Towards a Geometric Notion of Financial Market Volatility

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May 11, 2023

Abstract

Detecting anomalies and measuring volatility in financial markets is crucial for financial mathematicians, engineers, and analysts. Traditional approaches often rely on statistical moments, such as mean and variance, which may not fully capture the behavior of financial time-series data. In this paper, we introduce a novel geometric volatility method based on optimal transport theory. This distribution-aware method accounts for higher moments like skewness and kurtosis, preserving the shape of the time-series data and providing more interpretable insights into market volatility by identifying meaningful relationships between stock price distributions across different time scales. Our geometric volatility method treats each financial asset as a finite-metric measure space, with empirical distributions sampled for each market-open day. We evaluate the performance of this method by implementing an algorithmic trading strategy and comparing it to the average rate of return for a given financial asset and a traditional trading algorithm based on variance, namely Bollinger Bands. Our results, derived from the analysis of financial time-series data for some of the top 7 highest-volume traded assets in the S&P500 index over the year 2022 - 2023, demonstrate that geometric volatility significantly outperforms the traditional approach for some distribution widths and performs at least as well for others. This innovative method offers a promising alternative for capturing the intricacies of market volatility and enhancing financial market analysis.

Keywords: Optimal transport theory, Bollinger Bands

1 Introduction

The global financial markets are dynamic, complex systems that exhibit fluctuations in asset prices and trading volumes. Analyzing these markets and understanding their inherent volatility is of paramount importance for both financial mathematicians and practitioners. Financial time series data derived from asset prices are known to exhibit certain properties. For example, a traditional trading strategy employed by many financial practitioners is mean-reversion, which relies on the idea that there is an expected value that asset prices will eventually return to. Additionally, the canonical measure of volatility in financial markets has been variance. Both of these variables are relatively easy to compute, interpret and provide a straightforward way of analzying the central tendency and dispersion of the data, making them the intuitive choice for parameterizing financial time-series data. The rolling-mean of financial asset price data can help identify trends in the time-series by smoothing out short-term fluctuations and highlighting longer-term patterns; similarly, the rolling-standard deviation can help detect changes in market volatility over time, with respect to the rolling-mean. Together, these parameters have become the basis for several other technical indicators employed by financial analysts and traders. In particular, Bollinger Bands are envelopes plotted as standard deviation levels above and below a simple moving average of an asset's price data. It is employed widely by traders and analysts for decision-making purposes or trade signals.

To some extent, this reduces the analysis of financial price data to a visual exercise of pattern matching and trend-following to identify anomalies or signals in the time-series data. There are limitations to the information captured by the second moment of the stock price distribution. Higher moments such as skewness and kurtosis are not accounted for. These moments provide insights into the shape and tail behavior of the underlying distributions, which are key information for financial practitioners to identify anomalies or signals. Rolling means and standard deviations also often implicitly assume that the underlying financial time-series data follows a Gaussian distribution, which is an assumption that may not always hold, as financial data can exhibit non-Gaussian behavior such as heavy tails and asymmetry. This can limit the effectiveness of these measures in capturing the true characteristics of the data. These factors motivate the need for new measures of volatility that can preserve information from the underlying distribution's shape. Thus, the method of utilizing the Wasserstein metric, which arises from the idea of optimal transport theory, is being proposed as a new perspective to interpret volatility. A proxy for a rolling mean is determined by computing the barycenter over a sliding window of time. For a given time interval, the volatility is then measured by the distance with respect to the barycenter in the Wasserstein space. This method operates under a similar assumption of financial time-series data properties, which is that the behavior of the data will revert to some expectation after a given period of time.

The remainder of this paper is organized as follows: the remainder of Section 1 summarizes the contributions of this project and related work in the field. Section 2 introduces mathematical foundations of optimal transport and the Wasserstein metric. Section 3 motivates the use of optimal transport theory in financial time-series data. Section 4 provides details on the experimental setup and results. Sections 5 and 6 discuss results, future work, and closing remarks.

1.1 Contributions

This project's main contribution is to propose a novel method of computing market volatility based on financial time-series data. The method that this project discusses can be used to detect anomalies or implemented as an algorithmic trading strategy. Section 5 discusses the results of back-tests over the previous year from 2022 to 2023 on 7 of the highest volume traded tickers in the S&P500 index. The proposed method is based on optimal transport theory and leverages the Wasserstein metric, also known as Earth Mover's Distance (EMD). This paper aims to provide a theoretical explanation for why a financial asset can be modeled as a metric measure space with each market-open day's data representing a sampled point. The paper also provides empirical evidence validating the mathematical formulation for the proposed method.

1.2 Related Works

As of date, there is no major work to our knowledge that applies the Wasserstein metric to financial time-series data with the purpose of anomaly detection or generating trade signals. Modeling a financial market as a metric measure space with discretely sampled points is not unique. The Wasserstein metric has also been applied under this problem framework for clustering market regimes, which is the problem of detecting distinct periods in a given financial asset's time-series data that reflects a distinct underlying distribution (Horvath et al., 2021). The mathematical framework for which the proposed method is based on relies on notions of barycenters in the Wasserstein space (Agueh et al., 2010). An issue that Wasserstein distance faces is a lack of robustness, which will be discussed in more detail in Section 2.1. Researchers have proposed adaptations to Wasserstein distance to resolve some of these stability issues specifically for mathematical finance under the objective of pricing some financial asset or optimizing some expected utility (Backhoff-Veraguas et al., 2020). There has been significant work in developing flexible models that can compute dynamical Wasserstein barycenters for time-series modeling given a discrete samples (Cheng et al., 2021). The work in these fields provided a foundation for developing the mathematical framework for the proposed method in this paper.

2 Preliminaries

This section provides some preliminaries for the proposed method's mathematical formulation. It provides background on Bollinger Bands, a traditional technical indicator based on variance as well as for the Wasserstein metric.

2.1 Bollinger Bands

Bollinger Bands are a widely-used technical analysis tool in the financial industry for assessing market volatility and generating potential trade signals. They are based on the concept of rolling mean and standard deviation, providing a dynamic framework that adapts to changing market conditions. Bollinger Bands are an envelop containing 3 lines: the middle line representing a simple moving average (SMA) of a given asset's price, and the upper and lower bands which are computed by adding or subtracting a multiple of the rolling standard deviation from the middle line. For a given time t, a price P_t at t, and a rolling sequence of n discrete points, we observe that the SMA and bounds for the Bollinger Bands are given by

$$\begin{aligned} &SMA_t = (P_{t-n+1} + P_{t-n+2} + \ldots + P_t)/n \\ &UB_t = SMA_t + m \cdot \sigma_t \\ &LB_t = SMA_t - m \cdot \sigma_t \end{aligned} \tag{1}$$

The primary function of Bollinger Bands is to measure the relative position of an asset's price within a historical range of volatility. Financial practitioners commonly use Bollinger Bands to build technical indicators and chart patterns to generate corresponding trade signals. The exercise of identifying technical indicators by chart patterns and trend-following motivates the need for another measure of volatility that captures the shape of the data as well.

2.2 Wasserstein Distance

Given a set X and a σ-algebra Σ on X, define a measure $\mu : \Sigma \to \mathbb{R}^{\geq 0}$. The measure μ must contain the empty set and support countable union such that $\mu(\sqcup A_i) = \sum \mu(A_i)$. Suppose we have a fixed metric measure space (X, d_X, μ_X) and a fixed Borel σ-algebra, it follows that points from X can be sampled according to the measure, inducing a probability measure. The Wasserstein distance, also called the Kantorovich-Rubenstein metric or Earth Mover's Distance, is a distance measure in optimal transport theory that quantifies the dissimilarity between two probability distributions by considering the minimal cost of transporting mass between them. The Wasserstein distance of order p between two probability distributions μ, υ defined on a continuous and finite metric measure space, respectively (X, d_X, μ) is given by

$$\begin{aligned} &W_p(\mu, \upsilon) = \inf_C \Big(\int_{\mathbb{R}^d \times \mathbb{R}^d} d(x_i, x_j)^p \, d\pi(x_i, x_j) \Big)^{1/p}, \ \pi \in C(\mu, \upsilon) \\ &W_p(\mu, \upsilon) = \inf_C \Big(\sum_{x_i, x_j} d(x_i, x_j)^p \, \pi(x_i, x_j) \Big)^{1/p}, \ \pi \in C(\mu, \upsilon) \end{aligned} \tag{2}$$

, where the infimum is taken over all joint distributions produced by all couplings $\pi(x_i, x_j) \in C$ with marginals μ, υ. In this formulation, $d(x_i, x_j)$ represents the cost of moving mass from location (or discrete point in X) x_i to x_j, and $\pi(x_i, x_j)$ denotes the amount of mass being transported. The Wasserstein distance identifies the optimal joint distribution that minimizes the total transportation cost. There are many other popular discrepancy measures such as Kulbach-Leiller Divergence. However, it is non-symmetric, meaning it does not qualify as a metric. Wasserstein distance is in fact a metric, meaning it satisfies the properties of non-negativity, identity of indiscernibles, symmetry, and triangle inequality. The Wasserstein distance provides a natural way to incorporate the geometry of an the underlying space into the comparison of probability distributions, which can be particularly useful for capturing the structure and dynamics of financial time-series data. This motivated the case for measuring market volatility in a geometric perspective by projecting the time-series data into the Wasserstein space and computing some measure of loss for a given interval, with respect to an equivalent notion of a mean.

3 Financial Market Geometric Volatility

Our proposed method of measuring volatility applies optimal transport theory to the financial time-series data for a given asset's price. Similar to having a rolling mean and standard deviation, we substitute these indicators with a rolling Wasserstein-barycenter and corresponding loss. For this to apply, one must substantiate the claim that an asset's financial time-series data can be modeled as a finite-measure space (X, d_X, μ_X). For sampling from the space, we can apply the uniform measure to the financial time-series data. Another option is to apply the counting measure, which is defined as

$$\mu(A) = \frac{|A|}{|X|} \tag{3}$$

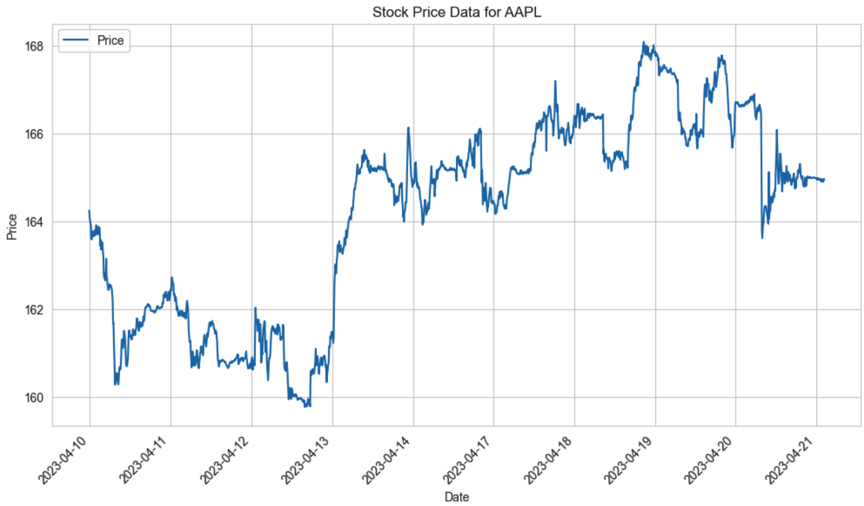
In the context of financial time-series data, X represents the collection of prices for a given financial asset over the course of time, T, defined as $X = \{x(t) \mid t \in T\}$. The set A is the collection of subsets of X that are Borel sets and the σ-algebra is generated by taking all possible unions, intersections, and complements of open intervals in the real line. These subsets can be thought of as representing price intervals or ranges in the asset's price data. Thus, in this framework, each day is modeled as a probability distribution and intervals are sampled by the corresponding measure from the metric space for a given financial asset. If the uniform measure is applied then all intervals are sampled equally; if the counting measure is applied, the size of each subset A and the total set X is dictated by the trade volume over the interval. This is a consequence of discretized financial time-series data by intervals of t, where t could be 5 minutes, 30 minutes, 1 hour, 1 day, etc. Each day of price data is regarded as an empirical distribution. Now, to compute a notion of loss for the volatility measure, we must identify an equivalent notion of a mean. If we induce each $x \in X$ to a probability measure and project it to the Wasserstein space, the problem at hand is to identify the distance minimizer or centroid. We can compute the Wasserstein barycenter, which is the probability distribution μ that minimizes

$$J(\mu) = \frac{1}{N} \sum_{j=1}^{N} W_p(\mu, \upsilon_j) \tag{4}$$

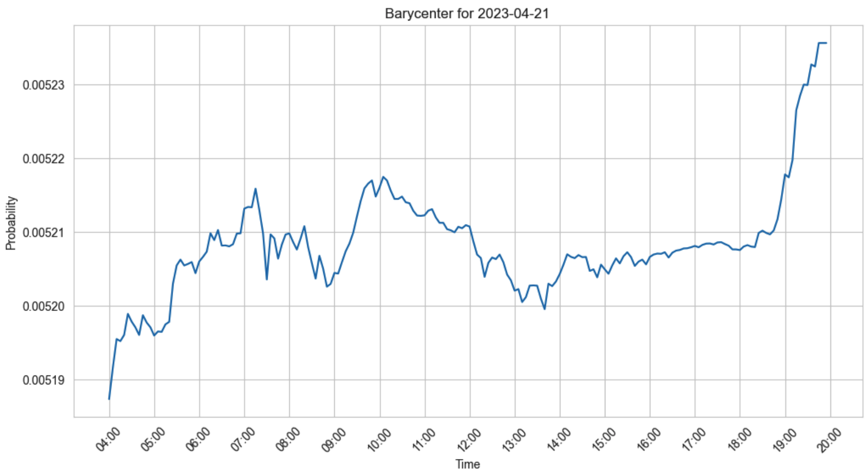
Thus, for a given period size window d, we collect d points $x \in X$, project each of them to the Wasserstein space and compute the d-rolling barycenter. This effectively yields the "average" shape data for a given window of width d. For each subsequent interval of time of size w in the current day, the w-width slice of the current day is compared to the corresponding w-width slice of the (re-normalized) d-rolling barycenter. The volatility is computed by finding the Wasserstein distance between these two w-width distributions. For example, if the current moment is 10:00 and $w = 6$ for 5 minute interval data, then we look at the previous 30 minutes to produce a distribution ranging from 09:30 to 10:00. The same corresponding slice of time from the d-rolling barycenter for 09:30—10:00 is re-normalized and used to compute the Wasserstein distance. Thus, we have the following formulation for geometric volatility

$$GV\left(\mu_w(t)\right) = W_p\left(J_d,\ \mu_w(t)\right),\ |\mu_w| = w \tag{5}$$

where $GV\left(\mu_w(t)\right)$ is our proposed measure of volatility, represented as the Wasserstein distance for a current distribution $\mu_w(t)$ at current time t with w prior points, compared against the d-rolling Wasserstein barycenter. The figures below capture this idea, where the first figure shows the time-series data spanning a 10 day window and the subsequent figure shows the Wasserstein barycenter projected as Euclidean 1-d.



(a) Stock price data over 10 days from 2023-04-10 to 2023-04-21



(b) Wasserstein barycenter projected to 1-d Euclidean for 10-day rolling window

Fig. 1: Comparison of 10-day rolling window to corresponding Wasserstein barycenter

4 Experiments

In this project, we propose a new method of measuring asset price volatility that employs a geometric notion by projecting financial time-series data to the Wasserstein space and computing the barycenter and loss. To evaluate the performance of this measure, we compare it to its traditional counterpart, Bollinger Bands. As mentioned before, Bollinger Bands utilizes the second moment (variance) to construct bounds for triggering trade signals. The experiments ran on simulated data from 2022–2023 and included 7 of some of the highest volume traded stocks in the S&P500 index. The companies included in the experiments were Apple (AAPL), Amazon (AMZN), Google (GOOGL), Nvidia (NVDA), AT&T (T), Tesla (TSLA), and Uber (UBER). The data was queried via the free API available at Alpha Vantage. The data collected included the following features in 5 minute intervals (albeit some data was missing for intervals sporadically throughout the data set): date, time, open, high, low, close, and volume. The price used to represent each interval was the simple average of the high and low. For the rolling day averages, both a 10 day and 30 day window was evaluated. The size of the sliding window used to induce a probability distribution and compute geometric volatility, i.e. distance from its corresponding barycenter, was 30 minutes ($w = 6$).

For the simulations, a uniform measure was assumed. When computing the rolling Wasserstein barycenter, the time-series data for the stock's price was converted to a probability distribution by applying the soft-max activation function. Because some data for periods throughout the day for different days was missing from the data set, the distribution sizes among days were different lengths. The distributions were padded to maximum length of 192, which is the most number of open-market 5 minute intervals available. The identity matrix was used as the cost basis, a regularization term of 0.001 was applied, with uniform weights between samples. We compute the geometric volatility for every point (represented by a 5 minute interval) by finding the Wasserstein distance between the current w-width of time and the w-width of the corresponding date's Wasserstein barycenter, which was re-normalized by $J_i = \frac{j_i}{\sum_i^w j_i}$. An alternative to using a sliding window of w-width was considered: comparing a cumulatively growing window of the current day's distribution to the barycenter. However, in practice, we found that the time complexity of computing the Wasserstein distance between the growing distribution sizes was unsatisfactorily large.

We back-test the performance of a trade signal based on our proposed geometric volatility. It is based on an ϵ-net in the Wasserstein space with respect to the corresponding barycenter. **A buy signal is triggered when the distance is greater than or equal to ϵ and the current price is lower than its rolling mean. A sell signal is triggered similarly but when the current price is higher than its rolling mean**. We determine ϵ on a per-asset basis by iterating over a scale range because the shape of each stock's price data is unique to itself. The back-test simulation begins with $C = \$10,000$ and $S = 0$ shares of the asset. We pre-compute the buy-sell signals for a given stock. We then iterate over the full time-series data: if there is a buy signal and the remaining liquid capital $C > 0$, a buy signal is executed by deducting $r = 0.01$ from C and incrementing S by $r \cdot C/P(t)$, where $P(t)$ is the current price. A similar procedure is applied for sell signals: sell trade executes if $S > 0$, capital is incremented by $r \cdot S \cdot P(t)$, and S deducted by $r \cdot S$. The final equity value, F, is determined by $F = C + S \cdot P(t_n)$, where $P(t_n)$ is the real-time, current market value for the asset.

5 Results

Our simulated back-tests found that our proposed measure of volatility significantly outperformed both the average rate of return for each financial asset as well as the traditional indicator, Bollinger Bands, on a 10 day rolling basis for 6 of the 7 assets, with the last asset only narrowly beating the proposed method. The performances were more comparable for the 30 day rolling basis; however,

Table 1: Average rate of return and ϵ values used for each stock

Ticker	Avg. RoR	$\epsilon_{d=10}$	$\epsilon_{d=30}$
AAPL	5.43%	0.12	0.1
AMZN	-23.93%	0.1	0.14
GOOGL	-9.13%	0.1	0.05
NVDA	44.17%	0.2	0.2
T	-0.22%	0.008	0.008
TSLA	-44.36%	0.25	0.25
UBER	-2.77%	0.04	0.04

we suspect that it could be a matter of hyper-parameter tuning to close this gap, since the settings were designed for a more local scale. Regardless, the performance was at least as comparable. Table 1 summarize the average rate of returns for the studied assets. It also shows the corresponding ϵ threshold values used to trigger signals for our method. They are separated by $d = 10$ and $d = 30$ day rolling windows.

In Tables 2 and 3, the results for the back-test simulations are summarized. The results are separated by rolling window size ($d = 10$ or $d = 30$). For both, the same size window $w = 6$ was used to project the time-series data to the Wasserstein space.

Table 2: Back-test simulation results for $d = 10$ rolling Barycenter

	Bollinger Bands[1]				Geometric Volatility[2]			
Ticker	Capital	Equity	Total	Profit	Capital	Equity	Total	Profit
AAPL	**\$10,705**	**\$294**	**\$10,998**	**9.98%**	\$7,227	\$3,596	\$10,822	8.22%
AMZN	\$8,404	\$187	\$8,591	-14.09%	**\$6,515**	**\$3,728**	**\$10,243**	**2.43%**
GOOGL	\$10,430	\$0	\$10,430	4.30%	**\$7,370**	**\$4,403**	**\$11,773**	**17.73%**
NVDA	\$11,936	\$386	\$12,322	23.22%	**\$8,541**	**\$4,669**	**\$13,210**	**32.10%**
T	\$743	\$8,715	\$9,458	-5.42%	**\$6,537**	**\$3,775**	**\$10,312**	**3.12%**
TSLA	\$306	\$5,799	\$6,105	-38.95%	**\$3,857**	**\$4,580**	**\$8,437**	**-15.63%**
UBER	\$3,788	\$8,259	\$12,047	20.47%	**\$5,301**	**\$7,395**	**\$12,695**	**26.95%**

Note: The bold rows mean that the resulting profit margin was higher.

[1]The multiplier was consistent $m = 2.5$ and determined by standard industry level.

[2]The scale for ϵ was adjusted per-asset. Values are shown in Table 1.

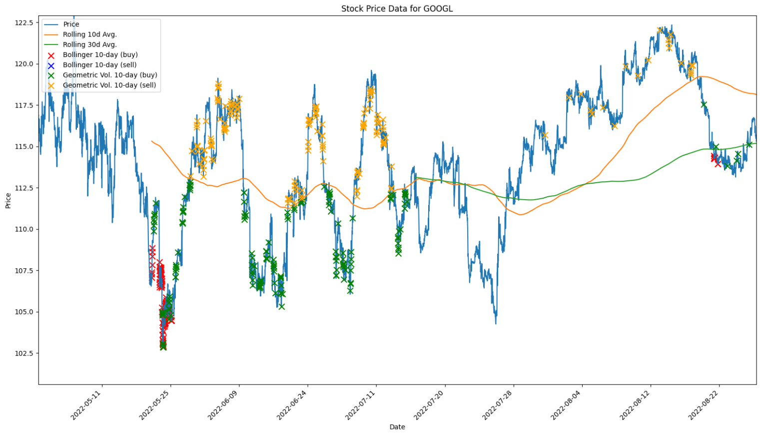
Table 3: Back-test simulation results for $d = 30$ rolling Barycenter

	Bollinger Bands[1]				Geometric Volatility[2]			
Ticker	Capital	Equity	Total	Profit	Capital	Equity	Total	Profit
AAPL	\$10,281	\$600	\$10,881	8.81%	**\$10,512**	**\$654**	**\$11,166**	**11.66%**
AMZN	**\$9,421**	**\$1,720**	**\$11,141**	**11.41%**	\$7,419	\$2,361	\$9,780	-2.20%
GOOGL	**\$10,446**	**\$1,002**	**\$11,448**	**14.48%**	\$11,305	\$0	\$11,305	13.05%
NVDA	**\$14,024**	**\$190**	**\$14,214**	**42.14%**	\$11,164	\$2,212	\$13,375	33.75%
T	\$9,906	\$209	\$10,115	1.15%	**\$5,905**	**\$4,903**	**\$10,808**	**8.08%**
TSLA	\$958	\$6,248	\$7,206	-27.94%	**\$3,717**	**\$4,987**	**\$8,704**	**-12.96%**
UBER	**\$14,142**	**\$184**	**\$14,326**	**43.26%**	\$3,051	\$10,540	\$13,592	35.92%

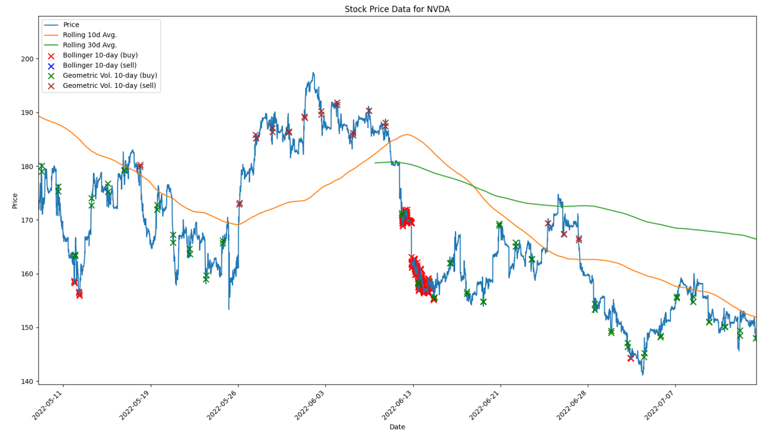
Note: The bold rows mean that the resulting profit margin was higher.

[1]The multiplier was consistent $m = 2.5$ and determined by standard industry level.

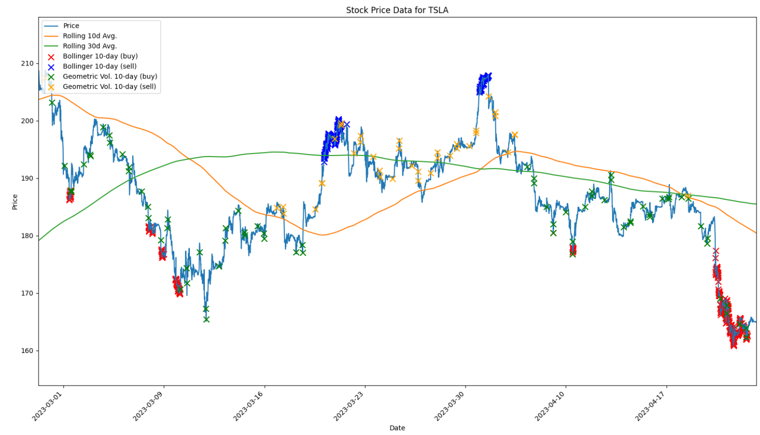
[2]The scale for ϵ was adjusted per-asset. Values are shown in Table 1.



(a) Google asset anomaly detection and trade signals

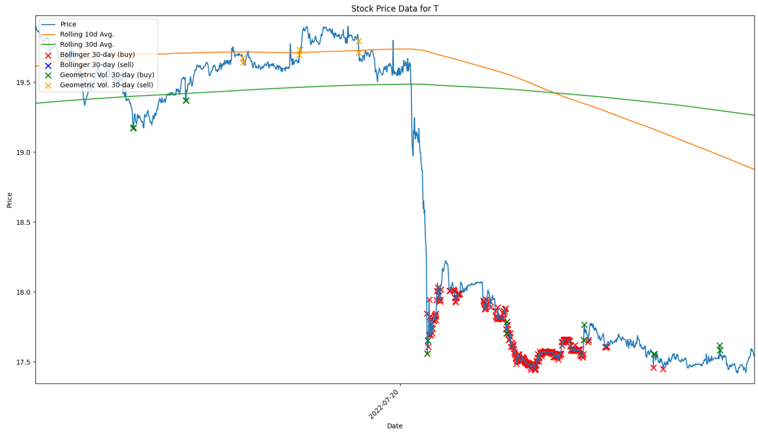


(b) Nvidia asset anomaly detection and trade signals

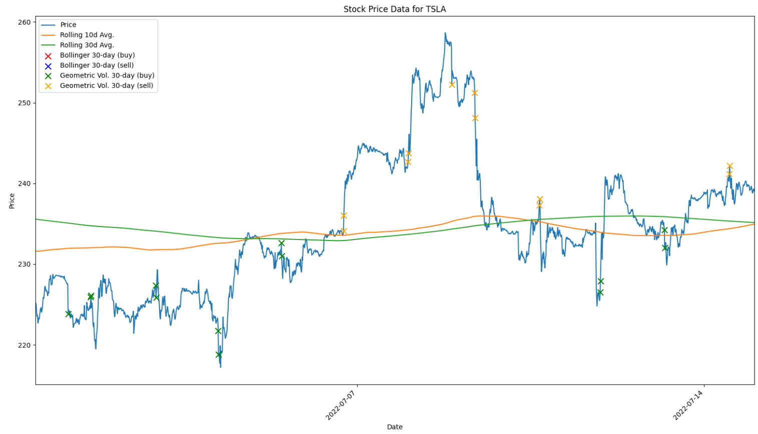


(c) Tesla asset anomaly detection and trade signals

Fig. 2: Examples of trades signals for both Bollinger Bands and Geometric Volatility using a 10-day rolling window



(a) AT&T asset anomaly detection and trade signals



(b) Tesla asset anomaly detection and trade signals

Fig. 3: Examples of trade signals for both Bollinger Bands and Geometric Volatility using a 30-day rolling window

Because each stock in itself is its own measure space, meaning the nature of sampled points and distances in it can be different from those of another stock, the ϵ used to obtain the results in Tables 2 and 3 varied. Having a lower ϵ as the threshold implies that the asset's time-series data exhibited higher similarity between days, i.e., the points are more densely packed around the Wasserstein barycenter. Additionally, as d increases, the Wasserstein barycenter becomes diluted with samples and the spread of data is likely to increase. This can cause a lower threshold ϵ required. The ϵ values used to obtain the results in Tables 2 and 3 are summarized in Table 1. Identifying trades often regresses to a visual exercise and pattern matching game, as mentioned before. We wanted to validate the behavior of the proposed method with respect to the financial time-series data. Figure 2 captures slices from the past year (with 10-day rolling window) to demonstrate that our proposed method successfully triggers at localized peaks and troughs in the data, i.e. it operates as an anomaly detector. The red and blue marks on the plots represent trade signals generated by Bollinger Band bounds and the green and yellow marks represent trade signals generated by our proposed method. Note that our proposed method's signals occur at the localized peaks and troughs for each time interval, correctly corresponding to incidents of different probability distribution shapes and thus opportunities to capitalize on. We also provide examples of our proposed method's trade signals for a 30-day rolling window in Figure 3.

6 Discussion and Future Work

The results from our experiments demonstrate that our proposed method outperforms both the average rate of return and the canonical measure of volatility, variance. This can first be seen from the figures providing examples of the trade signals generated by both volatility measures. In general, a satisfactory interpretation of volatility should be able to identify incidents of anomalies in the financial time-series data. In Figures 2 and 3, one can see that the trade signals produced by our geometric volatility measure concentrate around regions of atypical behavior. For example, in Figure 3b, the buy and sell trade signals generally lie on the local peaks and troughs of the data, allowing for rapid, optimal entry and exit opportunities. By identifying these local, optimal entry points, our method is able to take advantage of more opportunities; its behavior has higher spread than the concentrated trade entries and exits of the Bollinger Bands. Furthermore, this substantiates our proposition that financial assets can be modeled as a metric measure space using the Wasserstein distance as the metric between sampled points. Regarding the back-test results for the 30 day rolling window, we hypothesized that the performance decreased compared to the 10 day because the barycenter captured more sample points. The current window of w-width that is compared to the rolling Wasserstein barycenter captures very localized information regarding the shape of the time-series data. Thus, with respect to this granular information, the distance to the rolling 30 day barycenter fails to capture as much accuracy. A solution to this problem could be to increase w when comparing the current distribution to the barycenter.

Despite the positive outcomes from these experiments, there is plenty of future work available to improve and further develop our key claim: financial assets can be modeled as a metric measure space using Wasserstein distance as the metric. We identified the scale for ϵ by spanning over a range of distances for a given financial asset's time-series data. However, there is an opportunity to develop a heuristic that could compute a satisfactory scale ϵ. We did not do any hyper-parameter tuning to find optimal results, so the experiment results could have an even larger disparity. Related works have developed new techniques of computing dynamic Wasserstein barycenters designed to resolve the stability issue (Muskulus et al., 2009). Implementing these techniques to compute a more stable, dynamic Wasserstein barycenter as the mean proxy could also improve performance as these techniques were designed for continuous, time-series data. Lastly, these experiments provided evidence supporting the idea that financial asset data can be modeled as some metric measure space. There is an opportunity to apply topological data analysis (TDA) to this problem setup. This could be done by projecting each distribution to the Wasserstein space, constructing a simplicial complex from the point cloud, spanning different scales ϵ to explore connectivity properties such as the Betti number to identify different dimension holes in the data. If there are loops in the simplicial complex, this could potentially suggest that there are cyclic behaviors in the financial time-series data based on the shape of the distributions. Following our problem formulation of an asset as a measure space, distances between different financial assets could be explored by applying Gromov-Wasserstein distance, a metric between measure spaces.

7 Conclusion

In conclusion, this study has presented a novel approach to measuring financial market volatility by modeling financial assets as a metric measure space using the uniform or counting measure and the Wasserstein metric. By treating each day's data as a sample from this space, we have developed a method for quantifying volatility through projecting sample points onto the Wasserstein space, computing a rolling d day barycenter, and measuring the distance to this minimizer. Our proposed method was rigorously tested by implementing a trading algorithm based on the geometric volatility concept and back-testing it on 2022–2023 data for seven of the highest-volume stocks. The results of this analysis provided strong evidence supporting our hypothesis, as our method consistently outperformed the traditional measure of volatility, variance. This success not only demonstrates the potential of our approach but also opens up new avenues for further research and applications in the realm of financial market analysis.

Acknowledgements

I want to thank Professor Blumberg and the TA, Pedro Lazcano, for providing an academically enriching semester. I thoroughly enjoyed the material, philosophical viewpoints on data analysis, and historical connections in lecture.

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