# Dynamic delegation with a persistent state

Yi Chen, Theoretical Economics, 2022

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# Introduction

# Motivation

- A firm's headquarters (**P**) allocates resources to a division manager (**A**) over time
- **P**'s goal: allocating resources to hit a target amount depending on some states slowly evolving over time
  - Examples of state: profitability, consumer taste, or technical parameters...
- State is only observed by **A** (not by **P**)
- **A** wants to receive more resources *regardless* of the state

  $\Rightarrow$ Conflicts of interest between **P** and **A** arise
- **Main Question**: Does **P** benefit from **A**'s information with a dynamic contract?

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# Preview of the Setting

- Dynamic delegation problem
  1. **A** privately observes a persistently evolving state and report it to **P**
  2. **P** commits to actions based on the agent's reported state
  3. Preferences
     - **A**: state independent
     - **P**: matching a state-dependent target
  4. No monetary transfer

## Preview of the Results

- Quota mechanism
  - **P** commits to a fixed *quota*, which is the discounted sum of actions
  - It implies that **A**'s continuation payoff is independent of his current report
  - **P** intertemporally reallocates the quota to make the best use of **A**'s information

- How would **P** respond to information?
  1. **Conformist**: an action moves in the same direction as the target
  2. **Contrarian**: an action moves in the opposite direction from the target
  3. **Unresponsive**: an action does not reflect any information
     - **Babbling**: if the contract is unresponsive for all states

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# A Two-Period Example

# A Two-Period Example: Preliminaries

- Time: $t = 1,\ 2$
- States: $\theta_1,\ \theta_2 \in \mathbb{R}$
  - $\theta_1 \sim \mathcal{N}(0, 1)$
  - $\theta_2 = \theta_1 + \epsilon$ where $\epsilon \sim \mathcal{N}(0, 1)$
- **P**'s total cost:

$$(x_1 - f(\theta_1))^2 + (x_2 - f(\theta_2))^2$$

  where $f(\cdot)$ is a time-invariant *target function*

- **A**'s total payoff:

$$x_1 + x_2$$

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## A Two-Period Example: Principal's Problem

- In each period $t$,
  1. **A** privately learns $\theta_t$
  2. **A** reports $\hat{\theta}_t$ to **P**

- Contract: $(x_1(\hat{\theta}_1), x_2(\hat{\theta}_1, \hat{\theta}_2))$
  - Focus on truthful contracts

- **P**'s problem:

$$\min_{x_1(\cdot), x_2(\cdot,\cdot)} \quad \mathbb{E}\left[(x_1 - f(\theta_1))^2 + (x_2 - f(\theta_2))^2\right]$$

$$\text{s.to.} \quad x_1(\theta_1) + \mathbb{E}\left[x_2(\theta_1, \theta_2) | \theta_1\right] \geq x_1(\hat{\theta}_1) + x_2(\hat{\theta}_1, \hat{\theta}_2), \quad \forall \theta_1, \hat{\theta}_1, \hat{\theta}_2, \tag{1}$$

$$x_2(\theta_1, \theta_2) \geq x_2(\theta_1, \hat{\theta}_2), \quad \forall \theta_1, \theta_2, \hat{\theta}_2. \tag{2}$$

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& x_2(\theta_1, \theta_2) \geq x_2(\theta_1, \hat{\theta}_2), \quad \forall \theta_1, \theta_2, \hat{\theta}_2. \qquad (2)
\end{aligned}
$$

## A Two-Period Example: Quota Mechanism

- (2) says that for all $\theta_2$ and $\hat{\theta}_2$, $x_2(\theta_1, \theta_2) \geq x_2(\theta_1, \hat{\theta}_2)$:

$$x_2(\theta_1, \theta_2) \geq x_2(\theta_1, \hat{\theta}_2) \geq x_2(\theta_1, \theta_2) \;\Rightarrow\; x_2(\theta_1, \theta_2) = x_2(\theta_1, \hat{\theta}_2)$$

$\Rightarrow$ $x_2(\theta_1, \theta_2)$ does not depend on $\theta_2$, and we can write $x_2(\theta_1, \theta_2) = x_2(\theta_1)$ for short.

- Now plug this into (1): for all $\theta_1$, $\hat{\theta}_1$,

$$x_1(\theta_1) + x_2(\theta_1) \geq x_1(\hat{\theta}_1) + x_2(\hat{\theta}_1). \tag{3}$$

$\Rightarrow$ by using the similar trick, we have that for some constant $W$,

$$x_1(\theta_1) + x_2(\theta_1) = W$$

$W$ can be interpreted as the *quota* (total payoff) promised to the agent.

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# A Two-Period Example: Optimal Quota Mechanism

- By using $x_1(\theta_1) + x_2(\theta_1) = W$, **P**'s problem can be rewritten as follows:

$$\min_{x_1(\cdot),W} \mathbb{E}\left[(x_1(\theta_1) - f(\theta_1))^2 + \mathbb{E}[(W - x_1(\theta_1) - f(\theta_2))^2|\theta_1]\right] \tag{4}$$

  - F.O.C. for $x_1(\theta_1)$:

$$\begin{aligned} & x_1(\theta_1) = \frac{1}{2}W + \frac{1}{2}\left(f(\theta_1) - \mathbb{E}[f(\theta_2)|\theta_1]\right) \\ \iff \quad & x_1(\theta_1) - f(\theta_1) = \frac{1}{2}\left(W - \mathbb{E}\left[f(\theta_1) + f(\theta_2)|\theta_1\right]\right) \end{aligned}$$

  - Plug it into (4), F.O.C. for $W$:

$$W = \mathbb{E}\left[f(\theta_1) + f(\theta_2)\right]. \tag{5}$$

  - We also have

$$x_2(\theta_1) = \frac{1}{2}W - \frac{1}{2}(f(\theta_1) - \mathbb{E}[f(\theta_2)|\theta_1]). \tag{6}$$

# A Two-Period Example: Linear Target

- **Example 1**: $f(\theta) = \theta$

$$
\begin{aligned}
W =& \mathbb{E}[\theta_1 + \theta_2] = 0 \\
x_1(\theta_1) =& \frac{1}{2} W + \frac{1}{2} \left(\theta_1 - \mathbb{E}[\theta_2 | \theta_1]\right) = 0 \\
x_2(\theta_1) =& \frac{1}{2} W - \frac{1}{2} \left(\theta_1 - \mathbb{E}[\theta_2 | \theta_1]\right) = 0
\end{aligned}
$$

- The outcome is "babbling" as the actions do not reflect information about the state

# A Two-Period Example: Quadratic Target

- **Example 2**: $f(\theta) = \theta^2$

$$
\begin{aligned}
W =& \mathbb{E}[\theta_1^2 + \theta_2^2] = 3 \\
x_1(\theta_1) =& \frac{1}{2} W + \frac{1}{2}\left(\theta_1^2 - \mathbb{E}[\theta_2^2 | \theta_1]\right) = \frac{1}{2}(3-1) = 1 \\
x_2(\theta_1) =& \frac{1}{2} W - \frac{1}{2}\left(\theta_1^2 - \mathbb{E}[\theta_2^2 | \theta_1]\right) = \frac{1}{2}(3+1) = 2
\end{aligned}
$$

- This contract is also babbling

# A Two-Period Example: Exponential Target

- **Example 3**: $f(\theta) = e^{\theta}$

$$W = \mathbb{E}[e^{\theta_1} + e^{\theta_2}] = \sqrt{e} + e$$

$$x_1(\theta_1) = \frac{1}{2}W + \frac{1}{2}\left(e^{\theta_1} - \mathbb{E}[e^{\theta_2}|\theta_1]\right) = \frac{1}{2}(e + \sqrt{e}) - \frac{1}{2}(\sqrt{e} - 1)e^{\theta_1}$$

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- As the first-period target $e^{\theta_1}$ increases, the corresponding action $x_1$ decreases, in order for $x_2$ to increase in the next period.

- **Why? P** lowers the 1st-period action to increase the 2nd-period action (sacrificing the 1st-period precision to have a better 2nd-period precision)

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# A Two-Period Example: General Target

- Recall that

$$x_1(\theta_1) = \frac{1}{2}W + \frac{1}{2}(f(\theta_1) - \mathbb{E}[f(\theta_2)|\theta_1]) \tag{7}$$

- Then, we have

$$x_1'(\theta) \propto f'(\theta_1) - \frac{\partial}{\partial\theta_1}\mathbb{E}[f(\theta_2) \mid \theta_1] = f'(\theta_1) - \mathbb{E}[f'(\theta_2) \mid \theta_1] \tag{8}$$

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**Note**:

$$\begin{aligned}
\frac{\partial}{\partial\theta_1}\mathbb{E}[f(\theta_2) \mid \theta_1] &= \frac{\partial}{\partial\theta_1}\int_{-\infty}^{\infty} f(\theta_1+\epsilon)\cdot\phi(\epsilon)\ d\epsilon \\
&= \int_{-\infty}^{\infty} f'(\theta_1+\epsilon)\cdot\phi(\epsilon)\ d\epsilon = \mathbb{E}[f'(\theta_2)|\theta_1]
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- $f'''$ matters:
    - If $f'$ is convex, $\mathbb{E}[f'(\theta_2)|\theta_1] > f'(\mathbb{E}[\theta_2|\theta_1]) = f'(\theta_1) \quad \Rightarrow \quad x'(\theta) < 0$
    - If $f'$ is concave, $\mathbb{E}[f'(\theta_2)|\theta_1] < f'(\mathbb{E}[\theta_2|\theta_1]) = f'(\theta_1) \quad \Rightarrow \quad x'(\theta) > 0$

# Continuous Time Model

# Continuous Time Setup

- State evolution: when $Z$ is the standard Brownian motion,

$$\theta_t = \mu t + Z_t \tag{9}$$

- **A** reports $\hat{\theta}_t$ to **P**

$$d\hat{\theta}_t = m_t dt + d\theta_t \tag{10}$$

- **P**'s problem:

$$\min_{x_t(\cdot)} \quad \mathbb{E}\left[\int_0^\infty re^{-rt}(x_t(\theta^t) - f(\theta_t))^2 \ dt\right]$$
$$\text{s.to.} \quad \mathbb{E}\left[\int_0^\infty re^{-rt}x_t(\theta^t)dt\right] \geq \mathbb{E}\left[\int_0^\infty re^{-rt}x_t(\hat{\theta}^t)dt\right]$$

# Quota Mechanism

- Define the continuation payoff process:

$$W_t \equiv \mathbb{E}_t \left[ \int_t^{\infty} r e^{-r(s-t)} x_s ds \right], \tag{11}$$

then we have

$$dW_t = r(W_t - x_t)dt + r\beta_t(d\theta_t - \mu dt) \tag{12}$$

and we can interpret $\beta$ as $W$'s responsiveness to information

- **Propositions:** Roughly speaking, IC constraints $\Leftrightarrow \beta = 0$

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# Optimal Quota Mechanism

- Recall the F.O.C. in the two period model:

$$x_1(\theta_1) - f(\theta_1) = \frac{1}{2}\left(W - \mathbb{E}[f(\theta_1) + f(\theta_2)|\theta_1]\right) \tag{13}$$

- F.O.C. in the continuous model:

$$x(\theta, W) - f(\theta) = W - \gamma \star f(\theta) \tag{14}$$

where

$$\gamma \star f(\theta) = \mathbb{E}\left[\int_0^\infty re^{-rt} f(\theta_t) dt \;\middle|\; \theta_0 = \theta\right] \tag{15}$$

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## Response to Information

- Contract $x$ is called *conformist* at state $\theta$, if $sgn\,[\partial x(\theta, W)/\partial\theta] = sgn[f'(\theta)]$
- Contract $x$ is called *contrarian* at state $\theta$, if $sgn\,[\partial x(\theta, W)/\partial\theta] = -sgn[f'(\theta)]$
- **Theorem**
  1. the optimal contract is conformist at state $\theta$ iff $(\gamma \star f)'/f' < 1$ at state $\theta$
  2. the optimal contract is contrarian at state $\theta$ iff $(\gamma \star f)'/f' > 1$ at state $\theta$

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# Conclusion

- Dynamic delegation problem where a state is persistently evolving
- **Quota mechanism** should be employed to induce the agent's truthful reports
- **Contrarian** pattern may arise depending on the shape of the target function