Strategic Concealment in Innovation Races *

WORKING VERSION

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Abstract

We investigate firms' incentives to conceal intermediate research discoveries in innovation races. To study this, we introduce an innovation game where two racing firms dynamically allocate their resources between two distinct research and development (R&D) paths towards a final innovation: (i) developing it with the currently available but slower technology; (ii) conducting research to discover a faster new technology for developing it. We fully characterize the equilibrium behavior of the firms in the cases where their research progress is public and private information. Then, we extend the private information setting by allowing firms to conceal or license their intermediate discoveries. We demonstrate that firms may have an incentive to conceal their interim discoveries during innovation races, which can lead to a slower pace of innovation that is inefficient, particularly when the reward for winning the race is high.

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To-Do Lists

1. Whether to assume that $\lambda_H \geq \lambda_L$ to focus on the case where the new technology is about 'speed.' ***YG: I think that we should do this. If $\lambda_H < \lambda_L$, under some parameter values, even if both firms have the new technology, the firms may choose to use the old technology. They both know that using the new technology would give higher expected social welfare, but they may prefer to win the race with lower reward. ***

2. Check model section

3. (General reward) Public information case proof

4. (Symmetric reward) Private information case main text

5.

1 Introduction

In the course of research and development (R&D), firms often discover interim knowledge that brings them closer to successfully producing a final innovation. When multiple firms race towards such innovation, a firm's optimal R&D strategy is likely to be influenced by the information about whether its rivals have made intermediate breakthroughs. Thus, a firm may want to conceal intermediate discoveries in order to hinder its rivals from adjusting their R&D strategies. On the other hand, it may prefer to disclose an intermediate discovery because this can open the opportunity for monetization via licensing the technological breakthrough. In this paper, we introduce and study an innovation race model that captures the tradeoffs between licensing and concealing interim discoveries and characterize firms' equilibrium behavior.

We consider a situation where two firms race towards developing an innovative product, such as a COVID-19 vaccine or a full self-driving (FSD) vehicle. The first firm to develop the product receives a reward (e.g., a transitory flow of monopoly profit) and the other firm does not. At each point in time, the firms allocate their limited resources between two routes for developing the product and incur constant flow costs. One route is to conduct basic *research* to discover a new technology that does not directly deliver the product but makes developing it faster, e.g., messenger RNA (mRNA) or light detection and ranging (LIDAR) technology.[1][2] This route requires two breakthroughs: discovering the new technology and developing the product with it. The other route is to *develop* the product with a currently available but slow technology, namely the old technology. For example, the viral vector method for developing

[1] The mRNA technology was not utilized in practice before the COVID-19 outbreak. Thus, pharmaceutical firms had to first acquire basic knowledge in order to employ this new methodology. The advantage of possessing this intermediate technology is that firms can develop vaccines in a laboratory by using readily available materials. Hence vaccines can be developed faster with mRNA technology than with older methods. Moderna and Pfizer-BioNTech utilize mRNA technology to develop COVID-19 vaccines. For more information, see the web page of the Centers for Disease Control and Prevention (CDC): https://www.cdc.gov/coronavirus/2019-ncov/vaccines/different-vaccines/mrna.html.

[2] LIDAR is a laser radar that can provide extensive and reliable information surrounding a vehicle including an object's distance, size, position, and velocity if it is moving. Most FSD vehicle developers including Waymo—formerly the Google self-driving car project—use LIDAR combined with cameras. The main drawback of LIDAR is its current high cost. Thus, to develop a commercializable FSD vehicle, firms first need to discover a way to make LIDAR less expensive. Once LIDAR becomes affordable, it will be relatively easy to develop a commercializable FSD vehicle. In this sense, successfully developing an FSD vehicle with the LIDAR technology can be understood as a route requiring two breakthroughs.

a COVID vaccine and the camera-based vision technology for developing an FSD vehicle can be considered old technologies.[3][4] This path requires a single breakthrough but the arrival rate is relatively low. We assume that the path with the new technology is more efficient: the total expected completion time of doing research for the new technology and developing the product with it is shorter than that of developing with the old strategy. Thus, the socially efficient policy is to have both firms allocate all their resources to research, and once one of them discovers the new technology, have it share the breakthrough with the other firm to prevent duplication of research costs.

We investigate three different settings in the context of this framework. First, we consider the case where it is public information whether a firm has discovered the new technology or not. In this setting, a firm can condition its strategy not only on its own technological breakthrough but also on its rival's progress. We show that there exists a unique equilibrium and its form is determined by the relative efficiency of the new technology. The efficiency measure is defined to be inversely proportional to the expected total completion time of the path with the new technology, i.e., doing research is more attractive when efficiency is high. It is shown that when efficiency is extreme (high or low), a firm's equilibrium strategy does not depend on its rival's progress. Specifically, when the new technology is highly efficient, both firms allocate all their resources to research (i.e., perform research only); and when the new technology is not much more efficient, both firms allocate all their resources to development (i.e., develop with the old technology only) regardless of their rival's status. On the contrary, when efficiency is intermediate, the equilibrium strategy of each firm does depend on its rival's progress. In this case, both firms begin by conducting research, but once one firm makes the intermediate technological breakthrough, the other switches to developing with the old technology, namely it pursues a *fall-back* strategy.

[3] The viral vector technology was used during recent disease outbreaks including the 2014-2016 Ebola outbreak in West Africa. Many pharmaceutical firms had access to this methodology when the COVID-19 outbreak began. Indeed, this technology was utilized to develop COVID-19 vaccines by Oxford-AstraZeneca and Janssen (Johnson&Johnson). For more information, see the web page of the CDC: `https://www.cdc.gov/coronavirus/2019-ncov/vaccines/different-vaccines/viralvector.html`.

[4] Unlike other companies, Tesla's approach towards developing an FSD vehicle is to use only cameras without LIDAR (Templeton, 2019). Since camera technology is already very cheap, no cost-saving breakthrough is needed to implement it. However, the quality of information attained from cameras is inferior to that attained from LIDAR, thus it will take more time to develop an FSD vehicle utilizing only cameras.

Next, we analyze the setting where technological discoveries are private information, i.e., a firm cannot observe its rivals' technological progress. As in the public information setting, when efficiency is high, each firm conducts research until it succeeds or its rival produces the final innovation. Similarly, when efficiency is low, both firms endeavor to develop with the old technology. This invariance occurs because, in the extreme cases of very high and very low efficiency, firms do not use the information about their rival's progress even when it is observable. However, in the case of intermediate efficiency, the firms cannot use the fall-back strategy as in the public information setting since they are no longer able to make their resource allocations contingent on their rivals' state of technology. Instead, their resource allocations must depend on their 'beliefs' about their rivals' progress. We characterize the unique symmetric equilibrium that is Markov with respect to these beliefs. The equilibrium strategy has a cutoff structure: firms conduct research exclusively up to a certain date (belief), then they start allocating their resources between developing with the old technology and researching the new one, namely they employ a *stationary fall-back* strategy. The most intriguing feature of this equilibrium is that beliefs remain constant once the allocation of resources to development begins. This stationarity derives from two conflicting effects in the belief evolution. First, as time passes, it becomes more likely that one's rival has found the new technology (the *duration effect*). On the other hand, the lack of one's rival producing the final innovation (which is publically observable) implies that it is less likely that the new technology has been discovered (the *still-in-the-race effect*).

Last, we extend the private information setting by allowing firms to protect their discoveries by using either a *patent* or a *trade secret*. First, when a firm treats the new technology as a trade secret, it conceals the discovery, i.e., its rival still cannot observe its progress. However, this does not prohibit the firm's rival from discovering the new technology independently. Second, when a firm files a patent, it discloses the discovery of the new technology. On the one hand, if its rival has not yet made the technological breakthrough, then the exclusive right to use the new technology is bestowed on the patenting firm. In addition, the patenting firm may *license* the new technology, i.e., it may permit its rival to use the new technology for a fee. Once the licensee pays the fee, both firms race for the final innovation employing the new technology. On the other hand, if the rival firm has already discovered the new technology, i.e., it was protected as a trade secret. Then, the patenting firm cannot claim the exclusive right—rather, the new technology is now considered common property—and both firms can use it without making transfers.[5][6]

We first show that if a firm files a patent and the rival firm does not possess the new technology, the patenting firm always licenses. Thus, both firms develop the final innovation with the new technology, which is socially efficient. Once a firm files a patent, its rival can only try to develop the product with the old slow technology. Given this, the patenting firm can extract rent from its rival by allowing it to use the new technology for a fee. This is an application of the classical result of Coase (1960) in the sense that the socially efficient outcome can be achieved when the property right of the new technology is given to a firm and trade involves no transaction costs. Therefore, disclosing the new technology implies licensing it.

Finally, we explore whether a firm prefers to disclose or conceal the new technology. We show that this decision crucially depends on the size of the reward of winning the race: when the reward is high, firms may prefer to conceal their discoveries, whereas when the reward is low, they disclose and license them. Intuitively, this is because concealment involves a higher chance of winning the race, which is more attractive when the reward is high. Whereas, disclosure delivers an immediate payment from licensing, which is more appealing when the reward is low. More specifically, when a firm conceals a discovery, its rival does not know whether it possesses the new technology. Thus, per the results from the private information setting, the rival firm continues allocating some of its resources to researching the new technology. This is not desirable for the rival, especially when efficiency of the new technology is intermediate, because if it knew that the other firm already possessed the new technology, then its best response would be to allocate all its resources to development with the old technology (i.e., to employ the fall-back strategy). In this sense, concealing the new technology hinders the rival firm from strategically responding to its discovery.

Concealment is detrimental not only to the rival firm but also to social surplus because

[5] When a firm files a patent, the firm with the trade secret can dispute the patent based on 35 U.S. Code §273 - Defense to infringement based on prior commercial use.

[6] For more information about trade secrets and patents, see the web page of the World Intellectual Property Organization: https://www.wipo.int/about-ip/en/. Also, see Lobel (2013) for examples.

it generates duplicate research efforts. This slows down the pace of innovation. On the contrary, the socially efficient outcome could be achieved by disclosing and licensing the new technology. These results on firms' incentives for concealment imply a simple policy intervention. Reducing the reward of winning the race (e.g., weakening the transitory monopoly power in the innovative product market by imposing a tax,) reduces incentives to conceal and promotes licensing, thus speeding up the pace of innovation.

Related Literature

This paper primarily contributes to the literature on patent vs. secrecy by introducing a novel incentive to conceal a firm's discovery: hindering its rival's strategic response. Previous studies mainly focused on the limited protection power of patents. For example, the seminal article by Horstmann et al. (1985) posits that "patent coverage may not exclude profitable imitation." Thus, in their framework, the main reason why a firm may choose secrecy over a patent is not to be imitated.[7] Another limitation of a patent is that it expires in a finite time. For instance, Denicolò and Franzoni (2004) consider a framework where a patent gives the patenting firm monopoly power only for a certain period of time (and no profit after expiration), whereas secrecy can give indefinite monopoly power to a firm but it can be leaked or duplicated by a rival with some probability. On the contrary, in this paper, we abstract from the restrictions of patents and focus analysis on the potential advantages of concealment.

Another hallmark of this paper is its consideration of 'interim' discoveries. Therefore, it is naturally related to the literature on licensing of interim R&D knowledge, e.g., Bhattacharya et al. (1992); d'Aspremont et al. (2000); Bhattacharya and Guriev (2006); Spiegel (2008). In these papers it is assumed that firms already know which of them has superior knowledge, i.e., the firm that will license the technology is exogenously given. Unlike in those studies, we allow firms to choose when to license (and even allow them not to license), i.e., the licensing decision is endogenous.

We also contribute to the innovation literature by introducing a model with two char-

[7] Many subsequent papers study the imitation threat and potential patent infringement, e.g., Gallini (1992); Takalo (1998); Anton and Yao (2004); Kultti et al. (2007); Kwon (2012); Zhang (2012).

acteristics. First, there are different avenues towards innovation: developing with the old technology and doing research for the new technology. Second, one of the paths involves multiple stages: once a firm discovers the new technology, then the firm develops the innovative product with it.

With respect to the first characteristic, there is a recent branch of the literature that studies races where there are different routes to achieve a final objective. Das and Klein (2020) and Akcigit and Liu (2016) study a patent race where two firms compete for a breakthrough and there are two methods to get the breakthrough: a safe method and a risky method. In Das and Klein (2020) the safe method has a known constant arrival intensity while the risky method has an unknown constant arrival intensity. In Akcigit and Liu (2016), instead, the safe method has a known payoff associated with breakthrough arrival, while there is uncertainty about the payoff if the risky method is used. In this paper, firms face no uncertainty about whether the innovation is feasible. Instead, they are uncertain whether their rival possesses the new and faster technology.

The second characteristic, multi-stage innovation, is also widely studied in the literature, e.g., Scotchmer and Green (1990); Denicolò (2000); Green and Taylor (2016); Song and Zhao (2021). Our paper shares the framework with these in that we use two sequential Poisson discovery processes and ask whether a firm would patent the first discovery or not. A feature setting apart from their works is that there is another path that only requires one but slower breakthrough toward innovation. This feature connects our model to Carnehl and Schneider (2022) and Kim (2022) in the sense that players can choose between a sequential approach—which requires two breakthroughs—and a direct approach, which requires only one breakthrough, but its riskier or slower.[8] Our model mainly differs from theirs in that multiple players compete by choosing between these approaches, whereas Carnehl and Schneider (2022) considers a problem by a single decision maker and Kim (2022) studies a contracting setup between a principal and an agent. In their studies, a key factor for a player to choose the direct approach is a deadline that is either exogenously given or endogenously determined

[8]In Carnehl and Schneider (2022), an agent is uncertain whether the direct approach is feasible or not, i.e., this approach is risky. On the other hand, in Kim (2022), there is no uncertainty on the feasibility of the direct approach, but its completion rate is slower than the ones for the sequential approach. In this sense, our framework is closer to Kim (2022).

to reduce moral hazard. In contrast to these, a deadline is not involved in our model. Rather, the race with the rival firm may induce a firm to develop with the old technology, which can be considered as a direct approach.

Last, this paper is related to the recent literature on information disclosure in priority races, e.g., Hopenhayn and Squintani (2016); Bobtcheff et al. (2017). In those papers, once a firm makes a breakthrough, the innovation value grows as time passes until one of the firms files a patent. Thus, firms face a tradeoff between disclosing to claim the priority and delaying in order to grow the innovation value. On the contrary, in this paper, the value of innovation is fixed and the discovery of the new technology only allows the firm to develop the innovative product faster. Therefore, a firm may delay the disclosure purely to confound the rival's R&D decisions.

Roadmap

We introduce the model in the next section, then characterize equilibria in the private and the public information settings in Section 3 and 4. In Section 5, we extend the private information setting by allowing firms to disclose their discoveries. We conclude in Section 6. All proofs appear in the appendix.

2 Model

We consider a race between two firms, A and B, to develop an *innovative product*. Time is continuous and infinite $t \in [0, \infty)$. The innovative product can be developed using two different technologies: at the start of the race, both firms have access to an old technology, but they can gain access to a new technology by conducting research.

Each firm owns one unit of resources per unit of time, which can be used either for research to discover the new technology or for developing the innovative product. We denote by $\sigma_t^i \in [0, 1]$ the resources at time t that Firm i allocates to 'research.' Then, $1-\sigma_t^i$ is the amount of resources that Firm i allocates to 'develop' the innovative product. Firm i stochastically gains access to the new technology at rate $\sigma_t^i \cdot \mu$, where μ is a constant parameter. We call that the research progress is made once the firm discovers the new technology, and this progress is irreversible. Firms can develop the innovative product stochastically at rate $(1 - \sigma_t^i) \cdot \lambda_L$ with the old technology, and $(1 - \sigma_t^i) \cdot \lambda_H$ with the new technology. Obviously, the parameters μ, λ_L, and λ_H are positive.

The race ends once one of the firms develops the innovative product. During the race, firms pay a flow cost c. The first firm to develop the innovative product receives a lump-sum reward worth The race ends once one of the firms develops the innovative product. During the race, firms pay a flow cost $c > 0$. The first firm to develop the innovative product receives a lump-sum reward worth Π when the product is developed.[9]Firms do not discount the future and maximize their expected total payoff.[10] The successful development of the innovative product is publicly observable. Thus, firms know at all times if they are still on the race. However, firms do not observe their opponents' resource allocations over time. Regarding the research progress, we explore different setups where they are publicly or privately observed by rival firms.

For the rest of the paper, we make the following two parametric assumptions:

$$\Pi - \frac{c}{\mu} - \frac{c}{\lambda_H} > \Pi - \frac{c}{\lambda_L} > 0. \tag{2.1}$$

The first inequality implies that when there is only one firm, doing research and developing with the new technology is more efficient than developing with the old technology. Then, the second inequality implies that developing with the old technology is profitable.

First-best Resource Allocation. Consider the social planner whose goal is to maximize the joint expected profit of the firms. In addition, assume that the planner can observe the research progress and force firms to share the new technology to others. Note that the expected cost is constant with respect to the number of firms, i.e., though the flow cost is doubled, the expected completion time will become a half. Then, by (2.1), the first-best resource allocation is characterized as follows.

Observation 1. *If the join-profit-maximizing planner can observe firms' research progress*

[9] We model the race as winner-takes-all competition. This payoff structure has been commonly used in the innovation race literature, e.g., Loury (1979); Lee and Wilde (1980); Denicolò and Franzoni (2010).

[10] With discounting the firms are not risk-neutral over the duration of the race conditional on the outcome. This complicates the closed-form solutions without affecting the qualitative results of the paper.

and force them to share the new technology to others, the optimal policy is to

(i) allocate all resources to research, so that the new technology is discovered at rate 2μ*;*

(ii) and once a firm discovers the new technology, force the firm to share it immediately to the opponent so that both firms start developing the product at rate $2\lambda_H$*.*

Parameter normalization. To facilitate the interpretation of the results, we introduce normalized parameters as follows:

$$\pi \equiv \lambda_L \Pi / c, \qquad \rho \equiv \lambda_L / \lambda_H, \qquad \text{and} \qquad \gamma \equiv \lambda_L / \mu. \tag{2.2}$$

Note that the assumptions can be rewritten as follows:

$$1 - \rho > \gamma \qquad \text{and} \qquad \pi > 1. \tag{2.3}$$

In the rest of paper, most results are written in terms of γ, which measures the difficulty of research.

3 Public Information Setting

We begin our analysis by exploring a setting where the research progress of firms is publicly available information, i.e., each firm can observe the research progress made by its competitor. In this case, we can define the set of firms that have successfully obtained the new technology as common knowledge, which we can represent as a state variable denoted by $\omega \in \Omega \equiv \{\{A, B\}, \{A\}, \{B\}, \emptyset\}$.

We focus on firms' Markov strategies. Specifically, Firm i's Markov strategy is defined by $\mathbf{s}^i : \Omega \to [0, 1]$. A pair of Markov strategies $(\mathbf{s}^A, \mathbf{s}^B)$ constitutes a Markov perfect equilibrium (MPE) if, for any given state, each firm's strategy is the best response to the opponent's strategy.

Next, we introduce three benchmark Markov strategies and provide a theorem demonstrating that firms adopt one of these strategies in the MPE.

Definition 3.1. (a) The *research strategy* $\mathbf{s}_R^i$ for firm i fully allocates resources to research regardless of the opponent's research progress ($\mathbf{s}_R^i(\omega) \equiv \mathbb{1}_{\{i \notin \omega\}}$).

(b) The *fall-back strategy* $\mathbf{s}_F^i$ fully allocates resources to research if neither firm has the new technology. If one of the firms has obtained the new technology, it fully allocates resources to development ($\mathbf{s}_F^i(\omega) \equiv \mathbb{1}_{\{\omega=\emptyset\}}$).

(c) The *direct-development strategy* fully allocates the resources to development regardless of the state ($\mathbf{s}_D^i(\omega) \equiv 0$).

Theorem 1. *Suppose that firms' research progress is public information. Let* $\overline{\gamma} \equiv \frac{1-\rho}{1+\rho}$ *and* $\underline{\gamma} \equiv \frac{1-\rho}{2}$. *Then, the Markov perfect equilibrium is uniquely characterized as follows:*

(a) if $\gamma < \underline{\gamma}$*, both firms play their respective research strategies* $(\mathbf{s}_R^A, \mathbf{s}_R^B)$*;*

(b) if $\gamma \in (\underline{\gamma}, \overline{\gamma})$*, both firms play the fall-back strategies* $(\mathbf{s}_F^A, \mathbf{s}_F^B)$*;*

(c) if $\gamma > \overline{\gamma}$*, both firms play the direct-development strategies* $(\mathbf{s}_D^A, \mathbf{s}_D^B)$*.*

The above theorem provides a clear understanding of how the difficulty of research (γ) influences the firms' R&D decisions. When the research is relatively easy ($\gamma < \underline{\gamma}$), firms choose to conduct research regardless of the opponent's research progress. On the other extreme, when the research is relatively difficult ($\gamma > \overline{\gamma}$), firms do not engage in research at all. In the intermediate case ($\gamma \in (\underline{\gamma}, \overline{\gamma})$), firms' R&D decisions are influenced by their rival's research progress. If neither firm has discovered the new technology, both firms engage in research. However, once a firm obtains the new technology, the other firm switches to developing with the old technology.

3.1 Proof of Theorem 1

A MPE consists of a profile of Markov strategies such that each of the players is best responding to the strategy of their opponent. Lemma A.2 proves that we only need to consider Markov deviations to construct the set of Markov Perfect Equilibria.[11]

[11] Intuitively, when the opponent is using a Markov strategy, the problem of a firm is independent of calendar time. Thus, there exists a best response that is Markov and, therefore, the best Markov response

Given a Markov strategy profile of the firms, we can define U^i_ω as the continuation payoff of Firm i in state ω. Next, we provide some intuition for the proof of Theorem 1 by splitting the problem of the firms in two: On one hand, we solve the problem of the firms before any research progress has been made (and fixing the continuation payoffs). On the other hand, we compute the best responses of the firms after one of them obtains the new technology, and therefore the equilibrium continuation payoffs. Finally, by plugging these continuation payoffs into the problem of the firms at the initial state, we prove the theorem.

Best Responses under no Research Progress We first consider the case where neither firm has made research progress, i.e., $\omega = \emptyset$. The conventional approach is to solve the problem with backward induction. However, in order to facilitate the analysis in various extensions, we present the problem under the state $\omega = \emptyset$ in a general manner by treating the continuation payoffs $U^i_{\{i\}}$ and $U^i_{\{j\}}$ as exogenous values.

When Firm i and j play $\mathbf{s}(\emptyset) = x$ and $\hat{\mathbf{s}}(\emptyset) = y$, Firm i's expected payoff at the state $\emptyset$ is

$$u_0(x, y) \equiv \frac{x\mu U^i_{\{i\}} + (1-x)\lambda_L \Pi + y\mu U^i_{\{j\}} - c}{x\mu + (1-x)\lambda_L + y\mu + (1-y)\lambda_L}. \tag{3.1}$$

Define $\Delta_y := u_0(1, y) - u_0(0, y)$. The following proposition characterizes the equilibrium allocations at state $\emptyset$ in any MPE.

Proposition 1. *The equilibrium allocations at state $\emptyset$ are characterized as follows:*

(a) when $\Delta_0, \Delta_1 > 0$, both firms do research, i.e., $(\mathbf{s}^A(\emptyset), \mathbf{s}^B(\emptyset)) = (1, 1)$;

(b) when $\Delta_0, \Delta_1 < 0$, both firms develop with the old technology, i.e., $(\mathbf{s}^A(\emptyset), \mathbf{s}^B(\emptyset)) = (0, 0)$;

(c) when $\Delta_0 > 0 > \Delta_1$, there are three possible equilibrium allocations:

- *one firm does research and the other firm develops with the old technology, i.e., $(\mathbf{s}^A(\emptyset), \mathbf{s}^B(\emptyset)) = (1, 0)$ or $(0, 1)$,*

is also a best response. For general treatment on the existence of Markov equilibria in a larger class of continuous-time stochastic games with finite states and actions, see Neyman (2017).

- *both firms allocate* $z^* = \Delta_0/(\Delta_0 - \Delta_1)$ *amount of resources to research and the remainder to the development with the old technology, i.e.,* $(\mathbf{s}^A(\emptyset), \mathbf{s}^B(\emptyset)) = (z^*, z^*)$*;*

(d) when $\Delta_1 > 0 > \Delta_0$*, there are three possible equilibrium allocations:*

- *both firms do research, i.e.,* $(\mathbf{s}^A(\emptyset), \mathbf{s}^B(\emptyset)) = (1, 1)$*,*
- *both firms develop with the old technology, i.e.,* $(\mathbf{s}^A(\emptyset), \mathbf{s}^B(\emptyset)) = (0, 0)$
- *both firms allocate* $z^* = -\Delta_0/(\Delta_1 - \Delta_0)$ *amount of resources to research and the remainder to the development with the old technology, i.e.,* $(\mathbf{s}^A(\emptyset), \mathbf{s}^B(\emptyset)) = (z^*, z^*)$*.*

The set of equilibria as a function of Δ_1 and Δ_0 characterized in Proposition 1 is summarized in Figure 1. The proof, in Appendix, boils down to showing that the derivative of u_0 with respect to x shares the same sign with $\lambda_L \cdot \Delta_0 \cdot (1 - y) + \mu \cdot \Delta_1 \cdot y$ (Lemma A.3). As this sign is independent of x, the best response function exhibits a 'bang-bang' characteristic: the optimal response to the allocation y of the opponent is either 0, 1, or any value in $[0, 1]$.

In scenarios where Δ_0 and Δ_1 share the same sign, the best response is independent of the opponent's resource allocation. Specifically, when both Δ_0 and Δ_1 are positive, it is optimal to assign all resources to research. Conversely, when both Δ_0 and Δ_1 are negative, it is optimal to develop with the old technology.

When Δ_0 and Δ_1 have different signs, the optimal response depends on the resource allocation y of the opponent. When Δ_1 is positive and Δ_0 is negative, the function u_0 satisfies the single-crossing property from Milgrom and Shannon (1994) and the best-response functions of firms are increasing with respect to the opponent's resource allocation y. Thus, we can interpret the firms' allocations as strategic complements. This complementarity explains the two symmetric equilibria at the extreme allocations and the equilibrium with interior allocation.

When Δ_0 is positive and Δ_1 is negative, the negative of u_0 satisfies the single-crossing property and the best response functions are decreasing. In this context, we can interpret the firms' allocations as strategic substitutes. This substitutability explains why we obtain two asymmetric extreme equilibria and one symmetric equilibrium with interior allocations.

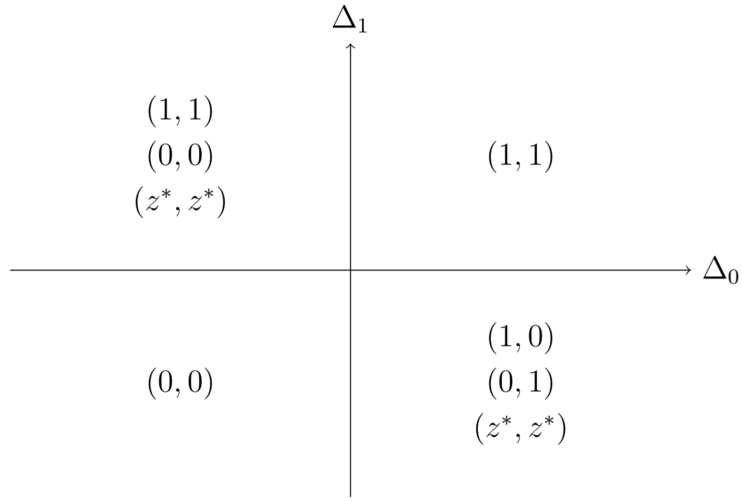


Figure 1: Equilibrium Allocations under No Research Progress

Best Responses under Research Progress We now consider the cases where at least one of the firms has made the research progress.

When both firms have made the research progress ($\omega = \{i,j\}$), they will develop with the new technology and their expected payoffs are $U^i_{\{i,j\}} = U^j_{\{i,j\}} = V_C \equiv \frac{\lambda_H \Pi - c}{2\lambda_H}$ (Lemma A.4).

Next, suppose that only one of the firms, say Firm i, has made the research progress, i.e., $\omega = \{i\}$. Intuitively, it is optimal for Firm i to develop with the new technology (Lemma A.5). The following lemma shows that Firm j's best response depends on the relative difficulty of the research ($\gamma = \lambda_L/\mu$).

Lemma 3.1. *In any MPE, when the state is $\omega = \{i\}$, Firm j does research ($\mathbf{s}^j(\{i\}) = 1$) when research is relatively easy ($\gamma < \underline{\gamma} \equiv \frac{1-\rho}{2}$), and develops with the old technology ($\mathbf{s}^j(\{i\})) = 0$) when research is not easy ($\gamma > \underline{\gamma}$).*

Based on the above result, we can derive the continuation values as follows:

(i) if $\gamma < \underline{\gamma}$,

$$U^i_{\{i\}} = U^j_{\{j\}} = \frac{\lambda_H \Pi + \mu V_C - c}{\mu + \lambda_H}, \qquad U^i_{\{j\}} = U^j_{\{i\}} = \frac{\mu V_c - c}{\mu + \lambda_H}, \tag{3.2}$$

(ii) if $\gamma > \underline{\gamma}$,

$$U^i_{\{i\}} = U^j_{\{j\}} = \frac{\lambda_H \Pi - c}{\lambda_L + \lambda_H}, \qquad U^i_{\{j\}} = U^j_{\{i\}} = \frac{\lambda_L \Pi - c}{\lambda_L + \lambda_H}. \tag{3.3}$$

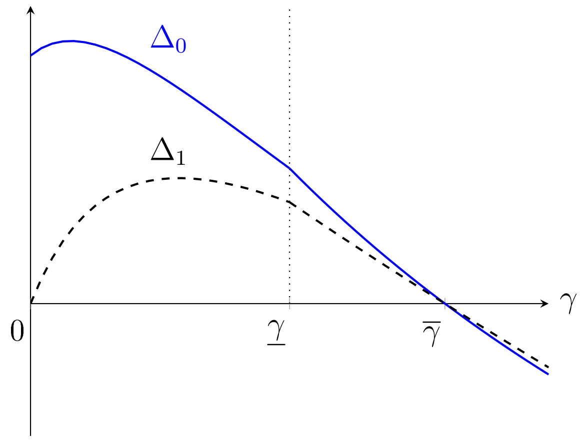


Figure 2: Comparative Statics with respect to γ

Equilibrium Characterization Now that we have derived the continuation values, we can finalize the proof of Theorem 1 by plugging these values into Proposition 1.

First, when $\gamma < \underline{\gamma}$, in Lemma A.6, we show that the following equations hold:

$$\Delta_0 = \frac{c}{\lambda_L} \cdot \frac{(\pi+1)\gamma(2\underline{\gamma}-\gamma)+2\rho(\underline{\gamma}-\gamma)}{2(\rho+\gamma)(\gamma+1)}, \tag{3.4}$$

$$\Delta_1 = \frac{c}{\lambda_L} \cdot \frac{\gamma \cdot \{\gamma(2\underline{\gamma}-\gamma)+\pi\gamma+2(\pi+\rho)(\underline{\gamma}-\gamma)\}}{2(\rho+\gamma)(\gamma+1)}. \tag{3.5}$$

From the above equations and $\gamma < \underline{\gamma}$, we can see that $\Delta_0, \Delta_1 > 0$, as illustrated in Figure 2. By applying Proposition 1 (a), both firms do research at the state $\emptyset$. Then, when one of the firms, say Firm j, succeeds in research, by Lemma 3.1, Firm i will keep doing research. Therefore, the unique MPE is for firms to follow the research strategy (Theorem 1 (a)).

Next, when $\gamma > \underline{\gamma}$, in Lemma A.7, we show that the following equations hold:

$$\Delta_0 = \frac{c}{\lambda_L} \cdot \frac{(\overline{\gamma}-\gamma)(1+\pi)}{2(\gamma+1)}, \tag{3.6}$$

$$\Delta_1 = \frac{c}{\lambda_L} \cdot \frac{(\overline{\gamma}-\gamma)(\gamma+\pi)}{2(\gamma+1)}. \tag{3.7}$$

Note that $\overline{\gamma} = \frac{1-\rho}{1+\rho} > \frac{1-\rho}{2} = \underline{\gamma}$ from the parametric assumption (2.3).

When $\gamma \in (\underline{\gamma}, \overline{\gamma})$, (3.6) and (3.7) imply that Δ_0 and Δ_1 are positive, as illustrated in Figure 2. Thus, by Proposition 1 (a), both firms do research at the state $\emptyset$. Then, when one of the firms, say Firm j, succeeds in research, by Lemma 3.1, Firm i will switch to develop with the old technology. Therefore, the unique MPE is for firms to follow the fall-back strategy (Theorem 1 (b)).

Last, when $\gamma > \overline{\gamma}$, we can see that Δ_0 and Δ_1 are negative. Then, by Proposition 1 (b), both firms develop with the old technology at the state $\emptyset$. Additionally, even if a firm happens to succeed in research, the other firm will keep developing with the old technology due to Lemma 3.1. Thus, the unique MPE is for firms to employ the direct-development strategy (Theorem 1 (c)).

4 Private Information Setting

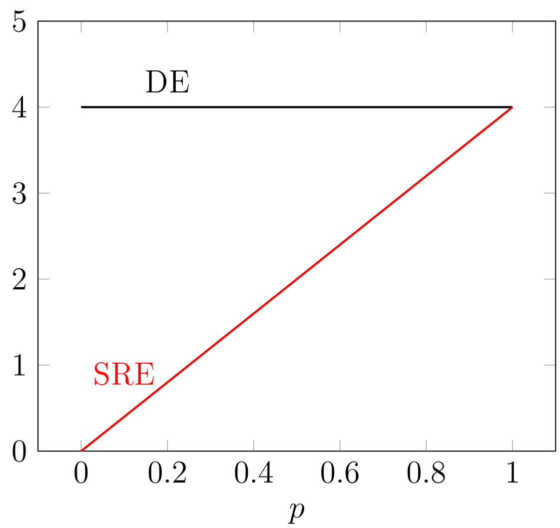
We now consider the case in which research progress is private information, i.e., firms cannot observe whether their opponents have the new technology or not. In this case, the firms can only condition their resource allocations on their own progress and calendar time t. As before, a firm with the new technology will fully allocate resources to development with it. Thus, we focus on the dynamic resource allocation problem of a firm that has not discovered the new technology yet. Thus, an allocation policy for a firm can be therefore described by a function $\sigma : \mathbb{R}_+ \to [0, 1]$ that represents the research allocation at a given time conditional on the new technology not being discovered. We assume that σ is right-continuous.

Given an allocation policy σ, let τ_D be the arrival time of the successful development with either technology. Likewise, let τ_R be the arrival time of the new technology discovery. We provide mathematical details about these arrival times including survival function $S^D_\sigma(t) \equiv \Pr[t < \tau_D]$ in Appendix B.1.

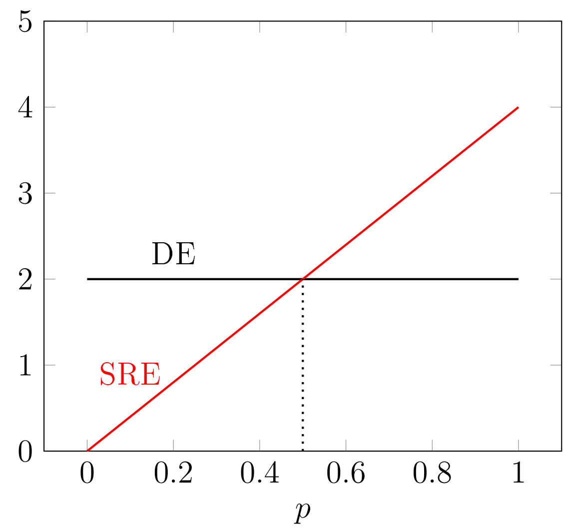
The probability that Firm i has discovered the new technology by time t conditional that it has not developed the product yet can be considered as follows:

$$p_\sigma(t) \equiv \frac{\Pr[\tau_R < t < \tau_D]}{\Pr[t < \tau_D]}. \tag{4.1}$$

Proposition 2. *Suppose that a firm follows an allocation policy $\sigma : \mathbb{R}_+ \to [0, 1]$. Then,*



(a) $\sigma = 1$, $\gamma = \rho = 1/2$ and $\lambda_L = 2$.



(b) $\sigma = 1$, $\gamma = 1$, $\rho = 1/2$ and $\lambda_L = 2$.

Figure 3: Duration Effect (Black) and Still-in-the-Race Effect (Red)

$p_\sigma(t)$ evolves according to the following differential equation:

$$\frac{p'_\sigma(t)}{1 - p_\sigma(t)} = \mu \cdot \sigma_t - \{\lambda_H - (1 - \sigma_t) \cdot \lambda_L\} \cdot p_\sigma(t). \tag{4.2}$$

The proof is provided in Appendix B.2. Since none of the firms has access to the new technology at the beginning of the race, we have for any allocation policy σ the initial probability p_0 is zero, which serves as an initial condition for the evolution of $p_\sigma(t)$.

The left-hand side of (4.2) is the opposite of the time derivative for the log of $(1 - p_\sigma(t))$. The right-hand side of (4.2) captures two distinct effects in the evolution of the conditional probability. First, given that the firm has not yet attained the new technology by time t, the research succeeds at rate $\mu \cdot \sigma_t$ and it may raise $p_\sigma(t)$. The first term of (4.2) represents this positive effect, which we dub the duration effect (DE). On the other hand, the fact that the race is still ongoing indicates that the firm has not succeeded yet in development and therefore it is less likely to have the new technology in hand. The second term of (4.2) reflects this effect, which we dub the still-in-the-race effect (SRE).[12] Notice that this term

[12]Similar types of the belief updating can be found in the strategic experimentation literature, e.g., Keller et al. (2005); Bonatti and Hörner (2011). The main difference is that the agents form beliefs about whether the project is good or bad in those papers, whereas the firms form beliefs about whereas in our model, firms only form beliefs about the technology access of the rival.

is proportional to $\lambda_H - (1-\sigma_t)\lambda_L$, which is the rate of successful innovation development given the new technology net of that without the new technology.

In Figure 3, we illustrate these effects when a firm fully allocates its resources to research ($\sigma_t = 1$ for all $t \geq 0$). Specifically, we provide the graphs of the terms of each effect divided by $(1-p)$: μ (DE), $\lambda_H p$ (SRE). In Figure 3a, we depict the case where $\mu = \lambda_H$, i.e., $\rho = \gamma$. Observe that, in this case, the duration effect is larger than the still-in-the-race effect for every p. If we fix λ_H and increase μ, we observe that the duration effect continues to dominate the still-in-the-race effect. Hence, when $\gamma < \rho$, the probability converges to 1. On the other hand, in Figure 3b, we illustrate the case where $\mu < \lambda_H$, i.e., $\gamma > \rho$. In this case, the duration effect is greater than the still-in-the-race effect only when $p < \mu/\lambda_H = \rho/\gamma$. This induces the probability to converge to ρ/γ.

Next, note that Firm i's expected payoff at time 0 is:

$$\mathcal{U}(\sigma, \hat{\sigma}) = \Pr[\tau_D < \hat{\tau}_D] \cdot \Pi - c \cdot \mathbb{E}[\tau_D \wedge \hat{\tau}_D].$$

In Appendix B.1, we show that

$$\mathcal{U}(\sigma, \hat{\sigma}) = \int_0^\infty [\{\lambda_L(1-\sigma_t) \cdot (1 - p_\sigma(t)) + \lambda_H \cdot p_\sigma(t)\} \cdot \Pi - c] \cdot S^D_\sigma(t) \cdot S^D_{\hat{\sigma}}(t)\ dt. \tag{4.3}$$

Intuitively, given that neither firm has developed the product by time t, represented by the product of survival functions $S^D_\sigma(t) \cdot S^D_{\hat{\sigma}}(t)$, with the probability $p_\sigma(t)$, Firm i has the new technology and completes the development at rate λ_H, and with the probability $(1 - p_\sigma(t))$, Firm i does not have the new technology by then, and completes the development at rate $\lambda_L(1-\sigma_t)$. Also, the flow cost is incurred.

As in the literature on dynamic games with unobservable actions (e.g., Bonatti and Hörner, 2011), we aim to characterize the symmetric Nash equilibria (SNE) in this game. Especially, we focus on the SNE with the following property.

Definition 4.1. An allocation policy σ exhibits an *escalating tension* if $h^D_\sigma(t) = \lambda_L(1 - \sigma_t)(1 - p_\sigma(t)) + \lambda_H p_\sigma(t)$ is weakly increasing in t.

Definition 4.2. ***YG: how about htis*** An allocation policy σ exhibits a *monotone* *hazard rate (MHR) property* if $h^D_\sigma(t) = \lambda_L(1-\sigma_t)(1-p_\sigma(t)) + \lambda_H p_\sigma(t)$ is weakly increasing in t.

YG: explain:

1. $h_\star$ satisfies:
$$\frac{\lambda_L \Pi - c}{\lambda_L + h_\star} = \frac{\mu \frac{\lambda_H \Pi - c}{\lambda_H + h_\star} - c}{\mu + h_\star}$$
Meaning: when the opponent's hazard rate is fixed at $h_\star$, the firm is indifferent between research and development (w/ old tech).

   FP: add arguments

2. Define
$$p_\star \equiv \frac{\mu}{2\lambda_L} - \frac{\mu}{2h_\star}.$$
We have
$$\frac{1}{2h_\star} = p_\star \cdot \left(\frac{1}{\lambda_H + h_\star}\right) + (1-p_\star) \cdot \left(\frac{1}{\mu + h_\star} + \frac{\mu}{\mu + h_\star} \cdot \frac{1}{\lambda_H + h_\star}\right)$$
Meaning: $p_\star$ is the 'stationary' belief such that the average hazard rate is equal to $h_\star$ when the rival's hazard rate is $h_\star$

3. Define
$$\sigma_\star \equiv \frac{(\lambda_H - \lambda_L)p_\star}{\mu - \lambda_L p_\star}.$$
By definition, we have
$$\dot{p} = \mu\sigma_\star - \{\lambda_H - (1-\sigma_\star)\lambda_L\} \cdot p_\star = 0.$$
Meaning: $\sigma_\star$ keeps the belief stationary at $p_\star$

The following theorem characterizes the unique SNE with the above property.

Theorem 2. *Suppose that firms' research progress is private information. Let* $\tilde{\gamma} \equiv \max\{\underline{\gamma},\ 1-2\rho\}$. *The unique symmetric Nash equilibrium allocation policy exhibiting an escalating tension is characterized as follows.*

*(i) If $\gamma < \tilde{\gamma}$, firms play a **research policy**: $\sigma_t^A = \sigma_t^B = 1$ for all $t \geq 0$.*

*(ii) If $\gamma > \overline{\gamma}$, firms play a **direct-development policy** $\sigma_t^A = \sigma_t^B = 0$ for all $t \geq 0$.*

*(iii) If $\gamma \in (\tilde{\gamma}, \overline{\gamma})$, firms play a **stationary fall-back policy**:*

$$\sigma_t^A = \sigma_t^B = \begin{cases} 1, & \text{if } t < T, \\ \sigma_\star \in (0,1), & \text{if } t \geq T. \end{cases}$$

In addition, the belief is stationary onward: $p_{\sigma^A}(t) = p_{\sigma^B}(t) = p_\star$ for all $t \geq T$.

4.1 Proof of Theorem 2

Recursive Formulation Suppose that Firm j employs $\hat{\sigma}$. Let $V_1(t;\hat{\sigma})$ be the continuation payoff of Firm i on the event that Firm i discovers the new technology at time t and neither firm succeeds in development by then.

Proposition 3. *Suppose that σ^* exhibits an escalating tension and constitutes a symmetric Nash equilibrium. If $\sigma_t^* \in (0,1)$ for some t, then $h_s = h_\star$, $\sigma_s = \sigma_\star$, and $p_s = p_\star$ for all $s \geq t$.*

Proposition 4. *Suppose that σ^* exhibits an escalating tension and constitutes a symmetric Nash equilibrium. Then, there exists $T \in \mathbb{R}_+ \cup \{\infty\}$ and a stationary allocation $\sigma_\circ \in [0,1)$ such that $\sigma_t^* = 1$ for all $t < T$, and $\sigma_t^* = \sigma_\circ$ for all $t \geq T$.*

5 Patents

In this section, we extend the model by allowing the firms to patent and license the new technology at no cost. The model remains the same as the one described in Section 2, with two key differences: (1) firms can apply for a patent at any time once they possess the new technology. The patent is granted only if no competitor has achieved the same breakthrough before the application; (2) a granted patent provides the firm with exclusive usage rights and the right to license the technology to a competitor for a fee.

We assume that, immediately after a patent is granted, the patent holder offers a take-it-or-leave-it fee, x, to its competitor. Accepting the offer grants the competitor access to the new technology, requires them to pay the fee x to the patent holder, and allows both firms to use the new technology going forward. If the competitor rejects the offer, only the patent holder can use the new technology, and the competitor must develop using the old technology. Patent applications are assumed to be publicly observable, and we explore both the case in which breakthroughs are public and private information.

5.1 Patents under Public Information

As in Section 3, we consider a setting where research breakthroughs are public. The solution concept is Markov Perfect Bayesian Equilibrium. As it turns out, patents with public breakthroughs provide the incentives for the firms to share the new technology and, under the TIOLI protocol where the patent holder captures all surplus, the resource allocation is efficient. This is formalized in the following proposition.

Definition 5.1. A MPBE is *efficient* if firms license the new technology immediately and allocate all resources to research before the technology is patented.

Proposition 5. *When technological breakthroughs are public, the new technology is always immediately patented and licensed at at fee*

$$x = \frac{(\lambda_H - \lambda_L)(\lambda_H \Pi + c)}{2\lambda_H(\lambda_H + \lambda_L)}.$$

Moreover, there exists $\bar{\pi} > 1$ such that there is an efficient MPBE if and only if $\gamma \leq \underline{\gamma}$ or $\frac{\lambda_L \Pi}{c} < \bar{\pi}$. The efficient PBE is unique when it exists.

5.2 Patents with private information

When the research progress is private information, a firm that obtains the new technology can choose to conceal its discovery—treating it as a trade secret—or disclose it by filing a patent. Additionally, a firm that files a patent can choose whether to license the new technology or not. As before, we assume that the patent holder can make a take-it-or-leave-it licensing fee offer to its competitor. We solve the equilibrium by working backward, beginning with the subgame in which one of the firms has filed a patent.

5.2.1 The Subgame after Patenting

We begin by describing the subgame that occurs after a firm files a patent. There are two possible scenarios, depending on whether the competitor had the new technology and was concealing it, or didn't have the new technology.

First, consider the case where the competitor already had the new technology. In this situation, the patent is not granted.[13] Consequently, both firms can develop using the new technology, and their continuation value is V_C.

Now, consider the case where the competitor doesn't have the new technology. In this scenario, the patent is granted. As in the case of public information, a firm that has patented the new technology can always obtain a higher payoff by licensing the technology to its competitor. The equilibrium TIOLI offer and continuation payoffs remain the same as in the public information case: $x^* = \frac{\lambda_H - \lambda_L}{\lambda_H + \lambda_L}\left(V_C + \frac{c}{\lambda_H}\right)$.

5.2.2 Immediate-Patent Equilibrium

We first explore whether there exists an equilibrium in which firms patent—and subsequently license—the new technology as soon as the breakthrough occurs. We consider a strategy profile such that a firm with the new technology employs the *immediate-disclosure strategy*—a firm discloses (and licenses) the new technology as soon as it discovers—and a firm without the new technology employs the research strategy ($\sigma_t = 1$ for all $t \geq 0$). Then, we ask whether both firms playing this strategy can be sustained as an equilibrium.

Suppose that a firm (say Firm A) just discovered the new technology and Firm B has not disclosed it yet. Given that Firm B sticks to the immediate disclosure and research strategy, Firm A's belief that Firm B has the new technology is zero. Then, by disclosing the new technology, Firm A expects to license it, i.e., the expected payoff for Firm A after disclosure is V_L. Now consider Firm A's deviation to delay the disclosure by time dt. With the probability $\lambda_H dt$, Firm A wins the race and receives Π. But with the probability μdt,

[13] The trade secret protection allows the rival firm to dispute the patent.

Firm B will discover the new technology and files a patent, but it will be disputed by Firm A's trade secret right. Thus, both firms will race with the new technology from then on and the expected payoff is V_C. With the probability $(1 - \lambda_H dt - \mu dt)$, neither of the events happens and Firm A licenses, then the expected payoff is V_L. Last, the flow cost cdt will be paid. To sum up, Firm A's expected payoff from delaying the disclosure is

$$\begin{aligned}\Pi \cdot \lambda_H dt + V_C \cdot \mu dt + (1 - \lambda_H dt - \mu dt) \cdot V_L - cdt &= V_L + [(\mu + 2\lambda_H)V_C - (\mu + \lambda_H)V_L]\, dt \\ &= V_L + [\lambda_H V_C - (\mu + \lambda_H)x^*]\, dt.\end{aligned}$$

Let $\delta \equiv \lambda_H/(\mu+\lambda_H)$. Then, the immediate-disclosure and research strategy can be sustained as an equilibrium if and only if $\delta V_C \leq x^*$. By using (C.1) and some algebra, this inequality is equivalent to:

$$\frac{\rho(\gamma - \underline{\gamma})}{(1-\rho)(\gamma + \rho)} \leq \frac{c}{\lambda_H \Pi - c}. \tag{5.1}$$

From the assumption that $\lambda_L \Pi \geq c$, observe that (5.1) always holds if $\gamma \leq \underline{\gamma}$. Recall that firms do research regardless of the rival's progress. It implies that there does not exist any incentive for a firm to conceal its progress. Therefore, the firms would monetize the new technology by licensing it as soon as it discovers, and the first-best outcome would be achieved.

Next, suppose that $\underline{\gamma} < \gamma < \overline{\gamma}$. Then, (5.1) is equivalent to:

$$\pi = \frac{\lambda_L \Pi}{c} \leq 1 + \frac{(1+\rho)\underline{\gamma}}{\gamma - \underline{\gamma}} \equiv \underline{\pi}. \tag{5.2}$$

Also note that $\underline{\pi} > 1$ since $\gamma > \underline{\gamma}$. Therefore, when the reward of winning the race is sufficiently low $(1 \leq \pi \leq \underline{\pi})$, the firms would license the new technology as soon as it discovers. The following proposition formally summarizes the above results.

Proposition 6. *Suppose that one of the following conditions holds: (i)* $\gamma \leq \underline{\gamma}$*; or (ii)* $\gamma \in (\underline{\gamma}, \underline{\gamma})$ *and* $1 \leq \pi \leq \underline{\pi}$*. Then, there exists an equilibrium in which firms fully allocate resources to research and license the new technology as soon as they access it.*

5.2.3 No-Patenting Equilibrium

In this subsection, we explore conditions under which it is an equilibrium for firms to never patent the new technology. In this cases, the resource allocation and the expected completion time must correspond to the private information setting from Section 4.

First, we consider the case where $\tilde{\gamma} = \max\{\underline{\gamma}, 1-2\rho\} \geq \gamma > \underline{\gamma}$. By Theorem 2, in the equilibrium under the private information setting, firms do research until it succeeds ($T^* = \infty$ and $\sigma_t = 1$ for all $t \geq 0$). Suppose that both firms stick to this resource allocation strategy and never disclose their discoveries. When Firm A discovers the new technology at time t and never discloses it, the expected payoff of Firm A is $V_1^t = \{1 + \delta(1-q_t)\} \cdot V_C$ by Theorem 2 (i). If Firm A discloses the discovery at time t, Firm B has the new technology with the probability q_t. Thus, the expected payoff from the disclosure is $V_C \cdot q_t + V_L \cdot (1-q_t) = V_C + (1-q_t)x^*$. Therefore, the firm will not disclose if $x^* < \delta V_C$. We can also consider the case where Firm A discovers at time t but conceals until t' and decides to disclose or not at time t'. Even in this case, Firm A faces the same problem as before and will not disclose if $x^* < \delta V_C$. Recall that $x^* < \delta V_C$ is equivalent to $\pi > \underline{\pi}$. Therefore, if $\pi > \underline{\pi}$, there exists an equilibrium such that firms never disclose their discoveries and do research until it succeeds.

Next, we consider the case where $\gamma \in (\tilde{\gamma}, \overline{\gamma})$. By Theorem 2 (iii), in the equilibrium under private information, firms employ the stationary fall-back strategy (for some $T^* \in (0, \infty)$ and $\sigma^* \in (0,1)$, $\sigma_t = 1$ for all $0 \leq t < T^*$ and $\sigma_t = \sigma^*$ for all $t \geq T^*$). Suppose that Firm A discovers the new technology at $t \geq T^*$. If Firm A keeps the discovery secret, the expected payoff of Firm A is $V_1^* = \dfrac{2\delta}{\eta - 1 + \delta} V_C$. In addition, $V_1^* < \overline{V}_1(p^*) = \{1 + \delta(1-p^*)\} V_C$ (see Lemma B.14). On the other hand, if Firm A discloses the discovery, the expected payoff from the disclosure is $V_C + (1-p^*)x^*$. Then, Firm A does not disclose under the condition stronger than $x^* < \delta V_C$. In this case, there exists $\overline{\pi} > \underline{\pi}$ such that Firm A does not disclose when $\pi > \overline{\pi}$. The following proposition formally states this result.

Proposition 7. *Suppose that $\gamma \in (\underline{\gamma}, \overline{\gamma})$. Then, there exists $\overline{\pi} > \underline{\pi}$ such that for all $\pi > \overline{\pi}$, there is an equilibrium in which (i) firms never patent the new technology (ii) firms employ the equilibrium resource allocations from Theorem 2.*

6 Conclusion

In this article, we study the long-lasting question of patent vs. secrecy by highlighting the firm's incentives to conceal breakthroughs to hinder the rival's strategic response. To do so, we introduce an innovation race model with multiple paths and show that firms' disclosing decisions depend on the reward for winning the race.

We show that when interim breakthroughs are public, patent protection is effective in inducing a more efficient allocation of R&D resources. However, when interim breakthroughs are private and stakes are high, patent protection has a limited effect. Based on this result, we can argue that, in some situations, higher stakes may reduce patenting and licensing which would decrease the pace of innovation.

There are many avenues open for further research. For example, we assume that there are exogenously given two paths towards innovation, and one of the paths requires two breakthroughs. However, in practice, there are numerous ways to make an innovation, and it often requires more than two breakthroughs. We also assume that a firm's R&D resources are fixed over time, but we could also allow firms to endogenously choose how much effort to put into each point in time. Finally, we assume the contest structure is given by the winner-takes-all competition, but we might consider a contest designing problem. We leave these intriguing questions and others for future work.

References

Akcigit, U. and Liu, Q. (2016). The role of information in innovation and competition. *Journal of the European Economic Association*, 14(4):828–870.

Anton, J. J. and Yao, D. A. (2004). Little patents and big secrets: managing intellectual property. *RAND Journal of Economics*, pages 1–22.

Bhattacharya, S., Glazer, J., and Sappington, D. E. (1992). Licensing and the sharing of knowledge in research joint ventures. *Journal of Economic Theory*, 56(1):43–69.

Bhattacharya, S. and Guriev, S. (2006). Patents vs. trade secrets: Knowledge licensing and spillover. *Journal of the European Economic Association*, 4(6):1112–1147.

Bobtcheff, C., Bolte, J., and Mariotti, T. (2017). Researcher's dilemma. *The Review of Economic Studies*, 84(3):969–1014.

Bonatti, A. and Hörner, J. (2011). Collaborating. *American Economic Review*, 101(2):632–63.

Carnehl, C. and Schneider, J. (2022). on Risk and Time Pressure: When to Think and When to Do. *Journal of the European Economic Association*.

Coase, R. (1960). The problem of social cost. *The Journal of Law & Economics*, 3:1–44.

Das, K. and Klein, N. (2020). Do stronger patents lead to faster innovation? the effect of duplicative search.

d'Aspremont, C., Bhattacharya, S., and Gerard-Varet, L.-A. (2000). Bargaining and sharing innovative knowledge. *The Review of Economic Studies*, 67(2):255–271.

Denicolò, V. (2000). Two-stage patent races and patent policy. *the RAND Journal of Economics*, pages 488–501.

Denicolò, V. and Franzoni, L. A. (2004). Patents, secrets, and the first-inventor defense. *Journal of Economics & Management Strategy*, 13(3):517–538.

Denicolò, V. and Franzoni, L. A. (2010). On the winner-take-all principle in innovation races. *Journal of the European Economic Association*, 8(5):1133–1158.

Gallini, N. T. (1992). Patent policy and costly imitation. *The RAND Journal of Economics*, pages 52–63.

Green, B. and Taylor, C. R. (2016). Breakthroughs, deadlines, and self-reported progress: Contracting for multistage projects. *American Economic Review*, 106(12):3660–99.

Hopenhayn, H. A. and Squintani, F. (2016). Patent rights and innovation disclosure. *The Review of Economic Studies*, 83(1):199–230.

Horstmann, I., MacDonald, G. M., and Slivinski, A. (1985). Patents as information transfer mechanisms: To patent or (maybe) not to patent. *Journal of Political Economy*, 93(5):837–858.

Keller, G., Rady, S., and Cripps, M. (2005). Strategic experimentation with exponential bandits. *Econometrica*, 73(1):39–68.

Kim, Y. (2022). Managing a project by splitting it into pieces. Available at SSRN: https://ssrn.com/abstract=3450802.

Kultti, K., Takalo, T., and Toikka, J. (2007). Secrecy versus patenting. *The RAND Journal of Economics*, 38(1):22–42.

Kwon, I. (2012). Patent races with secrecy. *The Journal of Industrial Economics*, 60(3):499–516.

Lee, T. and Wilde, L. L. (1980). Market structure and innovation: A reformulation. *The Quarterly Journal of Economics*, 94(2):429–436.

Lobel, O. (2013). Filing for a patent versus keeping your invention a trade secret. *Harvard Business Review*, 21.

Loury, G. C. (1979). Market structure and innovation. *The quarterly journal of economics*, pages 395–410.

Milgrom, P. and Shannon, C. (1994). Monotone comparative statics. *Econometrica*, 62:157–180.

Neyman, A. (2017). Continuous-time stochastic games. *Games and Economic Behavior*, pages 92–130.

Scotchmer, S. and Green, J. (1990). Novelty and disclosure in patent law. *The RAND Journal of Economics*, pages 131–146.

Seierstad, A. and Sydsaeter, K. (1987). *Optimal control theory with economic applications.* Elsevier North-Holland, Inc.

Song, Y. and Zhao, M. (2021). Dynamic r&d competition under uncertainty and strategic disclosure. *Journal of Economic Behavior & Organization*, 181:169–210.

Spiegel, Y. (2008). Licensing interim r&d knowledge. Technical report.

Takalo, T. (1998). Innovation and imitation under imperfect patent protection. *Journal of Economics*, 67(3):229–241.

Templeton, B. (2019). Elon musk's war on lidar: who is right and why do they think that. *Forbes https://www. forbes. com/sites/bradtempleton/2019/05/06/elon-musks-war-on-lidar-who-is-right-and-why-do-they-think-that/7fe42c4f2a3b.*

Zhang, T. (2012). Patenting in the shadow of independent discoveries by rivals. *International Journal of Industrial Organization*, 30(1):41–49.

Appendix

A Proofs for the Public Information Setting

A.1 Useful Observations

Let τ be a random variable on $\mathbb{R}_+$. Suppose that it has a continuous and differentiable cumulative distribution function $F : \mathbb{R}_+ \to [0,1]$. Let $S(t)$ denote the survival function of τ, i.e., $S(t) = 1 - F(t)$. If $\lim_{t\to\infty} t \cdot S(t) = 0$, the following equation holds:

$$\mathbb{E}[\tau] = \int_0^\infty t \cdot F'(t) dt = -t \cdot S(t)\Big|_0^\infty + \int_0^\infty S(t) dt = \int_0^\infty S(t) dt. \tag{A.1}$$

Consider another random variable $\hat{\tau}$ independent to τ. Let $\hat{S}$ be its survival function. Observe that

$$\Pr[\tau < \hat{\tau}] = \int_0^\infty \hat{S}(t)\ dF(t) = -\int_0^\infty S'(t) \cdot \hat{S}(t)\ dt. \tag{A.2}$$

Let h be a hazard rate function of τ: $h(t) = -S'(t)/S(t)$. Then, (A.2) can be rewritten as follows:

$$\Pr[\tau < \hat{\tau}] = \int_0^\infty h(t) \cdot S(t) \cdot \hat{S}(t)\ dt. \tag{A.3}$$

Now consider another random variable which is a minimum of τ and $\hat{\tau}$, denoted by $(\tau \wedge \hat{\tau})$. Then, the survival function of $(\tau \wedge \hat{\tau})$ is $S(t) \cdot \hat{S}(t)$. By applying (A.1), when $\lim_{t\to\infty} t \cdot S(t) \cdot \hat{S}(t) = 0$, we have

$$\mathbb{E}[\tau \wedge \hat{\tau}] = \int_0^\infty S(t) \cdot \hat{S}(t)\ dt. \tag{A.4}$$

A.2 Formal Definitions of Arrival Times

Given an allocation policy $\sigma : \mathbb{R}_+ \to [0,1]$, we define the following random variables:

1. τ_L: the arrival time of successful development with the old technology;

2. τ_R: the arrival time of the new technology discovery.

Let $\Sigma_t \equiv \int_0^t \sigma_s ds$. Then, the survival functions of τ_L and τ_R are given as follows: for all $t \geq 0$,

$$S_\sigma^L(t) = e^{-\lambda_L(t-\Sigma_t)} \quad \text{and} \quad S_\sigma^R(t) = e^{-\mu\Sigma_t}. \tag{A.5}$$

In addition, the hazard rate functions can be derived as follows:

$$h_\sigma^L(t) = \lambda_L(1-\sigma_t) \quad \text{and} \quad h_\sigma^R(t) = \mu\sigma_t. \tag{A.6}$$

Intuitively, the product is developed with the old technology at the rate $h_\sigma^L(t) = \lambda_L(1-\sigma_t)$ and the new technology is discovered at the rate $h_\sigma^R(t) = \mu\sigma_t$.

A.3 Omitted Lemmas

Lemma A.1. *Suppose that Firm i and j employ allocation policies σ and $\hat{\sigma}$ at the state $\emptyset$. Let $U_{\{i\}}^i$ and $U_{\{j\}}^i$ be Firm i's continuation payoffs at the states $\{i\}$ and $\{j\}$. Then, Firm i's expected payoffs are given as follows:*

$$U_0(\sigma,\hat{\sigma}) = \int_0^\infty \left(\lambda_L(1-\sigma_t)\cdot\Pi + \mu\ \sigma_t \cdot U_{\{i\}}^i + \mu\ \hat{\sigma}_t \cdot U_{\{j\}}^i - c\right)\cdot S_{\sigma,\hat{\sigma}}^M(t)\ dt, \tag{A.7}$$

where $S_{\sigma,\hat{\sigma}}^M(t) \equiv S_\sigma^L(t)\cdot S_\sigma^R(t)\cdot S_{\hat{\sigma}}^L(t)\cdot S_{\hat{\sigma}}^R(t)$.

Proof. When any of the arrival times $\tau_L,\ \tau_R,\ \hat{\tau}_L$ and $\hat{\tau}_R$ occurs, the Firm i's payoff is realized. Furthermore, it incurs a flow cost c until one of these arrival times takes place. Thus, Firm i's expected payoff can be written as follows:

$$\begin{aligned} U_0(\sigma,\hat{\sigma}) = &\Pr[\tau_L < (\tau_R \wedge \hat{\tau}_L \wedge \hat{\tau}_R)]\cdot\Pi + \Pr[\tau_R < (\tau_L \wedge \hat{\tau}_L \wedge \hat{\tau}_R)]\cdot U_{\{i\}}^i \\ &+ \Pr[\hat{\tau}_R < (\tau_L \wedge \tau_R \wedge \hat{\tau}_L)]\cdot U_{\{j\}}^i - \mathbb{E}[(\tau_L \wedge \tau_R \wedge \hat{\tau}_L \wedge \hat{\tau}_R)]\cdot c. \end{aligned} \tag{A.8}$$

Note that the survival function of $(\tau_R \wedge \hat{\tau}_L \wedge \hat{\tau}_R)$ is $S_\sigma^R \cdot S_{\hat{\sigma}}^L \cdot S_{\hat{\sigma}}^R$. By using (A.3) and (A.6), we have

$$\Pr[\tau_L < (\tau_R \wedge \hat{\tau}_L \wedge \hat{\tau}_R)] = \int_0^\infty \lambda_L(1-\sigma_t)\cdot S_{\sigma,\hat{\sigma}}^M(t)\ dt.$$

Likewise, we can derive that

$$\Pr[\tau_R < (\tau_L \wedge \hat{\tau}_L \wedge \hat{\tau}_R)] = \int_0^\infty \mu\ \sigma_t \cdot S^M_{\sigma,\hat{\sigma}}(t)\ dt,$$
$$\Pr[\hat{\tau}_R < (\hat{\tau}_L \wedge \tau_L \wedge \tau_R)] = \int_0^\infty \mu\ \hat{\sigma}_t \cdot S^M_{\sigma,\hat{\sigma}}(t)\ dt.$$

Next, observe that the survival function of $(\tau_L \wedge \tau_R \wedge \hat{\tau}_L \wedge \hat{\tau}_R)$ is

$$S^M_{\sigma,\hat{\sigma}}(t) = e^{-\lambda_L(t-\Sigma_t)-\mu\Sigma_t-\lambda_L(t-\hat{\Sigma}_t)-\mu\Sigma_t} = e^{-2\lambda_L t-(\mu-\lambda_L)(\Sigma_t+\hat{\Sigma}_t)}.$$

Then, from $\mu \geq \lambda_L$ and $\Sigma_t + \hat{\Sigma}_t \geq 0$, we have $\lim_{t\to\infty} t \cdot S^M_{\sigma,\hat{\sigma}}(t) = 0$. By applying (A.1), we have

$$\mathbb{E}[(\tau_L \wedge \tau_R \wedge \hat{\tau}_L \wedge \hat{\tau}_R)] = \int_0^\infty S^M_{\sigma,\hat{\sigma}}(t)\ dt.$$

By plugging the above equations into (A.8), we obtain (A.7). □

Lemma A.2. *Suppose that $(x_0, y_0) \in [0,1]^2$ satisfies $x_0 \in \arg\max_{x\in[0,1]} u_0(x, y_0)$. Let $\sigma^*, \hat{\sigma}^* : \mathbb{R}_+ \to [0,1]$ be $\sigma^*_t = x_0$ and $\hat{\sigma}^*_t = y_0$ for all $t \geq 0$. Then, σ^* is a best response to $\hat{\sigma}^*$.*

Proof of Lemma A.2. Let r_t denote $S^M_{\sigma,\hat{\sigma}^*}(t)$. By taking a derivative, we have

$$\dot{r}_t = -\left\{\mu(\sigma_t + y_0) + \lambda_L(2 - \sigma_t - y_0)\right\} \cdot r_t. \tag{A.9}$$

By Lemma A.1, given Firm j's allocation profile $\hat{\sigma}^*$, Firm i's problem is

$$\max_{\sigma} \int_0^\infty \left\{\lambda_L(1-\sigma_t)\cdot\Pi + \mu\sigma_t \cdot U^i_{\{i\}} + \mu y_0 \cdot U^i_{\{j\}} - c\right\} \cdot r_t\ dt \tag{A.10}$$

subject to (A.9).

Observe that the Hamiltonian of this optimal control problem is

$$\begin{aligned} H(\sigma_t, r_t, \eta_t) =& \left\{\lambda_L(1-\sigma_t)\cdot\Pi + \mu\sigma_t \cdot U^i_{\{i\}} + \mu y_0 \cdot U^i_{\{j\}} - c\right\} \cdot r_t \\ &- \eta_t \left\{\mu(\sigma_t + y_0) + \lambda_L(2 - \sigma_t - y_0)\right\} \cdot r_t \\ =& \left\{u_0(\sigma_t, y_0) - \eta_t\right\} \cdot \left\{\mu(\sigma_t + y_0) + \lambda_L(2 - \sigma_t - y_0)\right\} \cdot r_t, \end{aligned} \tag{A.11}$$

where η_t is a co-state variable.

To show that σ^* is a solution of (A.10) subject to (A.9) by using the Arrow sufficiency condition (Seierstad and Sydsaeter, 1987, Theorem 3.14), we consider (η^*, r^*) defined as follows: for all $t \geq 0$, $\eta_t^* = u_0(x_0, y_0)$ and $r_t^* = e^{-\{\mu(x_0+y_0)+\lambda_L(2-x_0-y_0)\}\cdot t}$.

Then, we need to check following four primitive conditions:

1. Maximum principle: for all $t \geq 0$,

$$\sigma_t^* = x_0 \in \arg\max_{\sigma_t \in [0,1]} H(\sigma_t, r_t^*, \eta_t^*). \tag{A.12}$$

2. Evolution of the co-state variable:

$$\dot{\eta}_t^* = -\frac{\partial H}{\partial r_t} = -\{u_0(\sigma_t^*, y_0) - \eta_t^*\} \cdot \{\mu(\sigma_t^* + y_0) + \lambda_L(2 - \sigma_t^* - y_0)\}. \tag{A.13}$$

3. Transversality condition: If r^* is the optimal trajectory, i.e., $r_t^* = S^M_{\sigma^*,\hat{\sigma}^*}(t)$, $\lim_{t\to\infty} \eta_t^*(r_t^* - r_t) \leq 0$ for all feasible trajectories r_t.

4. $\hat{H}(r_t, \eta_t) = \max_{\sigma_t \in [0,1]} H(\sigma_t, r_t, \eta_t)$ is concave in r_t.

First, by plugging r_t^* and η_t^* into (A.11), we have

$$H(\sigma_t, r_t^*, \eta_t^*) = \{u_0(\sigma_t, y_0) - u_0(x_0, y_0)\} \cdot \{\mu(\sigma_t + y_0) + \lambda_L(2 - \sigma_t - y_0)\} \cdot r_t \tag{A.14}$$

Recall that $x_0 \in \arg\max_{x \in [0,1]} u_0(x, y_0)$. Thus, $H(\sigma_t, r_t^*, \eta_t^*) \leq 0$ for all $\sigma_t \in [0,1]$. In addition, $H(x_0, r_t^*, \eta_t^*) = 0$. Therefore, $x_0 \in \arg\max_{\sigma_t \in [0,1]} H(\sigma_t, r_t, \eta_t)$, i.e., (A.12) holds.

Second, by the definition of η^*, (A.13) holds.

Third, note that for any admissible allocation σ,

$$r_t = e^{-\{\mu(\Sigma_t + y_0 t) + \lambda_L(2t - \Sigma_t - y_0 t)\}} = r_t^* \cdot e^{(\mu - \lambda_L)\cdot(x_0 t - \Sigma_t)}.$$

Then, we have

$$\lim_{t\to\infty} \eta_t^* \cdot (r_t^* - r_t) = \lim_{t\to\infty} u_0(x_0, y_0) \cdot r_t^* \cdot \left(1 - e^{(\mu - \lambda_L)\cdot(x_0 t - \Sigma_t)}\right) = 0.$$

Last, we can see that $\hat{H}$ is linear in r_t, thus, the fourth condition holds. Hence, by the Arrow sufficiency condition, σ^* is the best response to $\hat{\sigma}^*$. $\square$

Lemma A.3. *The following equation holds:*

$$\frac{\partial u_0}{\partial x} = \mathcal{C}(x,y) \cdot \{\lambda_L \cdot \Delta_0 \cdot (1-y) + \mu \cdot \Delta_1 \cdot y\}\,, \tag{A.15}$$

where

$$\mathcal{C}(x,y) = \frac{2(\lambda_L+\mu)}{\{\mu x + \lambda_L(1-x) + \mu y + \lambda_L(1-y)\}^2} > 0.$$

Proof of Lemma A.3. Observe that

$$\begin{aligned} \Delta_0 =& \frac{\mu U^i_{\{i\}} - c}{\mu + \lambda_L} - \frac{\lambda_L \Pi - c}{2\lambda_L}, \\ \Delta_1 =& \frac{\mu(U^i_{\{i\}} + U^i_{\{j\}}) - c}{2\mu} - \frac{\lambda_L \Pi + \mu U^i_{\{j\}} - c}{\lambda_L + \mu}. \end{aligned}$$

Thus, we have

$$2(\lambda_L+\mu)\lambda_L \cdot \Delta_0 = 2\lambda_L \mu U^i_{\{i\}} - \lambda_L(\lambda_L+\mu)\Pi + (\mu - \lambda_L)c, \tag{A.16}$$

$$2(\lambda_L+\mu)\mu \cdot \Delta_1 = (\lambda_L+\mu)\mu U^i_{\{i\}} - (\mu-\lambda_L)\mu U^i_{\{j\}} - 2\lambda_L \mu \Pi + (\mu - \lambda_L)c, \tag{A.17}$$

and

$$2(\lambda_L+\mu)\mu \cdot \Delta_1 - 2(\lambda_L+\mu)\lambda_L \cdot \Delta_0 = (\mu - \lambda_L)\left(\mu U^i_{\{i\}} - \mu U^i_{\{j\}} - \lambda_L \Pi\right).$$

Also note that

$$\frac{\partial u_0}{\partial x} = \frac{NUM_0}{\{\mu x + \lambda_L(1-x) + \mu y + \lambda_L(1-y)\}^2}$$

where

$$\begin{aligned} NUM_0 =& (\mu U^i_{\{i\}} - \lambda_L \Pi) \cdot (\mu x + \lambda_L(1-x) + \mu y + \lambda_L(1-y)) \\ &- (x\mu U^i_{\{i\}} + (1-x)\lambda_L \Pi + y\mu U^i_{\{j\}} - c) \cdot (\mu - \lambda_L). \end{aligned}$$

With some algebra, we can show that

$$
\begin{aligned}
NUM_0 = & \left\{2\lambda_L \mu U^i_{\{i\}} - \lambda_L(\lambda_L+\mu)\Pi + (\mu-\lambda_L)c\right\} \\
& + (\mu-\lambda_L)(\mu U^i_{\{i\}} - \mu U^i_{\{j\}} - \lambda_L \Pi)y \\
= & 2(\lambda_L+\mu)\lambda_L \cdot \Delta_0 + (2(\lambda_L+\mu)\mu \cdot \Delta_1 - 2(\lambda_L+\mu)\lambda_L \cdot \Delta_0) \cdot y.
\end{aligned}
$$

By plugging this in, we can show that (A.15) holds. □

Lemma A.4. *In any MPE, when the state is $\omega = \{A, B\}$, both firms develop with the new technology ($\mathbf{s}^A(\{A,B\}) = \mathbf{s}^B(\{A,B\}) = 1$). In addition, the expected payoffs for both firms under this state is*

$$U^A_{\{A,B\}} = U^B_{\{A,B\}} = V_C \equiv \frac{1}{2}\left(\Pi - \frac{c}{\lambda_H}\right) = \frac{c}{2\lambda_L}(\pi-\rho). \tag{A.18}$$

Proof of Lemma A.4. Suppose that Firm j allocates y units of resources toward developing with the new technology, and $1-y$ units of resources toward developing with the old technology. Then, Firm j develops the product at rate $\hat{\lambda} \equiv \lambda_H y + \lambda_L(1-y)$. Then, Firm i's problem is to choose the resource allocation towards developing with the new technology $x \in [0,1]$ to maximize his continuation value: $U^i_{\{j\}} = \max_{x\in[0,1]} u_2(x)$ where

$$u_2(x) \equiv \frac{x\lambda_H \Pi + (1-x)\lambda_L \Pi - c}{x\lambda_H + (1-x)\lambda_L + \hat{\lambda}}.$$

By taking a derivative, we have

$$u_2'(x) = \frac{(\lambda_H - \lambda_L)(\hat{\lambda}\Pi + c)}{(x\lambda_H + (1-x)\lambda_L + \hat{\lambda})^2}.$$

Thus, we have $u_2'(x) > 0$. Therefore, $x = 1$ maximizes Firm i's expected payoff regardless of Firm j's strategy, i.e., Firm i develops with the new technology. Likewise, Firm j develops with the new technology. Then, since both firms choose $x = 1$ under this state, and the expected payoffs are $\frac{\lambda_H \Pi - c}{2\lambda_H} = V_C$. □

Lemma A.5. *In any MPE, when the state is $\omega = \{i\}$, Firm i develops with the new technology ($\mathbf{s}^i(\{i\}) = 1$).*

Proof of Lemma A.5. Suppose that Firm j allocates y units of resources toward doing research for the new technology, and $1-y$ units of resources toward developing with the old technology. Then, Firm i's problem is to choose $x \in [0,1]$ to maximize his continuation value: $U^i_{\{j\}} = \max_{x\in[0,1]} u_1^w(x)$ where

$$u_1^w(x) \equiv \frac{x\lambda_H\Pi + (1-x)\lambda_L\Pi + y\mu V_C - c}{x\lambda_H + (1-x)\lambda_L + y\mu + (1-y)\lambda_L}.$$

By taking a derivative, we have

$$u_1^{w\prime}(x) = \frac{(\lambda_H - \lambda_L)\cdot\left\{c + \lambda_L\Pi(1-y) + \mu\left(\frac{\lambda_H\Pi + c}{2\lambda_H}\right)y\right\}}{(x\lambda_H + (1-x)\lambda_L + y\mu + (1-y)\lambda_L)^2}.$$

Therefore, we have $u_1^{w\prime}(x) > 0$. Therefore, $x = 1$ maximizes Firm i's expected payoff regardless of Firm j's strategy, i.e., Firm i develops with the new technology. $\square$

Lemma A.6. *When $\gamma < \underline{\gamma}$, the equations (3.4) and (3.5) hold.*

Proof of Lemma A.6. By plugging (3.2) into (A.16),

$$\begin{aligned} 2(\lambda_L + \mu)\lambda_L \cdot \Delta_0 =& 2\lambda_L\mu \cdot U^i_{\{i\}} - \lambda_L(\lambda_L + \mu)\cdot\Pi + (\mu - \lambda_L)\cdot c \\ =& \frac{\lambda_L\mu}{\lambda_H}\cdot\frac{2\lambda_H + \mu}{\lambda_H + \mu}\cdot(\lambda_H\Pi - c) - \lambda_L(\lambda_L + \mu)\cdot\Pi + (\mu - \lambda_L)c. \end{aligned}$$

By using (2.2), we have

$$\begin{aligned} 2(1+\gamma)\lambda_L\mu\cdot\Delta_0 =& \left[\left(\frac{2\gamma + \rho}{\gamma + \rho}\right)(\pi - \rho) - (1+\gamma)\pi + (1-\gamma)\right]\mu c \\ =& \frac{\mu c}{\rho + \gamma}\left[(\pi + 1)\gamma(1 - \rho - \gamma) + \rho(1 - \rho - 2\gamma)\right]. \end{aligned}$$

Then, from $\underline{\gamma} = (1-\rho)/2$, (3.4) holds.

Next, by plugging (3.2) into (A.17),

$$\begin{aligned}2(\lambda_L+\mu)\mu\cdot\Delta_1 =&(\lambda_L+\mu)\mu\cdot U^i_{\{i\}}-(\mu-\lambda_L)\mu\cdot U^i_{\{j\}}-2\lambda_L\mu\cdot\Pi+(\mu-\lambda_L)\cdot c\\ =&\mu\frac{2(\lambda_H\lambda_L+\lambda_H\mu+\lambda_L\mu)}{\mu+\lambda_H}\cdot\frac{\lambda_H\Pi-c}{2\lambda_H}-2\lambda_L\mu\cdot\Pi+(\mu-\lambda_L)\frac{2\mu+\lambda_H}{\mu+\lambda_H}\cdot c.\end{aligned}$$

By using (2.2), we have

$$\begin{aligned}2(1+\gamma)\mu^2\cdot\Delta_1 =&\left[\frac{(\gamma+\rho+1)(\pi-\rho)}{\rho+\gamma}-2\pi+\frac{(1-\gamma)(2\rho+\gamma)}{\rho+\gamma}\right]\mu c\\ =&\frac{\mu c}{\rho+\gamma}\cdot\left[\gamma(1-\rho-\gamma)+\pi\gamma+(\pi+\rho)(1-\rho-2\gamma)\right].\end{aligned}$$

Then, from $\underline{\gamma}=(1-\rho)/2$ and $\gamma=\lambda_L/\mu$, (3.5) holds. $\square$

Lemma A.7. *When $\gamma>\underline{\gamma}$, the equations (3.6) and (3.7) hold.*

Proof of Lemma A.7. By plugging (3.3) into (A.16),

$$\begin{aligned}2(\lambda_L+\mu)\lambda_L\cdot\Delta_0 =&2\lambda_L\mu\cdot U^i_{\{i\}}-\lambda_L(\lambda_L+\mu)\cdot\Pi+(\mu-\lambda_L)\cdot c\\ =&\frac{2\lambda_L\mu}{\lambda_L+\lambda_H}(\lambda_H\Pi-c)-\lambda_L(\lambda_L+\mu)\cdot\Pi+(\mu-\lambda_L)c.\end{aligned}$$

With some algebra and (2.2), we have

$$\begin{aligned}2(1+\gamma)\lambda_L\mu\cdot\Delta_0 =&\left[\frac{2}{\rho+1}(\pi-\rho)-(1+\gamma)\pi+(1-\gamma)\right]\mu c\\ =&\frac{\{1-\rho-(1+\rho)\gamma\}(\pi+1)\mu c}{1+\rho}.\end{aligned}$$

Then, from $\overline{\gamma}=(1-\rho)/(1+\rho)$, (3.6) holds.

Next, by plugging (3.3) into (A.17),

$$\begin{aligned}2(\lambda_L+\mu)\mu\cdot\Delta_1 =&(\lambda_L+\mu)\mu\cdot U^i_{\{i\}}-(\mu-\lambda_L)\mu\cdot U^i_{\{j\}}-2\lambda_L\mu\cdot\Pi+(\mu-\lambda_L)\cdot c\\ =&(\lambda_L+\mu)\mu\cdot\frac{\lambda_H\Pi-c}{\lambda_L+\lambda_H}-(\mu-\lambda_L)\mu\cdot\frac{\lambda_L\Pi-c}{\lambda_L+\lambda_H}-2\lambda_L\mu\cdot\Pi+(\mu-\lambda_L)\cdot c.\end{aligned}$$

With some algebra and (2.2), we have

$$
\begin{aligned}
2(1+\gamma)\mu^2 \cdot \Delta_1 &= \left[\frac{1-\rho-(1+\rho)\gamma}{\gamma(\rho+1)}\pi + 1 - \gamma - \frac{2\rho}{1+\rho}\right]\mu c \\
&= \frac{\{1-\rho-(1+\rho)\gamma\}(\pi+\gamma)\mu c}{\gamma(1+\rho)}.
\end{aligned}
$$

Then, from $\overline{\gamma} = (1-\rho)/(1+\rho)$ and $\gamma = \lambda_L/\mu$, (3.7) holds. □

A.4 Omitted Proofs

Proof of Proposition 1. (a) When $\Delta_0, \Delta_1 > 0$, from (A.15), $\frac{\partial u_0}{\partial x} > 0$ for all $y \in [0,1]$, i.e., $x = 1$ is optimal. Thus, both firms play $\mathbf{s}(\emptyset) = 1$ in any MPE.

(b) When $\Delta_0, \Delta_1 < 0$, from (A.15), $\frac{\partial u_0}{\partial x} < 0$ for all $y \in [0,1]$, i.e., $x = 0$ is optimal. Thus, both firms play $\mathbf{s}(\emptyset) = 0$ in any MPE.

(c) From $\Delta_0 > 0$ and (A.15), we have $\frac{\partial u_0}{\partial x}|_{y=0} > 0$, i.e., $x = 1$ is the best response for $y = 0$. In addition, from $0 > \Delta_1$ and (A.15), we have $\frac{\partial u_0}{\partial x}|_{y=1} < 0$, i.e., $x = 0$ is the best response for $y = 1$. Therefore, $(1,0)$ and $(0,1)$ can be supported equilibrium allocations at $\omega = \emptyset$.

Next, note that $z^* \in (0,1)$ and $\frac{\partial u_0}{\partial x}|_{y=z^*} = 0$, i.e., any $x \in [0,1]$ is the best response for $y = z^*$. Thus, (z^*, z^*) can be supported as an equilibrium allocation.

Last, consider any $\tilde{y} \in (0,1)$ with $\tilde{y} \neq z^*$. Then, $\frac{\partial u_0}{\partial x}|_{y=\tilde{y}} \neq 0$, i.e., the best response is $x = 1$ or $x = 0$. Recall that the best response of $x = 1$ ($x = 0$) is $y = 0$ ($y = 1$), thus, $y = \tilde{y}$ cannot be a part of an equilibrium allocation.

(d) From $\Delta_0 < 0$ and (A.15), we have $\frac{\partial u_0}{\partial x}|_{y=0} < 0$, i.e., $x = 0$ is the best response for $y = 0$. Thus, $(0,0)$ can be supported as an equilibrium allocation.

Similarly, from $0 < \Delta_1$ and (A.15), we have $\frac{\partial u_0}{\partial x}|_{y=1} > 0$, i.e., $x = 1$ is the best response for $y = 1$. Therefore, $(1,1)$ can also be supported as an equilibrium allocation.

Next, note that $z^* \in (0,1)$ and $\frac{\partial u_0}{\partial x}|_{y=z^*} = 0$, i.e., any $x \in [0,1]$ is the best response for $y = z^*$. Thus, (z^*, z^*) can be supported as an equilibrium allocation.

Last, by using the similar argument as in the previous case, $\tilde{y} \in (0,1)$ with $\tilde{y} \neq z^*$ cannot be a part of an equilibrium allocation.

□

Proof of Lemma 3.1. Let $\omega = \{i\}$, i.e., only firm i possesses the new technology. By Lemma A.5, Firm i develops with the new technology. For firm j, the problem is to choose $x \in [0,1]$ to maximize his continuation value: $U^j_{\{i\}} = \max_{x\in[0,1]} u_1(x)$ where

$$u_1(x) \equiv \frac{x\mu V_C + (1-x)\lambda_L \Pi_L - c}{x\mu + (1-x)\lambda_L + \lambda_H} = \frac{\rho c}{2\lambda_L} \cdot \frac{x(\pi + \phi - \rho) + 2\{(1-x)\pi - 1\}\gamma}{x\rho(1-\gamma) + \gamma(1+\rho)}. \tag{A.19}$$

With some algebra, we can derive that

$$u_1'(x) = \frac{\rho\gamma c}{2\lambda_L} \cdot \frac{(\rho+\pi)(1-\rho) + \phi(1+\rho) - 2\gamma(\rho+\pi)}{(x\rho(1-\gamma) + \gamma(1+\rho))^2}. \tag{A.20}$$

Therefore, $u_1'(x) > 0$ if and only if $\underline{\gamma} > \gamma$. Hence, if $\gamma > \underline{\gamma}$, $x = 0$ is optimal, and if $\gamma < \underline{\gamma}$, $x = 1$ is optimal. □

B Proofs for the Private Information Setting

B.1 Preliminaries

Given an allocation policy σ, we define the following two arrival times in addition to τ_L and τ_R defined in Section A.2:

1. τ_H: the arrival time of successful development with the new technology;
2. $\tau_D = \tau_L \wedge \tau_H$: the arrival time of successful development with either technology.

Let $\tau_M \equiv \tau_L \wedge \tau_R$ and note that the survival function of τ_M is:

$$S^M_\sigma(t) \equiv \Pr[\tau_M > t] = e^{-\lambda_L(t-\Sigma_t)-\mu\Sigma_t}. \tag{B.1}$$

Since $\tau_H \geq \tau_R$, $(\tau_D \wedge \tau_R) > t$ is equivalent to $(\tau_M \wedge \tau_R) > t$, thus, $S_\sigma^M(t) = \Pr[\tau_D \wedge \tau_R > t]$. In addition, we implicitly assume that once a firm discovers the new technology, the firm develops with it. From this, for any $t > s$,

$$\Pr[\tau_D > t \mid \tau_R = s < \tau_L] = \Pr[\tau_H > t \mid \tau_R = s < \tau_L] = e^{-\lambda_H (t-s)}.$$

Then, we can derive that

$$L_\sigma(t) \equiv \Pr[\tau_D > t > \tau_R] = \int_0^t \mu\sigma_s \cdot S_\sigma^M(s) \cdot e^{-\lambda_H(t-s)} ds. \tag{B.2}$$

Then, the survival function of τ_D can be written as follows:

$$S_\sigma^D(t) \equiv \Pr[\tau_D > t] = S_\sigma^M(t) + L_\sigma(t). \tag{B.3}$$

In addition, by plugging (B.2) and (B.3) into (4.1), we have

$$p_\sigma(t) = \frac{L_\sigma(t)}{S_\sigma^D(t)} = \frac{L_\sigma(t)}{S_\sigma^M(t) + L_\sigma(t)}. \tag{B.4}$$

Note that

$$S_\sigma^{M\prime}(t) = -\left\{\lambda_L(1-\sigma_t) + \mu\sigma_t\right\} \cdot S_\sigma^M(t), \tag{B.5}$$

$$L_\sigma'(t) = \mu\sigma_t \cdot S_\sigma^M(t) - \lambda_H \cdot L_\sigma(t) \tag{B.6}$$

Then, the hazard rate of τ_D can be derived as follows:

$$\begin{aligned} h_\sigma^D(t) = -\frac{S_\sigma^{D\prime}(t)}{S_\sigma^D(t)} &= \frac{\lambda_L(1-\sigma_t) \cdot S_\sigma^M(t) + \lambda_H \cdot L_\sigma(t)}{S_\sigma^M(t) + L_\sigma(t)} \\ &= \lambda_L(1-\sigma_t) \cdot (1 - p_\sigma(t)) + \lambda_H \cdot p_\sigma(t). \end{aligned} \tag{B.7}$$

By using (A.3), (A.4) and (B.7), we can show that the expected payoff of Firm i takes a form of (4.3).

B.2 Evolution of Beliefs

Proof of Proposition 2. From (B.4), we can derive that

$$\log(p_\sigma(t)) - \log(1 - p_\sigma(t)) = \log(L_\sigma(t)) - \log(S_\sigma^M(t)).$$

By differentiating this equation and using (B.5) and (B.6), we have

$$\begin{aligned}\frac{p'_\sigma(t)}{p_\sigma(t)(1-p_\sigma(t))} =& \frac{L'_\sigma(t)}{L_\sigma(t)} - \frac{S_\sigma^{M\prime}(t)}{S_\sigma^M(t)} = \mu\sigma_t \cdot \frac{S_\sigma^M(t)}{L_\sigma(t)} - \lambda_H + \{\lambda_L(1-\sigma_t) + \mu\sigma_t\} \\ =& \mu\sigma_t \cdot \frac{1 - p_\sigma(t)}{p_\sigma(t)} - \lambda_H + \{\lambda_L(1-\sigma_t) + \mu\sigma_t\}\end{aligned}$$

By multiplying this equation to $p_\sigma(t)$, we can see that (4.2) holds. □

Lemma B.1. *Suppose that a firm follows an allocation policy σ, with $\sigma_s = 1$ for $s \in [0, t)$. Then, the conditional probability $p_\sigma(t)$ of having access to the technology by time t given that the race is ongoing is:*

$$p_\sigma(t) = q_t \equiv \begin{cases} \dfrac{\frac{1}{\lambda_H}\left(e^{-\mu t} - e^{-\lambda_H t}\right)}{\frac{1}{\mu}e^{-\mu t} - \frac{1}{\lambda_H}e^{-\lambda_H t}}, & \text{if } \mu \neq \lambda_H, \\ \dfrac{\mu t}{1+\mu t}, & \text{if } \mu = \lambda_H. \end{cases} \tag{B.8}$$

In addition, q is increasing, with $\lim_{t\to\infty} q_t = \min\{1, \mu/\lambda_H\}$.

Proof of Lemma B.1. By plugging $\sigma_t = 1$ to (4.2), we have $p'_\sigma(t) = (\mu - \lambda_H p_\sigma(t))(1 - p_\sigma(t))$. By rearranging the differential equation, we can derive that

$$\begin{cases} \lambda_H - \mu = \dfrac{d}{dt}\log\left(\dfrac{\lambda_H - \lambda_H p_\sigma(t)}{\mu - \lambda_H p_\sigma(t)}\right), & \text{if } \mu \neq \lambda_H, \\ \mu = \dfrac{d}{dt}\dfrac{1}{1 - p_\sigma(t)}, & \text{if } \mu = \lambda_H. \end{cases}$$

Then, from $p_\sigma(0) = 0$, we can derive that

$$\begin{cases} \dfrac{\lambda_H(1-p_\sigma(t))}{\mu - \lambda_H p_\sigma(t)} = \dfrac{\lambda_H}{\mu} e^{(\lambda_H-\mu)t}, & \text{if } \mu \neq \lambda_H, \\ \dfrac{1}{1-p_\sigma(t)} - 1 = \mu t, & \text{if } \mu = \lambda_H. \end{cases}$$

By rearranging the above equation, we have (B.8).

Observe that

$$\dot{q}_t = \begin{cases} \dfrac{\mu(\lambda_H-\mu)^2 e^{(\lambda_H+\mu)t}}{(\lambda_H e^{\lambda_H t} - \mu e^{\mu t})^2} > 0, & \text{if } \mu \neq \lambda_H, \\ \dfrac{\mu}{(1+\mu t)^2} > 0, & \text{if } \mu = \lambda_H. \end{cases}$$

Thus, q is increasing in t.

When $\mu > \lambda_H$,

$$\lim_{t\to\infty} q_t = \lim_{t\to\infty} \frac{\frac{1}{\lambda_H}\left(e^{(\lambda_H-\mu)t} - 1\right)}{\frac{1}{\mu}e^{(\lambda_H-\mu)t} - \frac{1}{\lambda_H}} = 1.$$

When $\mu < \lambda_H$,

$$\lim_{t\to\infty} q_t = \lim_{t\to\infty} \frac{\frac{1}{\lambda_H}\left(1 - e^{(\mu-\lambda_H)t}\right)}{\frac{1}{\mu} - \frac{1}{\lambda_H}e^{(\mu-\lambda_H)t}} = \frac{\mu}{\lambda_H}.$$

When $\mu = \lambda_H$,

$$\lim_{t\to\infty} q_t = \lim_{t\to\infty} \frac{\mu t}{1+\mu t} = 1 = \frac{\mu}{\lambda_H}.$$

$\square$

B.3 Recursive Formulation

B.3.1 Reformulations of Value Functions

Lemma B.2. *Let $V_1(t;\hat{\sigma})$ be the continuation payoff of Firm i at time t where Firm i discovers the new technology at time t, neither firm succeeds in development by then, and*

Firm j employs the allocation policy $\hat{\sigma}$. Then, $V_1(t;\hat{\sigma})$ can be written as follows:

$$\begin{aligned} V_1(t;\hat{\sigma}) =& (\lambda_H \Pi - c) \cdot \int_t^\infty e^{-\lambda_H(s-t)} \cdot \frac{S^D_{\hat{\sigma}}(s)}{S^D_{\hat{\sigma}}(t)} ds \\ =& (\lambda_H \Pi - c) \cdot \int_t^\infty e^{-\int_t^s (\lambda_H + h^D_{\hat{\sigma}}(u))du} ds. \end{aligned} \tag{B.9}$$

In addition, the following differential equation holds:

$$0 = V_1'(t;\hat{\sigma}) + (\lambda_H \Pi - c) - (\lambda_H + h^D_{\hat{\sigma}}(t)) \cdot V_1(t;\hat{\sigma}). \tag{HJB$_1$}$$

Proof. Note that the continuation payoffs can be written as follows.

$$\begin{aligned} V_1(t;\hat{\sigma}) =& \Pr[\tau_D < \hat{\tau}_D \mid \tau_R = t < (\tau_L \wedge \hat{\tau}_D)] \cdot \Pi \\ & - c \cdot \mathbb{E}[\tau_D \wedge \hat{\tau}_D - t \mid \tau_R = t < (\tau_L \wedge \hat{\tau}_D)]. \end{aligned} \tag{B.10}$$

Given that Firm i discovers the new technology at time t and the development has not made by then, the product will never be developed with the old technology by Firm i, i.e., $\tau_L = \infty$. Thus, under this circumstance, τ_D is equal to τ_H. Note that (conditional) survival functions of $\hat{\tau}_D$ and τ_D $(= \tau_H)$ can be written as follows:

$$\begin{aligned} \Pr[\hat{\tau}_D > s \mid \tau_R = t < (\tau_L \wedge \hat{\tau}_D)] =& \frac{S^D_{\hat{\sigma}}(s)}{S^D_{\hat{\sigma}}(t)}, \\ \Pr[\tau_D = \tau_H > s \mid \tau_R = t < (\tau_L \wedge \hat{\tau}_D)] =& e^{-\lambda_H(s-t)}. \end{aligned}$$

By using this, we have

$$\begin{aligned} \Pr[\tau_D < \hat{\tau}_D \mid \tau_R = t < (\tau_L \wedge \hat{\tau}_D)] &= \int_t^\infty \lambda_H e^{-\lambda_H(s-t)} \cdot \frac{S^D_{\hat{\sigma}}(s)}{S^D_{\hat{\sigma}}(t)} ds, \\ \mathbb{E}[\tau_D \wedge \hat{\tau}_D - t \mid \tau_R = t < (\tau_L \wedge \hat{\tau}_D)] &= \int_t^\infty e^{-\lambda_H(s-t)} \cdot \frac{S^D_{\hat{\sigma}}(s)}{S^D_{\hat{\sigma}}(t)} ds. \end{aligned}$$

By plugging these equations into (B.10), we can derive the first equation of (B.9). From

$h^D_{\hat{\sigma}}(t) = -S^{D\prime}_{\hat{\sigma}}(t)/S^D_{\hat{\sigma}}(t)$ and $S^D_{\hat{\sigma}}(0) = 1$, we have $S^D_{\hat{\sigma}}(t) = e^{-\int_0^t h^D_{\hat{\sigma}}(u)du}$. Note that

$$e^{-\lambda_H(s-t)} \cdot \frac{S^D_{\hat{\sigma}}(s)}{S^D_{\hat{\sigma}}(t)} = e^{-\lambda_H(s-t)-\int_s^t h^D_{\hat{\sigma}}(u)du} = e^{-\int_t^s(\lambda_H + h^D_{\hat{\sigma}}(u))du},$$

which gives the second equation of (B.9).

By taking a derivative of (B.9), we have

$$\begin{aligned} V_1'(t;\hat{\sigma}) =& -(\lambda_H\Pi - c) \cdot e^{-\int_t^t(\lambda_H + h^D_{\hat{\sigma}}(u))du} \\ & + (\lambda_H + h^D_{\hat{\sigma}}(t)) \cdot (\lambda_H\Pi - c) \cdot \int_t^\infty e^{-\int_t^s(\lambda_H + h^D_{\hat{\sigma}}(u))du} ds \\ =& -(\lambda_H\Pi - c) + (\lambda_H + h^D_{\hat{\sigma}}(t)) \cdot V_1(t;\hat{\sigma}), \end{aligned}$$

which is equivalent to (HJB_1). □

Lemma B.3. *When Firm i and j employ allocation policies σ and $\hat{\sigma}$, Firm i's continuation payoff at time t where neither research nor development has been completed by Firm i by then can be written as follows:*

$$V_0(t;\sigma,\hat{\sigma}) = \int_t^\infty [\lambda_L(1-\sigma_s)\cdot\Pi + \mu\sigma_s \cdot V_1(s;\hat{\sigma}) - c] \cdot \frac{S^M_\sigma(s)}{S^M_\sigma(t)} \cdot \frac{S^D_{\hat{\sigma}}(s)}{S^D_{\hat{\sigma}}(t)}\ ds. \tag{B.11}$$

Therefore, Firm i's expected payoff at time 0 can be rewritten as follows:

$$\mathcal{U}(\sigma,\hat{\sigma}) = V_0(0;\sigma,\hat{\sigma}) = \int_0^\infty [\lambda_L(1-\sigma_t)\cdot\Pi + \mu\sigma_t \cdot V_1(t;\hat{\sigma}) - c] \cdot S^M_\sigma(t) \cdot S^D_{\hat{\sigma}}(t)\ dt. \tag{B.12}$$

In addition, the following differential equation holds:

$$\begin{aligned} 0 = V_0'(t;\sigma,\hat{\sigma}) + \lambda_L(1-\sigma_t)\cdot\Pi + \mu\sigma_t \cdot V_1(t;\hat{\sigma}) - c \\ - \left\{h^M_\sigma(t) + h^D_{\hat{\sigma}}(t)\right\} \cdot V_0(t;\sigma,\hat{\sigma}). \end{aligned} \tag{HJB_0}$$

Proof. We focus on the event such that $(\tau_L \wedge \tau_R \wedge \hat{\tau}_D) = (\tau_M \wedge \hat{\tau}_D) > t$. The continuation payoff can be written as follows:

$$
\begin{aligned}
V_0(t;\sigma,\hat{\sigma}) = & \Pr[\tau_D < \hat{\tau}_D \mid (\tau_M \wedge \hat{\tau}_D) > t] \cdot \Pi \\
& - c \cdot \mathbb{E}[\tau_D \wedge \hat{\tau}_D - t \mid (\tau_M \wedge \hat{\tau}_D) > t].
\end{aligned}
\tag{B.13}
$$

Note that

$$
\begin{aligned}
\Pr[\tau_M > s \mid \tau_M > t] = & \frac{S_\sigma^M(s)}{S_\sigma^M(t)}, \\
\Pr[\tau_D > s > \tau_R > t \mid \tau_M > t] = & \int_t^s e^{-\lambda_H(s-u)} \cdot \mu\sigma_u \cdot \frac{S_\sigma^M(u)}{S_\sigma^M(t)} du = \frac{L_\sigma(s|t)}{S_\sigma^M(t)},
\end{aligned}
$$

where $L_\sigma(s|t) \equiv \int_t^s e^{-\lambda_H(s-u)} \cdot \mu\sigma_u \cdot S_\sigma^M(u)du$. Then, the survival function of τ_D conditional on $\tau_M > t$ can be written as follows:

$$
S_{\sigma|_t}^D(s) \equiv \Pr\left[\tau_D > s \mid \tau_M > t\right] = \frac{S_\sigma^M(s) + L_\sigma(s|t)}{S_\sigma^M(t)}
$$

Also note that $\Pr[\hat{\tau}_D > s \mid \hat{\tau}_D > t] = S_{\hat{\sigma}}^D(s)/S_{\hat{\sigma}}^D(t)$.

Observe that

$$
L'_\sigma(s|t) = \mu\sigma_s \cdot S_\sigma^M(s) - \lambda_H \cdot L_\sigma(s|t). \tag{B.14}
$$

Since τ_D and $\hat{\tau}_D$ are independent, we can apply (A.2) and (A.4) by resetting the initial time to t. Then, by using (B.5) and (B.14), we have

$$
\begin{aligned}
\Pr[\tau_D < \hat{\tau}_D \mid (\tau_M \wedge \hat{\tau}_D) > t] = & -\int_t^\infty {S_{\sigma|_t}^D}'(s) \cdot \frac{S_{\hat{\sigma}}^D(s)}{S_{\hat{\sigma}}^D(t)} ds \\
= & -\int_t^\infty \frac{{S_\sigma^M}'(s) + L'_\sigma(s|t)}{S_\sigma^M(t)} \cdot \frac{S_{\hat{\sigma}}^D(s)}{S_{\hat{\sigma}}^D(t)} ds \\
= & \int_t^\infty \frac{\lambda_L(1-\sigma_s) \cdot S_\sigma^M(s) + \lambda_H \cdot L_\sigma(s|t)}{S_\sigma^M(t)} \cdot \frac{S_{\hat{\sigma}}^D(s)}{S_{\hat{\sigma}}^D(t)} ds, \\
\mathbb{E}[\tau_D \wedge \hat{\tau}_D - t \mid (\tau_M \wedge \hat{\tau}_D) > t] = & \int_t^\infty \frac{S_\sigma^M(s) + L_\sigma(s|t)}{S_\sigma^M(t)} \cdot \frac{S_{\hat{\sigma}}^D(s)}{S_{\hat{\sigma}}^D(t)} ds.
\end{aligned}
$$

By plugging these into (B.13) and using (B.7), we can derive that

$$V_0(t;\sigma,\hat{\sigma}) = \frac{1}{S_\sigma^M(t) S_{\hat{\sigma}}^D(t)} \cdot \int_t^\infty \left[\{\lambda_L(1-\sigma_s)\Pi - c\} \cdot S_\sigma^M(s) + (\lambda_H \Pi - c) \cdot L_\sigma(s|t)\right] \cdot S_{\hat{\sigma}}^D(s) ds.$$

Thus, it remains to show that

$$\int_t^\infty \mu\sigma_s \cdot V_1(s;\hat{\sigma}) \cdot S_\sigma^M(s) \cdot S_{\hat{\sigma}}^D(s)\ ds = (\lambda_H \Pi - c) \cdot \int_0^\infty L_\sigma(s|t) \cdot S_{\hat{\sigma}}^D(s)\ ds. \tag{B.15}$$

By plugging (B.9) into the left hand side of (B.15), we have

$$\begin{aligned}
&\int_t^\infty \mu\sigma_s \left[\int_s^\infty (\lambda_H \Pi - c) \cdot e^{-\lambda_H(u-s)} \cdot S_{\hat{\sigma}}^D(u) du\right] \cdot S_\sigma^M(s)\ ds \\
=&(\lambda_H \Pi - c) \cdot \int_t^\infty \left[\int_s^\infty \mu\sigma_s \cdot e^{-\lambda_H(u-s)} \cdot S_{\hat{\sigma}}^D(s) \cdot S_\sigma^M(u)\ du\right]\ ds \\
=&(\lambda_H \Pi - c) \cdot \int_t^\infty \left[\int_t^u \mu\sigma_u \cdot e^{-\lambda_H(u-s)} \cdot S_\sigma^M(s)\ ds\right] \cdot S_{\hat{\sigma}}^D(u)\ du \\
=&(\lambda_H \Pi - c) \cdot \int_t^\infty L_\sigma(u|t) \cdot S_{\hat{\sigma}}^D(u)\ du.
\end{aligned}$$

Thus, (B.11) holds. When $t = 0$, we have $S_\sigma^M(0) = S_{\hat{\sigma}}^D(0) = 1$, thus, $\mathcal{U}(\sigma,\hat{\sigma})$ can be rewritten as in (B.12).

Last, to show that (HJB_0) holds, we multiply $S_\sigma^M(t) \cdot S_{\hat{\sigma}}^D(t)$ to (B.11) and take a derivative:

$$\begin{aligned}
&-\left[\lambda_L(1-\sigma_t) \cdot \Pi + \mu\sigma_t \cdot V_1(t;\hat{\sigma}) - c\right] \cdot S_\sigma^M(t) \cdot S_{\hat{\sigma}}^D(t) \\
&= \frac{d}{dt}\left[V_0(t;\sigma,\hat{\sigma}) \cdot S_\sigma^M(t) \cdot S_{\hat{\sigma}}^D(t)\right] \\
&= \left[V_0'(t;\sigma,\hat{\sigma}) - (h_\sigma^M(t) + h_{\hat{\sigma}}^D(t)) \cdot V_0(t;\sigma,\hat{\sigma})\right] \cdot S_\sigma^M(t) \cdot S_{\hat{\sigma}}^D(t).
\end{aligned}$$

Since $S_\sigma^M(t) \cdot S_{\hat{\sigma}}^D(t) > 0$, by dividing the above equation by this term, we can see that (HJB_0) holds. □

B.3.2 Optimality Condition

Proposition 8. *An allocation policy σ^* is a best response to the rival's allocation policy $\hat{\sigma}$, i.e., $\sigma^* \in \arg\max_\sigma \mathcal{U}(\sigma,\hat{\sigma})$, if and only if $V_0(t;\sigma^*,\hat{\sigma}) > 0$ for all $t \geq 0$, and the following* *inequality holds for any allocation policy σ and time $t \geq 0$:*

$$(\sigma_t^* - \sigma_t) \cdot [\mu \cdot (V_1(t;\hat{\sigma}) - V_0(t;\sigma^*,\hat{\sigma})) - \lambda_L \cdot (\Pi - V_0(t;\sigma^*,\hat{\sigma}))] \geq 0, \tag{B.16}$$

or equivalently,

$$\sigma_t^* = \begin{cases} 1, & \text{if } \mu(V_1(t;\hat{\sigma}) - V_0(t;\sigma^*,\hat{\sigma})) > \lambda_L(\Pi - V_0(t;\sigma^*,\hat{\sigma})), \\ \in [0,1], & \text{if } \mu(V_1(t;\hat{\sigma}) - V_0(t;\sigma^*,\hat{\sigma})) = \lambda_L(\Pi - V_0(t;\sigma^*,\hat{\sigma})), \\ 0, & \text{if } \mu(V_1(t;\hat{\sigma}) - V_0(t;\sigma^*,\hat{\sigma})) < \lambda_L(\Pi - V_0(t;\sigma^*,\hat{\sigma})). \end{cases} \tag{B.17}$$

We introduce two useful lemmas and then prove Proposition 8.

Lemma B.4. *For any $\hat{\sigma}$ and $t \geq 0$, the following inequalities hold:*

$$\frac{\lambda_H}{\mu + \lambda_H}\left(\Pi - \frac{c}{\lambda_H}\right) \cdot e^{-\mu t} < S_{\hat{\sigma}}^D(t) \cdot V_1(t;\hat{\sigma}) < \frac{\mu + \lambda_H}{\lambda_L + \lambda_H}\left(\Pi - \frac{c}{\lambda_H}\right) \cdot e^{-\lambda_L t}, \tag{B.18}$$

and $\lim_{t\to\infty} S_{\hat{\sigma}}^D(t) \cdot V_1(t;\hat{\sigma}) = 0.$

Proof of Lemma B.4. From $\lambda_H > \lambda_L$ and $\mu > \lambda_L$, we have

$$e^{-\mu t} \leq S_{\hat{\sigma}}^M(t) = e^{-\lambda_L(t-\hat{\Sigma}_t) - \mu\hat{\Sigma}_t} \leq e^{-\lambda_L t}, \tag{B.19}$$

$$\begin{aligned} 0 \leq L_{\hat{\sigma}}(t) &= \int_0^t \mu\hat{\sigma}_s \cdot S_{\hat{\sigma}}^M(s) \cdot e^{-\lambda_H(t-s)} ds \\ &< e^{-(\lambda_L+\lambda_H)t} \cdot \int_0^t \mu \cdot e^{\lambda_H s}\, ds < \frac{\mu}{\lambda_H} e^{-\lambda_L t}. \end{aligned} \tag{B.20}$$

Note that the left inequality of (B.19) binds when $\hat{\Sigma}_t = t$, and the left inequality of (B.20) binds when $\hat{\Sigma}_t = 0$. By (B.3), we have

$$e^{-\mu t} < S_{\hat{\sigma}}^D(t) = S_{\hat{\sigma}}^M(t) + L_{\hat{\sigma}}(t) < e^{-\lambda_L t} \cdot \left(\frac{\mu + \lambda_H}{\lambda_H}\right). \tag{B.21}$$

From (B.9), we have

$$S_{\hat{\sigma}}^D(t) \cdot V_1(t;\hat{\sigma}) = (\lambda_H \Pi - c) \cdot \int_t^\infty e^{-\lambda_H(s-t)} \cdot S_{\hat{\sigma}}^D(s)\, ds.$$

By applying (B.21) and $\lambda_H\Pi > \lambda_L\Pi > c$, we have

$$\begin{aligned}(\lambda_H\Pi - c)\cdot\int_t^\infty e^{-\lambda_H(s-t)}\cdot S^D_{\hat\sigma}(s)\ ds\ &>\ (\lambda_H\Pi - c)\cdot\int_t^\infty e^{-\lambda_H(s-t)}\cdot e^{-\mu s}ds\\ &= \frac{\lambda_H}{\mu+\lambda_H}\left(\Pi - \frac{c}{\lambda_H}\right)\cdot e^{-\mu t}\end{aligned}$$

and

$$\begin{aligned}(\lambda_H\Pi - c)\cdot\int_t^\infty e^{-\lambda_H(s-t)}\cdot S^D_{\hat\sigma}(s)\ ds\ &<\ (\lambda_H\Pi - c)\cdot\int_t^\infty e^{-\lambda_H(s-t)}\cdot \frac{\mu+\lambda_H}{\lambda_H}e^{-\lambda_L s}ds\\ &= \frac{\mu+\lambda_H}{\lambda_L+\lambda_H}\left(\Pi - \frac{c}{\lambda_H}\right)\cdot e^{-\lambda_L t}.\end{aligned}$$

Therefore, (B.18) holds. Since the lower bound and the upper bound converge to 0 as t goes to infinity, $\lim_{t\to\infty} S^D_{\hat\sigma}(t)\cdot V_1(t;\hat\sigma) = 0$. □

Lemma B.5. *For any σ and $\hat\sigma$,*

$$\lim_{t\to\infty} V_0(t;\sigma,\hat\sigma)\cdot S^M_\sigma(t)\cdot S^D_{\hat\sigma}(t) = 0. \tag{B.22}$$

Proof of Lemma B.5. Note that for any $s\in\mathbb{R}_+$, $-c < \lambda_L(1-\sigma_s)\Pi - c < \lambda_L\Pi$. Since $\lambda_L\Pi > c$, we have $|\lambda_L(1-\sigma_s)\Pi - c| < \lambda_L\Pi$.

From (B.11), we have

$$\begin{aligned}\left|V_0(t;\sigma,\hat\sigma)\cdot S^M_\sigma(t)\cdot S^D_{\hat\sigma}(t)\right| <\ &\lambda_L\Pi\cdot\int_t^\infty S^M_\sigma(s)\cdot S^D_{\hat\sigma}(s)\ ds\\ &+\mu\cdot\int_t^\infty V_1(s;\hat\sigma)\cdot S^M_\sigma(s)\cdot S^D_{\hat\sigma}(s)\ ds.\end{aligned}$$

Observe that from (B.19) and (B.21), we have

$$\int_t^\infty S^M_\sigma(s)\cdot S^D_{\hat\sigma}(s)\ ds < \frac{\mu+\lambda_H}{\lambda_H}\cdot\int_t^\infty e^{-2\lambda_L s}ds = \frac{\mu+\lambda_H}{2\lambda_L\lambda_H}\cdot e^{-2\lambda_L t}.$$

In addition, from (B.18) and (B.21), we have

$$\int_t^\infty V_1(s;\hat\sigma)\cdot S_\sigma^M(s)\cdot S_{\hat\sigma}^D(s)\,ds < \frac{(\mu+\lambda_H)^2}{\lambda_H(\lambda_L+\lambda_H)}\cdot\left(\Pi-\frac{c}{\lambda_H}\right)\cdot\int_t^\infty e^{-2\lambda_L s}ds$$
$$= \frac{(\mu+\lambda_H)^2}{2\lambda_L\lambda_H(\lambda_L+\lambda_H)}\cdot\left(\Pi-\frac{c}{\lambda_H}\right)\cdot e^{-2\lambda_L t}.$$

Then, we have

$$\left|V_0(t;\sigma,\hat\sigma)\cdot S_\sigma^M(t)\cdot S_{\hat\sigma}^D(t)\right| < \frac{\mu+\lambda_H}{2\lambda_L\lambda_H}\left[\lambda_L\Pi+\frac{\mu(\mu+\lambda_H)}{\lambda_L+\lambda_H}\left(\Pi-\frac{c}{\lambda_H}\right)\right]\cdot e^{-2\lambda_L t}.$$

Since the right hand side of the above inequality converges to 0 as $t\to\infty$, (B.22) holds. $\square$

Proof of Proposition 8. ($\Longleftarrow$) Suppose that $V_0(t;\sigma^*,\hat\sigma)>0$ and (B.16) hold for all σ and $t\geq 0$. From (HJB$_0$) and $h_\sigma^M(t)=\lambda_L(1-\sigma_t)+\mu\sigma_t$, we have

$$\begin{aligned}0 =& V_0'(t;\sigma^*,\hat\sigma)-c-h_{\hat\sigma}^D(t)\cdot V_0(t;\sigma^*,\hat\sigma)+\lambda_L\cdot(\Pi-V_0(t;\sigma^*,\hat\sigma))\\ &+\sigma_t^*\cdot\left[\mu\cdot(V_1(t;\hat\sigma)-V_0(t;\sigma^*,\hat\sigma))-\lambda_L\cdot(\Pi-V_0(t;\sigma^*,\hat\sigma))\right].\end{aligned}$$

Then, (B.16) implies that for any σ and $t\geq 0$,

$$\left\{h_{\hat\sigma}^D(t)+h_\sigma^M(t)\right\}\cdot V_0(t;\sigma^*,\hat\sigma)-V_0'(t;\sigma^*,\hat\sigma)\geq \lambda_L(1-\sigma_t)\cdot\Pi+\mu\sigma_t\cdot V_1(t;\hat\sigma)-c.$$

By multiplying $S_\sigma^M(t)\cdot S_{\hat\sigma}^D(t)$, we have

$$-\frac{d}{dt}\left[V_0(t;\sigma^*,\hat\sigma)\cdot S_\sigma^M(t)\cdot S_{\hat\sigma}^D(t)\right]\geq\left[\lambda_L(1-\sigma_t)\cdot\Pi+\mu\sigma_t\cdot V_1(t;\hat\sigma)-c\right]\cdot S_\sigma^M(t)\cdot S_{\hat\sigma}^D(t)$$

for all $t\geq 0$. By integrating this inequality from 0 to ∞ and using Lemma B.3, we have

$$\begin{aligned}&V_0(0;\sigma^*,\hat\sigma)\cdot S_\sigma^M(0)\cdot S_{\hat\sigma}^D(0)-\lim_{t\to\infty}V_0(t;\sigma^*,\hat\sigma)\cdot S_\sigma^M(t)\cdot S_{\hat\sigma}^D(t)\\ \geq&\int_0^\infty\left[\lambda_L(1-\sigma_t)\cdot\Pi+\mu\sigma_t\cdot V_1(t;\hat\sigma)-c\right]\cdot S_\sigma^M(t)\cdot S_{\hat\sigma}^D(t)dt=\mathcal{U}(\sigma,\hat\sigma).\end{aligned}$$

Since $V_0(t;\sigma^*,\hat\sigma)>0$, $S_\sigma^M(t)>0$ and $S_{\hat\sigma}^D(t)>0$, we have $\lim_{t\to\infty}V_0(t;\sigma^*,\hat\sigma)\cdot S_\sigma^M(t)\cdot S_{\hat\sigma}^D(t)\geq$

0. By using this, $\mathcal{U}(\sigma^*, \hat{\sigma}) = V_0(0; \sigma^*, \hat{\sigma})$, and $S^M_\sigma(0) = S^D_{\hat{\sigma}}(0) = 1$, we have $\mathcal{U}(\sigma^*, \hat{\sigma}) \geq \mathcal{U}(\sigma, \hat{\sigma})$.

($\implies$) Suppose that $\sigma^* \in \arg\max_\sigma \mathcal{U}(\sigma, \hat{\sigma})$. From Lemma B.3, observe that for any $t \geq 0$, a firm's expected payoff can be rewritten as follows:

$$\begin{aligned}\mathcal{U}(\sigma, \hat{\sigma}) = &\int_0^t [\lambda_L(1-\sigma_s) \cdot \Pi + \mu\sigma_s \cdot V_1(s; \hat{\sigma}) - c] \cdot S^M_\sigma(s) \cdot S^D_{\hat{\sigma}}(s)\, ds \\ &+ S^M_\sigma(t) \cdot S^D_{\hat{\sigma}}(t) \cdot V_0(t; \sigma, \hat{\sigma}).\end{aligned}$$

Now consider the following allocation policy $\tilde{\sigma}$:

$$\tilde{\sigma}_s = \begin{cases} \sigma^*_s, & \text{if } s < t, \\ 0, & \text{if } s \geq t. \end{cases}$$

Then, $S^M_{\sigma^*}(s) \cdot S^D_{\hat{\sigma}}(s) = S^M_{\tilde{\sigma}}(s) \cdot S^D_{\hat{\sigma}}(s)$ for all $s \leq t$.[14] In addition, by using $\sigma^*_s = \tilde{\sigma}_s$ for all $s < t$ and $\mathcal{U}(\sigma^*, \hat{\sigma}) \geq \mathcal{U}(\tilde{\sigma}, \hat{\sigma})$, we have $V_0(t; \sigma^*, \hat{\sigma}) \geq V_0(t; \tilde{\sigma}, \hat{\sigma})$.

Note that

$$V_0(t; \tilde{\sigma}, \hat{\sigma}) = \int_t^\infty (\lambda_L \Pi - c) \cdot \frac{S^M_{\tilde{\sigma}}(s)}{S^M_{\tilde{\sigma}}(t)} \cdot \frac{S^D_{\hat{\sigma}}(s)}{S^D_{\hat{\sigma}}(t)}\, ds > 0$$

from $\lambda_L \Pi > c$, $S^M_{\tilde{\sigma}}(s) > 0$, and $S^D_{\hat{\sigma}}(s) > 0$. Therefore, $V_0(t; \sigma^*, \hat{\sigma}) > 0$ for all $t \geq 0$.

Now assume that there exists σ such that (B.16) does not hold for some $t \geq 0$. Observe that V_1 and V_0 are continuous in t. Since σ^* and σ are right-continuous, there exists $\epsilon > 0$ such that for all $s \in [t, t+\epsilon)$,

$$(\sigma^*_s - \sigma_s) \cdot [\mu \cdot (V_1(s; \hat{\sigma}) - V_0(s; \sigma^*, \hat{\sigma})) - \lambda_L \cdot (\Pi - V_0(s; \sigma^*, \hat{\sigma}))] < 0. \tag{B.23}$$

Consider the following allocation policy σ^{**}:

$$\sigma^{**}_s = \begin{cases} \sigma^*_s, & \text{if } s \notin [t, t+\epsilon), \\ \sigma_s, & \text{if } s \in [t, t+\epsilon). \end{cases}$$

[14] Note that the equality also holds at $s = t$, since σ^* and $\tilde{\sigma}$ differ only at $\{t\}$, which is negligible after integration.

By using the similar reformulation in the previous case, we have

$$
\begin{aligned}
&-\frac{d}{ds}\left[V_0(s;\sigma^*,\hat{\sigma})\cdot S^M_{\sigma^{**}}(s)\cdot S^D_{\hat{\sigma}}(s)\right] \\
&\le [\lambda_L(1-\sigma^{**}_s)\cdot\Pi+\mu\sigma^{**}_s\cdot V_1(s;\hat{\sigma})-c]\cdot S^M_{\sigma^{**}}(s)\cdot S^D_{\hat{\sigma}}(s)
\end{aligned}
\tag{B.24}
$$

for all $s \ge 0$, and the inequality strictly holds for $s \in [t, t+\epsilon)$. Also note that by Lemma B.5,

$$\lim_{s\to\infty} V_0(s;\sigma^*,\hat{\sigma})\cdot S^M_{\sigma^{**}}(s)\cdot S^D_{\hat{\sigma}}(s) = \lim_{s\to\infty} V_0(s;\sigma^*,\hat{\sigma})\cdot S^M_{\sigma^{*}}(s)\cdot S^D_{\hat{\sigma}}(s) = 0.$$

By integrating (B.24) from 0 to ∞, we have

$$
\begin{aligned}
\mathcal{U}(\sigma^*,\hat{\sigma}) &= V_0(0;\sigma^*,\hat{\sigma}) \\
&< \int_0^\infty [\lambda_L(1-\sigma^{**}_s)\cdot\Pi+\mu\sigma^{**}_s\cdot V_1(s;\hat{\sigma})-c]\cdot S^M_{\sigma^{**}}(s)\cdot S^D_{\hat{\sigma}}(s)\, ds = \mathcal{U}(\sigma^{**},\hat{\sigma}),
\end{aligned}
$$

which contradicts $\sigma^* \in \arg\max_\sigma \mathcal{U}(\sigma,\hat{\sigma})$. Therefore, (B.16) holds for all $t \ge 0$. □

B.4 Equilibrium Properties with Escalating Tension

From here on, we use abbreviated notions to ease notations:

$$
\begin{aligned}
V^1_t &\equiv V_1(t;\sigma^*), & \dot{V}^1_t &\equiv \tfrac{\partial V_1}{\partial t}(t;\sigma^*), \\
V^0_t &\equiv V_0(t;\sigma^*,\sigma^*), & \dot{V}^0_t &\equiv \tfrac{\partial V_0}{\partial t}(t;\sigma^*,\sigma^*), \\
p_t &\equiv p_{\sigma^*}(t), & h_t &\equiv h^D_{\sigma^*}(t),
\end{aligned}
\tag{B.25}
$$

i.e., $h_t = \lambda_L\cdot(1-\sigma^*_t)\cdot(1-p_t)+\lambda_H\cdot p_t$.

YG: steady-state candidate: Define

$$h_\star \equiv \frac{\mu(\lambda_H-\lambda_L)}{\lambda_L}-\lambda_H, \qquad V^1_\star \equiv \frac{\lambda_H\Pi - c}{\lambda_H + h_\star}, \qquad V^0_\star \equiv \frac{\mu V^1_\star - \lambda_L\Pi}{\mu-\lambda_L}$$

B.4.1 Hazard Rate Evolution

Lemma B.6. *Suppose that σ^* constitutes a symmetric Nash equilibrium. If $\mu(V_t^1 - V_t^0) < \lambda_L(\Pi - V_t^0)$, the following differential equation holds:*

$$\dot{h}_t = (\lambda_H - h_t) \cdot (\lambda_L - h_t). \tag{B.26}$$

Moreover, if σ^* exhibits an escalating tension and $\sigma_t^* = 0$, then $\sigma_s^* = 0$ for all $s < t$.

Proof. From the continuity of V^1 and V^0, there exists $\epsilon > 0$ such that $\mu(V_s^1 - V_s^0) < \lambda_L(\Pi - V_s^0)$ for all $s \in (t - \epsilon, t + \epsilon)$. Then, by Proposition 8, $\sigma_s^* = 0$ for all $s \in (t - \epsilon, t + \epsilon)$, thus, we have $\dot{\sigma}_t^* = 0$.

YG: think about mentioning the role of symmetry here

By taking a derivative of h at t, we have

$$\dot{h}_t = -\lambda_L(1 - \sigma_t^*)\dot{p}_t + \lambda_H \dot{p}_t - \lambda_L \dot{\sigma}_t^*(1 - p_t) = (\lambda_H - \lambda_L)\dot{p}_t. \tag{B.27}$$

Note that $\dot{p}_t = -(\lambda_H - \lambda_L) \cdot p_t(1 - p_t)$, and $h_t = \lambda_L(1 - p_t) + \lambda_H p_t$, or equivalently, $p_t = (h_t - \lambda_L)/(\lambda_H - \lambda_L)$. By plugging these into (B.27), we can see that (B.26) holds.

Observe that p_t is always less than 1, thus, from $\lambda_H > \lambda_L$, we have $\lambda_H > h_t$. In addition, $h_t \geq \lambda_L$ and the right equality holds if and only if $p_t = 0$. If σ^* exhibits an escalating tension, h_t should be equal to λ_L—if not, $\dot{h}_t$ is negative by (B.26). Therefore, p_t is equal to zero, which implies that $\Sigma_t^* = 0$ and $\sigma_s^* = 0$ for all $s < t$. □

Lemma B.7. *Suppose that σ^* constitutes a symmetric Nash equilibrium. If $\mu(V_t^1 - V_t^0) > \lambda_L(\Pi - V_t^0)$, the following differential equation holds:*

$$\dot{h}_t = (\lambda_H - h_t) \cdot (\mu - h_t). \tag{B.28}$$

Proof. From the continuity of V^1 and V^0, there exists $\epsilon > 0$ such that $\mu(V_s^1 - V_s^0) > \lambda_L(\Pi - V_s^0)$ for all $s \in (t - \epsilon, t + \epsilon)$. Then, by Proposition 8, $\sigma_s^* = 1$ for all $s \in (t - \epsilon, t + \epsilon)$, thus, we have $\dot{\sigma}_t^* = 0$.

By taking a derivative of h at t, we have

$$\dot{h}_t = -\lambda_L(1-\sigma_t^*)\dot{p}_t + \lambda_H\dot{p}_t - \lambda_L\dot{\sigma}_t^*(1-p_t) = \lambda_H \cdot \dot{p}_t. \tag{B.29}$$

Note that $\dot{p}_t = (1-p_t)\cdot(\mu - \lambda_H p_t)$, and $h_t = \lambda_H p_t$. By plugging these into (B.29), we can see that (B.28) holds. □

Lemma B.8. *Suppose that σ^* constitutes a symmetric Nash equilibrium and right-continuous. If $\sigma_t^* \in (0,1)$, the following differential equation holds:*

$$\dot{h}_{t+} = (h_t - h_\star)(h_t + c/\Pi), \tag{B.30}$$

where $\dot{h}_{t+}$ is the right time derivative of the hazard rate h at time t.

Proof. By the right-continuity of σ^*, there exists $\epsilon > 0$ such that $\sigma_s^* \in (0,1)$ for all $s \in [t, t+\epsilon)$. By Proposition 8, we have $\mu(V_s^1 - V_s^0) = \lambda_L(\Pi - V_s^0)$ for all $s \in [t, t+\epsilon)$. Thus, we have $\mu(\dot{V}_{t+}^1 - \dot{V}_{t+}^0) = -\lambda_L \dot{V}_{t+}^0$.

Note that from (HJB$_0$), (HJB$_1$), and $\mu(V_t^1 - V_t^0) = \lambda_L(\Pi - V_t^0)$, we have

$$\begin{aligned}
\dot{V}_{t+}^0 =& h_t \cdot V_t^0 - \mu(V_t^1 - V_t^0) + c, \\
\dot{V}_{t+}^1 =& -(\lambda_H\Pi - c) + (\lambda_H + h_t)\cdot V_t^1.
\end{aligned}$$

Then, from $\mu\dot{V}_{t+}^1 - (\mu - \lambda_L)\dot{V}_{t+}^0 = 0$, $\mu(\lambda_H - \lambda_L) = \lambda_L(\lambda_H + h_\star)$ and $V_\star^1 = \frac{\lambda_H\Pi - c}{\lambda_H + h_\star}$, we can derive that

$$\begin{aligned}
0 =& h_t \cdot \left\{\mu(V_t^1 - V_t^0) + \lambda_L V_t^0\right\} + \mu(\mu - \lambda_L)(V_t^1 - V_t^0) - \mu\lambda_H(\Pi - V_t^1) + \lambda_L c \\
=& \lambda_L\Pi \cdot h_t - \mu(\lambda_H - \lambda_L)(\Pi - V_t^1) + \lambda_L c \\
=& \lambda_L\Pi \cdot h_t - \lambda_L(\lambda_H + h_\star)(\Pi - V_t^1) + \lambda_L c \\
=& \lambda_L(\lambda_H + h_\star)\left[V_t^1 - \left\{V_\star^1 + \frac{(h_\star - h_t)\cdot\Pi}{\lambda_H + h_\star}\right\}\right].
\end{aligned}$$

Therefore, we have $V_t^1 = V_\star^1 + \frac{(h_\star - h_t)\cdot\Pi}{\lambda_H + h_\star}$. By plugging this into (HJB$_1$), we can derive that

$$\begin{aligned}
\dot{h}_{t^+} =& -\frac{\lambda_H + h_\star}{\Pi}\cdot \dot{V}_{t^+}^1 = -\frac{\lambda_H + h_\star}{\Pi}\cdot\left[-(\lambda_H\Pi - c) + (\lambda_H + h_t)\cdot V_t^1\right] \\
=& \frac{\lambda_H + h_\star}{\Pi}\cdot\left[(\lambda_H + h_\star)\cdot V_\star^1 - (\lambda_H + h_t)\cdot\left\{V_\star^1 + \frac{(h_\star - h_t)\cdot\Pi}{\lambda_H + h_\star}\right\}\right] \\
=& \frac{1}{\Pi}\cdot\left[-(\lambda_H + h_\star)\cdot V_\star^1 + (\lambda_H + h_t)\Pi\right]\cdot(h_t - h_\star) = (h_t + c/\Pi)\cdot(h_t - h_\star).
\end{aligned}$$

□

B.4.2 Proof of Proposition 3

We introduce a useful lemma and then provide the proof of Proposition 3. Observe that $h_t \le \lambda_H$, then, under the escalating tension assumption, there exists $\overline{h} \equiv \lim_{t\to\infty} h_t$.

Lemma B.9. *Suppose that σ^* exhibits an escalating tension and constitutes a symmetric Nash equilibrium and $\overline{h} = \lim_{t\to\infty} h_t$. Then, we have*

$$\lim_{t\to\infty} V_t^1 = V_\infty^1 \equiv \frac{\lambda_H\Pi - c}{\lambda_H + \overline{h}}, \tag{B.31}$$

and for any $t \in \mathbb{R}_+$,

$$V_t^0 \ge \frac{\mu V_\infty^1 - c}{\mu + \overline{h}}. \tag{B.32}$$

Proof. For any $t \in \mathbb{R}_+$, $h_t \le \overline{h}$, thus, we have

$$V_t^1 = (\lambda_H\Pi - c)\int_t^\infty e^{-\int_t^s(\lambda_H + h_u)du}ds \;\ge\; (\lambda_H\Pi - c)\int_t^\infty e^{-(\lambda_H+\overline{h})(s-t)}ds = \frac{\lambda_H\Pi - c}{\lambda_H + \overline{h}}.$$

By $\lim_{t\to\infty} h_t = \overline{h}$, for any $\epsilon > 0$, there exists T_ϵ such that $\overline{h} \ge h_t > \overline{h} - \epsilon$ for all $t > T_\epsilon$. Then, by (B.9), for any $t > T_\epsilon$, we have

$$V_t^1 \;\le\; (\lambda_H\Pi - c)\int_t^\infty e^{-(\lambda_H+\overline{h}-\epsilon)(s-t)}ds = \frac{\lambda_H\Pi - c}{\lambda_H + \overline{h} - \epsilon}.$$

By sending ϵ to zero, we can see that as t goes to infinity, V_t^1 converges to V_∞^1.

Next, since σ^* constitutes an equilibrium, it should return a continuation payoff at time t greater than or equal to that of another allocation policy such that a firm keeps doing research from time t on:

$$V_t^0 \geq \int_t^\infty (\mu \cdot V_s^1 - c) \cdot e^{-\int_t^s (\mu + h_u) du} ds.$$

Then, by using $V_s^1 \geq V_\infty^1$ and $h_u \leq \overline{h}$ for any $s, u \geq t$, we can derive that (B.32) holds. $\square$

Proof of Proposition 3. By Lemma B.8, h_t is greater than or equal to $h_\star$. If not, $\dot{h}_{t+}$ is negative, which violates an escalating tension. Therefore, we have $\overline{h} \geq h_\star$.

If $\sigma_s^* = 0$ for some $s > t$, $\sigma_t^* = 0$ by Lemma B.6. Therefore, $\sigma_s^* > 0$ for all $s \geq t$, which implies that $\mu(V_s^1 - V_s^0) \geq \lambda_L(\Pi - V_s^0)$. By sending s to infinity and using (B.32), we have

$$\mu V_\infty^1 \geq \lambda_L \Pi + (\mu - \lambda_L) \cdot \frac{\mu V_\infty^1 - c}{\mu + \overline{h}}.$$

By rearranging the above inequality, we can derive the followings:

$$\begin{aligned}
& \mu(\mu + \overline{h}) V_\infty^1 \geq (\mu + \overline{h}) \lambda_L \Pi + (\mu - \lambda_L) \cdot (\mu V_\infty^1 - c) \\
\iff \quad & \mu(\lambda_L + \overline{h}) V_\infty^1 \geq \lambda_L \Pi (\mu + \overline{h}) - (\mu - \lambda_L) c \\
\iff \quad & \mu(\lambda_L + \overline{h})(\lambda_H \Pi - c) \geq \left[\lambda_L \Pi (\mu + \overline{h}) - (\mu - \lambda_L) c\right] \cdot (\lambda_H + \overline{h}) \\
\iff \quad & 0 \geq \left[\lambda_L \overline{h} - \mu(\lambda_H - \lambda_L) + \lambda_L \lambda_H\right] \cdot (\overline{h} \cdot \Pi + c) \\
\iff \quad & 0 \geq \lambda_L (\overline{h} - h_\star) \cdot (\overline{h} \cdot \Pi + c).
\end{aligned}$$

Since $\lambda_L(\overline{h} \cdot \Pi + c) > 0$, we have $h_\star \geq \overline{h}$. Therefore, we have $\overline{h} = h_\star$. By $h_t \geq h_\star$ and an escalating tension, we also have that $h_s = h_\star$ for all $s \geq t$.

By the definition of h_s, we have

$$\lambda_H p_s + \lambda_L (1 - p_s)(1 - \sigma_s^*) = h_\star,$$

or equivalently,

$$1 - \sigma_s^* = \frac{h_\star - \lambda_H p_s}{\lambda_L (1 - p_s)}. \tag{B.33}$$

From (4.2), we have

$$
\begin{aligned}
\dot{p}_s =& (1-p_s)\left[\mu - \lambda_H p_s - (1-\sigma_s)(\mu - \lambda_L p_s)\right] \\
=& (1-p_s)(\mu - \lambda_H p_s) - \{h_\star - \lambda_H p_s\}\left(\frac{\mu}{\lambda_L} - p_s\right) \\
=& \mu - \frac{\mu}{\lambda_L} h_\star + \left(h_\star + \frac{\lambda_H \mu}{\lambda_L} - \mu - \lambda_H\right) p_s \\
=& -\frac{\mu}{\lambda_L}(h_\star - \lambda_L) + 2h_\star p_s = 2h_\star(p_s - p_\star).
\end{aligned}
$$

If $p_t \neq p_\star$, then the solution of the above differential equation diverges and contradicts $0 \leq p_s \leq 1$ for all $s \geq t$. Therefore, $p_s = p_\star$ and it also gives $p_s = p_\star$ for all $s \geq t$. In addition, from $\dot{p}_s = 0$ for all $s \geq t$, we can derive that

$$
\sigma_s = \frac{(\lambda_H - \lambda_L)p_s}{\mu - \lambda_L p_s} = \frac{(\lambda_H - \lambda_L)p_\star}{\mu - \lambda_L p_\star} = \sigma_\star
$$

for all $s \geq t$. In addition, $\sigma_t \in (0,1)$ implies $\sigma_\star \in (0,1)$. $\square$

B.4.3 Proof of Proposition 4

Proof of Proposition 4. First, assume that $\sigma^*_t \in (0,1)$ for some $t \geq 0$. Define a time T as follows: $T \equiv \inf\{s \mid \sigma^*_s \in (0,1)\}$. By Proposition 3 and the right-continuity of σ^*, we have $\sigma^*_s = \sigma_\star$ for all $s \geq T$. By the definition of T, for any $s < T$, σ^*_s is equal to 0 or 1. Suppose that there exists $s \in [0,T)$ such that $\sigma^*_s = 0$. Then, we can define $S \equiv \sup\{s \mid \sigma^*_s = 0\}$ and it is less than or equal to T. For any $s \geq S$, the only available values for σ^*_s are $\sigma_\star$ and 1. Then, by the right-continuity of σ^*, $\sigma^*_S > 0$. On the other hand, by Lemma B.6, $\sigma^*_{S^-} = 0 < \sigma^*_S$. Then, by the continuity of p, we have

$$
h_{S^-} = \lambda_L(1-\sigma_{S^-})(1-p_S) + \lambda_H p_S > \lambda_L(1-\sigma_S)(1-p_S) + \lambda_H p_S = h_S,
$$

which contradicts the escalating tension property. Therefore, by setting $\sigma_\circ = \sigma_\star$, we have that $\sigma^*_s = 1$ for all $s < T$ and $\sigma^*_s = \sigma_\circ$ for all $s \geq T$.

Next, consider the case where $\sigma^*_t \notin (0,1)$ for all $t \geq 0$. Suppose that $\sigma^*_s = 0$ for some

$s \geq 0$. If $S \equiv \{s \mid \sigma_s^* = 0\}$ is finite, it contradicts to the escalating tension property by using the same argument as above. Thus, we have $S = \infty$, and from Lemma B.6, $\sigma_t^* = 0$ for all $t \geq 0$. In this case, we can set $T = 0$ and $\sigma_\circ = 0$, then $\sigma_t^* = \sigma_t$ for all $t \geq T$. The remaining case is $\sigma_s^* = 1$ for all $s \geq 0$, and in this case, we can set $T = \infty$, then, $\sigma_t^* = 1$ for all $t < T$. □

B.5 Equilibrium Characterization

B.5.1 The Direct-Development Equilibrium

Lemma B.10. *Suppose that σ^* is the direct-development strategy, i.e., $\sigma_t^* = 0$ for all $t \geq 0$. Then, $\sigma^A = \sigma^B = \sigma^*$ constitutes a symmetric Nash equilibrium if and only if $\gamma \geq \overline{\gamma}$.*

Proof of Lemma B.10. ($\Longrightarrow$) Suppose that the direct-development strategy constitutes an equilibrium. Since neither firm conducts research, the belief that the other firm possesses the new technology is 0, i.e., $p_t = 0$ for all $t \geq 0$. Observe that $V_t^0 = \frac{\lambda_L \Pi - c}{2\lambda_L}$ since both firms develop with the old technology. If a firm happens to have the new technology and the other firm sticks with the strategy, the expected payoff is $\frac{\lambda_H \Pi - c}{\lambda_H + \lambda_L}$, i.e., $V_t^1 = \frac{\lambda_H \Pi - c}{\lambda_H + \lambda_L}$ for all $t \geq 0$. To support this equilibrium, by Proposition 8, $\mu(V_t^1 - V_t^0) \leq \lambda_L(\Pi - V_t^0)$ needs to hold. By plugging V_t^1 and V_t^0 in, we have

$$
\begin{aligned}
& \mu\left(\frac{\lambda_H \Pi - c}{\lambda_H + \lambda_L} - \frac{\lambda_L \Pi - c}{2\lambda_L}\right) \leq \lambda_L\left(\Pi - \frac{\lambda_L \Pi - c}{2\lambda_L}\right) \\
\iff \quad & \frac{\mu(\lambda_L \Pi + c)(\lambda_H - \lambda_L)}{2\lambda_L(\lambda_H + \lambda_L)} \leq \frac{\lambda_L \Pi + c}{2} \\
\iff \quad & \overline{\gamma} = \frac{1-\rho}{1+\rho} = \frac{\lambda_H - \lambda_L}{\lambda_H + \lambda_L} \leq \frac{\lambda_L}{\mu} = \gamma.
\end{aligned}
$$

($\Longleftarrow$) Now suppose that $\gamma \geq \overline{\gamma}$. By the above inequality, $\mu(V_t^1 - V_t^0) \leq \lambda_L(\Pi - V_t^0)$ holds for all $t \geq 0$, thus, (B.17) of Proposition 8 is satisfied. In addition, $V_0(t; \sigma^*, \sigma^*) = V_t^0 = \frac{\lambda_L \Pi - c}{2\lambda_L} > 0$ holds for all $t \geq 0$. By Proposition 8, σ^* is a best response to σ^*, thus, it constitutes a symmetric Nash equilibrium. □

B.5.2 The Research Equilibrium

Lemma B.11. *Suppose that for some T, both firms play $\sigma_t = 1$ for all $0 \leq t \leq T$. Then, there exist $C_0, C_1 \in \mathbb{R}$ such that the expected payoffs of the firm with and without the new technology at time $t \in [0, T]$ is given as follows:*

$$V_t^1 = \overline{V}_1(q_t) + C_1 \cdot (1 - q_t) \cdot \left(\frac{\mu - \lambda_H q_t}{1 - q_t} \right)^{\frac{\mu + \lambda_H}{\mu - \lambda_H}}, \tag{B.34}$$

$$V_t^0 = \overline{V}_0(q_t) + \left(C_0 \left(\frac{\mu}{\lambda_H} - q_t \right) - C_1 \frac{\mu}{\lambda_H} \right) \cdot \left(\frac{\mu - \lambda_H q_t}{1 - q_t} \right)^{\frac{\mu + \lambda_H}{\mu - \lambda_H}}, \tag{B.35}$$

[15]

where q_t is the belief defined in (B.8) and

$$\overline{V}_1(q) \equiv \frac{1}{2} \left(\Pi - \frac{c}{\lambda_H} \right) \left(1 + \frac{\lambda_H}{\lambda_H + \mu} (1 - q) \right), \tag{B.36}$$

$$\overline{V}_0(q) \equiv \frac{1}{2} \left(\Pi - \frac{c}{\mu} - \frac{c}{\lambda_H} \right) \left(1 - \frac{\lambda_H}{\lambda_H + \mu} q \right) - \frac{c}{2(\lambda_H + \mu)}. \tag{B.37}$$

Moreover, if both firms play the research strategy ($T = \infty$), $C_1 = C_0 = 0$, i.e., $V_t^1 = \overline{V}_1(q_t)$ and $V_t^0 = \overline{V}_0(q_t)$.

Proof of Lemma B.11. Note that q_t is increasing in t. Then, the value functions V_t^1 and V_t^2 for $t \in [0, T]$ can be written as functions of q_t: $V_t^1 = V_1(q_t)$ and $V_t^0 = V_0(q_t)$. Observe that $\dot{V}_t^n = V_n'(q_t)\dot{q}_t = V_n'(q_t)(\mu - \lambda_H q_t)(1 - q_t)$ for $n \in \{0, 1\}$. By plugging this into (HJB$_1$), we have

$$0 = V_1'(q)(\mu - \lambda_H q)(1 - q) - \lambda_H(1 + q)V_1(q) + \lambda_H \Pi - c. \tag{B.38}$$

By multiplying $(\mu - \lambda_H q)^{-\frac{2\mu}{\mu - \lambda_H}} (1 - q)^{\frac{3\lambda_H - \mu}{\mu - \lambda_H}}$ and rearranging the equation, for all $0 = q_0 \leq q \leq q_T$, we can derive that

$$0 = \frac{d}{dq} \left[\frac{(1 - q)^{\frac{2\lambda_H}{\mu - \lambda_H}}}{(\mu - \lambda_H q)^{\frac{\mu + \lambda_H}{\mu - \lambda_H}}} \left\{ V_1(q) - \overline{V}_1(q) \right\} \right]. \tag{B.39}$$

[15] If $\mu = \lambda_H$, we need to replace $\left(\frac{\mu - \lambda_H q_t}{1 - q_t} \right)^{\frac{\mu + \lambda_H}{\mu - \lambda_H}}$ to $e^{\frac{2}{1 - q_t}}$.

Therefore, for all $0 = q_0 \le q \le q_T$, we have

$$V_1(q) = \overline{V}_1(q) + C_1 \cdot (1-q) \cdot \left(\frac{\mu - \lambda_H q}{1-q}\right)^{\frac{\mu+\lambda_H}{\mu-\lambda_H}} \tag{B.40}$$

for some $C_1 \in \mathbb{R}$. By $V_t^1 = V_1(q_t)$ for all $0 \le t \le T$, (B.34) holds.

Next, plug $\dot{V}_t^0 = V_0'(q_t)(\mu - \lambda_H q_t)(1-q_t)$ into (HJB$_0$):

$$\begin{aligned} 0 =& V_0'(q)(\mu - \lambda_H q)(1-q) - \lambda_H q V_0(q) - c + \mu(V_1(q) - V_0(q)) \\ =& V_0'(q)(\mu - \lambda_H q)(1-q) - V_0(q)(\lambda_H q + \mu) - c \\ &+ \mu \left(\Pi - \frac{c}{\lambda_H}\right)\left(\frac{1}{2} + \frac{\lambda_H(1-q)}{2(\lambda_H + \mu)}\right) + \mu \cdot C_1 \cdot (1-q) \cdot \left(\frac{\mu - \lambda_H q}{1-q}\right)^{\frac{\mu+\lambda_H}{\mu-\lambda_H}}. \end{aligned} \tag{B.41}$$

By multiplying $(1-q)^{\frac{2\lambda_H}{\mu-\lambda_H}}(\mu - \lambda_H q)^{-\frac{3\mu-\lambda_H}{\mu-\lambda_H}}$ and rearranging the equation, $0 \le q \le q_T$, we have

$$0 = \frac{d}{dq}\left[\frac{(1-q)^{\frac{\mu+\lambda_H}{\mu-\lambda_H}}}{(\mu-\lambda_H q)^{\frac{2\mu}{\mu-\lambda_H}}}\left\{V_0(q) - \overline{V}_0(q) + C_1 \cdot \frac{\mu}{\lambda_H} \cdot \left(\frac{\mu - \lambda_H q}{1-q}\right)^{\frac{\mu+\lambda_H}{\mu-\lambda_H}}\right\}\right].$$

Therefore, we have

$$V_0(q) = \overline{V}_0(q) + \left(C_0\left(\frac{\mu}{\lambda_H} - q\right) - C_1 \frac{\mu}{\lambda_H}\right) \cdot \left(\frac{\mu - \lambda_H q}{1-q}\right)^{\frac{\mu+\lambda_H}{\mu-\lambda_H}}. \tag{B.42}$$

for some $C_0 \in \mathbb{R}$. By $V_t^0 = V_0(q_t)$ for all $0 \le t \le T$, (B.35) holds.

Now suppose that both firms play research-first strategy. Then, (B.34) and (B.35) hold for all $t \ge 0$. When $\mu \ge \lambda_H$, by Lemma B.1, $\lim_{t\to\infty} q_t = 1$. Since $\lim_{t\to\infty}(1-q_t)\left(\frac{\mu-\lambda_H q_t}{1-q_t}\right)^{\frac{\mu+\lambda_H}{\mu-\lambda_H}} = \infty$ and $\lim_{t\to\infty}\left(\frac{\mu-\lambda_H q_t}{1-q_t}\right)^{\frac{\mu+\lambda_H}{\mu-\lambda_H}} = \infty$, to make the value functions converge, $C_1 = C_0 = 0$. When $\mu < \lambda_H$, by Lemma B.1, $\lim_{t\to\infty} q_t = \mu/\lambda_H$, which also implies $\lim_{t\to\infty}(1 - q_t)\left(\frac{\mu-\lambda_H q_t}{1-q_t}\right)^{\frac{\mu+\lambda_H}{\mu-\lambda_H}} = \infty$ and $\lim_{t\to\infty}\left(\frac{\mu-\lambda_H q_t}{1-q_t}\right)^{\frac{\mu+\lambda_H}{\mu-\lambda_H}} = \infty$. Likewise, we also have $C_1 = C_0 = 0$ in this case to make the value functions converge. □

Lemma B.12. *Suppose that σ is the research strategy, i.e., $\sigma_t^* = 1$ for all $t \ge 0$. Then,*

$\sigma^A = \sigma^B = \sigma^*$ *constitutes a symmetric Nash equilibrium if and only if* $\gamma \leq \tilde{\gamma} \equiv \max\{(1-\rho)/2, 1-2\rho\}$.

Proof of Lemma B.12. Suppose that both firms play the research-first strategy. By Lemma B.11, the expected payoffs at time t with and without the new technology are $V_t^1 = \overline{V}_1(q_t)$ and $V_t^0 = \overline{V}_0(q_t)$.

By Proposition 8, σ^* constitutes a symmetric Nash equilibrium if and only if $\overline{V}_0(q_t) \geq 0$ and $\mu(\overline{V}_1(q_t) - \overline{V}_0(q_t)) \geq \lambda_L(\Pi - \overline{V}_0(q_t))$ for all $t \geq 0$. Note that

$$\frac{d}{dq}\overline{V}_0(q) = -\frac{\lambda_H}{2(\lambda_H+\mu)}\left(\Pi - \frac{c}{\mu} - \frac{c}{\lambda_H}\right) < 0,$$

$$\frac{d}{dq}\left[\mu(\overline{V}_1(q) - \overline{V}_0(q)) - \lambda_L(\Pi - \overline{V}_0(q))\right] = -\frac{\left(\Pi - \frac{c}{\mu} - \frac{c}{\lambda_H}\right) + \frac{c}{\lambda_L}}{2\left(\frac{\lambda_H+\mu}{\lambda_H\lambda_L}\right)} < 0.$$

Therefore, it is enough to check whether the following inequalities holds:

$$\lim_{t\to\infty} \overline{V}_0(q_t) \geq 0, \tag{B.43}$$

$$\lim_{t\to\infty} \left[\mu(\overline{V}_1(q_t) - \overline{V}_0(q_t)) - \lambda_L(\Pi - \overline{V}_0(q_t))\right] \geq 0. \tag{B.44}$$

When $\mu \geq \lambda_H$, or equivalently, $\rho \geq \gamma$, by $\lim_{t\to\infty} q_t = 1$, (B.44) is equivalent to

$$\mu(\overline{V}_1(1) - \overline{V}_0(1)) \geq \lambda_L(\Pi - \overline{V}_0(1)). \tag{B.45}$$

Note that

$$\begin{aligned}
\overline{V}_1(1) =& \frac{1}{2}\left(\Pi - \frac{c}{\lambda_H}\right), \\
\overline{V}_0(1) =& \frac{1}{2}\left(\Pi - \frac{c}{\mu} - \frac{c}{\lambda_H}\right)\cdot\frac{\mu}{\lambda_H+\mu} - \frac{c}{2(\lambda_H+\mu)} \\
=& \frac{\mu}{2(\lambda_H+\mu)}\Pi - \frac{c}{2\lambda_H} - \frac{c}{2(\lambda_H+\mu)}.
\end{aligned}$$

Then, we have

$$\begin{aligned}
\mu(\overline{V}_1(1) - \overline{V}_0(1)) =& \mu \cdot \frac{\lambda_H \Pi + c}{2(\lambda_H + \mu)}, \\
\lambda_L(\Pi - \overline{V}_0(1)) =& \frac{\lambda_L(\mu + 2\lambda_H)}{\lambda_H} \cdot \frac{\lambda_H \Pi + c}{2(\lambda_H + \mu)}.
\end{aligned}$$

Therefore, (B.45) is equivalent to

$$\mu \geq \frac{\lambda_L(\mu + 2\lambda_H)}{\lambda_H} \quad \Longleftrightarrow \quad \underline{\gamma} = \frac{1-\rho}{2} \geq \gamma. \tag{B.46}$$

In addition, if (B.46) holds, by using $\Pi > c/\lambda_L$, we have

$$\lim_{t\to\infty} \overline{V}_0(q_t) = \overline{V}_0(1) = \frac{c}{2(\lambda_H + \mu)} \left(\frac{\mu}{\lambda_L} - \frac{\lambda_H + \mu}{\lambda_H} - 1 \right) > 0.$$

Therefore, both (B.43) and (B.44) hold if and only if (B.46) holds.

When $\lambda_H > \mu$, by $\lim_{t\to\infty} q_t = \mu/\lambda_H$, (B.44) is equivalent to

$$\mu \left(\overline{V}_1 (\mu/\lambda_H) - \overline{V}_0 (\mu/\lambda_H) \right) \geq \lambda_L \left(\Pi - \overline{V}_0 (\mu/\lambda_H) \right). \tag{B.47}$$

Note that

$$\begin{aligned}
\overline{V}_1(\mu/\lambda_H) =& \frac{\lambda_H \Pi - c}{\lambda_H + \mu}, \\
\overline{V}_0(\mu/\lambda_H) =& \frac{\lambda_H}{2(\lambda_H + \mu)} \Pi - \frac{c}{2\mu} - \frac{c}{2(\lambda_H + \mu)}.
\end{aligned}$$

Then, we have

$$\begin{aligned}
\mu(\overline{V}_1(\mu/\lambda_H) - \overline{V}_0(\mu/\lambda_H)) =& \lambda_H \cdot \frac{\mu \Pi + c}{2(\lambda_H + \mu)}, \\
\lambda_L(\Pi - \overline{V}_0(\mu/\lambda_H)) =& \frac{\lambda_L(\lambda_H + 2\mu)}{\mu} \cdot \frac{\mu \Pi + c}{2(\lambda_H + \mu)}.
\end{aligned}$$

Therefore, (B.47) is equivalent to

$$\lambda_H \geq \frac{\lambda_L(\lambda_H + 2\mu)}{\mu} \quad \iff \quad 1 - 2\rho \geq \gamma. \tag{B.48}$$

In addition, if (B.48) holds, by using $\Pi > c/\lambda_L$, we have

$$\lim_{t\to\infty} \overline{V}_0(q_t) = \overline{V}_0(\mu/\lambda_H) = \frac{c}{2(\lambda_H + \mu)} \left(\frac{\lambda_H}{\lambda_L} - \frac{\lambda_H + \mu}{\mu} - 1 \right) > 0.$$

Therefore, both (B.43) and (B.44) hold if and only if (B.48) holds.

Observe that when $\rho > 1/3$, $\rho > (1-\rho)/2 > 1 - 2\rho$. In this case, both (B.43) and (B.44) hold iff $\gamma \leq (1-\rho)/2 = \max\{(1-\rho)/2, 1-2\rho\}$. When $\rho \leq 1/3$, note that $1 - 2\rho \geq (1-\rho)/2 \geq \rho$. If $\rho \geq \gamma$, both (B.43) and (B.44) hold by $(1-\rho)/2 \geq \rho \geq \gamma$. If $\gamma > \rho$, both (B.43) and (B.44) hold if and only if $\gamma \leq 1 - 2\rho = \max\{(1-\rho)/2, 1-2\rho\}$. Therefore, (B.44) holds iff $\gamma \leq \tilde{\gamma}$. □

B.5.3 The Stationary Fall-Back Equilibrium

Lemma B.13. *The following statements hold:*

(a) $p_\star \in (0,1) \iff \gamma \in (\underline{\gamma}, \overline{\gamma})$,

(b) $\sigma_\star \in (0,1) \iff \gamma \in (1-2\rho, \overline{\gamma})$.

Proof of Lemma B.13. (a) Note that

$$h_\star = \frac{\mu(\lambda_H - \lambda_L)}{\lambda_L} - \lambda_H = \lambda_L \cdot \left[\frac{1 - \frac{\lambda_L}{\lambda_H}}{\frac{\lambda_L}{\mu} \cdot \frac{\lambda_L}{\lambda_H}} - \frac{1}{\frac{\lambda_L}{\lambda_H}} \right] = \lambda_L \cdot \frac{1 - \rho - \gamma}{\rho\gamma} \tag{B.49}$$

By the definition of $p_\star$, we have

$$p_\star = \frac{\mu}{2\lambda_L} - \frac{\mu}{2h_\star} = \frac{\mu}{2\lambda_L} \cdot \left[1 - \frac{\rho\gamma}{1 - \rho - \gamma} \right] = \frac{1 - \rho - \gamma - \rho\gamma}{2\gamma(1 - \rho - \gamma)}. \tag{B.50}$$

From (2.3), we have $1-\rho-\gamma > 0$ and $1 > \gamma$. Then, $0 < p_\star$ is equivalent to $\gamma < \frac{1-\rho}{1+\rho} = \overline{\gamma}$, and $p_\star < 1$ is equivalent to:

$$\begin{aligned}
& & 1-\rho-\gamma-\rho\gamma < 2\gamma(1-\rho-\gamma) \\
&\iff & 0 < (1-\gamma)\cdot\{2\gamma-(1-\rho)\} \\
&\iff & \underline{\gamma} = \frac{1-\rho}{2} < \gamma.
\end{aligned}$$

(b) By the definition of $\sigma_\star$, $0 < \sigma_\star < 1$ is equivalent to:

$$0 < \frac{(\lambda_H-\lambda_L)p_\star}{\mu-\lambda_L p_\star} < 1 \qquad \iff \qquad 0 < p_\star < \frac{\mu}{\lambda_H}.$$

From the result of (a), $0 < p_\star$ is equivalent to $\gamma < \overline{\gamma}$.

From (2.3), we have $1-\rho-\gamma > 0$ and $1 > \rho$. By using these and (B.50), we have that $p_\star \leq \frac{\mu}{\lambda_H}$ is equivalent to:

$$\begin{aligned}
& & \frac{1-\rho-\gamma-\rho\gamma}{2\gamma(1-\rho-\gamma)} < \frac{\mu}{\lambda_H} = \frac{\rho}{\gamma} \\
&\iff & (1-\rho)(1-2\rho) < (1-\rho)\gamma \\
&\iff & 1-2\rho < \gamma.
\end{aligned}$$

$\square$

Lemma B.14. *Suppose that σ^* is a stationary fall-back strategy, i.e., for some $T \geq 0$, $\sigma_t = 1$ for all $t < T$ and $\sigma_t = \sigma_\star \in (0,1)$ for all $t \geq T$. If $\sigma^A = \sigma^B = \sigma^*$ constitutes a symmetric Nash equilibrium if and only if $\gamma \in (\tilde{\gamma}, \overline{\gamma})$.*

Proof of Lemma B.14. ($\Longrightarrow$) Suppose that $\sigma^A = \sigma^B = \sigma^*$ constitutes an equilibrium. Then, we need to have $p_\star \in (0,1)$ and $\sigma_\star \in (0,1)$. By Lemma B.13, we have $\tilde{\gamma} = \max\{\underline{\gamma}, 1-2\rho\} < \gamma < \overline{\gamma}$.

($\Longleftarrow$) Assume that $\gamma \in (\tilde{\gamma}, \overline{\gamma})$. By the construction of the stationary fall-back strategy, for all $t \geq T$, $\mu(V_t^1 - V_t^0) = \lambda_L(\Pi - V_t^0)$, which supports $\sigma_t^* \in (0,1)$. Next, we need to show that $\mu(V_t^1 - V_t^0) \geq \lambda_L(\Pi - V_t^0)$ for all $0 \leq t < T$ to support $\sigma_t = 1$. Assume the contrary:

$\mu(V_s^1 - V_s^0) < \lambda_L(\Pi - V_s^0)$ for some $0 \leq s < T$. Since $\mu(V_T^1 - V_T^0) = \lambda_L(\Pi - V_T^0)$, there exists $s < t \leq T$ such that $\mu(V_t^1 - V_t^0) = \lambda_L(\Pi - V_t^0)$ and $\mu(\dot{V}_t^1 - \dot{V}_t^0) > -\lambda_L \dot{V}_t^0$, or equivalently,

$$\lambda_L \dot{V}_t^1 > (\mu - \lambda_L)(\dot{V}_t^0 - \dot{V}_t^1). \tag{B.51}$$

As a first step, we show that there exists $C_1 < 0$ such that V_t^1 is given as (B.34) in Lemma B.11 for all $0 \leq t < T$. By $V_T^1 = V1_\star$ and $q_T = p_\star$, we have

$$C_1 = \frac{1}{(1-p_\star)} \left(\frac{1-p_\star}{\mu - \lambda_H p_\star} \right)^{\frac{\mu+\lambda_H}{\mu-\lambda_H}} \left(V_\star^1 - \overline{V}_1(p_\star) \right)$$

where $\overline{V}_1$ is defined as in (B.36).

Observe that

$$\begin{aligned}
V_\star^1 =& \frac{\lambda_H \Pi - c}{\lambda_H + h_\star} = \left(\Pi - \frac{c}{\lambda_H} \right) \cdot \frac{\gamma}{1-\rho}, \\
\overline{V}_1(p_\star) =& \left(\Pi - \frac{c}{\lambda_H} \right) \cdot \left[\frac{1}{2} + \frac{\lambda_H}{2(\lambda_H + \mu)}(1 - p_\star) \right] \\
=& \left(\Pi - \frac{c}{\lambda_H} \right) \cdot \left[\frac{1}{2} + \frac{(1-\gamma)\left(\gamma - \frac{1-\rho}{2}\right)}{2(\gamma+\rho)(1-\rho-\gamma)} \right].
\end{aligned}$$

Then, with some algebra, we have

$$\begin{aligned}
\overline{V}_1(p_\star) - V_\star^1 =& \left(\Pi - \frac{c}{\lambda_H} \right) \cdot \left[\frac{1}{2} + \frac{(1-\gamma)\left(\gamma - \frac{1-\rho}{2}\right)}{2(\gamma+\rho)(1-\rho-\gamma)} - \frac{\gamma}{1-\rho} \right] \\
=& \left(\Pi - \frac{c}{\lambda_H} \right) \cdot \left(\gamma - \frac{1-\rho}{2} \right)^2 \cdot \frac{\gamma - (1-2\rho)}{(\gamma+\rho)(1-\rho-\gamma)}.
\end{aligned}$$

From $\gamma > \tilde{\gamma} \geq 1 - 2\rho$, we have $\overline{V}_1(p_\star) - V_\star^1 > 0$, thus, $C_1 < 0$. Then, for all $0 \leq t < T$, we have

$$\dot{V}_t^1 = \dot{q}_t \left[- \left(\Pi - \frac{c}{\lambda_H} \right) \frac{\lambda_H}{2(\lambda_H + \mu)} + C_1 \cdot \frac{\lambda_H(1+q_t)}{1-q_t} \left(\frac{1-q_t}{\mu - \lambda_H q_t} \right)^{\frac{2\lambda_H}{\lambda_H - \mu}} \right] < 0. \tag{B.52}$$

By (HJB$_1$) and (HJB$_0$), we have

$$\begin{aligned}\dot{V}_t^1 =&\lambda_H(1+q_t)V_t^1 + c - \lambda_H\Pi\\ \dot{V}_t^0 =&\lambda_H q_t V_t^0 + c - \mu(V_t^1 - V_t^0).\end{aligned}$$

By using $\lambda_L(\Pi - V_t^0) = \mu(V_t^1 - V_t^0)$, we can derive that

$$\begin{aligned}\dot{V}_t^0 - \dot{V}_t^1 =&\lambda_H(1+q_t)(V_t^0 - V_t^1) + \mu(V_t^0 - V_t^1) + \lambda_H(\Pi - V_t^0)\\ =&\left[(\lambda_H - \lambda_L)\mu - \lambda_H\lambda_L(1+q_t)\right]\left(\frac{\Pi - V_t^0}{\mu}\right).\end{aligned}$$

Note that $\Pi > V_t^0$ since the expected payoff cannot exceed the rent Π. By using $\Pi > V_t^0$, $p_\star \geq q_t$ and $\gamma > \max\{1-2\rho, \frac{1-\rho}{2}\}$, we can derive that

$$\begin{aligned}\dot{V}_t^0 - \dot{V}_t^1 \geq&\left[(\lambda_H - \lambda_L)\mu - \lambda_H\lambda_L(1+p_\star)\right]\left(\frac{\Pi - V_t^0}{\mu}\right)\\ =&\mu\lambda_H\frac{\{\gamma - (1-2\rho)\}\cdot\left(\gamma - \frac{1-\rho}{2}\right)}{1-\rho-\gamma}\left(\frac{\Pi - V_t^0}{\mu}\right) > 0.\end{aligned} \tag{B.53}$$

Then, (B.52) and (B.53) contradict (B.51). Therefore, $\mu(V_t^1 - V_t^0) \geq \lambda_L(\Pi - V_t^0)$ for all $0 \leq t < T$, and the stationary fall-back strategy constitutes an equilibrium. □

B.5.4 Proof of Theorem 2

C Proofs of Section 5.1

C.1 Proof of Proposition 5

We divide the proof in two parts. First, we study what is the equilibrium behavior of the firms in the patent fee negotiation phase, i.e. after the first research breakthrough is obtained and publicly observed. We then move to the equilibrium allocation of resources before the first research breakthrough occurs.

Subgame after Research Breakthrough

After a research breakthrough occurs, the firm that had the breakthrough has to decide whether to patent the technology or not and which fee to offer to its competitor. The competitor observes the offer and decides whether to accept or not.

Lemma C.1. *In any PBE, the new technology is immediately patented after breakthough. Moreover, the license fee is equal to* $\dfrac{(\lambda_H - \lambda_L)(\lambda_H \Pi + c)}{2\lambda_H(\lambda_H + \lambda_L)}$ *and accepted.*

Proof. Once a patent is granted and a fee x is offered, the competitor must decide whether to accept or reject it. If the competitor rejects the offer, it must continue developing using the old technology. In this case, its continuation value of rejecting is $V_R = \frac{\lambda_L \Pi - c}{\lambda_H + \lambda_L}$. If the competitor accepts the offer, both firms can use the new technology moving forward, and the expected continuation payoff is $V_C = \frac{\lambda_H \Pi - c}{2\lambda_H}$. Thus, it is sequentially rational for the competitor to accept the offer if $x \leq x^*$, where

$$x^* = V_C - V_R = \frac{(\lambda_H - \lambda_L)(\lambda_H \Pi + c)}{2\lambda_H(\lambda_H + \lambda_L)} > 0.$$

The patent holder will never choose a fee strictly below x^*, since higher fees would also be accepted. Moreover, he will not choose a fee that he expects to be rejected, since the total surplus of the firms is larger when the competitor accepts.[16] To see this, note that the expected payoff for the patent holder when the offer x^* is accepted is:

$$V_L \equiv V_C + x^* = \left(1 + \frac{(\lambda_H - \lambda_L)c}{\lambda_H(\lambda_H \Pi - c)}\right) \frac{\lambda_H \Pi - c}{\lambda_H + \lambda_L} > \frac{\lambda_H \Pi - c}{\lambda_H + \lambda_L}. \tag{C.1}$$

Where the right-hand side corresponds to the payoff of the patent holder when the offer is rejected. Thus, in any PBE, the patent holder chooses a fee x^* and the competitor accepts.

Next, we consider the patent application decision of the first firm that obtains the research breakthrough and show that it always pays to apply for a patent. Note that the continuation payoff for a firm that applies for a patent is always above V_P, since the firm can always guarantee that continuation payoff by making an offer equal to V_P, that when rejected generates a continuation payoff V_P.

[16] Moreover, even if we consider a different bargaining protocol, a mutually agreeable licensing fee always exists.

Suppose that $\eta \geq \bar{\eta}(\delta)$. Then, $\mu \geq \frac{2\lambda_H \lambda_L}{\lambda_H - \lambda_L}$. Rearranging we get that $(\mu - \lambda_L) \geq \frac{\mu(\lambda_H + \lambda_L)}{2\lambda_H}$. Multiplying side by side by $\lambda_H \Pi - c$, and rearranging we obtain $\frac{\lambda_H \Pi - c}{\lambda_H + \lambda_L} \geq \frac{\lambda_H + \mu V_C - c}{\lambda_H + \mu}$. The left-hand side is V_P and the right-hand side the payoff when the firm decides not to patent the innovation, since the competitor best responds by choosing $\sigma = 1$. When $\eta < \bar{\eta}(\delta)$ the competitor reacts to the breakthrough by developing with the old technology. Thus, the continuation payoff of a firm that does not patent, in equilibrium, is equal to V_P, which is lower than the payoff of patenting. □

Pre Breakthrough

In this subsection, we characterize the allocation of resources before the first research breakthrough. The equilibrium continuation value of having a research breakthrough is equal to

$$V^{\text{Patent}} = V_C + x^* = 2V_C - V_R = \frac{\lambda_H \Pi - \frac{\lambda_L}{\lambda_H} c}{\lambda_H + \lambda_L} \tag{C.2}$$

By definition, in a MPBE, the firms use a constant allocation policy before the first breakthrough. Let σ be the allocation policy of a firm and let $\hat{\sigma}$ allocation of the opponent. We can express the expected payoff of the firm as:

$$V(\sigma, \hat{\sigma}) := \frac{\sigma \mu V^{\text{Patent}} + (1-\sigma)\lambda_L \Pi + \hat{\sigma} V_R - c}{\sigma \mu + \hat{\sigma}\mu + (1-\sigma)\lambda_L + (1-\hat{\sigma})\lambda_L} \tag{C.3}$$

Taking the partial derivative of V from Equation C.3 with respect to σ we obtain

$$V_1(\sigma, \hat{\sigma}) = \frac{-\lambda_L(\mu((1+\hat{\sigma})\Pi + c - (2-\hat{\sigma})V^{\text{Patent}} - \hat{\sigma} V_R) + \mu(\mu\hat{\sigma}(V^{\text{Patent}} - V_R) + c) - \Pi(1-\hat{\sigma})\lambda_L^2}{(\mu(\sigma + \hat{\sigma}) + \lambda_L(2 - \sigma - \hat{\sigma}))^2} \tag{C.4}$$

Note that the denominator of the right-hand side is strictly positive. Thus, the best response to any $\hat{\sigma}$ is pinned down by the sign of the numerator. The next two lemmata will help characterize the equilibrium allocation by determining when is that research allocation of firms present strategic complementarities and what is the best response to direct development.

Lemma C.2. *Research allocation presents strategic complementarities, i.e. $V(\sigma, \hat{\sigma})$ satisfies Milgrom-Shannon single crossing condition, if and only if $\mu \geq \bar{m} := \frac{\lambda_H \Pi}{c + \lambda_H \Pi} \cdot \underline{\mu}$.*

Proof. The numerator of the right-hand ride of Eq. C.4 is increasing in $\hat{\sigma}$ if and only if

$$
\begin{aligned}
& \mu(V^{\text{Patent}} - V_R) - \lambda_L \Pi \geq 0 \\
\iff \qquad & \mu \geq \frac{\lambda_L \Pi}{V^{\text{Patent}} - V_R} = \frac{\lambda_L \lambda_H \Pi(\lambda_L + \lambda_H)}{(c + \lambda_H \Pi)(\lambda_H - \lambda_L)} = \frac{\lambda_H \Pi}{c + \lambda_H \Pi} \cdot \underline{\mu}.
\end{aligned}
$$

□

Lemma C.3. *The derivative $V_1(\sigma, 0)$ is (strictly) positive if and only if $\mu \geq (>) \underline{m}$, where*

$$
\underline{m} := \frac{\lambda_L(c + \lambda_L \Pi)}{\lambda_H(c + \lambda_L \Pi) + 2\lambda_L c} \bar{\mu}
$$

Proof. Evaluating Eq. C.4 at $\hat{\sigma} = 0$, we obtain

$$
V_1(\sigma, 0) = \frac{c(\mu - \lambda_L) - \mu \lambda_L(\Pi - 2V^{\text{Patent}}) - \lambda_L^2 \Pi}{(\mu\sigma + \lambda_L(2 - \sigma))^2}
$$

Which is strictly positive if and only if the numerator is strictly positive, i.e. if

$$
c(\mu - \lambda_L) - \mu \lambda_L(\Pi - 2V^{\text{Patent}}) - \lambda_L^2 \Pi > 0
$$

Multiplying by $\lambda_H(\lambda_L + \lambda_H)$ and replacing V^{Patent} using Equation C.2, the inequality is equivalent to

$$
\lambda_H(c + \lambda_L \Pi)[\mu(\lambda_H - \lambda_L) - \lambda_L(\lambda_L + \lambda_H)] + 2\mu\lambda_L c(\lambda_H - \lambda_L) \geq 0
$$

Or, equivalently,

$$
\mu(\lambda_H - \lambda_L) - \lambda_H(\lambda_L + \lambda_H) \geq -\frac{2\mu\lambda_L c(\lambda_H - \lambda_L)}{\lambda_H(c + \lambda_L \Pi)} \qquad \Leftrightarrow \qquad \underline{\mu} - \mu \leq \frac{2\mu\lambda_L c}{\lambda_H(c + \lambda_L \Pi)}
$$

Solving for μ we obtain the desired result. □

Lemma C.4. *For all $(c, \Pi, \lambda_L, \lambda_H)$ that satisfy Assumption A2, $\bar{m} > \underline{m}$. Moreover, the following holds.*

1. *If $\mu \geq \bar{m}$, then both firms do research in any MPBE.*

2. *If $\mu \in (\underline{m}, \bar{m})$, there is a MPBE in which only one of the firms does research.*

3. *If $\mu \leq \underline{m}$ both firms do direct development in any MPBE.*

We start by proving that $\bar{m} > \underline{m}$. From the definitions, we get that

$$\underline{m} < \bar{m} \qquad \Leftrightarrow \qquad \frac{c + \lambda_L \Pi}{\lambda_H(c + \lambda_L \Pi) + 2\lambda_L c} < \frac{\Pi}{c + \lambda_H \Pi}$$

Rearranging we obtain that this is equivalent to

$$(\lambda_L \Pi - c)c + (\lambda_H - \lambda_L)\lambda_H \Pi^2 > 0$$

Which holds by Assumption A2.

We now move to prove the first point. When $\mu > \bar{m}$, we have that

$$V_1(\sigma, \hat{\sigma}) \geq V_1(\sigma, 0) > 0$$

Where the first inequality follows from Lemma C.2 and the second inequality holds since $\mu \geq \bar{\mu} > \underline{m}$ and Lemma C.3. Thus, research is a dominant allocation policy. Thus, in any MPBE, both firms do research.

To prove point 2, we will show that, when $\mu \in (\underline{m}, \bar{m})$, $V_1(\sigma, 0)$ is strictly positive and $V_1(\sigma, 1)$ is strictly negative for all σ. The first part is true by Lemma C.3. To get the second part, note that evaluating Equation C.4 at $\hat{\sigma} = 1$ we obtain that the numerator of $V_1(\sigma, 1)$ is,

$$(c + \mu\Pi)\lambda_H(\lambda_H - \lambda_L)(\mu - \bar{\mu}) - c\mu(\mu + \lambda_L)(\lambda_H - \lambda_L) \tag{C.5}$$

This is strictly negative since $\mu < \bar{m} < \bar{\mu}$. Thus, $V_1(\sigma, 1)$ is strictly negative.

Finally, to prove the third point, note that for $\mu < \underline{m}$ we have that

$$V_1(\sigma, \hat{\sigma}) < V_1(\sigma, 1) < 0$$

Where the first inequality holds because $\mu < \underline{m} < \bar{m}$ and Lemma C.2 and the second inequality by Eq. C.5.