmann's condition by using the noise notion: under the MLRP, F is more Lehmann accurate than G if and only if G is generated by adding reversely monotone noise to F.

The second question of this paper is whether we can extend Lehmann's analysis to general classes of monotone decision problems. In the seminal paper on monotone comparative statics, Milgrom and Shannon (1994) posit a partial order on a multidimensional action set to study the direction of change of the optimal actions in response to exogenous changes in parameter values. Nevertheless, in most extant work studying information in monotone decision problems, the common assumption is that the action set is unidimensional and the order of actions is simply inherited from the real line (Quah and Strulovici, 2009; Chi, 2015; Athey and Levin, 2017; Li and Zhou, 2020). Because of this restriction, we cannot directly apply these earlier results to problems involving multidimensional actions such as an investment decision with Arrow-Debreu securities or a nonlinear monopoly pricing mechanism involving a menu of tariffs and quantities. I show that the introduction of the monotone quasi-garbling order helps extend information comparisons to general monotone decision problems exhibiting multidimensional actions in applications such as these.

As a first step towards this characterization, I introduce the notion of general monotone decision problems, where the action space can be multidimensional. Since there is no natural order in a multidimensional set, the action set needs to have a partial order that determines which actions are higher or lower. To ensure the meaningfulness of this partial order, I impose a condition called the *dominated decreasing decision rule* (DDDR). Specifically, this condition requires that for any state-contingent decision rule that decreases in states with respect to the partial order, there exists an action dominating the decision rule independently of states. In addition, given an information structure, to say that a decision problem is monotone, an optimal action under a higher signal realization needs to be higher in the partial order, namely the *monotone comparative statics* (MCS) condition. Given a decision maker's payoff function and an information structure, if there exists a partial order on the action set satisfying the DDDR and MCS conditions, I say that the decision problem is *generally monotone*.[1]

One of the key results of this paper is that if G is a monotone quasi-garbling of F, for any general monotone decision problem, the decision maker obtains a higher ex-ante expected payoff under F than under G (Theorem 2). That is, the monotone quasi-garbling order is a sufficient condition for informativeness on general monotone decision problems. Intuitively, under monotone decision problems, the information structure G is degraded by adding reversely monotone noise to F, thus, the expected payoff under G will be less than

[1] My analysis differs from Milgrom and Shannon (1994) in that they identify conditions of utility functions that make decision problems monotone whereas I assume that monotone comparative statics are already present and focus on the comparison of information structures.

under F. In addition, when a class of decision problems includes every simple hypothesis testing problem, I also show that monotone quasi-garbling can serve as a necessary condition for informativeness on the class of decision problems (Theorem 3).

I apply this result to analyze the value of information in monopoly pricing with second-degree price discrimination as in Maskin and Riley (1984). At the beginning of the game, the seller chooses between two sources of information about the buyer's type. Unlike a standard monopoly pricing problem where the seller makes a unidimensional choice of price or quantity, in a nonlinear pricing problem, the seller needs to provide a menu of tariffs and quantities–which is multidimensional. Although this problem is usually modeled as a two-player game, it can be cast as a multidimensional decision problem for a seller constrained by incentive compatibility and individual rationality by regarding the buyer's type as a state. Moreover, with fairly mild assumptions, I show that this application satisfies the DDDR condition and the MCS condition, thus the main result of this paper is also applicable in this setting. In Appendix C, I also provide another application on optimal insurance with Arrow-Debreu securities.

The rest of the paper is organized as follows. I review the literature in Section 2. Section 3 sets up the preliminary notions. In Section 4, I introduce the monotone quasi-garbling order and compare it to other orderings such as Blackwell's garbling condition and Lehmann's accuracy condition. In Section 5, I define general monotone decision problems and show that the monotone quasi-garbling order is a necessary and sufficient condition for informativeness. I apply this result to a nonlinear pricing problem in Section 6. Proofs and examples omitted from the text are in the Appendix.

2 Related Literature

The comparison of information structures has been applied in numerous economic situations: Bayesian games (Gossner, 2000; Mekonnen and Leal Vizcaíno, 2021), investment decisions (Cabrales et al., 2013), auctions (Persico, 2000; Ganuza and Penalva, 2010), matching markets (Roesler, 2015), principal agent models with moral hazard (Kim, 1995), competitive markets with adverse selection (Levin, 2001), strategic sampling (Di Tillio et al., 2021), and monopoly pricing (Athey and Levin, 2017; Ottaviani and Prat, 2001).

Since Blackwell (1951, 1953) introduced a criterion to compare information structures in general decision problems, subsequent studies have refined this criterion by restricting it to monotone decision problems. In a seminal paper, Lehmann (1988) mentions location experiments to point out the limit of Blackwell's condition. Then, he restricts decision problems to have the MLRP for information structures and monotone utility introduced by

Karlin and Rubin (1956) and establishes a theorem showing that his accuracy condition is a necessary and sufficient condition for informativeness. Persico (2000) utilizes this accuracy condition for decision problems with single crossing utility.[2] Quah and Strulovici (2009) introduce the interval dominance order property which is weaker than the single crossing property and shows that Lehmann's condition can also be utilized as a criterion for decision problems with this property. These studies are well summarized by a unified framework provided by Chi (2015). He establishes an equivalence of accuracy, informativeness, and posterior dispersion under decision problems with supermodular, single crossing, and interval dominance order preferences. Last but not least, Li and Zhou (2020) show that Lehmann's ordering is robust under monotone decision problems with ambiguity-averse decision makers. However, as mentioned in the introduction, all of these studies assume that the action set is unidimensional whereas this paper allows the decision maker to choose a multidimensional action.[3]

To my knowledge, Jewitt (2007) is the only paper that links Blackwell's condition and Lehmann's condition. He shows that under the MLRP condition, Lehmann's condition is equivalent to Blackwell on Dichotomies, which means that for any pair of states, an information structure restricted to those states is more Blackwell sufficient than the other. Although this characterization helps us to understand better the relationship between the conditions by Blackwell and Lehmann, it still does not provide a 'garbling' interpretation of Lehmann's condition.

A property of the monotone quasi-garbling criterion is that the ordering is independent of prior beliefs. It is also true for the conditions of Blackwell and Lehmann. On the other hand, several recent studies exploit prior beliefs to refine Lehmann's condition. Athey and Levin (2017) restrict utility functions to satisfy certain conditions such as supermodularity and fix a prior belief, then introduce a criterion called monotone information order. Ganuza and Penalva (2010) apply integral and supermodular orders, which are defined on probability measures on expectations of states, to auction problems. Note that the expectation of states largely depends on the prior belief. Last, Cabrales et al. (2013) restricts attention to an investment decision problem and shows that an entropy ordering, which is prior dependent, gives a complete informative ordering. Since these orderings utilize prior beliefs as additional sources for information comparisons, they perform better than prior independent orderings if the prior is known. Nevertheless, it is still important to establish prior independent orderings because it is essential for applications with unknown or multiple prior beliefs. For example, in

[2] Jewitt (2007) compares Karlin Rubin monotonicity and the single crossing property.

[3] Quah and Strulovici (2007) extend their comparative statics results in Quah and Strulovici (2009) to the multidimensional action case, but they do not provide an information comparison result. In Section 5.1, I discuss how their extended results can be applied in this paper's framework.

the presence of ambiguity, Li and Zhou (2016, 2020) emphasize the role of prior-free criteria such as the conditions of Blackwell and Lehmann.

3 Preliminaries

Let $\Omega \equiv [\underline{\omega}, \overline{\omega}] \subset \mathbb{R}$ be the set of states of nature. Denote $\omega \in \Omega$ as a generic state. The decision maker, hereafter the DM, has a prior belief $\Lambda \in \Delta(\Omega)$, where $\Delta(Z)$ is the set of cumulative distribution functions on any compact set $Z \subset \mathbb{R}$.[4][5]

Let the set of actions or feasible decisions be $A \in \mathcal{A}$ where $\mathcal{A}$ is a collection of multidimensional real-valued compact sets, i.e., A is closed, bounded, and $A \subset \mathbb{R}^n$ for some $n \in \mathbb{N}$. Note that I allow an action to be multidimensional. Assume that the DM's payoff function $u : A \times \Omega \to \mathbb{R}$ is continuous in $a \in A$. Let $\overline{\mathcal{U}}_A$ denote the set of all continuous real-valued payoff functions on $A \times \Omega$.

An information structure is composed of (i) a closed interval of signals $X \equiv [\underline{x}, \overline{x}] \subset \mathbb{R}$; and (ii) a collection of (cumulative) distribution functions $F : \Omega \to \Delta(X)$, i.e., $F(\cdot|\omega) \in \Delta(X)$ for all $\omega \in \Omega$. Simply denote this information structure as F. For another information structure G, let the set of signals be $Y \equiv [\underline{y}, \overline{y}] \subset \mathbb{R}$, i.e., $G : \Omega \to \Delta(Y)$.

Without loss of generality, we assume that Λ is continuous in $\omega \in \Omega$, and F and G are continuous in $x \in X$ and $y \in Y$ for all $\omega \in \Omega$, i.e., there is no mass point. To achieve this, we can use the construction introduced in Theorem 5.1 of Lehmann (1988): if $H \in \Delta([\underline{z}, \overline{z}])$ is discontinuous at z_0, i.e., $p \equiv H(z_0) - H(z_0^-) > 0$ where $H(z_0^-) \equiv \lim_{y \uparrow z_0} H(y)$, we can define another distribution $\tilde{H} \in \Delta([\underline{z}, \overline{z}+1])$ such that

$$\tilde{H}(z) = \begin{cases} H(z), & \text{if } z < z_0, \\ H(z_0^-) + p \cdot (z - z_0), & \text{if } z \in [z_0, z_0 + 1], \\ H(z-1), & \text{if } z > z_0 + 1. \end{cases}$$

Then, $\tilde{H}$ is continuous on $[z_0, z_0 + 1]$, and from the DM's perspective, the random variables generated from H and $\tilde{H}$ are statistically equivalent.[6] From the continuity, there exist the probability distribution functions corresponding to Λ, $F(\cdot|\omega)$, and $G(\cdot|\omega)$: i.e., there exist

[4] By Theorem 1.2.1 of Durrett (2019), for any $H \in \Delta(Z)$, the following statements hold: (i) $H(\cdot)$ is increasing; (ii) $H(\cdot)$ is right continuous, i.e., $\lim_{y \downarrow z} H(y) = H(z)$; and (iii) $H(\overline{z}) = 1$ where $\overline{z} \equiv \max Z$.

[5] The terms 'increasing' and 'decreasing' are used in a weak sense, unless noted otherwise.

[6] Let $\mathcal{X}$ and $\tilde{\mathcal{X}}$ be random variables generated from H and $\tilde{H}$, and U be a uniformly distributed random variable independent to $\mathcal{X}$. Then, observing $\tilde{\mathcal{X}}$ is equivalent to observing $\mathcal{X}$ and U (in the case of $\tilde{\mathcal{X}} \in [z_0, z_0 + 1]$). Since U is an independent random variable which does not affect the DM's utility, observing $\tilde{\mathcal{X}}$ is essentially equivalent to observing $\mathcal{X}$.

λ, $f(\cdot|\omega)$ and $g(\cdot|\omega)$ such that $\Lambda(\omega) = \int_{\underline{\omega}}^{\omega} \lambda(z)dz$, $F(x|\omega) = \int_{\underline{x}}^{x} f(z|\omega)dz$, and $G(y|\omega) = \int_{\underline{y}}^{y} g(z|\omega)dz$.[7]

When a prior belief is Λ and an information structure is F, the marginal signal distribution, denoted by $F_\Lambda \in \Delta(X)$, is defined as $F_\Lambda(x) = \int_{\underline{\omega}}^{\overline{\omega}} F(x|\omega)\lambda(\omega)d\omega$ for all $x \in X$. Suppose that a signal x is realized, i.e., $f_\Lambda(x) > 0$ where f_Λ is the probability distribution function corresponding to F_Λ. Then, the posterior belief, $\Lambda_F^x \in \Delta(\Omega)$, is defined as $\Lambda_F^x(\omega) = f(x|\omega) \cdot \Lambda(\omega)/f_\Lambda(x)$, or equivalently $d\Lambda_F^x(\omega)dF_\Lambda(x) = dF(x|\omega)d\Lambda(\omega)$.

The decision making process is given as follows.

1. The DM chooses between information structures F and G.
2. From the chosen info structure (say F), the DM receives a signal $x \in X$.
3. The DM updates a belief on the state to $\Lambda_F^x(\omega)$ and chooses $a \in A$.
4. The state ω is revealed and the payoff $u(a, \omega)$ is realized.

If the payoff function is u, when a signal is realized in the third stage, the DM chooses $a \in A$ to maximize the interim expected payoff function $U : A \times X \to \mathbb{R}$ defined as follows:

$$U(a; x) \equiv \int_\Omega u\left(a, \omega\right) d\Lambda_F^x(\omega). \tag{3.1}$$

Let $A^*_{F,\Lambda}(x) \subseteq A$ denote the set of solutions for the interim maximization problem, i.e., $A^*_{F,\Lambda}(x) \equiv \arg\max_{a \in A} U(a, x)$. Note that $A^*_{F,\Lambda}(x)$ is nonempty since u is continuous and A is compact. The expected utility of the DM given the information structure F at the first stage is that for any $a^*(x) \in A^*_{F,\Lambda}(x)$,

$$\begin{aligned} V(F; u, \Lambda) &= \int_X U(a^*(x); x) dF_\Lambda(x) \\ &= \int_X \int_\Omega u\left(a^*(x), \omega\right) d\Lambda_F^x(\omega) dF_\Lambda(x) \\ &= \int_\Omega \int_X u\left(a^*(x), \omega\right) dF(x|\omega) d\Lambda(\omega). \end{aligned} \tag{3.2}$$

[7] By using the similar construction, the case with discrete state and signals can be transformed to the continuous state and signal case. For example, when a discrete set of states $\Omega = \{\omega_1, \cdots, \omega_N\}$ with $0 < \omega_1 < \cdots < \omega_N$ and the probability mass function $\lambda : \Omega \to [0, 1]$ with $\sum_{n=1}^N \lambda(\omega_n) = 1$ are given, we can construct the corresponding continuous state space and the probability distribution function $\overline{\lambda}$ on $\overline{\Omega} \equiv [0, \omega_n]$ as follows: if $\omega \in (\omega_{n-1}, \omega_n]$ for some $n \geq 2$, $\overline{\lambda}(\omega) = \lambda(\omega_n)/(\omega_n - \omega_{n-1})$, and if $\omega \in [0, \omega_1]$, $\overline{\lambda}(\omega) = \lambda(\omega_1)/\omega_1$. In addition, the DM's utility function can be simply extended: if $\omega \in (\omega_{n-1}, \omega_n]$ for some $n \geq 2$, $\overline{u}(a, \omega) = u(a, \omega_n)$, and if $\omega \in [0, \omega_1]$, $\overline{u}(a, \omega) = u(a, \omega_1)$.

Then, in the first stage, the DM chooses an information structure by comparing $V(F;u,\Lambda)$ and $V(G;u,\Lambda)$. Based on this comparison, the informativeness for decision problems in $\mathcal{U} \subseteq \bigcup_{A\in\mathcal{A}} \overline{\mathcal{U}}_A \equiv \overline{\mathcal{U}}$ is defined as follows.

Definition 1 (Informativeness on $\mathcal{U}$)**.** An information structure F is *more informative* than another information structure G on $\mathcal{U} \subseteq \overline{\mathcal{U}}$ if and only if for all priors $\Lambda \in \Delta(\Omega)$ and all $u \in \mathcal{U}$, $V(F;u,\Lambda) \geq V(G;u,\Lambda)$ holds.

4 Information Ranking Criteria

In this section, I introduce a novel condition called the monotone quasi-garbling order. Then, I explore how it is related to classical information ranking criteria such as Blackwell's garbling condition and Lehmann's accuracy condition. Since the monotone quasi-garbling order is motivated by Blackwell's garbling notion, I begin the section by reviewing the garbling condition.

Definition 2 (Garbling Condition by Blackwell (1951, 1953))**.** An information structure G is said to be a *garbling* of another information structure F, denoted $F \succeq_B G$, if there exists a function $\Gamma : X \to \Delta(Y)$ such that $G(y|\omega) = \int_X \Gamma(y|x) dF(x|\omega)$ for all $\omega \in \Omega$.

Blackwell shows that an information structure F is more informative than G for all decision problems if and only if G is a garbling of F. In other words, $V(F;u,\Lambda) \geq V(G;u,\Lambda)$ for any set of actions A, payoff function $u \in \overline{\mathcal{U}}_A$ and prior belief $\Lambda \in \Delta(\Omega)$ if and only if $F \succeq_B G$. Blackwell's garbling condition is widely used by economists not only because it has a powerful equivalence condition but also because it has a nice interpretation. We can imagine that a signal y from information G is constructed by adding *noise* Γ to signals in X from information F and the noise deteriorates the quality of information so that F gives better information than G.

Even though the garbling condition is useful, this criterion is restrictive because it requires preserving the order for 'all' decision problems. If we restrict attention to 'monotone' decision problems, there is room to improve Blackwell's ordering. I now introduce the monotone quasi-garbling order by adding a monotone structure to the garbling condition.

Definition 3 (Monotone Quasi-Garbling)**.** An information structure G is said to be a *monotone quasi-garbling* of another information structure F, denoted $F \succeq_{MQG} G$, if there exists a function $\Gamma : X \times \Omega \to \Delta(Y)$ such that

1. $G(y|\omega) = \int_X \Gamma(y|x,\omega) dF(x|\omega)$ for all $y \in Y$, $x \in X$ and $\omega \in \Omega$,

2. $\Gamma(\cdot|x,\omega) \geq_{FOSD} \Gamma(\cdot|x,\omega')$ for all $\omega' > \omega$ and $x \in X$, i.e., $\Gamma(y|x,\omega) \leq \Gamma(y|x,\omega')$ for all $y \in Y$.

We can easily observe that the garbling order implies the monotone quasi-garbling order. If G is a garbling of F via a function $\hat{\Gamma} : X \to \Delta(Y)$, we can see that G is also a monotone quasi-garbling of F by setting $\Gamma(y|x,\omega) = \hat{\Gamma}(y|x)$ for all $\omega \in \Omega$, because a probability distribution is first order stochastic dominant over itself. However, the converse is not true. In Appendix A.2, I provide an example of a pair of information structure (F, G) such that $F \succeq_{MQG} G$ but $F \not\succeq_B G$. This observation is formally stated in the following proposition.

Proposition 1. *For any pair of information structures (F, G), $F \succeq_B G$ implies $F \succeq_{MQG} G$, but $F \succeq_{MQG} G$ does not imply $F \succeq_B G$.*

The key difference between these two criteria is that the noise may depend on the state ω under the monotone quasi-garbling order. Note that if there were no restriction on Γ, any information structure G could be generated by adding state-contingent noise to F and it would be a meaningless criterion.[8] The second condition restricts state-contingent noise to be *reversely FOSD-ordered*. This means that the noise is more likely to return a higher y for a lower state and lower y for a higher state, thus, it can be interpreted as a *reversely monotone* noise. We can guess that the reversely monotone noise would deteriorate the quality of information under monotone decision problems.

Next, I review the accuracy condition by Lehmann (1988), which is widely used in monotone decision problems. Then, I explore how it is related to the monotone quasi-garbling order.

Definition 4 (Accuracy Condition by Lehmann (1988))**.** An information structure F is said to be *more Lehmann accurate* than another information structure G, denoted $F \succeq_L G$, if for all $y \in Y$, a function $\Phi : \Omega \times Y \to X$ defined by $\Phi(\omega; y) \equiv \sup\{x \mid F(x|\omega) \leq G(y|\omega)\}$ is increasing in ω.[9]

An assumption that is usually paired with Lehmann's condition is the monotone likelihood ratio property.

Definition 5 (Monotone Likelihood Ratio Property)**.** An information structure F is said to satisfy the monotone likelihood ratio property (MLRP) if and only if $f(x_0|\omega_0)f(x_1|\omega_1) \geq f(x_1|\omega_0)f(x_0|\omega_1)$ holds for all $x_1 > x_0$ and $\omega_1 > \omega_0$.

[8] For example, when there is no restriction on Γ, we can use $\Gamma(y|x,\omega) = G(y|\omega)$ for all $x \in X$, $y \in Y$ and $\omega \in \Omega$ to satisfy the first condition.

[9] In Lehmann (1988), he implicitly assumes that $F(x|\omega)$ is continuous and strictly increasing in x, thus $F^{-1}(\cdot|\omega)$ is properly defined and set $\Phi(\omega; y) \equiv F^{-1}(G(y|\omega)|\omega)$. Under the continuity and strict increasingness assumptions on $F(x|\omega)$, we have $F(\Phi(\omega; y)|\omega) = G(y|\omega)$, thus, Definition 4 coincides with Lehmann's original condition.

First, I show that Lehmann's accurate condition implies the monotone quasi-garbling order, i.e., $F \succeq_L G$ implies $F \succeq_{MQG} G$. Observe that $\Phi(\omega; y)$ is increasing in y since $G(y|\omega)$ is increasing in y for any $\omega \in \Omega$. Construct a function $\Gamma : X \times \Omega \to \Delta(Y)$ as follows:

$$\Gamma(y|x,\omega) \equiv \begin{cases} 1, & \text{if } x \leq \Phi(\omega; y), \\ 0, & \text{if } x > \Phi(\omega; y).^{10} \end{cases}$$

Then, we can easily check that $\Gamma(y|x,\omega)$ is well defined.[11] In addition, since $\Phi(\omega; y) \leq \Phi(\omega'; y)$ for all $\omega' > \omega$ and $y \in Y$, we have $\Gamma(y|x,\omega) \leq \Gamma(y|x,\omega')$ for all $x \in X$, i.e., Γ is reversely monotone.

Last, observe that for all $y \in Y$ and $\omega \in \Omega$,

$$G(y|\omega) = F(\Phi(\omega; y)|\omega) = \int_{\underline{x}}^{\Phi(\omega;y)} dF(x|\omega) = \int_X \Gamma(y|x,\omega) dF(x|\omega).$$

Therefore, G is a monotone quasi-garbling of F. The following proposition formally states this result.

Proposition 2. *Suppose that $F \succeq_L G$. Then, $F \succeq_{MQG} G$.*

Next, I explore whether $F \succeq_{MQG} G$ implies $F \succeq_L G$. The following proposition shows that the monotone quasi-garbling order implies Lehmann's accuracy condition when F satisfies the MLRP. The proof is relegated to Appendix A.1.

Proposition 3. *Suppose that an information structure F satisfies the MLRP. Then, $F \succeq_{MQG} G$ implies $F \succeq_L G$.*

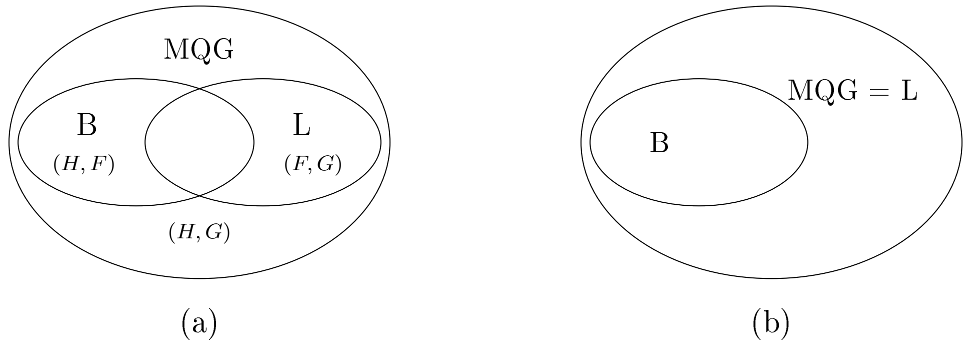
Based on the above results, we can establish the equivalence between the monotone quasi-garbling order and Lehmann's order under the MLRP condition.

Theorem 1. *Suppose that an information structure F satisfies the MLRP. Then, $F \succeq_{MQG} G$ and $F \succeq_L G$ are equivalent.*

Last, I show that the MLRP plays a crucial role in the equivalence between the monotone quasi-garbling order and Lehmann's accuracy condition. In Appendix A.2, in addition to the

[10] Although I assume that the cumulative distribution functions for the prior belief and the information structures are continuous, I allow the garbling functions are discontinuous.

[11] First, $\Gamma(y|x,\omega)$ is increasing in y because $\Phi(\omega; y)$ is increasing in y. Next, $\Gamma(y|x,\omega)$ is right-continuous in y because (i) if $\Phi(\omega; y) \geq x$, since $\Phi(\omega; y)$ is increasing in y and $\Phi(\omega; y') \geq x$ for all $y' > y$, thus $\lim_{\tilde{y} \downarrow y} \Gamma(\tilde{y}|x,\omega) = 1 = \Gamma(y|x,\omega)$; (ii) if $\Phi(\omega; y) < x$, by the definition of Φ, we have $G(y|\omega) < x$, then since $G(y|\omega)$ is continuous in y, there exist $\epsilon > 0$ and $\delta > 0$ such that $G(y + \epsilon|\omega) < x - \delta$ and $\Phi(\omega; y + \epsilon) < x$, thus, we have $\lim_{\tilde{y} \downarrow y} \Gamma(\tilde{y}|x,\omega) = 0 = \Gamma(y|x,\omega)$.



(a) Without the MLRP

(b) With the MLRP

Figure 1: The relationship among criteria

information structures F and G which are used in Proposition 1, I introduce an information structure H which does not satisfy the MLRP. In this example, I show that (i) $H \succeq_B F$ but $H \not\succeq_L F$; and (ii) $H \succeq_{MQG} G$ but $H \not\succeq_B G$ and $H \not\succeq_L G$. The first result implies that when the MLRP is not assumed, Lehmann's accuracy condition is no longer a sufficient condition for Blackwell's garbling condition. The second result means that there exists a pair of information structures that can be comparable by the monotone quasi-garbling order but neither by Blackwell's garbling condition nor by Lehmann's accuracy condition. Therefore, the monotone quasi-garbling order generally permits more comparisons than the conditions by Blackwell and Lehmann. The following proposition formally states these arguments.

Proposition 4. *Without the assumption of the MLRP on F, $F \succeq_B G$ does not imply $F \succeq_L G$, and $F \succeq_{MQG} G$ does not imply $F \succeq_L G$.*

The results in this section are summarized in Figure 1. Figure 1a illustrates the relationship among Blackwell's garbling condition, Lehmann's accuracy condition, and the monotone quasi-garbling condition when the MLRP is not assumed on the supposedly higher information structure. On the other hand, if the MLRP is assumed on the supposedly better information structure, the Lehmann order and the monotone quasi-garbling order are equivalent and refine the Blackwell garbling condition as shown in Figure 1b.

5 General Monotone Decision Problems

In this section, I show that the monotone quasi-garbling order can serve as a criterion for comparing information structures under monotone decision problems. Specifically, I allow the set of feasible decisions, A, to be multidimensional. Unlike the unidimensional action space case, there is no generic order in this setup. Hence, to establish a monotone decision problem under this general action space, an order on the set of actions needs to be properly defined. Since we will only use this order to compare the optimal actions, it does not need to be a complete order over A. Rather, a partial order, which can compare the potential optimal actions, would be enough. To say that a decision problem is monotone, (i) this partial order needs to be sensible; and (ii) an optimal action under a higher signal realization needs to be higher in this order.

Definition 6 (Dominated Decreasing Decision Rule Condition)**.** A payoff function $u : A \times \Omega \to \mathbb{R}$ satisfies the *dominated decreasing decision rule (DDDR) condition* with respect to a partial order $\geq_A$ of the set of feasible decisions A if for any decreasing (in terms of $\geq_A$) state-contingent decision rule $h : \Omega \to A$,[12] there exists $\hat{a} \in A$ such that

$$u(\hat{a}, \omega) \geq u(h(\omega), \omega), \quad \forall \omega \in \Omega.^{13} \tag{5.1}$$

This condition is about whether the partial order $\geq_A$ is sensible or not. Loosely speaking, under a given payoff function, a highly ranked action needs to be better for a higher state and worse for a lower state and vice versa for a lowly ranked action. According to this interpretation, a state-contingent decision rule, which is decreasing in states, should not be an optimal way of establishing a decision rule. This is because a highly ranked action (which is worse for a lower state) corresponds to a lower state and a lowly ranked action (which is worse for a higher state) corresponds to a higher state. The DDDR condition implies that any decreasing state-contingent decision rule is dominated by a constant decision rule. In this sense, the order $\geq_A$ can be considered sensible.

Definition 7 (Monotone Comparative Statics Condition)**.** An information structure F satisfies the *monotone comparative statics (MCS) condition* with respect to a payoff function $u : A \times \Omega \to \mathbb{R}$ and a partial order $\geq_A$, if for any prior belief $\Lambda \in \Delta(\Omega)$, there exists a function $a^* : X \to A$ such that $a^*(x) \in A^*_{F,\Lambda}(x)$ and $a^*(x) \geq_A a^*(x')$ for all $x, x' \in X$ with $x \geq x'$.

[12] A state-contingent decision rule $h : \Omega \to A$ is decreasing in terms of $\geq_A$ if $h(\omega) \geq_A h(\omega')$ for all $\omega' > \omega$.

[13] The DDDR condition was originally introduced as an intermediate result for the information comparison characterization in the context of unidimensional action spaces with utility functions satisfying the interval dominance order (IDO). Specifically, Quah and Strulovici (2009) presented this concept in Lemma 3, and utilized it to show the efficacy of Lehmann's criterion under the IDO condition. Chi (2015) provided simple proofs to establish that the DDDR condition is valid when the utility function satisfies the single crossing property (Lemma 3.2) or the IDO property (Lemma B.3). Building upon these findings, I extend their results by showing that general IDO condition, which is defined under the multidimensional action spaces, is strictly weaker than the DDDR condition (Proposition 5, Appendix B).

This condition simply implies that there exists an optimal decision rule, which is increasing (in terms of the given partial order) with respect to the signal. I call a class of decision problems $\mathcal{U}$ is *generally monotone* with respect to an information structure F, if for any $u \in \mathcal{U}$, there exists a partial order $\geq_A$ such that u satisfies the DDDR condition with respect to $\geq_A$ and the information structure F satisfies the MCS condition with respect to u and $\geq_A$.[14] I also call a decision problem u is generally monotone with respect to F if $\mathcal{U} = \{u\}$ is generally monotone with respect to F.

The following theorem stipulates that the monotone quasi-garbling order can serve as a sufficient condition for informativeness on general monotone decision problems.

Theorem 2. *Suppose that $\mathcal{U}$ is generally monotone with respect to an information structure G and G is a monotone quasi-garbling of F. Then, F is more informative than G on $\mathcal{U}$.*

The formal proof is in Section 5.2, but I briefly delineate the idea of this result. Consider any increasing decision rule $\psi : Y \to A$, i.e., $\psi(y'') \geq_A \psi(y')$ for all $y'' \geq_A y'$. When G is a monotone quasi-garbling of F, i.e., G is generated from F by adding a reversely monotone noise, I show that there exists $\tilde{\psi} : (0,1) \times X \times \Omega \to A$ such that $\tilde{\psi}$ is decreasing in ω and for any $\omega \in \Omega$,

$$\int_Y u(\psi(y), \omega) dG(y|\omega) = \int_0^1 \int_X u(\tilde{\psi}(t; x, \omega), \omega) dF(x|\omega) dt. \tag{5.2}$$

Then, by applying the DDDR condition, we can construct a mixed-action decision rules $\phi : X \times (0,1) \to A$ on the information structure F which dominates ψ on G (Proposition 6). Now assume that u is generally monotone with respect to G, i.e., the optimal decision rule under G, $a^*_G : Y \to A$, is increasing in y. By applying the above result, there exists a mixed-action decision rules dominating a^*_G, but this decision rule is dominated by a^*_F which eventually gives the expected payoff under F. Therefore, the expected payoff under F will be greater than that under G.

Next, to establish the necessary condition for informativeness, I introduce a bare-bones class of decision problems. Consider a binary action set $\underline{A} = \{0, 1\}$ and the generic order on $\underline{A}$: $1 > 0$. I define that $u : \underline{A} \times \Omega \to \mathbb{R}$ is a simple hypothesis testing problem if there exist $0 < \kappa < 1$ and $\hat{\omega} \in \Omega$ such that

$$u(a, \omega) = \begin{cases} -a \cdot \kappa, & \text{if } \omega < \hat{\omega}. \\ a \cdot (1 - \kappa), & \text{if } \omega \geq \hat{\omega}.^{15} \end{cases} \tag{5.3}$$

[14] The partial order $\geq_A$ is not necessarily fixed across the choice of payoff function $u \in \mathcal{U}$, i.e., for any $u_1, u_2 \in \mathcal{U}$, the corresponding partial orders $\geq^1_A$ and $\geq^2_A$ are not necessarily the same.

I also refer to $\underline{\mathcal{U}}$ as a class of such utility functions. The following theorem shows that the monotone quasi-garbling order is a necessary condition for informativeness when the class of decision problems $\mathcal{U}$ includes $\underline{\mathcal{U}}$, i.e., every possible simple hypothesis problems. The proof is provided in Appendix D.1.

Theorem 3. *Suppose that $\underline{\mathcal{U}} \subseteq \mathcal{U}$, $\mathcal{U}$ is generally monotone with respect to information structures F, and F is more informative than G on $\mathcal{U}$. Then, (i) F satisfies the MLRP; and (ii) F is more Lehmann accurate than G, thus, G is a monotone quasi-garbling of F.*

In this theorem, I show that the informativeness implies the monotone quasi-garbling order by showing that an information structure is more Lehmann accurate than the other (recall that $F \succeq_L G$ implies $F \succeq_{MQG} G$ by Proposition 2). As an intermediate step for the result, it is also shown that F satisfies the MLRP when every simple hypothesis testing problems are generally monotone with respect to F.

Note that I impose different assumptions on the above results: in Theorem 2, I assume that the decision problems with a supposedly 'worse' information structure are generally monotone, whereas, in Theorem 3, I assume that the decision problems with a supposedly 'better' information structure are generally monotone and the class of decision problems includes $\underline{\mathcal{U}}$. Therefore, if we assume that the decision problems with both information structures are monotone (and include $\underline{\mathcal{U}}$), we can see that the monotone quasi-garbling order can serve as a necessary and sufficient condition for informativeness. In addition, under these assumptions, both information structures will satisfy the MLRP and the monotone quasi-garbling condition and Lehmann accuracy condition are equivalent.

Corollary 1. *Suppose that $\underline{\mathcal{U}} \subseteq \mathcal{U}$ and $\mathcal{U}$ is generally monotone with respect to information structures F and G. Then, F is more informative than G on $\mathcal{U}$ if and only if G is a monotone quasi-garbling of F, or equivalently, F is more Lehmann accurate than G.*

Remark 1. Corollary 1 extends the results of Chi (2015) that Lehmann's accuracy condition can serve as a necessary and sufficient condition for the informativeness in some monotone decision problems. First, he assumes that both information structures have the MLRP, but I derive the MLRP as a result. Next, he restricts attention to the 'unidimensional' class of decision problems with well-known preferences such as supermodular, single crossing, and interval dominance order preferences. On the other hand, Corollary 1 can be applied to any (potentially 'multidimensional') monotone decision problems including $\underline{\mathcal{U}}$.

[15] This choice of utility function is inspired by Chi (2015). He shows that the class of monotone decision problems with certain properties, such as super modular, single crossing, or interval dominance order preferences, includes $\underline{\mathcal{U}}$.

5.1 General Interval Dominance Order

Before presenting the proofs of the main theorems, I demonstrate how the concept of the general monotone decision problem is connected to the general version of the interval dominance order condition introduced in Quah and Strulovici (2007).

I assume that the state space is discrete as described in Footnote 7: $\Omega = \{\omega_1, \cdots, \omega_N\}$ with $0 < \omega_1 < \cdots < \omega_N$. Suppose that a compact set of action $A \subset \mathbb{R}^n$ has a partial order $\geq_A$ with (i) transitivity ($a'' \geq_A a'$ & $a' \geq_A a \Rightarrow a'' \geq_A a$); (ii) reflexivity ($a \geq_A a$); and (iii) anti-symmetry ($a' \geq_A a$ & $a \geq_A a' \Rightarrow a = a'$). I also assume that the partial order is continuous: for all sequences $\{a_m\}_{m=1}^{\infty}$ and $\{b_m\}_{m=1}^{\infty}$ with $\lim_{m\to\infty} a_m = a$, $\lim_{m\to\infty} b_m = b$, and $a_m \geq_A b_m$ for all $m \geq 1$, then, $a \geq_A b$. Let $[a', a'']$ denote the set $\{a \in A \mid a' \leq_A a \leq_A a''\}$. Then, if a sequence $\{a_m\}_{m=1}^{\infty}$ in $[a', a'']$ converges to a, by the continuity of the partial order, $a' \leq_A a \leq_A a''$, i.e., $a \in [a', a'']$. Therefore, $[a', a'']$ is compact since it is a closed subset of A.

Now I briefly review the interval dominance order condition, then show that it is a sufficient condition for the DDDR condition in Proposition 5.

Definition 8 (Interval Dominance Order (IDO))**.** Let v' and v'' be two real-valued functions defined on A. We say that v'' dominates v' by the *interval dominance order* (or, for short, v'' I-dominates v') if, for any $a'' >_A a'$, whenever $v'(a'') \geq v'(a)$ for all $a \in [a', a'']$,

$$v'(a'') \;\geq (>)\; v'(a') \implies v''(a'') \;\geq (>)\; v''(a'). \tag{5.4}$$

In addition, a payoff function $u : A \times \Omega \to \mathbb{R}$ is said to be *IDO-ordered* with respect to $\geq_A$ if $u(\cdot;\omega'')$ I-dominates $u(\cdot;\omega')$ for all $\omega'' > \omega'$.

Proposition 5. *Suppose that the set of actions A has a transitive, reflexive, anti-symmetric, and continuous partial order $\geq_A$, a payoff function $u : A \times \Omega \to \mathbb{R}$ is continuous in $a \in A$ and IDO-ordered. Then, u satisfies the DDDR condition with respect to $\geq_A$.*[16]

Proof of Proposition 5. First, I show that the payoff function u satisfies the DDDR condition with respect to the partial order. Consider a decreasing state-contingent decision rule $h : \Omega \to A$. I inductively define a sequence $\{a_i\}_{i=1}^N$ as follows: $a_N = h(\omega_N)$, and for all $1 \leq n \leq N-1$,

$$a_n = \operatorname*{arg\,max}_{a \in [a_{n+1}, h(\omega_n)]} u(a, \omega_n). \tag{5.5}$$

[16] I thank Referee 1 (to be named) for suggesting arguments with the IDO conditions that shorten the proof.

Note that a_n exists since u is continuous in $a \in A$ and $[a_{n+1}, h(\omega_n)]$ is compact.

Observe that $u(a_n, \omega_n) \geq u(a, \omega_n)$ for all $a \in [a_{n+1}, a_n]$. Then, by the IDO condition, we have

$$u(a_n, \omega_{n'}) \geq u(a_{n+1}, \omega_{n'}) \tag{5.6}$$

for all $n' > n$.

Now set $\hat{a} = a_1$. Then, for all $1 \leq n \leq N$, by (5.6), we have $u(\hat{a}, \omega_n) \geq u(a_n, \omega_n)$. In addition, by (5.5), $u(a_n, \omega_n) \geq u(h(\omega_n), \omega_n)$. Therefore, $u(\hat{a}, \omega_n) \geq u(h(\omega_n), \omega_n)$ for all $1 \leq n \leq N$, i.e., u satisfies the DDDR condition with respect to the partial order. □

A natural subsequent question is whether the IDO condition can serve as a necessary condition. In Appendix B, I provide a simple (even unidimensional) example of a decision problem such that it is a general monotone decision problem but it does not satisfy the IDO condition. Thus, the IDO condition cannot be a necessary condition for the DDDR condition.

Next, to establish the MCS condition, additional conditions are needed. Following Topkis (1998), let $a' \vee a''$ denote (if exists) the least upper bound of $\{a', a''\}$, that is, $(a' \vee a'') \geq_A a'$, $(a' \vee a'') \geq_A a''$, and $a \geq_A (a' \vee a'')$ for all a with $a \geq_A a'$ and $a \geq_A a''$. Likewise, let $a' \wedge a''$ denote the greatest lower bound of $\{a', a''\}$, i.e., $a' \geq_A (a' \wedge a'')$, $a'' \geq_A (a' \wedge a'')$, and $(a' \wedge a'') \geq_A a$ for all a with $a' \geq_A a$ and $a'' \geq_A a$. A partially ordered set A is called to be a *lattice* if $a' \vee a''$ and $a' \wedge a''$ exist for all $a', a'' \in A$. A payoff function $u : A \times \Omega \to \mathbb{R}$ is *supermodular* in A if for all $a', a'' \in A$ and $\omega \in \Omega$,

$$u(a' \vee a'', \omega) - u(a', \omega) \geq u(a'', \omega) - u(a' \wedge a'', \omega).$$

When an information structure F satisfies the MLRP, for any $x'' > x'$, $\Lambda_F^{x''}$ is the monotone likelihood ratio (MLR) shift of $\Lambda_F^{x'}$, that is, $\lambda_F^{x''}(\omega)/\lambda_F^{x'}(\omega)$ is increasing in ω where λ_F^x is the probability mass function of Λ_F^x. Also note that $A^*_{F,\Lambda}(x) = \arg\max_{a \in A} U(a; x)$ is nonempty since U is continuous in a and A is compact. Under the IDO conditions on u and the MLRP of F, by Proposition 2 of Quah and Strulovici (2007), we have $U(\cdot; x'')$ I-dominates $U(\cdot; x')$. Observe that U is supermodular in A when u is supermodular in A. Then, by Theorem 1 of Quah and Strulovici (2007), we have $A^*_{F,\Lambda}(x'')$ dominates $A^*_{F,\Lambda}(x')$ in the strong set order, i.e., for any $a'' \in A^*_{F,\Lambda}(x'')$ and $a' \in A^*_{F,\Lambda}(x')$, $a'' \vee a' \in A^*_{F,\Lambda}(x'')$ and $a'' \wedge a' \in A^*_{F,\Lambda}(x')$.[17] Therefore, there exists $a^* : X \to A$ such that $a^*(x'') \geq_A a^*(x')$ for all $x'' > x'$, i.e., the MCS condition holds, and u is generally monotone with respect to F. The

[17] Quah and Strulovici (2007) actually impose I-quasisupermodular condition which is weaker than the supermodular condition.

following corollary formally states these findings.

Corollary 2. *Assume that the set of actions A is compact and a lattice with a transitive, reflexive, anti-symmetric, and continuous partial order $\geq_A$. In addition, a payoff function $u : A \times \Omega \to \mathbb{R}$ is continuous, IDO-ordered, and supermodular in A, and an information structure F satisfies the MLRP. Then, u is generally monotone with respect to F.*

5.2 Proof of Theorem 2

I begin by stating a useful lemma.

Lemma 1. *For any reversely monotone noise $\Gamma : X \times \Omega \to \Delta(Y)$ and random variable τ uniformly distributed on $(0,1)$, there exists a mapping $m : (0,1) \times X \times \Omega \to Y$ such that the cumulative distribution induced by $m(\tau|x,\omega)$ is $\Gamma(\cdot|x,\omega)$ and $m(t|x,\omega) \geq m(t|x,\omega')$ for all $\omega' > \omega$ and $t \in (0,1)$.*[18]

Proof of Lemma 1. The proof is a slight modification of Theorem 1.2.2 of Durrett (2019). For any $t \in (0,1)$, $x \in X$ and $\omega \in \Omega$, define $m(t|x,\omega) \equiv \sup \mathcal{M}(t|x,\omega)$ where

$$\mathcal{M}(t|x,\omega) \equiv \big\{ y \in Y = [\underline{y}, \overline{y}] \ : \ \Gamma(y|x,\omega) < t \big\} \cup \{\underline{y}\}.$$

Note that $\mathcal{M}(t|x,\omega)$ is nonempty and bounded above, thus, $m(t|x,\omega)$ is properly defined.

Now suppose that a random variable τ is uniformly distributed on $(0,1)$. Observe that $\Pr(m(\tau|x,\omega) \leq \hat{y}) = \Gamma(\hat{y}|x,\omega)$ for any $\hat{y} \in Y$ if and only if $L(\hat{y}|x,\omega) = R(\hat{y}|x,\omega)$ where

$$L(\hat{y}|x,\omega) \equiv \{t \in (0,1) \ : \ m(t|x,\omega) \leq \hat{y}\},$$
$$R(\hat{y}|x,\omega) \equiv \{t \in (0,1) \ : \ t \leq \Gamma(\hat{y}|x,\omega)\}.$$

First, when $\hat{y} = \overline{y}$, by the definition of $\mathcal{M}(t|x,\omega)$ and $\Gamma(\overline{y}|x,\omega) = 1$, we have $L(\overline{y}|x,\omega) = R(\overline{y}|x,\omega) = (0,1)$.

Next, consider the case where $\hat{y} \in [\underline{y}, \overline{y})$. Suppose that $t \in L(\hat{y}|x,\omega)$, i.e., $m(t|x,\omega) \leq \hat{y}$. If $\Gamma(\hat{y}|x,\omega) < t$, by the right continuity of Γ, there exists $y \in (\hat{y}, \overline{y})$ such that $\Gamma(y|x,\omega) < t$. Then, $y \in \mathcal{M}(t|x,\omega)$ and it contradicts $\hat{y} = \sup \mathcal{M}(t|x,\omega)$. Therefore, we have $\Gamma(\hat{y}|x,\omega) \geq t$, which implies $t \in R(\hat{y}|x,\omega)$, and $L(\hat{y}|x,\omega) \subseteq R(\hat{y}|x,\omega)$. Conversely, if $t \in R(\hat{y}|x,\omega)$, we have $\Gamma(y|x,\omega) \geq \Gamma(\hat{y}|x,\omega) \geq t$ for all $y \geq \hat{y}$. Thus, the upper bound of $\mathcal{M}(t|x,\omega)$ is at most $\hat{y}$, i.e., $\hat{y} \geq m(t|x,\omega)$. Hence, we have $t \in L(\hat{y}|x,\omega)$, and $R(\hat{y}|x,\omega) \subseteq L(\hat{y}|x,\omega)$. Therefore, $R(\hat{y}|x,\omega) = L(\hat{y}|x,\omega)$ for any $\hat{y} \in Y$ and the distribution induced by $m(\tau|x,\omega)$ is $\Gamma(\cdot|x,\omega)$.

[18] I thank Referee 1 (to be named) for providing this lemma.

By the reverse monotonicity, $\Gamma(y|x,\omega) \leq \Gamma(y|x,\omega')$ for all $y \in Y$, $x \in X$ and $\omega, \omega' \in \Omega$ with $\omega' > \omega$. Therefore, we have $\mathcal{M}(t|x,\omega') \subseteq \mathcal{M}(t|x,\omega)$, and it implies $m(t|x,\omega) \geq m(t|x,\omega')$ for all $t \in (0,1)$ and $\omega' > \omega$. $\square$

Next, by using this lemma, for any increasing decision rule $\psi : Y \to A$ on G, I construct a mixed-action decision rules $\phi : X \times (0,1) \to A$ on F which gives a higher expected payoff than ψ for any state.

Proposition 6. *Suppose that G is a monotone quasi-garbling of F and u satisfies the DDDR condition with respect to $\geq_A$. For any increasing decision rule $\psi : Y \to A$, there exists a mixed-action decision rules $\phi : X \times (0,1) \to A$ such that for all $\omega \in \Omega$,*

$$\int_X \left[\int_0^1 u(\phi(x;t),\omega) dt \right] dF(x|\omega) \geq \int_Y u(\psi(y),\omega) dG(y|\omega). \tag{5.7}$$

Proof of Proposition 6. By the definition of $F \succeq_{MQG} G$, we have

$$\int_Y u(\psi(y),\omega) dG(y|\omega) = \int_X \int_Y u(\psi(y),\omega) d\Gamma(y|x,\omega) dF(x|\omega).$$

From Lemma 1, the right hand side is equal to

$$\int_X \left[\int_0^1 u(\psi(m(t|x,\omega)),\omega) dt \right] dF(x|\omega).$$

Since ψ is increasing and m is decreasing in ω, $\psi(m(t|x,\omega))$ is decreasing in ω. (Therefore, $\tilde{\psi}(t;x,\omega) \equiv \psi(m(t|x,\omega))$ satisfies (5.2) for all $\omega \in \Omega$ and is decreasing in ω.)

Given t and x, by the DDDR condition, there exists an action $a \in A$ such that $u(a,\omega) \geq u(\psi(m(t|x,\omega)),\omega)$ for all ω, and denote $\phi(x;t)$. Then, by using the inequalities for each t and x, we can derive (5.7). $\square$

Now we are ready to prove Theorem 2.

Proof of Theorem 2. Fix a prior belief $\Lambda \in \Delta(\Omega)$ and a payoff function $u \in \mathcal{U}$. Let $a^*_G : Y \to A$ be the optimal decision function defined in the MCS condition: $a^*_G(y) \in A^*_{G,\Lambda}(y)$ and $a^*_G(y) \geq_A a^*_G(y')$ for all $y \geq y'$. By Proposition 6, there exists a mixed-action decision rules

$\phi : X \times (0,1)$ satisfying (5.7). Then, we have

$$\begin{aligned} V(G; u, \Lambda) &= \int_\Omega \int_Y u(a^*_G(y), \omega) dG(y|\omega)\, d\Lambda(\omega) \\ &\leq \int_\Omega \int_X \left[\int_0^1 u(\phi(x;t), \omega) dt \right] dF(x|\omega)\ d\Lambda(\omega) \\ &= \int_0^1 \left[\int_X \int_\Omega u(\phi(x;t), \omega) d\Lambda_F^x(\omega) dF_\Lambda(x) \right] dt \\ &\leq \int_X \int_\Omega u(a^*_F(x), \omega) d\Lambda^x_F(\omega) dF_\Lambda(x) = V(F; u, \Lambda). \end{aligned}$$

The third equality is simply from rearranging the integration order. The fourth inequality holds from the optimality of $a^*_F(x)$, and the last equality is straightforward. Therefore, F is more informative than G on $\mathcal{U}$. □

Remark 2. Proposition 6 is closely related to Proposition 9 of Quah and Strulovici (2009). Their result is stronger in the sense that they show that there exists an increasing pure-action decision rule $\phi : X \to A$ (rather than a mixed-action decision rules which are not necessarily increasing) such that $\int_X u(\phi(x), \omega) dF(x|\omega) \geq \int_Y u(\psi(y), \omega) dG(y|\omega)$ for all $\omega \in \Omega$. This is because their assumptions are stronger than the ones on Proposition 6: they impose the interval dominance order condition on the payoff function which is stronger than the DDDR condition; and they also assume that F is more Lehmann accurate than G, which is also stronger than the monotone quasi-garbling order in the absence of the MLRP.

6 Application: Nonlinear Monopoly Pricing

As an application of general monotone decision problems, I investigate information ranking in the context of monopoly pricing with second-degree price discrimination as in Maskin and Riley (1984). In this section, I review their model and basic results and rewrite them in the grammar of Section 5.[19]

A risk-neutral monopolistic seller that possesses full power of commitment wishes to sell output to a single buyer. The buyer has a privately known type $\omega \in \Omega = \{\omega_1, \cdots, \omega_N\}$

[19] Comparison of information structures for monopoly pricing has also been studied by Athey and Levin (2017) and Ottaviani and Prat (2001). Athey and Levin applied their result in a setting where the seller receives a signal about the buyer's type and determines the simple monopoly price and there is no second-degree price discrimination. In the study by Ottaviani and Prat, both the seller and the buyer receive a signal about the buyer's type and the seller offers a nonlinear price. The setup in this section is a mixture of these two models: the buyer is fully aware of her type, the seller receives a signal about the buyer's type and then a menu of nonlinear prices is offered.

with $\omega_n < \omega_{n+1}$. In the first stage of the game, the seller chooses between two sources of information about the buyer's type. Next, the signal is received from the chosen information source and the seller updates the belief on the buyer's type. Then, the seller posts a menu of quantities and tariffs (non-linear prices). In the last stage, the buyer selects an element from the menu or decides not to participate. The outside option of both parties yields a payoff of zero.

Let the utility of a type ω buyer from consuming q units of the good and paying p units of money be $v(q, \omega) - p$. Let the cost function of the seller be some $c : \mathbb{R}_+ \to \mathbb{R}_+$, then the seller's payoff would be $p - c(q)$.[20] I assume some standard conditions on c and v.

Assumption. The seller's cost function $c : \mathbb{R}_+ \to \mathbb{R}_+$ and the buyer's utility function $v : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ satisfy the following properties:

1. v is continuous, increasing in ω, and supermodular in (q, ω), i.e., $v(q', \omega') - v(q, \omega') \geq v(q', \omega) - v(q, \omega)$ for all $q' \geq q \geq 0$ and $\omega' > \omega$. In addition, $v(0, \omega) = 0$ for all $\omega \in \Omega$.

2. c is continuous in q with $c(0) = 0$;

3. there exists $Q > 0$ such that $v(Q, \omega) - c(Q) < 0$ for all $\omega \in \Omega$.

The seller has a prior belief $\Lambda \in \Delta(\Omega)$ with a corresponding probability mass function λ. After receiving a signal about the buyer's type, the seller forms a posterior belief and sets pricing options $\{\vec{q}(\omega), \vec{p}(\omega)\}_{\omega \in \Omega}$ based on the updated belief. The set of actions can be viewed as a subset of $[0, Q]^N \times \mathbb{R}_+^N$.

A pricing option $(\vec{q}, \vec{p})$ can be implemented if and only if it satisfies the following familiar constraints: for all $\omega, \omega' \in \Omega$,

$$v(\vec{q}(\omega), \omega) - \vec{p}(\omega) \geq v(\vec{q}(\omega'), \omega) - \vec{p}(\omega'), \tag{IC}$$

$$v(\vec{q}(\omega), \omega) - \vec{p}(\omega) \geq 0. \tag{IR}$$

Therefore, the set of feasible decisions is

$$A_M = \left\{ (\vec{q}, \vec{p}) \in [0, Q]^N \times \mathbb{R}_+^N \,\middle|\, (\vec{q}, \vec{p}) \text{ satisfies ICs and IRs} \right\}. \tag{6.1}$$

I call that an information F has the full support if $f(x|\omega) > 0$ for all $x \in X$ and $\omega \in \Omega$. Likewise, I call that a prior belief Λ has the full support if $\lambda(\omega) > 0$ for all $\omega \in \Omega$. Then, upon receiving a signal x, the probability mass of the posterior belief is $\lambda_F^x(\omega) > 0$ for all

[20] Note that in this example, the seller's utility itself is not dependent on ω. However, different payoffs across states arise from price discrimination.

$\omega \in \Omega$. This is essential for the further argument because if $\lambda_F^x(\omega) = 0$ for some $\omega \in \Omega$, the seller would ignore some ICs and IRs and suggest a menu of pricing which might not be in A_M. Under the full support assumption, the seller's problem is

$$\max_{(\vec{q},\vec{p})\in A_M} \sum_{n=1}^{N} [-c(\vec{q}(\omega_n)) + \vec{p}(\omega_n)] \cdot \lambda_F^x(\omega_n). \tag{6.2}$$

It is well known that an allocation $\vec{q}$ is implementable iff $\vec{q}$ is monotone, and all the local downward ICs and IR for the lowest type are binding at the solution of (6.2) (Maskin and Riley (1984), Guesnerie and Laffont (1984)). That is, the optimal pricing policy given the quantity allocation $\vec{q}$ is as follows: for all $1 \leq n \leq N$,

$$\vec{p}(\omega_n) = v(\vec{q}(\omega_n), \omega_n) - \sum_{i=1}^{n-1} [v(\vec{q}(\omega_i), \omega_{i+1}) - v(\vec{q}(\omega_i), \omega_i)]\,.^{21} \tag{6.3}$$

In addition, we can restrict attention to the set of decisions $\mathcal{Q}$:

$$\mathcal{Q} = \left\{\vec{q} \in [0, Q]^N \;\middle|\; \text{for all } 1 \leq i \leq j \leq N,\ \vec{q}(\omega_j) \geq \vec{q}(\omega_i)\right\}. \tag{6.4}$$

By plugging (6.2) into the seller's payoff function, define

$$u(\vec{q}, \omega_n) \equiv v(\vec{q}(\omega_n), \omega_n) - \sum_{i=1}^{n-1} [v(\vec{q}(\omega_i), \omega_{i+1}) - v(\vec{q}(\omega_i), \omega_i)] - c(\vec{q}(\omega_n)). \tag{6.5}$$

Then, we can rewrite (6.2) as $\max_{\vec{q}\in\mathcal{Q}} U(\vec{q}; x)$ where

$$U(\vec{q}\,; x) = \sum_{n=1}^{N} u(\vec{q}, \omega_n) \cdot \lambda_F^x(\omega_n). \tag{6.6}$$

Under the optimal mechanism, no matter what the seller's belief is, quantities for higher types are almost as high as the first best quantity. However, quantities for lower types serve as a tool for incentivizing not only the lower types of the buyer but also higher types through the pricing policy (6.3). From the supermodularity of v, we observe that reducing a quantity for a low type buyer would induce a price increase for a high type buyer. Note that if the seller believes that the buyer is more likely to be a high type, then the seller would be less concerned about the possibility of distorting the quantities designed for low types in order to raise the prices for high types.

[21] Note that the IR for the lowest type binds, i.e., $\vec{p}(\omega_1) = v(\vec{q}(\omega_1), \omega_1)$.

Using this intuition, consider a partial order $\geq_{\mathcal{Q}}$ on $\mathcal{Q}$ defined as follows: for any $\vec{q}_i,\ \vec{q}_j \in \mathcal{Q}$,

$$\vec{q}_j \geq_{\mathcal{Q}} \vec{q}_i \quad \Longleftrightarrow \quad \vec{q}_j(\omega) \leq \vec{q}_i(\omega) \ \ \forall\, \omega \in \Omega. \tag{6.7}$$

The next lemma justifies this partial order by showing that the payoff function satisfies the DDDR condition with respect to this partial order.

Lemma 2. *Assume that v is supermodular in (q, ω). The payoff function defined in (6.5) satisfies the DDDR condition with respect to the partial order $\geq_{\mathcal{Q}}$ defined in (6.7).*

The next task is to check the MCS condition. I use a similar argument as in the latter part of Section 5.1: (i) show that $U(\vec{q}\,; x)$ is supermodular in $\mathcal{Q}$; (ii) also show that $U(\vec{q}\,; x'')$ I-dominates $U(\vec{q}\,; x')$ for all $x'' > x'$; then (iii) apply Theorem 1 of Quah and Strulovici (2007).[22][23]

I begin by showing that $\mathcal{Q}$ is a lattice. Observe that $\vec{q}_i \vee \vec{q}_j = (\min\{\vec{q}_i(\omega), \vec{q}_j(\omega)\})_{\omega\in\Omega}$ and $\vec{q}_i \wedge \vec{q}_j = (\max\{\vec{q}_i(\omega), \vec{q}_j(\omega)\})_{\omega\in\Omega}$ under this partial order.[24] When $\vec{q}_i$ and $\vec{q}_j$ are in $\mathcal{Q}$, i.e., they are increasing in ω, $\min\{\vec{q}_i, \vec{q}_j\}$ and $\max\{\vec{q}_i, \vec{q}_j\}$ are increasing in ω as well. Thus, $\vec{q}_i \vee \vec{q}_j$ and $\vec{q}_i \wedge \vec{q}_j$ are in $\mathcal{Q}$, i.e., $\mathcal{Q}$ is a lattice. For any function $w : \mathbb{R} \to \mathbb{R}$, we have $w(a) + w(b) = w(\max\{a, b\}) + h(\min\{a, b\})$. By using this to $v(\cdot, \omega)$ and $c(\cdot)$, we can easily derive that $u(\vec{q}_i, \omega_n) + u(\vec{q}_j, \omega_n) = u(\vec{q}_i \vee \vec{q}_j, \omega_n) + u(\vec{q}_i \wedge \vec{q}_j, \omega_n)$, thus, u is supermodular in $\mathcal{Q}$. Since $U(\vec{q}\,; x)$ is a convex combination of $u(\vec{q}, \omega)$, $U(\vec{q}\,; x)$ is also supermodular in $\mathcal{Q}$.

Next, the following lemma shows that $U(\vec{q}\,; x'')$ I-dominates $U(\vec{q}\,; x')$ when the MLRP condition is imposed. The proof is provided in Appendix D.3.

Lemma 3. *Suppose that an information structure F satisfies the MLRP and has the full support, and the prior belief Λ has the full support. Then, for any $x'' > x'$, $U(\vec{q}\,; x'')$ I-dominates $U(\vec{q}\,; x')$.*

Now we are ready to show that the monotone quasi-garbling order can be served as a criterion for comparing information in the nonlinear monopoly pricing problem.

Proposition 7. *Suppose that assumptions 1-4 hold for (v, c), F and G have full support, G satisfies the MLRP, and G is a monotone quasi-garbling of F. Then, the seller prefers F to G for any prior belief Λ with full support.*

[22] This argument is slightly different from Section 5.1 in the second step. In Section 5.1, I use the IDO condition for the payoff function u to show that u satisfies the DDDR condition and $U(\vec{q}\,; x'')$ I-dominates $U(\vec{q}\,; x')$. In this example, I directly show these properties in Lemma 2 and 3.

[23] The proof of the MCS result in the previous version of the paper was based on the first-order condition and the ironing technique with stronger conditions on the primitives. The interval-dominance-order-based argument presented in this version is due to Referee 1 (to be named), which not only weakens the conditions in the previous version, but also substantially simplifies the proof.

[24] Note that it is opposite from the case with the usual product order of which inequality is reversed from (6.7).

Proof. By Lemma 2, the payoff function u defined in (6.5) satisfies the DDDR condition. Next, under the assumption that G satisfies the MLRP, since $U(\vec{q}\,;y)$ is supermodular in $\mathcal{Q}$ and $U(\vec{q}\,;y'')$ I-dominates $U(\vec{q}\,;y')$ for all $y'' > y'$, by Theorem 1 of Quah and Strulovici (2007), the MCS condition holds: there exists $\vec{q}_* : Y \to \mathcal{Q}$ such that $\vec{q}_*(y'') \geq_A \vec{q}_*(y')$ for all $y'' > y'$. Therefore, the seller's problem is generally monotone with respect to G. Then, when G is a monotone quasi-garbling of F, by Theorem 2, we have $V(F; u, \Lambda) \geq V(G; u, \Lambda)$. □

7 Conclusion

In this paper, I develop the concept of the monotone quasi-garbling order, which implies that one information structure is obtained from another by adding reversely monotone noise. I show that this ordering is equivalent to Lehmann's accuracy condition under the MLRP. I also show that this order is a necessary and sufficient condition for informativeness in general classes of monotone decision problems where the decision maker is allowed to choose a multidimensional action. I illustrate this result in decision problems of optimal insurance and nonlinear monopoly pricing.

The general setup presented here can be applied to conduct research on information comparisons in many other economic contexts. For example, as in the nonlinear monopoly pricing application, many mechanism design problems can be recast as general monotone decision problems amenable to analysis via the methods presented in this paper.

Appendix

A Details for Relationship among Criteria

A.1 Proof of Proposition 3

Proof of Proposition 3. Suppose that $F \succeq_L G$ does not hold, i.e., there exists $\omega' > \omega$ and $y \in Y$ such that $\Phi(\omega'; y) < \Phi(\omega; y)$. Since $F \succeq_{MQG} G$, there exists Γ such that $G(y|\omega) = \int_{\underline{x}}^{\overline{x}} \Gamma(y|x,\omega)dF(x|\omega)$ and $\Gamma(\cdot|x,\omega) \geq_{FOSD} \Gamma(\cdot|x,\omega')$ for all $\omega' > \omega$. Also note that $G(y|\omega) = F(\Phi(\omega; y)|\omega) = \int_{\underline{x}}^{\Phi(\omega;y)} dF(x|\omega)$. Then, we can rewrite $G(y|\omega) = \int_{\underline{x}}^{\overline{x}} \Gamma(y|x,\omega)dF(x|\omega) = \int_{\underline{x}}^{\Phi(\omega;y)} dF(x|\omega)$ as follows:

$$\int_{\underline{x}}^{\Phi(\omega;y)} (1 - \Gamma(y|x,\omega))dF(x|\omega) = \int_{\Phi(\omega;y)}^{\overline{x}} \Gamma(y|x,\omega)dF(x|\omega). \tag{A.1}$$

Suppose that (A.1) is equal to zero. Since $1 \geq \Gamma(y|x,\omega) \geq 0$ for all x, it implies that $\Gamma(y|x,\omega) = 1$ a.e. for $x \leq \Phi(\omega; y)$, and $\Gamma(y|x,\omega) = 0$ a.e. for $x > \Phi(\omega; y)$. Observe that, by $\Gamma(\cdot|x,\omega) \geq_{FOSD} \Gamma(\cdot|x,\omega')$, $\Gamma(y|x,\omega') = 1$ a.e. for $x \leq \Phi(\omega; y)$. Then, we have

$$G(y|\omega') = \int_{\underline{x}}^{\overline{x}} \Gamma(y|x,\omega')dF(x|\omega') \geq F(\Phi(\omega; y)|\omega'),$$

which implies $\Phi(\omega'; y) = \sup\{x \mid F(x|\omega') \leq G(y|\omega')\} \geq \Phi(\omega; y)$ contradicting $\Phi(\omega'; y) < \Phi(\omega; y)$. Therefore (A.1) is nonzero.

Note that

$$\begin{aligned}
&\left(\int_{\Phi(\omega;y)}^{\overline{x}} \Gamma(y|x,\omega)dF(x|\omega)\right) \cdot \left(\int_{\underline{x}}^{\Phi(\omega;y)} (1 - \Gamma(y|x,\omega))dF(x|\omega')\right) \\
&= \int_{\Phi(\omega;y)}^{\overline{x}} \int_{\underline{x}}^{\Phi(\omega;y)} \Gamma(y|x,\omega)(1 - \Gamma(y|x,\omega))dF(x|\omega)dF(x'|\omega') \\
&\leq \int_{\Phi(\omega;y)}^{\overline{x}} \int_{\underline{x}}^{\Phi(\omega;y)} \Gamma(y|x,\omega)(1 - \Gamma(y|x,\omega))dF(x'|\omega)dF(x|\omega') \\
&= \left(\int_{\Phi(\omega;y)}^{\overline{x}} \Gamma(y|x,\omega)dF(x|\omega')\right) \cdot \left(\int_{\underline{x}}^{\Phi(\omega;y)} (1 - \Gamma(y|x,\omega))dF(x|\omega)\right)
\end{aligned}$$

where the inequality is derived by doubly integrating the MLRP conditions multiplied by $\Gamma(y|x,\omega)(1 - \Gamma(y|x',\omega))$.

By canceling terms by using (A.1), we can derive that

$$F(\Phi(\omega;y)|\omega') = \int_{\underline{x}}^{\Phi(\omega;y)} dF(x|\omega') \leq \int_{\underline{x}}^{\overline{x}} \Gamma(y|x,\omega) dF(x|\omega').$$

Also from $\Gamma(y|x,\omega) \leq \Gamma(y|x,\omega')$ for all $x \in X$,

$$\int_{\underline{x}}^{\overline{x}} \Gamma(y|x,\omega) dF(x|\omega') \leq \int_{\underline{x}}^{\overline{x}} \Gamma(y|x,\omega') dF(x|\omega') = G(y|\omega').$$

Then, the above two inequalities imply $F(\Phi(\omega;y)|\omega') \leq G(y|\omega')$, thus, $\Phi(\omega';y) \geq \Phi(\omega;y)$ contradicting $\Phi(\omega';y) < \Phi(\omega;y)$. Therefore, $F \succeq_L G$ holds. □

A.2 An Illustrative Example on Information Ranking

In this section, I provide an illustrative example that is discussed in Section 4. Let the state space be $\Omega \equiv [0,4]$. I construct a triplet of information structures, F, G and H where their signal spaces are $X = Y = [-1,1]$ and $Z = [-2,2]$, and their probability distribution functions are given as in Figure 2. Observe that F and G satisfy the MLRP, but H does not.

	$h(z\|\omega)$				$f(x\|\omega)$		$g(y\|\omega)$	
$\omega\backslash$ signal	$[-2,-1)$	$[-1,0)$	$[0,1)$	$[1,2]$	$[-1,0)$	$[0,1]$	$[-1,0)$	$[0,1]$
$\omega \in [0,1)$	1/2	1/2	0	0	1	0	5/6	1/6
$\omega \in [1,2)$	1/2	0	1/2	0	1/2	1/2	2/3	1/3
$\omega \in [2,3)$	0	1/2	0	1/2	1/2	1/2	2/3	1/3
$\omega \in [3,4]$	0	0	1/2	1/2	0	1	1/3	2/3

Figure 2: An example of information structures

In the subsequent subsections, I show the following statements which are summarized in Figure 3.

1. G is neither a garbling of F nor H, but F is a garbling of G ($F \not\succeq_B G$, $H \not\succeq_B G$, but $F \succeq_B G$);

2. H is more Lehmann accurate than neither F nor G, but F is more Lehmann accurate than G ($H \not\succeq_L F$, $H \not\succeq_L G$, but $F \succeq_L G$);

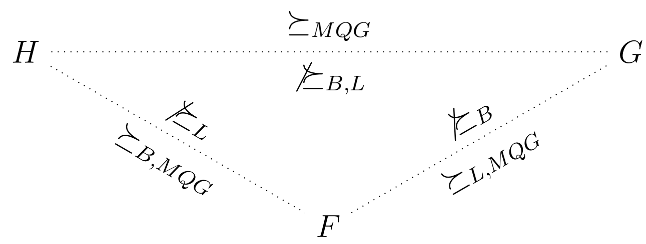


Figure 3: Relationship among criteria in the example

3. G is a monotone quasi-garbling of H and F, and F is a monotone quasi-garbling of H ($H \succeq_{MQG} F$, $H \succeq_{MQG} G$, and $F \succeq_{MQG} G$).

A.2.1 Blackwell's Garbling Condition

First, I show that G is a garbling of neither F nor H by showing that there exists a prior belief Λ and a utility function u such that $V(G; u, \Lambda) > V(F; u, \Lambda) = V(H; u, \Lambda)$. Consider a case where a prior Λ is uniformly distributed, $A = [0, 1]$ and utility function $u : A \times \Omega \to \mathbb{R}$ is linear in a, i.e., $u(a, \omega) = a \cdot u(1, \omega) + (1 - a) \cdot u(0, \omega)$, and $u(1, \omega)$ and $u(0, \omega)$ are defined as follows:

$$u(1,\omega) = \begin{cases} 1, & \text{if } \omega \in [1,3), \\ 0, & \text{otherwise,} \end{cases} \qquad u(0,\omega) = \begin{cases} 0, & \text{if } \omega \in [1,3), \\ 1, & \text{otherwise.} \end{cases}$$

Observe that the DM's optimal decision is determined by the posterior belief that the state is in $[1, 3)$. Under the information structure F and H, the posterior belief that the state is in $[1, 3)$ is $1/2$ for any signal realization. Therefore, $V(F; u, \Lambda) = V(H; u, \Lambda) = 1/2$. Under the information structure G, if he receives a negative signal, which happens at probability $15/24$, the posterior belief that the state is in $[1, 3)$ is $8/15$. In this case, it is optimal to choose $a = 1$ and the expected utility is $8/15$. If he receives a nonnegative signal, which happens at probability $9/24$, the posterior belief that the state is in $[1, 3)$ is $4/9$. In this case, it is optimal to choose $a = 0$ and the expected utility is $5/9$. Then, we have

$$V(G; u, \Lambda) = \frac{15}{24} \cdot \frac{8}{15} + \frac{9}{24} \cdot \frac{5}{9} = \frac{13}{24} > \frac{1}{2} = V(F; u, \Lambda) = V(H; u, \Lambda).$$

Therefore, G is a garbling of neither F nor H, i.e., $F \not\succeq_B G$ and $H \not\succeq_B G$.

Next, I show that F is a garbling of H by constructing a noise $\Gamma : Z \to \Delta(X)$. Consider Γ derived from the following probability distribution function:

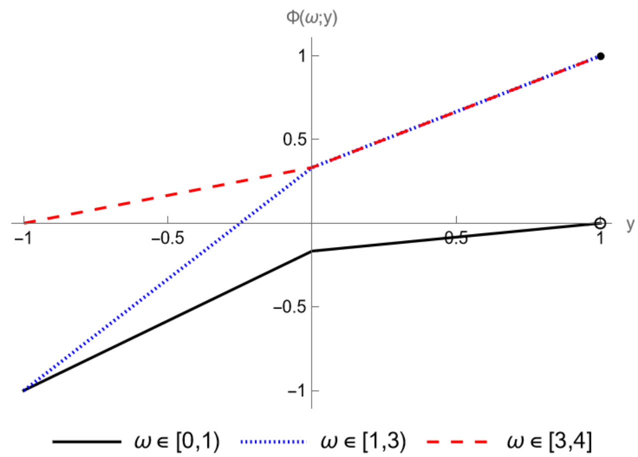


Figure 4: Lehmann Effectiveness Condition for F and G

$$\gamma(x|z) = \begin{array}{c|cc} & x<0 & x\geq 0 \\ \hline z<0 & 1 & 0 \\ z\geq 0 & 0 & 1 \end{array}.$$

By simple algebra, we can see that $f(x|\omega) = \int_{-2}^{2} \gamma(x|z)h(z|\omega)dz$ for all $\omega \in \Omega$. Therefore, F is a garbling of H, i.e., $H \succeq_B F$.

A.2.2 Lehmann's Effectiveness Condition

I show that H is not more Lehmann accurate than F by finding a signal x and a pair of states (ω, ω') such that $\omega' > \omega$ and $\Phi(\omega; x) > \Phi(\omega'; x)$. Consider $x = 0$, $\omega = 0$, and $\omega' = 1$. Observe that

$$\Phi(0;0) = \sup\{z \,|\, H(z|0) \leq F(0|0) = 1\} = 2,$$
$$\Phi(1;0) = \sup\{z \,|\, H(z|1) \leq F(0|1) = 1/2\} = 0,$$

thus, $\Phi(0;0) > \Phi(1;0)$ and $H \not\succeq_L F$.

Likewise, I show that H is not more Lehmann accurate than G by considering $y = 0$, $\omega = 2$ and $\omega' = 3$:

$$\Phi(2;0) = \sup\{z \,|\, H(z|2) \leq G(0|2) = 2/3\} = 4/3,$$
$$\Phi(3;0) = \sup\{z \,|\, H(z|3) \leq G(0|3) = 1/3\} = 2/3.$$

Last, I show that F is more Lehmann accurate than G. With simple algebra, we can derive Φ as follows.

1. When $\omega \in [0, 1)$,

$$\Phi(\omega; y) = \begin{cases} \frac{5}{6}y - \frac{1}{6}, & \text{if } -1 \leq y < 0, \\ \frac{1}{6}y - \frac{1}{6}, & \text{if } 0 \leq y < 1, \\ 2, & \text{if } y = 1. \end{cases}$$

2. When $\omega \in [1, 3)$,

$$\Phi(\omega; y) = \begin{cases} \frac{4}{3}y + \frac{1}{3}, & \text{if } -1 \leq y < 0, \\ \frac{2}{3}y + \frac{1}{3}, & \text{if } 0 \leq y \leq 1. \end{cases}$$

3. When $\omega \in [3, 4]$,

$$\Phi(\omega; y) = \begin{cases} \frac{1}{3}y + \frac{1}{3}, & \text{if } -1 \leq y < 0, \\ \frac{2}{3}y + \frac{1}{3}, & \text{if } 0 \leq y \leq 1. \end{cases}$$

Figure 4 illustrates the above results. As we can see in the figure, $\Phi(\omega; y)$ is increasing in ω. Therefore, F is more Lehmann accurate than G.

A.2.3 Monotone Quasi-Garbling Order

Since Blackwell's garbling condition implies the monotone quasi-garbling order, we have $H \succeq_{MQG} F$. We also have $F \succeq_{MQG} G$ because Lehmann's accuracy condition also implies the monotone quasi-garbling order (Proposition 2). It remains to show that G is a monotone quasi-garbling of H. I construct a reversely monotone noise $\Gamma : Z \times \Omega \to \Delta(Y)$ derived from the following probability distribution functions:

1. if $\omega \in [0, 1)$,

$$\gamma(y|z,\omega) = \begin{array}{c|cc} & y < 0 & y \geq 0 \\ \hline z \in [-2,-1) & 1 & 0 \\ z \in [-1,0) & 2/3 & 1/3 \\ z \in [0,1) & 0 & 1 \\ z \in [1,2] & 0 & 1 \end{array},$$

2. if $\omega \in [1, 2)$,

$$\gamma(y|z,\omega) = \begin{array}{c|cc} & y < 0 & y \geq 0 \\ \hline z \in [-2,-1) & 1 & 0 \\ z \in [-1,0) & 2/3 & 1/3 \\ z \in [0,1) & 1/3 & 2/3 \\ z \in [1,2] & 0 & 1 \end{array},$$

3. if $\omega \in [2,4]$,

$$\gamma(y|z,\omega) = \begin{array}{c|cc} & y<0 & y\geq 0 \\ \hline z\in[-2,-1) & 1 & 0 \\ z\in[-1,0) & 1 & 0 \\ z\in[0,1) & 1/3 & 2/3 \\ z\in[1,2] & 1/3 & 2/3 \end{array}.$$

With simple algebra, we can show that $g(y|\omega) = \int_{-2}^{2} \gamma(y|z,\omega)h(z|\omega)dz$ holds for all $y \in Y$ and $\omega \in \Omega$. In addition, we can also see that $\Gamma(y|x,\omega) \leq \Gamma(y|x,\omega')$ for all $y \in Y$, $x \in X$, and $\omega < \omega'$, i.e., Γ is a reversely monotone noise. Therefore, G is a monotone quasi-garbling of H, i.e., $H \succeq_{MQG} G$.

B An Example on IDO and DDDR Conditions

In this section, I provide an example of a decision problem that does not satisfy the IDO condition but is generally monotone with respect to any information structure with the MLRP. Let $\Omega = \{\omega_1, \omega_2\}$ and $A = [0,6]$. Consider a usual order in A. Define a payoff function $u : A \times \Omega \to \mathbb{R}$ as follows:

$$u(a,\omega_1) \equiv \begin{cases} a+1, & \text{if } a \leq 2, \\ 5-a, & \text{if } 2 < a \leq 5, \\ a-5, & \text{if } 5 < a, \end{cases} \quad u(a,\omega_2) \equiv \begin{cases} 0, & \text{if } a \leq 1, \\ a-1, & \text{if } 1 < a \leq 4, \\ 5-a, & \text{if } 4 < a. \end{cases}$$

This function is illustrated in Figure 5.

First, observe that this function does not satisfy IDO condition: $u(6,\omega_1) > u(a,\omega_1)$ for all $5 \leq a < 6$, but $u(6,\omega_2) < u(5,\omega_2)$. Now I show that it satisfies the DDDR condition. Consider any decreasing function $h : \Omega \to A$. When $h(\omega_1) \leq 4$, set $\hat{a} = h(\omega_1)$. Then, $u(\hat{a},\omega_1) = u(h(\omega_1),\omega_1)$ by definition of $\hat{a}$, and $u(\hat{a},\omega_2) = u(h(\omega_1),\omega_2) \geq u(h(\omega_2),\omega_2)$ since $u(a|\omega_2)$ is increasing in a when $a < 4$. When $h(\omega_1) > 4$, set $\hat{a} = 4$. Then, $u(\hat{a},\omega_1) \geq u(h(\omega_1),\omega_1)$ since $u(4,\omega_1) \geq u(a,\omega_1)$ for all $a \geq 4$. In addition, $u(a,\omega_2)$ is maximized at $a = 4$, thus, $u(\hat{a},\omega_2) \geq u(h(\omega_2),\omega_2)$. Therefore, this decision problem satisfies the DDDR condition with respect to the generic order.

Now I show that it satisfies the MCS condition. Observe that for any $a < 2$ and ω, $u(a,\omega) < u(2,\omega)$. Similarly, for any $a \geq 4$, $u(a,\omega) \leq u(4,\omega)$. Therefore, it is without loss to focus on $a \in [2,4]$. Let the posterior belief given a signal x be $\psi_x = \Pr(\omega_2|x)$ and

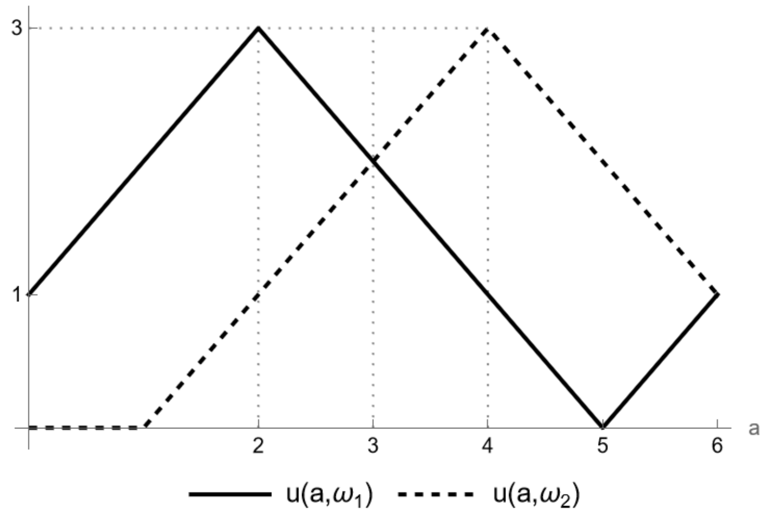


Figure 5: Illustration of dominated decreasing decision rules, where the interval dominance property is violated.

$1 - \psi_x = \Pr(\omega_1|x)$. Then, the expected payoff given the signal x is

$$\psi_x(a-1) + (1-\psi_x)(5-a) = (2\psi_x - 1)a + 5 - 6\psi_x.$$

Therefore, the optimal action given x is

$$a^*(x) = \begin{cases} 4, & \text{if } \psi_x \geq 1/2, \\ 2, & \text{if } \psi_x \leq 1/2. \end{cases}$$

Also observe that for any information structure with the MLRP, prior belief and pair of signals (x', x) with $x' > x$, the posterior belief on ω_2 is increasing, i.e., $\psi_{x'} \geq \psi_x$. Therefore, $a^*(x)$ is increasing in x, thus, the MCS condition holds.

C Application: Optimal Insurance

I now present a familiar economic application that can be considered a general monotone decision problem. Specifically, I consider an optimal insurance problem similar to the investment problem studied by Cabrales et al. (2013). An agent (DM) privately acquires information about a future state $\omega \in \Omega = \{\omega_1, \cdots, \omega_N\}$, which will be his realized income, with $\omega_N > \cdots > \omega_1 > 0$. At the beginning of the investment decision, the agent chooses between two sources of information, say F and G. The agent receives a signal from the chosen source. After receiving a signal (and before the realization of the state), the agent can buy or sell Arrow securities. Assume that the Arrow security market is large enough so that the agent's demand does not affect prices. Let the price vector be $p = (p_1, \cdots, p_N)$ and assume that $\sum_{i=1}^N p_i = 1$ and $p_i > 0$.

In this example, an action by the agent is the number of securities bought (or sold) for each state, thus, the action space is multidimensional. For any $1 \leq i \leq N$, denote the amount of securities bought for a state ω_i by q_i. By allowing a short sale, q_i can be negative. Assume that the agent is not allowed to execute a short sale more than his realized income, i.e., $q_i + \omega_i \geq 0$ for all $1 \leq i \leq N$. Along with these constraints, the agent's budget constraint determines the set of feasible decisions. By assuming the agent's budget is zero, i.e., he must finance purchases through short sales, we get the set of feasible decisions as follows:

$$A = \{q = (q_1, \cdots, q_n) \in \mathbb{R}^n \,|\, p \cdot q \leq 0 \;\text{ and }\; \omega_i + q_i \geq 0 \;\; \forall i \in N\}\,. \tag{C.1}$$

Also note that the budget constraint will bind at the optimum and define the set of budget binding decisions as follows:

$$A' = \{q = (q_1, \cdots, q_n) \in \mathbb{R}^n \,|\, p \cdot q = 0 \;\text{ and }\; \omega_i + q_i \geq 0 \;\; \forall i \in N\}\,. \tag{C.2}$$

For the agent's utility function, assume constant relative risk aversion (CRRA) preferences with respect to the realized net income:

$$\mathcal{U} \equiv \left\{ u : A_2 \times \Omega \to \mathbb{R} \;:\; \begin{array}{l} u(q, \omega_i) = v(\omega_i + q_i) \;\; \forall 1 \leq i \leq N, \\ \exists\, \rho > 0 \;\text{ s.th. }\; v(z) = (z^{1-\rho} - 1)/(1-\rho) \end{array} \right\}.^{25} \tag{C.3}$$

Next I define a partial order $\geq_A$ on A. Recall that the purpose of this order is to compare the optimal actions, thus it only needs to be defined over $A' \subset A$. I construct the partial order as follows: for any $q, q' \in A'$,

$$q' \geq_A q \;\Leftrightarrow\; \frac{\omega_j + q'_j}{\omega_i + q'_i} \geq \frac{\omega_j + q_j}{\omega_i + q_i} \quad \text{for all } j > i. \tag{C.4}$$

This definition is inspired by the following equation holding at the optimum bundle: for any $i < j$,

$$\frac{\Pr(\omega_i) \cdot v'(\omega_i + q_i)}{\Pr(\omega_j) \cdot v'(\omega_j + q_j)} = \frac{p_i}{p_j}. \tag{C.5}$$

This equation says that the (expected) marginal rate of substitution of every pair of states is equal to the price ratio. Observe that when a bundle q' is higher than another bundle q in

[25] When $\rho = 1$, $v(z) = \log(z)$.

the order $\geq_A$, the left hand side of (C.5) increases. Suppose that the DM receives a higher signal that leads to a fall in the likelihood ratio $\Pr(\omega_i)/\Pr(\omega_j)$ in (C.5). Since the price ratio is assumed to be fixed, i.e., the right hand side of (C.5) is unchanged, the DM would choose a higher bundle in terms of $\geq_A$. In this sense, the order $\geq_A$ is reasonable in this example. The following proposition shows that this insurance problem is a general monotone decision problem with this order.

Proposition 8. *The following statements hold:*

(a) The partial order $\geq_A$ defined on (C.4) satisfies the DDDR condition for the set of payoff functions $\mathcal{U}$ defined on (C.3);

(b) Suppose that an information structure f satisfies the MLRP. Then, f satisfies the MCS condition with respect to $\mathcal{U}$ and $\geq_A$.

Proof of Proposition 8. (a) Consider a decreasing decision rule $\{q^j\}_{j\in N} \in (A')^n$ where $q^1 \geq_A \cdots \geq_A q^n$. Define $\hat{q}_i \equiv q_i^i$ and $\hat{q} = (\hat{q}_1, \cdots, \hat{q}_n)$. To check whether $\geq_A$ satisfies the DDDR condition for $\mathcal{U}$, it is enough to show that $\hat{q} \in A$ since for any $u \in \mathcal{U}$, there exists a function v such that $u(\hat{q}, \omega_i) = v(\hat{q}_i + \omega_i) = v(a_i^i + \omega_i) = u(q^i, \omega_i)$ for all $1 \leq i \leq N$. Note that $\omega_i + \hat{q}_i = \omega_i + q_i^i \geq 0$ from $q^i \in A'$. Therefore, it suffices to show that $p \cdot \hat{q} \leq 0$.

For all $1 \leq i, j \leq N$, define $r_i^j \equiv p_i(\omega_i + q_i^j)/(\sum_{s=1}^n p_s\omega_s)$ and $r^j \equiv \{r_i^j\}_{1\leq i\leq N}$. Note that $\sum_{i\in N} r_i^j = 1$ from $q^j \in A'$, thus, r^j can be considered an element of $\Delta(\Omega)$. Moreover, the condition (C.4) makes $\{r^j\}_{j\in N}$ reversely MLRP ordered, i.e., $r^1 \geq_{MLRP} \cdots \geq_{MLRP} r^n$. Therefore, $r^1 \geq_{FOSD} \cdots \geq_{FOSD} r^n$. Then, the following inequalities hold:

$$
\begin{aligned}
1 &= r_1^n + r_2^n + \cdots + r_{n-2}^n + r_{n-1}^n + r_n^n \\
&\geq (r_1^{n-1} + r_2^{n-1} + \cdots + r_{n-2}^{n-1} + r_{n-1}^{n-1}) + r_n^n && (\text{by } \ r^{n-1} \geq_{FOSD} r^n) \\
&\geq (r_1^{n-2} + r_2^{n-2} + \cdots + r_{n-2}^{n-2}) + r_{n-1}^{n-1} + r_n^n && (\text{by } \ r^{n-2} \geq_{FOSD} r^{n-1}) \\
&\ \ \vdots \\
&\geq r_1^1 + r_2^2 + \cdots r_{n-2}^{n-2} + r_{n-1}^{n-1} + r_n^n && (\text{by } \ r^1 \geq_{FOSD} r^2) \\
&= \sum_{i\in N} \frac{p_i(\omega_i + \hat{q}_i)}{\sum_{s=1}^n p_s\omega_s}
\end{aligned}
$$

From the last inequality, $0 \geq \sum_{i\in N} p_i\hat{q}_i = p \cdot \hat{q}$. Therefore, $\geq_A$ satisfies the DDDR condition for $\mathcal{U}$.

(b) Fix a prior belief $\Lambda \in \Delta(\Omega)$ and a utility function $u \in \mathcal{U}$ with $u(q, \omega_i) = v(\omega_i + q_i)$ and $v(z) = z^{1-\rho}/(1-\rho)$ for some $\rho > 0$. In the third stage with the signal x, DM's problem is

$$\max_{q\in A}\sum_{i\in N}\lambda_F^x(\omega_i)v(\omega_i+q_i). \tag{C.6}$$

When an information structure is $\{f(\cdot|\omega)\}_{\omega\in\Omega}$ and the agent receives x as a signal, let the solution of (C.6) be $\{q_i(x)\}_{i\in N}$. Note that the solution can be obtained as follows:

$$\omega_i+q_i(x)=\left(\frac{\lambda_F^x(\omega_i)}{p_i}\right)^{1/\rho}\cdot\frac{\sum_{s=1}^n\omega_s\cdot p_s}{\sum_{s=1}^n\lambda_F^x(\omega_s)^{1/\rho}\cdot {p_s}^{1-1/\rho}}, \tag{C.7}$$

for all $i\in N$.

Assume that an information structure F satisfies the MLRP. By Milgrom (1981), the posterior belief also shows the MLRP, that is, for all $x'>x$ and $j>i$,

$$\lambda_F^{x'}(\omega_j)\lambda_F^{x}(\omega_i)-\lambda_F^{x}(\omega_j)\lambda_F^{x'}(\omega_i)\geq 0 \quad\Leftrightarrow\quad \frac{\lambda_F^{x'}(\omega_j)}{\lambda_F^{x}(\omega_j)}\geq\frac{\lambda_F^{x'}(\omega_i)}{\lambda_F^{x}(\omega_i)}. \tag{C.8}$$

We see that this condition represents the intuition that with a lower signal, a lower state is more likely to happen and with a higher signal, a higher state is more likely to happen. Note that from (C.7),

$$\frac{\omega_j+q_j(x')}{\omega_j+q_j(x)}\geq\frac{\omega_i+q_i(x')}{\omega_i+q_i(x)} \quad\Leftrightarrow\quad \frac{\lambda_F^{x'}(\omega_j)^{1/\rho}}{\lambda_F^{x}(\omega_j)^{1/\rho}}\geq\frac{\lambda_F^{x'}(\omega_i)^{1/\rho}}{\lambda_F^{x}(\omega_i)^{1/\rho}}, \tag{C.9}$$

and it holds from (C.8). Therefore, we can see that

$$\{q_i(x')\}_{1\leq i\leq N}\geq_A\{q_i(x)\}_{1\leq i\leq N}$$

for all $x'>x$ and the MCS condition holds.

□

Proposition 9 (Informativeness in the optimal insurance problem). *In the optimal insurance problem, if G satisfies the MLRP and G is a monotone quasi-garbling of F, then the decision maker obtains greater expected utility in the first stage from F than from G for all priors.*

Proof of Proposition 9. When G satisfies the MLRP, by Proposition 8, $\mathcal{U}$ is a general monotone decision problem with respect to G. Since G is a monotone quasi-garbling of F, by Theorem 2, the expected payoff from F is greater than that from G. □

D Omitted Proofs

D.1 Proof of Theorem 3

I focus on information structures with which decision problems in $\mathcal{U}$ are generally monotone. To qualify that a decision problem in $\underline{\mathcal{U}}$ is generally monotone, we first need to check whether the DDDR condition holds. The following lemma confirms this, and the proof of Theorem 3 follows.

Lemma. *The generic order on $\underline{A}$ satisfies the DDDR condition with respect to $\underline{\mathcal{U}}$.*

Proof. Observe that any decreasing decision rule $h : \Omega \to A$ has a following structure:

$$h(\omega) = \begin{cases} 1, & \omega < \tilde{\omega}, \\ 0, & \omega \geq \tilde{\omega}. \end{cases}$$

When $\tilde{\omega} \leq \hat{\omega}$, observe that

$$u(h(\omega), \omega) = \begin{cases} u(1, \omega) = -\kappa \leq 0 = u(0, \omega), & \text{if } \omega < \tilde{\omega} \leq \hat{\omega}, \\ u(0, \omega) = 0 \leq 0 = u(0, \omega), & \text{if } \tilde{\omega} \leq \omega. \end{cases}$$

Thus, $h(\omega)$ is dominated by the constant action $a = 0$.

When $\tilde{\omega} > \hat{\omega}$, observe that

$$u(h(\omega), \omega) = \begin{cases} u(1, \omega) = -\kappa \leq -\kappa = u(1, \omega), & \text{if } \omega < \hat{\omega} < \tilde{\omega}, \\ u(1, \omega) = 1 - \kappa \leq 1 - \kappa = u(1, \omega), & \text{if } \hat{\omega} \leq \omega < \tilde{\omega}, \\ u(0, \omega) = 0 \leq 1 - \kappa = u(1, \omega), & \text{if } \hat{\omega} < \tilde{\omega} \leq \omega. \end{cases}$$

Thus, $h(\omega)$ is dominated by the constant action $a = 1$. □

Proof of Theorem 3. First, I show that F satisfies the MLRP. I fix a pair of signals $x_0 < x_1$ and a pair of states $\omega_0 < \omega_1$ and show that $f(x_1|\omega_1) \cdot f(x_0|\omega_0) \geq f(x_1|\omega_0) \cdot f(x_0|\omega_1)$. Consider a prior belief $\Lambda \in \Delta(\Omega)$ such that there are probability masses $1/2$ on ω_0 and ω_1. Observe that the posterior belief has probability masses $\tilde{\lambda}_F(x) \equiv f(x|\omega_1)/(f(x|\omega_0) + f(x|\omega_1))$ on ω_1 and $1 - \tilde{\lambda}_F(x)$ on ω_0.[26] For the utility function, set $\hat{\omega} = \omega_1$. Then, upon receiving a signal

[26] This is the case when there is no probability mass at x neither at the state ω_0 nor at the state ω_1. If there is a probability mass at either state, $\tilde{\lambda}_F$ should be defined by $\tilde{\lambda}_F(x) = \Pr(x|\omega_1)/(\Pr(x|\omega_0) + \Pr(x|\omega_0))$ where $\Pr(x|\omega) \equiv F(x|\omega) - F(x^-|\omega)$.

x, the DM solves the following problem:

$$\max_{a\in\{0,1\}} \int_{\Omega} u(a,\omega)d\Lambda_F^x(\omega) = \max_{a\in\{0,1\}} a\left[-\kappa\cdot(1-\tilde{\lambda}_F(x)) + (1-\kappa)\cdot\tilde{\lambda}_F(x)\right]$$
$$= \max_{a\in\{0,1\}} a\left(\tilde{\lambda}_F(x)-\kappa\right).$$

Note that $a=1$ solves the above maximization problem if and only if $\kappa \leq \tilde{\lambda}_F(x)$. Consider $\kappa = \tilde{\lambda}_F(x_0) - \epsilon$ for some $\epsilon > 0$. Then, $a=1$ is the only optimal choice under a signal x_0. To satisfy the MCS condition, $a=1$ also needs to be a solution under a signal x_1. It implies that $\tilde{\lambda}_F(x_1) \geq \kappa = \tilde{\lambda}_F(x_0) - \epsilon$. By sending ϵ to zero, we can derive that $\tilde{\lambda}_F(x_1) \geq \tilde{\lambda}_F(x_0)$. By rearranging the terms, we can derive that $f(x_1|\omega_1)\cdot f(x_0|\omega_0) \geq f(x_0|\omega_1)\cdot f(x_1|\omega_0)$. Therefore, F satisfies the MLRP.

Now I show that $F \succeq_L G$, i.e., $\Phi(\omega_1;\hat{y}) \geq \Phi(\omega_0;\hat{y})$ for all $\omega_1 > \omega_0$ and $\hat{y} \in Y$. Set $\hat{\omega} = \omega_1$, and $\kappa = \tilde{\lambda}_F(\Phi(\omega_0;\hat{y}))$. By the MLRP of F, we have

$$\begin{cases} \tilde{\lambda}_F(x) \geq \kappa, & \text{if } x \geq \Phi(\omega_0;\hat{y}), \\ \tilde{\lambda}_F(x) < \kappa, & \text{otherwise.} \end{cases}$$

Recall that $a=1$ will be chosen if and only if $\kappa \leq \tilde{\lambda}_F(x)$, i.e., $x \geq \Phi(\omega_0;\hat{y})$. Also note that $F_\Lambda(x) = (F(x|\omega_0)+F(x|\omega_1))/2$ and $\tilde{\lambda}_F(x)\cdot dF_\Lambda(x) = f(x|\omega_1)/2$. Then, we can derive that

$$V(F;u,\Lambda) = \int_{\Phi(\omega_0;y)}^{\overline{x}} \left(\tilde{\lambda}_F(x)-\kappa\right) dF_\Lambda(x)$$
$$= \frac{1-\kappa}{2}\left(1-F(\Phi(\omega_0;\hat{y})|\omega_1)\right) - \frac{\kappa}{2}\left(1-F(\Phi(\omega_0;\hat{y})|\omega_0)\right).$$

Next, observe that

$$V(G;u,\Lambda) = \int_{\underline{y}}^{\overline{y}} \left[\max_{a\in\{0,1\}} a\cdot\left(\tilde{\lambda}_G(y)-\kappa\right)\right] dG_\Lambda(y)$$
$$\geq \int_{\hat{y}}^{\overline{y}} \left(\tilde{\lambda}_G(y)-\kappa\right) dG_\Lambda(y)$$
$$= \frac{1-\kappa}{2}\left(1-G(\hat{y}|\omega_1)\right) - \frac{\kappa}{2}\left(1-G(\hat{y}|\omega_0)\right).$$

Since F is more informative than G on $\mathcal{U}$ and $u \in \underline{\mathcal{U}} \subseteq \mathcal{U}$, we have $V(F;u,\Lambda) \geq V(G;u,\Lambda)$

which implies

$$(1-\kappa)\left(G(\hat{y}|\omega_1)-F(\Phi(\omega_0;\hat{y})|\omega_1)\right)\geq\kappa\left(G(\hat{y}|\omega_0)-F(\Phi(\omega_0;\hat{y})|\omega_0)\right).$$

Note that the right hand side is equal to zero by the definition of $\Phi(\omega_0;\hat{y})$. Then, we have $G(\hat{y}|\omega_1)\geq F(\Phi(\omega_0;\hat{y})|\omega_1)$, which implies $\Phi(\omega_1;\hat{y})\geq\Phi(\omega_0;\hat{y})$. Therefore, $F\succeq_L G$, and by Proposition 2, we have $F\succeq_{MQG}G$. $\square$

D.2 Proof of Lemma 2

Proof of Lemma 2. Consider any $\{q(\omega;s)\}_{(\omega,s)\in\Omega\times\Omega}\in Q^N$ such that for all $s'>s$

$$\{q(\cdot;s)\}\geq_Q\{q(\cdot;s')\},$$

i.e., $q(\omega;s')\geq q(\omega;s)$ for all $s'>s$ and $\omega\in\Omega$. Then, I consider $\hat{q}(\cdot)$ defined by $\hat{q}(\omega)\equiv q(\omega;\omega)$. Observe that for any $\omega'>\omega$, we have

$$\hat{q}(\omega')=q(\omega';\omega')\geq q(\omega';\omega)\geq q(\omega;\omega)=\hat{q}(\omega).$$

The first inequality holds from the definition of $\geq_Q$ and the second inequality holds since $q(\tilde{\omega};\omega)$ is increasing in $\tilde{\omega}$. Therefore, $\hat{q}(\cdot)$ is increasing and $\hat{q}(\cdot)\in Q$.

Next, I show that $u(\hat{q}(\cdot),\omega)\geq u(q(\cdot;\omega),\omega)$ for all $\omega\in\Omega$. From the definition of u and $\hat{q}(\omega)=q(\omega;\omega)$, we can derive that

$$\begin{aligned}&u(\hat{q}(\cdot),\omega_n)-u(q(\cdot;\omega_n),\omega_n)\\&=\sum_{j=1}^{n-1}\left[v(q(\omega_j;\omega_n),\omega_{j+1})-v(q(\omega_j;\omega_j),\omega_{j+1})-v(q(\omega_j;\omega_n),\omega_j)+v(q(\omega_j;\omega_j),\omega_j)\right].\end{aligned}$$

Then, from $q(\omega_j;\omega_n)\geq q(\omega_j;\omega_j)$ for $j<n$ and the supermodularity of v, we have $u(\hat{q}(\cdot),\omega)\geq u(q(\cdot;\omega),\omega)$. Therefore, $\geq_Q$ satisfies the DDDR condition for u. $\square$

D.3 Proof of Lemma 3

Proof of Lemma 3 . Assume the contrary, that is, $U(\vec{q}\ ;x'')$ does not I-dominate $U(\vec{q}\ ;x')$. Then, there exist $\vec{q}_a,\vec{q}_b\in\mathcal{Q}$ such that $\vec{q}_a\geq_{\mathcal{Q}}\vec{q}_b$ and $U(\vec{q}_a\ ;x')\geq U(\vec{q}\ ;x')$ for all $\vec{q}\in[\vec{q}_b,\vec{q}_a]$, but (i) $U(\vec{q}_a\ ;x'')<U(\vec{q}_b\ ;x'')$; or (ii) $U(\vec{q}_a\ ;x')>U(\vec{q}_b\ ;x')$ and $U(\vec{q}_a\ ;x'')=U(\vec{q}_b\ ;x'')$.

Consider the first case: $U(\vec{q}_a\ ; x'') < U(\vec{q}_b\ x'')$. Let $\vec{q}_c \in \arg\max_{\vec{q}\in[\vec{q}_b,\vec{q}_a]} U(\vec{q}\ ; x'')$. Then, we have $U(\vec{q}_c\ ; x'') > U(\vec{q}_a\ ; x'')$. Now I show that for each $n \le N$, the following inequality holds:

$$S_n \equiv \sum_{i=1}^{n} [u(\vec{q}_c, \omega_i) - u(\vec{q}_a, \omega_i)] \cdot \lambda_F^{x''}(\omega_i) \geq 0. \tag{D.1}$$

Define $\vec{q}_n \in [\vec{q}_c, \vec{q}_a]$ as follows:

$$\vec{q}_n(\omega_i) = \begin{cases} \vec{q}_a(\omega_i), & \text{if } i \le n, \\ \vec{q}_c(\omega_i), & \text{if } i > n. \end{cases}$$

Observe that for any $i \le n$,

$$u(\vec{q}_n(\omega_i), \omega_i) = v(\vec{q}_a(\omega_i), \omega_i) - \sum_{k=1}^{i-1} (v(\vec{q}_a(\omega_k), \omega_{k+1}) - v(\vec{q}_a(\omega_k), \omega_k)) - c(\vec{q}_a(\omega_i)) = u(\vec{q}_a(\omega_i), \omega_i).$$

Next, for any $i > n$, we have

$$\begin{aligned} &u(\vec{q}_n(\omega_i), \omega_i) - u(\vec{q}_c(\omega_i), \omega_i) \\ =&\sum_{k=1}^{n} \left\{ (v(\vec{q}_c(\omega_k), \omega_{k+1}) - v(\vec{q}_a(\omega_k), \omega_{k+1})) - (v(\vec{q}_c(\omega_k), \omega_k) - v(\vec{q}_a(\omega_k), \omega_k)) \right\}. \end{aligned}$$

Since $\vec{q}_c(\omega_k) \ge \vec{q}_a(\omega_k)$ for all $1 \le k \le n$ (from $\vec{q}_a \ge_{\mathcal{Q}} \vec{q}_c$) and v is supermodular, we have $u(\vec{q}_n(\omega_i), \omega_i) \ge u(\vec{q}_c(\omega_i), \omega_i)$. From the optimality of $\hat{q}_c$, we have $U(\vec{q}_c\ ; x'') \ge U(\vec{q}_n\ ; x'')$, which is equivalent to

$$0 \le S_n + \sum_{l=n+1}^{N} [u(\vec{q}_c(\omega_k), \omega_k) - u(\vec{q}_n(\omega_k), \omega_k)]\, \lambda_F^{x''}(\omega_l).$$

By using $u(\vec{q}_n(\omega_l), \omega_l) \ge u(\vec{q}_c(\omega_l), \omega_l)$ for all $l \ge n+1$, we have $S_n \ge 0$.

Observe that

$$U(\vec{q}_c\ ; x') - U(\vec{q}_a\ ; x') = \frac{\lambda_F^{x'}(\omega_N)}{\lambda_F^{x''}(\omega_N)} S_N + \sum_{n=1}^{N-1} \left(\frac{\lambda_F^{x'}(\omega_n)}{\lambda_F^{x''}(\omega_n)} - \frac{\lambda_F^{x'}(\omega_{n+1})}{\lambda_F^{x''}(\omega_{n+1})} \right) \cdot S_n.$$

By the MLRP of F, we have $\lambda_F^{x'}(\omega_n)/\lambda_F^{x''}(\omega_n) \ge \lambda_F^{x'}(\omega_{n+1})/\lambda_F^{x''}(\omega_{n+1})$ for all $1 \le n \le N-1$. Then, from (D.1), we have

$$U(\vec{q}_c\ ; x') - U(\vec{q}_a\ ; x') \ge \frac{\lambda_F^{x'}(\omega_N)}{\lambda_F^{x''}(\omega_N)} S_N = \frac{\lambda_F^{x'}(\omega_N)}{\lambda_F^{x''}(\omega_N)} (U(\vec{q}_c\ ; x'') - U(\vec{q}_a\ ; x'')) > 0,$$

which contradicts the assumption that $\vec{q}_c \in [\vec{q}_b, \vec{q}_a]$ and $U(\vec{q}_a\ ; x') \geq U(\vec{q}\ ; x')$ for all $\vec{q} \in [\vec{q}_b, \vec{q}_a]$.

Consider the second case: $U(\vec{q}_a\ ; x') > U(\vec{q}_b\ ; x')$ and $U(\vec{q}_a\ ; x'') = U(\vec{q}_b\ ; x'')$. If there exists $\vec{q}_c \in [\vec{q}_b, \vec{q}_a]$ such that $U(\vec{q}_c\ ; x'') > U(\vec{q}_a\ ; x'')$, we can use the same argument as in the first case. If not, for all $\vec{q} \in [\vec{q}_b, \vec{q}_a]$, $U(\vec{q}_a\ ; x'') = U(\vec{q}_b\ ; x'') \geq U(\vec{q}\ ; x'')$, i.e., $\vec{q}_b \in \arg\max_{\vec{q} \in [\vec{q}_b, \vec{q}_a]} U(\vec{q}\ ; x'')$. Then, we can use the similar argument as in the first case by substituting $\vec{q}_c$ to $\vec{q}_b$. The only difference is that the last inequality in (D.3) should be replaced to equality, thus we have $U(\vec{q}_b\ ; x') \geq U(\vec{q}_a\ ; x')$. But it contradicts the assumption that $U(\vec{q}_a\ ; x') > U(\vec{q}_b\ ; x')$. Therefore, $U(\vec{q}\ ; x'')$ I-dominates $U(\vec{q}\ ; x')$. $\square$

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