Research or Development in Innovation Races*

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Abstract

We introduce an innovation game in which two firms dynamically allocate resources across two distinct research and development (R&D) paths: (i) developing an innovative product with the currently available technology; (ii) conducting research to discover a faster technology for posterior development. Firms' optimal R&D strategies depend on the information about their rivals' progress. This creates an incentive for firms to conceal their technological discoveries, thereby slowing down the overall pace of social innovation.

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1 Introduction

In the highly competitive landscape of modern markets, firms face various considerations when determining the direction of their research and development (R&D). One crucial factor is the technology available to competitors. Although recent literature on the direction of innovation often focuses on cases where technology is shared among all firms (e.g., Bryan and Lemus, 2017; Hopenhayn and Squintani, 2021), in practice, much of it is acquired privately.

Consider, for example, the development of COVID-19 vaccines during the recent global pandemic. Pharmaceutical companies relied on two alternative technologies: messenger RNA (mRNA), as utilized by Moderna and Pfizer-BioNTech, and viral vector, pursued by Oxford-AstraZeneca and Janssen (Johnson&Johnson). The viral vector technology was available to most firms at the outset of COVID-19 outbreak.[1] In contrast, the mRNA technology was not in practical use before the COVID-19 outbreak. Therefore, pharmaceutical firms needed to acquire fundamental knowledge (e.g., a method to protect the mRNA sequence in the bloodstream during delivery) to utilize this methodology.[2] The acquisition of such technology is private.

This raises the question of how the private acquisition of technology impacts research and development dynamics in innovation races. Moreover, it leads to another question: does the patent system effectively incentivize firms to disclose their interim technology in pursuit of ultimate innovation?

To address these questions, we consider a model in which two firms race to develop an innovative product. The firm that achieves the first successful product development receives a fixed reward, such as a temporary monopoly profit. Firms allocate their limited resources across various pathways. One such approach involves developing the innovative product utilizing presently *available technologies*. We assume that successful development

[1] Viral vector technology was used during recent disease outbreaks including the 2014-2016 Ebola outbreak in West Africa. For more information, see the web page of the Centers for Disease Control and Prevention (CDC): `https://www.cdc.gov/coronavirus/2019-ncov/vaccines/different-vaccines/viralvector.html`.

[2] The mRNA technology offers the advantage of enabling firms to develop vaccines using readily available materials. Hence, vaccines can be developed faster compared to methods such as viral vector. For more information, see the web page of the Centers for Disease Control and Prevention (CDC): `https://www.cdc.gov/coronavirus/2019-ncov/vaccines/different-vaccines/mrna.html`.

with an available technology requires a single breakthrough. Furthermore, companies can allocate resources to *research*, which enables them to discover a faster new technology. Thus, developing a product via this path requires two steps: first, acquiring the technology, and second, developing the product using it.

By considering these different pathways, our model highlights the trade-off faced by resource-constrained firms. On one hand, allocating more resources to researching a new technology reduces the rate of short-term development, as fewer resources are available for developing with currently available technologies. On the other hand, this approach enhances the probability of obtaining a superior technology, thereby increasing the expected rate of development in the future.

We examine two distinct informational settings in the model: a firm's *research progress*, represented by the set of available technologies, can either be public or private information. In cases where firms' research progress is public, a firm can base its choice of an R&D path not only on its own progress but also on that of its competitors. We characterize the unique Markov perfect equilibrium (MPE) in this case. Under certain parametric conditions, we show that the unique MPE is the 'fall-back' equilibrium, wherein both firms begin by allocating their resources to research, and once one discovers new technology, the other firm switches to developing with existing technologies.

When research progress is private, firms cannot condition their resource allocations on the technologies available to their competitors. Hence, the fall-back equilibrium is not feasible in this case. Firms, however, form beliefs about technology access of competitors. These beliefs are shaped by two key forces. First, as time passes and more resources are allocated to research, it becomes more likely that opponents have a better technology available. Second, the absence of successful product developments indicates that opponents are less likely to have access to the highly effective technology.

The main challenge of equilibrium characterization lies in expressing a firm's problem as an optimal control problem. The solution to this problem is dependent not only on current beliefs but also on the competitor's future allocations, which influence the probability of winning the race. This interdependence complicates the characterization all fixed points of the game's best response correspondence in the infinite-dimensional space of strategies.

Nonetheless, by concentrating on strategies that generate monotone expected development rates, we can derive structural properties of the best response correspondence. Specifically, we establish a single-crossing property of the relative incentives to do research and exploit this property to uniquely characterize the equilibrium that features monotone expected development rates. This equilibrium presents two phases: in the first phase, firms without the new technology conduct research and the beliefs about competitors having discovered new technology increase over time. In the second phase, which starts at a deterministic time, firms without the new technology partially allocate their resources to developing with current technology and conducting research in a way such that consistent beliefs remain constant.

Lastly, we explore firms' incentives to patent and license new technologies. In this extension, firms that obtained the new technology have the option to publicly file a patent application. Patents are attractive because, when granted, they assign the recipient the exclusive right to use the technology and the option to license the technology to a competitor for a fee. The patent is granted with certainty to the first firm to apply for it, provided that no other firm has previously obtained the new technology. However, if another firm obtained the new technology before the patent application, the probability of the patent being granted drops to $1 - \alpha$.[3]

By patenting and licensing new technology, firms avoid duplicating efforts—meaning that they do not waste resources on research for technology that has already been discovered—and develop at the technological frontier. We first show that if a firm obtains a patent, there is always a licensing fee such that both the licensor and the licensee are willing to accept. Thus, in the spirit of Coase (1960), the efficient allocation of resources is achieved. Moreover, we find that when the progress of firm research is public information, firms benefit from patenting the new technology independently of the level of trade secret protection α.

When research progress is private, however, firms face a trade-off between licensing and concealing the discovery of the new technology. By concealing the discovery, a firm can

[3]The parameter α captures the possibility that the patent application successfully challenged under an argument of trade secret protection or prior commercial use. Thus, we focus on first-to-file patent systems with some level of trade secret protection, such as the protection given by the defense to infringement based on prior commercial use (US Code §273). However, it is possible to extend our framework to capture *first-to-invent* patent systems as well. For more information on trade secrets and patents, see the World Intellectual Property Organization's website: `https://www.wipo.int/about-ip/en/`. Also, see Lobel (2013) for examples.

prevent the rival firm from adjusting its R&D path and, in this way, increase its probability of winning the race. Based on this intuition, we show that when the trade secret protection level and the *stake*—the size of the reward of winning the race relative to the cost associated with the duration of the race—are sufficiently high, there is an equilibrium where firms conduct research and conceal their discoveries of the new technology, even when patent holders have all the bargaining power in licensing negotiations. This equilibrium behavior prevents the spillage of new technology, which is detrimental to the overall speed of innovation.

Related Literature

This paper contributes to the innovation literature by introducing a model with two characteristics. First, there are different avenues towards innovation: developing with the old technology and doing research for the new technology. Second, one of the paths involves multiple stages: once a firm discovers the new technology, then the firm develops the innovative product with it.

With respect to the first characteristic, there is a recent branch of the literature that studies races where there are different routes to achieve a final objective. Das and Klein (2020) and Akcigit and Liu (2016) study a patent race where two firms compete for a breakthrough and there are two methods to get the breakthrough: a safe method and a risky method. In Das and Klein (2020) the safe method has a known constant arrival intensity while the risky method has an unknown constant arrival intensity. In Akcigit and Liu (2016), instead, the safe method has a known payoff associated with breakthrough arrival, while there is uncertainty about the payoff if the risky method is used. In this paper, firms face no uncertainty about whether the innovation is feasible. Instead, they are uncertain whether their rivals possess the new and faster technology. Another related paper concerning this characteristic is the study by Bryan and Lemus (2017). They introduce a general model of direction of innovation using acyclic graphs, where a node denotes a set of available inventions in society, and an edge represents a feasible innovation path. They assume that whenever a new invention is discovered, the first firm to invent it receives the prize, and the access to the invention is given to all the other firms. In contrast, in our model, interim discoveries can remain private.

The second characteristic, multi-stage innovation, is also widely studied in the literature, e.g., Scotchmer and Green (1990); Denicolò (2000); Green and Taylor (2016); Song and Zhao (2021). Our paper shares the framework with these in that we use two sequential Poisson discovery processes and ask whether a firm would patent the first discovery or not. A feature setting apart from their works is that there is another path that only requires one but slower breakthrough toward innovation. This feature connects our model to Carnehl and Schneider (2022) and Kim (2022) in the sense that players can choose between a sequential approach—which requires two breakthroughs—and a direct approach, which requires only one breakthrough, but its riskier or slower. Our model mainly differs from theirs in that multiple players compete by choosing between these approaches, whereas Carnehl and Schneider (2022) considers a problem by a single decision maker and Kim (2022) studies a contracting setup between a principal and an agent. In their studies, a key factor for a player to choose the direct approach is a deadline that is either exogenously given or endogenously determined to reduce moral hazard. In contrast to these, a deadline is not imposed in our model. Rather, the race with the rival firm may induce a firm to develop with the old technology, which can be considered as a direct approach.

Another hallmark of this paper is its consideration of 'interim' discoveries. Therefore, it is naturally related to the literature on licensing of interim R&D technology, e.g., Bhattacharya et al. (1992); d'Aspremont et al. (2000); Bhattacharya and Guriev (2006); Spiegel (2008). In these papers it is assumed that firms already know which of them has superior technology, i.e., the firm that will license the technology is exogenously given. Unlike in those studies, we allow firms to choose when to license (and even allow them not to license), i.e., the licensing decision is endogenous.

We also contribute to the literature on patent vs. secrecy by introducing a novel incentive to conceal a firm's discovery: hindering its rival's strategic response.[4] Previous studies mainly focused on the limited protection power of patents. For example, the seminal article by Horstmann et al. (1985) posits that "patent coverage may not exclude profitable imitation." Thus, in their framework, the main reason why a firm may choose secrecy over a patent is not

[4]There exists an extensive body of literature addressing both the empirical and theoretical aspects of the patent vs. secrecy discussion. A comprehensive overview of this literature can be found in the excellent survey paper authored by Hall et al. (2014).

to be imitated.[5] Another limitation of a patent is that it expires in a finite time. For instance, Denicolò and Franzoni (2004) consider a framework where a patent gives the patenting firm monopoly power only for a certain period of time (and no profit after expiration), whereas secrecy can give indefinite monopoly power to a firm but it can be leaked or duplicated by a rival with some probability. On the contrary, in this paper, we abstract from the restrictions of patents and focus analysis on the potential advantages of concealment.

This paper is related to the recent studies on information disclosure in priority races, e.g., Hopenhayn and Squintani (2016); Bobtcheff et al. (2017).[6] In those papers, once a firm makes a breakthrough, the innovation value grows as time passes until one of the firms files a patent. Thus, firms face a tradeoff between disclosing to claim the priority and delaying in order to grow the innovation value. On the contrary, in this paper, the value of innovation is fixed and the discovery of the new technology only allows the firm to develop the innovative product faster. Therefore, a firm may delay the disclosure purely to confound the rival's R&D decisions.

Lastly, a closely related study is the recent paper by Chatterjee et al. (2023). They also explore a disclosure problem concerning an intermediate research finding in a two-step project. The key distinction lies in their assumption of an exogenous payoff from disclosing the intermediate discovery, whereas in our paper, the payoff is endogenously determined, considering the option to develop with the old technology. As in our paper, they also find that a high reward of the final discovery may induce firms to conceal their intermediate discoveries, resulting in socially inefficiency.

2 Model

We consider a race between two firms, A and B, trying to develop an *innovative product*. Time is continuous and infinite: $t \in [0, \infty)$. Firms can develop the innovative product using

[5]Many subsequent papers study the imitation threat and potential patent infringement, e.g., Gallini (1992); Takalo (1998); Anton and Yao (2004); Kultti et al. (2007); Kwon (2012); Zhang (2012); Krasteva (2014); Krasteva et al. (2020).

[6]There is a strand of literature on strategic disclosure, e.g., Lichtman et al. (2000); Baker and Mezzetti (2005); Gill (2008); Baker et al. (2011); Ponce (2011). These works are well summarized in Section 3.3 of Hall et al. (2014).

either *old* or *new* technology, each with a different development speed. At the outset of the race, both firms have access to an old technology, but they can gain access to a new technology by conducting research.

Each firm owns one unit of resources per unit of time, which can be allocated for either conducting research to discover the new technology or developing the innovative product. When a firm gains access to the new technology, it directs all its resources towards product development, resulting in a development rate of λ_H. When Firm i does not yet possess the new technology, it allocates a fraction $\sigma_t^i \in [0,1]$ to 'research' at time t. Then, $1-\sigma_t^i$ is the amount of resources that Firm i allocates to 'develop' the innovative product using the old technology, and the product can be stochastically developed at rate $\lambda_L \cdot (1-\sigma_t^i)$. In addition, Firm i stochastically discovers the new technology at rate $\sigma_t^i \cdot \mu$, where μ is a constant parameter. Firm i can observe its own discovery of the new technology. We consider two different settings regarding whether Firm i can observe Firm j's research progress, whether it has discovered the new technology or not. The parameters μ, λ_L, and λ_H are positive.

The race ends once one of the firms develops the innovative product. During the race, firms pay a flow cost $c > 0$. The first firm to develop the innovative product receives a lump-sum reward worth Π.[7] Firms do not discount the future and maximize their expected total payoff. The successful development of the innovative product is publicly observable. Thus, firms always know whether they are still on the race. However, firms do not observe their opponents' resource allocations over time.

For the rest of the paper, we make the following parametric assumption:

$$\Pi - \frac{c}{\mu} - \frac{c}{\lambda_H} > \Pi - \frac{c}{\lambda_L} > 0. \tag{2.1}$$

The first inequality states that when there is only one firm, conducting research and developing with the new technology is more efficient than developing with the old technology. Note that this condition is equivalent to $\frac{1}{\mu} + \frac{1}{\lambda_H} < \frac{1}{\lambda_L}$, implying that in expectation, the product can be developed faster by conducting research and developing with the new technology.

[7] We model the race as winner-takes-all competition. This payoff structure has been commonly used in the innovation race literature, e.g., Loury (1979); Lee and Wilde (1980); Denicolò and Franzoni (2010).

Then, the second inequality implies that developing with the old technology is profitable.[8]

3 Benchmark: Constant Development Rate

As a benchmark, imagine a scenario where Firm j does not engage in the resource allocation problem and instead its rate of development is held constant at λ. Solving this benchmark problem for Firm i provides valuable insights for the main analysis of the paper.

In the following proposition, we show that Firm i's resource allocation is determined by the following threshold:

$$\lambda_\star := \mu\lambda_H\left(\frac{1}{\lambda_L} - \frac{1}{\mu} - \frac{1}{\lambda_H}\right) > 0.^{9} \tag{3.1}$$

The proof is in Appendix A.

Proposition 3.1. *Suppose that Firm j has a constant development rate λ.*

(a) When $\lambda < \lambda_\star$, Firm i conducts research.

(b) When $\lambda > \lambda_\star$, Firm i develops with the old technology.

(c) When $\lambda = \lambda_\star$, Firm i is indifferent between conducting research and developing with the old technology.

To illustrate the intuition behind this proposition, in Figure 1, we depict the probability distributions of development times under Firm i's two different policies: (i) developing with the old technology (red dotted curve); and (ii) conducting research and then developing with the new technology (blue solid curve). The old technology is more likely to result in product development in a short time frame, as it requires only one breakthrough. In contrast, development with the new technology requires two breakthroughs. Although it has a lower expected development time, is less likely to lead to quick development. Therefore, when competing against a fast-development rival, a firm may choose the old technology to enhance its chances of winning the race.

[8]This assumption leads us to abstract away from firms' exit decisions: the flow expected payoff of staying in the race is at least $\lambda_L\Pi$, which is greater than the flow cost (c). If this assumption is violated, firms completely disregard the old technology and, therefore, there is no strategic choice of innovation path.

[9]Note that $\lambda_\star$ is a function of λ_L, μ and λ_H, but we suppress it to ease the notation.

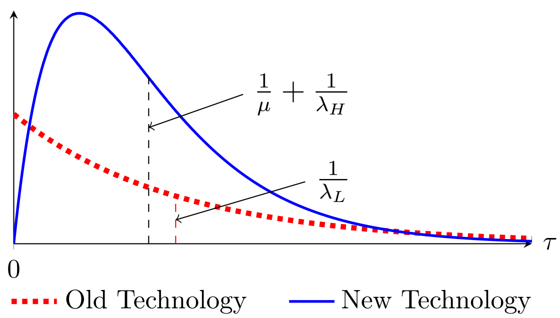

Figure 1: Probability distribution functions of a firm's development time

4 Public Information Setting

We explore a setting where firms' research progress is publicly available information. In this case, the set of firms that have successfully obtained the new technology is common knowledge, and we represent it as a state variable denoted by $\omega \in \Omega := \{\{A, B\}, \{A\}, \{B\}, \emptyset\}$.

We focus on equilibria in Markov strategies. Specifically, Firm i's Markov strategy is defined as $\mathbf{s}^i : \Omega \to [0, 1]$, where $\mathbf{s}^i(\omega)$ denotes the amount of resources allocated by Firm i to research in state ω. A pair of Markov strategies $(\mathbf{s}^A, \mathbf{s}^B)$ constitutes a Markov perfect equilibrium (MPE) if, for any given state, each firm's strategy is the best response to the opponent's strategy. Next, we introduce three benchmark Markov strategies.

Definition 1. (a) The *research strategy* $\mathbf{s}^i_R$ for Firm i fully allocates resources to research regardless of the opponent's progress ($\mathbf{s}^i_R := \mathbb{1}_{\{\omega | i \notin \omega\}}$).[10]

(b) The *fall-back strategy* $\mathbf{s}^i_F$ fully allocates resources to research if neither firm has the new technology. If one of the firms has obtained the new technology, it fully allocates resources to development ($\mathbf{s}^i_F := \mathbb{1}_{\{\emptyset\}}$).

(c) The *direct-development strategy* $\mathbf{s}^i_D$ fully allocates the resources to development regardless of the state ($\mathbf{s}^i_D := 0$).

The following theorem shows that a unique MPE exists, with firms adopting one of the benchmark Markov strategies based on parameters.

[10] The function $\mathbb{1}_X$ is an indicator function: $\mathbb{1}_X(\omega) = 1$ if $\omega \in X$ and $\mathbb{1}_X(\omega) = 0$ if $\omega \notin X$.

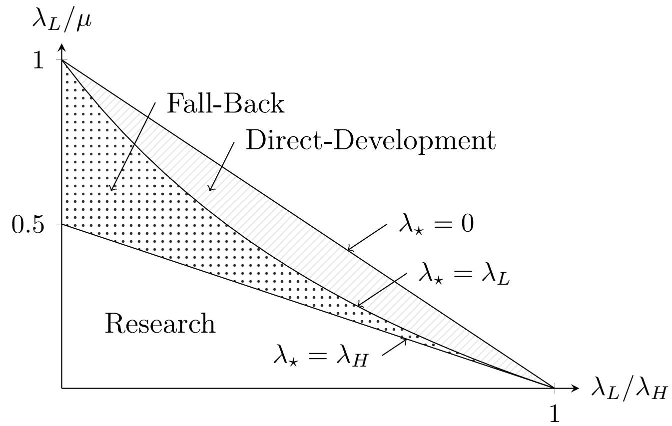

Figure 2: Markov Perfect Equilibrium in the Public Information Setting

Theorem 1. *Suppose that firms' research progress is public information. Then, the Markov perfect equilibrium is uniquely characterized as follows:*

(a) if $\lambda_\star > \lambda_H$, both firms play their respective research strategies $(\mathbf{s}_R^A, \mathbf{s}_R^B)$;

(b) if $\lambda_H > \lambda_\star > \lambda_L$, both firms play the fall-back strategies $(\mathbf{s}_F^A, \mathbf{s}_F^B)$;

(c) if $\lambda_L > \lambda_\star$, both firms play the direct-development strategies $(\mathbf{s}_D^A, \mathbf{s}_D^B)$.

This result primarily stems from Proposition 3.1. First, assume that $\lambda_\star > \lambda_H$. Note that the development rate never exceeds λ_H for any state and strategy, thus, it is always lower than $\lambda_\star$. Referring to (a) in Proposition 3.1, we can infer that firms would conduct research regardless of the rival's strategy. Therefore, both firms employing the research strategy would constitute an equilibrium.

Next, suppose that $\lambda_L > \lambda_\star$. If a firm develops with the old technology, its development rate is λ_L, which is greater than $\lambda_\star$. Then, by (b) of Proposition 3.1, the rival firm would also develop with the old technology. Therefore, both firms adopting the direct-development strategy would constitute an equilibrium.

Last, assume that $\lambda_H > \lambda_\star > \lambda_L$. Consider the case where only Firm j has discovered the new technology, i.e., $\omega = \{j\}$. Then, Firm j will develop with the new technology, i.e., the development rate of Firm j is λ_H, which is higher than $\lambda_\star$. Then, by (b) of Proposition 3.1, Firm i develops with the old technology. Since $\lambda_\star > \lambda_L$, the direct-development strategy cannot constitute an equilibrium. Thus, among the benchmark strategies, the fall-back strategy is the only candidate for an equilibrium strategy under this parametric region, and it indeed is.

It is worth noting that we do not limit our analysis solely to symmetric equilibrium; instead, symmetry emerges as a result of our analysis. Figure 2 illustrates the relevant parametric regions in the above theorem.

An intuitive way of interpreting our result is to fix λ_L and λ_H and perform a comparative statics with respect to the research rate μ. Then, in Figure 2, given an x coordinate, the y coordinate decreases as μ increases. Specifically, let $\underline{\mu}$, μ_L and μ_H be the solutions to the equations $\lambda_\star = 0$, $\lambda_\star = \lambda_L$ and $\lambda_\star = \lambda_H$, respectively. Then, Theorem 1 can be rewritten as follows.

Corollary 1. *Under the public information setting, the unique MPE is characterized as follows: (i) when the rate of research is high ($\mu > \mu_H$), both firms play their respective research strategies; (ii) when the rate of research is intermediate ($\mu_H > \mu > \mu_L$), both firms play the fall-back strategies; and (iii) when the rate of research is low ($\mu_L > \mu > \underline{\mu}$), both firms play the direct-development strategies.*

5 Private Information Setting

In this section, we consider the private information framework, in which firms do not observe whether their opponents have the new technology. In this setting, as before, a firm with the new technology fully allocates the resources to development. However, a firm without the new technology can only condition its resource allocation on the calendar time t. An *allocation policy* is a right-continuous function $\boldsymbol{\sigma} : \mathbb{R}_+ \to [0, 1]$ that represents the research allocation at a given time, conditional on not having obtained the new technology. We denote $\mathcal{S}$ as the set of allocation policies.

5.1 Preliminaries

We begin by laying out some essential elements for the equilibrium characterization.

New Technology Access and Development Rate Let $\mathbf{p}_{\boldsymbol{\sigma}}$ be the probability that a firm following allocation $\boldsymbol{\sigma}$ obtains the new technology by time t, conditional on not having developed the product yet.[11] In other words, when Firm A is committed to the policy $\boldsymbol{\sigma}$, Firm B's belief that Firm A has the access for the new technology at time t is $\mathbf{p}_{\boldsymbol{\sigma}}(t)$. The following proposition characterizes, the evolution of $\mathbf{p}_{\boldsymbol{\sigma}}$ over time for any $\boldsymbol{\sigma} \in \mathcal{S}$.

Proposition 5.1. *For any allocation policy $\boldsymbol{\sigma} \in \mathcal{S}$, the conditional probability $\mathbf{p}_{\boldsymbol{\sigma}}(t)$ satisfies the initial condition $\mathbf{p}_{\boldsymbol{\sigma}}(0) = 0$ and evolves according to the differential equation $\dot{\mathbf{p}}_{\boldsymbol{\sigma}}(t) = \delta(\mathbf{p}_{\boldsymbol{\sigma}}(t), \boldsymbol{\sigma}(t))$, where $\delta : [0,1] \times [0,1] \to \mathbb{R}$ is given by:*

$$\delta(p, \sigma) := \underbrace{\mu \cdot \sigma \cdot (1-p)}_{DE} - \underbrace{(\lambda_H - (1-\sigma) \cdot \lambda_L) \cdot p \cdot (1-p)}_{SRE}. \tag{5.1}$$

The function δ highlights two distinct effects of the resource allocation $\boldsymbol{\sigma}(t)$ on the evolution of $\mathbf{p}_{\boldsymbol{\sigma}}$, captured by the two terms in (5.1). First, if the firm does not have the new technology—which happens with probability $(1 - \mathbf{p}_{\boldsymbol{\sigma}}(t))$—the new technology is discovered at rate $\mu \cdot \boldsymbol{\sigma}(t)$. We dub the effect of this arrival rate the *duration effect* (DE). On the other hand, the lack of development success indicates that it is less likely that the firm has the new technology. This second effect, which we dub the *still-in-the-race* effect (SRE), is reflected in the second term.[12] Notice that the SRE is proportional to $\lambda_H - (1 - \boldsymbol{\sigma}(t))\lambda_L$, which is the difference in the rate of development of the firm with and without the new technology.

The access to the new technology and the allocation of resources determine the development rate of the firm. We can define the development rate of a policy as follows.

Definition 2. Given a policy $\boldsymbol{\sigma} \in \mathcal{S}$, the associated *development rate function* $\mathbf{h}_{\boldsymbol{\sigma}}$ is defined as $\mathbf{h}_{\boldsymbol{\sigma}}(t) = \xi(\mathbf{p}_{\boldsymbol{\sigma}}(t), \boldsymbol{\sigma}(t))$ where $\xi : [0,1] \times [0,1] \to \mathbb{R}$ is given by:

$$\xi(p, \sigma) := p \cdot \lambda_H + (1-p) \cdot (1-\sigma) \cdot \lambda_L. \tag{5.2}$$

The first term of (5.2) captures that a firm with the new technology develops at rate λ_H.

[11] See Appendix C.1 for the formal definition of $\mathbf{p}_{\boldsymbol{\sigma}}$.

[12] Similar types of belief updating can be found in the strategic experimentation literature, e.g., Keller et al. (2005); Bonatti and Hörner (2011). The main difference is that, in that literature, the agents form beliefs about whether a project is good or bad. In this paper, on the other hand, firms only form beliefs about the technology access of the rival.

If the firm does not have the new technology, they only develop at rate $(1-\boldsymbol{\sigma}(t))\lambda_L$.

Expected Payoffs and Solution Concept Given a firm and its rival's allocation policies $\boldsymbol{\sigma}$ and $\hat{\boldsymbol{\sigma}}$, we can express the firm's expected payoff in terms of the associated development rates, $\mathbf{h}_{\boldsymbol{\sigma}}$ and $\mathbf{h}_{\hat{\boldsymbol{\sigma}}}$, as follows:

$$\mathcal{U}(\boldsymbol{\sigma}, \hat{\boldsymbol{\sigma}}) = \int_0^{\infty} e^{-\int_0^t \{\mathbf{h}_{\boldsymbol{\sigma}}(s)+\mathbf{h}_{\hat{\boldsymbol{\sigma}}}(s)\}\ ds} \cdot (\mathbf{h}_{\boldsymbol{\sigma}}(t)\cdot \Pi - c)\, dt. \tag{5.3}$$

Intuitively, the exponential term captures the probability that no firm has developed the innovative product by time t, i.e. the probability that the race is still ongoing. In that case, the firm captures an expected flow payoff equal to $\mathbf{h}_{\boldsymbol{\sigma}}(t)\cdot\Pi$, due to the potential development of the innovative product, while incurring the flow cost c.

As in the literature on dynamic games with unobservable actions (e.g., Bonatti and Hörner, 2011), we aim to characterize the Nash equilibria (NE) in this game. A unique feature that makes solving the NE challenging in this model is the possibility of a non-monotonic belief. For instance, a firm that has been conducting research may suddenly allocate all the resources to direct development ($\boldsymbol{\sigma}(t) = 0$), making (5.1) negative. To overcome this challenge, we focus on the NE with the following property.

Definition 3. An allocation policy $\boldsymbol{\sigma} \in \mathcal{S}$ exhibits the *monotone development rate (MDR) property* if $h_{\boldsymbol{\sigma}}$ is weakly increasing. An allocation policy profile $(\boldsymbol{\sigma}_A, \boldsymbol{\sigma}_B)$ is a *Nash equilibrium with monotone development* (MDNE) if (i) $(\boldsymbol{\sigma}_A, \boldsymbol{\sigma}_B)$ is a Nash equilibrium; and (ii) $\boldsymbol{\sigma}_A$ and $\boldsymbol{\sigma}_B$ have the MDR.

5.2 Equilibrium Chacterization

We begin by defining a pair consisting of a probability and a resource allocation that can emerge in an MDNE.

Definition 4. A pair of a probability and an allocation, $(p_\star, \sigma_\star) \in (0,1)^2$, is called a *steady state* if (i) $\xi(p_\star, \sigma_\star) = \lambda_\star$; and (ii) $\delta(p_\star, \sigma_\star) = 0$.

Under the steady state, the belief is stationary ($\delta(p_\star, \sigma_\star) = 0$) and the development rate is $\lambda_\star$, implying that firms are indifferent between conducting research and developing with the old technology (Proposition 3.1 (c)). Thus, once both firms reach the steady state belief $p_\star$, allocating $\sigma_\star$ onward can be part of an MDNE. The following proposition provides a condition under which the steady state exists.

Proposition 5.2. *There exists a steady state $(p_\star, \sigma_\star)$ if and only if $\lambda_\star \in (\lambda_L, \min\{\lambda_H, \mu\})$.*

Now we provide the characterization of the MDNE.

Theorem 2. *Suppose that firms' research progress is private information. Then, an MDNE exists and is uniquely characterized as follows:*

(a) if $\lambda_\star < \lambda_L$, firms play ***direct-development policies****: $\boldsymbol{\sigma}_A = \boldsymbol{\sigma}_B = 0$;*

(b) if $\lambda_\star > \min\{\lambda_H, \mu\}$, firms play ***research policies****: $\boldsymbol{\sigma}_A = \boldsymbol{\sigma}_B = 1$;*

(c) if $\lambda_\star \in (\lambda_L, \min\{\lambda_H, \mu\})$, firms play ***stationary fall-back policies****: $\boldsymbol{\sigma}_A = \boldsymbol{\sigma}_B = \boldsymbol{\sigma}^{SF}$, which is defined as follows: for some $T_\star$, (i) $\boldsymbol{\sigma}^{SF}(t) = 1$ if $t < T_\star$; (ii) $\boldsymbol{\sigma}^{SF}(t) = \sigma_\star$ if $t \geq T_\star$. Moreover, $\mathbf{p}_{\boldsymbol{\sigma}^{SF}}(t) = p_\star$ for all $t \geq T_\star$.*

We provide the proof sketch in Appendix C.3, and the formal proof is in Online Appendix. As in the public information setting, symmetry is obtained as a result.

When the parameters are such that $\lambda_\star > \lambda_H$ or $\lambda_\star < \lambda_L$, we know from Theorem 1, specifically from points (a) and (c), that firms do not tailor their allocation to the opponent's progress even when this information is publicly available. Thus, under the private information setting, it is intuitive that the firms adopt the same equilibrium allocations as in the public information setting for those regions.

The more interesting case occurs when $\lambda_\star \in (\lambda_L, \lambda_H)$. As shown in Theorem 1 (b), for these parameters, firms employ the fall-back strategy under the public information setting. However, this strategy is no longer feasible under the private information setting, as it requires adjusting resource allocations based on information about the rival's access to the new technology. Despite this infeasibility, the optimality of the fall-back strategy under the public information setting suggests that if a firm believes that the rival likely has the new technology, it may allocate more resources to developing with the old technology.

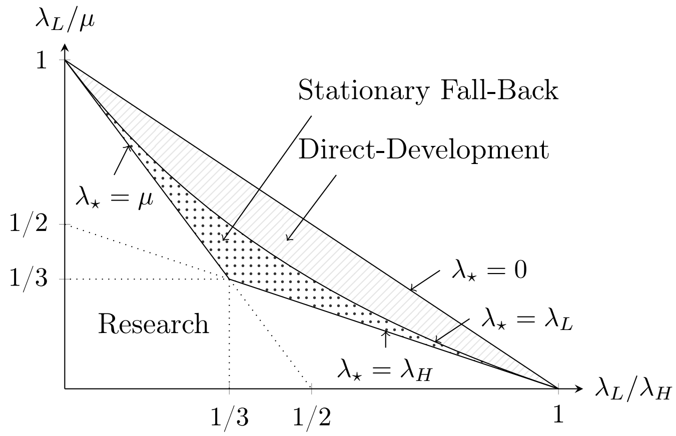

Figure 3: Nash Equilibrium with Monotone Development in the Private Information Setting

Based on these insights, our theorem shows that when the steady state exists ($\lambda_\star \in (\lambda_L, \min\{\mu, \lambda_H\})$), both firms conduct research until their beliefs reach the steady state probability $p_\star$. Then, both firms implement the steady state allocation $\sigma_\star$ from that point on, making the belief stationary at $p_\star$. Last, if $\mu < \lambda_\star < \lambda_H$, the still-in-the-race effect prevents the belief from exceeding a certain level.[13] This keeps the development rate lower than $\lambda_\star$, thus, by Proposition 3.1, it is optimal for both firms to conduct research indefinitely.

5.3 Expected Development Times

Now that we have characterized the equilibrium allocations, we can compare how the expected development times differ across the different settings.[14] Figure 4 illustrates how the expected development times change with respect to the research rate μ under the first-best scenario, the public information setting, and the private information setting.

Consider the first-best scenario where the planner allocates firms' resources and obtains the property right for the new technology upon research success of either firm. The planner directs firms to conduct research, and once a firm discovers the new technology, the planner shares it with another firm. Both firms then begin developing the product with the new technology. Thus, the expected completion time is $\frac{1}{2\mu} + \frac{1}{2\lambda_H}$.

[13]Specifically, it cannot exceed μ/λ_H. See Lemma OA.2.1.

[14]Since there is no discounting and the reward will be received by one of the firms, having a lower expected development time is equivalent to having a higher expected total surplus.

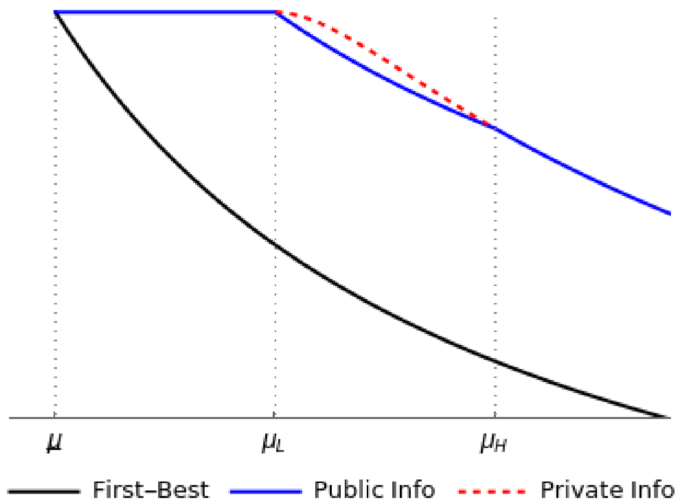

Figure 4: Expected Development Times

Under the public information setting, the expected development time is determined by the equilibrium strategy in Corollary 1. Since the research success of a firm is not spilled over to another firm, the expected development time is significantly longer compared to the first-best scenario.

The expected development time under the private information setting differs from that under the public information setting only when the research rate is intermediate ($\mu \in (\mu_L, \mu_H)$, or equivalently, $\lambda_\star \in (\lambda_L, \lambda_H)$). In this region, as shown by Theorem 2, firms employ stationary fall-back or research policy. This results in a longer expected development time compared to the fall-back equilibrium under the public information setting. This is because the lack of information about the rival's progress hampers the firms' ability to effectively choose the appropriate R&D approach.

6 Patent, License and Trade Secret

In this section, we extend the model by allowing the firms to patent and license the new technology, namely the patent game. The main components of the model remain the same as in the baseline model, with one key difference: once a firm discovers the new technology, it has the option to apply for a patent. If the patent is granted, the patent holder can license the new technology to the rival in exchange for a fee.

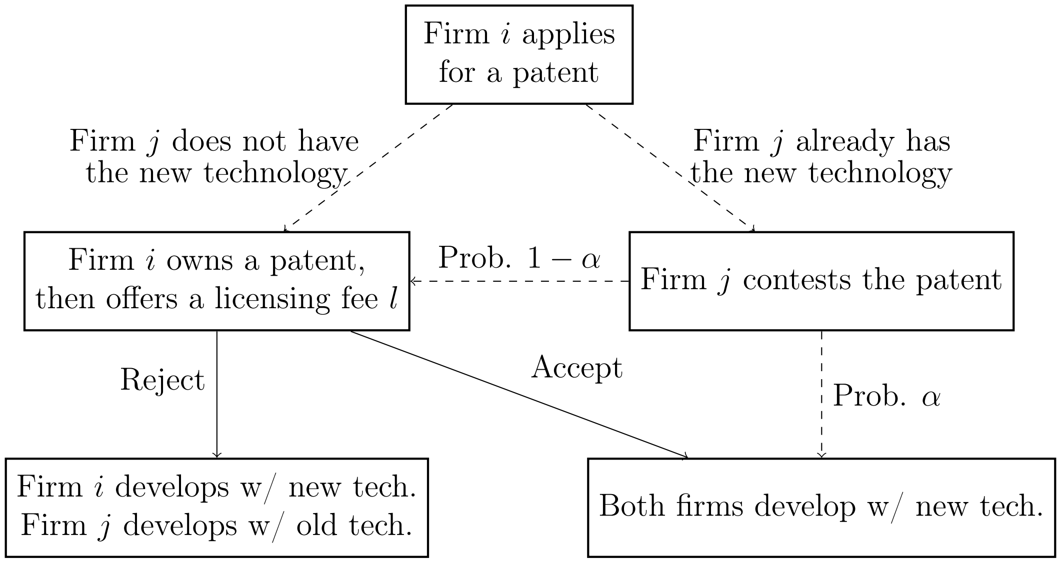

Figure 5: Timing of the patent game after the patent application

When a firm obtains the new technology, it can choose to apply for a patent or not.[15] The probability of the patent being granted depends on the opponent's research progress. If the opponent does not possess the new technology at the time of the application, the patent is granted with certainty. If the opponent does have the new technology at the time of application, it contests the patent, and the patent is granted with probability $1-\alpha \in [0,1]$. In other words, the opponent's new technology is protected with probability α. Thus, α captures the level of *trade secret protection*. When the patent is not granted, both firms have the right to use the new technology.

When the patent is granted, the patent holder offers a take-it-or-leave-it (TIOLI) licensing fee to the opponent. If the offer is accepted, the opponent pays the licensing fee, and both firms continue the race to develop the product using the new technology. If the offer is rejected, only the patent holder can use the new technology, while the opponent continues developing the product using the old technology. The subgame after the patent application is summarized in Figure 5.[16]

[15] To simplify the discussion, we assume that a firm can only apply for a patent right after the discovery of the new technology. In practice, it is possible to delay the patent application, e.g., a firm can protect the new technology by trade secret for six months, and then apply for a patent.

[16] This setup can also be interpreted in the context of patent infringement. When Firm i obtains the patent for the new technology, Firm j can decide whether to infringe the patent and use the new technology to develop the product or to utilize the old technology. If the patent is infringed, Firm i brings the case to

6.1 Optimal Licensing Fee

We begin by deriving the optimal licensing fee offer. Consider the subgame where Firm i has obtained the patent for the new technology. If the offer is accepted, both firms develop with the new technology, resulting in each firm obtaining the expected continuation payoff of $V_{11} := \frac{\lambda_H \Pi - c}{2\lambda_H}$. If the offer is rejected, Firm i develops with the new technology and Firm j develops with the old technology, obtaining the expected continuation payoffs of $V_{10} := \frac{\lambda_H \Pi - c}{\lambda_H + \lambda_L}$ and $V_{01} := \frac{\lambda_L \Pi - c}{\lambda_H + \lambda_L}$ respectively. Thus, Firm j accepts the licensing fee offer l iff $V_{11} - l \geq V_{01}$. The next proposition characterizes the optimal licensing fee offered by the patent holder.

Proposition 6.1. *When that Firm i has obtained the patent for the new technology, it offers a licensing fee*

$$l^* := V_{11} - V_{01} = \frac{\lambda_H - \lambda_L}{\lambda_H + \lambda_L} \cdot \frac{\lambda_H \Pi + c}{2\lambda_H} \tag{6.1}$$

to Firm j, and Firm j accepts the offer. Then, Firm i's expected continuation payoff is $U_{Licensor} = V_{11} + l^*$ and Firm j's expected continuation payoff is $U_{Licensee} = V_{11} - l^* = V_{01}$.

Intuitively, the total surplus when the two firms compete with different technologies is $V_{10} + V_{01} = \Pi - \frac{2c}{\lambda_L + \lambda_H}$, which is lower than that under licensing, $2 \cdot V_{11}$. The extra surplus represents the savings in costs associated with a shorter development time that can be achieved by allowing Firm j to use the new technology rather than the old technology. Since the patent holder has all the bargaining power in the licensing negotiations, it is able to capture the entire extra surplus that is generated through licensing. In other words, once the property right is given via the patent, the efficient allocation is achieved á la Coase (1960).

Patent Application Given these expected payoffs after patent application, consider Firm i's expected payoff from applying for a patent. After the patent application, both firms will have access to the new technology. If the patent is granted, Firm j will license it from Firm i as in Proposition 6.1; and if the patent is not granted, it must be the case that Firm j already possesses the new technology.

court, resulting in Firm j being ordered to compensate Firm i with a fee equal to the licensing fee.

Let p be the probability that Firm j has the new technology. Then, the probability that the license fee is paid is $1 - \alpha \cdot p$. Thus, the expected continuation payoff of applying for a patent is $V_{11} + (1 - \alpha \cdot p) \cdot l^*$. If firm i decides not to apply for a patent, the expected continuation payoff depends on the allocation policy and patenting decisions of the opponent.

6.2 First-Best Implementation

In this section, we identify parametric conditions under which the first-best outcome is implemented in the patent games under the public and the private information settings, respectively. As described in Section 5.3, the first-best outcome occurs when both firms conduct research, and once one of the firms discovers the new technology, it should be shared with the other firm through patenting and licensing.

The Patent Game under Public Information When research progress is public information, there is no downside to applying for a patent because the only advantage of not patenting is to keep research progress concealed (Proposition D.1). Therefore, on the equilibrium path, the patent application is never challenged.

Given this and the optimal licensing fee from Proposition 6.1, we can determine the continuation payoffs of each firm after the discovery of the new technology. Using these continuation payoffs, we can analyze the equilibrium resource allocations prior to the new technology discovery. The following proposition identifies the condition that the first-best outcome can be implemented. We present the full equilibrium characterization in Appendix D.2.1.

Proposition 6.2. *Suppose that firms' research progress is public information. There exists* $\tilde{\pi}_1 > 1$ *such that the first-best outcome can be implemented in the equilibrium if and only if (i)* $\lambda_\star > \lambda_L$*; or (ii)* $\lambda_L > \lambda_\star > \frac{\lambda_H \lambda_L}{2\lambda_H + \lambda_L}$ *and* $\tilde{\pi}_1 > \pi := \lambda_L \Pi / c$.

Intuitively, the possibility of patenting increases the incentives to conduct research. Recall that when $\lambda_\star > \lambda_L$, firms begin by conducting research in both public and private information settings without patenting. Therefore, firms will continue conducting research in this parametric region when patenting the new technology is possible (Part (i)).

When $\lambda_\star < \lambda_L$, recall that both firms directly develop the product using the old technology in the game without patents. Even in the patent game, the incentive to directly develop the product is strong especially when the stake of winning the race, π, is high enough. Therefore, the first-best outcome can be implemented when π is relatively small (Part (ii)). It is worth highlighting that, since patents are never challenged on the equilibrium path, the equilibrium research allocations are independent of the level of trade secret protection α.

The Patent Game under Private Information Next, assume that firms' research progress is private information. If both firms conduct research and apply for patents upon discovering the new technology, we call it an *efficient patent equilibrium.* On the equilibrium path, both firms behave as if research progress is public information. Therefore, the condition described in Proposition 6.2 is necessary for the existence of an efficient patent equilibrium.

In addition to this condition, we need another condition to incentivize a firm to disclose that it has discovered the new technology. The following lemma characterizes this condition.

Lemma 6.1. *Suppose that firms' research progress is private information, and Firm j's resource allocation strategy is to do research indefinitely ($\sigma_t = 1$ for all $t \geq 0$) and apply for a patent once the new technology is discovered. When Firm i discovers the new technology, it applies for a patent if and only if*

$$\frac{l^*}{V_{11}} > \frac{\lambda_H}{\lambda_H + \mu(2-\alpha)}. \tag{6.2}$$

This result is intuitive in that a firm is willing to apply for a patent if and only if the licensing fee l^* is attractive enough relative to the firm's expected payoff after licensing V_{11}. Observe that, as α increases, (6.2) becomes more difficult to hold. This result aligns with intuition: as the trade secret protection level increases, firms are less inclined to apply for patents. Also note that from (6.1) and $\pi = \lambda_L \Pi / c$, we have

$$\frac{l^*}{V_{11}} = \frac{\lambda_H - \lambda_L}{\lambda_H + \lambda_L} \cdot \frac{\lambda_H \Pi + c}{\lambda_H \Pi - c} = \frac{\lambda_H - \lambda_L}{\lambda_H + \lambda_L} \cdot \frac{\lambda_H \pi + \lambda_L}{\lambda_H \pi - \lambda_L}.$$

Therefore, the left hand side of (6.2) is decreasing in π, i.e., as π increases, (6.2) becomes more difficult to hold. Intuitively, since a part of the licensing fee comes from the saving of the cost, it does not increase proportionally with V_{11}. Equipped with this result, we can pin down the parametric conditions under which the efficient patent equilibrium exists.

Proposition 6.3. *Suppose that firms' research progress is private information. The efficient patent equilibrium exists if and only if the condition in Proposition 6.2 and one of the following conditions hold: (i)* $\alpha \leq \hat{\alpha} := \frac{2\lambda_\star}{\lambda_H + \lambda_\star}$*; or (ii)* $\alpha > \hat{\alpha}$ *and*

$$\pi < \hat{\pi}(\alpha) := 1 + \frac{\lambda_L + \lambda_H}{\lambda_H} \cdot \frac{2 - \alpha}{\alpha - \hat{\alpha}}. \tag{6.3}$$

Note that when $\lambda_\star > \lambda_H$, the efficient patent equilibrium exists, since $\hat{\alpha} > 1$. In this case, firms conduct research regardless of their rivals' progress. Therefore, when a firm discovers the new technology, there is no informational advantage to concealing it. Instead, firms can benefit from licensing the new technology to the rival firms, allowing the efficient patent equilibrium to be attained. On the other hand, when $\lambda_\star < \lambda_H$, it is possible that the efficient patent equilibrium does not exist. To illustrate this, consider a scenario where Firm A discovers the new technology. If Firm A patents and licenses the new technology, the licensing fee is determined based on the assumption that, if the offer is rejected, Firm B will develop with the old technology. Recall that, in the case of $\lambda_\star < \lambda_H$, developing with the old technology is the best response for Firm B when it knows that the rival has the new technology (Proposition 3.1). Therefore, by applying for a patent, Firm A provides an opportunity for Firm B to exercise its best response. In contrast, if Firm A keeps the discovery secret, it may induce Firm B to make suboptimal choices in R&D strategies, e.g., Firm B may squander its time in conducting research for the new technology, which Firm A already possesses. This trade-off creates the possibility that the efficient patent equilibrium does not exist.

6.3 Concealment Equilibrium

Now we consider another extreme equilibrium candidate in the patent game under the private information setting: both firms do not apply for patents, namely a *concealment equilibrium*. On the equilibrium path of the concealment equilibrium, firms do not observe rivals' research progress. Therefore, the equilibrium outcome corresponds to that under the private information setting without patenting.

To simplify the discussion, we focus on the parametric region where $\lambda_H > \lambda_\star > \mu$. Recall that in this region, both firms employ the fall-back strategy under the public information setting (Theorem 1 (b)), whereas they conduct research under the private information setting (Theorem 2 (b)). The following proposition shows that the concealment equilibrium exists when both the trade secret protection level and the stake of winning the race are high enough.

Proposition 6.4. *Suppose that firms' research progress is private information and $\lambda_H > \lambda_\star > \mu$. There exists $\tilde{\alpha} > \hat{\alpha}$ and $\tilde{\pi}(\alpha) \geq \hat{\pi}(\alpha)$ such that the concealment equilibrium exists if and only if $\alpha > \tilde{\alpha}$ and $\pi > \tilde{\pi}(\alpha)$.*

Intuitively, a substantial stake of winning the race and strong trade secret protection increase firms' incentives to conceal their discovery of the new technology. Instead of receiving the licensing fee after patenting—which involves another round of competition to develop the product with the new technology—firms would rather let their rivals squander time researching for the technology they already possess. Specifically, the licensing fee does not fully internalize the decreased chance of winning the race. This is because once Firm i applies for a patent, Firm j adjusts the R&D strategy based on the information that Firm i possesses the new technology, implying that Firm j's outside option is changed after the disclosure.

This concealment incentive slows down the social speed of development in two ways: (i) the discovery of the new technology is not shared with another firm, as described by the gap between the black and the blue curves in Figure 4; (ii) due to the lack of information, firms cannot appropriately adjust the R&D strategies, as described by the gap between the blue and the red dotted curves in Figure 4.

Last, note that the parametric regions of (α, π) in Proposition 6.3 and 6.4 do not overlap; that is, there exists an intermediate region where neither efficient patent equilibrium nor concealment equilibrium exists. In this region, as in Chatterjee et al. (2023), firms would engage in partial disclosure—applying for patents at some rate.

7 Discussion

In this article, we investigate how information about firms' research progress—particularly regarding the acquisition of the new technology that accelerates innovation—influences R&D dynamics by introducing an innovation race model with multiple paths. By extending the model to include the option to patent, we also study when the patent system is effective in promoting the social speed of innovation.

We highlight that the private acquisition of the new technology affects the innovation speed in two key ways. Firstly, it prevents the new technology from being shared with other firms. Secondly, the lack of information about research progress hinders firms' ability to adapt their R&D strategies effectively.

To counteract this slowdown, we propose implementing a patent system with an option to license. When the new technology is patented and licensed, the first-best outcome can be achieved. The socially optimal outcome is realized when firms' research progress is publicly observable, unless the direct development with the old technology is attractive enough (Proposition 6.2 (i)).

On the contrary, with private information about their progress, firms may choose to conceal their discovery of the new technology if the right to use it is well protected by trade secret laws and the stake from winning the race is high (Proposition 6.4). This concealment delays the innovation not only because the discovered new technology cannot be used by competing firms, but also because competing firms waste time independently discovering the new technology. Proposition 6.3 suggests two potential policies to resolve this inefficiency. One approach is to reduce the trade secret protection level as this would discourage firms from concealing their discoveries. Another is to decrease the stake of winning the race as this would make licensing more appealing. A caveat of this policy is that lowering the stake too much could discourage firms from participating in the innovation races in the first place.

We can also modify the model to reflect the first-to-invest patent system. For instance, when Firm i applies for a patent and Firm j contests it, with probability α, the firm that discovered the new technology earlier obtains the patent, and with probability $1-\alpha$, Firm i obtains the patent. Then, the first-to-invent system can be represented by $\alpha = 0$, while the first-to-file system can be represented by $\alpha = 1$. With this modification, firms have more incentives to conceal their discoveries as they now have a chance of becoming a patentee by contesting the other firms' patent application. Nevertheless, when α is low enough, these incentives cannot outweigh the advantage of licensing the new technology. Therefore, the socially efficient outcome can also be attained by decreasing α.

There are many avenues open for further research. For example, we assume that there are exogenously given two paths towards innovation, and one of the paths requires two breakthroughs. However, in practice, there are numerous ways to innovate, and it often requires more than two breakthroughs. We also assume that a firm's R&D resources are fixed over time, but we could also allow firms to endogenously choose how much effort to put into each point in time. Finally, we assume the contest structure is given by the winner-takes-all competition, but we might consider a contest design problem. We leave these intriguing questions and others for future work.

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Appendix

A Proofs for Benchmark

A.1 Proof of Proposition 3.1

Proof of Proposition 3.1 . Suppose that Firm i has already discovered the new technology. Then, Firm i develops with rate λ_H and Firm j develops with the rate λ. Firm i's probability of winning the race is $\frac{\lambda_H}{\lambda_H+\lambda}$ and the expected duration of the remaining race is $\frac{1}{\lambda_H+\lambda}$. Therefore, Firm i's expected continuation payoff is given by

$$\mathcal{V}_\lambda^1 := \frac{\lambda_H}{\lambda_H+\lambda} \cdot \Pi - \frac{1}{\lambda_H+\lambda} \cdot c = \frac{\lambda_H \Pi - c}{\lambda_H+\lambda}. \tag{A.1}$$

Now suppose that Firm i has yet to discover the new technology. Consider constant research allocation strategies, which allocate a fixed amount of resources to research until either the new technology is discovered or the race ends, i.e., for some $x \in [0,1]$, $\sigma_t^i = x$ for all $t \geq 0$.[17] When Firm i allocates x amount of resources towards research, there are three potential outcomes: (i) Firm i develops the product with the old technology at rate $\lambda_L(1-x)$; (ii) Firm i discovers the new technology at rate μx; (iii) Firm j develops the product at rate λ. In the first scenario, Firm i wins the race and receives Π, and the probability of this event happening is $\frac{\lambda_L(1-x)}{\lambda_L(1-x)+\mu x+\lambda}$. In the second scenario, Firm i enters the post-research phase, and its expected payoff is $\mathcal{V}_\lambda^1$. The probability of this event occurring is $\frac{\mu x}{\lambda_L(1-x)+\mu x+\lambda}$. In the third scenario, Firm i receives nothing, and the probability of this event happening is $\frac{\lambda}{\lambda_L(1-x)+\mu x+\lambda}$. The expected remaining duration of the game is $\frac{1}{\lambda_L(1-x)+\mu x+\lambda}$. Therefore, Firm i's expected payoff is given by

$$u(x) := \frac{\lambda_L(1-x)\cdot \Pi + \mu x \cdot \mathcal{V}_\lambda^1 - c}{\lambda_L(1-x)+\mu x+\lambda}. \tag{A.2}$$

[17] In Online Appendix OA.1.3, we show that it is without loss to focus on these strategies (Lemma OA.1.2).

After taking the first derivative of u, with some algebra, we can derive that

$$u'(x) = \frac{\lambda_L(\lambda\Pi + c)(\lambda_\star - \lambda)}{(\lambda + \lambda_H)(\lambda + (1-x)\lambda_L + x\mu)^2}. \tag{A.3}$$

Therefore, from $x \in [0,1]$, $x = 1$ is optimal when $\lambda < \lambda_\star$, $x = 0$ is optimal when $\lambda > \lambda_\star$, and any $x \in [0,1]$ is optimal when $\lambda = \lambda_\star$. □

B Proofs for the Public Information Setting

B.1 Proof of Theorem 1

We prove Theorem 1 using a sequence of Lemmas which we separately prove in Section B.2. Moreover, we restrict ourselves to Markov deviations, shown to be sufficient to do in Lemma OA.1.4 of Online Appendix OA.1.4.

Given a Markov strategy profile of the firms, we can define U^i_ω as the continuation payoff of Firm i in state ω. Next, we provide some intuition for the proof of Theorem 1 by splitting the problem of the firms in two: On one hand, we solve the problem of the firms before any research progress has been made (and fixing the continuation payoffs). On the other hand, we compute the best responses of the firms after one of them obtains the new technology, and therefore the equilibrium continuation payoffs. Finally, by plugging these continuation payoffs into the problem of the firms at the initial state, we prove the theorem.

Best Responses under no New Technology Discovery We first consider the case where neither firm has discovered the new technology, i.e., $\omega = \emptyset$. The conventional approach is to solve the problem with backward induction. However, in order to facilitate the analysis in various extensions, we present the problem under the state $\omega = \emptyset$ in a general manner by treating the continuation payoffs $U^i_{\{i\}}$ and $U^i_{\{j\}}$ as exogenous values.

When Firm i and j play $\mathbf{s}(\emptyset) = x$ and $\hat{\mathbf{s}}(\emptyset) = y$, Firm i's expected payoff at the state $\emptyset$ is

$$u_0(x,y) := \frac{x\mu U^i_{\{i\}} + (1-x)\lambda_L\Pi + y\mu U^i_{\{j\}} - c}{x\mu + (1-x)\lambda_L + y\mu + (1-y)\lambda_L}. \tag{B.1}$$

Define $\Delta_y := u_0(1, y) - u_0(0, y)$.

Lemma B.1. *The following equation holds:*

$$\frac{\partial u_0}{\partial x} = \mathcal{C}(x, y) \cdot \{\lambda_L \cdot \Delta_0 \cdot (1 - y) + \mu \cdot \Delta_1 \cdot y\}, \tag{B.2}$$

where

$$\mathcal{C}(x, y) = \frac{2(\lambda_L + \mu)}{\{\mu x + \lambda_L(1 - x) + \mu y + \lambda_L(1 - y)\}^2} > 0.$$

The following lemma characterizes the equilibrium allocations at state $\emptyset$ in any MPE.

Lemma B.2. *The equilibrium allocations at state $\emptyset$ are characterized as follows:*

(a) when $\Delta_0, \Delta_1 > 0$, both firms do research, i.e., $(\mathbf{s}^A(\emptyset), \mathbf{s}^B(\emptyset)) = (1, 1)$;

(b) when $\Delta_0, \Delta_1 < 0$, both firms develop with the old technology, i.e., $(\mathbf{s}^A(\emptyset), \mathbf{s}^B(\emptyset)) = (0, 0)$.

In scenarios where Δ_0 and Δ_1 share the same sign, the best response is independent of the opponent's resource allocation. Specifically, when both Δ_0 and Δ_1 are positive, it is optimal to assign all resources to research. Conversely, when both Δ_0 and Δ_1 are negative, it is optimal to develop with the old technology.

Best Responses under New Technology Discovery We now consider the cases where at least one of the firms has discovered the new technology.

When both firms have discovered the new technology ($\omega = \{i, j\}$), they will develop with the new technology and their expected payoffs are $U^i_{\{i,j\}} = U^j_{\{i,j\}} = V_{11} := \frac{\lambda_H \Pi - c}{2\lambda_H}$. Next, suppose that only one of the firms, say Firm i, has discovered the new technology, i.e., $\omega = \{i\}$. In this case, Firm i develops the product at rate λ_H with the new technology. Then, we can derive the continuation values by applying Proposition 3.1:

(i) if $\lambda_\star > \lambda_H$, Firm j keeps conducting research:

$$U^i_{\{i\}} = U^j_{\{j\}} = \frac{\lambda_H \Pi + \mu V_{11} - c}{\mu + \lambda_H} = \frac{\mu + 2\lambda_H}{\mu + \lambda_H} V_{11}, \qquad U^i_{\{j\}} = U^j_{\{i\}} = \frac{\mu V_{11} - c}{\mu + \lambda_H}, \tag{B.3}$$

(ii) if $\lambda_\star < \lambda_H$, Firm j develops with the old technology:

$$U^i_{\{i\}} = U^j_{\{j\}} = V_H := \frac{\lambda_H \Pi - c}{\lambda_L + \lambda_H}, \qquad U^i_{\{j\}} = U^j_{\{i\}} = V_L := \frac{\lambda_L \Pi - c}{\lambda_L + \lambda_H}. \tag{B.4}$$

Equilibrium Characterization Now that we have derived the continuation values, we can compute Δ_0 and Δ_1.

Lemma B.3. *When $\lambda_\star > \lambda_H$, the following equations hold:*

$$\Delta_0 = \frac{\lambda_H \cdot \lambda_\star \cdot (\lambda_L \Pi + c) + \mu \cdot (\lambda_\star - \lambda_H) \cdot c}{2\lambda_H(\lambda_H + \mu)(\lambda_L + \mu)}, \tag{B.5}$$

$$\Delta_1 = \frac{\lambda_L \cdot \{\lambda_H \cdot \lambda_\star \cdot (\mu \Pi + c) + \mu \cdot (\lambda_\star - \lambda_H) \cdot c\}}{2\mu\lambda_H(\lambda_H + \mu)(\lambda_L + \mu)}. \tag{B.6}$$

Lemma B.4. *When $\lambda_\star < \lambda_H$, the following equations hold:*

$$\Delta_0 = \frac{(\lambda_L \Pi + c) \cdot (\lambda_\star - \lambda_L)}{2(\lambda_L + \mu)(\lambda_L + \lambda_H)}, \tag{B.7}$$

$$\Delta_1 = \frac{(\mu \Pi + c) \cdot \lambda_L \cdot (\lambda_\star - \lambda_L)}{2\mu(\lambda_L + \mu)(\lambda_L + \lambda_H)}. \tag{B.8}$$

We can finalize the proof of Theorem 1 by using the above lemmas and Lemma B.2.

Proof of Theorem 1. First, when $\lambda_\star > \lambda_H$, by Lemma B.3, we have that $\Delta_0, \Delta_1 > 0$. By applying Lemma B.2 (a), both firms do research at the state $\emptyset$. Then, when one of the firms, say Firm j, succeeds in research, by Proposition 3.1 (a), Firm i will keep doing research. Therefore, the unique MPE is for firms to follow the research strategy (Theorem 1 (a)).

When $\lambda_\star \in (\lambda_L, \lambda_H)$, (B.7) and (B.8) imply that Δ_0 and Δ_1 are positive. Thus, by Lemma B.2 (a), both firms do research at the state $\emptyset$. Then, when one of the firms, say Firm j, succeeds in research, by Proposition 3.1 (b), Firm i will switch to develop with the old technology. Therefore, the unique MPE is for firms to follow the fall-back strategy (Theorem 1 (b)).

Last, when $\lambda_\star < \lambda_L$, we can see that Δ_0 and Δ_1 are negative. Then, by Lemma B.2 (b), both firms develop with the old technology at the state $\emptyset$. Additionally, even if a firm happens to succeed in research, the other firm will keep developing with the old technology due to

Proposition 3.1 (b). Thus, the unique MPE is for firms to employ the direct-development strategy (Theorem 1 (c)). □

B.2 Proofs of Lemmas

Proof of Lemma B.1. Observe that

$$\Delta_0 = \frac{\mu U^i_{\{i\}} - c}{\mu + \lambda_L} - \frac{\lambda_L \Pi - c}{2\lambda_L}, \quad \Delta_1 = \frac{\mu(U^i_{\{i\}} + U^i_{\{j\}}) - c}{2\mu} - \frac{\lambda_L \Pi + \mu U^i_{\{j\}} - c}{\lambda_L + \mu}.$$

Thus, we have

$$2(\lambda_L + \mu)\lambda_L \cdot \Delta_0 = 2\lambda_L \mu U^i_{\{i\}} - \lambda_L(\lambda_L + \mu)\Pi + (\mu - \lambda_L)c, \tag{B.9}$$

$$2(\lambda_L + \mu)\mu \cdot \Delta_1 = (\lambda_L + \mu)\mu U^i_{\{i\}} - (\mu - \lambda_L)\mu U^i_{\{j\}} - 2\lambda_L \mu \Pi + (\mu - \lambda_L)c, \tag{B.10}$$

and

$$2(\lambda_L + \mu)\mu \cdot \Delta_1 - 2(\lambda_L + \mu)\lambda_L \cdot \Delta_0 = (\mu - \lambda_L)\left(\mu U^i_{\{i\}} - \mu U^i_{\{j\}} - \lambda_L \Pi\right).$$

Also note that

$$\frac{\partial u_0}{\partial x} = \frac{NUM_0}{\{\mu x + \lambda_L(1-x) + \mu y + \lambda_L(1-y)\}^2}$$

where

$$\begin{aligned} NUM_0 =& (\mu U^i_{\{i\}} - \lambda_L \Pi) \cdot (\mu x + \lambda_L(1-x) + \mu y + \lambda_L(1-y)) \\ &- (x\mu U^i_{\{i\}} + (1-x)\lambda_L \Pi + y\mu U^i_{\{j\}} - c) \cdot (\mu - \lambda_L). \end{aligned}$$

With some algebra, we can show that

$$NUM_0 = 2(\lambda_L + \mu)\lambda_L \cdot \Delta_0 + (2(\lambda_L + \mu)\mu \cdot \Delta_1 - 2(\lambda_L + \mu)\lambda_L \cdot \Delta_0) \cdot y.$$

By plugging this in, we can show that (B.2) holds. □

Proof of Lemma B.2. (a) When $\Delta_0, \Delta_1 > 0$, from (B.2), $\frac{\partial u_0}{\partial x} > 0$ for all $y \in [0, 1]$, i.e.,

$x = 1$ is optimal. Thus, both firms play $\mathbf{s}(\emptyset) = 1$ in any MPE.

(b) When $\Delta_0, \Delta_1 < 0$, from (B.2), $\frac{\partial u_0}{\partial x} < 0$ for all $y \in [0, 1]$, i.e., $x = 0$ is optimal. Thus, both firms play $\mathbf{s}(\emptyset) = 0$ in any MPE.

$\square$

Proof of Lemma B.3. By plugging (B.3) into (B.9), with some algebra, we have

$$2(\lambda_L + \mu)\lambda_L \cdot \Delta_0 = \frac{\lambda_H\mu - \lambda_L\lambda_H - \mu\lambda_L}{\lambda_H + \mu} \cdot (\lambda_L \Pi + c) + \frac{\mu\left(\lambda_H\mu - \lambda_L\lambda_H - \mu\lambda_L - \lambda_L\lambda_H\right)}{\lambda_H(\lambda_H + \mu)} \cdot c.$$

By using (3.1), we have

$$2(\lambda_L + \mu)\lambda_L \cdot \Delta_0 = \frac{\lambda_L}{\lambda_H(\lambda_H + \mu)} \cdot \left[\lambda_H \cdot \lambda_\star \cdot (\lambda_L \Pi + c) + \mu \cdot (\lambda_\star - \lambda_H) \cdot c\right].$$

Then, by dividing both sides by $2(\lambda_L + \mu)\lambda_L$, we can show that (B.5) holds.

Next, by plugging (B.3) into (B.10),

$$2(\lambda_L + \mu)\mu \cdot \Delta_1 = \frac{\lambda_H\mu - \lambda_L\lambda_H - \mu\lambda_L}{\lambda_H + \mu} \cdot (\mu\Pi + c) + \frac{\mu(\lambda_H\mu - \lambda_L\lambda_H - \mu\lambda_L - \lambda_L\lambda_H)}{\lambda_H(\lambda_H + \mu)} \cdot c.$$

By using (3.1), we have

$$2(\lambda_L + \mu)\mu \cdot \Delta_1 = \frac{\lambda_L}{\lambda_H(\lambda_H + \mu)} \cdot \left[\lambda_H \cdot \lambda_\star \cdot (\mu\Pi + c) + \mu \cdot (\lambda_\star - \lambda_H) \cdot c\right]$$

Then, by dividing both sides by $2(\lambda_L + \mu)\mu$, we can show that (B.6) holds. $\square$

Proof of Lemma B.4. By plugging (B.4) into (B.9),

$$2(\lambda_L + \mu)\lambda_L \cdot \Delta_0 = \frac{\lambda_L \Pi + c}{\lambda_L + \lambda_H} \cdot \left\{\lambda_H\mu - \lambda_L\lambda_H - \mu\lambda_L - \lambda_L^2\right\}$$

By using (3.1), we have

$$2(\lambda_L + \mu)\lambda_L \cdot \Delta_0 = \frac{(\lambda_L \Pi + c) \cdot \lambda_L \cdot (\lambda_\star - \lambda_L)}{\lambda_L + \lambda_H}.$$

Then, by dividing both sides by $2(\lambda_L + \mu)\lambda_L$, we can show that (B.7) holds.

Next, by plugging (B.4) into (B.10),

$$2(\lambda_L + \mu)\mu \cdot \Delta_1 = \frac{\mu\Pi + c}{\lambda_L + \lambda_H} \cdot \left\{\lambda_H\mu - \lambda_L\lambda_H - \mu\lambda_L - \lambda_L^2\right\}.$$

By using (3.1), we have

$$2(\lambda_L + \mu)\mu \cdot \Delta_1 = \frac{(\mu\Pi + c) \cdot \lambda_L \cdot (\lambda_\star - \lambda_L)}{\lambda_L + \lambda_H}.$$

Then, by dividing both sides by $2(\lambda_L + \mu)\mu$, we can show that (B.6) holds. □

C Proofs for the Private Information Setting

C.1 Formal Definition of $\mathbf{p}_{\boldsymbol{\sigma}}$

Given an allocation policy $\boldsymbol{\sigma} \in \mathcal{S}$, we define two arrival times: (i) τ_M represents the time at which either the new technology is discovered or the product is developed by the old technology; (ii) τ_D represents the time of the product development. Observe that, τ_M must be less than or equal to τ_D by definition. This inequality is strict if and only if the new technology is discovered prior to the product development. Therefore, we use $(\tau_M = \tau_D)$ to indicate the event that the new technology is discovered before the product is developed using the old technology and $(\tau_M < \tau_D)$ to indicate the event that the product is developed before the new technology discovery.

Observe that $\mathbf{p}_{\boldsymbol{\sigma}}$ can be expressed in terms of τ_M and τ_D as follows: $\mathbf{p}_{\boldsymbol{\sigma}}(t) := \Pr(\tau_M < t < \tau_D \mid \tau_D > t)$. Let $\Sigma_t := \int_0^t \boldsymbol{\sigma}(s)\, ds$ represent the cumulative research. We begin by observing that the probability that neither new technology discovery nor product development is made by time t is given by

$$S^M_{\boldsymbol{\sigma}}(t) := \Pr(\tau_M > t) = e^{-\lambda_L(t-\Sigma_t)-\mu\Sigma_t}. \tag{C.1}$$

Additionally, we can derive the probability that new technology is discovered, but product is yet to be developed by time t:

$$L_{\boldsymbol{\sigma}}(t) := \Pr(\tau_M < t < \tau_D) = \int_0^t \mu\, \boldsymbol{\sigma}(s) e^{-\lambda_L(s-\Sigma_s)-\mu\Sigma_s} e^{-\lambda_H(t-s)}\, ds. \tag{C.2}$$

The probability $S^D_{\boldsymbol{\sigma}}(t)$ that neither product development nor new technology discovery was made by time t can be written as:

$$S^D_{\boldsymbol{\sigma}}(t) := \Pr(\tau_D > t) = \Pr(\tau_M > t) + \Pr(\tau_M < t < \tau_D) = S^M_{\boldsymbol{\sigma}}(t) + L_{\boldsymbol{\sigma}}(t). \tag{C.3}$$

Finally, we obtain an expression for our conditional probability $\mathbf{p}_{\boldsymbol{\sigma}}$ in terms of $L_{\boldsymbol{\sigma}}$ and $S^M_{\boldsymbol{\sigma}}$:

$$\mathbf{p}_{\boldsymbol{\sigma}}(t) = \Pr(\tau_M < t \mid \tau_D > t) = \frac{\Pr(\tau_M < t < \tau_D)}{S^D_{\boldsymbol{\sigma}}(t)} = \frac{L_{\boldsymbol{\sigma}}(t)}{S^M_{\boldsymbol{\sigma}}(t) + L_{\boldsymbol{\sigma}}(t)}. \tag{C.4}$$

C.2 Proofs of Propositions

C.2.1 Proof of Proposition 5.1

Proof of Proposition 5.1. From (C.4), we can derive that $\mathbf{p}_{\boldsymbol{\sigma}}(t)/(1-\mathbf{p}_{\boldsymbol{\sigma}}(t)) = L_{\boldsymbol{\sigma}}(t)/S^M_{\boldsymbol{\sigma}}(t)$. By differentiating this equation side-by-side, we have

$$\frac{\dot{\mathbf{p}}_{\boldsymbol{\sigma}}(t)}{(1-\mathbf{p}_{\boldsymbol{\sigma}}(t))^2} = \frac{L_{\boldsymbol{\sigma}}(t)}{S^M_{\boldsymbol{\sigma}}(t)}\left[\frac{L'_{\boldsymbol{\sigma}}(t)}{L_{\boldsymbol{\sigma}}(t)} - \frac{S^{M\prime}_{\boldsymbol{\sigma}}(t)}{S^M_{\boldsymbol{\sigma}}(t)}\right] = \frac{\mathbf{p}_{\boldsymbol{\sigma}}(t)}{1-\mathbf{p}_{\boldsymbol{\sigma}}(t)}\left[\frac{L'_{\boldsymbol{\sigma}}(t)}{L_{\boldsymbol{\sigma}}(t)} - \frac{S^{M\prime}_{\boldsymbol{\sigma}}(t)}{S^M_{\boldsymbol{\sigma}}(t)}\right]. \tag{C.5}$$

From deriving (C.1) and (C.2), we obtain that

$$S^{M\prime}_{\boldsymbol{\sigma}}(t) = -\left\{\lambda_L(1-\boldsymbol{\sigma}(t)) + \mu\,\boldsymbol{\sigma}(t)\right\}\cdot S^M_{\boldsymbol{\sigma}}(t), \tag{C.6}$$

$$L'_{\boldsymbol{\sigma}}(t) = \mu\cdot\boldsymbol{\sigma}(t)\cdot S^M_{\boldsymbol{\sigma}}(t) - \lambda_H\cdot L_{\boldsymbol{\sigma}}(t) \tag{C.7}$$

Using these expressions in (C.5) and multiplying side by side by $(1-\mathbf{p}_{\boldsymbol{\sigma}}(t))^2$, we obtain the desired result. □

C.2.2 Proof of Proposition 5.2

Proof of Proposition 5.2. Our goal is to show that the unique solution of $\xi(p_\star, \sigma_\star) = \lambda_\star$ and $\delta(p_\star, \sigma_\star) = 0$ is

$$p_\star = \frac{\mu(\lambda_\star - \lambda_L)}{2\lambda_L\lambda_\star} = 1 - \frac{(\mu - \lambda_L)(\lambda_H - \lambda_\star)}{2\lambda_L\lambda_\star}, \tag{C.8}$$

$$\sigma_\star = \frac{\lambda_\star - \lambda_L}{\mu - \lambda_L}. \tag{C.9}$$

Then, to have $(p_\star, \sigma_\star) \in (0,1)^2$, we need to have $\min\{\mu, \lambda_H\} > \lambda_\star > \lambda_L$.

From $\xi(p_\star, \sigma_\star) = \lambda_\star$, $\delta(p_\star, \sigma_\star) = 0$ and $p_\star < 1$, we have

$$\lambda_\star = p_\star\lambda_H + (1 - p_\star)(1 - \sigma_\star)\lambda_L, \tag{C.10}$$

$$0 = \mu\sigma_\star - \{\lambda_H - (1 - \sigma_\star)\lambda_L\}\, p_\star. \tag{C.11}$$

By rearranging (C.11), we have

$$\mu\sigma_\star = \lambda_H p_\star + (1 - \sigma_\star)\lambda_L(1 - p_\star) - \lambda_L(1 - \sigma_\star) = \lambda_\star - \lambda_L(1 - \sigma_\star).$$

By solving this, we can derive (C.9).

Next, from (C.11) and (C.9), we have

$$p_\star = \frac{\mu\sigma_\star}{\lambda_H - (1 - \sigma_\star)\lambda_L} = \frac{\mu(\lambda_\star - \lambda_L)}{(\mu - \lambda_L)\lambda_H - (\mu - \lambda_\star)\lambda_L}.$$

Note that $\lambda_L\lambda_\star = (\mu - \lambda_L)\lambda_H - \mu\lambda_L$. By plugging this into the above equation, we have the first equality of (C.8). Observe that

$$1 - p_\star = \frac{2\lambda_L\lambda_\star - \mu\lambda_\star + \mu\lambda_L}{2\lambda_L\lambda_\star} = \frac{\lambda_L(\mu + \lambda_\star) - (\mu - \lambda_L)\lambda_\star}{2\lambda_L\lambda_\star} = \frac{(\mu - \lambda_L)(\lambda_H - \lambda_L)}{2\lambda_L\lambda_\star},$$

which confirms the second equality of (C.8). $\square$

C.3 Proof Sketch of Theorem 2

Recursive Formulation Let $V_1(t;\mathbf{h})$ and $V_0(t;\mathbf{h})$ be the continuation payoffs of a firm with and without the new technology at time t, respectively, when the opponent employs an allocation policy with associated development rate $\mathbf{h}$, and no firm has succeeded in development so far. Formally, we define V_1 as follows:

$$V_1(t;\mathbf{h}) := \int_t^{\infty} \{\lambda_H \Pi - c\} \cdot e^{-\int_t^s (\mathbf{h}(u)+\lambda_H)du} \, ds \tag{C.12}$$

The exponential term captures the probability that the race is still on by time s, given that the race is on by time by time t. The term $\lambda_H \Pi - c$ captures the flow expected payoff of the firm with the new technology. On top of fixing the opponent's development rate $\mathbf{h}$, we can fix the firm's policy $\boldsymbol{\sigma} \in \mathcal{S}$ to compute the continuation value v_0 of the firm without the new technology as follows:

$$v_0(t;\boldsymbol{\sigma},\mathbf{h}) := \int_t^{\infty} \left\{\boldsymbol{\sigma}(s)\mu V^1(s;\mathbf{h}) + (1-\boldsymbol{\sigma}(s))\lambda_L \Pi - c\right\} \cdot \mathbf{r}_{\mathbf{h},\boldsymbol{\sigma}}(s;t) \, ds, \tag{C.13}$$

$$\mathbf{r}_{\mathbf{h},\boldsymbol{\sigma}}(s;t) := e^{-\int_t^s \{\mathbf{h}(u)+\boldsymbol{\sigma}(u)\mu+(1-\boldsymbol{\sigma}(u))\lambda_H\}du}.$$

In this expression, as before, the exponential term captures the probability that race is on and the firm does not have the new technology by time s, given that both hold at time t. Conditional on this event, the firm enjoys an expected flow payoff captured by the expression in brackets: the firm pays the cost c and, at rate $\boldsymbol{\sigma}(s)\mu$, the firm obtains the new technology which induces a continuation payoff $V_1(s,\mathbf{h})$. At rate $(1-\boldsymbol{\sigma}(s))\lambda_L$ the firm successfully develops, which induces a lump-sum payoff Π. By maximizing over all the allocation policies in $\mathcal{S}$, we obtain the continuation value of a firm without the new technology V_0.

$$V_0(t;\mathbf{h}) := \max_{\boldsymbol{\sigma}\in\mathcal{S}} v_0(t;\boldsymbol{\sigma},\mathbf{h}).$$

Best responses To characterize the optimal policy $\boldsymbol{\sigma}$ given the opponent's development rate $\mathbf{h}$, define $\mathcal{R}(x,t;\mathbf{h})$ and $R(t;\mathbf{h})$ as follows:

$$\mathcal{R}(x,t;\mathbf{h}) := \mu x(V_1(t;\mathbf{h}) - V_0(t;\mathbf{h})) + \lambda_L(1-x)(\Pi - V_0(t;\mathbf{h})), \tag{C.14}$$
$$R(t;\mathbf{h}) := \frac{\partial \mathcal{R}}{\partial x}(x,t;\mathbf{h}) = \mu(V_1(t;\mathbf{h}) - V_0(t;\mathbf{h})) - \lambda_L(\Pi - V_0(t;\mathbf{h})).$$

We can interpret $\mathcal{R}$ as the instantaneous payoff at time t by allocating x to research and $1-x$ to development with the old technology. The new technology is discovered at the rate μx, yielding the new continuation payoff $V_1(x;\mathbf{h})$ but losing the present continuation payoff $V_0(x;\mathbf{h})$. Similarly, the product is developed with the old technology at the rate $\lambda_L(1-x)$, resulting in the reward Π but losing $V_0(x;\mathbf{h})$. At each time t, the firm chooses a resource allocation to maximize $\mathcal{R}$. Therefore, we interpret R as capturing the relative incentives to conduct research: when R is positive, conducting research is preferred over developing with the old technology, conversely, when R is negative, developing with the old technology is preferred. The following proposition formalizes this verification arguments given the opponent's resource allocation policy $\hat{\sigma}$. The proof is in Appendix OA.2.4.

Proposition C.1. *An allocation policy $\boldsymbol{\sigma}^*$ is a best-response to $\hat{\boldsymbol{\sigma}}$, i.e. $\mathcal{U}(\boldsymbol{\sigma}^*, \hat{\boldsymbol{\sigma}}) \geq \mathcal{U}(\boldsymbol{\sigma}, \hat{\boldsymbol{\sigma}})$ for all $\boldsymbol{\sigma} \in \mathcal{S}$, if and only if the following two conditions hold for every time $t \geq 0$: (i) $v_0(t;\boldsymbol{\sigma}^*, \mathbf{h}_{\hat{\boldsymbol{\sigma}}}) > 0$; and (ii) $\boldsymbol{\sigma}^*(t) \in \arg\max_{x\in[0,1]} \mathcal{R}(x,t;\mathbf{h}_{\hat{\boldsymbol{\sigma}}})$.*

Properties with Monotone Development Rates We now highlight two features of the MDNE. The first feature arises when an allocation policy satisfies the MDR property. The proof is in Section OA.2.2.3.

Proposition C.2. *Suppose that $\boldsymbol{\sigma} \in \mathcal{S}$ satisfies the MDR property. If $\boldsymbol{\sigma}(s) = 0$, then $\boldsymbol{\sigma}(t) = 0$ for all $t < s$.*

The intuition for this result is as follows. Suppose that $\boldsymbol{\sigma}(s) = 0$ and $\mathbf{p}_{\boldsymbol{\sigma}}(s) > 0$. Then, the probability that this firm has discovered the new technology decreases as it does not conduct research at time s, which in turn decreases the development rate. This violates the MDR property. To eliminate this effect, we need to ensure $\mathbf{p}_{\boldsymbol{\sigma}} = 0$, which can only be achieved by $\boldsymbol{\sigma}(t) = 0$ for all $t < s$.

The next feature emerges when a firm faces a rival employing an allocation policy with the MDR property. The proof is in Section OA.2.5.2.

Proposition C.3 (Single-Crossing Property). *Suppose that* $\mathbf{h}$ *is increasing with* $\mathbf{h}(t) < \lambda_\star$ *for all* t*. Then,* $-\mathcal{R}(x, t; \mathbf{h})$ *satisfies the single-crossing property: for all* $x' > x$ *and* $t' > t$*,*

$$-\mathcal{R}(x', t; \mathbf{h}) \geq \ (>) \ -\mathcal{R}(x, t; \mathbf{h}) \quad \Rightarrow \quad -\mathcal{R}(x', t'; \mathbf{h}) \geq \ (>) \ -\mathcal{R}(x, t'; \mathbf{h}),$$

or equivalently, $0 \ \geq (>) \ R(t; \mathbf{h})$ *implies* $0 \ \geq (>) \ R(t'; \mathbf{h})$*.*

Roughly speaking, from the single-crossing property of $-\mathcal{R}$, we obtain that, under the best response policy, the resource allocation to research weakly decreases over time.[18]

Equilbirium Characterization Equipped with the verification result (Proposition C.1) and the properties derived from the MDR property (Proposition C.2 and C.3), we proceed to explain the intuition behind the proof of Theorem 2 concerning the different parameters.

Theorem 2 (a) When $\lambda_\star < \lambda_L$, $(\boldsymbol{\sigma}^A, \boldsymbol{\sigma}^B) = (\mathbf{0}, \mathbf{0})$ is the unique MDNE.

In Theorem 1, we obtained that when $\lambda_\star < \lambda_L$ and firms observe their rivals' research progress, the unique MPE involves both firms developing with the old technology. Intuitively, the equilibrium allocations from this MPE survive as an equilibrium in the unobservable case because the information about the opponent's technology was not used anyway. To show that this is the unique MDNE, first note that any optimal policy has to eventually generate development rates higher than λ_L. Otherwise, the policy would be dominated by developing with the old technology. Thus, the development rates must converge to a rate higher than $\lambda_\star$. In Online Appendix OA.2.8, we show that the incentives to do research, R, must therefore converge to a negative number. By Proposition C.1, there must be a time after which the firms stop allocating resources to research. However, Proposition C.2 implies that if a firm

[18] To be precise, the single-crossing property only guarantees monotonicity in the strong set of order (Milgrom and Shannon, 1994). We present additional arguments to substantiate this claim.

ever allocates resources to research, stopping would induce a decreasing development rate. Thus, the only possibility is that firms do not conduct research at all.

Theorem 2 (b) When $\lambda_\star > \min\{\mu, \lambda_H\}$, $(\boldsymbol{\sigma}^A, \boldsymbol{\sigma}^B) = (\mathbf{1}, \mathbf{1})$ is the unique MDNE.

First, we show that for any policy that satisfies monotone development rate, the development rates are bounded above by $\min\{\mu, \lambda_H\}$. This is true because maintaining a development rate higher than $\min\{\mu, \lambda_H\}$ requires a strictly decreasing $\boldsymbol{\sigma}$ to compensate for the decrease in beliefs $\mathbf{p}_{\boldsymbol{\sigma}}$. At some time, $\boldsymbol{\sigma}(t)$ must reach zero, and such development rate cannot be maintained anymore. Thus, the development rates of the firms must converge to some rate weakly lower than $\min\{\mu, \lambda_H\}$, which is lower than $\lambda_\star$. Therefore, for any $\boldsymbol{\sigma}$ satisfying MDR, there is a time T for which $R(t; \mathbf{h}_{\boldsymbol{\sigma}}) > 0$ for all $t > T$. However, we also show in Proposition C.3 that if a firm finds it optimal to allocate resources to development with the old technology ($R(t, \mathbf{h}) < 0$) then it must be that this is always optimal ($R(t, \mathbf{h}) < 0$ for all $s > t$). The only equilibrium candidate is therefore $(1, 1)$.

Theorem 2 (c) When $\lambda_\star \in (\lambda_L, \min\{\lambda_H, \mu\})$, the stationary fall-back policy profile is the unique MDNE.

First, we establish that in any MDNE both firms' development must converge precisely to $\lambda_\star$ (Lemma OA.2.13). Essentially, we show that any other converging limits lead to a contradiction. Next, we prove that, in any MDNE, the two firms must reach the development rate $\lambda_\star$ simultaneously. If one firm reaches $\lambda_\star$ first, we show using Lemma OA.2.11 that the firm has incentives to allocate all resources to research until the opponent reaches $\lambda_\star$. However, this allocation would necessarily elevate the development rate, pushing it beyond $\lambda_\star$.

Let's define $T_\star$ as the time when both firms reach the development rate $\lambda_\star$. From T onward, the firms develop at the rate $\lambda_\star$. We show that there is a unique constant probability and allocation, $p_\star$ and $\sigma_\star$, that can maintain the development rate at $\lambda_\star$, as any deviation from these levels would induce the development rates to diverge. To obtain the allocations before time $T_\star$, we apply Lemma OA.2.11 again to show that firms must strictly allocate all resources to research before $T_\star$. The continuity of the probability function pins down the time $T_\star$, since it must therefore be that $\mathbf{p_1}(T_\star) = p_\star$.

D Proofs for Patent, License and Trade Secret

D.1 Proof of Proposition 6.1

Proof of Proposition 6.1. When the offer is rejected, Firm j's expected payoff is $\frac{\lambda_L \Pi - c}{\lambda_H + \lambda_L}$. Note that V_{11} is the expected payoff when both firms race with the new technology. thus, when the license offer with the fee l is accepted, Firm j's expected payoff is $V_{11} - l$. Then, Firm i's optimal offer is $l^* = V_{11} - (\lambda_L \Pi - c)/(\lambda_H + \lambda_L)$, and we can derive (6.1) with simple algebra. Then, once the offer is accepted, Firm i's expected payoff is $V_{11} + l^*$ and Firm j's expected payoff is $V_{11} - l^*$. □

D.2 Patents under Public Information Setting

D.2.1 Equilibrium Characterization

In this section, we fully characterize the equilibrium of the patent game under the public information setting.

First, we show that firms always apply for patents upon discovering the new technology.

Proposition D.1. *Suppose that firms' research progress is public information. In any subgame perfect Nash Equilibrium (SPNE), the first firm to discover the new technology applies for a patent.*

Note that the patent application of the first firm to obtain the new technology cannot be challenged. With this result and the equilibrium licensing fee from Proposition 6.1, we can pin down the continuation payoffs of both firms after the new technology is first discovered. We use these continuation payoffs to analyze the resource allocation of the firms before the new technology is first discovered. As in Section 4, we focus on Markov strategies, i.e., allocations that are independent of calendar time. Let $\mathbf{s}_P^i$ denote the research allocation of Firm i in the absence of the new technology discovery by either firm. The following proposition characterizes the equilibrium resource allocations.

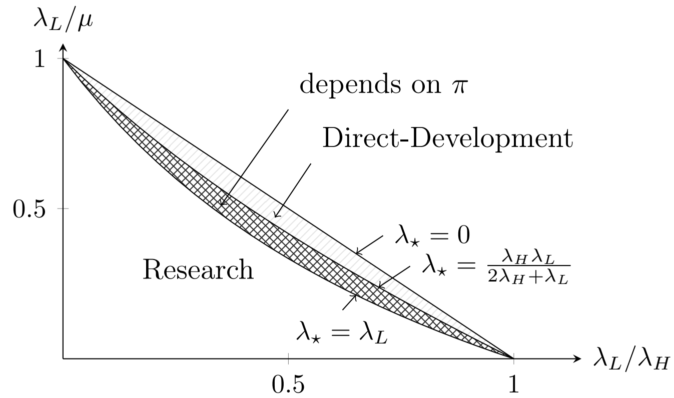

Figure 6: Equilibrium Resource Allocations in the Patent Game under Public Information

Proposition D.2. *Suppose that firms' research progress is public information. In any MPE, the resource allocations before the new technology is first discovered are characterized as follows:*

(a) if $\lambda_\star > \lambda_L$, both firms conduct research: $\mathbf{s}_P^A = \mathbf{s}_P^B = 1$;

(b) if $\frac{\lambda_H\lambda_L}{2\lambda_H+\lambda_L} > \lambda_\star$, both firms develop with the old technology: $\mathbf{s}_P^A = \mathbf{s}_P^B = 0$;

(c) if $\lambda_L > \lambda_\star > \frac{\lambda_H\lambda_L}{2\lambda_H+\lambda_L}$, there exist thresholds $\tilde{\pi}_0 > \tilde{\pi}_1 > 1$ such that

- *(i) when $\pi \equiv \lambda_L\Pi/c > \tilde{\pi}_0$, both firms develop with the old technology: $\mathbf{s}_P^A = \mathbf{s}_P^B = 0$;*
- *(ii) when $\tilde{\pi}_0 > \pi > \tilde{\pi}_1$, there are three equilibrium allocations: one firm does research and the other firm develops with the old technology, i.e., $(\mathbf{s}_P^A, \mathbf{s}_P^B) = (1,0)$ or $(0,1)$; both firms allocate some amount $z^* \in (0,1)$ resources to research: $\mathbf{s}_P^A = \mathbf{s}_P^B = z^*$;*
- *(iii) when $\tilde{\pi}_1 > \pi$, both firms do research: $\mathbf{s}_P^A = \mathbf{s}_P^B = 1$;*

Note that Proposition 6.2 corresponds to Proposition D.2 (a) and (c)-(i). Figure 6 summarizes the result. We can see that firms conduct research in a wider parametric region compared to the case without patents, as described in Figure 2. Intuitively, the option to patent increases the value of conducting research.

D.2.2 Proof of Proposition D.1

Proof of Proposition D.1. Suppose that Firm i has just discovered the new technology and Firm j does not have the patent for the new technology. If Firm j already has the patent, Firm i cannot apply for a patent in the first place.

First, consider the case where Firm j already has the new technology (not the patent). If Firm i does not apply for a patent, both firms race toward development with the new technology. Thus, Firm i's expected payoff is $\frac{\lambda_H \Pi - c}{2\lambda_H}$. If Firm i applies for a patent, with probability α, Firm j's right to use the new technology is protected, and with probability $1-\alpha$, Firm i acquires the patent. In either case, Firm i's expected payoff is at least $\frac{\lambda_H \Pi - c}{2\lambda_H}$, thus, Firm i prefers to apply for a patent.

Next, consider the case where Firm j does not have the new technology. Suppose that in equilibrium, Firm j allocates $x \in [0,1]$ to research and $1-x$ to development with the old technology, when it observes the new technology discovery by Firm i (without a patent). To maximize Firm j's expected payoff, we have

$$\frac{\mu x \cdot \tilde{U}^j + \lambda_L(1-x) \cdot \Pi - c}{\lambda_H + \mu x + \lambda_L(1-x)} \geq \frac{\lambda_L \Pi - c}{\lambda_H + \lambda_L}, \tag{D.1}$$

where $\tilde{U}^j$ is Firm j's expected payoff when it also discovers the new technology. To constitute an equilibrium, Firm i's expected payoff under this Firm j's strategy should be greater than or equal to Firm i's expected payoff from applying for a patent:

$$\frac{\lambda_H \cdot \Pi + \mu x \cdot \tilde{U}^i - c}{\lambda_H + \mu x + \lambda_L(1-x)} \geq U_{Licensor}, \tag{D.2}$$

where $\tilde{U}^i$ is Firm i's expected payoff when Firm j discovers the new technology.

Note that $\tilde{U}^i + \tilde{U}^j \leq \Pi - \frac{2c}{2\lambda_H}$ since the social welfare is maximized when both firms use the new technology, and $U_{Licensor} + \frac{\lambda_L \Pi - c}{\lambda_H + \lambda_L} = \Pi - \frac{c}{\lambda_H}$ from Proposition 6.1. By using these and summing (D.1) and (D.2) up, we have

$$\Pi - \frac{c}{\lambda_H} \leq \Pi - \frac{\frac{\mu x}{\lambda_H} + 2}{\lambda_H + \mu x + \lambda_L(1-x)} c.$$

However, this inequality is equivalent to $\lambda_H + \mu x + \lambda_L(1-x) \geq 2\lambda_H + \mu x$, which contradicts $\lambda_H > \lambda_L$ and $x \leq 1$. Therefore, in equilibrium, Firm i applies for a patent. $\square$

D.2.3 Proof of Proposition D.2

We begin by extending our MPE characterization in Lemma B.2 to the cases where Δ_1 and Δ_0 have different signs.

Lemma D.1. *The equilibrium allocations at state $\emptyset$ are characterized as follows:*

(a) when $\Delta_0 > 0 > \Delta_1$, there are three possible equilibrium allocations:

  (i) one firm does research and the other firm develops with the old technology, i.e., $(\mathbf{s}^A(\emptyset), \mathbf{s}^B(\emptyset)) = (1, 0)$ or $(0, 1)$,

  (ii) both firms allocate $z^* = \Delta_0/(\Delta_0 - \Delta_1)$ amount of resources to research and the remainder to the development with the old technology, i.e., $(\mathbf{s}^A(\emptyset), \mathbf{s}^B(\emptyset)) = (z^*, z^*)$;

(b) when $\Delta_1 > 0 > \Delta_0$, there are three possible equilibrium allocations:

  (i) both firms do research, i.e., $(\mathbf{s}^A(\emptyset), \mathbf{s}^B(\emptyset)) = (1, 1)$,

  (ii) both firms develop with the old technology, i.e., $(\mathbf{s}^A(\emptyset), \mathbf{s}^B(\emptyset)) = (0, 0)$,

  (iii) both firms allocate $z^* = -\Delta_0/(\Delta_1 - \Delta_0)$ amount of resources to research and the remainder to the development with the old technology, i.e., $(\mathbf{s}^A(\emptyset), \mathbf{s}^B(\emptyset)) = (z^*, z^*)$.

Proof of Lemma D.1. (a) From $\Delta_0 > 0$ and (B.2), we have $\frac{\partial u_0}{\partial x}|_{y=0} > 0$, i.e., $x = 1$ is the best response for $y = 0$. In addition, from $0 > \Delta_1$ and (B.2), we have $\frac{\partial u_0}{\partial x}|_{y=1} < 0$, i.e., $x = 0$ is the best response for $y = 1$. Therefore, $(1, 0)$ and $(0, 1)$ can be supported equilibrium allocations at $\omega = \emptyset$.

Next, note that $z^* \in (0, 1)$ and $\frac{\partial u_0}{\partial x}|_{y=z^*} = 0$, i.e., any $x \in [0, 1]$ is the best response for $y = z^*$. Thus, (z^*, z^*) can be supported as an equilibrium allocation.

Last, consider any $\tilde{y} \in (0, 1)$ with $\tilde{y} \neq z^*$. Then, $\frac{\partial u_0}{\partial x}|_{y=\tilde{y}} \neq 0$, i.e., the best response is $x = 1$ or $x = 0$. Recall that the best response of $x = 1$ $(x = 0)$ is $y = 0$ $(y = 1)$, thus, $y = \tilde{y}$ cannot be a part of an equilibrium allocation.

(b) From $\Delta_0 < 0$ and (B.2), we have $\frac{\partial u_0}{\partial x}|_{y=0} < 0$, i.e., $x = 0$ is the best response for $y = 0$. Thus, $(0, 0)$ can be supported as an equilibrium allocation.

Similarly, from $0 < \Delta_1$ and (B.2), we have $\frac{\partial u_0}{\partial x}|_{y=1} > 0$, i.e., $x = 1$ is the best response for $y = 1$. Therefore, $(1, 1)$ can also be supported as an equilibrium allocation.

Next, note that $z^* \in (0, 1)$ and $\frac{\partial u_0}{\partial x}|_{y=z^*} = 0$, i.e., any $x \in [0, 1]$ is the best response for $y = z^*$. Thus, (z^*, z^*) can be supported as an equilibrium allocation.

Last, by using the similar argument as in the previous case, $\tilde{y} \in (0, 1)$ with $\tilde{y} \neq z^*$ cannot be a part of an equilibrium allocation.

□

Proof of Proposition D.2. To apply Lemma D.1, we first compute $\hat{\Delta}_0$ and $\hat{\Delta}_1$ by replacing $(U^i_{\{i\}}, U^j_{\{i\}})$ to $(U_{Licensor}, U_{Licensee})$ in (B.1):

$$\hat{\Delta}_0 = \frac{\mu U_{Licensor} - c}{\mu + \lambda_L} - \frac{\lambda_L \Pi - c}{2\lambda_L},$$
$$\hat{\Delta}_1 = \frac{\mu U_{Licensor} + \mu U_{Licensee} - c}{2\mu} - \frac{\lambda_L \Pi + \mu U_{Licensee} - c}{\mu + \lambda_L}.$$

By using Proposition 6.1, we can derive that

$$\hat{\Delta}_0 = \frac{\lambda_H \lambda_L (\lambda_\star - \lambda_L)\Pi + (\lambda_H + \lambda_L)\lambda_\star c}{2\lambda_H(\lambda_H + \lambda_L)(\lambda_L + \mu)},$$
$$\hat{\Delta}_1 = \frac{\lambda_H \lambda_L (\lambda_\star - \lambda_L)\Pi + \frac{\lambda_L}{2\mu}\left\{(2\lambda_H + \mu + \lambda_L)\lambda_\star + (\mu - \lambda_L)\lambda_H\right\} c}{2\lambda_H(\lambda_H + \lambda_L)(\lambda_L + \mu)}.$$

First, observe that $\lambda_\star \geq \lambda_L$ implies $\hat{\Delta}_0,\ \hat{\Delta}_1 > 0$. Then, by Lemma D.1 (a), both firms do research, thus, Proposition D.2 (a) holds. Next, when $\lambda_L > \lambda_\star$, we have

$$\hat{\Delta}_0 > 0 \quad \Longleftrightarrow \quad \tilde{\pi}_0 \equiv \frac{\lambda_\star(\lambda_H + \lambda_L)}{\lambda_H(\lambda_L - \lambda_\star)} > \frac{\lambda_L \Pi}{c} = \pi,$$
$$\hat{\Delta}_1 > 0 \quad \Longleftrightarrow \quad \tilde{\pi}_1 \equiv \frac{\frac{\lambda_L}{2\mu}\left\{(2\lambda_H + \mu + \lambda_L)\lambda_\star + (\mu - \lambda_L)\lambda_H\right\}}{\lambda_H(\lambda_L - \lambda_\star)} > \pi.$$

Suppose that $\lambda_\star \in \left(\frac{\lambda_H \lambda_L}{2\lambda_H + \lambda_L}, \lambda_L\right)$. By using $\mu > \lambda_L$, we can show that $\tilde{\pi}_0 > \tilde{\pi}_1 > 1$.

(i) if $\pi > \tilde{\pi}_0 > \tilde{\pi}_1$, we have $\hat{\Delta}_0,\ \hat{\Delta}_1 < 0$, then, by Proposition 1 (b), both firms develop with old technology;

(ii) if $\tilde{\pi}_0 > \pi > \tilde{\pi}_1$, we have $\hat{\Delta}_0 > 0 > \hat{\Delta}_1$, then, by Proposition 1 (c), there are three equilibria including the asymmetric one;

(iii) if $\tilde{\pi}_1 > \pi > 1$, we have $\hat{\Delta}_0, \hat{\Delta}_1 > 0$, then, by Proposition 1 (a), both firms do research.

Thus, Proposition D.2 (b) holds.

Now suppose that $\lambda_\star \le \frac{\lambda_H \lambda_L}{2\lambda_H + \lambda_L}$. With some algebra, we have $1 \ge \tilde{\pi}_1 \ge \tilde{\pi}_0$. From $\pi > 1$, we have $\hat{\Delta}_0, \hat{\Delta}_1 < 0$, then, by Proposition 1 (b), both firms develop with old technology. Thus, Proposition D.2 (c) holds. □

D.3 Proofs for Efficient Patent Equilibrium

Proof of Lemma 6.1. Suppose that Firm i has discovered the new technology, and Firm j has not applied for a patent yet. Given Firm j's patent application strategy, the fact that Firm j has not applied for a patent implies that Firm j does not have the new technology yet. Therefore, if Firm i applies for a patent, it will attain the patent with probability one and its expected continuation payoff is $U_{Licensor} = V_{11} + l^*$. Suppose instead that Firm i decides not to apply for a patent. Firm i's payoff in the case in which Firm i finds the new technology before a successful development is $U^{\alpha}_{Challenger} = V_{11} - (1-\alpha)\cdot l^*$. Therefore, Firm i's expected payoff of not applying for a patent is

$$\frac{\lambda_H \Pi + \mu \cdot U^{\alpha}_{Challenger} - c}{\lambda_H + \mu} = \frac{(\mu + 2\lambda_H)V_{11} - \mu(1-\alpha)l^*}{\lambda_H + \mu}. \tag{D.3}$$

Firm i applies for a patent when $U_{Licensor}$ is greater than (D.3), which is equivalent to:

$$\begin{aligned} & (\lambda_H + \mu)V_{11} + (\lambda_H + \mu)l^* > (\mu + 2\lambda_H)V_{11} - \mu(1-\alpha)l^* \\ \iff \quad & \{\lambda_H + \mu(2-\alpha)\}\, l^* > \lambda_H V_{11}. \end{aligned}$$

Since $1 > \alpha$, $\lambda_H, \mu > 0$ and $V_{11} > 0$, it is equivalent to (6.2). □

Proof of Proposition 6.3. By plugging (6.1) in, we have that (6.2) is equivalent to:

$$
\begin{aligned}
& \frac{\lambda_H - \lambda_L}{\lambda_H + \lambda_L} \cdot \frac{\lambda_H \Pi + c}{\lambda_H \Pi - c} > \frac{\lambda_H}{\lambda_H + \mu(2-\alpha)} \\
\iff \quad & \{\lambda_H(\lambda_H - \lambda_L) + \mu(\lambda_H - \lambda_L)(2-\alpha)\}(\lambda_H \Pi + c) > \lambda_H(\lambda_H + \lambda_L)(\lambda_H \Pi - c) \\
\iff \quad & \{\mu(\lambda_H - \lambda_L)(2-\alpha) - 2\lambda_L\lambda_H\} \cdot \lambda_H \Pi + \left\{\mu(\lambda_H - \lambda_L)(2-\alpha) + 2\lambda_H^2\right\} \cdot c > 0.
\end{aligned}
$$

Note that $\mu(\lambda_H - \lambda_L) = \lambda_L(\lambda_\star + \lambda_H)$ from (3.1). By plugging this in, the above inequality is equivalent to:

$$
\begin{aligned}
& \{(2-\alpha)\lambda_\star - \alpha\lambda_H\} \cdot \lambda_H \lambda_L \Pi + \left\{(2-\alpha)\lambda_L(\lambda_\star + \lambda_H) + 2\lambda_H^2\right\} \cdot c > 0 \\
\iff \quad & (\lambda_\star + \lambda_H)(\hat{\alpha} - \alpha) \cdot \lambda_H \left(\frac{\lambda_L \Pi}{c} - 1\right) + (2-\alpha)(\lambda_L + \lambda_H)(\lambda_\star + \lambda_H) > 0.
\end{aligned}
$$

If $\alpha \leq \hat{\alpha}$, the first term in the above inequality is nonnegative and the second term is positive from $\alpha < 1$ and $\lambda_L,\ \lambda_H,\ \lambda_\star > 0$. If $\alpha > \hat{\alpha}$, by rearranging it and using $\pi = \frac{\lambda_L \Pi}{c}$, we can show that the above inequality is equivalent to (6.3). $\square$

D.4 Proofs for Concealment Equilibrium

Let $\boldsymbol{\sigma}^*$ denote the unique equilibrium policy in Theorem 2. From (C.12), $V_1(t; \mathbf{h}_{\boldsymbol{\sigma}^*})$ is the continuation value of firms in such equilibrium. A *concealment equilibirium* is an equilibrium of the game with patents such that the firms never patent the new technology and follow policy $\boldsymbol{\sigma}^*$.

Observation There is a concealment equilibrium if and only if, for all $t \geq 0$,

$$
V_1(t; \mathbf{h}_{\boldsymbol{\sigma}^*}) \geq V_{11} + (1 - \alpha\, \mathbf{p}_{\boldsymbol{\sigma}^*}(t)) \cdot l^*. \tag{D.4}
$$

To understand the observation, notice that (D.4) captures the trade-off in the patenting decision of a firm that discovers the new technology at time t, when the opponent follows policy $\boldsymbol{\sigma}^*$ and never patents. The left-hand-side denotes the payoff obtained by not patenting, i.e by keeping the discovery secret. The right-hand-side captures the expected payoff if the firm decides to patent at time t. If (D.4) holds for all t, then it is a best response to never patent.

Under $\lambda_H > \lambda_\star > \mu$, by Theorem 2 (b), firms employ the research policy in the private information setting, i.e., $\boldsymbol{\sigma}^* = \mathbf{1}$. The following lemma provides the closed form solution of $V_1(t; \mathbf{h_1})$.

Lemma D.2. *When $\lambda_H > \lambda_\star > \mu$, the following equation holds:*

$$V_1(t; \mathbf{h_1}) = \left\{1 + \frac{\lambda_H}{\lambda_H + \mu}(1 - \mathbf{p_1}(t))\right\} \cdot V_{11}. \tag{D.5}$$

Proof of Lemma D.2. By Lemma OA.2.1, $\mathbf{p_1}(t)$ is increasing in t. Then, $V_1(t; \mathbf{h_1})$ can be written as a function of $\mathbf{p_1}(t)$: $V_1(t; \mathbf{h_1}) = v_1(\mathbf{p_1}(t))$. Observe that

$$V_1'(t; \mathbf{h_1}) = v_1'(\mathbf{p}_1'(t)) \cdot \mathbf{p}_1'(t) = v_1'(\mathbf{p}_1'(t)))(\mu - \lambda_H \, \mathbf{p_1}(t))(1 - \mathbf{p_1}(t)).$$

By plugging this into (HJB_1), we have

$$0 = v_1'(p)(\mu - \lambda_H p)(1 - p) - \lambda_H(1 + p)v_1(p) + \lambda_H \Pi - c. \tag{D.6}$$

Define two function $g(p)$ and $k(p)$ as follows:

$$g(p) := \frac{(\mu - \lambda_H p)^{\frac{\mu + \lambda_H}{\lambda_H - \mu}}}{(1 - p)^{\frac{2\lambda_H}{\lambda_H - \mu}}} \quad \text{and} \quad k(p) := 1 + \frac{\lambda_H}{\lambda_H + \mu}(1 - p). \tag{D.7}$$

Observe that

$$\frac{g'(p)}{g(p)} = \frac{d \log(g(p))}{dp} = -\frac{\mu + \lambda_H}{\lambda_H - \mu} \cdot \frac{\lambda_H}{\mu - \lambda_H p} + \frac{2\lambda_H}{\lambda_H - \mu} \cdot \frac{1}{1 - p} = -\frac{\lambda_H(1 + p)}{(1 - p)(\mu - \lambda_H p)} \tag{D.8}$$

and

$$\frac{d}{dp}(g(p) \cdot k(p)) = -\frac{\lambda_H(1 + p)k(p)}{(1 - p)(\mu - \lambda_H p)} g(p) - \frac{\lambda_H}{\lambda_H + \mu} g(p) = -\frac{2\lambda_H}{(1 - p)(\mu - \lambda_H p)} g(p) \tag{D.9}$$

By multiplying (D.6) by $\frac{g(p)}{(\mu-\lambda_H p)(1-p)}$ and using above two equations, we have

$$\begin{aligned} 0 =& v_1'(p) \cdot g(p) + g'(p) \cdot v_1(p) + \frac{\lambda_H \Pi - c}{2\lambda_H} \cdot \frac{g(p)}{(1-p)(\mu - \lambda_H p)} \\ =& \frac{d}{dp}\left[(v_1(p) - V_{11} \cdot k(p)) \cdot g(p) \right]. \end{aligned}$$

Therefore, there exists $C \in \mathbb{R}$ such that

$$v_1(p) = V_{11} \cdot k(p) + \frac{C}{g(p)}. \tag{D.10}$$

In Lemma OA.2.1, we show that if $\mu \geq \lambda_H$, $\lim_{t\to\infty} \mathbf{p_1}(t) = 1$, and if $\mu < \lambda_H$, $\lim_{t\to\infty} \mathbf{p_1}(t) = \mu/\lambda_H$. By using these, we have that $\lim_{t\to\infty} g(\mathbf{p_1}(t)) = 0$. Then, to satisfy $V_1(t; \mathbf{h_1}) = v_1(\mathbf{p_1}(t))$ and (D.10), the constant C has to be zero, and (D.5) holds. $\square$

By using this lemma, (D.4) is equivalent to:

$$\frac{l^*}{V_{11}} < \frac{\lambda_H}{\lambda_H + \mu} \cdot \frac{1 - \mathbf{p_1}(t)}{1 - \alpha \cdot \mathbf{p_1}(t)}. \tag{D.11}$$

The right hand side is decreasing in $\mathbf{p_1}(t)$. Under $\lambda_H > \lambda_\star > \mu$, $\mathbf{p_1}(t)$ converges to μ/λ_H, thus, we can plug this into (D.11):

$$\frac{l^*}{V_{11}} < \frac{\lambda_H(\lambda_H - \mu)}{(\lambda_H + \mu)(\lambda_H - \alpha\mu)}. \tag{D.12}$$

With simple algebra, we can show that $\frac{\lambda_H(\lambda_H-\mu)}{(\lambda_H+\mu)(\lambda_H-\alpha\mu)} \leq \frac{\lambda_H}{\lambda_H+\mu(2-\alpha)}$. Therefore, the threshold for the concealment equilibrium is below the one for the efficient patent equilibrium, i.e., there is no parameter such that both the efficient patent equilibrium and the concealment equilibrium exist. By solving (D.12), we can pin down the parametric conditions under which the concealment equilibrium exists.

$$\alpha > \tilde{\alpha} := \frac{2\lambda_H(\mu + \lambda_\star)}{(\lambda_H + \mu)(\lambda_H + \lambda_\star)} \tag{D.13}$$

and

$$\pi > \tilde{\pi}(\alpha) := 1 + \frac{\lambda_H + \lambda_L}{\lambda_H + \mu} \cdot \frac{2\lambda_H - (\lambda_H + \mu)\alpha}{\lambda_H(\alpha - \tilde{\alpha})}. \tag{D.14}$$

Now we provide the proof of Proposition 6.4

Proof of Proposition 6.4. By using using $\mu(\lambda_H-\lambda_L)=\lambda_L(\lambda_\star+\lambda_H)$, $\lambda_H(\mu-\lambda_L)=\lambda_L(\lambda_\star+\mu)$ and (6.1), we have that (D.12) is equivalent to:

$$(\lambda_H+\lambda_L)(2\lambda_H-\alpha(\lambda_H+\mu))<(\lambda_H+\mu)(\alpha-\tilde{\alpha})\lambda_H\cdot(\pi-1)\,.$$

Note that $2\lambda_H-\alpha(\lambda_H+\mu)>0$ from $\lambda_H>\mu$ and $1\geq\alpha$. Therefore, if $\alpha\leq\tilde{\alpha}$, the above inequality cannot hold. When $\alpha>\tilde{\alpha}$, by rearranging the above inequality, we have (D.14).

Observe that $\tilde{\alpha}>\hat{\alpha}$ is equivalent to:

$$2\lambda_H(\mu+\lambda_\star)>2\lambda_\star(\lambda_H+\mu)$$

and it holds from the assumption that $\lambda_H>\lambda_\star$.

Next, observe that $\tilde{\pi}(\alpha)\geq\hat{\pi}(\alpha)$ is equivalent to:

$$\frac{\frac{2\lambda_H}{\lambda_H+\mu}-\alpha}{\alpha-\tilde{\alpha}}\geq\frac{2-\alpha}{\alpha-\hat{\alpha}}\qquad\Longleftrightarrow\qquad\frac{\frac{2\lambda_H}{\lambda_H+\mu}-\tilde{\alpha}}{\alpha-\tilde{\alpha}}\geq\frac{2-\hat{\alpha}}{\alpha-\hat{\alpha}}.\tag{D.15}$$

Also note that

$$\frac{2\lambda_H}{\lambda_H+\mu}-\tilde{\alpha}=\frac{2\lambda_H}{\lambda_H+\mu}\cdot\frac{\lambda_H-\mu}{\lambda_H+\lambda_\star}\qquad\text{and}\qquad 2-\hat{\alpha}=\frac{2\lambda_H}{\lambda_H+\lambda_\star}.$$

By plugging these in, (D.15) is equivalent to:

$$\tilde{\alpha}-\frac{\lambda_H-\mu}{\lambda_H+\mu}\hat{\alpha}\geq\frac{2\mu}{\lambda_H+\mu}\alpha.\tag{D.16}$$

Note that

$$\tilde{\alpha}-\frac{\lambda_H-\mu}{\lambda_H+\mu}\hat{\alpha}=\frac{2\lambda_H(\mu+\lambda_\star)}{(\lambda_H+\mu)(\lambda_H+\lambda_\star)}-\frac{\lambda_H-\mu}{\lambda_H+\mu}\cdot\frac{2\lambda_\star}{\lambda_H+\lambda_\star}=\frac{2\mu}{\lambda_H+\mu}.$$

Therefore, (D.16) is equivalent to $1\geq\alpha$. Therefore, $\tilde{\pi}(\alpha)\geq\hat{\pi}(\alpha)$ holds for all $1\geq\alpha>\tilde{\alpha}$ and the equality holds if and only if $\alpha=1$. ☐

Online Appendix for "*Strategic Concealment in Innovation Races*"

OA.1 Optimal Control Theory

OA.1.1 Useful Observations

Let τ be a random variable on $\mathbb{R}_+$. Suppose that it has a continuous and differentiable cumulative distribution function $F : \mathbb{R}_+ \to [0,1]$. Let $S(t)$ denote the survival function of τ, i.e., $S(t) = 1 - F(t)$. If $\lim_{t\to\infty} t \cdot S(t) = 0$, the following equation holds:

$$\mathbb{E}[\tau] = \int_0^\infty t \cdot F'(t)dt = -t \cdot S(t)\Big|_0^\infty + \int_0^\infty S(t)dt = \int_0^\infty S(t)dt. \tag{OA.1.1}$$

Let h be a development rate function of τ: $h(t) = -S'(t)/S(t)$.[19] Then, under the assumption that $F(0) = 0$, we can derive that $S(t) = e^{-\int_0^t h(s)ds}$. Then, (OA.1.1) can be rewritten as follows:

$$\mathbb{E}[\tau] = \int_0^\infty e^{-\int_0^t h(s)ds}dt. \tag{OA.1.2}$$

Consider another random variable $\hat{\tau}$ independent to τ. Let $\hat{S}$ and $\hat{h}$ be its survival and development rate functions. Observe that

$$\Pr[\tau < \hat{\tau}] = \int_0^\infty \hat{S}(t)\ dF(t) = -\int_0^\infty S'(t) \cdot \hat{S}(t)\ dt. \tag{OA.1.3}$$

Then, (OA.1.3) can be rewritten as follows:

$$\Pr[\tau < \hat{\tau}] = \int_0^\infty h(t) \cdot S(t) \cdot \hat{S}(t)\ dt = \int_0^\infty h(t) \cdot e^{-\int_0^t (h(s)+\hat{h}(s))ds}\ dt. \tag{OA.1.4}$$

Now consider another random variable which is a minimum of τ and $\hat{\tau}$, denoted by $(\tau \wedge \hat{\tau})$.

[19] In the literature, the function $h(t)$ is often referred to as a 'hazard rate' function. The term hazard rate originated from the tradition of describing arrivals as negative events such as failures. In our context, where we are analyzing the timing of product developments, we use the term 'development rate' instead of hazard rate.

Then, the survival function of $(\tau \wedge \hat{\tau})$ is $S(t) \cdot \hat{S}(t)$, and the development function of $(\tau \wedge \hat{\tau})$ is $h(t) + \hat{h}(t)$. By applying (OA.1.2), when $\lim_{t\to\infty} t \cdot S(t) \cdot \hat{S}(t) = 0$, we have

$$\mathbb{E}[\tau \wedge \hat{\tau}] = \int_0^\infty e^{-\int_0^t (h(s)+\hat{h}(s))ds}\, dt. \tag{OA.1.5}$$

OA.1.2 Formal Definitions of Arrival Times

Given an allocation policy $\sigma : \mathbb{R}_+ \to [0, 1]$, we define the following random variables:

1. τ_L: the arrival time of successful development with the old technology;

2. τ_R: the arrival time of the new technology discovery.

Recall that $\Sigma_t \equiv \int_0^t \sigma_s ds$. Then, the survival functions of τ_L and τ_R are given as follows: for all $t \geq 0$,

$$S_\sigma^L(t) = e^{-\lambda_L(t-\Sigma_t)} \quad \text{and} \quad S_\sigma^R(t) = e^{-\mu\Sigma_t}. \tag{OA.1.6}$$

In addition, the development rate functions can be derived as follows:

$$h_\sigma^L(t) = \lambda_L(1 - \sigma_t) \quad \text{and} \quad h_\sigma^R(t) = \mu\sigma_t. \tag{OA.1.7}$$

Intuitively, the product is developed with the old technology at the rate $h_\sigma^L(t) = \lambda_L(1 - \sigma_t)$ and the new technology is discovered at the rate $h_\sigma^R(t) = \mu\sigma_t$.

OA.1.3 Benchmark: Constant Development Rate

Lemma OA.1.1. *Suppose that Firm j has a constant development rate λ. When Firm i employs an allocation policy σ, its expected payoff is given as follows:*

$$V_\lambda^0(\sigma) = \int_0^\infty \Big(\lambda_L(1 - \sigma_t) \cdot \Pi + \mu\ \sigma_t \cdot \mathcal{V}_\lambda^1 - c\Big) \cdot e^{-\lambda_L(t-\Sigma_t)-\mu\Sigma_t-\lambda t}\, dt, \tag{OA.1.8}$$

where $\Sigma_t \equiv \int_0^t \sigma_s ds$.

Proof of Lemma OA.1.1. Let τ_λ be the arrival time of Firm j. When any of the arrival times τ_L, τ_R and τ_λ occurs, we can regard that Firm i's payoff is realized. Furthermore, it incurs a flow cost c until one of these arrival times takes place. Thus, Firm i's expected payoff can be written as follows:

$$V_\lambda^0(\sigma) = \Pr[\tau_L < (\tau_R \wedge \tau_\lambda)] \cdot \Pi + \Pr[\tau_R < (\tau_L \wedge \tau_\lambda)] \cdot \mathcal{V}_\lambda^1 - \mathbb{E}[(\tau_L \wedge \tau_R \wedge \tau_\lambda)] \cdot c. \tag{OA.1.9}$$

Note that the survival function of $(\tau_R \wedge \tau_\lambda)$ is $e^{-\int_0^t (\mu\sigma_s + \lambda) ds} = e^{-\mu\Sigma_t - \lambda t}$. By using (OA.1.4) and (OA.1.7), we have

$$\Pr[\tau_L < (\tau_R \wedge \tau_\lambda)] = \int_0^\infty \lambda_L (1 - \sigma_t) \cdot e^{-\lambda_L (t - \Sigma_t) - \mu\Sigma_t - \lambda t} \, dt.$$

Likewise, we can derive that

$$\Pr[\tau_R < (\tau_L \wedge \tau_\lambda)] = \int_0^\infty \mu \; \sigma_t \cdot e^{-\lambda_L (t - \Sigma_t) - \mu\Sigma_t - \lambda t} \, dt.$$

Next, observe that the survival function of $(\tau_L \wedge \tau_R \wedge \tau_\lambda)$ is

$$e^{-\lambda_L (t - \Sigma_t) - \mu\Sigma_t - \lambda t} = e^{-(\lambda_L + \lambda)t - (\mu - \lambda_L)\Sigma_t}.$$

Then, from $\mu \geq \lambda_L$ and $\Sigma_t + \hat{\Sigma}_t \geq 0$, we have $\lim_{t \to \infty} t \cdot e^{-\lambda_L (t - \Sigma_t) - \mu\Sigma_t - \lambda t} = 0$. By applying (OA.1.1), we have

$$\mathbb{E}[(\tau_L \wedge \tau_R \wedge \tau_\lambda)] = \int_0^\infty e^{-\lambda_L (t - \Sigma_t) - \mu\Sigma_t - \lambda t} \, dt.$$

By plugging the above equations into (OA.1.9), we obtain (OA.1.8). □

Lemma OA.1.2. *Suppose that $x_0 \in \arg\max_{x \in [0,1]} u(x)$ where u is a function defined in (A.2). Let $\sigma^* : \mathbb{R}_+ \to [0, 1]$ be $\sigma_t^* = x_0$ for all $t \geq 0$. Then, $\sigma^* \in \arg\max_\sigma V_\lambda^0(\sigma)$.*

Proof of Lemma OA.1.2. Let r_t denote $e^{-\lambda_L (t - \Sigma_t) - \mu\Sigma t - \lambda t}$. By taking a derivative, we have

$$\dot{r}_t = - \left\{ \lambda_L (1 - \sigma_t) + \mu\sigma_t + \lambda \right\} \cdot r_t. \tag{OA.1.10}$$

By Lemma OA.1.1, Firm i's problem is

$$\max_\sigma \int_0^\infty \left\{ \lambda_L (1 - \sigma_t) \cdot \Pi + \mu\sigma_t \cdot \mathcal{V}_\lambda^1 - c \right\} \cdot r_t \, dt \tag{OA.1.11}$$

subject to (OA.1.10).

Observe that the Hamiltonian of this optimal control problem is

$$\begin{aligned} H(\sigma_t, r_t, \eta_t) &= \left\{\lambda_L(1-\sigma_t)\cdot \Pi + \mu\sigma_t \cdot \mathcal{V}_\lambda^1 - c\right\}\cdot r_t - \eta_t\left\{\lambda_L(1-\sigma_t) + \mu\sigma_t + \lambda\right\}\cdot r_t \\ &= \{u(\sigma_t) - \eta_t\}\cdot\{\lambda_L(1-\sigma_t) + \mu\sigma_t + \lambda\}\cdot r_t, \end{aligned} \tag{OA.1.12}$$

where η_t is a co-state variable.

To show that σ^* is a solution of (OA.1.11) subject to (OA.1.10) by using the Arrow sufficiency condition (Seierstad and Sydsaeter, 1987, Theorem 3.14), we consider (η^*, r^*) defined as follows: for all $t \geq 0$, $\eta_t^* = u(x_0)$ and $r_t^* = e^{-\{\mu x_0 + \lambda_L(1-x_0)+\lambda\}\cdot t}$.

Then, we need to check following four primitive conditions:

1. Maximum principle: for all $t \geq 0$,

$$\sigma_t^* = x_0 \in \arg\max_{\sigma_t \in [0,1]} H(\sigma_t, r_t^*, \eta_t^*). \tag{OA.1.13}$$

2. Evolution of the co-state variable:

$$\dot{\eta}_t^* = -\frac{\partial H}{\partial r_t} = -\{u(\sigma_t^*) - \eta_t^*\}\cdot\{\lambda_L(1-\sigma_t^*) + \mu\sigma_t^* + \lambda\}. \tag{OA.1.14}$$

3. Transversality condition: If r^* is the optimal trajectory, i.e., $r_t^* = e^{-\{\mu x_0 + \lambda_L(1-x_0)+\lambda\}\cdot t}$, $\lim_{t\to\infty} \eta_t^*(r_t^* - r_t) \leq 0$ for all feasible trajectories r_t.

4. $\hat{H}(r_t, \eta_t) = \max_{\sigma_t\in[0,1]} H(\sigma_t, r_t, \eta_t)$ is concave in r_t.

First, by plugging r_t^* and η_t^* into (OA.1.12), we have

$$H(\sigma_t, r_t^*, \eta_t^*) = \{u(\sigma_t) - u(x_0)\}\cdot\{\lambda_L(1-\sigma_t^*) + \mu\sigma_t^* + \lambda\}\cdot r_t \tag{OA.1.15}$$

Recall that $x_0 \in \arg\max_{x\in[0,1]} u(x)$. Thus, $H(\sigma_t, r_t^*, \eta_t^*) \leq 0$ for all $\sigma_t \in [0,1]$. In addition, $H(x_0, r_t^*, \eta_t^*) = 0$. Therefore, $x_0 \in \arg\max_{\sigma_t\in[0,1]} H(\sigma_t, r_t, \eta_t)$, i.e., (OA.1.13) holds.

Second, by the definition of η^*, (OA.1.14) holds.

Third, note that for any admissible allocation σ,

$$r_t = e^{-\{\mu\Sigma_t+\lambda_L(t-\Sigma_t)+\lambda t\}} = r_t^* \cdot e^{(\mu-\lambda_L)\cdot(x_0 t-\Sigma_t)}.$$

Then, we have

$$\lim_{t\to\infty} \eta_t^* \cdot (r_t^* - r_t) = \lim_{t\to\infty} u(x_0)\cdot r_t^* \cdot \left(1 - e^{(\mu-\lambda_L)\cdot(x_0 t-\Sigma_t)}\right) = 0.$$

Last, we can see that $\hat{H}$ is linear in r_t, thus, the fourth condition holds. Hence, by the Arrow sufficiency condition, σ^* is the best response to $\hat{\sigma}^*$. $\square$

OA.1.4 Public Information Setting

Lemma OA.1.3. *Suppose that Firm i and j employ allocation policies σ and $\hat{\sigma}$ at the state $\emptyset$. Let $U^i_{\{i\}}$ and $U^i_{\{j\}}$ be Firm i's continuation payoffs at the states $\{i\}$ and $\{j\}$. Then, Firm i's expected payoffs are given as follows:*

$$U_0(\sigma,\hat{\sigma}) = \int_0^\infty \left(\lambda_L(1-\sigma_t)\cdot\Pi + \mu\ \sigma_t\cdot U^i_{\{i\}} + \mu\ \hat{\sigma}_t\cdot U^i_{\{j\}} - c\right)\cdot e^{-\lambda_L(2t-\Sigma_t-\hat{\Sigma}_t)-\mu(\Sigma_t+\hat{\Sigma}_t)}\ dt, \tag{OA.1.16}$$

where $\Sigma_t = \int_0^t \sigma_s ds$ and $\hat{\Sigma}_t = \int_0^t \hat{\sigma}_s ds$.

Proof. When any of the arrival times $\tau_L,\ \tau_R,\ \hat{\tau}_L$ and $\hat{\tau}_R$ occurs, the Firm i's payoff is realized. Furthermore, it incurs a flow cost c until one of these arrival times takes place. Thus, Firm i's expected payoff can be written as follows:

$$\begin{aligned} U_0(\sigma,\hat{\sigma}) =& \Pr[\tau_L < (\tau_R\wedge\hat{\tau}_L\wedge\hat{\tau}_R)]\cdot\Pi + \Pr[\tau_R < (\tau_L\wedge\hat{\tau}_L\wedge\hat{\tau}_R)]\cdot U^i_{\{i\}} \\ &+ \Pr[\hat{\tau}_R < (\tau_L\wedge\tau_R\wedge\hat{\tau}_L)]\cdot U^i_{\{j\}} - \mathbb{E}[(\tau_L\wedge\tau_R\wedge\hat{\tau}_L\wedge\hat{\tau}_R)]\cdot c. \end{aligned} \tag{OA.1.17}$$

Note that the survival function of $(\tau_R\wedge\hat{\tau}_L\wedge\hat{\tau}_R)$ is $e^{-\lambda_L(t-\hat{\Sigma}_t)-\mu(\Sigma_t+\hat{\Sigma}_t)}$. By using (OA.1.4) and (OA.1.7), we have

$$\Pr[\tau_L < (\tau_R\wedge\hat{\tau}_L\wedge\hat{\tau}_R)] = \int_0^\infty \lambda_L(1-\sigma_t)\cdot e^{-\lambda_L(2t-\Sigma_t-\hat{\Sigma}_t)-\mu(\Sigma_t+\hat{\Sigma}_t)}\ dt.$$

Likewise, we can derive that

$$\Pr[\tau_R < (\tau_L \wedge \hat{\tau}_L \wedge \hat{\tau}_R)] = \int_0^\infty \mu\ \sigma_t \cdot e^{-\lambda_L(2t-\Sigma_t-\hat{\Sigma}_t)-\mu(\Sigma_t+\hat{\Sigma}_t)}\ dt,$$
$$\Pr[\hat{\tau}_R < (\hat{\tau}_L \wedge \tau_L \wedge \tau_R)] = \int_0^\infty \mu\ \hat{\sigma}_t \cdot e^{-\lambda_L(2t-\Sigma_t-\hat{\Sigma}_t)-\mu(\Sigma_t+\hat{\Sigma}_t)}\ dt.$$

Next, observe that the survival function of $(\tau_L \wedge \tau_R \wedge \hat{\tau}_L \wedge \hat{\tau}_R)$ is

$$e^{-\lambda_L(2t-\Sigma_t-\hat{\Sigma}_t)-\mu(\Sigma_t+\hat{\Sigma}_t)} = e^{-2\lambda_L t-(\mu-\lambda_L)(\Sigma_t+\hat{\Sigma}_t)}.$$

Then, from $\mu \geq \lambda_L$ and $\Sigma_t + \hat{\Sigma}_t \geq 0$, we have $\lim_{t\to\infty} t \cdot e^{-\lambda_L(2t-\Sigma_t-\hat{\Sigma}_t)-\mu(\Sigma_t+\hat{\Sigma}_t)} = 0$. By applying (OA.1.1), we have

$$\mathbb{E}[(\tau_L \wedge \tau_R \wedge \hat{\tau}_L \wedge \hat{\tau}_R)] = \int_0^\infty e^{-\lambda_L(2t-\Sigma_t-\hat{\Sigma}_t)-\mu(\Sigma_t+\hat{\Sigma}_t)}\ dt.$$

By plugging the above equations into (OA.1.17), we obtain (OA.1.16). □

Lemma OA.1.4. *Suppose that $(x_0, y_0) \in [0,1]^2$ satisfies $x_0 \in \arg\max_{x\in[0,1]} u_0(x, y_0)$. Let $\sigma^*, \hat{\sigma}^* : \mathbb{R}_+ \to [0,1]$ be $\sigma_t^* = x_0$ and $\hat{\sigma}_t^* = y_0$ for all $t \geq 0$. Then, σ^* is a best response to $\hat{\sigma}^*$.*

Proof of Lemma OA.1.4. This can be proven by following the same steps of the proof of Lemma OA.1.2 by setting r_t denote $S^M_{\sigma,\hat{\sigma}^*}(t)$ and using Lemma OA.1.3. □

OA.2 Private Information Setting

OA.2.1 Roadmap

The formal proof of Theorem 2 is organized as follows. In Section OA.2.2, we characterize the closed form solution of the belief under the research policy and provide the proof of Proposition C.2. Next, we derive the HJB equations (Section OA.2.3), then provide the proof of Proposition C.1 (Section OA.2.4). Then, we prove Proposition C.3—the single-crossing property of the instantaenous payoff function $\mathcal{R}$—in Section OA.2.5, and we characterize the MDNE for each parametric regions in Section OA.2.6.

OA.2.2 Preliminary Results

OA.2.2.1 Closed form solution of $\mathbf{p_1}(t)$

Lemma OA.2.1. *Suppose that a firm follows an allocation policy $\boldsymbol{\sigma}$, with $\boldsymbol{\sigma}(s) = 1$ for $s \in [0, t)$. Then, the conditional probability $\mathbf{p}_{\boldsymbol{\sigma}}(t)$ of having access to the new technology by time t given that the race is ongoing is the same as that under the research policy ($\mathbf{p_1}(t)$), which is given as follows:*

$$\mathbf{p}_{\boldsymbol{\sigma}}(t) = \mathbf{p_1}(t) \equiv \frac{\frac{1}{\lambda_H}\left(e^{-\mu t} - e^{-\lambda_H t}\right)}{\frac{1}{\mu}e^{-\mu t} - \frac{1}{\lambda_H}e^{-\lambda_H t}}.\text{[20]} \tag{OA.2.1}$$

In addition, $\mathbf{p_1}(t)$ is increasing in t, with $\lim_{t\to\infty} \mathbf{p_1}(t) = \min\{1, \mu/\lambda_H\}$.

Proof of Lemma OA.2.1. Note that the conditional probability of having access to the new technology by time t only depends on the resource allocations prior to time t. Thus, since $\boldsymbol{\sigma}$ and $\mathbf{1}$ have the same resource allocation by time t, $\mathbf{p}_{\boldsymbol{\sigma}}(t)$ and $\mathbf{p_1}(t)$ are equal. By plugging $\boldsymbol{\sigma}(t) = 1$ to the result of Proposition 5.1, we have $\mathbf{p}'_{\boldsymbol{\sigma}}(t) = (\mu - \lambda_H \mathbf{p}_{\boldsymbol{\sigma}}(t))(1 - \mathbf{p}_{\boldsymbol{\sigma}}(t))$. By rearranging the differential equation, we can derive that

$$\lambda_H - \mu = \frac{d}{dt}\log\left(\frac{\lambda_H - \lambda_H \mathbf{p}_{\boldsymbol{\sigma}}(t)}{\mu - \lambda_H \mathbf{p}_{\boldsymbol{\sigma}}(t)}\right)$$

Then, from $\mathbf{p}_{\boldsymbol{\sigma}}(0) = 0$, we can derive that

$$\frac{\lambda_H(1 - \mathbf{p}_{\boldsymbol{\sigma}}(t))}{\mu - \lambda_H \mathbf{p}_{\boldsymbol{\sigma}}(t)} = \frac{\lambda_H}{\mu}e^{(\lambda_H - \mu)t}$$

By rearranging the above equation, we have (OA.2.1).

Observe that

$$\mathbf{p}'_{\mathbf{1}}(t) = \frac{\mu(\lambda_H - \mu)^2 e^{(\lambda_H + \mu)t}}{(\lambda_H e^{\lambda_H t} - \mu e^{\mu t})^2} > 0$$

Thus, $\mathbf{p_1}(t)$ is increasing in t.

When $\mu > \lambda_H$,

$$\lim_{t\to\infty} \mathbf{p_1}(t) = \lim_{t\to\infty} \frac{\frac{1}{\lambda_H}\left(e^{(\lambda_H - \mu)t} - 1\right)}{\frac{1}{\mu}e^{(\lambda_H - \mu)t} - \frac{1}{\lambda_H}} = 1.$$

[20] If $\mu = \lambda_H$, $\mathbf{p_1}(t) = \mu t/(1 + \mu t)$. All the results follow through with this case.

When $\mu < \lambda_H$,

$$\lim_{t\to\infty} \mathbf{p_1}(t) = \lim_{t\to\infty} \frac{\frac{1}{\lambda_H}\left(1 - e^{(\mu-\lambda_H)t}\right)}{\frac{1}{\mu} - \frac{1}{\lambda_H} e^{(\mu-\lambda_H)t}} = \frac{\mu}{\lambda_H}.$$

□

OA.2.2.2 Development rates

For any continuous random variable, the hazard rate can be expressed as the negative of the log of the survival function. The development rate of a firm that follows policy $\boldsymbol{\sigma} \in \mathcal{S}$ is the hazard rate associated with the random variable τ_D. Therefore, it can be derived as follows:

$$\begin{aligned}\mathbf{h}_{\boldsymbol{\sigma}}(t) = -\frac{\partial \log\left[S^D_{\boldsymbol{\sigma}}(t)\right]}{\partial t} = -\frac{S^{D\prime}_{\boldsymbol{\sigma}}(t)}{S^D_{\boldsymbol{\sigma}}(t)} =& \frac{\lambda_L(1-\boldsymbol{\sigma}(t)) \cdot S^M_{\boldsymbol{\sigma}}(t) + \lambda_H \cdot L_{\boldsymbol{\sigma}}(t)}{S^M_{\boldsymbol{\sigma}}(t) + L_{\boldsymbol{\sigma}}(t)} \\ =& \lambda_L(1-\boldsymbol{\sigma}(t)) \cdot (1 - \mathbf{p}_{\boldsymbol{\sigma}}(t)) + \lambda_H \cdot \mathbf{p}_{\boldsymbol{\sigma}}(t).\end{aligned} \tag{OA.2.2}$$

Also note that from $S^D_{\boldsymbol{\sigma}}(0) = 1$, $S^D_{\boldsymbol{\sigma}}(t)$ can be rewritten as follows:

$$S^D_{\boldsymbol{\sigma}}(t) = e^{-\int_0^t \mathbf{h}_{\boldsymbol{\sigma}}(s)ds}. \tag{OA.2.3}$$

OA.2.2.3 Proof of Proposition C.2

Proof of Proposition C.2. Since $\boldsymbol{\sigma}(s) = 0$ and $\boldsymbol{\sigma}$ is right-continuous, it must be that $\mathbf{h}_{\boldsymbol{\sigma}}(\tilde{s}) = \lambda_L \cdot (1 - \mathbf{p}_{\boldsymbol{\sigma}}(\tilde{s})) + \lambda_H \cdot \mathbf{p}_{\boldsymbol{\sigma}}(\tilde{s})$ for $\tilde{s}$ slightly above s. This means that

$$0 \leq \mathbf{h}'_{\boldsymbol{\sigma}}(s) = (\lambda_H - \lambda_L) \cdot \dot{\mathbf{p}}_{\boldsymbol{\sigma}}(s) = -(\lambda_H - \lambda_L)^2 \cdot \mathbf{p}_{\boldsymbol{\sigma}}(s)(1 - \mathbf{p}_{\boldsymbol{\sigma}}(s))$$

where the inequality holds since $\mathbf{h}_{\boldsymbol{\sigma}}$ is weakly increasing. Since $\mathbf{p}_{\boldsymbol{\sigma}}(s) < 1$, it must be the case that $\mathbf{p}_{\boldsymbol{\sigma}}(s) = 0$. This holds only if $\boldsymbol{\sigma}(t) = 0$ for all $t < s$. □

OA.2.3 Recursive Formulation

The opponent's allocation policy is only payoff-relevant for a firm through the distribution of development times. Thus, in this section, we focus on characterizing the continuation payoffs of firms fixing the development rate function $\mathbf{h}$ of the opponent.

Lemma OA.2.2. *Let $V_1(t;\mathbf{h})$ be the continuation payoff of a firm at time t when the firm has the new technology, neither firm had succeeded in development by time t, and the opponent employs an allocation policy with development rate $\mathbf{h}$. Then, $V_1(t;\mathbf{h})$ takes a form of (C.12). In addition, the following differential equation holds:*

$$0 = V_1'(t;\mathbf{h}) + (\lambda_H \Pi - c) - (\lambda_H + \mathbf{h}(t)) \cdot V_1(t;\mathbf{h}). \tag{HJB$_1$}$$

Proof of Lemma OA.2.2. Let $\hat{\tau}_D$ be the arrival time of the product development by the opponent whose development rate is $\mathbf{h}$. Note that the continuation payoffs can be written as follows.

$$\begin{aligned} V_1(t;\mathbf{h}) = & \Pr[\tau_D < \hat{\tau}_D \mid \tau_M = t < (\tau_D \wedge \hat{\tau}_D)] \cdot \Pi \\ & - c \cdot \mathbb{E}[\tau_D \wedge \hat{\tau}_D - t \mid \tau_M = t < (\tau_D \wedge \hat{\tau}_D)]. \end{aligned} \tag{OA.2.4}$$

Note that (conditional) survival functions of $\hat{\tau}_D$ and τ_D can be written as follows:

$$\begin{aligned} \Pr[\hat{\tau}_D > s \mid \tau_M = t < (\tau_D \wedge \hat{\tau}_D)] = & e^{-\int_t^s \mathbf{h}(u)du}, \\ \Pr[\tau_D = \tau_H > s \mid \tau_R = t < (\tau_L \wedge \hat{\tau}_D)] = & e^{-\lambda_H (s-t)}. \end{aligned}$$

By applying (OA.1.4) and (OA.1.5), we have

$$\begin{aligned} \Pr[\tau_D < \hat{\tau}_D \mid \tau_R = t < (\tau_L \wedge \hat{\tau}_D)] &= \int_t^\infty \lambda_H e^{-\int_t^s (\lambda_H + \mathbf{h}(u))du} ds, \\ \mathbb{E}[\tau_D \wedge \hat{\tau}_D - t \mid \tau_R = t < (\tau_L \wedge \hat{\tau}_D)] &= \int_t^\infty e^{-\int_t^s (\lambda_H + \mathbf{h}(u))du} ds. \end{aligned}$$

By plugging these equations into (OA.2.4), we can derive that (C.12) holds.

By taking a derivative of (OA.2.4), we have

$$\begin{aligned} V_1'(t;\mathbf{h}) = & -(\lambda_H \Pi - c) \cdot e^{-\int_t^t (\lambda_H + \mathbf{h}(u))du} + (\lambda_H + \mathbf{h}(t)) \cdot (\lambda_H \Pi - c) \cdot \int_t^\infty e^{-\int_t^s (\lambda_H + \mathbf{h}(u))du} ds \\ = & -(\lambda_H \Pi - c) + (\lambda_H + \mathbf{h}(t)) \cdot V_1(t;\hat{\sigma}), \end{aligned}$$

which is equivalent to (HJB$_1$). $\square$

Lemma OA.2.3. *Let v_0 be the continuation payoff at time t of a firm that does not have the new technology and employs allocation policy $\boldsymbol{\sigma} \in \mathcal{S}$ when the opponent has a development rate $\mathbf{h} \in \mathcal{H}$. Then, v_0 takes a form of* (C.13)*. In addition, the following differential equation holds:*

$$0 = v_0'(t; \boldsymbol{\sigma}, \mathbf{h}) + \lambda_L(1 - \boldsymbol{\sigma}(t)) \cdot \Pi + \mu\boldsymbol{\sigma}(t) \cdot V_1(t; \mathbf{h}) - c \\ - \{\lambda_L(1 - \boldsymbol{\sigma}(t)) + \mu\boldsymbol{\sigma}(t) + \mathbf{h}(t)\} \cdot v_0(t; \sigma, \hat{\sigma}). \tag{HJB$_0$}$$

Proof of Lemma OA.2.3. We focus on the event such that $(\tau_M \wedge \hat{\tau}_D) > t$. The continuation payoff can be written as follows:

$$v_0(t; \boldsymbol{\sigma}, \mathbf{h}) = \Pr[\tau_D < \hat{\tau}_D \mid (\tau_M \wedge \hat{\tau}_D) > t] \cdot \Pi - c \cdot \mathbb{E}[\tau_D \wedge \hat{\tau}_D - t \mid (\tau_M \wedge \hat{\tau}_D) > t]. \tag{OA.2.5}$$

Note that

$$\Pr[\tau_M > s \mid \tau_M > t] = \frac{S^M_{\boldsymbol{\sigma}}(s)}{S^M_{\boldsymbol{\sigma}}(t)},$$
$$\Pr[\tau_D > s > \tau_M > t \mid \tau_M > t] = \int_t^s e^{-\lambda_H(s-u)} \cdot \mu\,\boldsymbol{\sigma}(u) \cdot \frac{S^M_{\boldsymbol{\sigma}}(u)}{S^M_{\boldsymbol{\sigma}}(t)} du = \frac{L_{\boldsymbol{\sigma}}(s|t)}{S^M_{\boldsymbol{\sigma}}(t)},$$

where $L_{\boldsymbol{\sigma}}(s|t) \equiv \int_t^s e^{-\lambda_H(s-u)} \cdot \mu\,\boldsymbol{\sigma}(u) \cdot S^M_{\boldsymbol{\sigma}}(u) du$. Then, the survival function of τ_D conditional on $\tau_M > t$ can be written as follows:

$$S^D_{\boldsymbol{\sigma}|_t}(s) \equiv \Pr[\tau_D > s \mid \tau_M > t] = \frac{S^M_{\boldsymbol{\sigma}}(s) + L_{\boldsymbol{\sigma}}(s|t)}{S^M_{\boldsymbol{\sigma}}(t)}$$

Also note that $\Pr[\hat{\tau}_D > s \mid \hat{\tau}_D > t] = e^{-\int_t^s \mathbf{h}(u) du}$.

Observe that

$$L'_{\boldsymbol{\sigma}}(s|t) = \mu\,\boldsymbol{\sigma}(s) \cdot S^M_{\boldsymbol{\sigma}}(s) - \lambda_H \cdot L_{\boldsymbol{\sigma}}(s|t). \tag{OA.2.6}$$

Since τ_D and $\hat{\tau}_D$ are independent, we can apply (OA.1.3) and (OA.1.5) by resetting the initial time to t. Then, by using (C.6) and (OA.2.6), we have

$$\begin{aligned}\Pr[\tau_D < \hat{\tau}_D \mid (\tau_M \wedge \hat{\tau}_D) > t] &= -\int_t^\infty {S^D_{\boldsymbol{\sigma}|_t}}'(s) \cdot e^{-\int_t^s \mathbf{h}(u)du} ds \\ &= \int_t^\infty \frac{\lambda_L(1-\boldsymbol{\sigma}(s)) \cdot S^M_{\boldsymbol{\sigma}}(s) + \lambda_H \cdot L_{\boldsymbol{\sigma}}(s|t)}{S^M_{\boldsymbol{\sigma}}(t)} \cdot e^{-\int_t^s \mathbf{h}(u)du} ds,\end{aligned}$$

$$\mathbb{E}[\tau_D \wedge \hat{\tau}_D - t \mid (\tau_M \wedge \hat{\tau}_D) > t] = \int_t^\infty \frac{S^M_{\boldsymbol{\sigma}}(s) + L_{\boldsymbol{\sigma}}(s|t)}{S^M_{\boldsymbol{\sigma}}(t)} \cdot e^{-\int_t^s \mathbf{h}(u)du} ds.$$

By plugging these into (OA.2.5) and using (OA.2.2), we can derive that

$$v_0(t;\boldsymbol{\sigma},\mathbf{h}) = \int_t^\infty \left[\{\lambda_L(1-\boldsymbol{\sigma}(s))\Pi - c\} \cdot S^M_{\boldsymbol{\sigma}}(s) + (\lambda_H\Pi - c) \cdot L_{\boldsymbol{\sigma}}(s|t)\right] \cdot \frac{e^{-\int_t^s \mathbf{h}(u)du}}{S^M_{\boldsymbol{\sigma}}(t)} ds.$$

Thus, it remains to show that

$$\int_t^\infty \mu\,\boldsymbol{\sigma}(s) \cdot V_1(s;\mathbf{h}) \cdot S^M_{\boldsymbol{\sigma}}(s) \cdot e^{-\int_t^s \mathbf{h}(u)du}\, ds = (\lambda_H\Pi - c) \cdot \int_0^\infty L_{\boldsymbol{\sigma}}(s|t) \cdot e^{-\int_t^s \mathbf{h}(u)du}\, ds. \quad \text{(OA.2.7)}$$

By plugging (C.12) into the left hand side of (OA.2.7), we have

$$\begin{aligned}&\int_t^\infty \mu\,\boldsymbol{\sigma}(s) \cdot (\lambda_H\Pi - c) \cdot \left[\int_s^\infty e^{-\int_s^u (\lambda_H + \mathbf{h}(v))dv} du\right] \cdot S^M_{\boldsymbol{\sigma}}(s) \cdot e^{-\int_t^s \mathbf{h}(v)dv}\, ds \\ =&(\lambda_H\Pi - c) \cdot \int_t^\infty L_{\boldsymbol{\sigma}}(u|t) \cdot e^{-\int_t^u \mathbf{h}(v)dv}\, du.\end{aligned}$$

Thus, (C.13) holds.

Last, to show that (HJB$_0$) holds, we multiply $S^M_{\boldsymbol{\sigma}}(t) \cdot e^{-\int_0^t \mathbf{h}(u)du}$ to (C.13) and take a derivative:

$$\begin{aligned}&-[\lambda_L(1-\boldsymbol{\sigma}(t)) \cdot \Pi + \mu\,\boldsymbol{\sigma}(t) \cdot V_1(t;\mathbf{h}) - c] \cdot S^M_{\boldsymbol{\sigma}}(t) \cdot e^{-\int_0^t \mathbf{h}(u)du} \\ =&\left[v_0'(t;\sigma,\mathbf{h}) - \left(-\frac{{S^M_{\boldsymbol{\sigma}}}'(t)}{S^M_{\boldsymbol{\sigma}}(t)} + \mathbf{h}(t)\right) \cdot v_0(t;\boldsymbol{\sigma},\mathbf{h})\right] \cdot S^M_{\boldsymbol{\sigma}}(t) \cdot e^{-\int_0^t \mathbf{h}(u)du}.\end{aligned}$$

By using (C.6) and $S^M_\sigma(t) \cdot e^{-\int_0^t \mathbf{h}(u)du} > 0$, we can see that (HJB$_0$) holds. □

Corollary 2. *Let $\mathbf{h}, \hat{\mathbf{h}} \in \mathcal{H}$ be two development functions such that $\mathbf{h}(s) = \hat{\mathbf{h}}(s)$ for all $s > t$. Then $V_0(t;\mathbf{h}) = V_0(t;\hat{\mathbf{h}})$ and $V_1(t;\mathbf{h}) = V_1(t;\hat{\mathbf{h}})$.*

OA.2.4 Verification

In this subsection, we prove the verification result (Proposition C.1). To prove the verification result, it is useful to first introduce two convergence results.

Lemma OA.2.4. *For any $\boldsymbol{\sigma} \in \mathcal{S}$, the following holds:*

$$\lim_{t\to\infty} V_1(t; \mathbf{h}_{\boldsymbol{\sigma}}) \cdot S^D_{\boldsymbol{\sigma}}(t) = 0.$$

Proof. Recall that $\Sigma_t := \int_0^t \boldsymbol{\sigma}(s)\, ds$. From $\lambda_H > \lambda_L$ and $\mu > \lambda_L$, we have

$$e^{-\mu t} \leq S^M_{\boldsymbol{\sigma}}(t) = e^{-\lambda_L(t-\Sigma_t)-\mu\Sigma_t} \leq e^{-\lambda_L t}, \tag{OA.2.8}$$

$$\begin{aligned} 0 \leq L_{\boldsymbol{\sigma}}(t) =& \int_0^t \mu\, \boldsymbol{\sigma}(s) \cdot S^M_{\boldsymbol{\sigma}}(s) \cdot e^{-\lambda_H(t-s)}\, ds \\ <& e^{-(\lambda_L+\lambda_H)t} \cdot \int_0^t \mu \cdot e^{\lambda_H s}\, ds < \frac{\mu}{\lambda_H} e^{-\lambda_L t}. \end{aligned} \tag{OA.2.9}$$

Note that the left inequality of (OA.2.8) binds when $\Sigma_t = t$, and the left inequality of (OA.2.9) binds when $\Sigma_t = 0$. By (C.3), we have

$$e^{-\mu t} < S^D_{\boldsymbol{\sigma}}(t) = S^M_{\boldsymbol{\sigma}}(t) + L_{\boldsymbol{\sigma}}(t) < e^{-\lambda_L t} \cdot \left(\frac{\mu + \lambda_H}{\lambda_H}\right). \tag{OA.2.10}$$

From (OA.2.3) and (C.12), we have

$$S^D_{\boldsymbol{\sigma}}(t) \cdot V_1(t; \mathbf{h}_{\boldsymbol{\sigma}}) = (\lambda_H \Pi - c) \cdot \int_t^\infty e^{-\lambda_H(s-t)} \cdot S^D_{\boldsymbol{\sigma}}(s)\, ds.$$

By applying (OA.2.10) and since $\lambda_H \Pi > \lambda_L \Pi > c$, we have

$$\begin{aligned} (\lambda_H \Pi - c) \cdot \int_t^\infty e^{-\lambda_H(s-t)} \cdot S^D_{\boldsymbol{\sigma}}(s)\, ds \;>\; & (\lambda_H \Pi - c) \cdot \int_t^\infty e^{-\lambda_H(s-t)} \cdot e^{-\mu s}\, ds \\ =& \frac{\lambda_H}{\mu + \lambda_H}\left(\Pi - \frac{c}{\lambda_H}\right) \cdot e^{-\mu t} \end{aligned}$$

and

$$(\lambda_H \Pi - c) \cdot \int_t^\infty e^{-\lambda_H(s-t)} \cdot S^D_{\boldsymbol{\sigma}}(s)\ ds \ < \ (\lambda_H \Pi - c) \cdot \int_t^\infty e^{-\lambda_H(s-t)} \cdot \frac{\mu+\lambda_H}{\lambda_H} e^{-\lambda_L s}\ ds$$
$$= \frac{\mu+\lambda_H}{\lambda_L+\lambda_H}\left(\Pi - \frac{c}{\lambda_H}\right) \cdot e^{-\lambda_L t}.$$

Therefore, we have that

$$\frac{\lambda_H}{\mu+\lambda_H}\left(\Pi - \frac{c}{\lambda_H}\right) \cdot e^{-\mu t} < S^D_{\boldsymbol{\sigma}}(t) \cdot V_1(t;\mathbf{h}_{\boldsymbol{\sigma}}) < \frac{\mu+\lambda_H}{\lambda_L+\lambda_H}\left(\Pi - \frac{c}{\lambda_H}\right) \cdot e^{-\lambda_L t}. \qquad \text{(OA.2.11)}$$

Since the lower bound and the upper bound converge to 0 as t goes to infinity, we obtain the desired result. □

Lemma OA.2.5. *For any $\boldsymbol{\sigma}, \hat{\boldsymbol{\sigma}} \in \mathcal{S}$,*

$$\lim_{t\to\infty}\ v_0(t;\boldsymbol{\sigma},\mathbf{h}_{\hat{\boldsymbol{\sigma}}}) \cdot S^M_{\boldsymbol{\sigma}}(t) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(t) = 0. \qquad \text{(OA.2.12)}$$

Proof. Note that for any time $s \in \mathbb{R}_+$, $-c < \lambda_L(1-\boldsymbol{\sigma}(s))\Pi - c < \lambda_L \Pi$. Since $\lambda_L\Pi > c$, we have $|\lambda_L(1-\boldsymbol{\sigma}(s))\Pi - c| < \lambda_L \Pi$.

From (C.13), we have

$$\left|v_0(t;\boldsymbol{\sigma},\mathbf{h}_{\hat{\boldsymbol{\sigma}}}) \cdot S^M_{\boldsymbol{\sigma}}(t) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(t)\right| < \ \lambda_L \Pi \cdot \int_t^\infty S^M_{\boldsymbol{\sigma}}(s) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(s)\ ds$$
$$+ \mu \cdot \int_t^\infty V_1(s;\mathbf{h}_{\hat{\boldsymbol{\sigma}}}) \cdot S^M_{\boldsymbol{\sigma}}(s) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(s)\ ds.$$

Observe that from (OA.2.8) and (OA.2.10) in Lemma OA.2.4, we have

$$\int_t^\infty S^M_{\boldsymbol{\sigma}}(s) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(s)\ ds < \frac{\mu+\lambda_H}{\lambda_H} \cdot \int_t^\infty e^{-2\lambda_L s} ds = \frac{\mu+\lambda_H}{2\lambda_L\lambda_H} \cdot e^{-2\lambda_L t}.$$

In addition, from (OA.2.11) and (OA.2.10) in Lemma OA.2.4, we have

$$\int_t^\infty V_1(s;\mathbf{h}_{\hat{\boldsymbol{\sigma}}}) \cdot S^M_{\boldsymbol{\sigma}}(s) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(s)\ ds < \frac{(\mu+\lambda_H)^2}{\lambda_H(\lambda_L+\lambda_H)} \cdot \left(\Pi - \frac{c}{\lambda_H}\right) \cdot \int_t^\infty e^{-2\lambda_L s} ds$$
$$= \frac{(\mu+\lambda_H)^2}{2\lambda_L\lambda_H(\lambda_L+\lambda_H)} \cdot \left(\Pi - \frac{c}{\lambda_H}\right) \cdot e^{-2\lambda_L t}.$$

Then, we have

$$\left|v_0(t;\boldsymbol{\sigma},\mathbf{h}_{\hat{\sigma}})\cdot S^M_{\boldsymbol{\sigma}}(t)\cdot S^D_{\hat{\boldsymbol{\sigma}}}(t)\right| < \frac{\mu+\lambda_H}{2\lambda_L\lambda_H}\left[\lambda_L\Pi+\frac{\mu(\mu+\lambda_H)}{\lambda_L+\lambda_H}\left(\Pi-\frac{c}{\lambda_H}\right)\right]\cdot e^{-2\lambda_L t}.$$

Since the right-hand side of the above inequality converges to 0 as $t\to\infty$, (OA.2.12) holds. $\square$

OA.2.4.1 Proof of Proposition C.1

In this proof, we fix the policy of the opponent at $\hat{\boldsymbol{\sigma}}$. To save on notation, we will drop the dependency of the value and survival functions on $\hat{\boldsymbol{\sigma}}$ and the opponent's development rate $\mathbf{h}_{\hat{\boldsymbol{\sigma}}}$. Specifically, we will abuse notation and use $V_1(t)\equiv V_1(t;\mathbf{h}_{\hat{\boldsymbol{\sigma}}})$, $v_0(t;\boldsymbol{\sigma})\equiv v_0(t;\boldsymbol{\sigma},\mathbf{h}_{\hat{\boldsymbol{\sigma}}})$, $\hat{S}(t)\equiv S^D_{\hat{\boldsymbol{\sigma}}}(t)$.

Proof of Proposition C.1 ($\Longleftarrow$). From $\boldsymbol{\sigma}^*$, we have that for all $\boldsymbol{\sigma}\in\mathcal{S}$ and $t\in R_+$

$$(\boldsymbol{\sigma}^*(t)-\boldsymbol{\sigma}(t))\cdot[\mu\cdot(V_1(t)-v_0(t;\boldsymbol{\sigma}^*))-\lambda_L\cdot(\Pi-v_0(t;\boldsymbol{\sigma}^*))]\geq 0 \tag{OA.2.13}$$

Suppose that $v_0(t;\boldsymbol{\sigma}^*)>0$. From (HJB$_0$), we have

$$\begin{aligned}0=&v_0'(t;\boldsymbol{\sigma}^*)-c-\mathbf{h}_{\hat{\boldsymbol{\sigma}}}(t)\cdot v_0(t;\boldsymbol{\sigma}^*)+\lambda_L\cdot(\Pi-v_0(t;\boldsymbol{\sigma}^*))\\&+\boldsymbol{\sigma}^*(t)\cdot[\mu\cdot(V_1(t)-v_0(t;\boldsymbol{\sigma}^*))-\lambda_L\cdot(\Pi-v_0(t;\boldsymbol{\sigma}^*))].\end{aligned}$$

Then, (OA.2.13) implies that, for any $\boldsymbol{\sigma}\in\mathcal{S}$ and $t\geq 0$,

$$\left\{h^D_{\hat{\boldsymbol{\sigma}}}(t)+h^M_{\boldsymbol{\sigma}}(t)\right\}\cdot v_0(t;\boldsymbol{\sigma}^*)-v_0'(t;\boldsymbol{\sigma}^*)\geq\lambda_L(1-\boldsymbol{\sigma}(t))\cdot\Pi+\mu\,\boldsymbol{\sigma}(t)\cdot V_1(t)-c.$$

Multiplying side-by-side by $S^M_{\boldsymbol{\sigma}}(t)\cdot S^D_{\hat{\boldsymbol{\sigma}}}(t)$, we have

$$-\frac{d}{dt}\left[v_0(t;\boldsymbol{\sigma}^*)\cdot S^M_{\boldsymbol{\sigma}}(t)\cdot S^D_{\hat{\boldsymbol{\sigma}}}(t)\right]\geq[\lambda_L(1-\boldsymbol{\sigma}(t))\cdot\Pi+\mu\,\boldsymbol{\sigma}(t)\cdot V_1(t)-c]\cdot S^M_{\boldsymbol{\sigma}}(t)\cdot S^D_{\hat{\boldsymbol{\sigma}}}(t)$$

for all $t \geq 0$. Integrating this inequality from 0 to ∞ and using Lemma OA.2.3, we have

$$\begin{aligned} & v_0(0;\boldsymbol{\sigma}^*) \cdot S^M_{\boldsymbol{\sigma}}(0) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(0) - \lim_{t\to\infty} v_0(t;\boldsymbol{\sigma}^*) \cdot S^M_{\boldsymbol{\sigma}}(t) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(t) \\ \geq & \int_0^\infty [\lambda_L(1-\boldsymbol{\sigma}(t)) \cdot \Pi + \mu\,\boldsymbol{\sigma}(t) \cdot V_1(t) - c] \cdot S^M_{\boldsymbol{\sigma}}(t) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(t) dt = \mathcal{U}(\boldsymbol{\sigma}, \hat{\boldsymbol{\sigma}}). \end{aligned}$$

Since $v_0(t;\boldsymbol{\sigma}^*)$, $S^M_{\boldsymbol{\sigma}}(t)$ and $S^D_{\hat{\boldsymbol{\sigma}}}(t)$ are strictly positive, we have

$$\lim_{t\to\infty} v_0(t;\boldsymbol{\sigma}^*) \cdot S^M_{\boldsymbol{\sigma}}(t) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(t) \geq 0.$$

By using this, $\mathcal{U}(\boldsymbol{\sigma}^*, \hat{\boldsymbol{\sigma}}) = v_0(0;\boldsymbol{\sigma}^*)$, and $S^M_{\boldsymbol{\sigma}}(0) = S^D_{\hat{\boldsymbol{\sigma}}}(0) = 1$, we obtain $\mathcal{U}(\boldsymbol{\sigma}^*, \hat{\boldsymbol{\sigma}}) \geq \mathcal{U}(\boldsymbol{\sigma}, \hat{\boldsymbol{\sigma}})$. $\square$

Proof of Proposition C.1 ($\Longrightarrow$). Suppose that $\boldsymbol{\sigma}^* \in \arg\max_{\boldsymbol{\sigma}\in\mathcal{S}} \mathcal{U}(\boldsymbol{\sigma}, \hat{\boldsymbol{\sigma}})$. From Lemma OA.2.3, observe that for any $t \geq 0$, a firm's expected payoff can be rewritten as follows:

$$\begin{aligned} \mathcal{U}(\boldsymbol{\sigma}, \hat{\boldsymbol{\sigma}}) = & \int_0^t [\lambda_L(1-\boldsymbol{\sigma}(s)) \cdot \Pi + \mu\,\boldsymbol{\sigma}(s) \cdot V_1(s) - c] \cdot S^M_{\boldsymbol{\sigma}}(s) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(s)\; ds \\ & + S^M_{\boldsymbol{\sigma}}(t) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(t) \cdot v_0(t;\boldsymbol{\sigma}). \end{aligned}$$

Now consider the following allocation policy $\tilde{\boldsymbol{\sigma}}(s) := \boldsymbol{\sigma}^*(s)1_{s<t}$. Then, $S^M_{\boldsymbol{\sigma}^*}(s) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(s) = S^M_{\tilde{\boldsymbol{\sigma}}}(s) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(s)$ for all $s \leq t$.[21] In addition, by using $\sigma^*(s) = \tilde{\boldsymbol{\sigma}}(s)$ for all $s < t$ and $\mathcal{U}(\boldsymbol{\sigma}^*, \hat{\boldsymbol{\sigma}}) \geq \mathcal{U}(\tilde{\boldsymbol{\sigma}}, \hat{\boldsymbol{\sigma}})$, we have $v_0(t;\boldsymbol{\sigma}^*) \geq v_0(t;\tilde{\boldsymbol{\sigma}})$.

Note that

$$v_0(t;\tilde{\boldsymbol{\sigma}}) = \int_t^\infty (\lambda_L\Pi - c) \cdot \frac{S^M_{\tilde{\boldsymbol{\sigma}}}(s)}{S^M_{\tilde{\boldsymbol{\sigma}}}(t)} \cdot \frac{S^D_{\hat{\boldsymbol{\sigma}}}(s)}{S^D_{\hat{\boldsymbol{\sigma}}}(t)}\; ds > 0$$

from $\lambda_L\Pi > c$, $S^M_{\tilde{\boldsymbol{\sigma}}}(s) > 0$, and $S^D_{\hat{\boldsymbol{\sigma}}}(s) > 0$. Therefore, $v_0(t;\boldsymbol{\sigma}^*) > 0$ for all $t \geq 0$.

Now assume that there exists $\boldsymbol{\sigma} \in \mathcal{S}$ such that (OA.2.13) does not hold for some $t \geq 0$. Observe that $V_1(\cdot;\mathbf{h})$ and $v_0(\cdot;\boldsymbol{\sigma},\mathbf{h})$ are continuous. Since $\boldsymbol{\sigma}^*$ and $\boldsymbol{\sigma}$ are right-continuous, there exists $\epsilon > 0$ such that for all $s \in [t, t+\epsilon)$,

$$(\boldsymbol{\sigma}^*(s) - \boldsymbol{\sigma}(s)) \cdot [\mu \cdot (V_1(s) - v_0(s;\boldsymbol{\sigma}^*) - \lambda_L \cdot (\Pi - v_0(s;\boldsymbol{\sigma}^*))] < 0. \tag{OA.2.14}$$

[21] Note that the equality also holds at $s = t$, since $\boldsymbol{\sigma}^*$ and $\tilde{\boldsymbol{\sigma}}$ differ only at $\{t\}$, which is negligible after integration.

Consider the following allocation policy $\boldsymbol{\sigma}^{**}$ defined by:

$$\boldsymbol{\sigma}^{**}(s) := \begin{cases} \boldsymbol{\sigma}^{*}(s), & \text{if } s \notin [t, t+\epsilon), \\ \boldsymbol{\sigma}(s), & \text{if } s \in [t, t+\epsilon). \end{cases}$$

By using a similar reformulation as in the previous case, we have

$$\begin{aligned} & -\frac{d}{ds}\left[v_0(s;\boldsymbol{\sigma}^*)\cdot S^M_{\boldsymbol{\sigma}^{**}}(s)\cdot S^D_{\hat{\boldsymbol{\sigma}}}(s)\right] \\ & \leq \left[\lambda_L(1-\boldsymbol{\sigma}^{**}(s))\cdot\Pi+\mu\sigma^{**}(s)\cdot V_1(s)-c\right]\cdot S^M_{\boldsymbol{\sigma}^{**}}(s)\cdot S^D_{\hat{\boldsymbol{\sigma}}}(s) \end{aligned} \tag{OA.2.15}$$

for all $s \geq 0$, and the inequality strictly holds for $s \in [t, t+\epsilon)$. Also note that by Lemma OA.2.5,

$$\lim_{s\to\infty} v_0(s;\boldsymbol{\sigma}^*)\cdot S^M_{\boldsymbol{\sigma}^{**}}(s)\cdot S^D_{\hat{\boldsymbol{\sigma}}}(s) = \lim_{s\to\infty} v_0(s;\boldsymbol{\sigma}^*)\cdot S^M_{\boldsymbol{\sigma}^{*}}(s)\cdot S^D_{\hat{\boldsymbol{\sigma}}}(s) = 0.$$

By integrating (OA.2.15) from 0 to ∞, we have

$$\begin{aligned} \mathcal{U}(\boldsymbol{\sigma}^*,\hat{\boldsymbol{\sigma}}) &= v_0(0;\boldsymbol{\sigma}^*) \\ &< \int_0^\infty \left[\lambda_L(1-\boldsymbol{\sigma}^{**}(s))\cdot\Pi+\mu\,\boldsymbol{\sigma}^{**}(s)\cdot V_1(s)-c\right]\cdot S^M_{\boldsymbol{\sigma}^{**}}(s)\cdot S^D_{\hat{\boldsymbol{\sigma}}}(s)\,ds \\ &= \mathcal{U}(\boldsymbol{\sigma}^{**},\hat{\boldsymbol{\sigma}}), \end{aligned}$$

which contradicts $\boldsymbol{\sigma}^* \in \arg\max_{\boldsymbol{\sigma}\in\mathcal{S}}\mathcal{U}(\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}})$. Therefore, (OA.2.13) holds for all $t \geq 0$. □

OA.2.5 The Single-Crossing Property

OA.2.5.1 Monotonicity of V_1 and V_0

We start by considering an opponent with a constant development rate. The following lemma characterizes the best response in this case, in line with the best responses described for the case of public information in equations (B.3) and (B.4).

Lemma OA.2.6. *For any constant development rate $\lambda \in \mathbb{R}_+$, $V_1(t;\lambda)$, $V_0(t;\lambda)$, and $R(t;\lambda)$ are constant over time. Moreover, $sgn(R(t;\lambda)) = sgn(\lambda_\star - \lambda)$.*

Proof. Since the allocation problem of a firm when the opponent develops at a constant rate

λ is memoryless, there must be a constant research rate $\sigma^* \in [0,1]$ that is optimal. Then,

$$V_1(t;\lambda) = \frac{\lambda_H \Pi - c}{\lambda_H + \lambda} \qquad \text{and} \qquad V_0(t;\lambda) = \frac{(1-\sigma^*)\lambda_L \Pi + \sigma^* \mu V_1(t;\lambda) - c}{(1-\sigma^*)\lambda_L + \sigma^*\mu + \lambda}.$$

Observe that these two value functions are constant in t. Thus, using these expressions, we obtain:

$$R(t;\lambda) = \mu(V_1(0;\lambda) - V_0(0;\lambda)) - \lambda_L(\Pi - V_0(0;\lambda)) = \frac{(\lambda\Pi + c)\lambda_L(\lambda_\star - \lambda)}{(\lambda+\lambda_H)(\lambda + (1-\sigma^*)\lambda_L + \sigma^*\mu)},$$

which is also constant in t and shares the sign of $(\lambda_\star - \lambda)$. $\square$

We now consider an opponent with a weakly increasing development rate.

Lemma OA.2.7. *Let $\mathbf{h} \in \mathcal{H}$ be weakly increasing. Then, $V_1(t;\mathbf{h})$ and $V_0(t,\mathbf{h})$ are weakly decreasing in t.*

Proof. Note that

$$V_1(t;\mathbf{h}) = (\lambda_H\Pi - c) \cdot \int_t^\infty e^{-\int_t^s (\mathbf{h}(u)+\lambda_H)\ du}\ ds \le (\lambda_H\Pi - c) \cdot \int_t^\infty e^{-(\mathbf{h}(t)+\lambda_H)(s-t)} ds = \frac{\lambda_H \Pi - c}{\mathbf{h}(t) + \lambda_H}.$$

From (HJB_1), we have

$$V_1'(t;\mathbf{h}) = -(\lambda_H\Pi - c) + (\lambda_H + \mathbf{h}(t)) \cdot V_1(t;\mathbf{h}) \le 0. \tag{OA.2.16}$$

Therefore, $V_1(t;\mathbf{h})$ is decreasing in t.

Next, let $\boldsymbol{\sigma}^*$ be a policy satisfying $V_0(t;\mathbf{h}) = v_0(t;\boldsymbol{\sigma}^*,\mathbf{h})$. Note that for all $s \ge t$, $V_0(s;\mathbf{h}) \ge v_0(s;\mathbf{0},\mathbf{h}) > 0$ from $\Pi > c/\lambda_L$. Additionally, from (HJB_0),

$$\begin{aligned}
&(1-\boldsymbol{\sigma}^*(s))\lambda_L(\Pi - V_0(s;\mathbf{h})) + \boldsymbol{\sigma}^*(s)\mu(V_1(s;\mathbf{h}) - V_0(s;\mathbf{h})) \ge \lambda_L(\Pi - V_0(s;\mathbf{h})) \\
\Rightarrow \quad & \boldsymbol{\sigma}^*(s)\cdot\mu\cdot V_1(s;\mathbf{h}) + (1-\boldsymbol{\sigma}^*(s))\cdot\lambda_L\Pi - c \\
& \ge \boldsymbol{\sigma}^*(s)\cdot\mu\cdot V_0(s;\mathbf{h}) + (1-\boldsymbol{\sigma}^*(s))\cdot\lambda_L\cdot V_0(s;\mathbf{h}) + \lambda_L(\Pi - V_0(s;\mathbf{h})) - c \\
& = (\lambda_L\Pi - c) + \boldsymbol{\sigma}^*(s)\cdot(\mu - \lambda_L)\cdot V_0(s;\mathbf{h}) \ge 0.
\end{aligned}$$

Then, we have

$$\begin{aligned} v_0(t;\sigma^*,\mathbf{h}) &= \int_t^\infty \{\boldsymbol{\sigma}^*(s)\cdot\mu\cdot V_1(s;\mathbf{h}) + (1-\boldsymbol{\sigma}^*(s))\cdot\lambda_L\Pi - c\}\cdot \mathbf{r}_{\mathbf{h},\boldsymbol{\sigma}^*}(s;t)ds \\ &\le \int_t^\infty \{\boldsymbol{\sigma}^*(s)\cdot\mu\cdot V_1(t;\mathbf{h}) + (1-\boldsymbol{\sigma}^*(s))\cdot\lambda_L\Pi - c\}\cdot \mathbf{r}_{\mathbf{h},\boldsymbol{\sigma}^*}(s;t)ds \\ &\le \max_{\sigma\in[0,1]} \frac{\sigma\cdot\mu\cdot V_1(t;\mathbf{h}) + (1-\sigma)\cdot\lambda_L\Pi - c}{(1-\sigma)\lambda_L + \sigma\mu + \mathbf{h}(t)} \end{aligned}$$

Let the solution of the maximization problem of the right hand side is $\hat{\sigma}$. Then, we have

$$\begin{aligned} 0 \ge& c + \mathbf{h}(t)\cdot V_0(t;\mathbf{h}) - \{(1-\hat{\sigma})\lambda_L(\Pi - V_0(t;\mathbf{h})) + \hat{\sigma}\mu(V_1(t;\mathbf{h}) - V_0(t;\mathbf{h}))\} \\ \ge& c + \mathbf{h}(t)\cdot V_0(t;\mathbf{h}) - \max_{\sigma\in[0,1]} \{(1-\sigma)\lambda_L(\Pi - V_0(t;\mathbf{h})) + \sigma\mu(V_1(t;\mathbf{h}) - V_0(t;\mathbf{h}))\} = V_0'(t;\mathbf{h}). \end{aligned}$$

Therefore, $V_0(t;\mathbf{h})$ is decreasing in t. □

OA.2.5.2 Proof of Proposition C.3

Proof of Proposition C.3. It is sufficient to show that $R(t;\mathbf{h}) \le 0$ implies $\frac{\partial R}{\partial t}(t;\mathbf{h}) < 0$. Note that

$$\frac{\partial R}{\partial t}(t;\mathbf{h}) = \mu\cdot(V_1'(t;\mathbf{h}) - V_0'(t;\mathbf{h})) + \lambda_L\cdot V_0'(t;\mathbf{h}).$$

By Lemma OA.2.7, we have $V_0'(t;\mathbf{h}) \le 0$.

By subtracting (HJB_1) and (HJB_0), we have

$$V_1'(t;\mathbf{h}) - V_0'(t;\mathbf{h}) = (\lambda_H + \mathbf{h}(t))(V_1(t;\mathbf{h}) - V_0(t;\mathbf{h})) - (\lambda_H - \lambda_L)(\Pi - V_0(t;\mathbf{h})).$$

If $R(t;\mathbf{h}) \le 0$, we have

$$V_1(t;\mathbf{h}) - V_0(t;\mathbf{h}) \le \frac{\lambda_L}{\mu}(\Pi - V_0(t;\mathbf{h})).$$

By plugging this in, we have

$$
\begin{aligned}
V_1'(t;\mathbf{h}) - V_0'(t;\mathbf{h}) \leq & (\lambda_H + \mathbf{h}(t))\frac{\lambda_L}{\mu}[\Pi - V_0(t;\mathbf{h})] - (\lambda_H - \lambda_L)(\Pi - V_0(t;\mathbf{h})) \\
= & \frac{\lambda_L}{\mu}\left[\mathbf{h}(t) - \lambda_\star\right](\Pi - V_0(t;\mathbf{h})).
\end{aligned}
$$

From $\mathbf{h}(t) < \lambda_\star$, we have $V_1'(t;\mathbf{h}) - V_0'(t;\mathbf{h}) < 0$. Then, by (OA.2.5.2), we have $\frac{\partial R}{\partial t}(t;\mathbf{h}) < 0$. □

OA.2.6 Equilibrium Characterization

OA.2.6.1 Useful Properties

Lemma OA.2.8 (Limit incentives)**.** *Let $\mathbf{h} \in \mathcal{H}$ be increasing with $\mathbf{h}(t) \to \bar{h}$. Then $R(t;\mathbf{h}) \to R(t;\bar{h})$.*

Proof. First, we show that $V_1(t;\mathbf{h})$ converges to $V_1(0;\bar{h})$. Since $\mathbf{h}(t) \leq \mathbf{h}(s) \leq \bar{h}$ for all $s > t$, we can bound V_1 by the value when the opponent has constant hazard rates $\mathbf{h}(t)$ and $\bar{h}$.

$$
V_1(0;\bar{h}) = V_1(t;\bar{h}) \leq V_1(t;\mathbf{h}) \leq V_1(t;\mathbf{h}(t)) = V_1(0;\mathbf{h}(t))
$$

By continuity of $V_1(0;h)$ in h, and the fact that the upper-bound $V_1(0;\mathbf{h}(t))$ converges to the lower-bound $V_1(0;\bar{h})$, we can apply the squeeze theorem to get $V_1(t;\mathbf{h}) \to V_1(0;\bar{h})$. We can obtain bounds for $V_0(t;\mathbf{h})$ using a similar logic. Since $h(t) \leq h(s) \leq \bar{h}$,

$$
V_0(0;\bar{h}) = V_0(t;\bar{h}) \leq V_0(t;\mathbf{h}) \leq V_0(t;\mathbf{h}(t)) = V_0(0;\mathbf{h}(t))
$$

Using the continuity of $V_0(0;h)$ in h, as V_0 is the maximum of continuous functions, and applying the squeeze theorem, we obtain that $V_0(t;\mathbf{h}) \to V_0(0;\bar{h})$. □

Lemma OA.2.9. *Let T be a finite time and consider a policy $\boldsymbol{\sigma}$ such that $\boldsymbol{\sigma}(t) = 1$ for all $t > T$. Then, $\mathbf{h}_{\boldsymbol{\sigma}}(t) \to \min\{\lambda_H, \mu\}$.*

Proof. From the evolution of beliefs, using that $\boldsymbol{\sigma}(t) = 1$, we get that, for all $t > T$,

$$\dot{\mathbf{p}}_{\boldsymbol{\sigma}}(t) = (1 - \mathbf{p}_{\boldsymbol{\sigma}}(t))\,[\mu - \lambda_H\,\mathbf{p}_{\boldsymbol{\sigma}}(t)]$$

This evolution of beliefs gives us that $\mathbf{p}_{\boldsymbol{\sigma}}(t)$ converges to 1 when $\mu > \lambda_H$ and to μ/λ_H when $\mu \leq \lambda_H$. Using this, together with $\boldsymbol{\sigma}_t = 0$, in the hazard rate function and taking limits, we obtain that $\lim_{t\to\infty} \mathbf{h}_{\boldsymbol{\sigma}}(t) = \lim_{t\to\infty} \lambda_H\,\mathbf{p}_{\boldsymbol{\sigma}}(t) = \lambda_H \cdot \min\{1, \mu/\lambda_H\} = \min\{\lambda_H, \mu\}$. □

OA.2.6.2 Proof of Theorem 2 (a) : $\lambda_\star < \lambda_L$.

Proof of Theorem 2 (a). Let $(\boldsymbol{\sigma}_A, \boldsymbol{\sigma}_B)$ be a MDNE. Then, by MDR of $\boldsymbol{\sigma}_j$, it must be that $\mathbf{h}_{\boldsymbol{\sigma}_j}$ is increasing. $h_{\boldsymbol{\sigma}_j}$ is also bounded by λ_H, and therefore it converges. We denote $\bar{h}$ the limit of $h_{\boldsymbol{\sigma}_j}(t)$ when $t \to \infty$.

Note that $\bar{h} \geq \lambda_L$: otherwise $\mathbf{h}_{\boldsymbol{\sigma}_j}(t) < \bar{h} < \lambda_L$ for all t and, thus, it would be more profitable for the firm to choose $\boldsymbol{\sigma} = 0$, which induces a constant rate of development equal to λ_L. By continuity, the relative attractiveness of research $R(t; \mathbf{h}_{\boldsymbol{\sigma}_j})$ converges to $R_{\bar{h}} < 0$, where the inequality holds since $\bar{h} \geq \lambda_L > \lambda_\star$. This implies that there is a time T such that $R(t; \mathbf{h}_{\boldsymbol{\sigma}_j}) < 0$ for all $t \geq T$. By Proposition C.1, it must be that $\boldsymbol{\sigma}_i(t) = 0$ for all $t \geq T$. It remains to show that $\boldsymbol{\sigma}_i(t) = 0$ for all $t \leq T$, which follows immediately from applying Proposition C.2.

Summarizing, $(0, 0)$ is the unique candidate for MDNE. First notice that the policy 0 satisfies MDR since $\mathbf{h}_0$ is constant and equal to λ_L. Moreover, to check that $(0, 0)$ is a Nash equilibrium, notice that $\lambda_L > \lambda_\star$, which implies by Proposition 3.1 that developing with the old technology is a best response. □

OA.2.6.3 Proof of Theorem 2 (b): $\lambda_\star > \min\{\mu, \lambda_H\}$.

We begin by obtaining an upper bound for the development rate for any policy with monotone development rates.

Lemma OA.2.10. *Let $\boldsymbol{\sigma} \in \mathcal{S}$ be MDR. Then, $\mathbf{h}_{\boldsymbol{\sigma}} < \min\{\mu, \lambda_H\}$.*

Proof. First, observe that for any $\boldsymbol{\sigma} \in \mathcal{S}$ and $t \geq 0$, $\mathbf{p}_{\boldsymbol{\sigma}}(t) \leq \min\{\mu/\lambda_H, 1\}$. Suppose toward a contradiction that there is a T such that $\mathbf{p}_{\boldsymbol{\sigma}}(T) > \min\{\mu/\lambda_H, 1\}$. Then, by continuity of $\mathbf{p}_{\boldsymbol{\sigma}}$, there must be a $t < T$ such that $\mathbf{p}_{\boldsymbol{\sigma}}(t) \in (\min\{\mu/\lambda_H, 1\}, \mathbf{p}_{\boldsymbol{\sigma}}(T))$ and $\dot{\mathbf{p}}_{\boldsymbol{\sigma}}(t) > 0$. However,

$$\begin{aligned}\dot{\mathbf{p}}_{\boldsymbol{\sigma}}(t) &= \mu(1 - \mathbf{p}_{\boldsymbol{\sigma}}(t))\,\boldsymbol{\sigma}(t) - (\lambda_H - (1 - \boldsymbol{\sigma}(t))\lambda_L)\,\mathbf{p}_{\boldsymbol{\sigma}}(t)(1 - \mathbf{p}_{\boldsymbol{\sigma}}(t)) \\ &\leq [\mu - \lambda_H\,\mathbf{p}_{\boldsymbol{\sigma}}(t)]\,(1 - \mathbf{p}_{\boldsymbol{\sigma}}(t)) < 0\end{aligned}$$

Where the first inequality holds because the $\delta(\sigma, p)$, as defined in (5.1), is increasing in σ and the second inequality holds because if $p_{\boldsymbol{\sigma}}(t) > \min\{\mu/\lambda_H, 1\}$ is only possible if $\mu < \lambda_H$ and $p_{\boldsymbol{\sigma}}(t) > \mu/\lambda_H$.

Next we prove that for any policy $\boldsymbol{\sigma}$ satisfying MDR, the hazard rate $\mathbf{h}_{\boldsymbol{\sigma}}$ never exceeds $\min\{\mu, \lambda_H\}$. First,

$$\mathbf{h}_{\boldsymbol{\sigma}}(t) = \mathbf{p}_{\boldsymbol{\sigma}}(t) \cdot \lambda_H + (1 - \mathbf{p}_{\boldsymbol{\sigma}}(t))\,\underbrace{(1 - \boldsymbol{\sigma}(t))\lambda_L}_{<\lambda_H} < \lambda_H$$

It remains to show that, when $\mu < \lambda_H$, $\mathbf{h}_{\boldsymbol{\sigma}}(t) < \mu$. First, we can see that $\dot{\mathbf{p}}_{\boldsymbol{\sigma}}(t) \geq 0$ implies $\mathbf{h}_{\boldsymbol{\sigma}}(t) \leq \mu$.

$$\dot{\mathbf{p}}_{\boldsymbol{\sigma}}(t) = [\mu\,\boldsymbol{\sigma}(t) - (\lambda_H - (1 - \boldsymbol{\sigma}(t))\lambda_L)\,\mathbf{p}_{\boldsymbol{\sigma}}(t)](1 - \mathbf{p}_{\boldsymbol{\sigma}}(t)) \geq 0$$

Since $\mathbf{p}_{\boldsymbol{\sigma}}(t) < 1$, this holds if and only if $\mu\,\boldsymbol{\sigma}(t) \geq (\lambda_H - (1 - \boldsymbol{\sigma}(t))\lambda_L)\,\mathbf{p}_{\boldsymbol{\sigma}}(t)$. In this case,

$$\mathbf{h}_{\boldsymbol{\sigma}}(t) = (1 - \boldsymbol{\sigma}(t))\lambda_L + p(\lambda_H - (1 - \boldsymbol{\sigma}(t))\lambda_L) \leq (1 - \boldsymbol{\sigma}(t))\lambda_L + \boldsymbol{\sigma}(t)\mu \leq \mu$$

Thus, $\mathbf{h}_{\boldsymbol{\sigma}}(T) > \mu$ implies $\dot{\mathbf{p}}_{\boldsymbol{\sigma}}(t) < 0$ for all $t > T$. $\mathbf{p}_{\boldsymbol{\sigma}}$ is bounded below by 0, thus it must converge. Let $\bar{p}$ be the limit of $\mathbf{p}_{\boldsymbol{\sigma}}(t)$ when $t \to \infty$. Moreover, $\mathbf{h}_{\boldsymbol{\sigma}}$ increasing with decreasing $\mathbf{p}_{\boldsymbol{\sigma}}$ implies that $\boldsymbol{\sigma}$ has to be decreasing as well. Since $\boldsymbol{\sigma}$ is bounded, it must converge as well. Let $\bar{\sigma}$ be the limit of $\boldsymbol{\sigma}(t)$ when $t \to \infty$. However, δ is continuous at $(\bar{p}, \bar{\sigma})$ and $\delta(\bar{p}, \bar{\sigma})$ is bounded away from zero, which contradicts the limit of $\mathbf{p}_{\boldsymbol{\sigma}}$. $\square$

Proof of Theorem 2 (b). Let $\mathbf{h}$ be the opponent's equilibrium hazard rate. Since $\mathbf{h}$ is increasing and bounded, it must be that it converges. Let $\bar{h}$ be the limit of $\mathbf{h}(t)$ when $t \to \infty$. Note that, for all t, $\mathbf{h}(t) < \bar{h} < \min\{\mu, \lambda_H\} < \lambda_\star$, where the first inequality holds by monotonicity of $\mathbf{h}$, the second inequality by Proposition C.2, and the third inequality by assumption. By applying Lemma OA.2.6, we obtain that $R(t;\bar{h}) > 0$. Thus, by Lemma OA.2.8 there is a time T such that $R(t;\mathbf{h}) > 0$ for all $t > T$. Suppose toward a contradiction that $R(s;\mathbf{h}) < 0$ for some $s \in \mathbb{R}_+$. Then, by Proposition C.3, it must be that $R(s;\mathbf{h}) < 0$ for all $s > T$. Thus, there is no such s and $R(t;\mathbf{h}) \geq 0$ for all t. This is true for both firms, so using Proposition C.1, we have that $(1,1)$ is the only equilibrium candidate.

It remains to check that $(1,1)$ is a MDNE. First, observe that $\mathbf{h}_1$ is increasing, since $\dot{\mathbf{h}}_1(t) = \lambda_H \dot{\mathbf{p}}_1(t) = \lambda_H(\mu - \lambda_H \, \mathbf{p_1}(t))(1 - \mathbf{p_1}(t))$ and $(\mu - \lambda_H \, \mathbf{p_1}(t)) > 0$ by Lemma OA.2.10. By Lemma OA.2.9, $\mathbf{h}_1$ converges to $\min\{\mu, \lambda_H\}$, which is lower than $\lambda_\star$. Therefore, there is a time T such that $R(t;\mathbf{h}_1) > 0$ for all $t > T$. Moreover, suppose toward a contradiction that $R(s;\mathbf{h}) < 0$ for some $s \in \mathbb{R}_+$. Then, by Proposition C.3, it must be that $R(s;\mathbf{h}) < 0$ for all $s > T$. Thus, there is no such s and $R(t;\mathbf{h}) \geq 0$ for all t. Thus, by the verification result, $\boldsymbol{\sigma} = 1$ is a best response to $\mathbf{h}_1$ and $(1,1)$ is a NE. □

OA.2.6.4 Proof of Theorem 2 (c): $\lambda_\star \in (\lambda_L, \min\{\mu, \lambda_H\})$.

Lemma OA.2.11. *Let $\lambda_\star \in (\lambda_L, \lambda_H)$, and let $\mathbf{h}$ be increasing with $\mathbf{h}(t) \to \lambda_\star$. Let T be the first time at which $\mathbf{h}(T) = \lambda_\star$. Then $R(t;\mathbf{h}) > 0$ for all $t < T$ and $R(t;\mathbf{h}) = 0$ for all $t \geq T$.*

Proof. First, note that $\mathbf{h}(s) = \lambda_\star$ for all $s \geq T$. Therefore, by Corollary 2, $R(t;\mathbf{h}) = R(0;\lambda_\star) = 0$ for all $t \geq T$. Let $\hat{T}$ be the first time it is profitable to use the old technology, i.e. $\hat{T} := \inf\{t \in [0,\infty] : R(t;\mathbf{h}) \leq 0\}$. Observe that, since $R(T;\mathbf{h}) = 0$, it must be that $\hat{T} \leq T$. Next, we show that $\hat{T} < T$ leads to a contradiction.

Suppose towards a contradiction that $\hat{T} < T$. By Proposition C.3, $R(t,\mathbf{h}) \leq 0$ for all $t \geq \hat{T}$. Additionally, in the proof of Proposition C.3, we show that $R(t,\mathbf{h}) \leq 0$ implies $R'(t,\mathbf{h}) < 0$, which gives $R(T,\mathbf{h}) < 0$ which contradicts $R(T,\mathbf{h}) = 0$. □

The next lemma shows that if the opponent does research first ($\boldsymbol{\sigma}_j(t) = 1$ for all t) it is not a best-response to do direct development.

Lemma OA.2.12. *Let $\lambda_\star \in (\lambda_L, \min\{\lambda_H, \mu\})$. Then $R(0,\mathbf{h}_1) > 0$.*

Proof. $\mathbf{h}_1$ is the development rate associated with the research policy ($\boldsymbol{\sigma} = 1$). We can compute the continuation value, at time zero, of doing direct development $v_0(0;0,\mathbf{h}_1)$.

$$v_0(0;0,\mathbf{h}_1) = \Pi\left[\frac{\lambda_L}{\lambda_L+\mu} + \frac{\mu}{\lambda_L+\mu}\cdot\frac{\lambda_L}{\lambda_L+\lambda_H}\right] - c\left[\frac{1}{\lambda_L+\mu} + \frac{\mu}{\lambda_L+\mu}\cdot\frac{1}{\lambda_L+\lambda_H}\right]$$

The first bracket captures the probability of the firm winning the race. The firm can win by developing before the opponent finds the new technology—which happens with probability $\lambda_L/(\lambda_L+\mu)$—or the opponent can find the new technology first, in which case the firm wins with probability $\lambda_L/\lambda_L+\lambda_H$. The second bracket captures the expected duration of the race. The expected time before the first breakthrough in the race is $1/(\lambda_L+\mu)$. If the opponent finds the new technology—which happens with probability $\mu/(\lambda_L+\mu)$—the race is extended by $1/(\lambda_L+\lambda_H)$ in expectation. By doing some algebra, we obtain that:

$$v_0(0;0,\mathbf{h}_1) = \frac{\lambda_L\Pi - c}{\lambda_L+\mu}\cdot\frac{\lambda_L+\lambda_H+\mu}{\lambda_L+\lambda_H}$$

We can obtain $V_1(0,h_1)$ by using the same logic, but replacing the development rate of the incumbent technology λ_L with the development rate of the new technology λ_H.

$$V_1(0;\mathbf{h}_1) = \frac{\lambda_H\Pi - c}{\lambda_H+\mu}\cdot\frac{\lambda_H+\lambda_H+\mu}{\lambda_H+\lambda_H}$$

Suppose toward a contradiction that direct development ($\sigma=0$) is a best response toward research first ($\sigma=1$). This implies that $V_0(t;\mathbf{h}_1) = v_0(t;0,\mathbf{h}_1)$ and that $R(t;h_1)\leq 0$ for all t. However,

$$\begin{aligned}
R(0;\mathbf{h}_1) &= \mu(V_1(0;\mathbf{h}_1) - V_0(0;\mathbf{h}_1)) - \lambda_L(\Pi - V_0(0,\mathbf{h}_1)) \\
&= \frac{c\left(\left(\lambda_\star-\lambda_L\right)\left(2\lambda_H+\mu\right)+\lambda_L\left(\lambda_H-\lambda_\star\right)\right)}{2\lambda_H\left(\lambda_H+\mu\right)} \\
&\qquad + \mu\cdot(\lambda_L\Pi - c)\cdot\frac{\left(\lambda_\star-\lambda_L\right)\left(2\lambda_H+\mu+\lambda_L\right)+\lambda_L\left(2\lambda_H+\lambda_L-\lambda_\star\right)}{2\left(\lambda_H+\mu\right)\left(\lambda_H+\lambda_L\right)\left(\mu+\lambda_L\right)} > 0
\end{aligned}$$

Where the inequality uses that $\lambda_L\Pi - c > 0$ and that $\lambda_\star\in(\lambda_L,\lambda_H)$. □

Lemma OA.2.13. *Let $\lambda_\star\in(\lambda_L,\min\{\lambda_H,\mu\})$ and let $(\boldsymbol{\sigma}_A,\boldsymbol{\sigma}_B)$ be a MDNE. Then $\mathbf{h}_{\boldsymbol{\sigma}_A}(t)$ and $\mathbf{h}_{\boldsymbol{\sigma}_B}(t)$ converge to $\lambda_\star$.*

Proof. First, note that $\mathbf{h}_{\boldsymbol{\sigma}_i}$ is weakly increasing and bounded above by λ_H. Thus, $\mathbf{h}_{\boldsymbol{\sigma}_i}(t)$ must converge. Let $\bar{h}_i$ be the limit of $\mathbf{h}_{\boldsymbol{\sigma}_i}(t)$ when t goes to infinity.

Suppose towards a contradiction that $\bar{h}_i > \lambda_\star$. Then, by Lemma OA.2.8, $R(t; \mathbf{h}_{\boldsymbol{\sigma}_i})$ converges $R(0; \bar{h}_i)$. Since $\bar{h} > \lambda_\star$, applying Lemma OA.2.6, we get that $R(0; \bar{h}_i) < 0$. Thus, there is a time T for which $R(t; \mathbf{h}_{\boldsymbol{\sigma}_i}) < 0$ for all $t > T$. This implies that $\boldsymbol{\sigma}_j = 0$ for all $t > T$ and moreover, by Proposition C.2, $\boldsymbol{\sigma}_j = 0$. Therefore, $\mathbf{h}_{\boldsymbol{\sigma}_j} = \mathbf{h}_0 = \lambda_L$. Since $\lambda_L < \lambda_\star$, it must be, by OA.2.6, that $R(t; \lambda_L) > 0$. Thus, since $\boldsymbol{\sigma}_i$ is a best-response, $\boldsymbol{\sigma}_i = 1$. However, $(\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_j) = (0, 1)$ is ruled out as an equilibrium by Lemma OA.2.12. Therefore, there cannot be an equilibrium in which one of the development rates converges to a rate greater than $\lambda_\star$.

Suppose towards a contradiction that $\bar{h}_i < \lambda_\star$. Then, by Lemma OA.2.8, $R(t; \mathbf{h}_{\boldsymbol{\sigma}_i})$ converges to $R(0; \bar{h}) > 0$. Thus, there is a time T such that $R(t, \mathbf{h}_{\boldsymbol{\sigma}_i}) > 0$ for all $t > T$. Since $\boldsymbol{\sigma}_j$ is a best-response, it must be that $\boldsymbol{\sigma}_j(t) = 1$ for all $t > T$. By Lemma OA.2.9, $\mathbf{h}_j$ converges to $\min\{\mu, \lambda_H\} > \lambda_\star$. However, we showed that this was not possible. $\square$

Proof of Theorem 2 (b). Suppose that $\lambda_\star \in (\lambda_L, \min\{\lambda_H, \mu\})$ and $(\boldsymbol{\sigma}_A, \boldsymbol{\sigma}_B)$ is a MDNE.

By Lemma OA.2.13, it must be that $\mathbf{h}_{\boldsymbol{\sigma}_A}$ and $\mathbf{h}_{\boldsymbol{\sigma}_B}$ converge to $\lambda_\star$. For $i = A, B$, let $T_i = \sup\{t \;:\; \mathbf{h}_{\boldsymbol{\sigma}_i}(t) < \lambda_\star\}$ and let $T = \min\{T_A, T_B\}$.

Suppose that $T_A < T_B$. By Lemma OA.2.11, we know that $R(t; \mathbf{h}_B) > 0$ for all $t < T_B$. This means that $\boldsymbol{\sigma}_A(t) = 1$ for $t \in (T_A, T_B)$. This, however, contradicts the fact that $\mathbf{h}_{\boldsymbol{\sigma}_A}$ is constant and equal to $\lambda_\star$ on that interval:

$$\dot{\mathbf{h}}_{\boldsymbol{\sigma}_A} = \dot{\mathbf{p}}_{\boldsymbol{\sigma}_A} = (\mu - \lambda_H \, \mathbf{p}_{\boldsymbol{\sigma}_A}(t))(1 - \mathbf{p}_{\boldsymbol{\sigma}_A}(t))\lambda_H > 0 \qquad \forall t \in (T_A, T_B)$$

Thus, $T_A = T_B = T$ with $\boldsymbol{\sigma}_i(t) = 1$ for $t < T$.

For $s > T$ and $i \in \{A, B\}$, we have $\mathbf{h}_{\boldsymbol{\sigma}_i}(s) = \lambda_\star$. Using the definition of $\mathbf{h}_{\boldsymbol{\sigma}_i}$, we have that

$$\lambda_\star = \lambda_H \, \mathbf{p}_{\boldsymbol{\sigma}_i}(s) + \lambda_L(1 - \mathbf{p}_{\boldsymbol{\sigma}_i}(s))(1 - \boldsymbol{\sigma}^*(s)),$$

or equivalently,

$$1 - \sigma^*(s) = \frac{\lambda_\star - \lambda_H \, \mathbf{p}_{\boldsymbol{\sigma}_i}(s)}{\lambda_L(1 - \mathbf{p}_{\boldsymbol{\sigma}_i}(s))}. \tag{OA.2.17}$$

From the evolution of beliefs, we have that for every $s > T$

$$\begin{aligned}\dot{\mathbf{p}}_{\boldsymbol{\sigma}_i}(s) =& (1-\mathbf{p}_{\boldsymbol{\sigma}_i}(s))\left[\mu - \lambda_H\,\mathbf{p}_{\boldsymbol{\sigma}_i}(s) - (1-\boldsymbol{\sigma}_i(s))(\mu - \lambda_L\,\mathbf{p}_{\boldsymbol{\sigma}_i}(s))\right]\\ =& -\frac{\mu}{\lambda_L}(\lambda_\star - \lambda_L) + 2\lambda_\star\,\mathbf{p}_{\boldsymbol{\sigma}_i}(s) = 2\lambda_\star(\mathbf{p}_{\boldsymbol{\sigma}_i}(s) - p_\star).\end{aligned}$$

If there is an $s > T$ such that $\mathbf{p}_{\boldsymbol{\sigma}_i}(s) \neq p_\star$, then the solution of the above differential equation diverges. Therefore, $\mathbf{p}_{\boldsymbol{\sigma}_i}(s) = p_\star$ for all $s \geq T$. Using this, in conjunction with $\dot{\mathbf{p}}_{\boldsymbol{\sigma}_i}(s) = 0$, we obtain

$$\boldsymbol{\sigma}_i(s) = \frac{(\lambda_H - \lambda_L)\,\mathbf{p}_{\boldsymbol{\sigma}_i}(s)}{\mu - \lambda_L\,\mathbf{p}_{\boldsymbol{\sigma}_i}(s)} = \frac{(\lambda_H - \lambda_L)p_\star}{\mu - \lambda_L p_\star} = \sigma_\star \qquad \text{for all } s \geq T.$$

By Lemma OA.2.11, we have that $\boldsymbol{\sigma}_A(s) = \boldsymbol{\sigma}_B(s) = 1$ for all $s < T$. Finally, the fact that $\mathbf{p}_{\mathbf{1}}(T) = p_\star$ is given by the continuity of the probability function $\mathbf{p}_{\boldsymbol{\sigma}}$. Therefore, both firms playing the stationary fall-back policies is the only candidate for a MDNE. Moreover, Lemma OA.2.11 implies that the stationary fall-back policy is the best response when the rival plays the same policy, thereby constituting a MDNE. □