Strategic Concealment in Innovation Races*

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Abstract

Firms racing to innovate often achieve interim technological breakthroughs that can accelerate final innovation. When these advances are acquired privately, firms face a choice: disclose the discovery or keep it secret. This paper studies how this trade-off is shaped by the race structure and the intellectual property system. We develop a dynamic model where firms allocate resources between direct development and research for new technology. Our results show that excessively strong reward from winning the race or strong prior-use protections can discourage disclosure, thereby impeding knowledge spillovers and slowing the social speed of innovation.

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1 Introduction

Firms racing to innovate often achieve intermediate breakthroughs—partial advances that do not fully solve the underlying problem but bring them significantly closer to the final goal. For example, the development of mRNA technology before the COVID-19 pandemic provided a crucial building block for the rapid creation of vaccines. Similarly, advances in battery chemistry have accelerated innovation in electric vehicle design. In software, new frameworks or tools can simplify later stages of development and speed up the creation of a finished product. Although these discoveries are not the final innovation, they can significantly alter the dynamics of the race.

When firms *privately* acquire such interim technologies, they face a choice: keep the discovery secret or disclose it. This decision plays a crucial role in shaping knowledge spillovers and the social speed of innovation. This paper studies the trade-off firms face between disclosure and concealment, and how it is influenced by the structure of the race and features of the intellectual property system, such as patents and prior-use defenses.

To illustrate high-level intuition, consider two firms, A and B, competing to innovate. Firm A privately discovers an interim technology that could accelerate innovation, while Firm B remains uncertain whether Firm A holds this advantage. Firm A must decide whether to conceal or disclose the technology. The table below summarizes Firm B's expected payoff and total welfare—the combined expected payoffs of both firms—under each case:

	Concealment	**Disclosure**
Firm B's payoff	V_C^B	V_D^B
Total welfare	W_C	W_D

Disclosure can enhance overall welfare by enabling faster innovation through knowledge spillover, implying $W_D > W_C$. However, disclosure also reveals to Firm B that Firm A possesses the new technology, potentially prompting Firm B to adjust its R&D strategies based on this information. This may raise Firm B's outside option to $V_D^B > V_D^C$, thereby weakening Firm A's strategic position. Thus, Firm A's disclosure decision hinges on whether the net gain from disclosure $(W_D - V_D^B)$ exceeds the benefit of concealment $(W_C - V_C^B)$, which is not immediately clear. As a result, a **hold-up problem** may arise: although disclosure and cooperation could maximize social welfare, Firm A may hesitate to share the technology due to the risk of strengthening Firm B's outside option and reducing its own payoff.

To analyze this trade-off in the context of race structure and the patent system, we consider a model in which firms race to develop a new product, such as a vaccine. The firm to succeed receives a fixed reward, such as temporary monopoly profits. Firms can allocate their limited resources across various pathways. One option is to develop the product using currently *available technologies*. We assume that success along the path requires a single breakthrough and refer to this approach as *direct development*. Alternatively, firms can allocate resources to *research*, seeking to discover a faster new technology. Success along this path requires two steps: first, discovering the new technology, and second, developing the product using it. After discovering the interim technology, a firm chooses to disclose it through patenting or keep it secret.

We begin by abstracting away from the patenting decision, assuming that the new technology is non-patentable. In this case, our model highlights the trade-off faced by resource-constrained firms. On one hand, allocating more resources to researching a new technology slows short-term development because fewer resources are deployed for direct development. On the other hand, this approach increases the probability of obtaining a superior technology, thereby raising the expected future development rate.

Based on this intuition, if firms can observe rival firms' research progress—the acquisition of a new technology—a firm's optimal resource allocation may depend on information about the competitor's progress. Under certain parametric conditions, firms adopt a *fall-back strategy* in equilibrium: they engage in research when there is no progress from rival, but switch to direct development once the rival firm discovers the new technology, anticipating the rival's product development is imminent (Proposition 1 (b)). In contrast, when research progress is private information, firms cannot condition their strategies on rivals' discoveries, making the fall-back strategy infeasible. Instead, firms either continue researching or, after some time, partially reallocate resources from research to direct development (Proposition 3).

These differences in equilibrium behaviors between public and private information environments highlight how a rival's outside option is shaped by the disclosure or concealment of the new technology. In particular, the ability to react to discovery information—enabled by disclosure—becomes a key channel through which the rival's outside option is elevated.

Next, we incorporate firms' patenting decisions into the model. When a firm patents the new technology, the firm secures an exclusive right to use it and can license it to the rival, enabling controlled spillovers through licensing. However, patenting also signals to the rival that the firm holds the new technology, allowing the rival to adjust its R&D strategy and potentially raising its outside option in the licensing process. Thus, the patenting decision captures the trade-off between disclosure and concealment highlighted earlier.

Our main results show that firms' patenting decisions crucially depend on the stake of winning the race (π) and the strength of prior-use defense (β)—the probability that a patent application is challenged on the grounds of trade secret protection or prior commercial use.[1] We show that when both π and β are sufficiently high, firms conceal their discoveries of the new technology in equilibrium, thereby preventing knowledge spillover and slowing innovation (Proposition 5). In contrast, if β is sufficiently small, or if π lies within an intermediate range when β is high, firms choose to disclose and license the new technology, leading to knowledge spillovers that accelerate the social speed of innovation (Proposition 4).

Our results yield empirical implications for how the stakes of winning the race and the strength of prior-use protections shape innovation dynamics. When β is sufficiently high, the model predicts a non-monotonic relationship between π and social speed of innovation. If π is too low, firms opt out of R&D altogether, resulting in no innovation. As π rises to an intermediate range, firms engage in R&D and share interim discoveries, boosting the overall rate of innovation. However, if π becomes excessively high, the incentive to conceal technological breakthroughs in order to preserve competitive advantage dominates, suppressing knowledge spillovers and slowing social innovation. Similarly, a stronger prior-

[1] Specifically, suppose Firm A applies for a patent on a new technology, while Firm B has independently discovered the same technology but has not patented it. Firm B challenges Firm A's patent, the challenge succeeds with probability β. In that case, both firms retain the right to use the new technology based on prior commercial use (US Code §273). However, Firm B cannot claim exclusive rights—even if it discovered the technology earlier—as it did not file for a patent, and trade secret does not extend to independent discoveries by rival firms.

use defense can discourage patenting and public disclosure, leading firms to conceal their discoveries and impeding the diffusion of technological advances.

These findings also offer policy insights. While strong rewards are necessary to spur R&D investment, excessively high stakes in the product market can undermine knowledge diffusion and slow cumulative innovation. Competition policy and limits on monopoly power thus play an important role in balancing incentives for discovery and disclosure. Likewise, while prior-use defenses protect early innovators, overly strong protections may weaken patenting incentives and encourage secrecy, resulting in duplicative efforts and inefficiencies. Our results underscore that policies intended to promote innovation along one margin can unintentionally suppress it along another, highlighting the need for careful institutional design to foster long-term technological progress.

2 Model

Race Setup We consider a race between two firms, A and B, aiming to develop a new product. Time is continuous and infinite: $t \in [0, \infty)$. The race ends when one of the firms successfully develops the product. Throughout the race, firms incur a constant flow cost $c > 0$, and the first firm to develop the product receives a lump-sum reward worth Π.[2] Firms do not discount the future and maximize expected total payoffs.

R&D Paths and Resource Allocation There are two technologies for product development: the *old* and *new* technologies, with development rates λ_L and λ_H, respectively, where $\lambda_L < \lambda_H$. Initially, both firms have access only to the old technology. The new technology can be acquired either (i) through independent research or (ii) by licensing it from a competitor that has patented it—we elaborate this later in the section.

To discover the new technology through independent research, a firm must invest time and resources. At each point in time, a firm without the new technology can flexibly allocate its unit of *resources* between researching the new technology and developing the product using the old technology. If Firm i allocates a fraction $\sigma \in [0, 1]$ of resources to 'research' at time

[2] This winner-takes-it-all payoff structure has been commonly used in the innovation race literature, e.g., Loury (1979); Lee and Wilde (1980); Denicolò and Franzoni (2010).

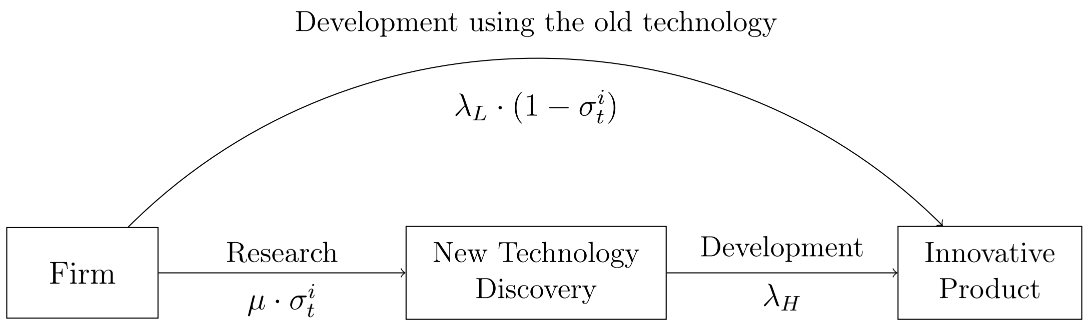

Figure 1: Development Paths and Resource Allocations

t, it discovers the new technology at rate $\mu \cdot \sigma$. The remaining fraction $1 - \sigma$ of resources contributes to the product development using the old technology at rate $\lambda_L \cdot (1 - \sigma)$. Figure 1 illustrates the development paths and resource allocation framework.

Informational Setup Let $N^i = \{N_t^i\}_{t \geq 0}$ be a point process indicating whether Firm i has acquired the new technology through independent research, where $N_t^i = 1$ if discovery has occurred by time t and $N_t^i = 0$ otherwise. We refer to N^i as the *research progress* of Firm i.

We assume that firms observe their own research progress but not their competitors' resource allocations. Regarding rivals' research progress, we consider two settings: (i) **public research progress**, where Firm i observes N^j, and (ii) **private research progress**, where Firm i cannot observe N^j. These settings are analyzed in Section 3 and 4, respectively.

Patenting and Licensing Firms can patent the new technology upon discovery, provided that no rival has already done so.[3] A patent grants (i) exclusive rights to use the new technology and (ii) the ability to license it to the rival. If Firm i patents the technology, then Firm j cannot use the new technology without obtaining a license, even if Firm j had independently discovered it.

A patent application by Firm i is publicly observable. If Firm j has also independently discovered the technology, it can credibly and costlessly appeal the patent by presenting verifiable research progress. If the appeal is successful, both firms are granted rights to use the new technology; otherwise, Firm i holds the patent. Thus, the absence of an appeal from Firm j reveals $N^j = 0$, making its research progress publicly observable.

[3] For simplicity, we assume firms must apply immediately after discovery.

Let $\alpha(N)$ denote the probability of patent approval given the rival's research progress N. It is natural to assume that $\alpha(0) \geq \alpha(1)$: the patent is more likely to be granted when there is no appeal. Here, $\alpha(0)$ denotes the patentability of the new technology: (i) it is *non-patentable* if $\alpha(0) = 0$; (ii) it is *patentable* if $\alpha(0) = 1$. We define $\beta := 1 - \alpha(1)$ as the probability that an appeal is successful, which can be interpreted as the level of *prior-use defense*. For example, when $\alpha(0) = \alpha(1) = 1$, it corresponds to a *pure first-to-file* patent system.

Once the patent is approved, the holder can license the technology to its rival using a take-it-or-leave-it bargaining process, i.e., the holder has the full bargaining power in negotiations.[4]

Parametric Assumptions We impose the following two parametric assumptions for the remainder of the paper:

$$\frac{1}{\mu} + \frac{1}{\lambda_H} < \frac{1}{\lambda_L} \qquad \& \qquad \Pi - \frac{c}{\lambda_L} > 0. \tag{2.1}$$

The first inequality implies that the new technology path is faster on average than the old one: discovering the new technology takes expected time $1/\mu$, followed by development time $1/\lambda_H$, while developing with the old technology alone takes $1/\lambda_L$. The second inequality ensures that using the old technology is profitable, allowing us to abstract from firms' exit decisions. Without this condition, firms would never allocate resources to development using the old technology.

First-Best Outcome Consider a social planner who aims to maximize the firms' joint expected profit. The planner controls firms' resource allocations, observe their research progress, and makes the patent and license decisions—including the patentability of the technology. If a firm discovers the new technology, it is socially optimal to license it to the other. When neither had made a discovery, (2.1) implies that it is socially efficient for both

[4]An equivalent interpretation is as follows. Suppose Firm i patents the new technology, thereby the technology is disclosed. Firm j may then infringe on the patent by using Firm i's technology to develop the product. In that case, Firm i can sue for patent infringement, and the court would order Firm j to pay Firm i an amount equivalent to what Firm i would have received if it had full bargaining power and had negotiated a license with Firm j.

to continue researching. Therefore, the first-best outcome is to allocate both firms' resources to research until a discovery is made, after which the discovering firm patents and licenses the technology for joint development.

3 Public Research Progress

In this section, we assume that firms' research progress is public information. Thus, firms can condition their allocation not only on their own research progress but also on that of the competitor. Formally, at any time t, the set of firms that have discovered the new technology, $\{i \mid N_t^i = 1\}$, is common knowledge. We represent this as a state variable, ω, which belongs to the state space $\Omega := \{\{A, B\}, \{A\}, \{B\}, \emptyset\}$.

Fix parameters λ_H and λ_L, and define

$$\lambda_\star(\mu) := \mu \cdot \lambda_H \cdot \left(\frac{1}{\lambda_L} - \frac{1}{\mu} - \frac{1}{\lambda_H} \right). \tag{3.1}$$

Under our parametric assumptions, the function $\lambda_\star(\mu)$ takes positive values and is strictly increasing. It captures the relative attractiveness of pursuing research to acquire the new technology, compared to direct development with the old technology. In Appendix B, we provide an interpretation of $\lambda_\star(\mu)$ as the threshold that governs optimal allocation in a setting where a single firm faces a constant exogenous rate of termination.

This threshold turns out to be central for characterizing equilibrium behavior of the firms in the innovation race, both when the new technology is patentable or not.

3.1 Non-Patentable Technology

When the new technology is not patentable, we can assume that firms do not apply for patents without loss of generality. Thus, the only way for firms to access the new technology is through independent research. We focus our analysis on how firms decide to allocate their resources to research or development with the old technology. In particular, given the stationarity of the problem, we consider Markov strategies, where a firm's allocation depends only on the state variable ω. Firm i's Markov strategy is given by $\mathbf{s}^i : \Omega \to [0, 1]$, specifying

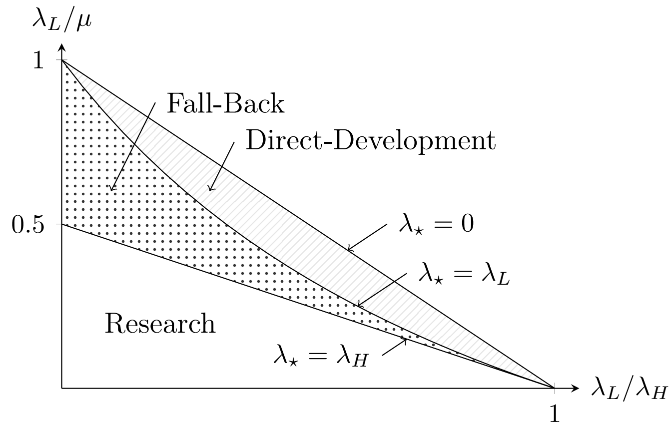

Figure 2: Markov Perfect Equilibrium in the Public Research Progress Setting

the fraction of resources allocated to research for each state. A pair of strategies $(\mathbf{s}^A, \mathbf{s}^B)$ constitutes a Markov perfect equilibrium (MPE) if each firm's Markov strategy is the best response to the opponent's strategy. Next, we introduce three relevant Markov strategies.

Definition 1. (a) The *research strategy* $\mathbf{s}^i_R$ for Firm i fully allocates resources to research regardless of the opponent's progress ($\mathbf{s}^i_R := \mathbb{1}_{\{i \notin \omega\}}$).[5]

(b) The *fall-back strategy* $\mathbf{s}^i_F$ fully allocates resources to research if neither firm has the new technology. If one of the firms has obtained the new technology, it fully allocates resources to development ($\mathbf{s}^i_F := \mathbb{1}_{\{\omega = \emptyset\}}$).

(c) The *direct-development strategy* $\mathbf{s}^i_D$ fully allocates the resources to development regardless of the state ($\mathbf{s}^i_D := 0$).

The following proposition shows that when the new technology is not patentable, there exists a unique MPE in which firms adopt one of the relevant Markov strategies defined above.

Proposition 1. *Suppose that firms' research progress is public information and the new technology is not patentable. Then, the Markov perfect equilibrium is uniquely characterized as follows:*

[5]The function $\mathbb{1}_X$ is an indicator function: $\mathbb{1}_X(\omega) = 1$ if $\omega \in X$ and $\mathbb{1}_X(\omega) = 0$ if $\omega \notin X$.

(a) *if* $\lambda_\star(\mu) > \lambda_H$*, both firms play their respective research strategies* $(\mathbf{s}_R^A, \mathbf{s}_R^B)$*;*

(b) *if* $\lambda_\star(\mu) \in (\lambda_L, \lambda_H)$*, both firms play the fall-back strategies* $(\mathbf{s}_F^A, \mathbf{s}_F^B)$*;*

(c) *if* $\lambda_\star(\mu) < \lambda_L$*, both firms play the direct-development strategies* $(\mathbf{s}_D^A, \mathbf{s}_D^B)$*.*

The relevant parametric regions from the proposition above are illustrated in Figure 2. In this figure, the vertical axis represents the relatively difficulty of discovering the new technology, while the horizontal axis captures how efficient the new technology is compared to the old one. The boundaries between the different regions is determined by the combination for which $\lambda_\star(\mu)$ equal to 0, λ_L, and λ_H.

This result is intuitive: when the research rate is fast, firms focus exclusively on research; and when it is slow, they concentrate entirely on development using the old technology. In both cases, they disregard information about competitors' research progress. However, when the research rate falls within an intermediate range, firms adopt the fall-back strategy, and their allocation decisions depend on information about competitors' research progress.

3.2 Patentable Technology

When the new technology is patentable, firms' patenting decisions, in addition to their resource allocations, must be considered. The next lemma shows that firms always patent the new technology upon discovery.

Lemma 1. *Suppose that firms' research progress is public information and that the new technology is patentable. In any subgame perfect Nash Equilibrium (SPNE), the first firm to discover the new technology applies for a patent.*

Having established firms' patenting behavior, we now examine their licensing decisions and the optimal fee set by the patent holder. Consider the subgame in which Firm i holds the patent. If the licensing offer is accepted, both firms develop using the new technology, each receiving the continuation payoff $V_{11} := \frac{\lambda_H \Pi - c}{2\lambda_H}$. If the offer is rejected, Firm i uses the new technology while Firm j develops with the old one, yielding continuation payoffs

$V_{10} := \frac{\lambda_H \Pi - c}{\lambda_H + \lambda_L}$ and $V_{01} := \frac{\lambda_L \Pi - c}{\lambda_H + \lambda_L}$, respectively.[6] Firm j accepts the licensing fee offer l iff $V_{11} - l \geq V_{01}$. The next lemma characterizes the optimal licensing fee offered by the patent holder.

Lemma 2. *In any subgame perfect Nash equilibrium (SPNE), when a firm obtains the patent for the new technology, it offers a licensing fee*

$$l^* := V_{11} - V_{01} = \frac{\lambda_H - \lambda_L}{\lambda_H + \lambda_L} \cdot \frac{\lambda_H \Pi + c}{2\lambda_H} \tag{3.2}$$

to its competitor, and the competitor accepts the offer.

Intuitively, the total surplus of the firms is maximized when the patenting firm licenses the technology. This is because the expected development time is shorter when both firms develop using the new technology, compared to when one firm uses the new technology and the other uses the old one. Therefore, there exists a non-empty set of fees that the patent holder is willing to offer and the licensee is willing to accept, and a licensing agreement is achieved á la Coase (1960). The license fee l^* is simply the fee that leaves the licensee indifferent between accepting and rejecting the offer.

As we showed in Lemma 1, when the new technology is patentable, firms apply for patents as soon as they discover the new technology. Therefore, when the new technology is patentable, the patent is granted with probability one on the equilibrium path. Given this and the optimal licensing fee obtained in Lemma 2, we can determine the continuation payoffs of each firm after the discovery of the new technology. Using these continuation payoffs, we can analyze the equilibrium resource allocations prior to the new technology discovery. The following proposition identifies the condition that the first-best outcome can be implemented. We present the full equilibrium characterization in Appendix C.2.3.

Proposition 2. *Suppose that firms' research progress is public information and that the new technology is patentable. There exists a threshold $\tilde{\pi}_1 > 1$ such that the first-best outcome can be implemented in the equilibrium if and only if (i) $\lambda_\star(\mu) > \lambda_L$; or (ii) $\lambda_L > \lambda_\star(\mu) > \frac{\lambda_H \lambda_L}{2\lambda_H + \lambda_L}$ and $\tilde{\pi}_1 > \pi := \lambda_L \Pi / c$.*

[6] Firm j does not exit the market since $V_{01} > 0$ under the parametric assumption.

Intuitively, the possibility of patenting increases incentives for research. Recall that when $\lambda_\star(\mu) > \lambda_L$, firms engage in research even when the new technology is non-patentable. Thus, firms will continue researching in this parametric region when patenting the new technology becomes possible (Part (i)).

When $\lambda_\star(\mu) < \lambda_L$, recall that if the new technology is non-patentable, both firms develop the product using the old technology. Even if the new technology is patentable, the incentive to develop the product directly using the old technology remains strong—particularly when the stake of winning the race, π, is sufficiently high. Therefore, the first-best outcome can be achieved when π is relatively small (Part (ii)). It is important to note that since patents are never challenged on the equilibrium path, the equilibrium research allocations remain independent of the level of prior-use defense level, $\beta = 1 - \alpha(1)$.

4 Private Research Progress

Having characterized the equilibrium behavior of firms when research progress is public, we now turn to the main specification of our model, in which firms do not observe their opponents' research progress. Unlike in the public research progress case, the fall-back strategy is no longer feasible—firms cannot adjust their R&D strategies based on their opponents' progress.

4.1 Non-Patentable Technology

When the technology is non-patentable, firms do not apply for patents, and the only way to access the new technology is through independent discovery. As before, a firm that possesses the new technology will use it to develop at rate λ_H. A firm without the new technology, on the other hand, can allocate resources between development and research. For this firm, the only relevant history is the calendar time, as the rival's progress is unobservable. Thus, its strategy is to choose an *allocation policy*, a right-continuous function $\boldsymbol{\sigma} : \mathbb{R}_+ \to [0, 1]$ that represents the share of resources allocated to research at a given time, conditional on not having obtained the new technology so far. We denote $\mathcal{S}$ as the set of allocation policies.

Belief Evolution Let $\mathbf{p}_{\boldsymbol{\sigma}}$ be the probability that a firm following allocation policy $\boldsymbol{\sigma}$ obtains the new technology by time t, conditional on not having developed the product yet.[7] In other words, when Firm A follows policy $\boldsymbol{\sigma}$, Firm B's belief that Firm A obtained the new technology by time t is $\mathbf{p}_{\boldsymbol{\sigma}}(t)$. The following proposition characterizes, the evolution of $\mathbf{p}_{\boldsymbol{\sigma}}$ over time for any policy $\boldsymbol{\sigma} \in \mathcal{S}$.

Lemma 3. *For any allocation policy $\boldsymbol{\sigma} \in \mathcal{S}$, the conditional probability $\mathbf{p}_{\boldsymbol{\sigma}}(t)$ satisfies the initial condition $\mathbf{p}_{\boldsymbol{\sigma}}(0) = 0$ and evolves according to the differential equation $\dot{\mathbf{p}}_{\boldsymbol{\sigma}}(t) = BE(\mathbf{p}_{\boldsymbol{\sigma}}(t), \boldsymbol{\sigma}(t))$, where $BE: [0,1] \times [0,1] \to \mathbb{R}$ is given by:*

$$BE(p, \sigma) := \underbrace{\mu \cdot \sigma \cdot (1-p)}_{DE} - \underbrace{(\lambda_H - (1-\sigma) \cdot \lambda_L) \cdot p \cdot (1-p)}_{SRE}. \tag{4.1}$$

The belief evolution function BE highlights two distinct effects of resource allocation $\boldsymbol{\sigma}(t)$ on the evolution of $\mathbf{p}_{\boldsymbol{\sigma}}$, captured by the two terms in (4.1). First, if the firm has not yet discovered the new technology—occurring with probability $(1 - \mathbf{p}_{\boldsymbol{\sigma}}(t))$—it is discovered at rate $\mu \cdot \boldsymbol{\sigma}(t)$. We dub this effect the *duration effect* (DE). Second, the absence of a successful development indicates that it is less likely that the firm has obtained the new technology. This is captured in the second term, referred to as the *still-in-the-race* effect (SRE).[8] The SRE is proportional to $\lambda_H - (1 - \boldsymbol{\sigma}(t))\lambda_L$, the difference in the rate of development of the firm with and without the new technology.

Based on this belief evolution, the following lemma derives the probability that a firm has discovered the new technology by time t, given that it has researched up to t and no product development has occurred by then. Since allocation beyond t does not affect this conditional probability, it remains the same as under the research policy ($\boldsymbol{\sigma} = \mathbf{1}$).

Lemma 4. *Suppose that a firm follows an allocation policy $\boldsymbol{\sigma}$, with $\boldsymbol{\sigma}(s) = 1$ for $s \in [0, t)$. Then, the conditional probability $\mathbf{p}_{\boldsymbol{\sigma}}(t)$ of having access to the new technology by time t given*

[7] See Appendix D.1 for the formal definition of $\mathbf{p}_{\boldsymbol{\sigma}}$.

[8] Similar belief updating occurs in strategic experimentation literature, e.g., Keller et al. (2005); Bonatti and Hörner (2011), where agents form beliefs about whether projects are *good* or *bad*. However, in this paper, firms form beliefs about the research progress of their rivals.

that the race is ongoing is given as follows:

$$\mathbf{p}_{\boldsymbol{\sigma}}(t) = \mathbf{p}_{\mathbf{1}}(t) := \frac{\frac{1}{\lambda_H}\left(e^{-\mu t} - e^{-\lambda_H t}\right)}{\frac{1}{\mu}e^{-\mu t} - \frac{1}{\lambda_H}e^{-\lambda_H t}}.^{9} \tag{4.2}$$

Moreover, $\mathbf{p}_{\mathbf{1}}(t)$ *is increasing in* t*, with* $\lim_{t\to\infty} \mathbf{p}_{\mathbf{1}}(t) = \min\{1, \mu/\lambda_H\}$.

The last part of this lemma highlights that when the development rate under the new technology (λ_H) exceeds the research rate (μ), the conditional probability under the research policy converges to μ/λ_H, which remains below 1. This result stems from the SRE: since the development rate under the new technology is rapid, the firm's continued presence in the race suggests a lower likelihood of possessing the new technology, ultimately restricting the conditional probability exceeding a certain threshold.

Expected Payoffs and Equilibrium Concept To define expected payoffs of firms, we begin by introducing a function that specifies the rate of development under a given allocation policy.

Definition 2. Given a policy $\boldsymbol{\sigma} \in \mathcal{S}$, the associated *development rate function* $\mathbf{h}_{\boldsymbol{\sigma}}$ is defined as $\mathbf{h}_{\boldsymbol{\sigma}}(t) = \mathrm{DR}(\mathbf{p}_{\boldsymbol{\sigma}}(t), \boldsymbol{\sigma}(t))$ where $DR : [0,1] \times [0,1] \to \mathbb{R}$ is given by:

$$\mathrm{DR}(p, \sigma) := p \cdot \lambda_H + (1-p) \cdot (1-\sigma) \cdot \lambda_L. \tag{4.3}$$

When a firm employs a policy $\boldsymbol{\sigma}$, the first term of (4.3) captures that, if the firm has discovered the new technology by time t—which occurs with probability $\mathbf{p}_{\boldsymbol{\sigma}}(t)$—it develops at rate λ_H using the new technology. If the firm has not discovered the new technology by time t, occurring with probability $(1 - \mathbf{p}(t))$, it develops at rate $(1 - \boldsymbol{\sigma}(t))\lambda_L$.

Given a firm and its rival's allocation policies $\boldsymbol{\sigma}$ and $\hat{\boldsymbol{\sigma}}$, we can express the firm's expected payoff in terms of the associated development rates, $\mathbf{h}_{\boldsymbol{\sigma}}$ and $\mathbf{h}_{\hat{\boldsymbol{\sigma}}}$, as follows:

$$\mathcal{U}(\boldsymbol{\sigma}, \hat{\boldsymbol{\sigma}}) = \int_0^\infty e^{-\int_0^t \{\mathbf{h}_{\boldsymbol{\sigma}}(s) + \mathbf{h}_{\hat{\boldsymbol{\sigma}}}(s)\}\, ds} \cdot (\mathbf{h}_{\boldsymbol{\sigma}}(t) \cdot \Pi - c)\, dt. \tag{4.4}$$

[9] If $\mu = \lambda_H$, $\mathbf{p}_{\mathbf{1}}(t) = \mu t/(1 + \mu t)$. All the results follow through with this case.

Intuitively, the exponential term captures the probability that no firm has developed the product, i.e. the probability that the race is still ongoing by time t. In that case, the firm captures an expected flow payoff equal to $\mathbf{h}_{\boldsymbol{\sigma}}(t) \cdot \Pi$, due to the potential development of the product, while incurring the flow cost c. By integrating, we obtain the expected payoffs of the firms.

Building on the literature on dynamic games with unobservable actions (e.g., Bonatti and Hörner, 2011), we focus on symmetric Nash equilibria. Specifically, we characterize the one with the shorted expected duration.

Definition 3. An allocation policy $\boldsymbol{\sigma}$ is called the *shortest expected duration symmetric Nash equilibrium* (SDSNE) policy if it satisfies the following two conditions: (i) $(\boldsymbol{\sigma}, \boldsymbol{\sigma})$ constitutes a Nash equilibrium; (ii) it minimizes the expected duration among all symmetric Nash equilibria.

Equilibrium Characterization We begin by defining a pair consisting of a probability and a resource allocation that can emerge in the SDSNE policy.

Definition 4. A steady state is a pair $(p_\star, \sigma_\star) \in (0,1)^2$ satisfying (i) $\mathrm{DR}(p_\star, \sigma_\star) = \lambda_\star(\mu)$; and (ii) $\mathrm{BE}(p_\star, \sigma_\star) = 0$.

In a steady state, the belief is stationary ($\mathrm{BE}(p_\star, \sigma_\star) = 0$) and the development rate is $\lambda_\star$, implying that firms are indifferent between researching and developing with the old technology (Proposition B.1 (c)). Thus, once both firms reach the steady state belief $p_\star$, allocating $\sigma_\star$ onward can be part of a Nash equilibrium policy. The following lemma provides a condition under which the steady state exists.

Lemma 5. *There exists a steady state if and only if $\lambda_\star \in (\lambda_L, \min\{\lambda_H, \mu\})$. The steady state is unique.*

We consider a policy where a firm researches until its conditional probability reaches the steady-state belief $p_\star$, and then begins using the steady-state allocation $\sigma_\star$ thereafter. We can identify a unique time $T_\star$ such that $\mathbf{p_1}(T_\star) = p_\star$ and formally define this policy.

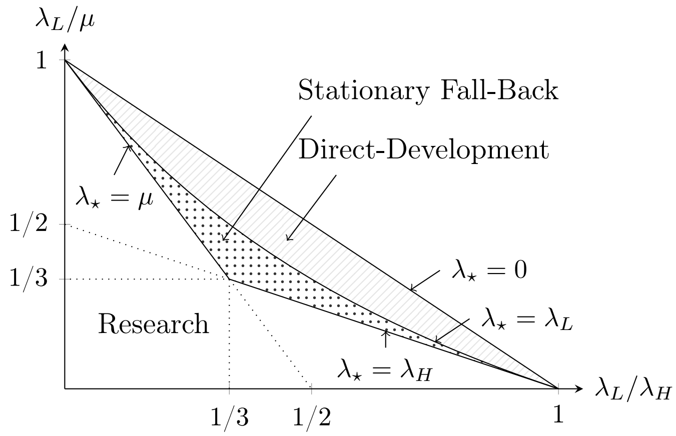

Figure 3: SDSNE policy in the Private Research Progress Setting

Definition 5. The *stationary fall-back policy*, $\boldsymbol{\sigma}^{SF}$, is defined as follows: (i) $\boldsymbol{\sigma}^{SF}(t) = 1$ if $t < T_\star$; (ii) $\boldsymbol{\sigma}^{SF}(t) = \sigma_\star$ if $t \geq T_\star$; and (iii) $\mathbf{p}_{\boldsymbol{\sigma}^{SF}}(t) = p_\star$ for all $t \geq T_\star$.

Now we provide the characterization of the SDSNE policy for each parametric region.

Proposition 3. *When the new technology is non patentable, the SDSNE policy is characterized as follows:*

(a) if $\lambda_\star(\mu) < \lambda_L$, firms employ the ***direct-development policy***, $\boldsymbol{\sigma} = \mathbf{0}$*;*

(b) if $\lambda_\star(\mu) > \min\{\lambda_H, \mu\}$, firms employ the ***research policy***, $\boldsymbol{\sigma} = \mathbf{1}$*;*

(c) if $\lambda_\star(\mu) \in (\lambda_L, \min\{\lambda_H, \mu\})$, firms employ the ***stationary fall-back policy***, $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{SF}$.

When the parameters satisfy $\lambda_\star(\mu) > \lambda_H$ or $\lambda_\star(\mu) < \lambda_L$, Proposition 1 (a) and 1 (c) establish that firms do not adjust their allocation based on the opponent's progress, even when this information is publicly available. Thus, in these parametric regions, equilibrium allocations remain unchanged regardless of whether research progress is public or private.

The more interesting case arises when $\lambda_\star(\mu) \in (\lambda_L, \lambda_H)$. As shown in Proposition 1 (b), under these parameters, firms employ the fall-back strategy in the public research progress case. However, this strategy is no longer feasible when research progress is private. Despite this limitation, the optimality of the fall-back strategy in the public case suggests that if a

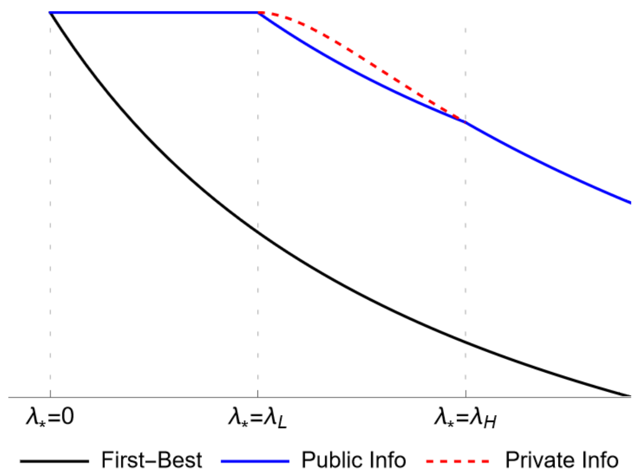

Figure 4: Expected Durations across Settings

firm believes that its rival likely possesses the new technology, it is inclined to allocate more resources toward direct development with the old technology.

Based on these insights, our main result shows that when the steady state exists ($\lambda_\star(\mu) \in (\lambda_L, \min\{\mu, \lambda_H\})$), both firms research until their beliefs reach the steady state probability $p_\star$. Then, both firms implement the steady state allocation $\sigma_\star$ from that point on, making the belief stationary at $p_\star$. Last, if $\mu < \lambda_\star(\mu) < \lambda_H$, the still-in-the-race effect prevents the belief from exceeding a certain level.[10] This keeps the development rate lower than $\lambda_\star(\mu)$, thus, by Proposition B.1, it is optimal for both firms to conduct research indefinitely.

Expected Durations across Settings Now that we have characterized the equilibrium allocations, we can compare how expected development times differ across settings.[11] Figure 4 illustrates how expected development time varies with the research rate μ under the first-best case and the cases of public and private research progress with non-patentable technology. In the first-best case, a planner allocates all the resources to research and shares the technology discovery between firms, yielding the shortest expected duration, $\frac{1}{2\mu} + \frac{1}{2\lambda_H}$. When the new technology is non-patentable, even with public research progress, firms cannot benefit from each other's progress, resulting in longer development times. The private information about research progress further prolongs development when μ is intermediate

[10]Specifically, it cannot exceed μ/λ_H. See Lemma 4.

[11]Since there is no discounting and the reward goes to one of the firms, a lower expected development time implies a higher expected total surplus.

($\lambda_L < \lambda_\star(\mu) < \lambda_H$), as uncertainty about rivals' progress leads firms to adopt stationary fall-back or research policies rather than the fall-back policy that would be optimal under public research progress.

4.2 Patentable Technology

Now assume that the new technology is patentable. We identify parametric conditions under which each of the following equilibria arises: (i) the *efficient patent equilibrium*—firms engage in research, upon discovery, patent and license the new technology, thereby achieving the first-best outcome; (ii) the *concealment equilibrium*—firms choose not patent, resulting in the same outcome as in the case of non-patentable technology.

4.2.1 First-Best Implementation

Recall that the first-best outcome occurs when both firms engage in research and, upon discovery, share the new technology with the other firm through patenting and licensing. If the first-best outcome is implemented, then on the equilibrium path, both firms behave as if research progress is public information. Therefore, the condition stated in Proposition 2 is necessary for the existence of an efficient patent equilibrium. In addition to this, another condition is required to ensure that a firm has the incentive to discloses the discovery of the new technology. The following lemma characterizes this condition.

Lemma 6. *Suppose that firms' research progress is private information, and Firm j's resource allocation strategy is to do research indefinitely ($\sigma_t = 1$ for all $t \geq 0$) and apply for a patent once the new technology is discovered. When Firm i discovers the new technology, it applies for a patent if and only if*

$$\frac{l^*}{V_{11}} > \frac{\lambda_H}{\lambda_H + \mu(2-\beta)}. \tag{4.5}$$

This result is intuitive in that a firm is willing to apply for a patent if and only if the licensing fee l^* is attractive enough relative to the firm's expected payoff after licensing V_{11}. Observe that, as β increases, (4.5) becomes more difficult to hold. This result aligns with intuition: as the prior-use defense level increases, firms are less inclined to apply for patents. Also note that from (3.2) and $\pi = \lambda_L \Pi / c$, we have

$$\frac{l^*}{V_{11}} = \frac{\lambda_H - \lambda_L}{\lambda_H + \lambda_L} \cdot \frac{\lambda_H \Pi + c}{\lambda_H \Pi - c} = \frac{\lambda_H - \lambda_L}{\lambda_H + \lambda_L} \cdot \frac{\lambda_H \pi + \lambda_L}{\lambda_H \pi - \lambda_L}.$$

Therefore, the left hand side of (4.5) is decreasing in π, i.e., as π increases, (4.5) becomes more difficult to hold. Intuitively, since a part of the licensing fee comes from the saving of the cost, it does not increase proportionally with V_{11}. Equipped with this result, we can pin down the parametric conditions under which the efficient patent equilibrium exists.

Proposition 4. *Suppose that firms' research progress is private information. The efficient patent equilibrium exists if and only if the condition in Proposition 2 and one of the following conditions hold: (i) $\beta \leq \hat{\beta} := \frac{2\lambda_\star}{\lambda_H + \lambda_\star(\mu)}$; or (ii) $\beta > \hat{\beta}$ and*

$$1 < \pi < \hat{\pi}(\beta) := 1 + \frac{\lambda_L + \lambda_H}{\lambda_H} \cdot \frac{2 - \beta}{\beta - \hat{\beta}}. \tag{4.6}$$

Note that when $\lambda_\star(\mu) > \lambda_H$, the efficient patent equilibrium exists, since $\hat{\beta} > 1$. In this case, firms conduct research regardless of their rivals' progress. Therefore, when a firm discovers the new technology, there is no informational advantage to concealing it. Instead, firms can benefit from licensing the new technology to the rival firms, allowing the efficient patent equilibrium to be attained. On the other hand, when $\lambda_\star(\mu) < \lambda_H$, it is possible that the efficient patent equilibrium does not exist. To illustrate this, consider a scenario where Firm A discovers the new technology. If Firm A patents and licenses the new technology, the licensing fee is determined based on the assumption that, if the offer is rejected, Firm B will develop with the old technology. Recall that, in the case of $\lambda_\star(\mu) < \lambda_H$, developing with the old technology is the best response for Firm B when it knows that the rival has the new technology (Proposition B.1). Therefore, by applying for a patent, Firm A provides an opportunity for Firm B to exercise its best response. In contrast, if Firm A keeps the discovery secret, it may induce Firm B to make suboptimal choices in R&D strategies, e.g., Firm B may squander its time in conducting research for the new technology, which Firm A already possesses. This trade-off creates the possibility that the efficient patent equilibrium does not exist.

4.2.2 Concealment Equilibrium

Now we consider another extreme equilibrium candidate where both firms do not apply for patents. On the equilibrium path of this concealment equilibrium, firms do not observe rivals' research progress. Therefore, the equilibrium outcome corresponds to that under the private research progress case with the non-patentable technology.

To simplify the discussion, we focus on the parametric region where $\lambda_H > \lambda_\star(\mu) > \mu$. Recall that in this region, both firms employ the fall-back strategy under the public information setting (Proposition 1 (b)), whereas they conduct research under the private information setting (Proposition 3 (b)). The following proposition shows that the concealment equilibrium exists when both the prior-use defense level and the stake of winning the race are high enough.

Proposition 5. *Suppose that firms' research progress is private information and $\lambda_H > \lambda_\star(\mu) > \mu$. There exists $\tilde{\beta} > \hat{\beta}$ and $\tilde{\pi}(\beta) \geq \hat{\pi}(\beta)$ such that the concealment equilibrium exists if and only if $\beta > \tilde{\beta}$ and $\pi > \tilde{\pi}(\beta)$.*

Intuitively, a substantial stake of winning the race and strong prior-use defense increase firms' incentives to conceal their discovery of the new technology. Instead of receiving the licensing fee after patenting—which involves another round of competition to develop the product with the new technology—firms would rather let their rivals squander time researching for the technology they already possess. Specifically, the licensing fee does not fully internalize the decreased chance of winning the race. This is because once Firm i applies for a patent, Firm j adjusts the R&D strategy based on the information that Firm i possesses the new technology, implying that Firm j's outside option is changed after the disclosure.

This concealment incentive slows down the social speed of development in two ways: (i) the discovery of the new technology is not shared with another firm, as described by the gap between the black and the blue curves in Figure 4; (ii) due to the lack of information, firms cannot appropriately adjust the R&D strategies, as described by the gap between the blue and the red dotted curves in Figure 4.

Last, note that the parametric regions of (β, π) in Proposition 4 and 5 do not overlap; that is, there exists an intermediate region where neither efficient patent equilibrium nor concealment equilibrium exists. In this region, as in Chatterjee et al. (2023), firms would engage in partial disclosure—applying for patents at some rate.

5 Discussion

5.1 Empirical Implications

Our results in Sections 4.2 yield empirical implications for how the stakes of winning the race (π) and the strength of prior-use defense protections (β) affect the social speed of innovation.

With respect to π, the model predicts a non-monotonic relationship with innovation speed when β is sufficiently high. When π is too low to violate the parametric assumption in (2.1), firms opt out of R&D races altogether, resulting in no innovation. As π rises to an intermediate range, firms are incentivized to engage in R&D and to share interim discoveries, which enhances the overall rate of innovation. However, when π becomes excessively high, the incentive to preserve competitive advantage through concealment dominates, reducing knowledge spillovers and thereby slowing innovation.

Regarding β, a stronger prior-use defense can discourage patenting and public disclosure. This behavior fosters concealment and impedes the diffusion of technological advances, ultimately slowing cumulative innovation. Notably, the strength and practical enforceability of prior-use defenses can vary across industries, depending on how easily firms can demonstrate independent development. For example, in industries such as manufacturing or hardware, it is often easier to document prior use due to physical prototypes and production records, whereas in fast-evolving sectors like software or biotechnology, prior-use claims can be more ambiguous. This variation opens a path for empirical investigation by comparing innovation patterns across sectors with differing de facto levels of prior-use protection.

5.2 Policy Implications

While significant rewards are essential to incentivize R&D, excessively high stakes in the product market may prompt firms to conceal technological breakthroughs to preserve their competitive advantage. Such concealment reduces opportunities for knowledge spillovers and can decelerate the overall pace of innovation. Thus, policies affecting monopoly power—such as competition regulation or exclusivity periods—must carefully weigh the trade-off between stimulating innovation and promoting knowledge diffusion.

Similarly, prior-use defenses—provisions that allow firms to continue using a technology they developed independently, even if subsequently patented by another—can protect early innovators but may also reduce the incentive to patent. This weakens public disclosure mechanisms and risks duplicative R&D on already-discovered technologies. This tension echoes the rationale for the U.S. patent system's shift from a first-to-invent to a first-to-file regime, aimed at reducing uncertainty and improving the efficiency of information disclosure.

Overall, our results highlight the complex trade-offs that define innovation policy. Policies designed to encourage innovation in one dimension may inadvertently hinder it in another. Policymakers should carefully weigh these tensions when designing institutions to foster long-term technological progress.

5.3 Related Literature

Patent vs. Secrecy We contribute to the literature on the choice between patenting and secrecy by introducing a novel incentive for concealment: preventing a rival's *strategic response*.[12]

Much of the earlier literature emphasizes the limitations of patent protection. For example, the seminal article by Horstmann et al. (1985) argues that "patent coverage may not exclude profitable imitation," suggesting that firms may prefer secrecy to avoid being copied.[13] Another concern is that patent protection is time-limited. For instance, Denicolò

[12]For a comprehensive review of the literature on patents vs. secrecy, see the excellent survey by Hall et al. (2014).

[13]Many subsequent papers study the imitation threat and patent infringement, e.g., Gallini (1992); Takalo (1998); Anton and Yao (2004); Kultti et al. (2007); Kwon (2012); Zhang (2012); Krasteva (2014); Krasteva et al. (2020).

and Franzoni (2004) model a setting where a patent grants temporary monopoly power, while secrecy offers potentially indefinite monopoly power but may be compromised through leaks or duplication.

In contrast, our analysis abstracts from these limitations and focuses on the strategic benefit of concealment in shaping rivals' outside options—depending on the information about a firm's technology level, a rival's strategy may shift, altering its outside option accordingly.

Information Disclosure in Races This novel trade-off between patenting and concealing naturally relates to recent work on information disclosure in innovation races, such as Hopenhayn and Squintani (2016); Bobtcheff et al. (2017).[14]

In these models, innovation value increases over time, and firms face a trade-off between early disclosure to secure priority and waiting to enhance value. In contrast, our model assumes a fixed innovation value, and disclosure decisions are motivated by strategic interactions around rival's R&D activity.

A particularly relevant study is Chatterjee et al. (2023), where they also examine disclosure in a two-stage project. Their model assumes an exogenous payoff from disclosing intermediate discovery. Our model endogenizes this payoff by incorporating the option to continue with the old technology. As in our paper, they find that high rewards of the final discovery can lead firms to withhold intermediate findings, resulting in inefficiency.

Multiple Avenues towards Innovation There is growing interest in models where firms can pursue multiple innovation paths. Das and Klein (2024) and Akcigit and Liu (2016) study patent races involving a safe and a risky method, differing in arrival intensity or payoff uncertainty. In our framework, firms face no uncertainty about feasibility of each path but are uncertain about rivals' progress.

Bryan and Lemus (2017) offer a related framework about direction of innovation using acyclic graph. They assume that whenever a new invention is discovered, the first firm to invent it receives the prize, and the access to the invention is given to all the other firms. In contrast, in our model allows interim discoveries to remain private.

[14] See also Lichtman et al. (2000); Baker and Mezzetti (2005); Gill (2008); Baker et al. (2011); Ponce (2011), reviewed in Section 3.3 of Hall et al. (2014).

Multi-stage Innovation Multi-stage innovation has been extensively studied, e.g., Scotchmer and Green (1990); Denicolò (2000); Green and Taylor (2016); Song and Zhao (2021). As in those works, we model innovation as a sequence of Poisson arrivals. A distinctive feature of our model is that firms may alternatively pursue a direct, slower path that requires only one discovery.

This setup connects to Carnehl and Schneider (2023) and Kim (2022), where players can choose between sequential and direct paths. Unlike their models, which involve a single decision-maker or a principal-agent structure, ours features strategic interactions between firms in the race. Moreover, we do not assume exogenous or endogenous deadlines; rather, the competitive race itself can induce firms to adopt the old technology, effectively choosing the direct path.

Interim Discoveries Our focus on interim discoveries links this work to the literature on licensing intermediate technologies, e.g., Bhattacharya et al. (1992); d'Aspremont et al. (2000); Bhattacharya and Guriev (2006); Spiegel (2008). These studies typically assume the holder of the superior technology is known. By contrast, our model incorporates endogenous licensing choices, including the decision not to license, shaped by strategic considerations.

5.4 Future Research

There are many avenues open for further research. For example, we assume that there are exogenously given two paths towards innovation, and one of the paths requires two breakthroughs. However, in practice, there are numerous ways to innovate, and it often requires more than two breakthroughs. We also assume that a firm's R&D resources are fixed over time, but we could also allow firms to endogenously choose how much effort to put into each point in time. Finally, we assume the contest structure is given by the winner-takes-all competition, but we might consider a contest design problem. We leave these intriguing questions and others for future work.

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Appendix

A Preliminaries: Optimal Control Theory

A.1 Useful Observations

Let τ be a random variable on $\mathbb{R}_+$. Suppose that it has a continuous and differentiable cumulative distribution function $F : \mathbb{R}_+ \to [0,1]$. Let $S(t)$ denote the survival function of τ, i.e., $S(t) = 1 - F(t)$. If $\lim_{t\to\infty} t \cdot S(t) = 0$, the following equation holds:

$$\mathbb{E}[\tau] = \int_0^\infty t \cdot F'(t)dt = -t \cdot S(t)\Big|_0^\infty + \int_0^\infty S(t)dt = \int_0^\infty S(t)dt. \tag{A.1}$$

Let h be a development rate function of τ: $h(t) = -S'(t)/S(t)$.[15] Then, under the assumption that $F(0) = 0$, we can derive that $S(t) = e^{-\int_0^t h(s)ds}$. Then, (A.1) can be rewritten as follows:

$$\mathbb{E}[\tau] = \int_0^\infty e^{-\int_0^t h(s)ds}dt. \tag{A.2}$$

Consider another random variable $\hat{\tau}$ independent to τ. Let $\hat{S}$ and $\hat{h}$ be its survival and development rate functions. Observe that

$$\Pr[\tau < \hat{\tau}] = \int_0^\infty \hat{S}(t)\ dF(t) = -\int_0^\infty S'(t) \cdot \hat{S}(t)\ dt. \tag{A.3}$$

Then, (A.3) can be rewritten as follows:

$$\Pr[\tau < \hat{\tau}] = \int_0^\infty h(t) \cdot S(t) \cdot \hat{S}(t)\ dt = \int_0^\infty h(t) \cdot e^{-\int_0^t (h(s)+\hat{h}(s))ds}\ dt. \tag{A.4}$$

Now consider another random variable which is a minimum of τ and $\hat{\tau}$, denoted by $(\tau \wedge \hat{\tau})$. Then, the survival function of $(\tau \wedge \hat{\tau})$ is $S(t) \cdot \hat{S}(t)$, and the development function of $(\tau \wedge \hat{\tau})$

[15] In the literature, the function $h(t)$ is often referred to as a 'hazard rate' function. The term hazard rate originated from the tradition of describing arrivals as negative events such as failures. In our context, where we are analyzing the timing of product developments, we use the term 'development rate' instead of hazard rate.

is $h(t) + \hat{h}(t)$. By applying (A.2), when $\lim_{t\to\infty} t \cdot S(t) \cdot \hat{S}(t) = 0$, we have

$$\mathbb{E}[\tau \wedge \hat{\tau}] = \int_0^\infty e^{-\int_0^t (h(s)+\hat{h}(s))ds}\, dt. \tag{A.5}$$

A.2 Formal Definitions of Arrival Times

Given an allocation policy $\boldsymbol{\sigma} : \mathbb{R}_+ \to [0, 1]$, we define the following random variables:

1. τ_L: the arrival time of successful development with the old technology;

2. τ_R: the arrival time of the new technology discovery.

Define $\Sigma_t := \int_0^t \boldsymbol{\sigma}(s)ds$. Then, the survival functions of τ_L and τ_R are given as follows: for all $t \geq 0$,

$$S^L_{\boldsymbol{\sigma}}(t) = e^{-\lambda_L(t-\Sigma_t)} \quad \text{and} \quad S^R_{\boldsymbol{\sigma}}(t) = e^{-\mu\Sigma_t}. \tag{A.6}$$

In addition, the development rate functions can be derived as follows:

$$h^L_{\boldsymbol{\sigma}}(t) = \lambda_L(1 - \boldsymbol{\sigma}(t)) \quad \text{and} \quad h^R_{\boldsymbol{\sigma}}(t) = \mu\, \boldsymbol{\sigma}(t). \tag{A.7}$$

Intuitively, the product is developed with the old technology at the rate $h^L_{\boldsymbol{\sigma}}(t) = \lambda_L(1-\boldsymbol{\sigma}(t))$ and the new technology is discovered at the rate $h^R_{\boldsymbol{\sigma}}(t) = \mu\, \boldsymbol{\sigma}(t)$.

B Benchmark: Single Firm Setting

To interpret $\lambda_\star$ as defined in (3.1), it is useful to consider a setting in which there is a single firm attempting to develop the product, and the game ends exogenously at a stochastic termination rate λ.[16] The firm receives the reward Π only if it completes development before the game ends, and incurs the cost c until either the game ends or the product is developed. Analyzing this case provides key insights that are useful in characterizing the solution in the race with two firms.

[16] As an alternative interpretation, one can equivalently consider that one of the firms, e.g. Firm B, maintains a constant development rate λ without engaging in resource allocation, and analyze the optimal allocation of Firm A.

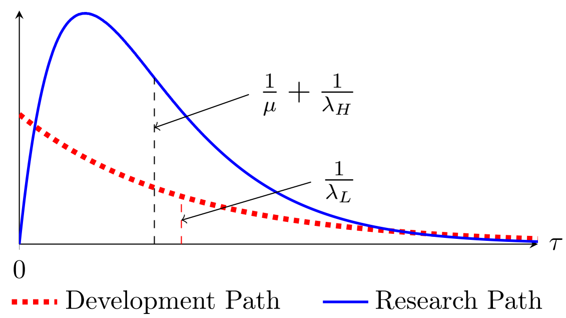

Figure 5: Probability distribution functions of a firm's development time

Once the firms discovers the new technology, it develops the product at rate λ_H. Before the firm obtains the new technology, allocation decisions depend solely on calendar time. We define the firm's *allocation policy* as a right-continuous function $\boldsymbol{\sigma} : \mathbb{R}_+ \to [0, 1]$ that represents the share of attention allocated to research at any time before discovery. Let $\mathcal{S}$ denote the set of such policies.

In the following proposition, we show that the firm's optimal allocation policy, i.e. the allocation policy that maximizes the firm's expected payoff, is determined by $\lambda_\star(\mu)$.

Proposition B.1. *In the benchmark case with exogenous termination λ, the firm's optimal allocation policy is characterized as follows:*

(a) if $\lambda < \lambda_\star(\mu)$, Firm i employs the research policy ($\boldsymbol{\sigma} = \mathbf{1}$);[17]

(b) if $\lambda > \lambda_\star(\mu)$, Firm i employs the direct-development policy ($\boldsymbol{\sigma} = \mathbf{0}$);

(c) if $\lambda = \lambda_\star(\mu)$, Firm i is indifferent between researching and developing with the old technology.

To illustrate the intuition behind this proposition, Figure 5 shows the probability distributions of development times with the direct development policy (red dotted curve) and the research policy (blue solid curve). The direct development policy is more likely to produce a successful product in a short time frame, as it requires only one breakthrough. In contrast, the research policy produces a successful product after two breakthroughs and, although it has a shorter expected development time, is less likely to lead to quick development.

[17] $\mathbf{1} : \mathbb{R} \to [0, 1]$ is defined as $\mathbf{1}(t) = 1$ for all $t \in \mathbb{R}_+$. Similarly, $\mathbf{0}$ is defined as $\mathbf{0}(t) = 1$ for all $t \in \mathbb{R}_+$.

B.1 Proof of Proposition B.1

Suppose that Firm i has already discovered the new technology. Then, Firm i develops with rate λ_H and Firm j develops with the rate λ. Firm i's probability of winning the race is $\frac{\lambda_H}{\lambda_H+\lambda}$ and the expected duration of the remaining race is $\frac{1}{\lambda_H+\lambda}$. Therefore, Firm i's expected continuation payoff is given by

$$\mathcal{V}^1_\lambda := \frac{\lambda_H}{\lambda_H+\lambda}\cdot\Pi - \frac{1}{\lambda_H+\lambda}\cdot c = \frac{\lambda_H\Pi - c}{\lambda_H+\lambda}. \tag{B.1}$$

Now suppose that Firm i has not yet discovered the new technology. The following lemma characterizes Firm i's expected payoff when it employs an allocation policy $\boldsymbol{\sigma}$.

Lemma B.1. *Suppose that Firm j has a constant development rate λ. When Firm i employs an allocation policy $\boldsymbol{\sigma}$, its expected payoff is given as follows:*

$$V^0_\lambda(\boldsymbol{\sigma}) = \int_0^\infty \big(\lambda_L(1-\boldsymbol{\sigma}(t))\cdot\Pi + \mu\ \boldsymbol{\sigma}(t)\cdot\mathcal{V}^1_\lambda - c\big)\cdot e^{-\lambda_L(t-\Sigma_t)-\mu\Sigma_t-\lambda t}\, dt, \tag{B.2}$$

where $\Sigma_t \equiv \int_0^t \boldsymbol{\sigma}(s)ds$.

Proof of Lemma B.1. Let τ_λ be the arrival time of Firm j. When any of the arrival times τ_L, τ_R and τ_λ occurs, we can regard that Firm i's payoff is realized. Furthermore, it incurs a flow cost c until one of these arrival times takes place. Thus, Firm i's expected payoff can be written as follows:

$$V^0_\lambda(\boldsymbol{\sigma}) = \Pr[\tau_L < (\tau_R\wedge\tau_\lambda)]\cdot\Pi + \Pr[\tau_R < (\tau_L\wedge\tau_\lambda)]\cdot\mathcal{V}^1_\lambda - \mathbb{E}[(\tau_L\wedge\tau_R\wedge\tau_\lambda)]\cdot c. \tag{B.3}$$

Note that the survival function of $(\tau_R\wedge\tau_\lambda)$ is $e^{-\int_0^t(\mu\boldsymbol{\sigma}(s)+\lambda)ds} = e^{-\mu\Sigma_t-\lambda t}$. By using (A.4) and (A.7), we have

$$\Pr[\tau_L < (\tau_R\wedge\tau_\lambda)] = \int_0^\infty \lambda_L(1-\boldsymbol{\sigma}(t))\cdot e^{-\lambda_L(t-\Sigma_t)-\mu\Sigma_t-\lambda t}\, dt.$$

Likewise, we can derive that

$$\Pr[\tau_R < (\tau_L \wedge \tau_\lambda)] = \int_0^\infty \mu \;\boldsymbol{\sigma}(t) \cdot e^{-\lambda_L(t-\Sigma_t)-\mu\Sigma_t-\lambda t}\, dt.$$

Next, observe that the survival function of $(\tau_L \wedge \tau_R \wedge \tau_\lambda)$ is

$$e^{-\lambda_L(t-\Sigma_t)-\mu\Sigma_t-\lambda t} = e^{-(\lambda_L+\lambda)t-(\mu-\lambda_L)\Sigma_t}.$$

Then, from $\mu \geq \lambda_L$ and $\Sigma_t + \hat{\Sigma}_t \geq 0$, we have $\lim_{t\to\infty} t \cdot e^{-\lambda_L(t-\Sigma_t)-\mu\Sigma_t-\lambda t} = 0$. By applying (A.1), we have

$$\mathbb{E}[(\tau_L \wedge \tau_R \wedge \tau_\lambda)] = \int_0^\infty e^{-\lambda_L(t-\Sigma_t)-\mu\Sigma_t-\lambda t}\, dt.$$

By plugging the above equations into (B.3), we obtain (B.2). □

Consider constant research allocation strategies, which allocate a fixed amount of resources to research until either the new technology is discovered or the race ends, i.e., for some $x \in [0,1]$, $\boldsymbol{\sigma}^i(t) = x$ for all $t \geq 0$.

When Firm i allocates x amount of resources towards research, there are three potential outcomes: (i) Firm i develops the product with the old technology at rate $\lambda_L(1-x)$; (ii) Firm i discovers the new technology at rate μx; (iii) Firm j develops the product at rate λ. In the first scenario, Firm i wins the race and receives Π, and the probability of this event happening is $\frac{\lambda_L(1-x)}{\lambda_L(1-x)+\mu x+\lambda}$. In the second scenario, Firm i enters the post-research phase, and its expected payoff is $\mathcal{V}^1_\lambda$. The probability of this event occurring is $\frac{\mu x}{\lambda_L(1-x)+\mu x+\lambda}$. In the third scenario, Firm i receives nothing, and the probability of this event happening is $\frac{\lambda}{\lambda_L(1-x)+\mu x+\lambda}$. The expected remaining duration of the game is $\frac{1}{\lambda_L(1-x)+\mu x+\lambda}$. Therefore, Firm i's expected payoff is given by

$$u(x) := \frac{\lambda_L(1-x)\cdot\Pi + \mu x \cdot \mathcal{V}^1_\lambda - c}{\lambda_L(1-x)+\mu x+\lambda}. \tag{B.4}$$

The following lemma shows that the constant research allocation strategy $\boldsymbol{\sigma}$ with x maximizing u maximizes V^0_λ. Thus, it is without loss to focus on constant research allocation strategies.

Lemma B.2. *Suppose that $x_0 \in \arg\max_{x\in[0,1]} u(x)$ where u is a function defined in (B.4). Let $\boldsymbol{\sigma}^* : \mathbb{R}_+ \to [0,1]$ be $\boldsymbol{\sigma}^*(t) = x_0$ for all $t \geq 0$. Then, $\boldsymbol{\sigma}^*(t) \in \arg\max_{\boldsymbol{\sigma}} V^0_\lambda(\boldsymbol{\sigma})$.*

Proof of Lemma B.2. Let r_t denote $e^{-\lambda_L(t-\Sigma_t)-\mu\Sigma t-\lambda t}$. By taking a derivative, we have

$$\dot{r}_t = -\left\{\lambda_L(1-\boldsymbol{\sigma}(t)) + \mu\,\boldsymbol{\sigma}(t) + \lambda\right\} \cdot r_t. \tag{B.5}$$

By Lemma B.1, Firm i's problem is

$$\max_{\boldsymbol{\sigma}\in\mathcal{S}} \int_0^\infty \left\{\lambda_L(1-\boldsymbol{\sigma}(t)) \cdot \Pi + \mu\boldsymbol{\sigma}(t) \cdot \mathcal{V}^1_\lambda - c\right\} \cdot r_t \; dt \tag{B.6}$$

subject to (B.5).

Observe that the Hamiltonian of this optimal control problem is

$$\begin{aligned} H(\boldsymbol{\sigma}(t), r_t, \eta_t) &= \left\{\lambda_L(1-\boldsymbol{\sigma}(t)) \cdot \Pi + \mu\,\boldsymbol{\sigma}(t) \cdot \mathcal{V}^1_\lambda - c\right\} \cdot r_t \\ &\quad - \eta_t \left\{\lambda_L(1-\boldsymbol{\sigma}(t)) + \mu\,\boldsymbol{\sigma}(t) + \lambda\right\} \cdot r_t \\ &= \left\{u(\boldsymbol{\sigma}(t)) - \eta_t\right\} \cdot \left\{\lambda_L(1-\boldsymbol{\sigma}(t)) + \mu\,\boldsymbol{\sigma}(t) + \lambda\right\} \cdot r_t, \end{aligned} \tag{B.7}$$

where η_t is a co-state variable.

To show that $\boldsymbol{\sigma}^*$ is a solution of (B.6) subject to (B.5) by using the Arrow sufficiency condition (Seierstad and Sydsaeter, 1987, Theorem 3.14), we consider (η^*, r^*) defined as follows: for all $t \geq 0$, $\eta^*_t = u(x_0)$ and $r^*_t = e^{-\{\mu x_0 + \lambda_L(1-x_0)+\lambda\}\cdot t}$.

Then, we need to check following four primitive conditions:

1. Maximum principle: for all $t \geq 0$,

$$\boldsymbol{\sigma}^*(t) = x_0 \in \arg\max_{x\in[0,1]} H(x, r^*_t, \eta^*_t). \tag{B.8}$$

2. Evolution of the co-state variable:

$$\dot{\eta}^*_t = -\frac{\partial H}{\partial r_t} = -\left\{u(\boldsymbol{\sigma}^*(t)) - \eta^*_t\right\} \cdot \left\{\lambda_L(1-\boldsymbol{\sigma}^*(t)) + \mu\,\boldsymbol{\sigma}^*(t) + \lambda\right\}. \tag{B.9}$$

3. Transversality condition: If r^* is the optimal trajectory, i.e., $r_t^* = e^{-\{\mu x_0 + \lambda_L(1-x_0)+\lambda\}\cdot t}$, $\lim_{t\to\infty} \eta_t^*(r_t^* - r_t) \le 0$ for all feasible trajectories r_t.

4. $\hat{H}(r_t, \eta_t) = \max_{x\in[0,1]} H(x, r_t, \eta_t)$ is concave in r_t.

First, by plugging r_t^* and η_t^* into (B.7), we have

$$H(\boldsymbol{\sigma}(t), r_t^*, \eta_t^*) = \{u(\boldsymbol{\sigma}(t)) - u(x_0)\} \cdot \{\lambda_L(1-\boldsymbol{\sigma}(t)) + \mu\,\boldsymbol{\sigma}(t) + \lambda\} \cdot r_t. \tag{B.10}$$

Recall that $x_0 \in \arg\max_{x\in[0,1]} u(x)$. Thus, $H(x, r_t^*, \eta_t^*) \le 0$ for all $x \in [0,1]$. In addition, $H(x_0, r_t^*, \eta_t^*) = 0$. Therefore, $x_0 \in \arg\max_{x\in[0,1]} H(x, r_t, \eta_t)$, i.e., (B.8) holds.

Second, by the definition of η^*, (B.9) holds.

Third, note that for any admissible allocation policy $\boldsymbol{\sigma}$,

$$r_t = e^{-\{\mu\Sigma_t + \lambda_L(t-\Sigma_t)+\lambda t\}} = r_t^* \cdot e^{(\mu-\lambda_L)\cdot(x_0 t - \Sigma_t)}.$$

Then, we have

$$\lim_{t\to\infty} \eta_t^* \cdot (r_t^* - r_t) = \lim_{t\to\infty} u(x_0) \cdot r_t^* \cdot \left(1 - e^{(\mu-\lambda_L)\cdot(x_0 t - \Sigma_t)}\right) = 0.$$

Last, we can see that $\hat{H}$ is linear in r_t, thus, the fourth condition holds. Hence, by the Arrow sufficiency condition, $\boldsymbol{\sigma}^*$ is the best response to $\hat{\boldsymbol{\sigma}}^*$. □

Now we are ready to prove Proposition B.1.

Proof of Proposition B.1 . After taking the first derivative of u, with some algebra, we can derive that

$$u'(x) = \frac{\lambda_L(\lambda\Pi + c)(\lambda_\star - \lambda)}{(\lambda+\lambda_H)(\lambda + (1-x)\lambda_L + x\mu)^2}. \tag{B.11}$$

Therefore, from $x \in [0,1]$, $x = 1$ is optimal when $\lambda < \lambda_\star$, $x = 0$ is optimal when $\lambda > \lambda_\star$, and any $x \in [0,1]$ is optimal when $\lambda = \lambda_\star$. □

C Proofs for the Public Research Progress

C.1 Non-Patentable Technology

In this section, we prove Proposition 1. We use a backward induction to characterize MPE.

C.1.1 Best Responses upon New Technology Discovery

We begin by considering the cases where at least one of the firms has discovered the new technology.

When both firms have discovered the new technology ($\omega = \{i, j\}$), they will develop with the new technology and their expected payoffs are $U^i_{\{i,j\}} = U^j_{\{i,j\}} = V_{11} := \frac{\lambda_H \Pi - c}{2\lambda_H}$.

Next, suppose that only one of the firms, say Firm i, has discovered the new technology, i.e., $\omega = \{i\}$. In this case, Firm i develops the product at rate λ_H with the new technology. Then, we can derive the continuation values by applying Proposition B.1:

(i) if $\lambda_\star > \lambda_H$, Firm j researches:

$$U^i_{\{i\}} = U^j_{\{j\}} = \frac{\lambda_H \Pi + \mu V_{11} - c}{\mu + \lambda_H} = \frac{\mu + 2\lambda_H}{\mu + \lambda_H} V_{11}, \qquad U^i_{\{j\}} = U^j_{\{i\}} = \frac{\mu V_{11} - c}{\mu + \lambda_H}, \tag{C.1}$$

(ii) if $\lambda_\star < \lambda_H$, Firm j develops with the old technology:

$$U^i_{\{i\}} = U^j_{\{j\}} = V_H := \frac{\lambda_H \Pi - c}{\lambda_L + \lambda_H}, \qquad U^i_{\{j\}} = U^j_{\{i\}} = V_L := \frac{\lambda_L \Pi - c}{\lambda_L + \lambda_H}. \tag{C.2}$$

C.1.2 Best Responses under no New Technology Discovery

Now, we consider the case where neither firm has discovered the new technology, i.e., $\omega = \emptyset$. To allow for flexibility in various extensions, we formulate the problem in a general way by treating the continuation payoffs $U^i_{\{i\}}$ and $U^i_{\{j\}}$ as exogenous values.

We start by expressing Firm i's expected payoff when it follows $\boldsymbol{\sigma}$ and Firm j follows $\boldsymbol{\sigma}$ and $\hat{\boldsymbol{\sigma}}$, which are not necessarily Markov strategies.

Lemma C.1. *Suppose that Firm i and j employ allocation policies $\boldsymbol{\sigma}$ and $\hat{\boldsymbol{\sigma}}$ at the state $\emptyset$. Let $U^i_{\{i\}}$ and $U^i_{\{j\}}$ be Firm i's continuation payoffs at the states $\{i\}$ and $\{j\}$. Then, Firm i's expected payoff, $U_0(\boldsymbol{\sigma}, \hat{\boldsymbol{\sigma}})$, is given as follows:*

$$\int_0^\infty \big(\lambda_L(1-\boldsymbol{\sigma}(t))\cdot\Pi + \mu\,\boldsymbol{\sigma}(t)\cdot U^i_{\{i\}} + \mu\hat{\boldsymbol{\sigma}}(t)\cdot U^i_{\{j\}} - c\big)\cdot e^{-\lambda_L(2t-\Sigma_t-\hat{\Sigma}_t)-\mu(\Sigma_t+\hat{\Sigma}_t)}\,dt, \quad \text{(C.3)}$$

where $\Sigma_t = \int_0^t \boldsymbol{\sigma}(s)ds$ and $\hat{\Sigma}_t = \int_0^t \hat{\boldsymbol{\sigma}}(s)ds$.

Proof. When any of the arrival times $\tau_L,\ \tau_R,\ \hat{\tau}_L$ and $\hat{\tau}_R$ occurs, the Firm i's payoff is realized. Furthermore, it incurs a flow cost c until one of these arrival times takes place. Thus, Firm i's expected payoff can be written as follows:

$$\begin{aligned} U_0(\boldsymbol{\sigma}, \hat{\boldsymbol{\sigma}}) = &\Pr[\tau_L < (\tau_R \wedge \hat{\tau}_L \wedge \hat{\tau}_R)]\cdot\Pi + \Pr[\tau_R < (\tau_L \wedge \hat{\tau}_L \wedge \hat{\tau}_R)]\cdot U^i_{\{i\}} \\ &+ \Pr[\hat{\tau}_R < (\tau_L \wedge \tau_R \wedge \hat{\tau}_L)]\cdot U^i_{\{j\}} - \mathbb{E}[(\tau_L \wedge \tau_R \wedge \hat{\tau}_L \wedge \hat{\tau}_R)]\cdot c. \end{aligned} \quad \text{(C.4)}$$

Note that the survival function of $(\tau_R \wedge \hat{\tau}_L \wedge \hat{\tau}_R)$ is $e^{-\lambda_L(t-\hat{\Sigma}_t)-\mu(\Sigma_t+\hat{\Sigma}_t)}$. By using (A.4) and (A.7), we have

$$\Pr[\tau_L < (\tau_R \wedge \hat{\tau}_L \wedge \hat{\tau}_R)] = \int_0^\infty \lambda_L(1-\boldsymbol{\sigma}(t))\cdot e^{-\lambda_L(2t-\Sigma_t-\hat{\Sigma}_t)-\mu(\Sigma_t+\hat{\Sigma}_t)}\,dt.$$

Likewise, we can derive that

$$\begin{aligned} \Pr[\tau_R < (\tau_L \wedge \hat{\tau}_L \wedge \hat{\tau}_R)] &= \int_0^\infty \mu\ \boldsymbol{\sigma}(t)\cdot e^{-\lambda_L(2t-\Sigma_t-\hat{\Sigma}_t)-\mu(\Sigma_t+\hat{\Sigma}_t)}\,dt, \\ \Pr[\hat{\tau}_R < (\hat{\tau}_L \wedge \tau_L \wedge \tau_R)] &= \int_0^\infty \mu\ \hat{\boldsymbol{\sigma}}(t)\cdot e^{-\lambda_L(2t-\Sigma_t-\hat{\Sigma}_t)-\mu(\Sigma_t+\hat{\Sigma}_t)}\,dt. \end{aligned}$$

Next, observe that the survival function of $(\tau_L \wedge \tau_R \wedge \hat{\tau}_L \wedge \hat{\tau}_R)$ is

$$e^{-\lambda_L(2t-\Sigma_t-\hat{\Sigma}_t)-\mu(\Sigma_t+\hat{\Sigma}_t)} = e^{-2\lambda_L t-(\mu-\lambda_L)(\Sigma_t+\hat{\Sigma}_t)}.$$

Then, from $\mu \geq \lambda_L$ and $\Sigma_t + \hat{\Sigma}_t \geq 0$, we have $\lim_{t\to\infty} t\cdot e^{-\lambda_L(2t-\Sigma_t-\hat{\Sigma}_t)-\mu(\Sigma_t+\hat{\Sigma}_t)} = 0$. By applying (A.1), we have

$$\mathbb{E}[(\tau_L \wedge \tau_R \wedge \hat{\tau}_L \wedge \hat{\tau}_R)] = \int_0^\infty e^{-\lambda_L(2t-\Sigma_t-\hat{\Sigma}_t)-\mu(\Sigma_t+\hat{\Sigma}_t)}\, dt.$$

By plugging the above equations into (C.4), we obtain (C.3). □

Next, consider the case where Firm i and j play Markov strategies, with $\mathbf{s}(\emptyset) = x$ and $\hat{\mathbf{s}}(\emptyset) = y$. Equivalently, they adopt constant allocation policies $\boldsymbol{\sigma}$ and $\hat{\boldsymbol{\sigma}}$ where $\boldsymbol{\sigma}(t) = x$ and $\hat{\boldsymbol{\sigma}}(t) = y$ for all $t \geq 0$. Firm i's expected payoff in state $\emptyset$ is then given by:

$$u_0(x,y) := \frac{x\mu U^i_{\{i\}} + (1-x)\lambda_L \Pi + y\mu U^i_{\{j\}} - c}{x\mu + (1-x)\lambda_L + y\mu + (1-y)\lambda_L}. \tag{C.5}$$

The next lemma shows that it is without loss of generality to focus on deviations to Markov strategies.

Lemma C.2. *Suppose that $(x_0, y_0) \in [0,1]^2$ satisfies $x_0 \in \arg\max_{x\in[0,1]} u_0(x, y_0)$. Let $\boldsymbol{\sigma}^*, \hat{\boldsymbol{\sigma}}^* : \mathbb{R}_+ \to [0,1]$ be $\boldsymbol{\sigma}^*(t) = x_0$ and $\hat{\boldsymbol{\sigma}}^*(t) = y_0$ for all $t \geq 0$. Then, $\boldsymbol{\sigma}^*$ is a best response to $\hat{\boldsymbol{\sigma}}^*$.*

Proof of Lemma C.2. This can be proven by following the same steps of the proof of Lemma B.2 by setting r_t denote $S^M_{\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}}^*}(t)$ and using Lemma C.1. □

Define $\Delta_y := u_0(1,y) - u_0(0,y)$. With some algebra, we can derive that

$$\frac{\partial u_0}{\partial x} = \mathcal{C}(x,y) \cdot \{\lambda_L \cdot \Delta_0 \cdot (1-y) + \mu \cdot \Delta_1 \cdot y\}, \tag{C.6}$$

where

$$\mathcal{C}(x,y) = \frac{2(\lambda_L + \mu)}{\{\mu x + \lambda_L(1-x) + \mu y + \lambda_L(1-y)\}^2} > 0.$$

The following lemma characterizes the equilibrium allocations at state $\emptyset$ in any MPE.

Lemma C.3. *The equilibrium allocations at state $\emptyset$ are characterized as follows:*

(a) when $\Delta_0, \Delta_1 > 0$, both firms do research, i.e., $(\mathbf{s}^A(\emptyset), \mathbf{s}^B(\emptyset)) = (1,1)$;

(b) when $\Delta_0, \Delta_1 < 0$, both firms develop with the old technology, i.e., $(\mathbf{s}^A(\emptyset), \mathbf{s}^B(\emptyset)) = (0,0)$.

Proof of Lemma C.3. (a) When $\Delta_0, \Delta_1 > 0$, from (C.6), $\frac{\partial u_0}{\partial x} > 0$ for all $y \in [0,1]$, i.e., $x = 1$ is optimal. Thus, both firms play $\mathbf{s}(\emptyset) = 1$ in any MPE.

(b) When $\Delta_0, \Delta_1 < 0$, from (C.6), $\frac{\partial u_0}{\partial x} < 0$ for all $y \in [0,1]$, i.e., $x = 0$ is optimal. Thus, both firms play $\mathbf{s}(\emptyset) = 0$ in any MPE.

$\square$

In scenarios where Δ_0 and Δ_1 share the same sign, the best response is independent of the opponent's resource allocation. Specifically, when both Δ_0 and Δ_1 are positive, it is optimal to assign all resources to research. Conversely, when both Δ_0 and Δ_1 are negative, it is optimal to develop with the old technology.

C.1.3 MPE Characterization

Proof of Proposition 1. First, when $\lambda_\star > \lambda_H$, by plugging (C.1) into (C.5) and the definition of Δ_y, we obtain the followings with some algebra:

$$\Delta_0 = \frac{\lambda_H \cdot \lambda_\star \cdot (\lambda_L \Pi + c) + \mu \cdot (\lambda_\star - \lambda_H) \cdot c}{2\lambda_H(\lambda_H + \mu)(\lambda_L + \mu)}, \tag{C.7}$$

$$\Delta_1 = \frac{\lambda_L \cdot \{\lambda_H \cdot \lambda_\star \cdot (\mu \Pi + c) + \mu \cdot (\lambda_\star - \lambda_H) \cdot c\}}{2\mu\lambda_H(\lambda_H + \mu)(\lambda_L + \mu)}. \tag{C.8}$$

Then, we have that $\Delta_0, \Delta_1 > 0$. By applying Lemma C.3 (a), both firms do research at the state $\emptyset$. Then, when one of the firms, say Firm j, succeeds in research, by Proposition B.1.(a), Firm i will keep doing research. Therefore, the unique MPE is for firms to follow the research strategy (Proposition 1 (a)).

Next, if $\lambda_\star < \lambda_H$, by plugging (C.2) into (C.5) and the definition of Δ_y, we obtain the followings with some algebra:

$$\Delta_0 = \frac{(\lambda_L \Pi + c) \cdot (\lambda_\star - \lambda_L)}{2(\lambda_L + \mu)(\lambda_L + \lambda_H)}, \tag{C.9}$$

$$\Delta_1 = \frac{(\mu \Pi + c) \cdot \lambda_L \cdot (\lambda_\star - \lambda_L)}{2\mu(\lambda_L + \mu)(\lambda_L + \lambda_H)}. \tag{C.10}$$

When $\lambda_\star \in (\lambda_L, \lambda_H)$, (C.9) and (C.10) imply that Δ_0 and Δ_1 are positive. Thus, by

Lemma C.3 (a), both firms do research at the state $\emptyset$. Then, when one of the firms, say Firm j, succeeds in research, by Proposition B.1.(b), Firm i will switch to develop with the old technology. Therefore, the unique MPE is for firms to follow the fall-back strategy (Proposition 1 (b)).

Last, when $\lambda_\star < \lambda_L$, we can see that Δ_0 and Δ_1 are negative. Then, by Lemma C.3 (b), both firms develop with the old technology at the state $\emptyset$. Additionally, even if a firm happens to succeed in research, the other firm will keep developing with the old technology due to Proposition B.1.(b). Thus, the unique MPE is for firms to employ the direct-development strategy (Proposition 1 (c)). □

C.2 Patentable Technology

C.2.1 Proof of Lemma 1

Proof of Lemma 1. Suppose that Firm i has just discovered the new technology and Firm j does not have the patent for the new technology. If Firm j already has the patent, Firm i cannot apply for a patent in the first place.

First, consider the case where Firm j already has the new technology (not the patent). If Firm i does not apply for a patent, both firms race toward development with the new technology. Thus, Firm i's expected payoff is $\frac{\lambda_H \Pi - c}{2\lambda_H}$. If Firm i applies for a patent, with probability α, Firm j's right to use the new technology is protected, and with probability $1-\alpha$, Firm i acquires the patent. In either case, Firm i's expected payoff is at least $\frac{\lambda_H \Pi - c}{2\lambda_H}$, thus, Firm i prefers to apply for a patent.

Next, consider the case where Firm j does not have the new technology. Suppose that in equilibrium, Firm j allocates $x \in [0,1]$ to research and $1-x$ to development with the old technology, when it observes the new technology discovery by Firm i (without a patent). To maximize Firm j's expected payoff, we have

$$\frac{\mu x \cdot \tilde{U}^j + \lambda_L(1-x) \cdot \Pi - c}{\lambda_H + \mu x + \lambda_L(1-x)} \geq \frac{\lambda_L \Pi - c}{\lambda_H + \lambda_L}, \tag{C.11}$$

where $\tilde{U}^j$ is Firm j's expected payoff when it also discovers the new technology. To constitute an equilibrium, Firm i's expected payoff under this Firm j's strategy should be greater than or equal to Firm i's expected payoff from applying for a patent:

$$\frac{\lambda_H \cdot \Pi + \mu x \cdot \tilde{U}^i - c}{\lambda_H + \mu x + \lambda_L(1-x)} \geq U_{Licensor}, \tag{C.12}$$

where $\tilde{U}^i$ is Firm i's expected payoff when Firm j discovers the new technology.

Note that $\tilde{U}^i + \tilde{U}^j \leq \Pi - \frac{2c}{2\lambda_H}$ since the social welfare is maximized when both firms use the new technology, and $U_{Licensor} + \frac{\lambda_L \Pi - c}{\lambda_H + \lambda_L} = \Pi - \frac{c}{\lambda_H}$ from Lemma 2. By using these and summing (C.11) and (C.12) up, we have

$$\Pi - \frac{c}{\lambda_H} \leq \Pi - \frac{\frac{\mu x}{\lambda_H} + 2}{\lambda_H + \mu x + \lambda_L(1-x)} c.$$

However, this inequality is equivalent to $\lambda_H + \mu x + \lambda_L(1-x) \geq 2\lambda_H + \mu x$, which contradicts $\lambda_H > \lambda_L$ and $x \leq 1$. Therefore, in equilibrium, Firm i applies for a patent. □

C.2.2 Proof of Lemma 2

Proof of Lemma 2. When the offer is rejected, Firm j's expected payoff is $\frac{\lambda_L \Pi - c}{\lambda_H + \lambda_L}$. Note that V_{11} is the expected payoff when both firms race with the new technology. thus, when the license offer with the fee l is accepted, Firm j's expected payoff is $V_{11} - l$. Then, Firm i's optimal offer is $l^* = V_{11} - (\lambda_L \Pi - c)/(\lambda_H + \lambda_L)$, and we can derive (3.2) with simple algebra. Then, once the offer is accepted, Firm i's expected payoff is $V_{11} + l^*$ and Firm j's expected payoff is $V_{11} - l^*$. □

C.2.3 Equilibrium Characterization

In this section, we fully characterize the equilibrium under public research progress with patentable technology. Using the equilibrium licensing fee from Lemma 2, we can pin down the continuation payoffs of both firms after the new technology is first discovered. We use these continuation payoffs to analyze the resource allocation of the firms before the new technology is first discovered. As in Section 3, we focus on Markov strategies, i.e., allocations that are independent of calendar time. Let $\mathbf{s}_P^i$ denote the research allocation

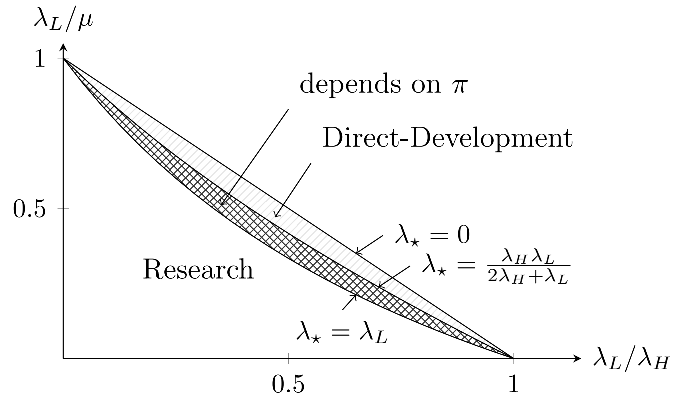

Figure 6: Equilibrium Resource Allocations in the Patent Game under Public Information

of Firm i in the absence of the new technology discovery by either firm. The following proposition characterizes the equilibrium resource allocations.

Proposition C.1. *Suppose that firms' research progress is public information. In any MPE, the resource allocations before the new technology is first discovered are characterized as follows:*

(a) if $\lambda_\star > \lambda_L$, both firms research: $\mathbf{s}_P^A = \mathbf{s}_P^B = 1$;

(b) if $\frac{\lambda_H\lambda_L}{2\lambda_H+\lambda_L} > \lambda_\star$, both firms develop with the old technology: $\mathbf{s}_P^A = \mathbf{s}_P^B = 0$;

(c) if $\lambda_L > \lambda_\star > \frac{\lambda_H\lambda_L}{2\lambda_H+\lambda_L}$, there exist thresholds $\tilde{\pi}_0 > \tilde{\pi}_1 > 1$ such that

(i) when $\pi/c > \tilde{\pi}_0$, both firms develop with the old technology: $\mathbf{s}_P^A = \mathbf{s}_P^B = 0$;

(ii) when $\tilde{\pi}_0 > \pi > \tilde{\pi}_1$, there are three equilibrium allocations: one firm does research and the other firm develops with the old technology, i.e., $(\mathbf{s}_P^A, \mathbf{s}_P^B) = (1, 0)$ or $(0, 1)$; both firms allocate some amount $z^* \in (0, 1)$ resources to research: $\mathbf{s}_P^A = \mathbf{s}_P^B = z^*$;

(iii) when $\tilde{\pi}_1 > \pi$, both firms research: $\mathbf{s}_P^A = \mathbf{s}_P^B = 1$;

Note that Proposition 2 corresponds to Proposition C.1 (a) and (c)-(i). Figure 6 summarizes the result. We can see that firms conduct research in a wider parametric region compared to the case without patents, as described in Figure 2. Intuitively, the option to patent increases the value of conducting research.

C.2.4 Proof of Proposition C.1

We begin by extending our MPE characterization in Lemma C.3 to the cases where Δ_1 and Δ_0 have different signs.

Lemma C.4. *The equilibrium allocations at state $\emptyset$ are characterized as follows:*

(a) when $\Delta_0 > 0 > \Delta_1$, there are three possible equilibrium allocations:

(i) one firm does research and the other firm develops with the old technology, i.e., $(\mathbf{s}^A(\emptyset), \mathbf{s}^B(\emptyset)) = (1, 0)$ or $(0, 1)$,

(ii) both firms allocate $z^* = \Delta_0/(\Delta_0 - \Delta_1)$ amount of resources to research and the remainder to the development with the old technology, i.e., $(\mathbf{s}^A(\emptyset), \mathbf{s}^B(\emptyset)) = (z^*, z^*)$;

(b) when $\Delta_1 > 0 > \Delta_0$, there are three possible equilibrium allocations:

(i) both firms do research, i.e., $(\mathbf{s}^A(\emptyset), \mathbf{s}^B(\emptyset)) = (1, 1)$,

(ii) both firms develop with the old technology, i.e., $(\mathbf{s}^A(\emptyset), \mathbf{s}^B(\emptyset)) = (0, 0)$,

(iii) both firms allocate $z^* = -\Delta_0/(\Delta_1 - \Delta_0)$ amount of resources to research and the remainder to the development with the old technology, i.e., $(\mathbf{s}^A(\emptyset), \mathbf{s}^B(\emptyset)) = (z^*, z^*)$.

Proof of Lemma C.4. (a) From $\Delta_0 > 0$ and (C.6), we have $\frac{\partial u_0}{\partial x}|_{y=0} > 0$, i.e., $x = 1$ is the best response for $y = 0$. In addition, from $0 > \Delta_1$ and (C.6), we have $\frac{\partial u_0}{\partial x}|_{y=1} < 0$, i.e., $x = 0$ is the best response for $y = 1$. Therefore, $(1, 0)$ and $(0, 1)$ can be supported equilibrium allocations at $\omega = \emptyset$.

Next, note that $z^* \in (0, 1)$ and $\frac{\partial u_0}{\partial x}|_{y=z^*} = 0$, i.e., any $x \in [0, 1]$ is the best response for $y = z^*$. Thus, (z^*, z^*) can be supported as an equilibrium allocation.

Last, consider any $\tilde{y} \in (0, 1)$ with $\tilde{y} \neq z^*$. Then, $\frac{\partial u_0}{\partial x}|_{y=\tilde{y}} \neq 0$, i.e., the best response is $x = 1$ or $x = 0$. Recall that the best response of $x = 1$ $(x = 0)$ is $y = 0$ $(y = 1)$, thus, $y = \tilde{y}$ cannot be a part of an equilibrium allocation.

(b) From $\Delta_0 < 0$ and (C.6), we have $\frac{\partial u_0}{\partial x}|_{y=0} < 0$, i.e., $x = 0$ is the best response for $y = 0$. Thus, $(0,0)$ can be supported as an equilibrium allocation.

Similarly, from $0 < \Delta_1$ and (C.6), we have $\frac{\partial u_0}{\partial x}|_{y=1} > 0$, i.e., $x = 1$ is the best response for $y = 1$. Therefore, $(1,1)$ can also be supported as an equilibrium allocation.

Next, note that $z^* \in (0,1)$ and $\frac{\partial u_0}{\partial x}|_{y=z^*} = 0$, i.e., any $x \in [0,1]$ is the best response for $y = z^*$. Thus, (z^*, z^*) can be supported as an equilibrium allocation.

Last, by using the similar argument as in the previous case, $\tilde{y} \in (0,1)$ with $\tilde{y} \neq z^*$ cannot be a part of an equilibrium allocation.

□

Proof of Proposition C.1. To apply Lemma C.4, we first compute $\hat{\Delta}_0$ and $\hat{\Delta}_1$ by replacing $(U^i_{\{i\}}, U^j_{\{i\}})$ to $(U_{Licensor}, U_{Licensee})$ in (C.5):

$$\hat{\Delta}_0 = \frac{\mu U_{Licensor} - c}{\mu + \lambda_L} - \frac{\lambda_L \Pi - c}{2\lambda_L},$$
$$\hat{\Delta}_1 = \frac{\mu U_{Licensor} + \mu U_{Licensee} - c}{2\mu} - \frac{\lambda_L \Pi + \mu U_{Licensee} - c}{\mu + \lambda_L}.$$

By using Lemma 2, we can derive that

$$\hat{\Delta}_0 = \frac{\lambda_H \lambda_L (\lambda_\star - \lambda_L)\Pi + (\lambda_H + \lambda_L)\lambda_\star c}{2\lambda_H(\lambda_H + \lambda_L)(\lambda_L + \mu)},$$
$$\hat{\Delta}_1 = \frac{\lambda_H \lambda_L (\lambda_\star - \lambda_L)\Pi + \frac{\lambda_L}{2\mu}\left\{(2\lambda_H + \mu + \lambda_L)\lambda_\star + (\mu - \lambda_L)\lambda_H\right\} c}{2\lambda_H(\lambda_H + \lambda_L)(\lambda_L + \mu)}.$$

First, observe that $\lambda_\star \geq \lambda_L$ implies $\hat{\Delta}_0,\ \hat{\Delta}_1 > 0$. Then, by Lemma C.4 (a), both firms do research, thus, Proposition C.1 (a) holds. Next, when $\lambda_L > \lambda_\star$, we have

$$\hat{\Delta}_0 > 0 \quad \Longleftrightarrow \quad \tilde{\pi}_0 \equiv \frac{\lambda_\star(\lambda_H + \lambda_L)}{\lambda_H(\lambda_L - \lambda_\star)} > \frac{\lambda_L \Pi}{c} = \pi,$$
$$\hat{\Delta}_1 > 0 \quad \Longleftrightarrow \quad \tilde{\pi}_1 \equiv \frac{\frac{\lambda_L}{2\mu}\left\{(2\lambda_H + \mu + \lambda_L)\lambda_\star + (\mu - \lambda_L)\lambda_H\right\}}{\lambda_H(\lambda_L - \lambda_\star)} > \pi.$$

Suppose that $\lambda_\star \in \left(\frac{\lambda_H \lambda_L}{2\lambda_H + \lambda_L}, \lambda_L\right)$. By using $\mu > \lambda_L$, we can show that $\tilde{\pi}_0 > \tilde{\pi}_1 > 1$.

(i) if $\pi > \tilde{\pi}_0 > \tilde{\pi}_1$, we have $\hat{\Delta}_0,\ \hat{\Delta}_1 < 0$, then, by Proposition 1 (b), both firms develop with old technology;

(ii) if $\tilde{\pi}_0 > \pi > \tilde{\pi}_1$, we have $\hat{\Delta}_0 > 0 > \hat{\Delta}_1$, then, by Proposition 1 (c), there are three equilibria including the asymmetric one;

(iii) if $\tilde{\pi}_1 > \pi > 1$, we have $\hat{\Delta}_0, \hat{\Delta}_1 > 0$, then, by Proposition 1 (a), both firms do research.

Thus, Proposition C.1 (b) holds.

Now suppose that $\lambda_\star \leq \frac{\lambda_H \lambda_L}{2\lambda_H + \lambda_L}$. With some algebra, we have $1 \geq \tilde{\pi}_1 \geq \tilde{\pi}_0$. From $\pi > 1$, we have $\hat{\Delta}_0, \hat{\Delta}_1 < 0$, then, by Proposition 1 (b), both firms develop with old technology. Thus, Proposition C.1 (c) holds. $\square$

D Proofs for the Private Research Progress

D.1 Formal Definition of $\mathbf{p}_{\boldsymbol{\sigma}}$ and $\mathbf{h}_{\boldsymbol{\sigma}}$

Given an allocation policy $\boldsymbol{\sigma} \in \mathcal{S}$, we define two arrival times: (i) τ_M represents the time at which either the new technology is discovered or the product is developed by the old technology; (ii) τ_D represents the time of the product development. Observe that, τ_M must be less than or equal to τ_D by definition. This inequality is strict if and only if the new technology is discovered prior to the product development. Therefore, we use $(\tau_M = \tau_D)$ to indicate the event that the new technology is discovered before the product is developed using the old technology and $(\tau_M < \tau_D)$ to indicate the event that the product is developed before the new technology discovery.

Observe that $\mathbf{p}_{\boldsymbol{\sigma}}$ can be expressed in terms of τ_M and τ_D as follows: $\mathbf{p}_{\boldsymbol{\sigma}}(t) := \Pr(\tau_M < t < \tau_D \mid \tau_D > t)$. Let $\Sigma_t := \int_0^t \boldsymbol{\sigma}(s)\, ds$ represent the cumulative research. We begin by observing that the probability that neither new technology discovery nor product development is made by time t is given by

$$S_{\boldsymbol{\sigma}}^M(t) := \Pr(\tau_M > t) = e^{-\lambda_L(t-\Sigma_t)-\mu\Sigma_t}. \tag{D.1}$$

Additionally, we can derive the probability that new technology is discovered, but product is yet to be developed by time t:

$$L_{\boldsymbol{\sigma}}(t) := \Pr(\tau_M < t < \tau_D) = \int_0^t \mu\, \boldsymbol{\sigma}(s) e^{-\lambda_L(s-\Sigma_s)-\mu\Sigma_s} e^{-\lambda_H(t-s)}\, ds. \tag{D.2}$$

The probability $S^D_{\boldsymbol{\sigma}}(t)$ that neither product development nor new technology discovery was made by time t can be written as:

$$S^D_{\boldsymbol{\sigma}}(t) := \Pr(\tau_D > t) = \Pr(\tau_M > t) + \Pr(\tau_M < t < \tau_D) = S^M_{\boldsymbol{\sigma}}(t) + L_{\boldsymbol{\sigma}}(t). \tag{D.3}$$

Finally, we obtain an expression for our conditional probability $\mathbf{p}_{\boldsymbol{\sigma}}$ in terms of $L_{\boldsymbol{\sigma}}$ and $S^M_{\boldsymbol{\sigma}}$:

$$\mathbf{p}_{\boldsymbol{\sigma}}(t) = \Pr(\tau_M < t \mid \tau_D > t) = \frac{\Pr(\tau_M < t < \tau_D)}{S^D_{\boldsymbol{\sigma}}(t)} = \frac{L_{\boldsymbol{\sigma}}(t)}{S^M_{\boldsymbol{\sigma}}(t) + L_{\boldsymbol{\sigma}}(t)}. \tag{D.4}$$

For any continuous random variable, the hazard rate can be expressed as the negative of the log of the survival function. The development rate of a firm that follows policy $\boldsymbol{\sigma} \in \mathcal{S}$ is the hazard rate associated with the random variable τ_D. Therefore, it can be derived as follows:

$$\begin{aligned}\mathbf{h}_{\boldsymbol{\sigma}}(t) = -\frac{\partial \log\left[S^D_{\boldsymbol{\sigma}}(t)\right]}{\partial t} = -\frac{S^{D\prime}_{\boldsymbol{\sigma}}(t)}{S^D_{\boldsymbol{\sigma}}(t)} &= \frac{\lambda_L(1-\boldsymbol{\sigma}(t)) \cdot S^M_{\boldsymbol{\sigma}}(t) + \lambda_H \cdot L_{\boldsymbol{\sigma}}(t)}{S^M_{\boldsymbol{\sigma}}(t) + L_{\boldsymbol{\sigma}}(t)} \\ &= \lambda_L(1-\boldsymbol{\sigma}(t)) \cdot (1-\mathbf{p}_{\boldsymbol{\sigma}}(t)) + \lambda_H \cdot \mathbf{p}_{\boldsymbol{\sigma}}(t).\end{aligned} \tag{D.5}$$

Also note that from $S^D_{\boldsymbol{\sigma}}(0) = 1$, $S^D_{\boldsymbol{\sigma}}(t)$ can be rewritten as follows:

$$S^D_{\boldsymbol{\sigma}}(t) = e^{-\int_0^t \mathbf{h}_{\boldsymbol{\sigma}}(s)ds}. \tag{D.6}$$

D.2 Proofs of Lemmas

D.2.1 Proof of Lemma 3

Proof of Lemma 3. From (D.4), we can derive that $\mathbf{p}_{\boldsymbol{\sigma}}(t)/(1-\mathbf{p}_{\boldsymbol{\sigma}}(t)) = L_{\boldsymbol{\sigma}}(t)/S^M_{\boldsymbol{\sigma}}(t)$. By differentiating this equation side-by-side, we have

$$\frac{\dot{\mathbf{p}}_{\boldsymbol{\sigma}}(t)}{(1-\mathbf{p}_{\boldsymbol{\sigma}}(t))^2} = \frac{L_{\boldsymbol{\sigma}}(t)}{S^M_{\boldsymbol{\sigma}}(t)}\left[\frac{L'_{\boldsymbol{\sigma}}(t)}{L_{\boldsymbol{\sigma}}(t)} - \frac{S^{M\prime}_{\boldsymbol{\sigma}}(t)}{S^M_{\boldsymbol{\sigma}}(t)}\right] = \frac{\mathbf{p}_{\boldsymbol{\sigma}}(t)}{1-\mathbf{p}_{\boldsymbol{\sigma}}(t)}\left[\frac{L'_{\boldsymbol{\sigma}}(t)}{L_{\boldsymbol{\sigma}}(t)} - \frac{S^{M\prime}_{\boldsymbol{\sigma}}(t)}{S^M_{\boldsymbol{\sigma}}(t)}\right]. \tag{D.7}$$

From deriving (D.1) and (D.2), we obtain that

$$\dot{S}^M_{\boldsymbol{\sigma}}(t) = -\left\{\lambda_L(1-\boldsymbol{\sigma}(t)) + \mu\,\boldsymbol{\sigma}(t)\right\}\cdot S^M_{\boldsymbol{\sigma}}(t), \tag{D.8}$$

$$\dot{L}_{\boldsymbol{\sigma}}(t) = \mu\cdot\boldsymbol{\sigma}(t)\cdot S^M_{\boldsymbol{\sigma}}(t) - \lambda_H\cdot L_{\boldsymbol{\sigma}}(t) \tag{D.9}$$

Using these expressions in (D.7) and multiplying side by side by $(1-\mathbf{p}_{\boldsymbol{\sigma}}(t))^2$, we obtain the desired result. □

D.2.2 Proof of Lemma 4

Proof of Lemma 4. Note that the conditional probability of having access to the new technology by time t only depends on the resource allocations prior to time t. Thus, since $\boldsymbol{\sigma}$ and $\mathbf{1}$ have the same resource allocation by time t, $\mathbf{p}_{\boldsymbol{\sigma}}(t)$ and $\mathbf{p}_{\mathbf{1}}(t)$ are equal. By plugging $\boldsymbol{\sigma}(t) = 1$ to the result of Lemma 3, we have $\mathbf{p}'_{\boldsymbol{\sigma}}(t) = (\mu - \lambda_H\,\mathbf{p}_{\boldsymbol{\sigma}}(t))(1-\mathbf{p}_{\boldsymbol{\sigma}}(t))$. By rearranging the differential equation, we can derive that

$$\lambda_H - \mu = \frac{d}{dt}\log\left(\frac{\lambda_H - \lambda_H\,\mathbf{p}_{\boldsymbol{\sigma}}(t)}{\mu - \lambda_H\,\mathbf{p}_{\boldsymbol{\sigma}}(t)}\right)$$

Then, from $\mathbf{p}_{\boldsymbol{\sigma}}(0) = 0$, we can derive that

$$\frac{\lambda_H(1-\mathbf{p}_{\boldsymbol{\sigma}}(t))}{\mu - \lambda_H\,\mathbf{p}_{\boldsymbol{\sigma}}(t)} = \frac{\lambda_H}{\mu}e^{(\lambda_H-\mu)t}$$

By rearranging the above equation, we have (4.2).

Observe that

$$\mathbf{p_1}'(t) = \frac{\mu(\lambda_H - \mu)^2 e^{(\lambda_H+\mu)t}}{(\lambda_H e^{\lambda_H t} - \mu e^{\mu t})^2} > 0$$

Thus, $\mathbf{p_1}(t)$ is increasing in t.

When $\mu > \lambda_H$,

$$\lim_{t\to\infty} \mathbf{p_1}(t) = \lim_{t\to\infty} \frac{\frac{1}{\lambda_H}\left(e^{(\lambda_H-\mu)t} - 1\right)}{\frac{1}{\mu}e^{(\lambda_H-\mu)t} - \frac{1}{\lambda_H}} = 1.$$

When $\mu < \lambda_H$,

$$\lim_{t\to\infty} \mathbf{p_1}(t) = \lim_{t\to\infty} \frac{\frac{1}{\lambda_H}\left(1 - e^{(\mu-\lambda_H)t}\right)}{\frac{1}{\mu} - \frac{1}{\lambda_H}e^{(\mu-\lambda_H)t}} = \frac{\mu}{\lambda_H}.$$

□

D.2.3 Proof of Lemma 5

Proof of Lemma 5. Our goal is to show that the unique solution of $DR(p_\star, \sigma_\star) = \lambda_\star$ and $BE(p_\star, \sigma_\star) = 0$ is

$$p_\star = \frac{\mu(\lambda_\star - \lambda_L)}{2\lambda_L\lambda_\star} = 1 - \frac{(\mu - \lambda_L)(\lambda_H - \lambda_\star)}{2\lambda_L\lambda_\star}, \tag{D.10}$$

$$\sigma_\star = \frac{\lambda_\star - \lambda_L}{\mu - \lambda_L}. \tag{D.11}$$

Then, to have $(p_\star, \sigma_\star) \in (0,1)^2$, we need to have $\min\{\mu, \lambda_H\} > \lambda_\star > \lambda_L$.

From $DR(p_\star, \sigma_\star) = \lambda_\star$, $BE(p_\star, \sigma_\star) = 0$ and $p_\star < 1$, we have

$$\lambda_\star = p_\star\lambda_H + (1 - p_\star)(1 - \sigma_\star)\lambda_L, \tag{D.12}$$

$$0 = \mu\sigma_\star - \{\lambda_H - (1 - \sigma_\star)\lambda_L\} p_\star. \tag{D.13}$$

By rearranging (D.13), we have

$$\mu\sigma_\star = \lambda_H p_\star + (1 - \sigma_\star)\lambda_L(1 - p_\star) - \lambda_L(1 - \sigma_\star) = \lambda_\star - \lambda_L(1 - \sigma_\star).$$

By solving this, we can derive (D.11).

Next, from (D.13) and (D.11), we have

$$p_\star = \frac{\mu\sigma_\star}{\lambda_H - (1-\sigma_\star)\lambda_L} = \frac{\mu(\lambda_\star - \lambda_L)}{(\mu-\lambda_L)\lambda_H - (\mu-\lambda_\star)\lambda_L}.$$

Note that $\lambda_L\lambda_\star = (\mu-\lambda_L)\lambda_H - \mu\lambda_L$. By plugging this into the above equation, we have the first equality of (D.10). Observe that

$$1-p_\star = \frac{2\lambda_L\lambda_\star - \mu\lambda_\star + \mu\lambda_L}{2\lambda_L\lambda_\star} = \frac{\lambda_L(\mu+\lambda_\star) - (\mu-\lambda_L)\lambda_\star}{2\lambda_L\lambda_\star} = \frac{(\mu-\lambda_L)(\lambda_H-\lambda_L)}{2\lambda_L\lambda_\star},$$

which confirms the second equality of (D.10). □

D.3 Proof of Proposition 3

D.3.1 Minimizing expected duration

When both firms employ $\boldsymbol{\sigma}$, the probability that the race has not been ended by t is $S^D_{\boldsymbol{\sigma}}(t)^2 = (S^M_{\boldsymbol{\sigma}}(t) + L_{\boldsymbol{\sigma}}(t))^2$. Therefore, the expected duration of the race is $\int_0^\infty (S^M_{\boldsymbol{\sigma}}(t) + L_{\boldsymbol{\sigma}}(t))^2 dt$.

Under the assumption that both firms employs the same allocation policy, the problem of minimizing the expected duration is:

$$\max_{\boldsymbol{\sigma}} \quad -\int_0^\infty (S^M_{\boldsymbol{\sigma}}(t) + L_{\boldsymbol{\sigma}}(t))^2 dt \tag{D.14}$$

subject to (D.8), (D.9), $S^M_{\boldsymbol{\sigma}}(0) = 1$ and $L_{\boldsymbol{\sigma}}(0) = 0$.

Lemma D.1. *Given an allocation policy $\boldsymbol{\sigma}^*$, define*

$$\xi_1^*(t) = -\int_t^\infty 2(S^M_{\boldsymbol{\sigma}^*}(s) + L_{\boldsymbol{\sigma}^*}(s)) \cdot \frac{e^{-\lambda_H s}}{e^{-\lambda_H t}} ds, \tag{D.15}$$

$$\xi_0^*(t) = -\int_t^\infty \left\{2(S^M_{\boldsymbol{\sigma}^*}(s) + L_{\boldsymbol{\sigma}^*}(s)) - \xi_1(s)\cdot\mu\,\boldsymbol{\sigma}^*(s)\right\} \cdot \frac{e^{-\lambda_L(s-\Sigma^*_s)-\mu\Sigma^*_s}}{e^{-\lambda_L(t-\Sigma^*_t)-\mu\Sigma^*_t}} ds.^{[18]} \tag{D.16}$$

Suppose that the following equation holds:

$$\boldsymbol{\sigma}^*(t) = \begin{cases} 1, & \textit{if} \;\; \mu\xi_1^*(t) > (\mu - \lambda_L)\xi_0^*(t), \\ \in [0,1], & \textit{if} \;\; \mu\xi_1^*(t) = (\mu - \lambda_L)\xi_0^*(t), \\ 0, & \textit{if} \;\; \mu\xi_1^*(t) < (\mu - \lambda_L)\xi_0^*(t). \end{cases} \tag{D.17}$$

Then, $\boldsymbol{\sigma}^*$ *solves* (D.14) *subject to* (D.8), (D.9), $S^M_{\boldsymbol{\sigma}}(0) = 1$ *and* $L_{\boldsymbol{\sigma}}(0) = 0$.

Proof of Lemma D.1. For simplicity, suppress subscripts of $S^M_{\boldsymbol{\sigma}}$ and $L_{\boldsymbol{\sigma}}$. Moreover, simply denote S for S^M. Let $S^*(t)$ and $L^*(t)$ denote $S^M_{\boldsymbol{\sigma}^*}(t)$ and $L_{\boldsymbol{\sigma}^*}(t)$.

Observe that the Hamiltonian of the optimal control problem (D.14) is

$$\begin{aligned} H(\boldsymbol{\sigma}(t), S(t), L(t), \xi_0(t), \xi_1(t)) := -(S(t) + L(t))^2 - \xi_0(t) \cdot (\lambda_L(1 - \boldsymbol{\sigma}(t)) + \mu\,\boldsymbol{\sigma}(t)) \cdot S(t) \\ + \xi_1(t) \cdot (\mu\,\boldsymbol{\sigma}(t) \cdot S(t) - \lambda_H \cdot L(t)) \end{aligned}$$

As in Lemma B.2, we need to check four primitive conditions.

1. For all $t \geq 0$, (D.17) implies that

$$\boldsymbol{\sigma}^*(t) \in \arg\max_{x \in [0,1]} H(x, S^*(t), L^*(t), \xi_0^*(t), \xi_1^*(t)),$$

thus, the maximum principle holds.

2. Note that the following differential equations hold:

$$\dot{\xi}_0^*(t) = -\frac{\partial H}{\partial S(t)} = \; 2(S(t) + L(t)) + \xi_0^*(t) \cdot (\lambda_L(1 - \boldsymbol{\sigma}_t) + \mu\,\boldsymbol{\sigma}_t) - \xi_1^*(t) \cdot \mu\,\boldsymbol{\sigma}_t, \tag{D.18}$$

$$\dot{\xi}_1^*(t) = -\frac{\partial H}{\partial L(t)} = \; 2(S(t) + L(t)) + \lambda_H \cdot \xi_1^*(t). \tag{D.19}$$

[18] Consider a firm (A) employs $\boldsymbol{\sigma}^*$ and A knows that the rival firm (B) also employs the same policy $\boldsymbol{\sigma}^*$. If A possesses the new technology and does not know about B's status and knows that the race has not yet ended by time t, the expected duration is $-\frac{\xi_1^*(t)}{2(S^M_{\boldsymbol{\sigma}^*}(t) + L_{\boldsymbol{\sigma}^*}(t))}$. Likewise, if A does not possess the new technology and does not know about B's status and knows that the race has not yet ended by time t, the expected duration is $-\frac{\xi_0^*(t)}{2(S^M_{\boldsymbol{\sigma}^*}(t) + L_{\boldsymbol{\sigma}^*}(t))}$.

Therefore, the conditions for the co-state variables evolutions are satisfied.

3. Note that $\lim_{t\to\infty} \xi_0^*(t) = \lim_{t\to\infty} \xi_1^*(t) = 0$. Therefore, the transversality condition holds.

4. Arrow sufficiency theorem (Seierstad, Sydsaeter, page 107, Theorem 5)

   $\hat{H}(S(t), L(t), \theta_0(t), \theta_1(t)) = \max_{\sigma_t \in [0,1]} H(\sigma_t, S(t), L(t), \theta_0(t), \theta_1(t))$ is concave in $(S(t), L(t))$.

   Note that

$$\begin{aligned}\hat{H}(S(t), L(t), \theta_0(t), \theta_1(t)) = &- (S(t) + L(t))^2 - \theta_1(t) \cdot \lambda_H \cdot L(t) \\ &- \{\theta_0(t) \cdot (\lambda_L(1 - \boldsymbol{\sigma}_t^*) + \mu\, \boldsymbol{\sigma}_t^*) - \theta_1(t) \cdot \mu\, \boldsymbol{\sigma}_t^*\} \cdot S(t).\end{aligned}$$

   Therefore, $\hat{H}$ is concave in $(S(t), L(t))$.

□

Proposition D.1. *For each parametric region, the following policies minimize the expected duration of the race among symmetric policies:*

*i. the **direct-development policy**, $\boldsymbol{\sigma} = \mathbf{0}$, when $\lambda_\star < \lambda_L$;*

*ii. the **research policy**, $\boldsymbol{\sigma} = \mathbf{1}$, when $\lambda_\star > \min\{\lambda_H, \mu\}$;*

*iii. the **stationary fall-back policy**, $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{SF}$, when $\lambda_\star \in (\lambda_L, \min\{\lambda_H, \mu\})$.*

Proof of Proposition D.1.i. Suppose that $\boldsymbol{\sigma}^* = \mathbf{0}$. Then, we can derive that

$$S^*(t) = e^{-\lambda_L t}, \quad L^*(t) = 0, \quad \xi_0^*(t) = -\frac{1}{\lambda_L} e^{-\lambda_L t}, \quad \text{and} \quad \xi_1^*(t) = -\frac{2}{\lambda_L + \lambda_H} e^{-\lambda_L t}.$$

With some algebra, we have

$$(\mu - \lambda_L)\xi_0(t) - \mu\xi_1(t) = \frac{\lambda_L - \lambda_\star}{\lambda_H + \lambda_L} e^{-\lambda_L t}.$$

Therefore, $\mu\xi_1^*(t) \leq (\mu - \lambda_L)\xi_0^*(t)$ is equivalent to $\lambda_L \geq \lambda_\star$. Then, by Lemma D.1, $\boldsymbol{\sigma}^* = \mathbf{0}$ solves the problem when $\lambda_L \geq \lambda_\star$. □

Proof of Proposition D.1.ii. Suppose that $\boldsymbol{\sigma}^* = \mathbf{1}$. We can derive that $S(t) = e^{-\mu t}$ and $L(t) = \frac{\mu}{\lambda_H - \mu}(e^{-\mu t} - e^{-\lambda_H t})$. By plugging these into (D.15) and (D.16), we have $\xi_0^*(t) = \psi_0(t)$ and $\xi_1^*(t) = \psi_1(t)$ where

$$\psi_0(t) := \frac{1}{(\lambda_H - \mu)(\lambda_H + \mu)}\left[-\frac{\lambda_H(\lambda_H + 2\mu)}{\mu}e^{-\mu t} + \frac{\mu(2\lambda_H + \mu)}{\lambda_H}e^{-\lambda_H t}\right], \tag{D.20}$$

$$\psi_1(t) := \frac{2}{\lambda_H - \mu}\left[-\frac{\lambda_H}{\lambda_H + \mu}e^{-\mu t} + \frac{\mu}{2\lambda_H}e^{-\lambda_H t}\right]. \tag{D.21}$$

Then, with some algebra, we can derive that

$$\mu\xi_1^*(t) - (\mu - \lambda_L)\xi_0^*(t) = \frac{\lambda_L(\lambda_H + \lambda_L)}{(\lambda_H + \lambda_L)(\lambda_H + \mu)}\left(\frac{\lambda_H(\lambda_\star - \mu)}{(\lambda_H - \mu)\mu}e^{-\mu t} - \frac{\mu(\lambda_\star - \lambda_H)}{(\lambda_H - \mu)\lambda_H}e^{-\lambda_H t}\right).$$

Observe that the right hand side is proportional to:

$$\begin{aligned}&(\lambda_\star - \min\{\mu, \lambda_H\}) \cdot \mu\lambda_H \cdot \frac{\left(-\frac{e^{-\lambda_H t}}{\lambda_H^2}\right) - \left(-\frac{e^{-\mu t}}{\mu^2}\right)}{\lambda_H - \mu} \\ &+ \frac{\lambda_H}{\mu} \cdot \left(\frac{\min\{\mu, \lambda_H\} - \mu}{\lambda_H - \mu}\right) e^{-\mu t} + \frac{\mu}{\lambda_H} \cdot \left(\frac{\lambda_H - \min\{\mu, \lambda_H\}}{\lambda_H - \mu}\right) e^{-\lambda_H t}\end{aligned}$$

Note that the second and third terms are nonnegative. Also note that if $\lambda_\star \geq \min\{\mu, \lambda_H\}$, the first term is nonnegative, since the derivative of $-\frac{e^{-\mu t}}{\mu^2}$ with respect to μ is positive. Then, by Lemma D.1, $\boldsymbol{\sigma}^* = \mathbf{1}$ solves the problem when $\lambda_\star \geq \min\{\lambda_H, \mu\}$. $\square$

Proof of Proposition D.1.iii. Let $T_\star$ be the solution of $\mathbf{p_1}(T) = p_\star$, which is given by $T_\star = \frac{1}{\lambda_H - \mu}\log\left(\frac{\mu(1-p_\star)}{\mu - \lambda_H p_\star}\right)$. Then, $\boldsymbol{\sigma}^*$ is defined as follows: $\boldsymbol{\sigma}^*(t) = 1$ for all $t < T_\star$ and $\boldsymbol{\sigma}^*(t) = \sigma_\star$ for all $t \geq T_\star$.

We can derive that

$$S(t)=\begin{cases} e^{-\mu t}, & \text{if } t<T_\star,\\ e^{-\mu T_\star-\{\lambda_L(1-\boldsymbol{\sigma}_\star)+\mu\boldsymbol{\sigma}_\star\}(t-T_\star)}, & \text{if } t\ge T_\star. \end{cases}$$

$$L(t)=\begin{cases} \dfrac{\mu}{\lambda_H-\mu}(e^{-\mu t}-e^{-\lambda_H t}), & \text{if } t<T_\star,\\ \dfrac{p_\star}{1-p_\star}\cdot S(t), & \text{if } t\ge T_\star. \end{cases}$$

Additionally, we can derive that

$$\xi_0^*(t)=\begin{cases} \psi_0(t)+S(T_\star-t)\cdot(\xi_0^*(T_\star)-\psi_0(T_\star))+L(T_\star-t)\cdot(\xi_1^*(T_\star)-\psi_1(T_\star)), & \text{if } t<T_\star,\\ -\dfrac{\lambda_H+\lambda_\star+\mu\,\boldsymbol{\sigma}_\star}{\lambda_\star(1-p_\star)(\lambda_H+\lambda_\star)}S(t), & \text{if } t\ge T_\star. \end{cases}$$

$$\xi_1^*(t)=\begin{cases} \psi_1(t)+e^{-\lambda_H(T_\star-t)}\cdot(\xi_1^*(T_\star)-\psi_1(T_\star))\,, & \text{if } t<T_\star,\\ -\dfrac{2}{(1-p_\star)(\lambda_H+\lambda_\star)}S(t), & \text{if } t\ge T_\star. \end{cases}$$

With some algebra, it can be shown that $\mu\theta_1(t)-(\mu-\lambda_L)\theta_0(t)=0$. Therefore, $\boldsymbol{\sigma}_t=\sigma_\star$ satisfies (D.17).

Next, for all $t\in[0,T_\star)$, we claim that $\mu\xi_1^*(t)\ge(\mu-\lambda_L)\xi_0^*(t)$ to satisfy (D.17). First, with some algebra, we have $\mu\xi_1^*(T_\star)=(\mu-\lambda_L)\xi_0^*(T_\star)$ and $\mu\xi_1^{*\prime}(T_\star)=(\mu-\lambda_L)\xi_0^{*\prime}(T_\star)$. Then, it suffices to show that $\mu\xi_1^{*\prime\prime}(T_\star)-(\mu-\lambda_L)\xi_0^{*\prime\prime}(T_\star)\ge 0$. Note that

$$\mu\xi_1^{*\prime\prime}(t)-(\mu-\lambda_L)\xi_0^{*\prime\prime}(t)=\frac{\lambda_H\lambda_L\mu e^{-\mu t}}{\lambda_H+\mu}\cdot\Psi_1(t)+\frac{\lambda_L^2(\mu-\lambda_\star)e^{-\mu T_\star-\mu(T_\star-t)}}{(\lambda_H+\mu)(\mu-\lambda_L)}\cdot\Psi_2(T_\star-t)$$

where

$$\begin{aligned}\Psi_1(t):=&\frac{(\lambda_H-\lambda_\star)e^{-(\lambda_H-\mu)t}-(\mu-\lambda_\star)}{\lambda_H-\mu},\\ \Psi_2(t):=&\frac{(\lambda_H-\lambda_\star)\lambda_H e^{-(\lambda_H-\mu)t}-(\mu-\lambda_\star)\mu}{\lambda_H-\mu}.\end{aligned}$$

Observe that both Ψ_1 and Ψ_2 are decreasing in t from $\lambda_H \geq \lambda_\star$. Additionally, we have

$$\Psi_1(T_\star) = \frac{\mu - \lambda_\star}{\mu - \lambda_L} \quad \text{and} \quad \Psi_2(T_\star) = \frac{(\lambda_H + \mu - \lambda_L)(\mu - \lambda_\star)}{\mu - \lambda_L}.$$

Since Ψ_1 and Ψ_2 are decreasing and $\mu \geq \lambda_\star$, $\Psi_1(t),\ \Psi_2(t) \geq 0$ for all $t \in [0, T_\star]$. Therefore, $\mu \xi_1^{*\prime\prime}(T_\star) - (\mu - \lambda_L)\xi_0^{*\prime\prime}(T_\star) \geq 0$. □

D.3.2 Recursive Formulation

Let $V_1(t;\mathbf{h})$ and $V_0(t;\mathbf{h})$ be the continuation payoffs of a firm with and without the new technology at time t, respectively, when the opponent employs an allocation policy with associated development rate $\mathbf{h}$, and no firm has succeeded in development so far.

In the following lemmas, we formally derive V_1, V_0 and the HJB equations of them. Proofs are relegated to the Online Appendix.

Lemma D.2. $V_1(t;\mathbf{h})$ *is derived as follows:*

$$V_1(t;\mathbf{h}) := \int_t^\infty \{\lambda_H \Pi - c\} \cdot e^{-\int_t^s (\mathbf{h}(u) + \lambda_H) du}\, ds. \tag{D.22}$$

In addition, the following differential equation holds:

$$0 = V_1'(t;\mathbf{h}) + (\lambda_H \Pi - c) - (\lambda_H + \mathbf{h}(t)) \cdot V_1(t;\mathbf{h}). \tag{HJB$_1$}$$

Lemma D.3. *Let v_0 be the continuation payoff at time t of a firm that does not have the new technology and employs allocation policy $\boldsymbol{\sigma} \in \mathcal{S}$ when the opponent has a development rate $\mathbf{h} \in \mathcal{H}$. Then, v_0 takes a form of* (D.23).

$$v_0(t;\boldsymbol{\sigma},\mathbf{h}) := \int_t^\infty \{\boldsymbol{\sigma}(s)\mu V_1(s;\mathbf{h}) + (1 - \boldsymbol{\sigma}(s))\lambda_L \Pi - c\} \cdot \mathbf{r}_{\mathbf{h},\boldsymbol{\sigma}}(s;t)\, ds, \tag{D.23}$$

$$\mathbf{r}_{\mathbf{h},\boldsymbol{\sigma}}(s;t) := e^{-\int_t^s \{\mathbf{h}(u) + \boldsymbol{\sigma}(u)\mu + (1-\boldsymbol{\sigma}(u))\lambda_L\} du}.$$

In addition, the following differential equation holds:

$$0 = v_0'(t;\boldsymbol{\sigma},\mathbf{h}) + \lambda_L(1-\boldsymbol{\sigma}(t))\cdot\Pi + \mu\boldsymbol{\sigma}(t)\cdot V_1(t;\mathbf{h}) - c \\ - \{\lambda_L(1-\boldsymbol{\sigma}(t)) + \mu\boldsymbol{\sigma}(t) + \mathbf{h}(t)\}\cdot v_0(t;\boldsymbol{\sigma},\mathbf{h}). \tag{HJB$_0$}$$

Then, $V_0(t;\mathbf{h})$ is derived as follows:

$$V_0(t;\mathbf{h}) \;:=\; \max_{\boldsymbol{\sigma}\in\mathcal{S}} \quad v_0(t;\boldsymbol{\sigma},\mathbf{h}). \tag{D.24}$$

Best responses To characterize the optimal policy $\boldsymbol{\sigma}$ given the opponent's development rate $\mathbf{h}$, define $\mathcal{R}(x,t;\boldsymbol{\sigma},\mathbf{h})$ and $R(t;\boldsymbol{\sigma},\mathbf{h})$ as follows:

$$\mathcal{R}(x,t;\boldsymbol{\sigma},\mathbf{h}) := \mu x(V_1(t;\mathbf{h}) - v_0(t;\boldsymbol{\sigma},\mathbf{h})) + \lambda_L(1-x)(\Pi - v_0(t;\boldsymbol{\sigma},\mathbf{h})), \tag{D.25}$$
$$R(t;\boldsymbol{\sigma},\mathbf{h}) := \frac{\partial\mathcal{R}}{\partial x}(x,t;\boldsymbol{\sigma},\mathbf{h}) = \mu(V_1(t;\mathbf{h}) - v_0(t;\boldsymbol{\sigma},\mathbf{h})) - \lambda_L(\Pi - v_0(t;\boldsymbol{\sigma},\mathbf{h})).$$

We can interpret $\mathcal{R}$ as the instantaneous payoff at time t by allocating x to research and $1-x$ to development with the old technology. The new technology is discovered at the rate μx, yielding the new continuation payoff $V_1(x;\mathbf{h})$ but losing the present continuation payoff $V_0(x;\mathbf{h})$. Similarly, the product is developed with the old technology at the rate $\lambda_L(1-x)$, resulting in the reward Π but losing $V_0(x;\mathbf{h})$. At each time t, the firm chooses a resource allocation to maximize $\mathcal{R}$. Therefore, we interpret R as capturing the relative incentives to conduct research: when R is positive, conducting research is preferred over developing with the old technology, conversely, when R is negative, developing with the old technology is preferred. The following proposition formalizes this verification arguments given the opponent's resource allocation policy $\hat{\sigma}$. The proof is in Appendix OA.1.1.3.

Lemma D.4. *An allocation policy $\boldsymbol{\sigma}^*$ is a best-response to $\hat{\boldsymbol{\sigma}}$, i.e. $\mathcal{U}(\boldsymbol{\sigma}^*,\hat{\boldsymbol{\sigma}}) \geq \mathcal{U}(\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}})$ for all $\boldsymbol{\sigma}\in\mathcal{S}$, if and only if the following two conditions hold for every time $t\geq 0$: (i) $v_0(t;\boldsymbol{\sigma}^*,\mathbf{h}_{\hat{\boldsymbol{\sigma}}}) > 0$; and (ii) $\boldsymbol{\sigma}^*(t)\in\arg\max_{x\in[0,1]}\mathcal{R}(x,t;\boldsymbol{\sigma}^*,\mathbf{h}_{\hat{\boldsymbol{\sigma}}})$.*

D.3.3 Equilibrium Characterization

Proof of Proposition 3 (a) We show that when $\lambda_\star < \lambda_L$, $\boldsymbol{\sigma}^* = \mathbf{0}$ is the best response to $\mathbf{h_0}$ by applying Lemma D.4. It suffices to show that $R(t; \mathbf{0}, \mathbf{h_0}) \leq 0$. Notice that for all $t \geq 0$, $\mathbf{h_0}(t) = \lambda_L$, $V_1(t; \mathbf{h_0}) = \frac{\lambda_H \Pi - c}{\lambda_H + \lambda_L}$, and $v_0(t; \mathbf{0}, \mathbf{h_0}) = \frac{\lambda_L \Pi - c}{2\lambda_L}$. With some algebra, we have

$$R(t; \mathbf{0}, \mathbf{h_0}) = \frac{\lambda_L \Pi + c}{2(\lambda_H + \lambda_L)}(\lambda_\star - \lambda_L).$$

Therefore, when $\lambda_\star < \lambda_L$, $R(t; \mathbf{0}, \mathbf{h_0}) \leq 0$. □

Proof of Proposition 3 (b) We show that when $\lambda_\star > \min\{\mu, \lambda_H\}$, $\boldsymbol{\sigma}^* = \mathbf{1}$ is the best response to $\mathbf{h_1}$ by applying Lemma D.4. It suffices to show that $R(t; \mathbf{1}, \mathbf{h_1}) \geq 0$. With some algebra, we can derive that for all $t \geq 0$, $\mathbf{h_1}(t) = \lambda_H\, \mathbf{p_1}(t)$, $V_1(t; \mathbf{h_1}) = \tilde{V}_1(\mathbf{p_1}(t))$ and $v_0(t; \mathbf{1}, \mathbf{h_1}) = \tilde{V}_0(\mathbf{p_1}(t))$, where

$$\tilde{V}_1(p) := \frac{1}{2}\left(\Pi - \frac{c}{\lambda_H}\right) \cdot \left[1 + \frac{\lambda_H}{\lambda_H + \mu}(1-p)\right], \tag{D.26}$$

$$\tilde{V}_0(p) := \frac{1}{2}\left(\Pi - \frac{c}{\mu} - \frac{c}{\lambda_H}\right) \cdot \left[1 - \frac{\lambda_H}{\lambda_H + \mu} p\right] - \frac{c}{2(\lambda_H + \mu)}. \tag{D.27}$$

Note that

$$\frac{d}{dp}\left[\mu(\tilde{V}_1(p) - \tilde{V}_0(p)) - \lambda_L(\Pi - \tilde{V}_0(p))\right] = -\frac{\left(\Pi - \frac{c}{\mu} - \frac{c}{\lambda_H}\right) + \frac{c}{\lambda_L}}{2\frac{\lambda_H + \mu}{\lambda_H \lambda_L}} < 0.$$

Since $\mathbf{p_1}(t)$ is increasing in t, it suffices to show the following inequlaity:

$$\lim_{t\to\infty} R(t; \mathbf{1}, \mathbf{h_1}) = \lim_{t\to\infty} \mu(\tilde{V}_1(\mathbf{p_1}(t)) - \tilde{V}_0(\mathbf{p_1}(t))) - \lambda_L(\Pi - \tilde{V}_0(\mathbf{p_1}(t))) \geq 0. \tag{D.28}$$

When $\mu \geq \lambda_H$, since $\lim_{t\to\infty} \mathbf{p_1}(t) = 1$, with some algebra, we have

$$\lim_{t\to\infty} R(t; \mathbf{1}, \mathbf{h_1}) = \mu(\tilde{V}_1(1) - \tilde{V}_0(1)) - \lambda_L(\Pi - \tilde{V}_0(1)) = \frac{(\lambda_H \Pi + c)\lambda_L(\lambda_\star - \lambda_H)}{2\lambda_H(\lambda_H + \mu)},$$

which is nonnegative from $\lambda_\star \geq \min\{\lambda_H, \mu\}$.

Next, when $\mu < \lambda_H$, since $\lim_{t\to\infty} \mathbf{p_1}(t) = \mu/\lambda_H$, with some algebra, we have

$$\lim_{t\to\infty} R(t; \mathbf{1}, \mathbf{h_1}) = \mu(\tilde{V}_1(\tfrac{\mu}{\lambda_H}) - \tilde{V}_0(\tfrac{\mu}{\lambda_H})) - \lambda_L(\Pi - \tilde{V}_0(\tfrac{\mu}{\lambda_H})) = \frac{(\mu\Pi + c)\lambda_L(\lambda_\star - \mu)}{2\mu(\lambda_H + \mu)},$$

which is also nonnegative from $\lambda_\star \geq \min\{\lambda_H, \mu\}$. □

Proof of Proposition 3 (c) We show that when $\min\{\mu, \lambda_H\} > \lambda_\star > \lambda_L$, $\boldsymbol{\sigma}^* = \boldsymbol{\sigma}^{SF}$ is the best response to $\mathbf{h}_{\boldsymbol{\sigma}^{SF}}$ by applying Lemma D.4. Specifically, it suffices to show that (i) $R(t; \boldsymbol{\sigma}^{SF}, \mathbf{h}_{\boldsymbol{\sigma}^{SF}}) = 0$ for all $t \geq T_\star$; and (ii) $R(t; \boldsymbol{\sigma}^{SF}, \mathbf{h}_{\boldsymbol{\sigma}^{SF}}) \geq 0$ for all $t < T_\star$.

When $t \geq T_\star$, by the definitions of the stationary fall-back policy and the steady state, we have $\mathbf{h}_{\boldsymbol{\sigma}^{SF}}(t) = DR(p_\star, \sigma_\star) = \lambda_\star$. With some algebra, we can derive that for all $t \geq T_\star$, $V_1(t; \mathbf{h}_{\boldsymbol{\sigma}^{SF}}) = V_\star^1$ and $v_0(t; \boldsymbol{\sigma}^{SF}, \mathbf{h}_{\boldsymbol{\sigma}^{SF}}) = V_\star^0$, where

$$V_\star^1 := \frac{\lambda_H\Pi - c}{\lambda_H + \lambda_\star} \qquad \text{and} \qquad V_\star^0 := \frac{\lambda_L(\lambda_L\Pi - c)}{(\mu - \lambda_L)(\lambda_H - \lambda_L)}.$$

Additionally, with some algebra, we have

$$R(t; \boldsymbol{\sigma}^{SF}, \mathbf{h}_{\boldsymbol{\sigma}^{SF}}) = \mu(V_\star^1 - V_\star^0) - \lambda_L(\Pi - V_\star^0) = 0$$

for all $t \geq T_\star$.

Next, since $\boldsymbol{\sigma}^{SF}(t) = 1$ for all $t < T_\star$, we have $\mathbf{h}_{\boldsymbol{\sigma}^{SF}}(t) = \lambda_H\, \mathbf{p_1}(t)$. By solving (HJB$_1$) and (HJB$_0$), we have $V_1(t; \mathbf{h}_{\boldsymbol{\sigma}^{SF}}) = \hat{V}_1(\mathbf{p_1}(t))$ and $v_0(t; \boldsymbol{\sigma}^{SF}, \mathbf{h}_{\boldsymbol{\sigma}^{SF}}) = \hat{V}_0(\mathbf{p_1}(t))$ where

$$\hat{V}_1(p) = \tilde{V}_1(p) + C_1 \cdot (1-p) \cdot \left(\frac{\mu - \lambda_H p}{1-p}\right)^{\frac{\mu+\lambda_H}{\mu-\lambda_H}}, \tag{D.29}$$

$$\hat{V}_0(p) = \tilde{V}_0(p) + \left(C_0 \cdot \left(\frac{\mu}{\lambda_H} - p\right) - C_1 \cdot \frac{\mu}{\lambda_H}\right) \cdot \left(\frac{\mu - \lambda_H p}{1-p}\right)^{\frac{\mu+\lambda_H}{\mu-\lambda_H}}, \tag{D.30}$$

for some constants C_1 and C_0. Using the terminal conditions, $\hat{V}_1(\mathbf{p_1}(T_\star)) = V_\star^1$ and $\hat{V}_0(\mathbf{p_1}(T_\star)) =$

$V_\star^0$, we can identify C_1 and C_0:

$$C_1 = \frac{1}{1-p_\star} \cdot \left(\frac{1-p_\star}{\mu - \lambda_H p_\star}\right)^{\frac{\mu+\lambda_H}{\mu-\lambda_H}} (V_\star^1 - \tilde{V}_1(p_\star)), \tag{D.31}$$

$$C_0 = C_1 \cdot \frac{\mu}{\mu - \lambda_H p_\star} + \frac{\lambda_H}{\mu - \lambda_H p_\star} \left(\frac{1-p_\star}{\mu - \lambda_H p_\star}\right)^{\frac{\mu+\lambda_H}{\mu-\lambda_H}} (V_\star^0 - \tilde{V}_0(p_\star)). \tag{D.32}$$

Define $\hat{R}(p) := \mu(\hat{V}_1(p) - \hat{V}_0(p)) - \lambda_L(\Pi - \hat{V}_0(p))$, then we have $R(t; \boldsymbol{\sigma}^{SF}, \mathbf{h}_{\boldsymbol{\sigma}^{SF}}) = \hat{R}(\mathbf{p_1}(t))$. Since $\mathbf{p_1}$ is increasing and $\mathbf{p_1}(T_\star) = p_\star$, $R(t; \boldsymbol{\sigma}^{SF}, \mathbf{h}_{\boldsymbol{\sigma}^{SF}}) \geq 0$ for all $t \in [0, T_\star]$ is equivalent to $\hat{R}(p) \geq 0$ for all $p \in [0, p_\star]$. Note that $\hat{R}(p_\star) = \mu(V_\star^1 - V_\star^0) - \lambda_L(\Pi - V_\star^0) = 0$. Therefore, it suffices to show that $\hat{R}''(p) < 0$ and $\hat{R}(0) \geq 0$.

With some algebra, we can derive that $\hat{R}''(p) = -\mathcal{A}(p) \cdot \mathcal{B}(p)$ where

$$\begin{aligned}
\mathcal{A}(p) =& \frac{\lambda_L^2(\lambda_H - \lambda_\star)^2 (1-p)^{-3+\frac{2\lambda_H}{\lambda_H-\mu}}}{\mu^2(\lambda_H - \lambda_L)^2(\mu - \lambda_H p)} \left(\frac{(\mu - \lambda_H p)(\mu - \lambda_L)(\lambda_H - \lambda_\star)}{(\lambda_H - \lambda_L)\mu(\mu - \lambda_\star)}\right)^{\frac{2\lambda_H}{\mu-\lambda_H}}, \\
\mathcal{B}(p) =& (\mu - \lambda_L p)(\mu\Pi - c) + (1-p)(\lambda_H - \lambda_L)\mu\Pi.
\end{aligned}$$

Since $\min\{\lambda_H, \mu\} > \lambda_\star > \lambda_L$ and $0 \leq p \leq p_\star < \lim_{t\to\infty} \mathbf{p_1}(t) = \min\{1, \mu/\lambda_H\}$, we have $\mathcal{A}(p) > 0$ and $\mathcal{B}(p) > 0$, thus, $\hat{R}''(p) < 0$.

Next, with some algebra, we can also derive that

$$\hat{R}(0) = \lambda_L \cdot \frac{\lambda_H \mu \Pi - \lambda_L c}{2\lambda_H \mu(\lambda_H + \mu)} \cdot G(\lambda_\star) + \frac{\lambda_L(\lambda_\star - \lambda_L)c}{2\lambda_H \mu}$$

where

$$G(x) := x - \lambda_L \left(1 - \frac{x - \lambda_L}{\lambda_H - \lambda_L}\right)^2 \cdot \left(\frac{1 - \frac{x-\lambda_L}{\lambda_H - \lambda_L}}{1 - \frac{x - \lambda_L}{\lambda_H - \mu_L}}\right)^{\frac{2\lambda_H}{\mu-\lambda_H}}.$$

Note that $G(\lambda_L) = 0$ and

$$G'(x) := 1 + \frac{2\lambda_L(\lambda_H - x)(\lambda_H + \mu - x)}{(\lambda_H - \lambda_L)^2(\mu - x)} \cdot \left(\frac{1 - \frac{x-\lambda_L}{\lambda_H - \lambda_L}}{1 - \frac{x - \lambda_L}{\lambda_H - \mu_L}}\right)^{\frac{2\lambda_H}{\mu-\lambda_H}} \geq 0$$

for all $\min\{\mu, \lambda_H\} > x \geq \lambda_L$. Therefore, from $\lambda_H \mu \Pi - \lambda_L c > 0$, $\lambda_\star \geq \lambda_L$ and $G(\lambda_\star) \geq 0$, we have $\hat{R}(0) \geq 0$. $\square$

D.4 Patentable Technology

D.4.1 Proofs for Efficient Patent Equilibrium

Proof of Lemma 6. Suppose that Firm i has discovered the new technology, and Firm j has not applied for a patent yet. Given Firm j's patent application strategy, the fact that Firm j has not applied for a patent implies that Firm j does not have the new technology yet. Therefore, if Firm i applies for a patent, it will attain the patent with probability one and its expected continuation payoff is $U_{Licensor} = V_{11} + l^*$. Suppose instead that Firm i decides not to apply for a patent. Firm i's payoff in the case in which Firm i finds the new technology before a successful development is $U^{\beta}_{Challenger} = V_{11} - (1-\beta) \cdot l^*$. Therefore, Firm i's expected payoff of not applying for a patent is

$$\frac{\lambda_H \Pi + \mu \cdot U^{\beta}_{Challenger} - c}{\lambda_H + \mu} = \frac{(\mu + 2\lambda_H)V_{11} - \mu(1-\beta)l^*}{\lambda_H + \mu}. \tag{D.33}$$

Firm i applies for a patent when $U_{Licensor}$ is greater than (D.33), which is equivalent to:

$$\begin{aligned}
& (\lambda_H + \mu)V_{11} + (\lambda_H + \mu)l^* > (\mu + 2\lambda_H)V_{11} - \mu(1-\beta)l^* \\
\iff \quad & \{\lambda_H + \mu(2-\beta)\}\, l^* > \lambda_H V_{11}.
\end{aligned}$$

Since $1 > \beta$, $\lambda_H, \mu > 0$ and $V_{11} > 0$, it is equivalent to (4.5). ☐

Proof of Proposition 4. By plugging (3.2) in, we have that (4.5) is equivalent to:

$$\begin{aligned}
& \frac{\lambda_H - \lambda_L}{\lambda_H + \lambda_L} \cdot \frac{\lambda_H \Pi + c}{\lambda_H \Pi - c} > \frac{\lambda_H}{\lambda_H + \mu(2-\beta)} \\
\iff \quad & \{\lambda_H(\lambda_H - \lambda_L) + \mu(\lambda_H - \lambda_L)(2-\beta)\}\,(\lambda_H \Pi + c) > \lambda_H(\lambda_H + \lambda_L)(\lambda_H \Pi - c) \\
\iff \quad & \{\mu(\lambda_H - \lambda_L)(2-\beta) - 2\lambda_L\lambda_H\} \cdot \lambda_H \Pi + \left\{\mu(\lambda_H - \lambda_L)(2-\beta) + 2\lambda_H^2\right\} \cdot c > 0.
\end{aligned}$$

Note that $\mu(\lambda_H - \lambda_L) = \lambda_L(\lambda_\star + \lambda_H)$ from (3.1). By plugging this in, the above inequality is equivalent to:

$$\begin{aligned}
&\left\{(2-\beta)\lambda_\star - \beta\lambda_H\right\} \cdot \lambda_H \lambda_L \Pi + \left\{(2-\beta)\lambda_L(\lambda_\star+\lambda_H) + 2\lambda_H^2\right\} \cdot c > 0 \\
\iff \qquad &(\lambda_\star+\lambda_H)\left(\hat{\beta}-\beta\right) \cdot \lambda_H \left(\frac{\lambda_L \Pi}{c} - 1\right) + (2-\beta)(\lambda_L+\lambda_H)(\lambda_\star+\lambda_H) > 0.
\end{aligned}$$

If $\beta \leq \hat{\beta}$, the first term in the above inequality is nonnegative and the second term is positive from $\beta < 1$ and $\lambda_L,\ \lambda_H,\ \lambda_\star > 0$. If $\beta > \hat{\beta}$, by rearranging it and using $\pi = \frac{\lambda_L \Pi}{c}$, we can show that the above inequality is equivalent to (4.6). □

D.4.2 Proofs for Concealment Equilibrium

Let $\boldsymbol{\sigma}^*$ denote the unique equilibrium policy in Proposition 3. From (D.22), $V_1(t; \mathbf{h}_{\boldsymbol{\sigma}^*})$ is the continuation value of firms in such equilibrium. A *concealment equilibirium* is an equilibrium of the game with patents such that the firms never patent the new technology and follow policy $\boldsymbol{\sigma}^*$.

Observation There is a concealment equilibrium if and only if, for all $t \geq 0$,

$$V_1(t; \mathbf{h}_{\boldsymbol{\sigma}^*}) \geq V_{11} + (1 - \beta\, \mathbf{p}_{\boldsymbol{\sigma}^*}(t)) \cdot l^*. \tag{D.34}$$

To understand the observation, notice that (D.34) captures the trade-off in the patenting decision of a firm that discovers the new technology at time t, when the opponent follows policy $\boldsymbol{\sigma}^*$ and never patents. The left-hand-side denotes the payoff obtained by not patenting, i.e by keeping the discovery secret. The right-hand-side captures the expected payoff if the firm decides to patent at time t. If (D.34) holds for all t, then it is a best response to never patent.

Under $\lambda_H > \lambda_\star > \mu$, by Proposition 3 (b), firms employ the research policy in the private information setting, i.e., $\boldsymbol{\sigma}^* = \mathbf{1}$. The following lemma provides the closed form solution of $V_1(t; \mathbf{h_1})$.

Lemma D.5. *When $\lambda_H > \lambda_\star > \mu$, the following equation holds:*

$$V_1(t;\mathbf{h_1}) = \left\{1 + \frac{\lambda_H}{\lambda_H+\mu}(1-\mathbf{p_1}(t))\right\} \cdot V_{11}. \tag{D.35}$$

Proof of Lemma D.5. By Lemma 4, $\mathbf{p_1}(t)$ is increasing in t. Then, $V_1(t;\mathbf{h_1})$ can be written as a function of $\mathbf{p_1}(t)$: $V_1(t;\mathbf{h_1}) = v_1(\mathbf{p_1}(t))$. Observe that

$$V_1'(t;\mathbf{h_1}) = v_1'(\mathbf{p_1}'(t)) \cdot \mathbf{p_1}'(t) = v_1'(\mathbf{p_1}'(t)))(\mu - \lambda_H\, \mathbf{p_1}(t))(1-\mathbf{p_1}(t)).$$

By plugging this into (HJB$_1$), we have

$$0 = v_1'(p)(\mu-\lambda_H p)(1-p) - \lambda_H(1+p)v_1(p) + \lambda_H \Pi - c. \tag{D.36}$$

Define two function $g(p)$ and $k(p)$ as follows:

$$g(p) := \frac{(\mu-\lambda_H p)^{\frac{\mu+\lambda_H}{\lambda_H-\mu}}}{(1-p)^{\frac{2\lambda_H}{\lambda_H-\mu}}} \quad \text{and} \quad k(p) := 1 + \frac{\lambda_H}{\lambda_H+\mu}(1-p). \tag{D.37}$$

Observe that

$$\frac{g'(p)}{g(p)} = \frac{d\log(g(p))}{dp} = -\frac{\mu+\lambda_H}{\lambda_H-\mu}\cdot\frac{\lambda_H}{\mu-\lambda_H p} + \frac{2\lambda_H}{\lambda_H-\mu}\cdot\frac{1}{1-p} = -\frac{\lambda_H(1+p)}{(1-p)(\mu-\lambda_H p)} \tag{D.38}$$

and

$$\frac{d}{dp}(g(p)\cdot k(p)) = -\frac{\lambda_H(1+p)k(p)}{(1-p)(\mu-\lambda_H p)}g(p) - \frac{\lambda_H}{\lambda_H+\mu}g(p) = -\frac{2\lambda_H}{(1-p)(\mu-\lambda_H p)}g(p) \tag{D.39}$$

By multiplying (D.36) by $\frac{g(p)}{(\mu-\lambda_H p)(1-p)}$ and using above two equations, we have

$$\begin{aligned} 0 =& v_1'(p)\cdot g(p) + g'(p)\cdot v_1(p) + \frac{\lambda_H \Pi - c}{2\lambda_H}\cdot\frac{g(p)}{(1-p)(\mu-\lambda_H p)} \\ =& \frac{d}{dp}\left[(v_1(p) - V_{11}\cdot k(p))\cdot g(p)\right]. \end{aligned}$$

Therefore, there exists $C \in \mathbb{R}$ such that

$$v_1(p) = V_{11} \cdot k(p) + \frac{C}{g(p)}. \tag{D.40}$$

In Lemma 4, we show that if $\mu \geq \lambda_H$, $\lim_{t\to\infty} \mathbf{p_1}(t) = 1$, and if $\mu < \lambda_H$, $\lim_{t\to\infty} \mathbf{p_1}(t) = \mu/\lambda_H$. By using these, we have that $\lim_{t\to\infty} g(\mathbf{p_1}(t)) = 0$. Then, to satisfy $V_1(t; \mathbf{h_1}) = v_1(\mathbf{p_1}(t))$ and (D.40), the constant C has to be zero, and (D.35) holds. $\square$

By using this lemma, (D.34) is equivalent to:

$$\frac{l^*}{V_{11}} < \frac{\lambda_H}{\lambda_H + \mu} \cdot \frac{1 - \mathbf{p_1}(t)}{1 - \beta \cdot \mathbf{p_1}(t)}. \tag{D.41}$$

The right hand side is decreasing in $\mathbf{p_1}(t)$. Under $\lambda_H > \lambda_\star > \mu$, $\mathbf{p_1}(t)$ converges to μ/λ_H, thus, we can plug this into (D.41):

$$\frac{l^*}{V_{11}} < \frac{\lambda_H(\lambda_H - \mu)}{(\lambda_H + \mu)(\lambda_H - \beta\mu)}. \tag{D.42}$$

With simple algebra, we can show that $\frac{\lambda_H(\lambda_H-\mu)}{(\lambda_H+\mu)(\lambda_H-\beta\mu)} \leq \frac{\lambda_H}{\lambda_H+\mu(2-\beta)}$. Therefore, the threshold for the concealment equilibrium is below the one for the efficient patent equilibrium, i.e., there is no parameter such that both the efficient patent equilibrium and the concealment equilibrium exist. By solving (D.42), we can pin down the parametric conditions under which the concealment equilibrium exists.

$$\beta > \tilde{\beta} := \frac{2\lambda_H(\mu + \lambda_\star)}{(\lambda_H + \mu)(\lambda_H + \lambda_\star)} \tag{D.43}$$

and

$$\pi > \tilde{\pi}(\beta) := 1 + \frac{\lambda_H + \lambda_L}{\lambda_H + \mu} \cdot \frac{2\lambda_H - (\lambda_H + \mu)\beta}{\lambda_H(\beta - \tilde{\beta})}. \tag{D.44}$$

Now we provide the proof of Proposition 5

Proof of Proposition 5. By using using $\mu(\lambda_H - \lambda_L) = \lambda_L(\lambda_\star + \lambda_H)$, $\lambda_H(\mu - \lambda_L) = \lambda_L(\lambda_\star + \mu)$

and (3.2), we have that (D.42) is equivalent to:

$$(\lambda_H + \lambda_L)(2\lambda_H - \beta(\lambda_H + \mu)) < (\lambda_H + \mu)(\beta - \tilde{\beta})\lambda_H \cdot (\pi - 1)\,.$$

Note that $2\lambda_H - \beta(\lambda_H + \mu) > 0$ from $\lambda_H > \mu$ and $1 \geq \beta$. Therefore, if $\beta \leq \tilde{\beta}$, the above inequality cannot hold. When $\beta > \tilde{\beta}$, by rearranging the above inequality, we have (D.44).

Observe that $\tilde{\beta} > \hat{\beta}$ is equivalent to:

$$2\lambda_H(\mu + \lambda_\star) > 2\lambda_\star(\lambda_H + \mu)$$

and it holds from the assumption that $\lambda_H > \lambda_\star$.

Next, observe that $\tilde{\pi}(\beta) \geq \hat{\pi}(\beta)$ is equivalent to:

$$\frac{\frac{2\lambda_H}{\lambda_H+\mu} - \beta}{\beta - \tilde{\beta}} \geq \frac{2 - \beta}{\beta - \hat{\beta}} \qquad \iff \qquad \frac{\frac{2\lambda_H}{\lambda_H+\mu} - \tilde{\beta}}{\beta - \tilde{\beta}} \geq \frac{2 - \hat{\beta}}{\beta - \hat{\beta}}. \tag{D.45}$$

Also note that

$$\frac{2\lambda_H}{\lambda_H + \mu} - \tilde{\beta} = \frac{2\lambda_H}{\lambda_H + \mu} \cdot \frac{\lambda_H - \mu}{\lambda_H + \lambda_\star} \qquad \text{and} \qquad 2 - \hat{\beta} = \frac{2\lambda_H}{\lambda_H + \lambda_\star}.$$

By plugging these in, (D.45) is equivalent to:

$$\tilde{\beta} - \frac{\lambda_H - \mu}{\lambda_H + \mu}\hat{\beta} \geq \frac{2\mu}{\lambda_H + \mu}\beta. \tag{D.46}$$

Note that

$$\tilde{\beta} - \frac{\lambda_H - \mu}{\lambda_H + \mu}\hat{\beta} = \frac{2\lambda_H(\mu + \lambda_\star)}{(\lambda_H + \mu)(\lambda_H + \lambda_\star)} - \frac{\lambda_H - \mu}{\lambda_H + \mu} \cdot \frac{2\lambda_\star}{\lambda_H + \lambda_\star} = \frac{2\mu}{\lambda_H + \mu}.$$

Therefore, (D.46) is equivalent to $1 \geq \beta$. Therefore, $\tilde{\pi}(\beta) \geq \hat{\pi}(\beta)$ holds for all $1 \geq \beta > \tilde{\beta}$ and the equality holds if and only if $\beta = 1$. □

Online Appendix for "*Strategic Concealment in Innovation Races*"

OA.1 Omitted Proofs

OA.1.1 Private Research Progress

OA.1.1.1 Proof of Lemma D.2

Proof of Lemma D.2. Let $\hat{\tau}_D$ be the arrival time of the product development by the opponent whose development rate is $\mathbf{h}$. Note that the continuation payoffs can be written as follows.

$$\begin{aligned} V_1(t;\mathbf{h}) =& \Pr[\tau_D < \hat{\tau}_D \mid \tau_M = t < (\tau_D \wedge \hat{\tau}_D)] \cdot \Pi \\ & - c \cdot \mathbb{E}[\tau_D \wedge \hat{\tau}_D - t \mid \tau_M = t < (\tau_D \wedge \hat{\tau}_D)]. \end{aligned} \tag{OA.1.1}$$

Note that (conditional) survival functions of $\hat{\tau}_D$ and τ_D can be written as follows:

$$\begin{aligned} \Pr[\hat{\tau}_D > s \mid \tau_M = t < (\tau_D \wedge \hat{\tau}_D)] =& e^{-\int_t^s \mathbf{h}(u)du}, \\ \Pr[\tau_D = \tau_H > s \mid \tau_R = t < (\tau_L \wedge \hat{\tau}_D)] =& e^{-\lambda_H(s-t)}. \end{aligned}$$

By applying (A.4) and (A.5), we have

$$\begin{aligned} \Pr[\tau_D < \hat{\tau}_D \mid \tau_R = t < (\tau_L \wedge \hat{\tau}_D)] =& \int_t^\infty \lambda_H e^{-\int_t^s (\lambda_H + \mathbf{h}(u))du} ds, \\ \mathbb{E}[\tau_D \wedge \hat{\tau}_D - t \mid \tau_R = t < (\tau_L \wedge \hat{\tau}_D)] =& \int_t^\infty e^{-\int_t^s (\lambda_H + \mathbf{h}(u))du} ds. \end{aligned}$$

By plugging these equations into (OA.1.1), we can derive that (D.22) holds.

By taking a derivative of (OA.1.1), we have

$$\begin{aligned} V_1'(t;\mathbf{h}) =& -(\lambda_H \Pi - c) \cdot e^{-\int_t^t (\lambda_H + \mathbf{h}(u))du} + (\lambda_H + \mathbf{h}(t)) \cdot (\lambda_H \Pi - c) \cdot \int_t^\infty e^{-\int_t^s (\lambda_H + \mathbf{h}(u))du} ds \\ =& -(\lambda_H \Pi - c) + (\lambda_H + \mathbf{h}(t)) \cdot V_1(t;\mathbf{h}), \end{aligned}$$

which is equivalent to (HJB$_1$). $\square$

OA.1.1.2 Proof of Lemma D.3

Proof of Lemma D.3. We focus on the event such that $(\tau_M \wedge \hat{\tau}_D) > t$. The continuation payoff can be written as follows:

$$v_0(t;\boldsymbol{\sigma},\mathbf{h}) = \Pr[\tau_D < \hat{\tau}_D \mid (\tau_M \wedge \hat{\tau}_D) > t] \cdot \Pi - c \cdot \mathbb{E}[\tau_D \wedge \hat{\tau}_D - t \mid (\tau_M \wedge \hat{\tau}_D) > t]. \quad \text{(OA.1.2)}$$

Note that

$$\begin{aligned}\Pr[\tau_M > s \mid \tau_M > t] =& \frac{S^M_{\boldsymbol{\sigma}}(s)}{S^M_{\boldsymbol{\sigma}}(t)},\\ \Pr[\tau_D > s > \tau_M > t \mid \tau_M > t] =& \int_t^s e^{-\lambda_H(s-u)} \cdot \mu\,\boldsymbol{\sigma}(u) \cdot \frac{S^M_{\boldsymbol{\sigma}}(u)}{S^M_{\boldsymbol{\sigma}}(t)} du = \frac{L_{\boldsymbol{\sigma}}(s|t)}{S^M_{\boldsymbol{\sigma}}(t)},\end{aligned}$$

where $L_{\boldsymbol{\sigma}}(s|t) \equiv \int_t^s e^{-\lambda_H(s-u)} \cdot \mu\,\boldsymbol{\sigma}(u) \cdot S^M_{\boldsymbol{\sigma}}(u) du$. Then, the survival function of τ_D conditional on $\tau_M > t$ can be written as follows:

$$S^D_{\boldsymbol{\sigma}|_t}(s) \equiv \Pr\left[\tau_D > s \mid \tau_M > t\right] = \frac{S^M_{\boldsymbol{\sigma}}(s) + L_{\boldsymbol{\sigma}}(s|t)}{S^M_{\boldsymbol{\sigma}}(t)}$$

Also note that $\Pr[\hat{\tau}_D > s \mid \hat{\tau}_D > t] = e^{-\int_t^s \mathbf{h}(u)du}$.

Observe that

$$L'_{\boldsymbol{\sigma}}(s|t) = \mu\,\boldsymbol{\sigma}(s) \cdot S^M_{\boldsymbol{\sigma}}(s) - \lambda_H \cdot L_{\boldsymbol{\sigma}}(s|t). \quad \text{(OA.1.3)}$$

Since τ_D and $\hat{\tau}_D$ are independent, we can apply (A.3) and (A.5) by resetting the initial time to t. Then, by using (D.8) and (OA.1.3), we have

$$\begin{aligned}\Pr[\tau_D < \hat{\tau}_D \mid (\tau_M \wedge \hat{\tau}_D) > t] =& -\int_t^\infty {S^D_{\boldsymbol{\sigma}|_t}}'(s) \cdot e^{-\int_t^s \mathbf{h}(u)du} ds\\ =& \int_t^\infty \frac{\lambda_L(1-\boldsymbol{\sigma}(s)) \cdot S^M_{\boldsymbol{\sigma}}(s) + \lambda_H \cdot L_{\boldsymbol{\sigma}}(s|t)}{S^M_{\boldsymbol{\sigma}}(t)} \cdot e^{-\int_t^s \mathbf{h}(u)du} ds,\\ \mathbb{E}[\tau_D \wedge \hat{\tau}_D - t \mid (\tau_M \wedge \hat{\tau}_D) > t] =& \int_t^\infty \frac{S^M_{\boldsymbol{\sigma}}(s) + L_{\boldsymbol{\sigma}}(s|t)}{S^M_{\boldsymbol{\sigma}}(t)} \cdot e^{-\int_t^s \mathbf{h}(u)du} ds.\end{aligned}$$

By plugging these into (OA.1.2) and using (D.5), we can derive that

$$v_0(t;\boldsymbol{\sigma},\mathbf{h}) = \int_t^\infty \left[\{\lambda_L(1-\boldsymbol{\sigma}(s))\Pi - c\}\cdot S^M_{\boldsymbol{\sigma}}(s) + (\lambda_H\Pi - c)\cdot L_{\boldsymbol{\sigma}}(s|t)\right]\cdot \frac{e^{-\int_t^s \mathbf{h}(u)du}}{S^M_{\boldsymbol{\sigma}}(t)} ds.$$

Thus, it remains to show that

$$\int_t^\infty \mu\,\boldsymbol{\sigma}(s)\cdot V_1(s;\mathbf{h})\cdot S^M_{\boldsymbol{\sigma}}(s)\cdot e^{-\int_t^s \mathbf{h}(u)du}\, ds = (\lambda_H\Pi - c)\cdot \int_0^\infty L_{\boldsymbol{\sigma}}(s|t)\cdot e^{-\int_t^s \mathbf{h}(u)du}\, ds. \quad \text{(OA.1.4)}$$

By plugging (D.22) into the left hand side of (OA.1.4), we have

$$\begin{aligned}&\int_t^\infty \mu\,\boldsymbol{\sigma}(s)\cdot(\lambda_H\Pi - c)\cdot\left[\int_s^\infty e^{-\int_s^u(\lambda_H+\mathbf{h}(v))dv}du\right]\cdot S^M_{\boldsymbol{\sigma}}(s)\cdot e^{-\int_t^s \mathbf{h}(v)dv}\, ds\\ =&(\lambda_H\Pi - c)\cdot\int_t^\infty L_{\boldsymbol{\sigma}}(u|t)\cdot e^{-\int_t^u \mathbf{h}(v)dv}\, du.\end{aligned}$$

Thus, (D.23) holds.

Last, to show that (HJB$_0$) holds, we multiply $S^M_{\boldsymbol{\sigma}}(t)\cdot e^{-\int_0^t \mathbf{h}(u)du}$ to (D.23) and take a derivative:

$$\begin{aligned}&-\left[\lambda_L(1-\boldsymbol{\sigma}(t))\cdot\Pi + \mu\,\boldsymbol{\sigma}(t)\cdot V_1(t;\mathbf{h}) - c\right]\cdot S^M_{\boldsymbol{\sigma}}(t)\cdot e^{-\int_0^t \mathbf{h}(u)du}\\ =&\left[v_0'(t;\sigma,\mathbf{h}) - \left(-\frac{S^{M\prime}_{\boldsymbol{\sigma}}(t)}{S^M_{\boldsymbol{\sigma}}(t)} + \mathbf{h}(t)\right)\cdot v_0(t;\boldsymbol{\sigma},\mathbf{h})\right]\cdot S^M_{\boldsymbol{\sigma}}(t)\cdot e^{-\int_0^t \mathbf{h}(u)du}.\end{aligned}$$

By using (D.8) and $S^M_\sigma(t)\cdot e^{-\int_0^t \mathbf{h}(u)du} > 0$, we can see that (HJB$_0$) holds. □

OA.1.1.3 Proof of Lemma D.4

In this subsection, we prove the verification result (Lemma D.4). To prove the verification result, it is useful to first introduce two convergence results.

Lemma OA.1.1. *For any $\boldsymbol{\sigma}\in\mathcal{S}$, the following holds:*

$$\lim_{t\to\infty} V_1(t;\mathbf{h}_{\boldsymbol{\sigma}})\cdot S^D_{\boldsymbol{\sigma}}(t) = 0.$$

Proof. Recall that $\Sigma_t := \int_0^t \boldsymbol{\sigma}(s)\, ds$. From $\lambda_H > \lambda_L$ and $\mu > \lambda_L$, we have

$$e^{-\mu t} \le S^M_{\boldsymbol{\sigma}}(t) = e^{-\lambda_L(t-\Sigma_t)-\mu\Sigma_t} \le e^{-\lambda_L t}, \tag{OA.1.5}$$

$$\begin{aligned} 0 \le L_{\boldsymbol{\sigma}}(t) &= \int_0^t \mu\, \boldsymbol{\sigma}(s) \cdot S^M_{\boldsymbol{\sigma}}(s) \cdot e^{-\lambda_H(t-s)}\, ds \\ &< e^{-(\lambda_L+\lambda_H)t} \cdot \int_0^t \mu \cdot e^{\lambda_H s}\, ds < \frac{\mu}{\lambda_H} e^{-\lambda_L t}. \end{aligned} \tag{OA.1.6}$$

Note that the left inequality of (OA.1.5) binds when $\Sigma_t = t$, and the left inequality of (OA.1.6) binds when $\Sigma_t = 0$. By (D.3), we have

$$e^{-\mu t} < S^D_{\boldsymbol{\sigma}}(t) = S^M_{\boldsymbol{\sigma}}(t) + L_{\boldsymbol{\sigma}}(t) < e^{-\lambda_L t} \cdot \left(\frac{\mu + \lambda_H}{\lambda_H}\right). \tag{OA.1.7}$$

From (D.6) and (D.22), we have

$$S^D_{\boldsymbol{\sigma}}(t) \cdot V_1(t; \mathbf{h}_{\boldsymbol{\sigma}}) = (\lambda_H \Pi - c) \cdot \int_t^\infty e^{-\lambda_H(s-t)} \cdot S^D_{\boldsymbol{\sigma}}(s)\, ds.$$

By applying (OA.1.7) and since $\lambda_H \Pi > \lambda_L \Pi > c$, we have

$$\begin{aligned} (\lambda_H \Pi - c) \cdot \int_t^\infty e^{-\lambda_H(s-t)} \cdot S^D_{\boldsymbol{\sigma}}(s)\, ds \ &> \ (\lambda_H \Pi - c) \cdot \int_t^\infty e^{-\lambda_H(s-t)} \cdot e^{-\mu s}\, ds \\ &= \frac{\lambda_H}{\mu + \lambda_H}\left(\Pi - \frac{c}{\lambda_H}\right) \cdot e^{-\mu t} \end{aligned}$$

and

$$\begin{aligned} (\lambda_H \Pi - c) \cdot \int_t^\infty e^{-\lambda_H(s-t)} \cdot S^D_{\boldsymbol{\sigma}}(s)\, ds \ &< \ (\lambda_H \Pi - c) \cdot \int_t^\infty e^{-\lambda_H(s-t)} \cdot \frac{\mu + \lambda_H}{\lambda_H} e^{-\lambda_L s}\, ds \\ &= \frac{\mu + \lambda_H}{\lambda_L + \lambda_H}\left(\Pi - \frac{c}{\lambda_H}\right) \cdot e^{-\lambda_L t}. \end{aligned}$$

Therefore, we have that

$$\frac{\lambda_H}{\mu + \lambda_H}\left(\Pi - \frac{c}{\lambda_H}\right) \cdot e^{-\mu t} < S^D_{\boldsymbol{\sigma}}(t) \cdot V_1(t; \mathbf{h}_{\boldsymbol{\sigma}}) < \frac{\mu + \lambda_H}{\lambda_L + \lambda_H}\left(\Pi - \frac{c}{\lambda_H}\right) \cdot e^{-\lambda_L t}. \tag{OA.1.8}$$

Since the lower bound and the upper bound converge to 0 as t goes to infinity, we obtain the desired result. $\square$

Lemma OA.1.2. *For any $\boldsymbol{\sigma}, \hat{\boldsymbol{\sigma}} \in \mathcal{S}$,*

$$\lim_{t\to\infty} v_0(t;\boldsymbol{\sigma},\mathbf{h}_{\hat{\boldsymbol{\sigma}}}) \cdot S^M_{\boldsymbol{\sigma}}(t) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(t) = 0. \tag{OA.1.9}$$

Proof. Note that for any time $s \in \mathbb{R}_+$, $-c < \lambda_L(1-\boldsymbol{\sigma}(s))\Pi - c < \lambda_L\Pi$. Since $\lambda_L\Pi > c$, we have $|\lambda_L(1-\boldsymbol{\sigma}(s))\Pi - c| < \lambda_L\Pi$.

From (D.23), we have

$$\begin{aligned}\left|v_0(t;\boldsymbol{\sigma},\mathbf{h}_{\hat{\boldsymbol{\sigma}}}) \cdot S^M_{\boldsymbol{\sigma}}(t) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(t)\right| < \ & \lambda_L\Pi \cdot \int_t^\infty S^M_{\boldsymbol{\sigma}}(s) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(s)\ ds \\ & + \mu \cdot \int_t^\infty V_1(s;\mathbf{h}_{\hat{\boldsymbol{\sigma}}}) \cdot S^M_{\boldsymbol{\sigma}}(s) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(s)\ ds.\end{aligned}$$

Observe that from (OA.1.5) and (OA.1.7) in Lemma OA.1.1, we have

$$\int_t^\infty S^M_{\boldsymbol{\sigma}}(s) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(s)\ ds < \frac{\mu+\lambda_H}{\lambda_H} \cdot \int_t^\infty e^{-2\lambda_L s} ds = \frac{\mu+\lambda_H}{2\lambda_L\lambda_H} \cdot e^{-2\lambda_L t}.$$

In addition, from (OA.1.8) and (OA.1.7) in Lemma OA.1.1, we have

$$\begin{aligned}\int_t^\infty V_1(s;\mathbf{h}_{\hat{\boldsymbol{\sigma}}}) \cdot S^M_{\boldsymbol{\sigma}}(s) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(s)\ ds <& \frac{(\mu+\lambda_H)^2}{\lambda_H(\lambda_L+\lambda_H)} \cdot \left(\Pi - \frac{c}{\lambda_H}\right) \cdot \int_t^\infty e^{-2\lambda_L s} ds \\ =& \frac{(\mu+\lambda_H)^2}{2\lambda_L\lambda_H(\lambda_L+\lambda_H)} \cdot \left(\Pi - \frac{c}{\lambda_H}\right) \cdot e^{-2\lambda_L t}.\end{aligned}$$

Then, we have

$$\left|v_0(t;\boldsymbol{\sigma},\mathbf{h}_{\hat{\sigma}}) \cdot S^M_{\boldsymbol{\sigma}}(t) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(t)\right| < \frac{\mu+\lambda_H}{2\lambda_L\lambda_H}\left[\lambda_L\Pi + \frac{\mu(\mu+\lambda_H)}{\lambda_L+\lambda_H}\left(\Pi - \frac{c}{\lambda_H}\right)\right] \cdot e^{-2\lambda_L t}.$$

Since the right-hand side of the above inequality converges to 0 as $t \to \infty$, (OA.1.9) holds. $\square$

In this proof, we fix the policy of the opponent at $\hat{\boldsymbol{\sigma}}$. To save on notation, we will drop the dependency of the value and survival functions on $\hat{\boldsymbol{\sigma}}$ and the opponent's development rate $\mathbf{h}_{\hat{\boldsymbol{\sigma}}}$. Specifically, we will abuse notation and use $V_1(t) \equiv V_1(t;\mathbf{h}_{\hat{\boldsymbol{\sigma}}})$, $v_0(t;\boldsymbol{\sigma}) \equiv v_0(t;\boldsymbol{\sigma},\mathbf{h}_{\hat{\boldsymbol{\sigma}}})$, $\hat{S}(t) \equiv S^D_{\hat{\boldsymbol{\sigma}}}(t)$.

Proof of Lemma D.4 ($\Longleftarrow$). From $\boldsymbol{\sigma}^*$, we have that for all $\boldsymbol{\sigma} \in \mathcal{S}$ and $t \in R_+$

$$(\boldsymbol{\sigma}^*(t) - \boldsymbol{\sigma}(t)) \cdot [\mu \cdot (V_1(t) - v_0(t;\boldsymbol{\sigma}^*)) - \lambda_L \cdot (\Pi - v_0(t;\boldsymbol{\sigma}^*))] \geq 0 \tag{OA.1.10}$$

Suppose that $v_0(t;\boldsymbol{\sigma}^*) > 0$. From (HJB$_0$), we have

$$\begin{aligned} 0 =& v_0'(t;\boldsymbol{\sigma}^*) - c - \mathbf{h}_{\hat{\boldsymbol{\sigma}}}(t) \cdot v_0(t;\boldsymbol{\sigma}^*) + \lambda_L \cdot (\Pi - v_0(t;\boldsymbol{\sigma}^*)) \\ &+ \boldsymbol{\sigma}^*(t) \cdot [\mu \cdot (V_1(t) - v_0(t;\boldsymbol{\sigma}^*)) - \lambda_L \cdot (\Pi - v_0(t;\boldsymbol{\sigma}^*))]. \end{aligned}$$

Then, (OA.1.10) implies that, for any $\boldsymbol{\sigma} \in \mathcal{S}$ and $t \geq 0$,

$$\left\{h^D_{\hat{\boldsymbol{\sigma}}}(t) + h^M_{\boldsymbol{\sigma}}(t)\right\} \cdot v_0(t;\boldsymbol{\sigma}^*) - v_0'(t;\boldsymbol{\sigma}^*) \geq \lambda_L(1 - \boldsymbol{\sigma}(t)) \cdot \Pi + \mu\,\boldsymbol{\sigma}(t) \cdot V_1(t) - c.$$

Multiplying side-by-side by $S^M_{\boldsymbol{\sigma}}(t) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(t)$, we have

$$-\frac{d}{dt}\left[v_0(t;\boldsymbol{\sigma}^*) \cdot S^M_{\boldsymbol{\sigma}}(t) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(t)\right] \geq [\lambda_L(1 - \boldsymbol{\sigma}(t)) \cdot \Pi + \mu\,\boldsymbol{\sigma}(t) \cdot V_1(t) - c] \cdot S^M_{\boldsymbol{\sigma}}(t) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(t)$$

for all $t \geq 0$. Integrating this inequality from 0 to ∞ and using Lemma D.3, we have

$$\begin{aligned} & v_0(0;\boldsymbol{\sigma}^*) \cdot S^M_{\boldsymbol{\sigma}}(0) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(0) - \lim_{t\to\infty} v_0(t;\boldsymbol{\sigma}^*) \cdot S^M_{\boldsymbol{\sigma}}(t) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(t) \\ \geq & \int_0^\infty [\lambda_L(1 - \boldsymbol{\sigma}(t)) \cdot \Pi + \mu\,\boldsymbol{\sigma}(t) \cdot V_1(t) - c] \cdot S^M_{\boldsymbol{\sigma}}(t) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(t)dt = \mathcal{U}(\boldsymbol{\sigma}, \hat{\boldsymbol{\sigma}}). \end{aligned}$$

Since $v_0(t;\boldsymbol{\sigma}^*)$, $S^M_{\boldsymbol{\sigma}}(t)$ and $S^D_{\hat{\boldsymbol{\sigma}}}(t)$ are strictly positive, we have

$$\lim_{t\to\infty} v_0(t;\boldsymbol{\sigma}^*) \cdot S^M_{\boldsymbol{\sigma}}(t) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(t) \geq 0.$$

By using this, $\mathcal{U}(\boldsymbol{\sigma}^*, \hat{\boldsymbol{\sigma}}) = v_0(0;\boldsymbol{\sigma}^*)$, and $S^M_{\boldsymbol{\sigma}}(0) = S^D_{\hat{\boldsymbol{\sigma}}}(0) = 1$, we obtain $\mathcal{U}(\boldsymbol{\sigma}^*, \hat{\boldsymbol{\sigma}}) \geq \mathcal{U}(\boldsymbol{\sigma}, \hat{\boldsymbol{\sigma}})$. □

Proof of Lemma D.4 ($\Longrightarrow$). Suppose that $\boldsymbol{\sigma}^* \in \arg\max_{\boldsymbol{\sigma}\in\mathcal{S}} \mathcal{U}(\boldsymbol{\sigma}, \hat{\boldsymbol{\sigma}})$. From Lemma D.3, observe that for any $t \geq 0$, a firm's expected payoff can be rewritten as follows:

$$\begin{aligned}\mathcal{U}(\boldsymbol{\sigma}, \hat{\boldsymbol{\sigma}}) = &\int_0^t \left[\lambda_L (1 - \boldsymbol{\sigma}(s)) \cdot \Pi + \mu\, \boldsymbol{\sigma}(s) \cdot V_1(s) - c\right] \cdot S^M_{\boldsymbol{\sigma}}(s) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(s)\; ds \\ &+ S^M_{\boldsymbol{\sigma}}(t) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(t) \cdot v_0(t; \boldsymbol{\sigma}).\end{aligned}$$

Now consider the following allocation policy $\tilde{\boldsymbol{\sigma}}(s) := \boldsymbol{\sigma}^*(s) 1_{s<t}$. Then, $S^M_{\boldsymbol{\sigma}^*}(s) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(s) = S^M_{\tilde{\boldsymbol{\sigma}}}(s) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(s)$ for all $s \leq t$.[19] In addition, by using $\sigma^*(s) = \tilde{\boldsymbol{\sigma}}(s)$ for all $s < t$ and $\mathcal{U}(\boldsymbol{\sigma}^*, \hat{\boldsymbol{\sigma}}) \geq \mathcal{U}(\tilde{\boldsymbol{\sigma}}, \hat{\boldsymbol{\sigma}})$, we have $v_0(t; \boldsymbol{\sigma}^*) \geq v_0(t; \tilde{\boldsymbol{\sigma}})$.

Note that

$$v_0(t; \tilde{\boldsymbol{\sigma}}) = \int_t^\infty (\lambda_L \Pi - c) \cdot \frac{S^M_{\tilde{\boldsymbol{\sigma}}}(s)}{S^M_{\tilde{\boldsymbol{\sigma}}}(t)} \cdot \frac{S^D_{\hat{\boldsymbol{\sigma}}}(s)}{S^D_{\hat{\boldsymbol{\sigma}}}(t)}\; ds > 0$$

from $\lambda_L \Pi > c$, $S^M_{\tilde{\boldsymbol{\sigma}}}(s) > 0$, and $S^D_{\hat{\boldsymbol{\sigma}}}(s) > 0$. Therefore, $v_0(t; \boldsymbol{\sigma}^*) > 0$ for all $t \geq 0$.

Now assume that there exists $\boldsymbol{\sigma} \in \mathcal{S}$ such that (OA.1.10) does not hold for some $t \geq 0$. Observe that $V_1(\cdot; \mathbf{h})$ and $v_0(\cdot; \boldsymbol{\sigma}, \mathbf{h})$ are continuous. Since $\boldsymbol{\sigma}^*$ and $\boldsymbol{\sigma}$ are right-continuous, there exists $\epsilon > 0$ such that for all $s \in [t, t+\epsilon)$,

$$(\boldsymbol{\sigma}^*(s) - \boldsymbol{\sigma}(s)) \cdot \left[\mu \cdot (V_1(s) - v_0(s; \boldsymbol{\sigma}^*) - \lambda_L \cdot (\Pi - v_0(s; \boldsymbol{\sigma}^*))\right] < 0. \tag{OA.1.11}$$

Consider the following allocation policy $\boldsymbol{\sigma}^{**}$ defined by:

$$\boldsymbol{\sigma}^{**}(s) := \begin{cases} \boldsymbol{\sigma}^*(s), & \text{if } s \notin [t, t+\epsilon), \\ \boldsymbol{\sigma}(s), & \text{if } s \in [t, t+\epsilon). \end{cases}$$

By using a similar reformulation as in the previous case, we have

$$\begin{aligned}&-\frac{d}{ds}\left[v_0(s; \boldsymbol{\sigma}^*) \cdot S^M_{\boldsymbol{\sigma}^{**}}(s) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(s)\right] \\ &\leq \left[\lambda_L (1 - \boldsymbol{\sigma}^{**}(s)) \cdot \Pi + \mu \sigma^{**}(s) \cdot V_1(s) - c\right] \cdot S^M_{\boldsymbol{\sigma}^{**}}(s) \cdot S^D_{\hat{\boldsymbol{\sigma}}}(s)\end{aligned} \tag{OA.1.12}$$

for all $s \geq 0$, and the inequality strictly holds for $s \in [t, t+\epsilon)$. Also note that by Lemma

[19] Note that the equality also holds at $s = t$, since $\boldsymbol{\sigma}^*$ and $\tilde{\boldsymbol{\sigma}}$ differ only at $\{t\}$, which is negligible after integration.

OA.1.2,

$$\lim_{s\to\infty} v_0(s;\boldsymbol{\sigma}^*)\cdot S^M_{\boldsymbol{\sigma}^{**}}(s)\cdot S^D_{\hat{\boldsymbol{\sigma}}}(s) = \lim_{s\to\infty} v_0(s;\boldsymbol{\sigma}^*)\cdot S^M_{\boldsymbol{\sigma}^{*}}(s)\cdot S^D_{\hat{\boldsymbol{\sigma}}}(s) = 0.$$

By integrating (OA.1.12) from 0 to ∞, we have

$$\begin{aligned}\mathcal{U}(\boldsymbol{\sigma}^*,\hat{\boldsymbol{\sigma}}) &= v_0(0;\boldsymbol{\sigma}^*)\\ &< \int_0^\infty [\lambda_L(1-\boldsymbol{\sigma}^{**}(s))\cdot\Pi + \mu\,\boldsymbol{\sigma}^{**}(s)\cdot V_1(s) - c]\cdot S^M_{\boldsymbol{\sigma}^{**}}(s)\cdot S^D_{\hat{\boldsymbol{\sigma}}}(s)\;ds\\ &= \mathcal{U}(\boldsymbol{\sigma}^{**},\hat{\boldsymbol{\sigma}}),\end{aligned}$$

which contradicts $\boldsymbol{\sigma}^* \in \arg\max_{\boldsymbol{\sigma}\in\mathcal{S}}\mathcal{U}(\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}})$. Therefore, (OA.1.10) holds for all $t \geq 0$. $\square$