Blackwell-Monotone Information Costs

ITAM

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October 25, 2024

Introduction

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- Agenda: integration of costly information across various fields
- Question: Which information cost function *should* or *could* be used
  - Information cost: a map from a statistical experiment to a cost
- Examples
  - Entropy Costs: Sims (2003); Matějka, McKay (2015)
  - Posterior Separable Costs: Caplin, Dean, Leahy (2022); Denti (2022)
  - Log-Likelihood Ratio Costs: Pomatto, Strack, Tamuz (2023)
- Common Principle: Blackwell Monotonicity
  - More informative in Blackwell's order $\Rightarrow$ higher cost
  - **Minimum requirement** for plausible information costs

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Illustration: Blackwell Monotonicity

- Consider consumers seeking to acquire information about their COVID-19 status
- Two tests are available in the competitive market:

Test A ($10)

state	signal: n	signal: p
$-$	80%	20%
$+$	20%	80%

Test B ($12)

state	signal: n	signal: p
$-$	60%	40%
$+$	15%	85%

- Test B can be replicated using test A, which costs $10
  - When n is realized, toss a coin twice and replace it with p if both are heads

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		$n_{oth.}$	n_{HH}	p
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- **Blackwell's Theorem**
    - A is more informative than $B \Leftrightarrow B$ is a *garbling* of A
- **Blackwell Monotonicity**
    - A should be more costly than B whenever A is Blackwell more informative than B
- **This paper**
    - identifies elementary necessary and sufficient conditions for Blackwell monotonicity
    - characterizes a practical and tractable class of information cost functions

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▸ Literature

Roadmap

1. Preliminaries
2. Binary Experiments
3. Finite Experiments: more than two signals
4. Likelihood Separable Costs
5. Application: Bargaining and Information Acquisition

Preliminaries

Experiments

- $\Omega = \{\omega_1, \cdots, \omega_n\}$: a finite set of states
- $\mathcal{S} = \{s_1, \cdots, s_m\}$: a finite set of signals
- A *statistical experiment* $f : \Omega \to \Delta(\mathcal{S})$ can be represented by an $n \times m$ matrix:

$$f = \begin{bmatrix} f_1^1 & \cdots & f_1^m \\ \vdots & \ddots & \vdots \\ f_n^1 & \cdots & f_n^m \end{bmatrix},$$

  where $f_i^j = \Pr(s_j|\omega_i)$, thus, $f_i^j \geq 0$ and $\sum_{j=1}^m f_i^j = 1$
- Let f^j denote j-th column vector, namely likelihood vector of signal j
- $\mathcal{E}_m \subset \mathbb{R}^{n \times m}$: the space of all experiments with m possible signals

Blackwell Informativeness

- $f \succeq_B g$: f is *Blackwell more informative* than g
  iff g is a garbling of f: $\exists$ a stochastic matrix M s.t. $g = f\,M$

- Examples of garbling under binary signal
  1. **Signal Replacement**: for some $\epsilon > 0$,
  $$M = \begin{bmatrix} 1-\epsilon & \epsilon \\ 0 & 1 \end{bmatrix}$$
  meaning that s_1 is replaced with s_2 with probability ϵ
  2. **Permutation**:
  $$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
  meaning that signals are relabeled
     - $f \sim_B f\,P$: relabeling signals does not change the informativeness

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Information Costs and Blackwell Monotonicity

- **Information Costs**
  - $C : \mathcal{E}_m \to \mathbb{R}_+$: an information cost function
  - $\mathcal{C}_m$: the set of all absolutely continuous information cost functions defined over $\mathcal{E}_m$
  - Absolute continuity ensures that a derivative exists a.e. and is integrable
  - In the talk, assume that C is differentiable and the gradient exists

- **Blackwell Monotonicity**
  - An information cost function $C \in \mathcal{C}_m$ is **Blackwell monotone** if for all $f, g \in \mathcal{E}_m$, $C(f) \geq C(g)$ whenever $f \succeq_B g$.

- **Permutation Invariance**
  - Any Blackwell-monotone information cost function is **permutation invariant**, i.e., $C(f) = C(f\,P)$ for any permutation matrix P

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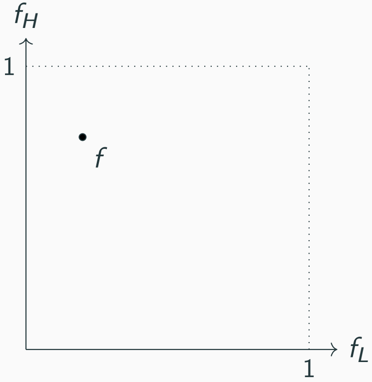
Binary Experiments

Binary Experiments

- Focus on the case where $n = m = 2$
- Any experiment can be represented by $f \equiv (f_L, f_H)^\intercal \in [0,1]^2$:

$$[1-f, f] = \begin{array}{c|cc} & s_L & s_H \\ \hline \omega_L & 1-f_L & f_L \\ \omega_H & 1-f_H & f_H \end{array}$$

- $1 - f$ is a permutation of f
- When $f_L = f_H$, it is completely uninformative



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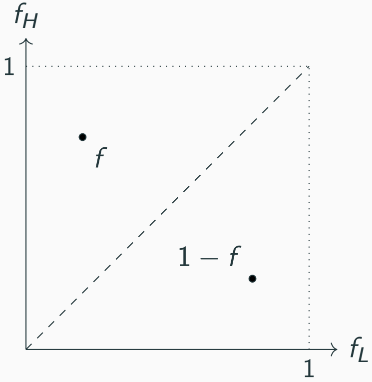

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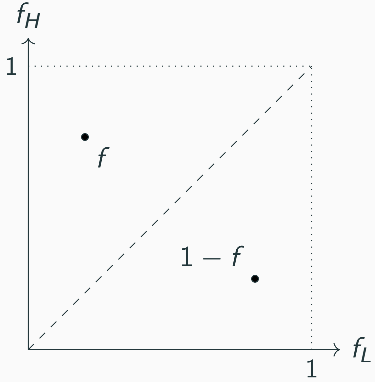
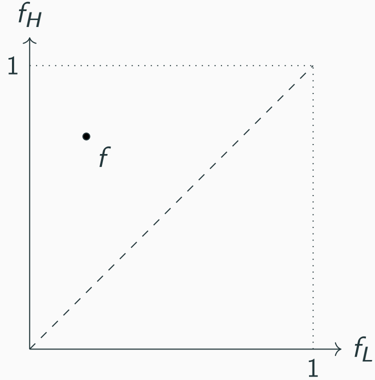


Blackwell Informativeness under Binary Experiments



- Recall that $f \succeq_B g$ iff

$$[1-g, g] = [1-f, f]\ M$$

  for some stochastic matrix M

- Extreme points of M:

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad M_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

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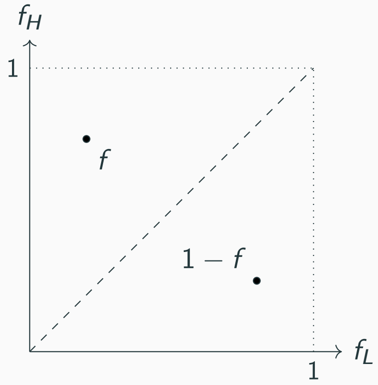

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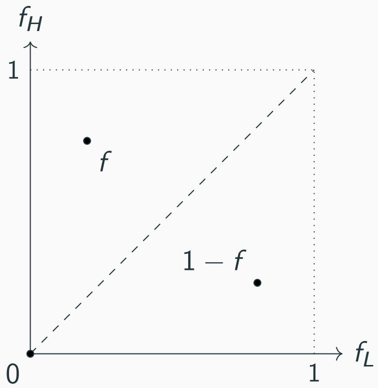

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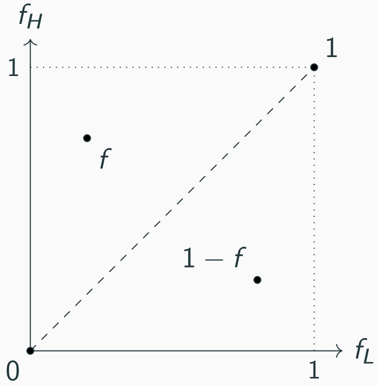

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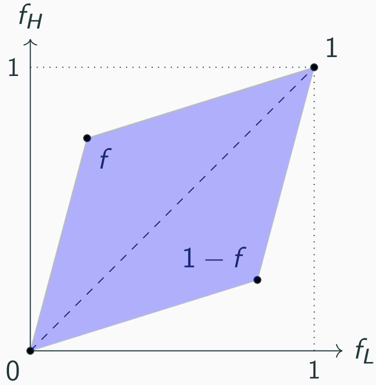

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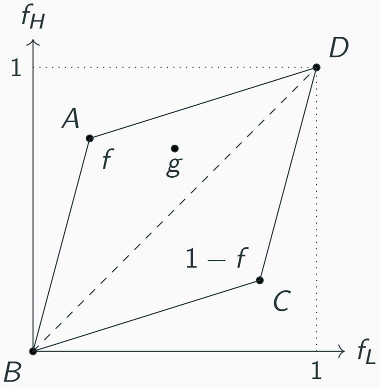
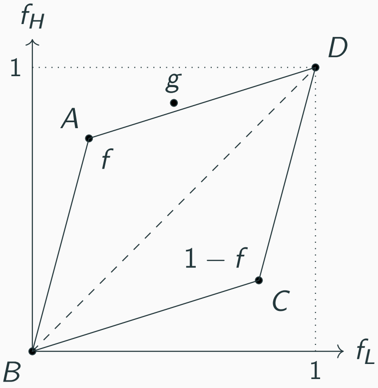


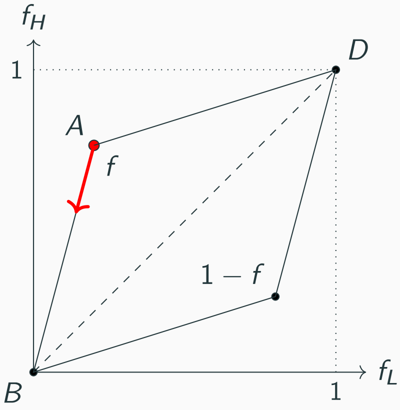
$f \succeq_B g$



$f \not\succeq_B g$

Necessary Conditions for Blackwell Monotonicity

When an information cost C is Blackwell monotone,



1. $\langle \nabla C(f), -f \rangle \leq 0$

$$\begin{tikzcd}
s_L \arrow[r, "1"] & s'_L \\
s_H \arrow[ur, "\epsilon"'] \arrow[r, "1-\epsilon"'] & s'_H
\end{tikzcd}$$

2. $\langle \nabla C(f), 1-f \rangle \leq 0$

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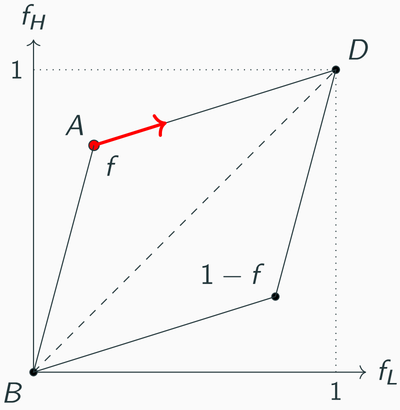

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$$\begin{array}{lcl} s_L & \xrightarrow{\quad 1 \quad} & s'_L \\ s_H & \xrightarrow{\quad \epsilon \quad} & s'_L \\ s_H & \xrightarrow[\quad 1-\epsilon \quad]{} & s'_H \end{array}$$

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Theorem for Binary Experiments

Theorem 1

$C \in \mathcal{C}_2$ is Blackwell monotone if and only if it is

1. permutation invariant;
2. for all $f \in \mathcal{E}_2$,

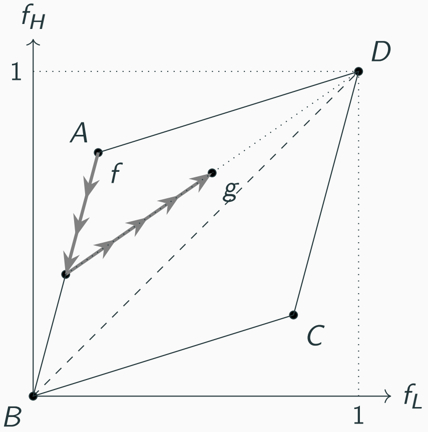
$$\langle \nabla C(f), f \rangle \geq 0 \geq \langle \nabla C(f), 1 - f \rangle. \tag{1}$$

- This theorem holds for the cases with more than two states, but the binary signal assumption is crucial.

Quiz

Proof for Sufficiency

For any $f \succeq_B g$, we can find a path from f to g (or the permutation of it) along which Blackwell informativeness decreases



Further Characterizations with Binary States

Quiz

Which of the followings (defined over $f_H > f_L$) are Blackwell-monotone information cost functions?

1. $C(f_L, f_H) = \dfrac{f_H(1 - f_H)}{f_L(1 - f_L)} - 1$
2. $C(f_L, f_H) = \dfrac{f_H}{f_L} + \dfrac{1 - f_L}{1 - f_H} - 2$
3. $C(f_L, f_H) = (f_H - f_L)^2$
4. $C(f_L, f_H) = f_H - 2f_L$

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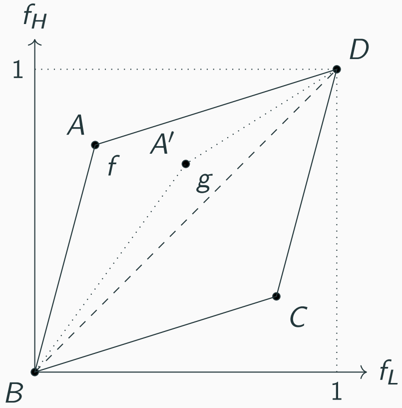

$f \succeq_B g$ is equivalent to:

1. AB steeper than $A'B$:

$$\alpha \equiv \frac{f_H}{f_L} \geq \frac{g_H}{g_L} \equiv \alpha'$$

α: likelihood ratio (LR) of receiving s_H

2. AD shallower than $A'D$:

$$\beta \equiv \frac{1-f_L}{1-f_H} \geq \frac{1-g_L}{1-g_H} \equiv \beta'$$

β: the inverse of LR of receiving s_L

Proposition

C is Blackwell monotone iff it is increasing in α and β after reparametrization

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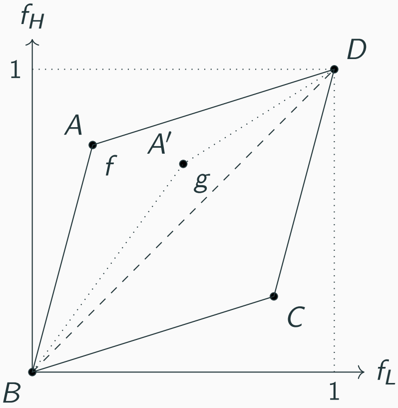

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Further Characterizations with Binary States

1. $C(f_L, f_H) = \frac{f_H(1-f_H)}{f_L(1-f_L)} - 1$ with $1 > f_H > f_L > 0$

$$\tilde{C}(\alpha, \beta) = \frac{\alpha}{\beta} - 1$$

   - $\tilde{C}$ is increasing in α but not in β, thus, $\tilde{C}$ is not Blackwell monotone.

2. $C(f_L, f_H) = \frac{f_H}{f_L} + \frac{1-f_L}{1-f_H} - 2$ with $1 > f_H > f_L > 0$

$$\tilde{C}(\alpha, \beta) = \alpha + \beta - 2$$

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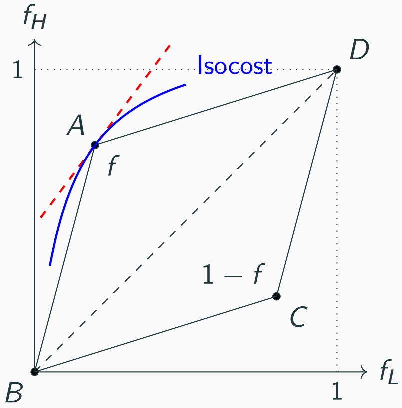
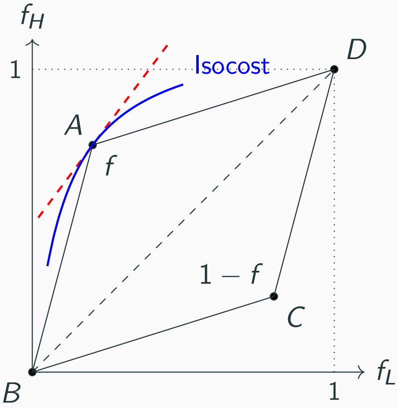


$\langle \nabla C(f), f \rangle \geq 0 \geq \langle \nabla C(f), 1 - f \rangle$
is equivalent to:

$$\underbrace{\frac{f_H}{f_L}}_{\substack{\text{the slope} \\ \text{of } \overline{AB}}} \geq \underbrace{-\frac{\partial C / \partial f_L}{\partial C / \partial f_H}}_{\substack{\text{the slope of} \\ \text{the isocost curve}}} \geq \underbrace{\frac{1 - f_H}{1 - f_L}}_{\substack{\text{the slope} \\ \text{of } \overline{AD}}}$$

- **Interpretation**: a *marignal rate of information transformation* (MRIT) lies between the two likelihood ratios provided by the experiment.



$$\langle \nabla C(f), f\rangle \geq 0 \geq \langle \nabla C(f), 1-f\rangle$$

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Further Characterizations with Binary States

3. $C(f_L, f_H) = (f_H - f_L)^2$ with $1 > f_H > f_L > 0$

$$\frac{f_H}{f_L} \geq -\frac{\partial C/\partial f_L}{\partial C/\partial f_H} = 1 \geq \frac{1 - f_H}{1 - f_L}$$

- The above inequalities hold for all $1 > f_H > f_L > 0$, thus, it is **Blackwell monotone**.

4. $C(f_L, f_H) = f_H - 2f_L$ with $1 > f_H > f_L > 0$

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Answer for the Quiz

Which of the followings are Blackwell-monotone information cost functions?

1. $C(f_L, f_H) = \dfrac{f_H(1-f_H)}{f_L(1-f_L)} - 1$
2. $C(f_L, f_H) = \dfrac{f_H}{f_L} + \dfrac{1-f_L}{1-f_H} - 2$
3. $C(f_L, f_H) = (f_H - f_L)^2$
4. $C(f_L, f_H) = f_H - 2f_L$

Finite Experiments: more than two signals

Necessary Conditions for Blackwell Monotonicity

Now assume that there are more than two signals.

- Permutation invariance is still necessary
- For any pair (i, j), the following garbling worsens the informativeness:

$$\begin{tikzcd} s_i \arrow[r, "1-\epsilon"] \arrow[dr, "\epsilon"] & s_i' \\ s_j \arrow[r, "1"'] & s_j' \end{tikzcd}$$

- This gives us $\langle \nabla^{i \to j} C(f), f^i \rangle \leq 0$, where

$$\langle \nabla^{i \to j} C(f), f^i \rangle \equiv -\sum_{s=1}^{n} \frac{\partial C}{\partial f_s^i} \cdot f_s^i + \sum_{s=1}^{n} \frac{\partial C}{\partial f_s^j} \cdot f_s^i$$

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Sufficient Conditions for Blackwell Monotonicity

- For binary experiments, sufficiency was established by finding a path between two experiments along which informativeness decreases
- However, when $m \geq 3$, there may not exist such path ▸ Illustrations
- To overcome this issue, we impose quasiconvexity on C: ▸ Illustration

$$C(\lambda f + (1-\lambda)g) \leq \max\{C(f),\ C(g)\}.$$

  With quasiconvexity, the first-order condition serves as a sufficient condition for Blackwell monotonicity
- **Remarks**
  - Quasiconvexity is not a necessary condition for Blackwell Monotonicity ▸ Details
  - We found a weaker (but less standard) version of Quasiconvexity serving as a necessary condition for Blackwell monotonicity ▸ Details

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Theorem for Finite Experiments

Theorem 2

Suppose that $C \in \mathcal{C}_m$ is absolutely continuous and quasiconvex. Then, C is Blackwell monotone if and only if it is

1. permutation invariant;
2. for all $f \in \mathcal{E}_m$ and $i \neq j$,
$$\langle \nabla^{i \to j} C(f), f \rangle \leq 0. \tag{2}$$

- $S_B(f)$: the set of experiments that are less Blackwell informative than f
- Two conditions ensure that extreme points of $S_B(f)$ are not more costly than f
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Likelihood Separable Costs

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C is *likelihood separable* if there exist a constant a and an absolutely continuous function $\psi : \mathbb{R}^n_+ \to \mathbb{R}_+$ such that, for all m and $f \in \mathcal{E}_m$,

$$C(f) = \sum_{j=1}^{m} \psi(f^j) + a.$$

Let $\mathcal{C}^{LS}$ be the class of likelihood separable costs

Groundedness

C is *grounded* if it assigns zero cost to uninformative experiments.

Let $\mathcal{C}^G$ be the class of grounded costs.

GSLS Costs

Theorem 3

When $C \in \mathcal{C}^{LS}$, C is Blackwell monotone if and only if ψ is sublinear: ▸ proof

1. positive homogeneity: $\psi(\alpha h) = \alpha\psi(h)$;
2. subadditivity: $\psi(k) + \psi(l) \geq \psi(k + l)$

Corollary: GSLS costs

C is called *grounded sublinear likelihood separable (GSLS)* if there exists a **sublinear** and absolutely continuous function ψ such that

$$C(f) = \sum_{j=1}^{m} \psi(f^j) - \psi(1).$$

Then,

$$\mathcal{C}^{GSLS} = \mathcal{C}^{LS} \cap \mathcal{C}^{G} \cap \mathcal{C}^{BM}$$

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Examples: GSLS Costs

1. Supnorm Costs

$$C(f) = \sum_{j=1}^{m} \max_i f_i^j - 1,$$

2. Absolute-Linear Costs

$$C(f) = \sum_{j=1}^{m} |\langle a, f^j \rangle| - |\langle a, 1 \rangle| = \sum_{j=1}^{m} \left| \sum_{i=1}^{n} a_i f_i^j \right| - \left| \sum_{i=1}^{n} a_i \right|.$$

3. Linear ϕ-divergence Costs (including LLR costs of Pomatto, Strack, Tamuz (2023))

$$C(f) = \sum_{j=1}^{m} \sum_{i,i'} \beta_{ii'} f_{i'}^j \phi_{ii'} \left(\frac{f_i^j}{f_{i'}^j} \right) = \sum_{i,i'} \beta_{ii'} \sum_{j=1}^{m} f_{i'}^j \phi_{ii'} \left(\frac{f_i^j}{f_{i'}^j} \right), \tag{3}$$

where $\phi_{ii'} : [0, \infty] \to \mathbb{R} \cup \{+\infty\}$ is a convex function with $\phi_{ii'}(1) = 0$ and $\beta_{ii'} \geq 0$

GSLS Costs and Posterior Separability

Posterior Separability

C has a *posterior separable (PS) representation* at a prior belief $\mu \in \Delta(\Omega)$ if there exists a concave and absolutely continuous function $H : \Delta(\Omega) \to \mathbb{R}$ such that

$$C(f) = H(\mu) - \sum_{j=1}^{m} \tau_\mu(f^j) \cdot H(q_\mu(f^j))$$

where $q_\mu(f^j)$ is the posterior belief upon receiving s_j and $\tau_\mu(f^j)$ is the probability of receiving s_j.

Let C_μ^{PS} denote the class of cost functions that have PS representations at μ.

Proposition

For any full support prior $\mu \in \Delta(\Omega)$, $\mathcal{C}^{GSLS} = \mathcal{C}_\mu^{PS}$.

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Application: Bargaining and Information Acquisition

Chatterjee, Dong, Hoshino (2023)

- Consider a bargaining problem with information acquisition
    - Players: **S**eller and **B**uyer
    - State (**B**'s valuation): $v \in \{L, H\}$ with $H > L > 0$
        - Prior belief: $\pi \equiv \Pr(v = H) \in (0, 1)$
    - Timing of the game
        1. Nature draws v and **S** observes v
        2. **S** offers p
        3. **B** costly acquires information about v and then accepts or rejects

- Chatterjee et al. focus on specific types of information acquisition

- We extend their analysis by allowing **B** to choose information flexibly

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Chatterjee, Dong, Hoshino (2023): H-focused information

B's cost: $\lambda \cdot c(f_H)$

Result 1: pooling eq'm

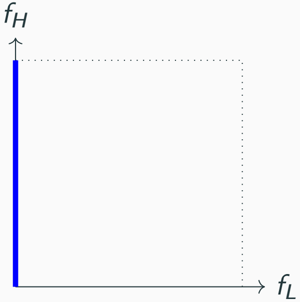
under H-focused signal structure, for any λ, there exists $\epsilon > 0$ such that every equilibrium is a pooling equilibrium where

1. both types of **S** offer $p^* \in [L, L + \epsilon)$;
2. **B** accepts without information acquisition.

Moreover, $\epsilon \to 0$ as $\lambda \to 0$, thus, **B** extracts full surplus as $\lambda \to 0$

H-focused Information

	s_L	s_H
L	1	0
H	$1 - f_H$	f_H



Chatterjee, Dong, Hoshino (2023): L-focused information

B's cost: $\lambda \cdot c(1 - f_L)$

Result 2: almost-separating eq'm

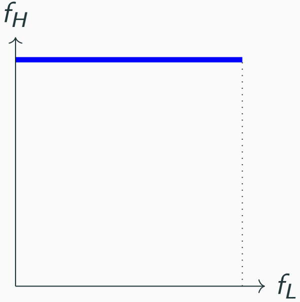
under L-focused signal structure, for any small enough λ, there exists an equilibrium such that

1. type H **S** offers $p^* \approx H$;
2. type L **S** offers L with prob. $1 - \epsilon$, p^* with prob. ϵ;
3. **B** acquires information and conditions her purchase decision on the signal realization

Moreover, **S**'s payoff is close to v and **B**'s payoff is close to zero

L-focused Information

	s_L	s_H
L	$1 - f_L$	f_L
H	0	1



Flexible Information Acquisition

- We extend to the full domain and consider $\lambda|f_2 - f_1|$ and $\lambda(f_2 - f_1)^2$

Result 1': when $C(f_1, f_2) = \lambda|f_2 - f_1|$, the unique equilibrium is the pooling equilibrium, and as $\lambda \to 0$, **B** extracts full surplus

Result 2': when $C(f_1, f_2) = \lambda(f_2 - f_1)^2$, there exists an almost-separating equilibrium, and **S**'s payoff is close to v and **B**'s payoff is close to zero

Flexible Information

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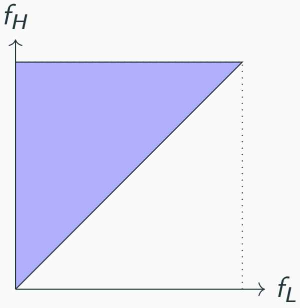
L	$1 - f_L$	f_L
H	$1 - f_H$	f_H



Conclusion

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- We identify necessary and sufficient conditions for Blackwell Monotonicity.
- Under likelihood separability, we show that the sublinearity of the component function is equivalent to Blackwell Monotonicity.
- **Applications**: we apply our results to extend
  1. Costly Persuasion (Gentzkow, Kamenica, 2014) ▸ Costly Persuasion
  2. Bargaining and Information Acquisition (Chatterjee, Dong, Hoshino, 2024)
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Costly Persuasion

Thank You!

Related Literature

- **Posterior-based information costs**
  - Entropy cost: Sims [2003]; Matějka, Mckay [2015]
  - Decision theory: Caplin, Dean [2015]; Caplin, Dean, Leahy [2022]; Chambers, Liu, Rehbeck [2020]; Denti [2022]
  - Applications: Ravid [2020]; Zhong [2022]; Gentzkow, Kamenica [2014]
- **Experiment-based information costs**
  - LLR cost: Pomatto, Strack, Tamuz [2023];
  - Applications: Denti, Marinacci, Rustichini [2022]; Ramos-Mercado [2023]

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Sufficient Conditions for Blackwell Monotonicity

When $m \geq 3$, there may not exist a path along which informativeness decreases

Proposition

Let

$$g = \begin{bmatrix} 4/5 & 1/5 & 0 \\ 0 & 4/5 & 1/5 \\ 1/5 & 0 & 4/5 \end{bmatrix} \in \mathcal{E}_3.$$

If $f \succeq_B g$ and $f \in \mathcal{E}_3$, then f is a permutation of I_3 or g.

- I_3 is Blackwell more informative than g, but we cannot find a path from I_3 to g along which Blackwell informativeness decreases

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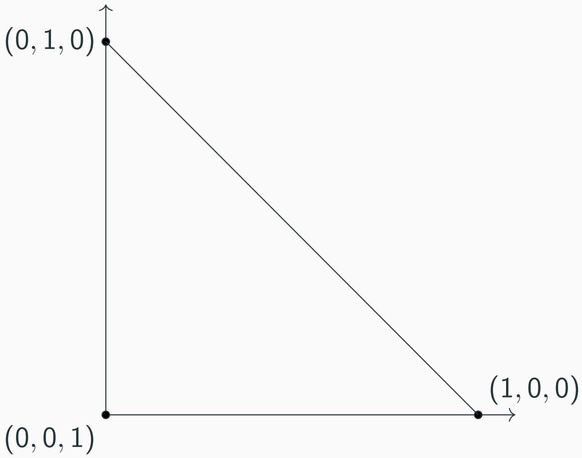
Go Back

- When $n = m = 3$, $f \succeq_B g$ iff the triangle generated by f^1, f^2, f^3 includes the one generated by g^1, g^2, g^3



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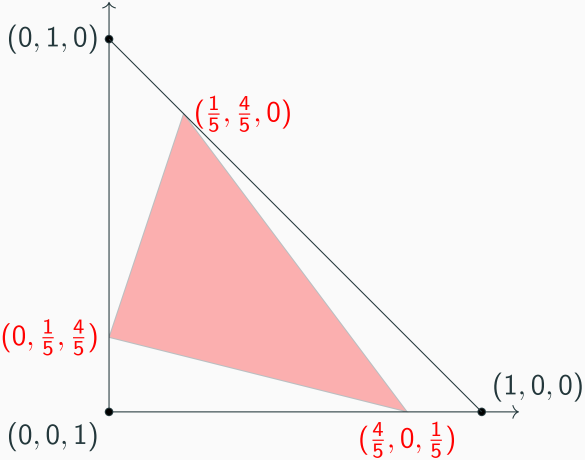

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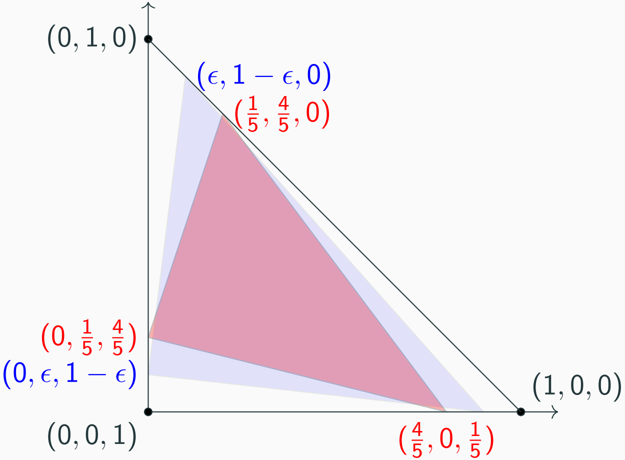

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Quasiconvexity

- Observe that there is a permutation of I_3 such that

$$g = \frac{4}{5} \cdot I_3 + \frac{1}{5} \cdot (I_3 \cdot P).$$

- If we impose **quasiconvexity**, with permutation invariance, we have

$$C(I_3) = C(I_3 \cdot P) \geq C\left(\frac{4}{5} \cdot I_3 + \frac{1}{5} \cdot I_3 \cdot P\right) = C(g).$$

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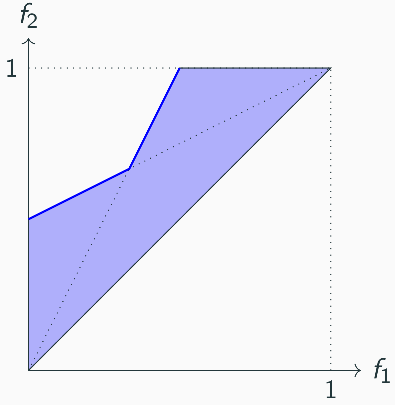
Quasiconvexity

- The following information cost function for binary experiments is not quasiconvex



$$C(f_1, f_2) = \min\left\{\frac{f_2}{f_1}, \frac{1-f_1}{1-f_2}\right\}$$
$$= \min\{\alpha, \beta\}$$

Go Back

Garbling Quasiconvexity

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$C \in \mathcal{C}_m$ is *garbling-quasiconvex* if for all $f \in \mathcal{E}_m$, any finite collection of its garblings, $\{g_1, \cdots, g_n\}$, and $\lambda_0, \cdots, \lambda_n \in [0,1]$ with $\sum_{i=0}^n \lambda_i = 1$,

$$C\Big(\lambda_0 f + \sum_{i=1}^n \lambda_i g_i\Big) \leq \max\{C(f), C(g_1), \cdots, C(g_n)\}$$

Theorem 4

$C \in \mathcal{C}_m$ is Blackwell monotone if and only if (i) C is permutation invariant; (ii) C is garbling-quasiconvex; and (iii) for all $f \in \mathcal{E}_m$,

Go Back

$$\langle \nabla^{i \to j} C(f), f \rangle \leq 0.$$

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Go Back

$$\langle \nabla^{i \to j} C(f), f \rangle \leq 0.$$

Proof of Theorem 3

[Sublinearity ⇒ Blackwell Monotonicity]

- From sublinearity, we can show that C is convex.

- Consider the garbling of replacing s_j to s_k with prob. ϵ:

$$\begin{aligned}
\Delta C &= \psi(f^k + \epsilon \cdot f^j) + \psi((1-\epsilon)f^j) - \Big[\psi(f^k) + \psi(f^j)\Big] \\
&= \psi(f^k + \epsilon \cdot f^j) + (1-\epsilon) \cdot \psi(f^j) - \psi(f^k) - \psi(f^j) \\
&= \psi(f^k + \epsilon \cdot f^j) - \psi(f^k) -
\end{aligned}$$

▸ Go Back

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[Sublinearity $\Rightarrow$ Blackwell Monotonicity]

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\Delta C &= \psi(f^k + \epsilon \cdot f^j) + \psi((1-\epsilon)f^j) - \left[\psi(f^k) + \psi(f^j)\right] \\
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Proof of Theorem 3

[Sublinearity $\Rightarrow$ Blackwell Monotonicity]

- From sublinearity, we can show that C is convex.

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Proof of Theorem 3

[Blackwell Monotonicity $\Rightarrow$ Sublinearity]

1. **Positive homegenity:** Note that $\psi(0) = 0$. For any $k \in \mathbb{N}$,

   $$[\hat{f}, 0, \cdots, 0, 1-\hat{f}] \sim_B [\hat{f}/k, \hat{f}/k, \cdots, \hat{f}/k, 1-\hat{f}] \quad \Rightarrow \quad \psi(\hat{f}) = k\ \psi(\hat{f}/k).$$

   Then, for any $(k, l) \in \mathbb{N}^2$, we also have

   $$\frac{l}{k}\ \psi(\hat{f}) = l\ \psi\left(\frac{\hat{f}}{k}\right) = \psi\left(\frac{l}{k}\ \hat{f}\right)$$

   By density of $\mathbb{Q}$ in $\mathbb{R}$ and the continuity of ψ, $\psi(\alpha\ \hat{f}) = \alpha\ \psi(\hat{f})$ for all $\alpha \in \mathbb{R}_+$

2. **Subadditivity**:

   $$[\hat{f}, \hat{g}, 1-\hat{f}-\hat{g}] \succeq_B [\hat{f}+\hat{g}, 0, 1-\hat{f}-\hat{g}] \quad \Rightarrow \quad \psi(\hat{f}) + \psi(\hat{g}) \geq \psi(\hat{f}+\hat{g})$$

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Illustration: Quasiconvexity

- Consider COVID tests again
- Test C' is a convex combination of A' and B'

	n	p
$-$	60%	40%
$+$	20%	80%

Test A' (\$10)

	n	p
$-$	80%	20%
$+$	40%	60%

Test B' (\$12)

	n	p
$-$	70%	30%
$+$	30%	70%

Test C' ($\leq$ \$12?)

- Pay someone \$12 to perform the following process:
  1. toss a coin and run Test A' for heads, Test B' for tails
  2. report the result without specifying which test was run

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Application: Costly Persuasion

Gentzkow, Kamenica (2014) Revisited

- Consider a costly persuasion problem with the standard example
  - State: $\{innocent, guilty\}$
  - Receiver's action: **A**cquit or **C**onvict
  - Sender's payoff: $u_S(C) = 1$, $u_S(A) = 0$
  - Receiver's payoff: $u_R(A, innocent) = u_R(C, guilty) = 1$
    $$u_R(C, innocent) = u_R(A, guilty) = 0$$
  - Sender commits to an experiment at some cost
- GK focuses on posterior separable costs (e.g., entropy cost) to utilize concavification technique
- Can we solve this problem with any Blackwell-monotone information cost function?

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Gentzkow, Kamenica (2014) Revisited

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Costly Persuasion with Blackwell-Monotone Information Cost

- It is without loss to consider binary experiments since **R**'s action is binary
  - $f_2 = \Pr(C|\textit{guilty})$ and $f_1 = \Pr(C|\textit{innocent})$

- When the prior is p, the sender's problem is

$$\max_{0 \le f_1 \le f_2 \le 1} pf_2 + (1-p)f_1 - C(f_1, f_2)$$

subject to

$$\frac{pf_2}{pf_2 + (1-p)f_1} \ge \frac{1}{2}.$$

- When $p \ge 1/2$, the solution is $f_1 = f_2 = 1$: always convict costlessly

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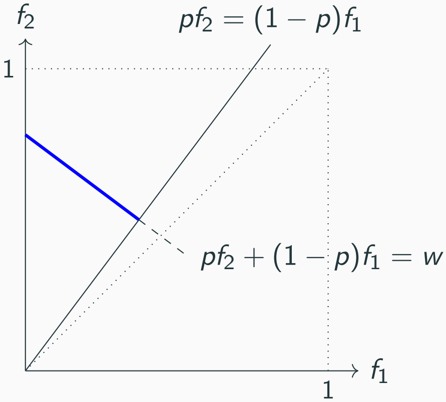
Cost Minimization

- Suppose $p < 1/2$.
- Cost minimization problem under $pf_2 + (1-p)f_1 = w$:

$$\min C(f_1, f_2) \quad \text{s.t.} \quad \begin{aligned} pf_2 + (1-p)f_1 &= w, \\ pf_2 &\geq (1-p)f_1 \end{aligned}$$



- **Proposition**: for any Blackwell-monotone information cost function, the cost is minimized when $pf_2 = (1-p)f_1$

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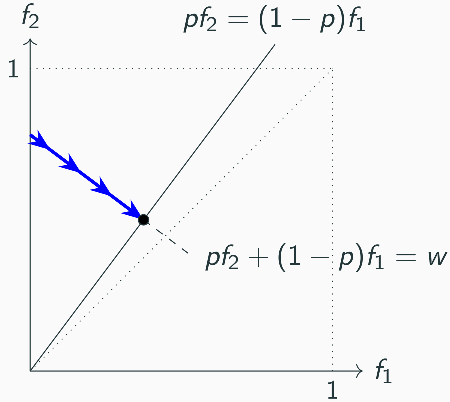

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Sender's Problem

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$$f_2 = \frac{w}{2p} \quad \text{and} \quad f_1 = \frac{w}{2(1-p)}.$$

- Now the sender's problem is

$$\max_{0 \le w \le 2p} w - C\left(\frac{w}{2(1-p)}, \frac{w}{2p}\right) \tag{4}$$

- From here on, a specific cost function is needed

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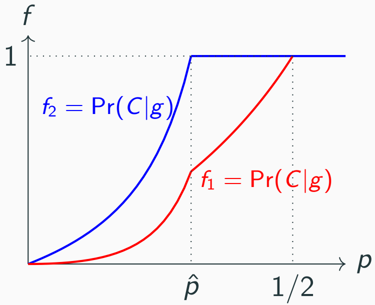
Costly Persuasion with Non-Posterior-Separable Cost

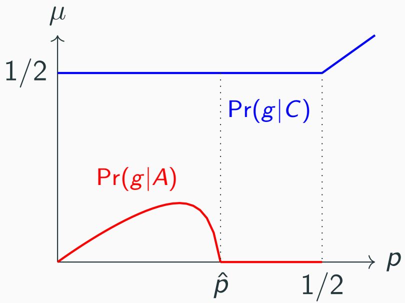
- When $C(f_1, f_2) = (f_2 - f_1)^2$, the solution for $p < 1/2$ is

$$f_2(p) = \min\left\{1,\ \frac{(1-p)^2 p}{(1-2p)^2}\right\} \quad \text{and} \quad f_1(p) = \frac{p}{1-p} \cdot f_2(p).$$

▸ Entropy



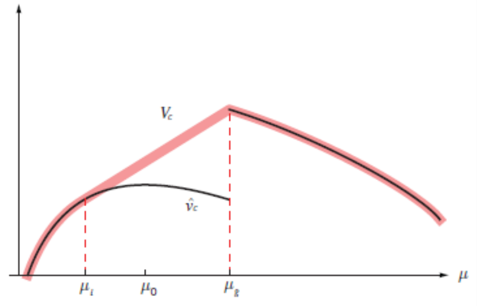
Optimal Experiments



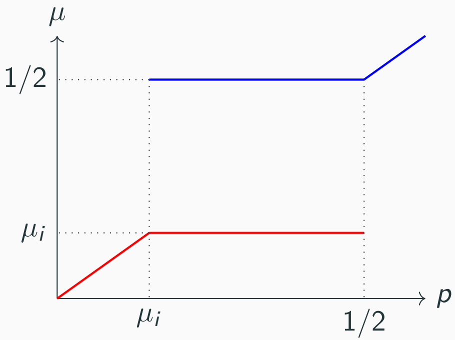
Posteriors

Gentzkow, Kamenica (2014) Revisited

- Entropy cost: $k \cdot \mathbb{E}_{\pi|p}[H(p) - H(\mu_s)]$ where $H(\mu) \equiv -\sum_\omega \mu(\omega)\log(\mu(\omega))$
    - p is prior, and μ_i and μ_g are posteriors from an experiment π



Concavification



Posteriors

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