Blackwell Monotonicity: A Marginal Characterization*

Xiaoyu Cheng† Yonggyun Kim‡

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Abstract

We establish necessary and sufficient conditions for Blackwell monotonicity: a more informative experiment has a higher cost. The key conditions are (i) the cost should decrease if a signal is replaced by another signal with a small probability; and (ii) the cost should remain unchanged after relabeling the signals. This characterization is based on a novel geometric characterization of Blackwell information order. Additionally, we provide a method to solve binary-action costly persuasion problems, which is applicable to a wide class of Blackwell monotone costs.

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†Department of Economics, Florida State University, Tallahassee, FL, USA. E-mail: xcheng@fsu.edu.

‡Department of Economics, Florida State University, Tallahassee, FL, USA. E-mail: ykim22@fsu.edu.

Recent developments in economic theory have expanded the scope of decision-making by treating information as a choice variable for economic agents. This approach positions information alongside other key variables in economic models: just as consumers select consumption bundles in consumer theory and producers choose input combinations like labor and capital in producer theory, agents actively decide which information to acquire, taking into account the cost or price of obtaining it.

In traditional consumer and producer frameworks, monotonicity is straightforward—greater consumption increases utility, and more inputs enhance production. Additionally, these relationships have clear marginal characterizations: the marginal utility of consumption and the marginal productivity of inputs like labor and capital are always non-negative.

When it comes to information, a widely accepted criterion of "more informative" is the classical information order introduced by Blackwell (1951, 1953). According to this criterion, a statistical experiment (A) is (Blackwell) more informative than another experiment (B) if and only if B can be replicated by adding noise to A, namely B is a garbling of A. An information cost function satisfies *Blackwell monotonicity* if it assigns higher costs to more informative experiments. This is considered the minimum requirement for plausible information cost functions in the literature.

A marginal characterization of Blackwell monotonicity, however, is not as straightforward as in consumer or producer theory. In this paper, we provide a novel geometric characterization of the Blackwell information order (Lemma 1 and 2), which identifies *signal replacements* as the key marginal operation on experiments that characterizes Blackwell monotonicity.

To illustrate this operation, consider the simplest case with two states and two signals, and the following statistical tests, A and B:

$$
\text{Test A} = \quad \begin{array}{cc|cc} & & \multicolumn{2}{c}{\text{signal}} \\ & & n & p \\ \hline \text{state} & - & 80\% & 20\% \\ & + & 20\% & 80\% \end{array}, \qquad \text{Test B} = \quad \begin{array}{cc|cc} & & \multicolumn{2}{c}{\text{signal}} \\ & & n & p \\ \hline \text{state} & - & 60\% & 40\% \\ & + & 15\% & 85\% \end{array}.
$$

In words, Test A has a 20% probability of either type 1 or type 2 error, while Test B has a slightly lower type 1 error rate (15%) but a much higher type 2 error rate (40%). While they are not directly comparable in error rates, Test A is actually more informative than Test B. To see this, notice that Test B can be replicated by the following process after performing

Test A: (i) if the result is 'p', report 'p'; (ii) if the result is 'n', flip a coin twice, report 'p' only if both flips are heads; otherwise report 'n'. In other words, Test B can be obtained as a garbling of Test A by replacing the negative signal with a positive signal 25% of the time, while retaining the original signal in all other cases.

We define this type of garbling a *signal replacement*: replacing one signal with another signal for some probability, while keeping the generation of other signals unchanged. And this change can be made arbitrarily small by letting the probability of replacement approach zero. As this marginal change of an experiment always decreases its informativeness, any Blackwell monotone information cost should decrease in these directions. In other words, the marginal cost of replacing a signal with another should be non-positive. We refer to this condition as *cost reduction in signal replacement*.

On the other hand, if we simply relabel the signals, e.g., switching n and p, the informativeness of the experiments remains unchanged. Therefore, the cost should remain the same under Blackwell monotonicity. We refer to this condition as *permutation invariance*.

Our main result shows that these two conditions are not only necessary but also sufficient for Blackwell monotonicity under certain circumstances. First, Theorem 1 establishes this equivalence when information cost functions are defined over experiments with two signals. Next, when there are more than two signals, we assume that costs are quasiconvex, i.e., the set of tests with costs below a certain level forms a convex set. In this case, permutation invariance and cost reduction in signal replacement remain necessary and sufficient for Blackwell monotonicity (Theorem 2).

As an application of our findings, we consider the costly persuasion problem proposed by Gentzkow and Kamenica (2014). They extend their celebrated Bayesian persuasion model (Kamenica and Gentzkow, 2011) by assuming that it is costly for the sender to commit to a persuasion policy, which takes the form of statistical experiments. To apply the concavification technique, they focus on cases where the information cost function is posterior separable, and a convention followed in the literature.

We propose a new method to solve binary-action costly persuasion problems (without relying on concavification) that can be applied to a wider class of Blackwell monotone information costs. Our technique takes a two-step approach, similar to that used in producer theory. Specifically, we first solve the cost minimization problem while fixing the sender's material utility, then solve the utility maximization problem incorporating the derived indirect cost. We show Blackwell monotone costs, in general, allow the posteriors under the optimal persuasion to change with the prior, while it is never the case when restricting attention to uniformly posterior separable costs.

Related Literature Our paper contributes to the study of Blackwell information order (Blackwell, 1951, 1953), the most fundamental concept in information economics. Specifically, our geometric characterization of Blackwell's information order is new and may be of independent interest.[1] Based on this characterization, we derive necessary and sufficient conditions for an information cost function (or any cardinal representation of experiments) to be monotone in the Blackwell order.

Blackwell monotonicity is essential in studying information costs. When formulating information costs, there are mainly two approaches: posterior-based costs (defined over distributions of posteriors); and experiment-based costs (defined over statistical experiments). We refer to Gentzkow and Kamenica (2014), Mensch (2018), Morris and Strack (2019), Denti et al. (2022a) and Bloedel and Zhong (2024) for very comprehensive discussions on their relations. Blackwell monotonicity of information costs formulated under the posterior-based approach has been established in de Oliveira et al. (2017), Denti et al. (2022a,b), and Ravid et al. (2022), among many others.[2] Specifically, Blackwell monotonicity of such costs (with a concave measure of uncertainty, e.g., entropy as in Sims (2003)) is implied by one of Blackwell's sufficient conditions related to the convex order.[3] Our result takes a different route—through the garbling representation of Blackwell's information order, and thus is more easily applicable to costs defined over experiments.

The existing literature on experiment-based costs usually considers Blackwell monotonicity as one of a set of axioms/properties that a cost function should satisfy and derives its representations. Along this direction, Mensch (2018), Denti et al. (2022b), Hébert and Woodford (2021), Pomatto et al. (2023), Baker (2023), and Bloedel and Zhong (2024) identify various classes of experiment-based costs that satisfy Blackwell monotonicity. Our paper contributes to this literature by isolating Blackwell monotonicity (with minimal technical assumptions) as the central property of interest and providing a direct characterization of Blackwell monotone costs. On a practical level, our results provide a simple way

[1] To the best of our knowledge, the closest geometric characterization to ours is the *zonotope order* in Bertschinger and Rauh (2014). The zonotope order coincides with the Blackwell order only if either the state or signal is binary. Our characterization holds for all finite experiments.

[2] In particular, Claim 1 of Ravid et al. (2022) provides a differential characterization based on directional derivatives, similar to ours in the techniques used. However, due to the difference in formulations, their directional derivatives in the space of random posteriors do not easily translate to directional derivatives in the space of experiments.

[3] See, for example, the discussion of Assumption 1 in Gentzkow and Kamenica (2014) and Lemma 6 of Denti et al. (2022b).

to verify whether a given cost function is Blackwell monotone, particularly useful for costs outside the classes identified in the literature, such as when it is not posterior separable.

Finally, it is well-understood that, for the purpose of rationalizing a decision-maker's optimal choice behavior under costly information, assuming Blackwell monotonicity of costs is without loss of generality (Caplin and Dean, 2015; de Oliveira et al., 2017). Specifically, starting with an arbitrary primitive cost function, the indirect cost function that arises as a consequence of the cost minimization problem is Blackwell monotone. Despite this, postulating a cost function and then studies the decision-maker's choice behavior requires deriving the indirect cost function as an intermediate step. Our paper provides a way to get around this problem by identifying a direct characterization of Blackwell monotone costs, in which case the indirect cost function equals the original cost function.[4] Alternatively, many recent papers have developed novel methods to study the relationship between the indirect and direct costs, for example, Lipnowski and Ravid (2023), Bloedel and Zhong (2024), and Bloedel et al. (2024).

1 Preliminaries

(Finite) Statistical Experiments Let $\Omega = \{\omega_1, \cdots, \omega_n\}$ be a finite set of states. Fix a finite set of signals $\mathcal{S} = \{s_1, \cdots, s_m\}$. A *statistical experiment* $f : \Omega \to \Delta(\mathcal{S})$ is represented by the $n \times m$ matrix

$$f = \begin{bmatrix} f_1^1 & \cdots & f_1^m \\ \vdots & \ddots & \vdots \\ f_n^1 & \cdots & f_n^m \end{bmatrix},$$

where $f_i^j = f(s_j|\omega_i)$ is the probability of generating signal s_j in state ω_i. Let $f^j = [f_1^j, \cdots, f_n^j]^\intercal \in \mathbb{R}_+^n$ denote the j-th column vector of f for $j = 1, \cdots m$. Using this notation, we can rewrite

$$f = [f^1, \cdots, f^m] \in \mathbb{R}_+^{n \times m}.$$

Notice that $\sum_{j=1}^m f^j = \mathbf{1}$ where $\mathbf{1} = [1, \cdots, 1]^\intercal$. Let $\mathcal{E}_m$ denote the set of all experiments that generate at most m signals. Any experiment with fewer than m signals can be represented by an $n \times m$ matrix by assigning zero vectors to unrealized signals. Thus, we can

[4]We thank an anonymous referee for suggesting this insightful explanation.

embed $\mathcal{E}_m$ into $\mathbb{R}^{n\times m}$ equipped with the Euclidean topology.

Information Costs We define an *information cost function* as $C : \mathcal{E}_m \to \mathbb{R}_+$, i.e., defined over the set of experiments with a fixed number of possible signals. Let $\mathcal{C}_m$ denote the set of all such functions. Under this formulation, each $C \in \mathcal{C}_m$ is a mapping on Euclidean space which facilitates analysis.

Blackwell Informativeness and Monotonicity An experiment f is *Blackwell more informative* than another experiment g, denoted by $f \succeq_B g$, if there exists a *stochastic matrix* M (i.e., $M_{ij} \geq 0$ and $\sum_j M_{ij} = 1$ for all i) such that $g = fM$. This matrix M is also called a *garbling matrix*. f and g are said to be *equally informative* and denoted by $f \simeq_B g$ if both $f \succeq_B g$ and $g \succeq_B f$ hold. When both f and g are in $\mathcal{E}_m$, any potential garbling matrix is an $m \times m$ square stochastic matrix. Let $\mathcal{M}_m$ denote the set of all such stochastic matrices. An information cost function $C \in \mathcal{C}_m$ is **Blackwell monotone** if for all $f, g \in \mathcal{E}_m$, $C(f) \geq C(g)$ whenever $f \succeq_B g$.

For any $C \in \mathcal{C}_m$, let $S_C(f) = \{g \in \mathcal{E}_m : C(f) \geq C(g)\}$ denote its sublevel set at $f \in \mathcal{E}_m$. In addition, let $S_B(f) = \{g \in \mathcal{E}_m : f \succeq_B g\}$ denote the sublevel set under the Blackwell information order. By definition, Blackwell monotonicity is equivalent to $S_C(f) \supseteq S_B(f)$ for all $f \in \mathcal{E}_m$.

Permutation Invariance A *permutation matrix* P is a stochastic matrix with exactly one non-zero entry in each row and each column. Observe that when P is a permutation matrix, its inverse, P^{-1}, is also a permutation matrix that restores the original experiment. Since both P and P^{-1} are stochastic matrices, it follows that $f \succeq_B fP \succeq_B fPP^{-1} = f$, implying $f \simeq_B fP$. Intuitively, permuting an experiment simply relabels the signals, preserving the information content. Therefore, for any Blackwell monotone cost function C, we have $C(f) = C(fP)$. We refer to this property as *permutation invariance* and it serves as a necessary condition for Blackwell monotonicity.[5]

Cost Reduction in Signal Replacement For each $C \in \mathcal{C}_m$, let $D^+C(f; h)$ denote its (one-sided) directional derivative at $f \in \mathcal{E}_m$ in the direction of $h \in \mathbb{R}^{n\times m}$, if the following

[5]The Bregman divergence cost, introduced in Fosgerau et al. (2020), provides an example of an information cost function in the literature that violates permutation invariance, and thus is not Blackwell monotone.

limit exists:

$$D^+C(f;h) \equiv \lim_{\epsilon \downarrow 0} \frac{C(f+\epsilon h) - C(f)}{\epsilon}.$$

For example, consider a matrix with f^j in the k-th column, $-f^j$ in the j-th column and zeros elsewhere, denoted by $f^{j\to k} \in \mathbb{R}^{n\times m}$:

$$f^{j\to k} \equiv \begin{bmatrix} 0 & \cdots & \underbrace{-f^j}_{j\text{-th column}} & \cdots & 0 & \cdots & \underbrace{f^j}_{k\text{-th column}} & \cdots & 0 \end{bmatrix}.$$

Observe that, for all $\epsilon \in [0,1]$, $f + \epsilon f^{j\to k} \in \mathcal{E}_m$ and represents the experiment obtained by replacing signal j with signal k with probability ϵ, i.e., it is a garbling of f. Therefore, $D^+C(f; f^{j\to k})$ can be interpreted as the marginal cost of replacing signal j with signal k. For any Blackwell monotone cost function C, it satisfies

$$D^+C(f; f^{j\to k}) \leq 0, \quad \forall j \neq k, \text{ whenever exists.} \tag{1}$$

We refer to (1) as *cost reduction in signal replacement* and it also serves as a necessary condition for Blackwell monotonicity.

Functional Assumption The weakest continuity assumption required for our results is absolute continuity. Say that $C \in \mathcal{C}_m$ is *absolutely continuous* if for all $f, g \in \mathcal{E}_m$ and $t \in [0,1]$, the function $\varphi(t) = C(f + t(g - f))$ is absolutely continuous in t over $[0,1]$.[6] Equivalently, it says that $\varphi(\cdot)$ is differentiable almost everywhere and the Fundamental Theorem of Calculus (FTC) holds, i.e.,

$$C(g) - C(f) = \varphi(1) - \varphi(0) = \int_0^1 \varphi'(t)dt = \int_0^1 D^+C(tg + (1-t)f; g - f)dt. \tag{2}$$

Notice that a sufficient condition for absolute continuity is Lipschitz continuity over $\mathcal{E}_m$.

2 Blackwell Monotonicity for Binary Experiments

In this section, we focus on experiments with two signals—hereafter referred to as binary experiments—and show that permutation invariance and cost reduction in signal replace-

[6]There are multiple generalizations of absolute continuity from $\mathbb{R}$ to $\mathbb{R}^n$ emphasizing different aspects. See Dymond et al. (2017) for a reference. We adopt the generalization which requires the restriction of C to any line segment is absolutely continuous, corresponding to their definition of 0-absolute continuity.

ment are not only necessary but also sufficient for Blackwell monotonicity.

For any $f \in \mathcal{E}_2$, since $f^1 + f^2 = \mathbf{1}$, f is uniquely identified by the vector f^1. For simplicity, we use the column vector to denote a binary experiment as $f = [f_1, \cdots, f_n]^\intercal \in [0,1]^n$. Similarly, for any $C \in \mathcal{C}_2$, we define $C : [0,1]^n \to \mathbb{R}_+$.

Observe that the unique permutation for f is $\mathbf{1} - f$, thus, permutation invariance is equivalent to $C(f) = C(\mathbf{1} - f)$. Next, note that $f^{1\to 2}$ and $f^{2\to 1}$ correspond to $-f$ and $\mathbf{1} - f$ respectively. Therefore, cost reduction in signal replacement can be rewritten as

$$D^+C(f; -f) \leq 0 \text{ and } D^+C(f; \mathbf{1} - f) \leq 0, \text{whenever exists.} \tag{3}$$

2.1 Parallelogram Hull

To establish necessary and sufficient conditions for Blackwell monotonicity, we begin by characterizing the sublevel set for binary experiments. For any $f, g \in \mathcal{E}_2$ with $f \succeq_B g$, there exists $M \in \mathcal{M}_2$ such that $[g, \mathbf{1} - g] = [f, \mathbf{1} - f]M$. For a stochastic matrix $M \in \mathcal{M}_2$, we can write it as, for some $(a, b) \in [0,1]^2$, that

$$M = \begin{bmatrix} a & 1-a \\ b & 1-b \end{bmatrix},$$

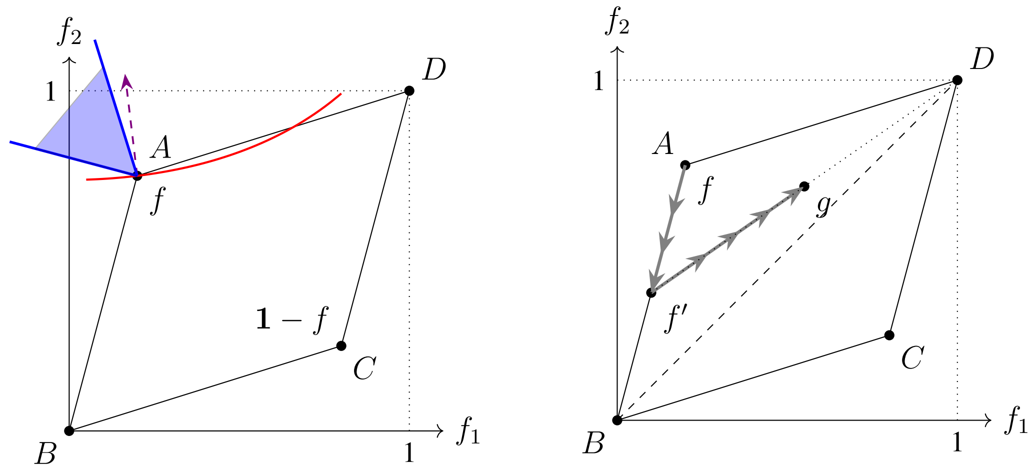
which implies $g = af + b(\mathbf{1} - f)$ and establishes the following lemma.

Lemma 1. *For any $f, g \in \mathcal{E}_2$, $f \succeq_B g$ if and only if g is in the* ***parallelogram hull*** *of f and $\mathbf{1} - f$, defined by*

$$\textit{PARL}(f, \mathbf{1} - f) \equiv \{af + b(\mathbf{1} - f) : a, b \in [0,1]\}.$$

In other words, $S_B(f) = \textit{PARL}(f, \mathbf{1} - f)$.

A parallelogram hull in the case of binary states, i.e., $n = 2$, is depicted by the parallelogram (ABCD) in Figure 1a. Specifically, when f and g are experiments with binary signals and states, f is Blackwell more informative than g if and only if g lies in the parallelogram (ABCD).



(a) A parallelogram hull and its polar cone

(b) A decreasing path from f to g

Figure 1: A Graphical Illustration with Binary States

2.2 The Characterization

Our first main result shows that for binary experiments, Blackwell monotonicity is equivalent to permutation invariance and cost reduction in signal replacement under absolute continuity. A sketch of the proof follows.

Theorem 1. *Suppose $C \in \mathcal{C}_2$ is absolutely continuous. C is Blackwell monotone if and only if C satisfies permutation invariance and cost reduction in signal replacement.*

Necessity The necessity of permutation invariance and cost reduction in signal replacement was established in Section 1. However, the parallelogram hull offers a geometrical interpretation of cost reduction in signal replacement. As depicted in Figure 1a, the parallelogram delineates two extreme directions of decreasing informativeness: $\overrightarrow{AB}$ and $\overrightarrow{AD}$. Observe that the vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$ correspond to $-f$ and $\mathbf{1}-f$. These vectors represent the processes of replacing s_1 with s_2 with a small probability while maintaining s_2, and replacing s_2 with s_1 with a small probability while maintaining s_1. If an experiment f moves in the direction of either $\overrightarrow{AB}$ or $\overrightarrow{AD}$, the Blackwell informativeness decreases, which in turn implies a reduction in cost for any Blackwell monotone cost.

Sufficiency Since cost reduction in signal replacement, described in (3), is only a local property, sufficiency requires additional regularity conditions on the cost function. Permutation invariance is necessarily needed and absolute continuity ensures the Fundamental

Theorem of Calculus (FTC) applies. Consider any experiment g lying inside the parallelogram $ABCD$, i.e., $f \succeq_B g$. If g is above the line BD, as illustrated in Figure 1b, we can find a two-segment path from f to g, which moves only in the extreme directions required by (3): moving from f in the direction of $-f$ to reach f' and then moving from f' in the direction of $\mathbf{1} - f'$ to reach g. Thus, applying (3) implies the directional derivatives are non-positive along this path, and applying (2) leads to $C(g) \leq C(f)$.

If g lies below the line BD, its permutation, namely gP, lies above the line BD and has the same cost as g, following from permutation invariance. Then, the same argument applies to gP implying $C(g) = C(gP) \leq C(f)$. Lemma A.1 formally shows this argument. □

Remark 1. When C is differentiable at f, $D^+C(f; h)$ is linear in h and equals $\langle \nabla C(f), h \rangle$ where $\nabla C(f)$ represents the gradient of the cost function C, and $\langle \cdot, \cdot \rangle$ denotes the inner product. Using the bilinearity of the inner product, note that (3) is equivalent to

$$\langle \nabla C(f), g - f \rangle \leq 0, \quad \forall g \in \text{PARL}(f, \mathbf{1} - f) = S_B(f).$$

Geometrically, this says that $\nabla C(f)$ lies in the *polar cone* of $S_B(f)$ at f, depicted in Figure 1a by the blue cone. In other words, when C is differentiable at f, Blackwell monotonicity imposes a constraint on the feasible directions of its gradient at f.

For a graphical demonstration, in Figure 1a, we draw a curve passing through the point A to illustrate a potential isocost curve, indicating the same information cost of a smooth cost function. As the gradient of such a function is tangent to its isocost curve, the gradient of this cost function (the purple arrow) lies outside the polar cone of $S_B(f)$ and thus violates Blackwell monotonicity. This is confirmed by noticing that the cost increases in the direction of $\overrightarrow{AD}$ near A.

Remark 2. In Online Appendix OA.1, we show that replacing the inequalities in (3) with strict inequalities provides a sufficient condition for strict Blackwell monotonicity.

2.3 A Further Characterization with Binary States

Consider the binary-binary case ($n = m = 2$) and restrict attention to the following set of experiments:[7,8]

$$\hat{\mathcal{E}}_2 \equiv \{(f_1, f_2) \ : 0 \le f_1 \le f_2 \le 1\}.$$

When C is differentiable and $\frac{\partial C}{\partial f_2} \neq 0$, (3) can be rewritten as follows:

$$\underbrace{\frac{f_2}{f_1}}_{\text{the slope of } \overline{AB}} \ge \underbrace{-\frac{\partial C/\partial f_1}{\partial C/\partial f_2}}_{\substack{\text{the slope of} \\ \text{the isocost curve}}} \ge \underbrace{\frac{1-f_2}{1-f_1}}_{\text{the slope of } \overline{AD}}.^{9} \tag{4}$$

The slope of the isocost curve can be considered as the *marginal rate of information transformation (MRIT)*. Thus, this inequality says that the MRIT of a Blackwell monotone cost function should fall between the two likelihood ratios provided by the experiment.

Example 1. Consider two information cost functions defined over $\hat{\mathcal{E}}_2$:

$$C_1(f_1, f_2) \equiv (f_2 - f_1)^2, \qquad C_2(f_1, f_2) \equiv f_2 - 2f_1.$$

Notice that

$$MRIT_1 \equiv -\frac{\partial C_1/\partial f_1}{\partial C_1/\partial f_2} = 1, \qquad MRIT_2 \equiv -\frac{\partial C_2/\partial f_1}{\partial C_2/\partial f_2} = 2.$$

Then, for C_1, (4) holds all $(f_1, f_2) \in \hat{\mathcal{E}}_2$, but not so for C_2, e.g., when $f_1 = .5$ and $f_2 = .6$. Therefore, we can conclude that C_1 is Blackwell monotone, but C_2 is not.

3 Blackwell Monotonicity under Quasiconvexity

In this section, we examine the properties of Blackwell monotonicity under the assumption that cost functions are quasiconvex. This assumption is motivated by two key reasons: (i) our characterization of Blackwell monotonicity under binary experiments does not extend to more general cases without this restriction, and (ii) quasiconvexity is a natural and

[7]By applying permutation invariance, the cost for the other half-piece will be properly defined.

[8]We provide another characterization of Blackwell monotonicity for this case using likelihood ratios in Online Appendix OA.2. This characterization does not require any continuity assumption on cost functions.

[9]With some algebra, we can show that $f_2 \ge f_1$ and (3) imply $\frac{\partial C}{\partial f_2} \ge 0 \ge \frac{\partial C}{\partial f_1}$.

reasonable property for cost functions to satisfy.

3.1 Quasiconvexity

Let $C \in \mathcal{C}_m$ be defined as *quasiconvex* if for any $f, g \in \mathcal{E}_m$ and $\lambda \in [0,1]$,

$$C(\lambda f + (1-\lambda)g) \leq \max\{C(f), C(g)\}.$$

In other words, a mixture of two experiments cannot be more costly than both of them.

By using a replication argument, we can provide a no-arbitrage justification for imposing quasiconvexity on information cost functions. A mixture of two experiments, $\lambda f + (1-\lambda)g$, can be replicated by running experiment f with probability λ, and experiment g with probability $1-\lambda$, then reporting the realized signal without indicating which experiment was conducted. Thus, if the cost of $\lambda f + (1-\lambda)g$ exceeds $\max\{C(f), C(g)\}$, arbitrage can be achieved through this replication no matter which experiment is conducted. An important geometric property of quasiconvexity is that the sublevel set of an experiment f, $S_C(f)$, is a convex set.

Many existing papers (e.g., de Oliveira et al. (2017)) argue that the cost of $\lambda f + (1-\lambda)g$ should not exceed the weighted cost $\lambda C(f) + (1-\lambda)C(g)$, which implies the convexity of C. Since quasiconvexity is a weaker assumption, our characterization continues to hold under convexity.

3.2 Blackwell Monotonicity for Finite Experiments

Geometric Characterization We now characterize Blackwell monotonicity for $\mathcal{E}_m$ with arbitrary m. We begin by extending the parallelogram hull characterization of the Blackwell order from Lemma 1. According to this characterization, for a pair of binary experiments (f, g) with $f \succeq_B g$, $g - f$ can be expressed as a positive linear combination of $f^{1\to 2}$ and $f^{2\to 1}$. For arbitrary m signals, we establish a similar characterization involving $f^{j\to k}$ for all $j \neq k$.

Lemma 2. *For any $f, g \in \mathcal{E}_m$, $f \succeq_B g$ if and only if*

$$g - f \in \left\{ \sum_{j=1}^{m} \lambda_j h_j : \lambda_j \in [0,1], h_j \in co(\{f^{j\to k} : k \neq j\}), \forall j \right\}, \tag{5}$$

where $co(\cdot)$ denotes the convex hull. In other words,

$$S_B(f) = \left\{ f + \sum_{j=1}^{m} \lambda_j h_j : \lambda_j \in [0,1], h_j \in co(\{f^{j\to k} : k \neq j\}), \forall j \right\}.$$

Necessity of cost reduction in signal replacement for Blackwell monotonicity can be reestablished by this characterization: for any $\epsilon \in [0,1]$ and $j \neq k$, $f + \epsilon f^{j\to k} \in \mathcal{E}_m$ is less Blackwell informative than f, thus, any Blackwell monotone cost should decrease in the direction of $f^{j\to k}$.

Challenges in Establishing Sufficiency A key step in proving the sufficiency for binary experiments is constructing a decreasing path connecting any $f \succeq_B g$. However, when $m \geq 3$, such a path within $\mathcal{E}_m$ may not exist, as demonstrated below—the proof is in the Online Appendix.

Proposition 1. *Suppose that $n = m = 3$ and let*

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \succeq_B g = \begin{bmatrix} 4/5 & 1/5 & 0 \\ 0 & 4/5 & 1/5 \\ 1/5 & 0 & 4/5 \end{bmatrix} \in \mathcal{E}_3.$$

If $f \in \mathcal{E}_3$ and $f \succeq_B g$, then f is a permutation of I_3 or g.

Proposition 1 suggests that there is no continuous path in $\mathcal{E}_3$ connecting I_3 and g along which Blackwell informativeness decreases. Because if such a path existed, there would have to be an experiment, other than permutations of I_3 or g, that is more informative than g but less informative than I_3, which is impossible according to the proposition.

Establishing Sufficiency under Quasiconvexity We next show that the above issue can be addressed by imposing quasiconvexity. Observe that $g = \frac{4}{5}I_3 + \frac{1}{5}I_3P$ for some permutation matrix P. When C is quasiconvex, $C(g) \leq \max\{C(I_3), C(I_3P)\} = C(I_3)$. Our next result shows that, with quasiconvexity, the same type of necessary and sufficient condition as in Theorem 1 can be established for all finite experiments.

Theorem 2. *Suppose $C \in \mathcal{C}_m$ is absolutely continuous and quasiconvex. Then, C is Blackwell monotone if and only if C satisfies permutation invariance and cost reduction in signal replacement.*

In the proof of Theorem 2, the key step is to show that any extreme point of $S_B(f)$ is either a permutation of f or can be reached from f via a sequence of segments in the directions of $f^{j\to k}$ (Lemma B.2). Once this is established, similar to the proof of sufficiency in Theorem 1, applying FTC along these segments shows that all extreme points of $S_B(f)$ are contained in $S_C(f)$. Finally, since $S_C(f)$ is a convex set due to quasiconvexity, the entire set $S_B(f)$ must also lie within $S_C(f)$, thereby proving that C is Blackwell monotone.

Remark 3. Given Theorem 2, checking Blackwell monotonicity over non-binary experiments requires one more step: verifying quasiconvexity of C. It is worth noting that when C is twice differentiable, quasiconvexity can be verified if the determinant of every order of its bordered Hessian matrices are non-positive, similar to checking convexity. See Arrow and Enthoven (1961) and also Proposition 3.4.4 in Osborne (2016) for references.

3.3 Likelihood Separable Costs

A widely studied class of information costs is *likelihood separable* costs, defined in Denti et al. (2022b) as those for which there exists an absolutely continuous and sublinear function $\psi : [0,1]^n \to \mathbb{R}_+$ such that, for any $f \in \mathcal{E}_m$ and any m,

$$C(f) = \sum_{j=1}^{m} \psi(f^j) - \psi(\mathbf{1}).$$

Likelihood separable costs encompass well-known costs such as entropy costs (Sims, 2003) and log-likelihood costs (Pomatto et al., 2023), and also admit a posterior separable representation. The latter observation can be used to establish Blackwell monotonicity of likelihood separable costs through the posterior characterization of Blackwell monotonicity. We show next that applying Theorem 2 provides a direct and simpler argument.[10]

First, sublinearity of ψ implies convexity of C. Permutation invariance holds by definition. Thus, it only remains to verify (1). For any $f \in \mathcal{E}_m$, any $j \neq k$, and $\epsilon > 0$, we have

$$\begin{aligned} C(f + \epsilon f^{j\to k}) - C(f) &= \psi(f^k + \epsilon f^j) - \psi(f^k) + \psi((1-\epsilon)f^j) - \psi(f^j) \\ &\leq \epsilon\psi(f^j) - \epsilon\psi(f^j) = 0, \end{aligned}$$

[10] Baker (2023) provides an alternative approach to establish Blackwell monotonicity for likelihood separable costs (Proposition 1). His proof involves a step of embedding an experiment to a higher-dimensional space, which is unnecessary in our argument.

where the inequality follows from sublinearity of ψ. As this holds for all $\epsilon > 0$, it follows that $D^+C(f; f^{j\to k}) \leq 0$ whenever exists, thus establishing Blackwell monotonicity by Theorem 2.

4 Application: Costly Persuasion

Consider a generalized prosecutor-judge problem with costly information provision, as studied in Gentzkow and Kamenica (2014). The judge (Receiver) chooses between two actions: acquitting ($a = 0$) or convicting ($a = 1$). The defendant's state is drawn from a set of possible states, $\Omega = \{\omega_1, \cdots, \omega_n\}$, with a prior belief $p \equiv (p_1, \cdots, p_n) \in \Delta(\Omega)$. The payoff of the prosecutor (Sender) is state-independent and is given by $u_S(a) = a$, whereas the judge's payoff depends on the state and is given by $u_R(a, \omega_i) = v_i \cdot a$, where $v_1 < \cdots < v_k < 0 < v_{k+1} < \cdots < v_n$ for some $k \in \{1, \cdots, n-1\}$.

Since the judge has only two possible actions, it is without loss of generality to focus on binary experiments, where the signal recommends either acquitting or convicting, using the standard argument with Blackwell monotonicity.[11] Let $f = [f_1, \cdots, f_n]^\intercal \in [0,1]^n$ denote the information structure to which the prosecutor commits, where f_i is the probability of recommending convicting in state ω_i. Assume that the prosecutor's cost function of committing to information, $C : [0,1]^n \to \mathbb{R}_+$, is lower semi-continuous: $\liminf_{f\to f_0} C(f) \geq C(f_0)$ for all $f_0 \in [0,1]^n$.

Observe that the probability of convicting is $\sum_{i=1}^n p_i \cdot f_i$, and the judge follows the recommendation if and only if the following obedience conditions hold:

$$\sum_{i=1}^n v_i \cdot p_i \cdot f_i \;\geq\; 0 \;\geq\; \sum_{i=1}^n v_i \cdot p_i \cdot (1 - f_i). \tag{OC}$$

Therefore, the prosecutor's problem is

$$\max_{f\in[0,1]^n} \; \sum_{i=1}^n p_i \cdot f_i - C(f) \tag{PP}$$

subject to (OC). The following lemma shows the existence of a solution to this problem which follows directly from the Weierstrass theorem.

[11] If s_i and s_j induce the same action by the judge, merging these two signals will induce the same action as the posteriors from both signals are either above or below the threshold. Since merging two signals makes the experiment less Blackwell informative, it incurs a lower cost under Blackwell monotonicity.

Lemma 3. *Suppose that C is lower semi-continuous. Then, there exists a solution to the problem* (PP) *subject to* (OC).

First, consider the case where $\sum_{i=1}^n v_i \cdot p_i \geq 0$. Then, $f = \mathbf{1}$ satisfies (OC). Note that $f = \mathbf{1}$ maximizes $\sum_{i=1}^n p_i \cdot f_i$, and minimizes $C(f)$ by Blackwell monotonicity. In other words, the prosecutor always recommends convicting, and the judge convicts as it is optimal under the prior belief, no persuasion needed.

Next, suppose $\sum_{i=1}^n v_i \cdot p_i < 0$, i.e., persuasion is required to induce the judge to convict. We solve the problem in two stages: (i) deriving the indirect cost function associated with the probability of conviction by solving a cost minimization problem; and (ii) solving the main problem of determining the optimal conviction probability using the indirect cost.

Let $\overline{w}$ be the maximum possible probability of conviction as follows

$$\overline{w} \equiv \max_{f \in [0,1]^n} p_i \cdot f_i \quad \text{s.to.} \quad \text{(OC)}.$$

Also note that the minimum probability of conviction is 0 obtained by setting $f = \mathbf{0}$. Consider the cost minimization problem given conviction probability $w \in [0, \overline{w}]$:

$$\min_{f \in [0,1]^n} C(f) \tag{6}$$

subject to (OC) and $\sum_{i=1}^n p_i \cdot f_i = w$. The following lemma shows that the first inequality of (OC) binds for any Blackwell monotone information costs. Its proof applies our geometric characterization.

Lemma 4. *Suppose that $\sum_{i=1}^n v_i \cdot p_i < 0$. If f satisfies $\sum_{i=1}^n p_i \cdot f_i = w$ and $\sum_{i=1}^n v_i \cdot p_i \cdot f_i > 0$, there exists $\tilde{f}$ such that $f \succeq_B \tilde{f}$, $\sum_{i=1}^n p_i \cdot \tilde{f}_i = w$ and $\sum_{i=1}^n v_i \cdot p_i \cdot \tilde{f}_i = 0$.*[12] *Therefore, if C is Blackwell monotone, $C(f) \geq C(\tilde{f})$, implying that the left hand side of* (OC) *should bind at the minimum.*

Now define the indirect cost function for probabilities of conviction as follows:

$$\tilde{C}(w) \equiv \min_{f \in [0,1]^n} C(f) \quad \text{s.to.} \quad \sum_{i=1}^n p_i \cdot f_i = w \ \text{ and } \ \sum_{i=1}^n v_i \cdot p_i \cdot f_i = 0. \tag{7}$$

By lower semi-continuity, this optimization problem always has a solution, which we de-

[12]Note that $\sum_{i=1}^n v_i \cdot p_i < 0$ and $\sum_{i=1}^n v_i \cdot p_i \cdot \tilde{f}_i = 0$ imply $0 > \sum_{i=1}^n v_i \cdot p_i \cdot (1 - \tilde{f}_i)$, i.e., (OC) holds.

note by $f^*(w)$.[13] Then, the prosecutor's problem can be rewritten as follows:

$$\max_{w\in[0,\overline{w}]} w - \tilde{C}(w). \tag{PP$'$}$$

Proposition 2. *Suppose that C is lower semi-continuous and Blackwell monotone. Let w^* be the solution of* (PP′)*. Then, $f^*(w^*)$ is a solution of* (PP) *subject to* (OC)*.*

This proposition shows that the solution of the binary-action costly persuasion problem can be obtained using the two-step approach. In Online Appendix OA.5, we study a two-state example with the quadratic cost in Example 1. The quadratic cost is not posterior separable and thus not uniformly posterior separable (UPS), so a concavification argument does not apply. More importantly, we show the optimal persuasion scheme under the quadratic cost is qualitatively different from that under UPS costs—the receiver's posterior beliefs under the optiaml persuasion changes with the prior while they are independent of the prior whenever information is provided under UPS costs. This observation offers a potential explanation when persuasions do not align with the predictions under UPS costs.

5 Conclusion

Our work deepens the understanding of Blackwell monotonicity, a fundamental requirement for plausible information cost functions. Specifically, we highlight the importance of signal replacement as offering a marginal characterization of Blackwell monotonicity. While additional properties or axioms may be necessary to characterize appropriate cost functions in specific economic settings, such extensions should be grounded in the principles of Blackwell monotonicity. We see our results as a stepping stone in this endeavor, providing a framework upon which further refinements can be developed.

[13] In the binary state case, two constraints in (7) uniquely determines the experiment, $f^*(w)$, so there is no need to solve the minimization problem explicitly. Thus, we can directly set $\tilde{C}(w) = C(f^*(w))$.

Appendix

A Proofs for Section 2

Lemma A.1. *For any $f, g \in \mathcal{E}_2$ such that $f \succeq_B g$, there exists $1 \geq a \geq b \geq 0$ such that either*

$$g = af + b(\mathbf{1} - f) \quad \textit{or} \quad \mathbf{1} - g = af + b(\mathbf{1} - f). \tag{A.1}$$

Let g satisfy the first equation of (A.1) and $f' = \frac{a-b}{1-b} f$.[14] *Then, for all $\lambda \in [0, 1]$,*

$$f \succeq_B (1 - \lambda)f + \lambda f' \succeq_B f', \textit{ and} \tag{A.2}$$

$$f' \succeq_B (1 - \lambda)f' + \lambda g \succeq_B g. \tag{A.3}$$

Proof of Lemma A.1. Recall that $f \succeq_B g$ implies that there exist $(a, b) \in [0, 1]^2$ such that $g = af + b(\mathbf{1} - f)$. If $a \geq b$, the first equation of (A.1) holds. If $a < b$, notice $\mathbf{1} - g = a'f + b'(\mathbf{1} - f)$ for $a' = 1 - a > 1 - b = b'$.

When $b = 1$, $a = 1$ and $g = f + (\mathbf{1} - f) = \mathbf{1} = f'$. (A.3) holds. Notice that $(1 - \lambda)f + \lambda\mathbf{1} = 1 \cdot f + \lambda(\mathbf{1} - f) \in \text{PARL}(f, \mathbf{1} - f)$, thus, $f \succeq_B (1 - \lambda)f + \lambda\mathbf{1}$. Similarly, $(1 - \lambda)f + \lambda\mathbf{1} \succeq_B \mathbf{1}$. (A.2) holds.

When $b < 1$, $\frac{a-b}{1-b} \in [0, 1]$ and $f \succeq_B f'$. For any $\lambda \in [0, 1]$, $f \succeq_B \lambda f + (1 - \lambda)f'$ follows from the convexity of $\text{PARL}(f, \mathbf{1} - f)$. Next, notice that

$$f' = \frac{\frac{a-b}{1-b}}{1 - \lambda + \lambda\frac{a-b}{1-b}}((1 - \lambda)f + \lambda f').$$

Since

$$\frac{\frac{a-b}{1-b}}{1 - \lambda + \lambda\frac{a-b}{1-b}} \in [0, 1],$$

we have $f' \in \text{PARL}(((1-\lambda)f + \lambda f'), \mathbf{1} - ((1-\lambda)f + \lambda f'))$, and thus $(1-\lambda)f + \lambda f' \succeq_B f'$.

From $g = af + b(\mathbf{1} - f)$,

$$g = \frac{a - b}{1 - b}f + b\left(\mathbf{1} - \frac{a - b}{1 - b}f\right) = f' + b(\mathbf{1} - f').$$

Thus $f' \succeq_B g$ and $g - f' = b(\mathbf{1} - f')$. Similarly, $f' \succeq_B (1 - \lambda)f' + \lambda g \succeq_B g$. $\square$

[14] When $b = 1$, define $f' = \mathbf{1}$.

Proof of Theorem 1. Necessity is proved in the main text.

For sufficiency, take any $f \succeq_B g$. First, permutate g if needed to have g satisfy the first equation of (A.1). Permutation invariance ensures the cost stays the same. Define $\varphi_1(\lambda) \equiv C((1-\lambda)f + \lambda f')$ and $\varphi_2(\lambda) \equiv C((1-\lambda)f' + \lambda g)$. Absolute continuity implies that φ_1 is differentiable almost everywhere and satisfies, when differentiable,

$$\varphi_1'(\lambda) = D^+C((1-\lambda)f + \lambda f'; -f + f').$$

On the other hand,

$$-f + f' = -\frac{\frac{1-a}{1-b}}{1 - \lambda + \lambda\frac{a-b}{1-b}}((1-\lambda)f + \lambda f').$$

Therefore, $\varphi_1'(\lambda)$ has the same sign as $D^+C((1-\lambda)f + \lambda f'; -((1-\lambda)f + \lambda f'))$ and is non-positive by (3). The FTC implies

$$C(f') = \varphi_1(1) = \varphi_1(0) + \int_0^1 \varphi_1'(\lambda)d\lambda \leq \varphi_1(0) = C(f).$$

Similarly, observe that

$$\begin{aligned}
\varphi_2'(\lambda) &= D^+C((1-\lambda)f' + \lambda g; -f' + g), \\
-f' + g = b(\mathbf{1} - f') &= \frac{b}{1-\lambda b}\left(\mathbf{1} - ((1-\lambda)f' + \lambda g)\right).
\end{aligned}$$

Then, $\varphi_2'(\lambda)$ is non-positive since it has the same sign as $D^+C((1-\lambda)f' + \lambda g; \mathbf{1} - ((1-\lambda)f' + \lambda g))$. By applying the FTC, $C(g) = \varphi_2(1) \leq \varphi_2(0) = C(f')$. Therefore, $C(g) \leq C(f)$. $\square$

B Proofs for Section 3

B.1 Proof of Lemma 2

Proof. Let $f, g \in \mathcal{E}_m$. If $f \succeq_B g$, then there exists a stochastic matrix $M \in \mathcal{M}_m$ such that $g = fM$. Then,

$$\begin{aligned}
g - f &= f(M - I) \\
&= \begin{bmatrix} f^1 & \cdots & f^m \end{bmatrix} \begin{bmatrix} m_1^1 - 1 & \cdots & m_1^m \\ \vdots & \ddots & \vdots \\ m_m^1 & \cdots & m_m^m - 1 \end{bmatrix} \\
&= \begin{bmatrix} f^1 & \cdots & f^m \end{bmatrix} \begin{bmatrix} -\sum_{k=2}^m m_1^k & \cdots & m_1^m \\ \vdots & \ddots & \vdots \\ m_m^1 & \cdots & -\sum_{k=1}^{m-1} m_m^k \end{bmatrix} \\
&= \sum_{j=1}^m \sum_{k \neq j} m_j^k f^{j \to k}
\end{aligned}$$

with $m_j^k \geq 0$ and $\sum_{k=1}^m m_j^k = 1$ for all j. Further write

$$g - f = \sum_{j=1}^k \left(\sum_{k \neq j} m_j^k \right) \left(\sum_{k \neq j} \frac{m_j^k}{\sum_{k \neq j} m_j^k} f^{j \to k} \right),$$

where if $\sum_{k \neq j} m_j^k = 0$, let the second term be zero. Notice $\sum_{k \neq j} m_j^k \in [0, 1]$ and the second term (if non-zero) is a convex combination of $f^{j \to k}$. Thus,

$$g - f = \sum_{j=1}^m \lambda_j h_j,$$

with $\lambda_j \in [0, 1]$ and $h_j \in \text{co}(\{f^{j \to k} : k \neq j\})$ where the latter requirement is vacuous when $\lambda_j = 0$.

Conversely, let

$$g - f = \sum_{j=1}^m \lambda_j h_j$$

where for each j,

$$h_j = \sum_{k \neq j} \mu_j^k f^{j \to k}.$$

Let $m_j^k = \lambda_j \mu_j^k$ and let $m_j^j = 1 - \sum_{k \neq j} m_j^k$. Then, M is a stochastic matrix and satisfies $g = fM$. □

B.2 Proof of Theorem 2

Recall $\mathcal{M}_m$ is the set of all $m \times m$ stochastic matrices. $\mathcal{M}_m$ is a convex subset of $\mathbb{R}_+^{m \times m}$ and its extreme points are those with exactly one non-zero entry in each row (Cao et al., 2022). Let $\mathbf{ext}(\mathcal{M}_m)$ denote the set of all extreme points of $\mathcal{M}_m$, and let $\mathbf{ext}_k(\mathcal{M}_m)$ denote the subset of $\mathcal{M}_m$ consisting of matrices with rank k.

First consider the following lemmas.

Lemma B.1. *Let B_j^k be an $m \times m$ matrix with $b_j^j = -1$, $b_j^k = 1$, and zeroes elsewhere. For any $f \in \mathcal{E}_m$, $fB_j^k = f^{j \to k}$.*

The proof is straightforward and thus omitted.

Lemma B.2. *Suppose $C \in \mathcal{C}_m$ is absolutely continuous and satisfies (1). Then for any $1 \leq k \leq m$ and $E \in \mathbf{ext}_{k-1}(\mathcal{M}_m)$, there exists $E' \in \mathbf{ext}_k(\mathcal{M}_m)$ such that for all $\lambda \in [0, 1]$,*

$$fE' \succeq_B (1 - \lambda) fE' + \lambda fE \succeq_B fE. \tag{B.1}$$

And it further implies $C(fE') \geq C(fE)$.

Proof of Lemma B.2. For any $1 \leq k \leq m$ and $E \in \mathbf{ext}_{k-1}(\mathcal{M}_m)$, since E is not a full rank matrix, there exists a column e^i with at least two entries being 1. Let $e_j^i = e_{j'}^i = 1$. Additionally, there are $n - k + 1$ columns such that all the entries are zero. Let one of such columns be $e^{i'}$. Let E' be a matrix such that $e'^{i'}_{j'} = 1$, $e'^{i}_{j'} = 0$ and all others the same as E. Note that E' has exactly $n - k$ empty columns, thus $E' \in \mathbf{ext}_k(\mathcal{M}_m)$.

Let B denote $B_{i'}^i$ as in Lemma B.1 and I_m the identity matrix of size m. Then $I_m + \lambda B$ is a stochastic matrix for all $\lambda \in [0, 1]$. Observe that $B^2 = -B$ and $(I_m + \lambda B) \cdot (I_m + B) = I_m + B$. Additionally, $E'(I_m + B) = E$ and $E'(I_m + \lambda B) = (1 - \lambda)E' + \lambda E$. Therefore,

$$\begin{aligned}
&(1 - \lambda) fE' + \lambda fE = fE'(I_m + \lambda B), \text{ and} \\
&fE = fE'(I_m + B) = fE'(I_m + \lambda B) \cdot (I_m + B).
\end{aligned}$$

Since $I_m + \lambda B$ and $I_m + B$ are stochastic matrices, (B.1) holds.

Recall Lemma B.1,

$$fB = f^{i' \to i}.$$

(1) implies that for all $\lambda \in [0, 1]$,

$$\begin{aligned} & D^+(C((1-\lambda)fE' + \lambda fE), fE - ((1-\lambda)fE' + \lambda fE)) \\ = & D^+(C((1-\lambda)fE' + \lambda fE), ((1-\lambda)fE' + \lambda fE)B) \quad \leq 0. \end{aligned} \tag{B.2}$$

Finally, we show for such E and E',

$$C(fE') \geq C(fE).$$

For $\lambda \in [0, 1]$, let $\varphi(\lambda) \equiv C((1-\lambda)fE' + \lambda fE)$. By absolute continuity, φ is differentiable almost everywhere on $[0, 1]$ and satisfy

$$\varphi'(\lambda) = D^+C((1-\lambda)fE' + \lambda fE; fE - fE').$$

Then, the FTC implies that

$$\begin{aligned} C(fE) - C(fE') = \varphi(1) - \varphi(0) = \int_0^1 \varphi'(\lambda) d\lambda &= \int_0^1 D^+((1-\lambda)fE' + \lambda fE; fE - fE') d\lambda \\ &= \int_0^1 \frac{1}{1-\lambda} D^+(C((1-\lambda)fE' + \lambda fE), fE - ((1-\lambda)fE' + \lambda fE)) d\lambda \\ &\leq 0, \end{aligned}$$

where the equalities use positive homogeneity of $D^+C(f; \cdot)$ and the last inequality follows from (B.2) holds for all $\lambda \in [0, 1]$. $\square$

Proof of Theorem 2. Necessity is proved in the main text.

For sufficiency, permutation invariance and Lemma B.2 together imply that $C(f) \geq C(fE)$ for all $E \in \textbf{ext}(\mathcal{M}_m)$. Take any $f, g \in \mathcal{E}_m$ with $f \succeq_B g$. Quasiconvexity of C implies

$$C(g) \leq \max\{C(fE) : E \in \textbf{ext}(\mathcal{M}_m)\} \leq C(f).$$

$\square$

C Proofs for Section 4

Proof of Lemma 4. Using Lemma 1, it suffices to find $(a, b) \in [0, 1]^2$ such that $\tilde{f} = a \cdot f + b \cdot (\mathbf{1} - f)$, $\sum_{i=1}^n p_i \cdot \tilde{f}_i = w$, and $\sum_{i=1}^n v_i \cdot p_i \cdot \tilde{f}_i = 0$.

Define $A \equiv \sum_{i=1}^n v_i \cdot p_i \cdot f_i$ and $B \equiv -\sum_{i=1}^n v_i \cdot p_i \cdot (1 - f_i)$. Since $\sum_{i=1}^n v_i \cdot p_i < 0$, it follows that $A < B$. Set a and b as follows

$$a = \frac{\frac{w}{1-w}}{\frac{w}{1-w} + \frac{A}{B}} \quad \text{and} \quad b = \frac{\frac{w}{1-w} \cdot \frac{A}{B}}{\frac{w}{1-w} + \frac{A}{B}}.$$

Since $B > A$ and $1 > w \geq 0$, we have $1 \geq a > b \geq 0$. The other conditions can be easily verified. $\square$

Proof of Proposition 2. Suppose $f^*(w^*)$ is not a solution of (PP) subject to (OC). Then there exists $\hat{f}$ such that (OC) holds and

$$\sum_{i=1}^n p_i \cdot \hat{f}_i - C(\hat{f}) > \sum_{i=1}^n p_i \cdot f^*(w^*)_i - C(f^*(w^*)) = w^* - \tilde{C}(w^*). \tag{C.1}$$

Note that $\hat{w} = \sum_{i=1}^n p_i \cdot \hat{f}_i \in [0, \overline{w}]$ and $\tilde{C}(\hat{w}) \leq C(\hat{f})$, a contradiction. $\square$

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Online Appendix for "Blackwell Monotonicity: A Marginal Characterization"

Xiaoyu Cheng and Yonggyun Kim

OA.1 Strict Blackwell Monotonicity

In this section, we provide a sufficient condition for strict Blackwell monotonicity in the case of binary experiments.

For any $f, g \in \mathcal{E}_m$, if $f \succeq_B g$ but $g \not\succeq_B f$, then f is *strictly more informative* than g, denoted by $f \succ_B g$. A Blackwell-monotone cost function $C \in \mathcal{C}_m$ is **strictly Blackwell monotone** if for all $f, g \in \mathcal{E}_m$, $C(f) > C(g)$ whenever $f \succ_B g$.

Theorem OA.1.1. *Suppose $C \in \mathcal{C}_2$ is absolutely continuous and Blackwell monotone. C is strictly Blackwell monotone if the inequalities in (3) hold strictly whenever $f \notin \{\lambda \mathbf{1} : \lambda \in [0,1]\}$.*

Proof of Theorem OA.1.1. First, the following lemma provides a characterization of when $f \succ_B g$.

Lemma OA.1.1. *For $f \neq g$, if $f \notin \{\lambda \mathbf{1} : \lambda \in [0,1]\}$, then $f \simeq_B g$ if and only if $g = \mathbf{1} - f$, otherwise $f \succeq_B g$ if and only if $f \simeq_B g$. In other words, $f \succ_B g$ if and only if $f \notin \{\lambda \mathbf{1} : \lambda \in [0,1]\}$ and $g \neq \mathbf{1} - f$.*

Proof of Lemma OA.1.1. Recall that $f \succeq_B g$ if and only if there exists $(a, b) \in [0,1]^2$ such that $g = af + b(\mathbf{1} - f)$. Thus, $f \simeq_B g$ if and only if there exists $(a, b) \in [0,1]^2$ and $(a', b') \in [0,1]^2$ such that

$$g = af + b(\mathbf{1} - f) \quad \text{and} \quad f = a'g + b'(\mathbf{1} - g).$$

Plugging the first equation into the second, we have

$$(1 - (a-b)(a'-b'))f = (a'b + b' - b'b)\mathbf{1}.$$

This equation holds only when either one of the following holds:

(i) $a = 1$ and $b = 0$, i.e., $g = f$; or

(ii) $a = 0$ and $b = 1$, i.e., $g = \mathbf{1} - f$; or

(iii) $f \in \{\lambda \mathbf{1} : \lambda \in [0, 1]\}$.

Notice in the third case, $f \succeq_B g$ if and only if $g \in \{\lambda \mathbf{1} : \lambda \in [0, 1]\}$. Consequently, it implies that $f \succeq_B g$ if and only if $f \simeq_B g$. □

Consider any $f \succ_B g$, Lemma OA.1.1 implies that $f \notin \{\lambda \mathbf{1} : \lambda \in [0, 1]\}$ and $g \neq \mathbf{1} - f$. Then, after a permutation if needed, there exists a path from f to g as proved in Lemma A.1 and that every experiment along this path is not in $\{\lambda \mathbf{1} : \lambda \in [0, 1]\}$. Since the inequalities in (3) are strict, FTC implies along this path implies $C(f) > C(g)$. □

OA.2 Likelihood Ratio Characterization

In this section, we provide another characterization of Blackwell monotonicity using likelihood ratios of experiments under the binary-binary case ($n = m = 2$).

For any $f, g \in \hat{\mathcal{E}}_2$, from the parallelogram in Figure 1b, we have $f \succeq_B g$ if and only if the slope of AB for f is steeper than that for g, and the slope of AD for f is shallower than that for g. In other words, $f \succeq_B g$ if and only if

$$\alpha \equiv \frac{f_2}{f_1} \geq \frac{g_2}{g_1} \equiv \alpha' \quad \text{and} \quad \beta \equiv \frac{1 - f_1}{1 - f_2} \geq \frac{1 - g_1}{1 - g_2} \equiv \beta'.\text{[15]} \tag{OA.2.1}$$

Note that α is the likelihood ratio for generating signal s_1 and $1/\beta$ is the likelihood ratio for signal s_2. Thus, (OA.2.1) implies that if both α and β increase, Blackwell informativeness increases. Also note that α and β can take any value in $[1, +\infty]$ and

$$f_1 = \frac{\beta - 1}{\alpha\beta - 1} \quad \text{and} \quad f_2 = \frac{\alpha(\beta - 1)}{\alpha\beta - 1}.\text{[16]}$$

Define $\tilde{C} : [1, \infty]^2 \to \mathbb{R}_+$ as follows:

$$\tilde{C}(\alpha, \beta) \equiv C\left(\tfrac{\beta - 1}{\alpha\beta - 1}, \tfrac{\alpha(\beta - 1)}{\alpha\beta - 1}\right). \tag{OA.2.2}$$

Thus, we obtain the following characterization of Blackwell monotonicity which does not require any continuity assumption.

[15] Let $x/0 = +\infty$ for all $x > 0$ and $0/0 = 1$.

[16] If $\alpha = +\infty$, then $f_1 = 0$ and $f_2 = \frac{\beta - 1}{\beta}$. If $\alpha = \beta = 1$, let $f_1 = f_2 = 0$.

Proposition OA.2.1. *For any $C : \hat{\mathcal{E}}_2 \to \mathbb{R}_+$, C is Blackwell monotone if and only if $\tilde{C}$ as defined in* (OA.2.2) *is increasing in α and β.*

Proposition OA.2.1 provides a simple way to check Blackwell monotonicity for binary-binary experiments. We illustrate this with two examples.

Example OA.2.1. Consider two information cost functions defined over $\hat{\mathcal{E}}_2$:

$$C_3(f_1, f_2) \equiv \left(\frac{f_2}{f_1} - 1\right)^2 \left(1 - \frac{1 - f_2}{1 - f_1}\right), \qquad C_4(f_1, f_2) \equiv \frac{f_2(1 - f_2)}{f_1(1 - f_1)} - 1.$$

By using (OA.2.2), we have

$$\tilde{C}_3(\alpha, \beta) \equiv (\alpha - 1)^2 \left(1 - \frac{1}{\beta}\right), \qquad \tilde{C}_4(\alpha, \beta) \equiv \frac{\alpha}{\beta} - 1.$$

Then, from $\alpha, \beta \geq 1$, $\tilde{C}_3$ is increasing in both α and β, whereas $\tilde{C}_4$ is not increasing in β. Therefore, it follows that C_3 is Blackwell monotone, but C_4 is not.

OA.3 Further Results regarding Quasiconvexity

OA.3.1 Non-necessity of Quasiconvexity

The following example illustrates a cost function over binary experiments that is Blackwell monotone but not quasiconvex.

Example OA.3.1. Suppose $n = m = 2$. Denote any experiment $f \in \mathcal{E}_2$ by $f = [f_1, f_2]^\intercal$. As before, we restrict attention to the set $\hat{\mathcal{E}}_2 = \{(f_1, f_2) \ : 0 \leq f_1 \leq f_2 \leq 1\}$. Consider $C : \hat{\mathcal{E}}_2 \to \mathbb{R}_+$ defined by

$$C(f) = \min\left\{\frac{f_2}{f_1}, \frac{1 - f_1}{1 - f_2}\right\}.$$

By using (4), we can easily see that $f \succeq_B g$ implies $C(f) \geq C(g)$, i.e., C is Blackwell monotone.

Consider $f = [0, 1/2]^\intercal$ and $g = [1/2, 1]^\intercal$ with costs $C(f) = C(g) = 2$. For the one-half mixture of them, given by $h = [1/4, 3/4]^\intercal$, the cost is $C(h) = 3 > C(f) = C(g)$. Hence, this cost function is not quasiconvex.

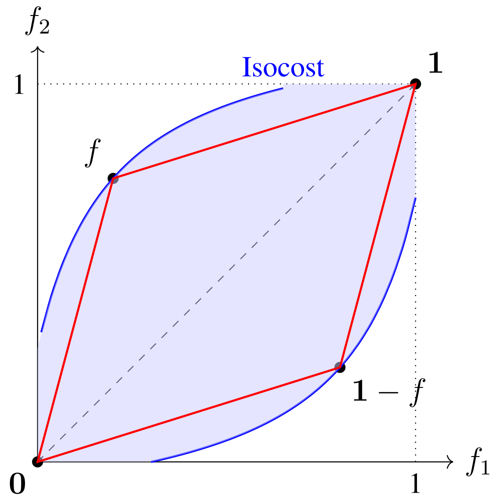


Figure 2: The sublevel set of a quasiconvex cost

OA.3.2 Binary Experiments with Quasiconvexity

Quasiconvexity is not necessary in establishing Blackwell monotonicity over binary experiments. However, when quasiconvexity is imposed in this case, it is almost sufficient for Blackwell monotonicity.

Recall that any binary experiment can be represented by $f = [f_1, \cdots, f_n]^\intercal \in [0,1]^n$, and $\mathbf{0}$ and $\mathbf{1}$ are completely uninformative experiments. Let C be *non-trivial* if for any $f \in [0,1]^n$, $C(f) \geq C(\mathbf{1}) = C(\mathbf{0})$.

Proposition OA.3.1. *If $C \in \mathcal{E}_2$ is quasiconvex, permutation invariant, and non-trivial, then C is Blackwell monotone.*

Proof of Proposition OA.3.1. By Lemma 1, $f \succeq_B g$ if and only if $g = af + b(\mathbf{1} - f)$ for $(a,b) \in [0,1]^2$. If $a \geq b$, $g = (1-a)\cdot\mathbf{0} + (a-b)\cdot f + b\cdot\mathbf{1}$; and if $a < b$, $g = (1-b)\cdot\mathbf{0} + (b-a)\cdot(\mathbf{1}-f) + a\cdot\mathbf{1}$. From quasiconvexity and non-nullness, we have $C(f) \geq C(g)$ or $C(\mathbf{1}-f) \geq C(g)$. Then, by permutation invariance, $C(f) = C(\mathbf{1}-f)$, thus, $C(f) \geq C(g)$. □

This proposition is illustrated in Figure 2. To satisfy Blackwell monotonicity, $S_B(f)$—represented by the red parallelogram—must be included in $S_C(f)$—represented by the blue region. Recall that $S_C(f)$ is a convex set due to the quasiconvexity of C. Furthermore, by non-triviality, $\mathbf{0}$ and $\mathbf{1}$ belong to $S_C(f)$. Therefore, the parallelogram spanned by f, $\mathbf{1}-f$, $\mathbf{0}$ and $\mathbf{1}$ must also be included by $S_C(f)$.

OA.4 Proof of Proposition 1

Proof of Proposition 1. Suppose that $f \succeq_B g$, i.e., there exists a 3×3 stochastic matrix $B = (b_i^j)$ such that $fB = g$.

Observe that at least one of $f_1^1,\ f_1^2$ and f_1^3 is positive—if not, every entry of the first row of fB is equal to zero. Without loss of generality, let f_1^1 be positive (we can obtain it by permuting f). Note that $f_1^1 b_1^3 + f_1^2 b_2^3 + f_1^3 b_3^3 = 0$. Since every entry of f and B are nonnegative, $b_1^3 = 0$.

Next, observe that $4/5 = f_2^1 b_1^3 + f_2^2 b_2^3 + f_2^3 b_3^3$. From $b_1^3 = 0$, at least one of f_2^2 and f_2^3 is positive. Without loss, let f_2^2 be positive. Then, from $g_2^1 = 0$, we have $b_2^1 = 0$. Then, it gives us $f_2^1 b_1^1 + f_2^3 b_3^1 = 0$. We consider two cases: $b_1^1 = 0$ or $f_2^1 = 0$.

1. $b_1^1 = 0$**:** From $b_1^1 = b_1^3 = 0$, we have $b_1^2 = 1$. From $g_3^2 = 0$ and $b_1^2 > 0$, we have $f_3^1 = 0$.

Additionally, we have $f_1^3 b_3^1 = 4/5$, $f_2^3 b_3^1 = 0$ and $f_3^3 b_3^1 = 1/5$. Therefore, $b_3^1, f_1^3, f_3^3 \neq 0$ and $f_2^3 = 0$. From $g_1^3 = 0$ and $f_1^3 \neq 0$, we have $b_3^3 = 0$. Likewise, from $g_3^2 = 0$ and $f_3^3 > 0$, $b_3^2 = 0$. Then, it gives us $b_3^1 = 1$.

From $b_1^1 = 0, b_2^1 = 0$ and $b_3^1 = 1$, we have $f_1^3 = 4/5$ and $f_3^3 = 1/5$. From $f_3^1 = 0$, $f_3^2 = 4/5$. From $g_3^2 = 0$ and $f_3^2 > 0$, we have $b_2^2 = 0$. It gives us $b_2^3 = 1$. Therefore, B is a permutation matrix and f is a permutation of g.

2. $b_1^1 > 0$ **and** $f_2^1 = 0$**:** Observe that $f_2^3 b_3^1 = 0$, $f_2^2 b_2^2 + f_2^3 b_3^2 = 4/5$ and $f_2^2 b_2^3 + f_2^3 b_3^3 = 1/5$. We consider two subcases: $f_2^3 = 0$ or $b_3^1 = 0$.

(a) $f_2^3 = 0$**:** From $f_2^1 = f_2^3 = 0$, we have $f_2^2 = 1$. Additionally, we have $b_2^2 = 4/5$ and $b_2^3 = 1/5$. From $0 = g_1^3 = f_1^2 b_2^3 + f_1^3 b_3^3$ and $0 = g_3^2 = f_3^1 b_1^2 + f_3^2 b_2^2 + f_3^3 b_3^2$, we have $f_1^2 = f_3^2 = 0$. Observe that $0 = g_1^3 = f_1^3 b_3^3$ and $4/5 = g_3^3 = f_3^3 b_3^3$. Then, we have $b_3^3 > 0$ and $f_1^3 = 0$. From $f_1^2 = f_1^3 = 0$, we have $f_1^1 = 1$. This also gives $b_1^1 = 4/5$ and $b_1^2 = 1/5$. Again from $0 = g_3^2$ and $b_1^2 = 1/5$, we have $f_3^1 = 0$. Therefore, from $f_3^1 = f_3^2 = 0$, we have $f_3^3 = 1$, i.e., f is I_3.

(b) $b_3^1 = 0$**:** From $b_2^1 = b_3^1 = 0$, we have $f_1^1 \cdot b_1^1 = 4/5$ and $f_3^1 \cdot b_1^1 = 1/5$. Therefore, $b_1^1, f_1^1, f_3^1 > 0$. Next, $0 = g_3^2 = f_3^1 b_1^2 + f_3^2 b_2^2 + f_3^3 b_3^2$ gives $b_1^2 = 0$. From $b_1^2 = b_1^3 = 0$, we have $b_1^1 = 1$.

Suppose that both b_2^2 and b_3^2 are positive. Then, from $0 = g_3^2 = f_3^2 b_2^2 + f_3^3 b_3^2$, we have $f_3^2 = f_3^3 = 0$. It contradicts $4/5 = g_3^3 = f_3^1 b_1^3 + f_3^2 b_2^3 + f_3^3 b_3^3$ since $b_1^3 = f_3^2 = f_3^3 = 0$.

Therefore, at least one of b_2^2 and b_3^2 is equal to zero. Likewise, if both b_2^3 and b_3^3 are positive, we have $f_1^2 = f_1^3 = 0$ from $g_1^3 = 0$, but it contradicts $g_1^2 = 1/5 > 0$. Thus, at least one of b_2^3 and b_3^3 is equal to zero. Also, note that B needs to be a full rank matrix (as g has a full rank). To have that, there are two possibilities: (i) $b_2^2 = b_3^3 = 1$ and $b_2^3 = b_3^2 = 0$; or (ii) $b_2^3 = b_3^2 = 1$ and $b_2^2 = b_3^3 = 0$. Then, B is either I_3 or a permutation of I_3. Therefore, f is g or a permutation of g. □

OA.5 Costly Persuasion Example

As an example, consider the binary state case with $v_1 = -1$ and $v_2 = 1$, i.e., the defendant is innocent under ω_1 and is guilty under ω_2. Let μ denote the prior belief $p_2 = \Pr(\omega_2)$. Observe that $v_1 \cdot p_1 + v_2 \cdot p_2 < 0$ is equivalent to $\mu < \frac{1}{2}$.

Lemma 4 is illustrated in Figure 3a. From the results of Section 2, as f_1 increases, Blackwell informativeness decreases along the line of $\mu f_2 + (1-\mu) f_1 = w$. In this binary case, we can uniquely pin down f satisfying both constraints of (7) as follows:

$$f_1 = \frac{w}{2(1-\mu)} \quad \text{and} \quad f_2 = \frac{w}{2\mu}. \tag{OA.5.1}$$

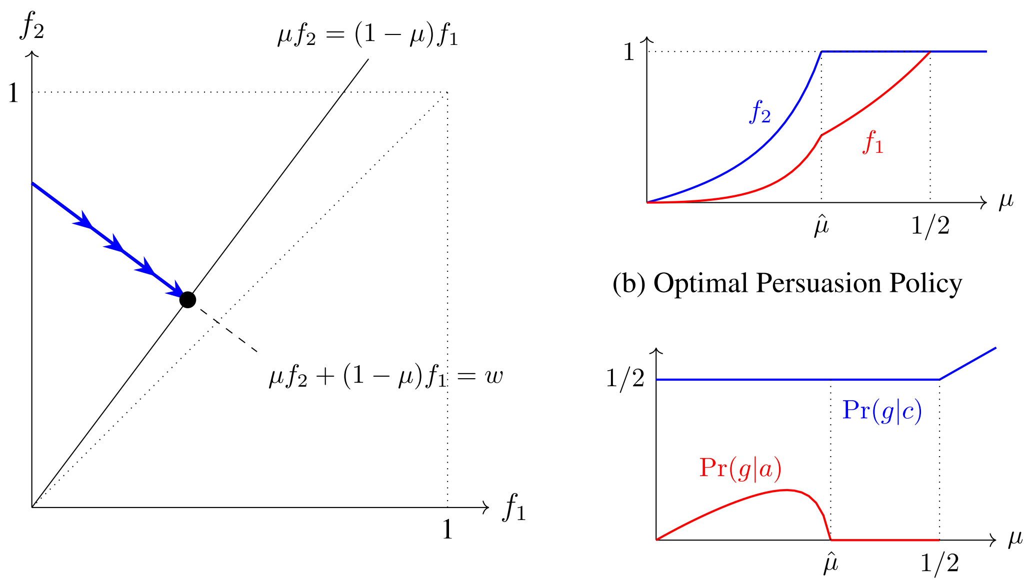
Next, by plugging in (OA.5.1), the prosecutor's problem becomes

$$\max_{0 \le w \le 2\mu} \; w - C\left(\frac{w}{2(1-\mu)}, \frac{w}{2\mu}\right). \tag{OA.5.2}$$

Therefore, given the cost function, the prosecutor's problem becomes a one-dimensional maximization problem. As an example, the following proposition characterizes the optimal persuasion policy under the quadratic cost function, $C(f_1, f_2) = (f_2 - f_1)^2$, which is shown to be Blackwell monotone in Example 1.

Proposition OA.5.1. *Suppose that $v_1 = -1$, $v_2 = 1$ and $C(f_1, f_2) = (f_2 - f_1)^2$. The prosecutor's optimal persuasion policy is given by: for some $0 < \hat{\mu} < 1/2$,*

$$f_1 = \begin{cases} 1, & \text{if } \mu \ge 1/2, \\ \frac{\mu}{1-\mu}, & \text{if } \mu \in (\hat{\mu}, 1/2), \\ \frac{\mu^2(1-\mu)}{(1-2\mu)^2} & \text{if } \mu \le \hat{\mu}, \end{cases} \quad \text{and} \quad f_2 = \begin{cases} 1, & \text{if } \mu \ge 1/2, \\ 1, & \text{if } \mu \in (\hat{\mu}, 1/2), \\ \frac{\mu(1-\mu)^2}{(1-2\mu)^2} & \text{if } \mu \le \hat{\mu}. \end{cases} \tag{OA.5.3}$$



(a) Cost Minimization

(b) Optimal Persuasion Policy

(c) Posteriors

Figure 3: Costly Persuasion with $C(f_1, f_2) = (f_2 - f_1)^2$

This result is illustrated in Figure 3b and 3c. When $\mu \geq \hat{\mu}$, the optimal persuasion policy is the same as the one without the cost: the posterior belief is either $1/2$ or 0. In this case, the prosecutor always convicts guilty defendants and, with some positive probability, convicts innocent defendants. When $\mu < \hat{\mu}$, this policy is no longer optimal as it becomes too expensive. Instead, the prosecutor sacrifices the probability of convicting the guilty defendant to lower the costs. Observe that the posterior belief upon receiving a depends on μ, while the posterior belief upon receiving c is constant $(1/2)$. This result differs qualitatively from the one with uniformly posterior separable costs, where the posterior beliefs are independent of the prior belief whenever the information is provided.

OA.5.1 Proof of Proposition OA.5.1

Proof of Proposition OA.5.1. For $\mu \geq 1/2$, we show that $f_1 = f_2 = 1$ is optimal in the main text.

Now assume that $\mu < 1/2$. By plugging the cost function in, (OA.5.2) is equivalent to

$$\max_{0 \leq w \leq 2\mu} w - \frac{w^2}{4\mu \cdot h(\mu)} \qquad \text{where} \quad h(\mu) \equiv \frac{\mu(1-\mu)^2}{(1-2\mu)^2}. \tag{OA.5.4}$$

Observe that for all $0 < \mu < 1/2$

$$h'(\mu) = \frac{2\mu + (1-2\mu)(1+\mu^2)}{(1-2\mu)^3} > 0.$$

Additionally, $h(0) = 0$ and $\lim_{\mu \to 1/2} h(\mu) = \infty$. Therefore, there exists $\hat{\mu}$ such that $h(\hat{\mu}) = 1$. Then, the solution of the minimization problem (OA.5.4) subject to $0 \leq w \leq 2\mu$ is

$$w^* = \begin{cases} 2\mu, & \text{if } \mu \in (\hat{\mu}, 1/2), \\ 2\mu \cdot h(\mu), & \text{if } \mu \leq \hat{\mu}. \end{cases}$$

By plugging this into (OA.5.1), we have (OA.5.3). $\square$