Blackwell-Monotone Information Costs

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Introduction

Introduction

- Consider a binary experiment with $1 > f_2 > f_1 > 0$:

	s_L	s_H
ω_L	$1 - f_1$	f_1
ω_H	$1 - f_2$	f_2

- Which of the followings are plausible information cost functions?

1. $C(f_1, f_2) = (f_2 - f_1)^2$
2. $C(f_1, f_2) = f_2 - 2f_1$
3. $C(f_1, f_2) = \frac{f_2(1 - f_2)}{f_1(1 - f_1)} - 1$
4. $C(f_1, f_2) = \frac{f_2}{f_1} + \frac{1 - f_1}{1 - f_2} - 2$

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- To say that an information cost function is *plausible*, it should at least satisfy **Blackwell Monotonicity** (higher cost for more Blackwell informative experiments)

- **Blackwell's Theorem**: the followings are equivalent
  1. For any Bayesian decision problem, the expected payoff under f is greater than or equal to that under g
  2. There exists a stochastic matrix M such that $g = f \cdot M$

- **Goal**: identify elementary necessary and sufficient conditions for Blackwell monotonicity

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Roadmap

1. Preliminaries
2. Blackwell Monotonicity under Binary Experiments
    - Answer for the motivating question
3. Blackwell Monotonicity under General Experiments
    - Examples: Additively Separable Costs
4. Application: Costly Persuasion

Preliminaries

Experiments

- $\Omega = \{\omega_1, \cdots, \omega_n\}$: a finite set of states
- $\mathcal{S} = \{s_1, \cdots, s_m\}$: a finite set of signals
- A *statistical experiment* $f : \Omega \to \Delta(\mathcal{S})$ can be represented by an $n \times m$ matrix:

$$f = \begin{bmatrix} f_{11} & \cdots & f_{1m} \\ \vdots & \ddots & \vdots \\ f_{n1} & \cdots & f_{nm} \end{bmatrix},$$

  where $f_{ij} = \Pr(s_j | \omega_i)$, thus, $f_{ij} \geq 0$ and $\sum_{j=1}^{m} f_{ij} = 1$

- $\mathcal{E}_m \subset \mathbb{R}^{n \times m}$: the space of all experiments with m possible signals

Blackwell Informativeness

- $f \succeq_B g$: f is *Blackwell more informative* than g if there exists a stochastic matrix M such that $g = f \cdot M$
  - M is a stochastic matrix iff $M_{ij} \geq 0$ and $\sum_j M_{ij} = 1$ for all i
- **Permutation**
  - A stochastic matrix P is called a *permutation matrix* if it has exactly one non-zero entry in each row and each column.
  - If P is a permutation matrix, so is P^{-1}.
  - **Observation**: f and $f \cdot P$ are equally Blackwell informative:

$$f \succeq_B f \cdot P \succeq f \cdot P \cdot P^{-1} = f \tag{1}$$

  - **Intuition**: relabeling signals does not change the informativeness

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Information Costs and Blackwell Monotonicity

- **Information Costs**
  - $C : \mathcal{E}_m \to \mathbb{R}_+$: an information cost function
  - $\mathcal{C}_m$: the set of all Lipschitz continuous information cost functions defined over $\mathcal{E}_m$
  - Lipschitz continuity ensures that a derivative exists a.e. and is integrable.
- **Blackwell Monotonicity**
  - An information cost function $C \in \mathcal{C}_m$ is **Blackwell monotone** if for all $f, g \in \mathcal{E}_m$, $C(f) \geq C(g)$ whenever $f \succeq_B g$.
- **Permutation Invariance**
  - Any Blackwell-monotone information cost function is **permutation invariant**, i.e., $C(f) = C(f \cdot P)$

$$f \succeq_B f \cdot P \succeq_B f \quad \Rightarrow \quad C(f) \geq C(f \cdot P) \geq C(f).$$

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Binary Experiments

Blackwell Informativeness under Binary Experiments

- Focus on the case where $n = m = 2$

- Any experiment can be represented by $f \equiv (f_1, f_2)^\intercal \in [0,1]^2$:

$$[\mathbf{1} - f, f] = \begin{array}{c|cc} & s_L & s_H \\ \hline \omega_L & 1 - f_1 & f_1 \\ \omega_H & 1 - f_1 & f_2 \end{array}$$

- Any stochastic matrix can also be represented by $(a, b) \in [0,1]^2$:

$$M = \begin{bmatrix} 1 - a & a \\ 1 - b & b \end{bmatrix}.$$

  Then, $[\mathbf{1} - g, g] = [\mathbf{1} - f, f] \cdot M$ implies

$$g = a \cdot (\mathbf{1} - f) + b \cdot f.$$

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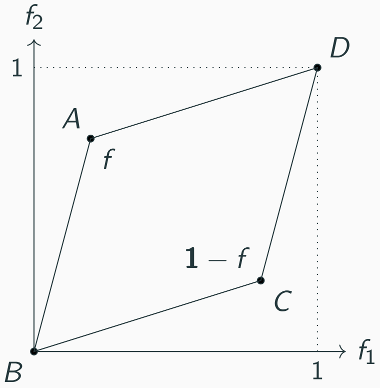
Parallelogram Hull

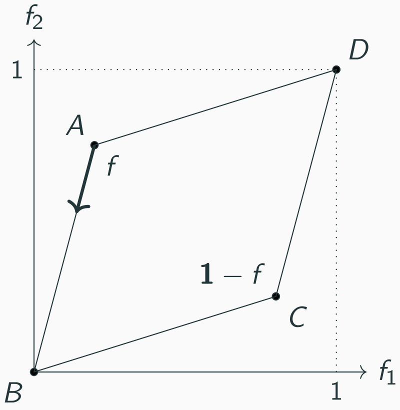
- $f \succeq_B g$ iff g is in the *parallelogram hull* of f and $\mathbf{1} - f$:

$$\mathrm{PARL}(f, \mathbf{1} - f) = \left\{ a \cdot (\mathbf{1} - f) + b \cdot f \in \mathbb{R}^2_+ : a, b \in [0, 1] \right\}.$$



Necessary Conditions for Blackwell Monotonicity

When C is Blackwell monotone,



1. $\langle \nabla C(f), -f \rangle \leq 0$

$$\begin{tikzcd}
s_L \arrow[r, "1"] & s'_L \\
s_H \arrow[ur, "\epsilon"'] \arrow[r, "1-\epsilon"'] & s'_H
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2. $\langle \nabla C(f), \mathbf{1} - f \rangle \leq 0$

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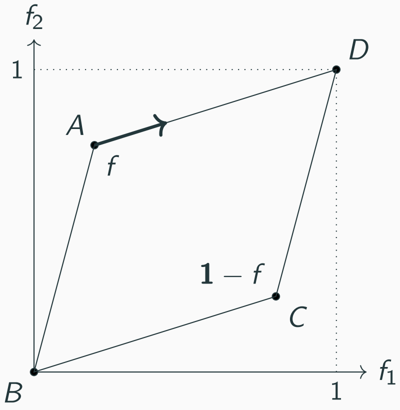

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Theorem for Binary Experiments

Theorem 1

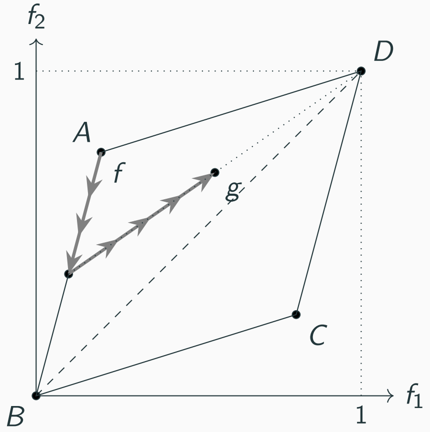
$C \in \mathcal{C}_2$ is Blackwell monotone if and only if it is

1. permutation invariant;
2. for all $f \in \mathcal{E}_2$,
$$\langle \nabla C(f), f \rangle \geq 0 \geq \langle \nabla C(f), \mathbf{1} - f \rangle. \tag{2}$$

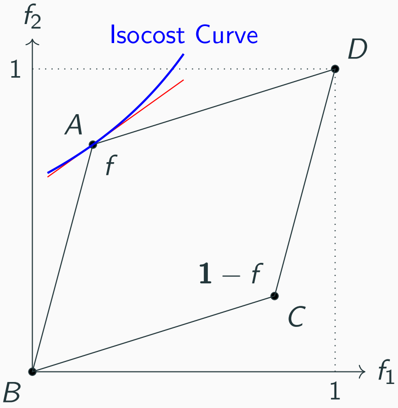
- This theorem holds for the cases with more than two states, but the binary signal assumption is curcial.

Proof for Sufficiency

For any $f \succeq_B g$, we can find a path from f to g (or the permutation of it) along which Blackwell informativeness decreases



Further Characterizations with Binary States



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is equivalent to:

$$\underbrace{\frac{f_2}{f_1}}_{\substack{\text{the slope} \\ \text{of } \overline{AB}}} \geq \underbrace{-\frac{\partial C / \partial f_1}{\partial C / \partial f_2}}_{\substack{\text{the slope of} \\ \text{the isocost curve}}} \geq \underbrace{\frac{1 - f_2}{1 - f_1}}_{\substack{\text{the slope} \\ \text{of } \overline{AD}}}$$

- **Interpretation**: a *marignal rate of information transformation* (MRIT) lies between the two likelihood ratios provided by the experiment.

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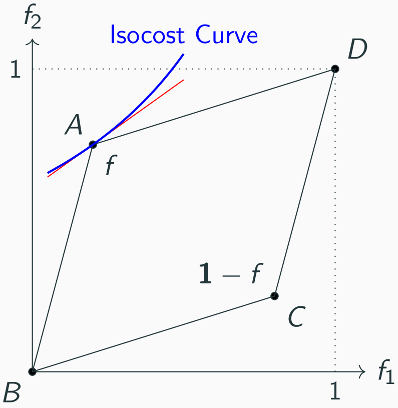

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Further Characterizations with Binary States

1. $C(f_1, f_2) = (f_2 - f_1)^2$ with $1 > f_2 > f_1 > 0$

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- The above inequalities hold for all $1 > f_2 > f_1 > 0$, thus, it is **Blackwell monotone**.

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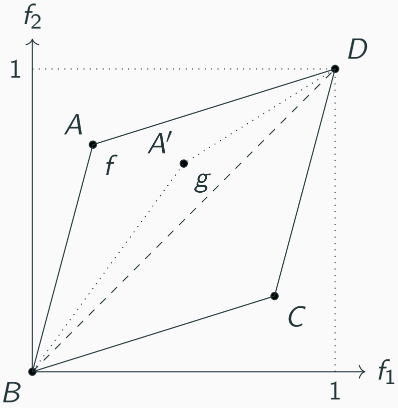

$f \succeq_B g$ is equivalent to:

1. AB steeper than $A'B$:

$$\alpha \equiv \frac{f_2}{f_1} \geq \frac{g_2}{g_1} \equiv \alpha'$$

2. AD slower than $A'D$:

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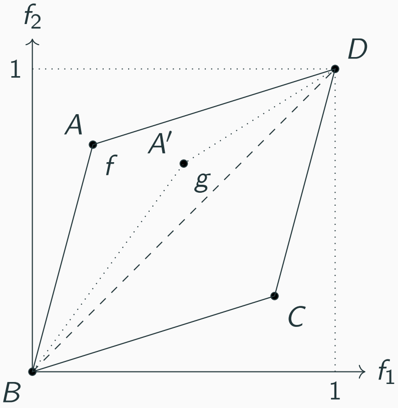

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$$\tilde{C}(\alpha, \beta) = \frac{\alpha}{\beta} - 1$$

- $\tilde{C}$ is increasing in α but not in β, thus, $\tilde{C}$ is not Blackwell monotone.

4. $C(f_1, f_2) = \dfrac{f_2}{f_1} + \dfrac{1-f_1}{1-f_2} - 2$ with $1 > f_2 > f_1 > 0$

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- $\tilde{C}$ is increasing in both α and β, thus, $\tilde{C}$ is **Blackwell monotone**.

Answer for the Motivating Question

Which of the followings are Blackwell-monotone information cost functions?

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General Experiments

Necessary Conditions for Blackwell Monotonicity

- Permutation invariance is still necessary

- For any pair (i, j), the following garbling worsens the informativeness:

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Sufficient Conditions for Blackwell Monotonicity

When $m \geq 3$, there may not exist a path along which informativeness decreases

Proposition

Let

$$g = \begin{bmatrix} 4/5 & 1/5 & 0 \\ 0 & 4/5 & 1/5 \\ 1/5 & 0 & 4/5 \end{bmatrix} \in \mathcal{E}_3.$$

If $f \succeq_B g$ and $f \in \mathcal{E}_3$, then f is a permutation of I_3 or g.

- I_3 is Blackwell more informative than g, but we cannot find a path from I_3 to g along which Blackwell informativeness decreases

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Quasiconvexity

- Observe that there is a permutation of I_3 such that

$$g = \frac{4}{5} \cdot I_3 + \frac{1}{5} \cdot (I_3 \cdot P).$$

- If we impose **quasiconvexity**, with permutation invariance, we have

$$C(I_3) = C(I_3 \cdot P) \geq C\left(\frac{4}{5} \cdot I_3 + \frac{1}{5} \cdot I_3 \cdot P\right) = C(g).$$

- Caveat: Quasiconvexity is not a necessary condition for Blackwell monotonicity

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Theorem for General Experiments

Theorem 2

Suppose that $C \in \mathcal{C}_m$ is Lipschitz continuous and quasiconvex. Then, C is Blackwell monotone if and only if it is

1. permutation invariant;
2. for all $f \in \mathcal{E}_2$ and $i \neq j$,

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- $S_B(f)$: the set of experiments that are less Blackwell informative than f
- Two conditions ensure that extreme points of $S_B(f)$ are not more costly than f
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Examples: Additively Separable Costs

Additively Separable Costs

C is additively separable if there exists Lipschitz continuous functions $\psi : \mathbb{R}^n_+ \to \mathbb{R}_+$ such that, for all m and $f \in \mathcal{E}_m$,

$$C(f) = \sum_{j=1}^{m} \psi(f^j).$$

Theorem 3

When C is additively separable, C is Blackwell monotone if and only if ψ is sublinear:

1. $\psi(\alpha h) = \alpha \psi(h)$;
2. $\psi(k + l) \geq \psi(k) + \psi(l)$

Examples: Additively Separable Costs

Additively Separable Costs

C is additively separable if there exists Lipschitz continuous functions $\psi : \mathbb{R}^n_+ \to \mathbb{R}_+$ such that, for all m and $f \in \mathcal{E}_m$,

$$C(f) = \sum_{j=1}^{m} \psi(f^j).$$

Theorem 3

When C is additively separable, C is Blackwell monotone if and only if ψ is sublinear:

1. $\psi(\alpha h) = \alpha \psi(h)$;
2. $\psi(k + l) \geq \psi(k) + \psi(l)$

Examples: Additively Separable Costs

1. Supnorm Costs

$$C(f) = \sum_{j=1}^{m} \max_i f_{ij}.$$

2. Linear Costs

$$C(f) = \sum_{j=1}^{m} |\langle a, f^j \rangle| = \sum_{j=1}^{m} \left| \sum_{i=1}^{n} a_i f_{ij} \right|.$$

3. Linear ϕ-divergence Costs

$$C(f) = \sum_{j=1}^{m} \sum_{i,i'} \beta_{ii'} f_{i'j} \phi_{ii'} \left(\frac{f_{ij}}{f_{i'j}} \right).$$

4. Entropy Costs

$$C_\mu(f) = \sum_{j=1}^{m} \lambda \left(\sum_{i=1}^{n} \mu_i f_{ij} \log \frac{\mu_i f_{ij}}{\sum_{i=1}^{n} \mu_i f_{ij}} \right) - \lambda \left(\sum_{i=1}^{n} \mu_i \log \mu_i \right).$$

Application: Costly Persuasion

Gentzkow, Kamenica (2014) Revisited

- Consider a costly persuasion problem with the standard example
  - State: $\{innocent, guilty\}$
  - Receiver's action: **A**cquit or **C**onvict
  - Sender's payoff: $u_S(C) = 1$, $u_S(A) = 0$
  - Receiver's payoff: $u_R(A, innocent) = u_R(C, guilty) = 1$
    $u_R(C, innocent) = u_R(A, guilty) = 0$
  - Sender commits to an experiment at some cost
- GK focuses on posterior separable costs (e.g., entropy cost) to utilize concavification technique
- Can we solve this problem with any Blackwell-monotone information cost function?

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Costly Persuasion with Blackwell-Monotone Information Cost

- It is without loss to consider binary experiments since **R**'s action is binary
  - $f_2 = \Pr(C|\mathit{guilty})$ and $f_1 = \Pr(C|\mathit{innocent})$

- Sender's problem is

$$\max_{0 \le f_1 \le f_2 \le 1} pf_2 + (1-p)f_1 - C(f_1, f_2)$$

subject to

$$\frac{pf_2}{pf_2 + (1-p)f_1} \ge \frac{1}{2}.$$

- When $p \ge 1/2$, the solution is $f_1 = f_2 = 1$: always convict costlessly

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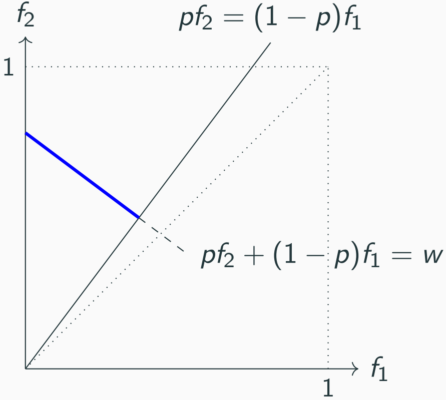
Cost Minimization

- Suppose $p < 1/2$.
- Cost minimization problem under $pf_2 + (1-p)f_1 = w$:

$$\min C(f_1, f_2) \quad \text{s.t.} \quad \begin{aligned} pf_2 + (1-p)f_1 &= w, \\ pf_2 &\geq (1-p)f_1 \end{aligned}$$



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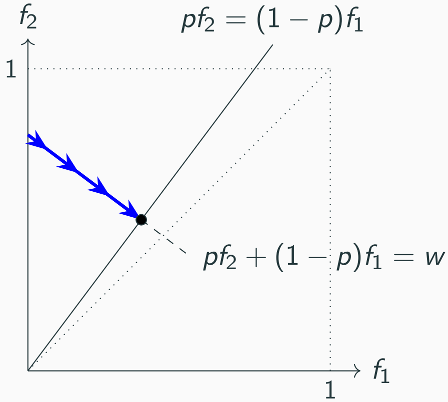

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Sender's Problem

- When $pf_2 + (1-p)f_1 = w$, the cost is minimized at

$$f_2 = \frac{w}{2p} \quad \text{and} \quad f_1 = \frac{w}{2(1-p)}.$$

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$$\max_{0 \leq w \leq 2p} w - C\left(\frac{w}{2p}, \frac{w}{2(1-p)}\right) \tag{4}$$

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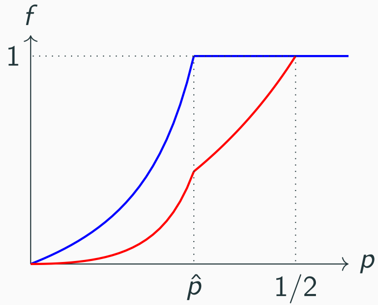
Costly Persuasion with Non-Posterior-Separable Cost

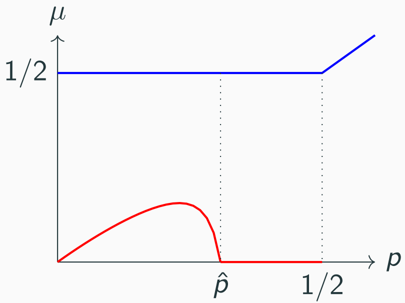
- When $C(f_1, f_2) = (f_2 - f_1)^2$, the solution for $p < 1/2$ is

$$f_2(p) = \min\left\{1,\ \frac{(1-p)^2 p}{(1-2p)^2}\right\} \quad \text{and} \quad f_1(p) = \frac{p}{1-p} \cdot f_2(p).$$

Entropy



Optimal Experiments



Posteriors

Conclusion

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- Our technique allows to solve the costly persuasion problem with any Blackwell-monotone information cost function
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Thank You!

Related Literature

- **Posterior-based information costs**
  - Entropy cost: Sims [2003]; Matějka, Mckay [2015]
  - Decision theory: Caplin, Dean [2015]; Caplin, Dean, Leahy [2022]; Chambers, Liu, Rehbeck [2020]; Denti [2022]
  - Applications: Ravid [2020]; Zhong [2022]; Gentzkow, Kamenica [2014]
- **Experiment-based information costs**
  - LLR cost: Pomatto, Strack, Tamuz [2023];
  - Applications: Denti, Marinacci, Rustichini [2022]; Ramos-Mercado [2023]

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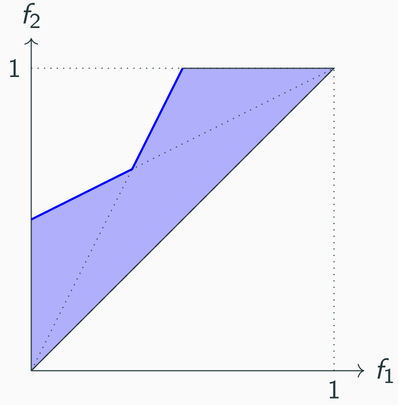
Quasiconvexity

- The following information cost function for binary experiments is not quasiconvex



$$C(f_1, f_2) = \min\left\{\frac{f_2}{f_1}, \frac{1-f_1}{1-f_2}\right\}$$
$$= \min\{\alpha, \beta\}$$

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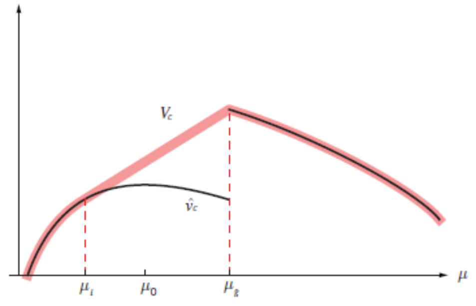
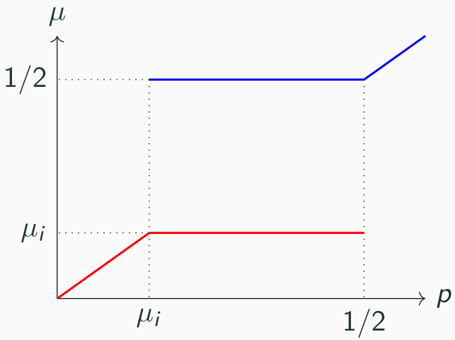
- Entropy cost: $k \cdot \mathbb{E}_{\pi|p}[H(p) - H(\mu_s)]$ where $H(\mu) \equiv -\sum_\omega \mu(\omega)\log(\mu(\omega))$
  - p is prior, and μ_i and μ_g are posteriors from an experiment π



Concavification



Posteriors

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